Fractals with Positive Length and Zero Buffon Needle Probability

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1. INTRODUCTION. The classical definition of the length of a curve, based on polygonal approximation, is not useful for measuring the "length" of more complicated sets. For a Borel (e.g., closed or open) subset F of $\mathbb{R}$, the Lebesgue measure $|F|$ of F, which is defined by

$$|F| = \inf\left\{\sum_i |I_i| : F \subset \bigcup_i I_i\right\} \tag{1}$$

using countable covers by intervals, works well (here $|I|$ signifies the length of an interval I). This definition was extended by Carathéodory [5] as follows.

Definition 1. For a Borel set F in $\mathbb{R}^d$ let

$$\mathcal{H}^1(F) = \lim_{\varepsilon \to 0} \inf\left\{\sum_i \operatorname{diam}(B_i) : F \subset \bigcup_i B_i, \ \operatorname{diam}(B_i) \le \varepsilon\right\},$$

where the infimum is extended over all countable covers of F by closed sets B_i of diameter at most ε.

It turns out that $\mathcal{H}^1$ is a measure, now called *one-dimensional Hausdorff measure* because it was generalized by Hausdorff [10] to the whole family of measures $\mathcal{H}^\alpha$, where α is any positive number (integer or noninteger). The modern theory of "fractals" is largely based on the notion of the *Hausdorff dimension* $\dim_H(F)$ of a set F, defined by $\dim_H(F) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(F) = 0\}$. We recommend the book by Falconer [8] for an introduction to fractals.

Here we consider only $\alpha = 1$, since we are interested in the generalizations of length. If F is a rectifiable curve, then $\mathcal{H}^1(F)$ is exactly its length; therefore in modern analysis, it is standard to refer to $\mathcal{H}^1(F)$ for any compact set F as the "length of F."

Another way to measure length goes back even further, to the eighteenth century. In 1733 Georges-Louis Leclerc, the "Comte de Buffon," posed the following problem, which became known as "Buffon's needle problem": *Given a collection of parallel lines in the plane, with distance d between adjacent lines, determine the probability that a needle of length $\ell < d$ will cross one of these lines when dropped at random on the plane.* The answer $2\ell/(\pi d)$ was given by Buffon himself in 1777 and can be found in many probability texts (for example, in [9]). It follows from Buffon's formula that if a polygon of perimeter p and diameter less than d is dropped on the same plane, then the expected number of points at which it will cross one of the parallel lines is $2p/(\pi d)$. This idea was formalized and extended in 1868 by Crofton (see [6] and [19]), as follows. Count the number of intersections of a given set F with a straight line, and then integrate this number over the space of all lines. The result is denoted by $I_1(F)$ and called the *integral-geometric measure* of F. More precisely, let Π_θ be

the orthogonal projection onto the line L_θ passing through the origin that makes an angle θ with the horizontal. Then

$$I_1(F) = \int_0^\pi \int_{L_\theta} \operatorname{card}(F \cap \Pi_\theta^{-1}(y)) \, dy \, d\theta.$$

When F is a rectifiable curve, $I_1(F)$ is twice its length (see [18]), but for other compact sets in the plane $I_1(F)$ and $2\mathcal{H}^1(F)$ can differ sharply; in the next section we provide examples for which $\mathcal{H}^1(F) > 0$, whereas $I_1(F) = 0$.

It is easy to see that $I_1(F) = 0$ if and only if $|\Pi_\theta(F)| = 0$ for almost all θ in $[0, \pi)$. Another notion of "length" in the Euclidean plane, called *Favard length* (see [3]), is defined for a Borel set F by

$$\operatorname{Fav}(F) = \int_0^\pi |\Pi_\theta(F)| \, d\theta.$$

Thus $\operatorname{Fav}(F) = 0$ if and only if $I_1(F) = 0$. Consequently, to say that a set has "zero Buffon needle probability" is another way of saying that almost every projection of the set onto a line has zero Lebesgue measure.

2. SELF-SIMILAR FRACTALS IN THE PLANE AND THEIR PROJEC-TIONS.
There is no universally agreed upon definition of a "fractal," but "we know one when we see one." The sets that we discuss in this article are certainly fractals, but they would be excluded if we used the naive definition of a fractal as a set having nonintegral Hausdorff dimension (our sets in the Euclidean plane have dimension one). When we say "fractal" we usually have several properties in mind, such as "fine structure on arbitrarily small scales" and some form of "self-similarity" (see [7] or [8]). The latter term will be used here in the following precise sense: a compact set in Euclidean space is said to be *self-similar* if it can be represented as a finite union of (not necessarily disjoint) sets, each of which can be obtained from the whole set by scaling (with a factor less than one), followed by an isometry.

The best known fractal is probably the *Cantor middle-third set*. We consider a variant of the Cantor set construction where at each stage we remove middle halves of intervals rather than their middle thirds.

