THE TEACHING OF MATHEMATICS

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Random Triangles

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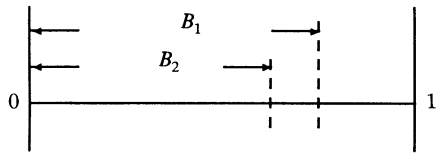
A question similar to the one that follows occurred on the final examination of a second-year course in probability at the University of Birmingham. One wonders if the setter got it right.

A chimpanzee named Euclid is teaching himself the geometry of the triangle by notching thin, straight jungle twigs with his finger nail at two independent random points, breaking the twig into three at these points, and then noting if it is possible to construct a triangle from the fragments. Nearby Pythagoras, a colleague of Euclid's, is learning geometry in a similar fashion; his procedure is the same except that after making the first notch he breaks the twig into two and then etches the second notch on the longer of the two resulting portions. After breaking this, he too tries to construct a triangle from the three fragments.

Archimedes, a rather learned chimpanzee, strolls through the jungle clearing and observes his two companions. After a moment's reflection he comments, "Pythagoras, you are 4/3 times as likely as Euclid to succeed in constructing a triangle."

Geometrically or otherwise evaluate the two probabilities and say whether Archimedes's calculation is correct.

Analysis. Consider an idealized twig of unit length and random break-points B_1, B_2 measured from one end of the twig.



The sample space for this experiment can be parameterized by the set of all pairs of break-points, (b_1, b_2), falling in the unit square (see Figure 1). A success, where a triangle is possible, occurs if and only if the sum of any two sides is greater than the third. When $b_2 < b_1$, the lengths of the three sides are b_2, $b_1 - b_2$ and $1 - b_1$, and a success implies: (i) $b_2 + b_1 - b_2 > 1 - b_1$, so that $b_1 > 1/2$; (ii)$b_2 + 1 - b_1 > b_1 - b_2$ so that $b_1 - b_2 < 1/2$; and (iii) $b_1 - b_2 + 1 - b_1 > b_2$, so that $b_2 < 1/2$. These three linear inequalities determine one of the two triangular regions enclosed by the dashed lines in Figure 1(a). The other is determined by consideration of the

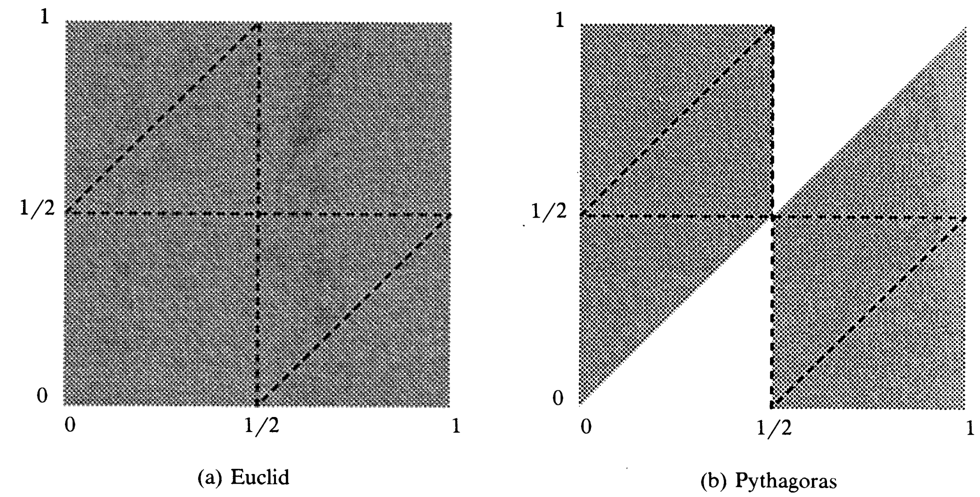


FIG. 1. The sample space.

case $b_2 > b_1$. Exterior to the dotted triangles, the square regions are failures as the two notches fall in the same half of the twig, and the triangular regions are failures as the two notches are too far apart. For Euclid, all points in the square are possible, and by symmetry equal areas are equally probable, so that his probability of success is $1/4$. Now Pythagoras's sample space is a subset of Euclid's modified to exclude the region corresponding to making the second break in the shorter fragment. So from Figure 1(b), Pythagoras's success probability is $1/3$ and Archimedes is right.

But there is a twist to the tale: under Pythagoras's sampling scheme, equal areas are *not* equally probable. To see this, suppose they are and calculate the areas in the vertical strips of width δ located on the horizontal axis at two points $0 < \alpha < \beta < 1/2$. From Figure 1(b) the probability $P(B_1$ lies in $\alpha \pm \delta/2)$ is greater than $P(B_1$ lies in $\beta \pm \delta/2)$, but this violates the uniform distribution of B_1 along the twig.

To ensure a uniform distribution we have to assign a larger probability to the shorter interval: so given $B_1 = b_1$ ($> 1/2$) say, then B_2 is uniformly distributed on the interval $(0, b_1)$ and not the interval $(0, 1)$. Hence

$$P\left(B_2 \text{ lies in } \alpha \pm \frac{\delta}{2} | B_1 = b_1 \right) = \frac{\delta}{(b_1 - 0)}$$

and the density function is illustrated in Figure 2. Consequently Pythagoras's success probability is

$$2\int_{b_1=1/2}^{1} \left\{ \int_{b_2=b_1-1/2}^{1/2} \frac{1}{b_1} db_2 \right\} db_1$$

which simplifies to

$$2\int_{1/2}^{1} \frac{1}{b_1}(1 - b_1)\, db_1 = 2[\log b_1 - b_1]_{1/2}^{1} = 2\log 2 - 1 = 0.38\ldots$$

and not $0.33 \cdots$, Archimedes's analysis was naive.

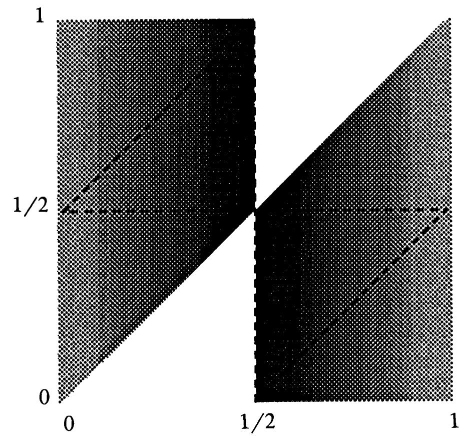


FIG. 2. Unequal probabilities. Pythagoras density function.

A New Extension of the Derivative

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How, in an elementary setting, can we go beyond calculating one example after another to learn about differentiation? Can we see what differentiation *is* rather than just how it *works*? What are the vital properties that an operator must have to be called a "derivative?" One way to consider these questions is to construct extensions of the derivative. In checking whether a proposed extension is what we want it to be, we learn more than we could by just doing examples. In this note we examine one such extension. The motivation for our definition is very simple, and might form the basis of other, similar experiments.

The relationship of the graph of a function to its tangent line is described in familiar words: "Look closer and closer at the point of tangency, and the curve and its tangent line look more and more alike." Most methods of extending the derivative retain the difference quotient as the measure of how much a curve "looks like" its tangent line, and proceed by manipulating the way the limit of the difference quotients is computed (see e.g., **[1, 2]**).

We take a different approach: Reinterpret the phrase "The curve looks like the line." There are many ways in which a set of points can be said to "look like" a line. The one given by the ordinary definition of the derivative may not be the most natural. On the other hand, most of us can recognize the similarity of two sets of points, even as we appreciate the difficulties in defining it.

One way in which a set of points may be said to look like a line is in the sense of least squares. Usually used to find the line that best fits a finite set of points, the