Start with the unit interval $F_0 = [0, 1]$. Remove the (open) middle half—resulting in $F_1 = [0, 1/4] \cup [3/4, 1]$. Then repeat the process removing the middle half of each of the intervals that remain (see Figure 1). At stage n we get a set F_n that is the union of 2^n intervals of length 4^{-n}. These sets are nested: $F_0 \supset F_1 \supset F_2 \supset \ldots$. Their intersection
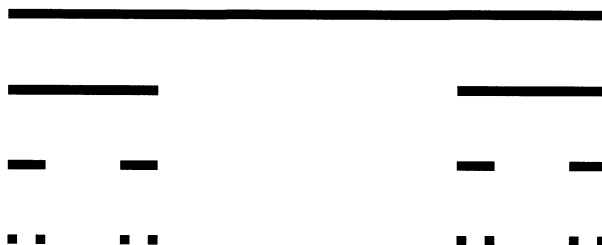

Figure 1. Construction of the Cantor middle-half set.

$$K = \bigcap_{n=0}^{\infty} F_n$$

(nonempty by the nested intersection theorem) is called the *Cantor middle-half set.*

Observe that $|F_n| = (1/2)^n \to 0$, so $|K| = 0$. Thus, the length measure is not very useful when applied to the set K. (It turns out that the 'natural' measure on K is the $1/2$-dimensional Hausdorff measure, but we do not discuss this here.)

Notice also that the Cantor middle-half set is self-similar in the sense defined earlier. Indeed, the recursive nature of the construction allows us to write

$$K = \tfrac{1}{4}K \cup \left(\tfrac{1}{4}K + \tfrac{3}{4}\right). \tag{2}$$

Iterating formula (2) leads to a more explicit expression for the Cantor middle-half set in terms of *expansions in base 4*. In fact, K is the set of all numbers in $[0, 1]$ admitting base 4 expansions in which the only allowable digits are 0 and 3 (or 0 and $3/4$, depending on whether we elect to index the series that gives the expansion starting with $n = 1$ or $n = 0$). Formally,

$$K = \left\{ \sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0, 3\} \right\} = \left\{ \sum_{n=0}^{\infty} a_n 4^{-n} : a_n \in \{0, 3/4\} \right\}. \tag{3}$$

More generally, a self-similar set K in the real line can be defined by a *set equation*

$$K = \bigcup_{i=1}^{m} (rK + d_i), \tag{4}$$

where r in $(0, 1)$ is called the *contraction ratio* and the m numbers $d_1, \ldots, d_m$ are said to be *admissible digits*. A nonempty compact subset of $\mathbb{R}$ satisfying (4) is *unique* [**8**, p.114] and can be expressed as follows:

$$K = \left\{ \sum_{n=0}^{\infty} a_n r^n : a_n \in \{d_1, d_2, \ldots, d_m\} \right\}. \tag{5}$$

(It seems natural to assume that the digits d_i are all distinct. However, if we have a *family* of self-similar sets depending on parameters, then for some parameter values two or more digits may coincide, so we do not preclude such a possibility.) In fact, we do not have to restrict ourselves to the real line—everything works the same way in the plane and in higher dimensions. The only change that needs to be made is to allow the digits in (4) and (5) to be *vectors*. Of particular interest to us will be the set $K^2 = K \times K$, where K is the Cantor middle-half set. It can be described directly by a Cantor-like construction in which we start with the unit square $[0, 1]^2 = [0, 1] \times [0, 1]$ at stage 0, and at stage n replace each of the squares from the previous stage by four corner squares of side-length 4^{-n} (see Figure 2). The alternative "digit" description of the set K^2 is:

$$K^2 = \left\{ \sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \left\{ \binom{0}{0}, \binom{0}{3}, \binom{3}{0}, \binom{3}{3} \right\} \right\}. \tag{6}$$

We remark that the "four corner" set K^2 is a union of four pieces, each of which is a translate of K^2 scaled by a factor of $1/4$. This self-similar structure suggests that K^2 is
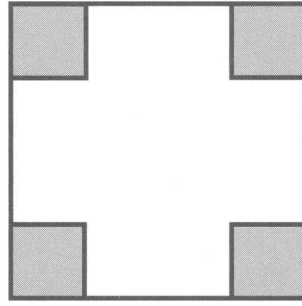
Figure 2. Construction of the "four corner" set.

a *one-dimensional* set. Indeed, one can show that $0 < \mathcal{H}^1(K^2) < \infty$. The upper bound follows easily from the definition of $\mathcal{H}^1$, since K^2 is covered by 4^n squares of diameter $\sqrt{2} \cdot 4^{-n}$ for each n. The lower bound can be derived by considering the probability measure μ on K^2 such that $\mu(K^2 \cap Q) = 4^{-n}$ for each of the 4^n squares Q arising at stage n of the construction (see [8, 1.3] for a justification that such a measure exists). This measure satisfies $\mu(B) \leq 9\,\mathrm{diam}(B)$ for any closed set B, since B can be covered by at most nine dyadic squares with side-length smaller than $\mathrm{diam}(B)$. Therefore, for any cover $\{B_i\}$ of K^2 by disks, we have $\sum_i \mathrm{diam}(B_i) \geq \sum_i \mu(B_i)/9 \geq \mu(K^2)/9 = 1/9$. For more details, we refer the reader to [8, chap. 4].

We would like to understand the *projections* of self-similar sets in the plane. Again, let $\Pi_\theta : R^2 \to \mathbb{R}$ denote the orthogonal projection on the line L_θ through the origin that makes an angle θ with the horizontal. It is more convenient, however, to consider all projections to be subsets of $\mathbb{R}$. To this end, we let

$$P_\theta \begin{pmatrix} x \\ y \end{pmatrix} = x\cos\theta + y\sin\theta \tag{7}$$

for θ in $[0, \pi)$. Clearly, $\Pi_\theta = R_\theta P_\theta$, where R_θ is the rotation through angle θ, so $|P_\theta(\Lambda)| = |\Pi_\theta(\Lambda)|$ for any set Λ of the plane. If Λ is a self-similar set in the plane given by (4) or (5), but now with vector digits $\mathbf{b}_i$, then $P_\theta(\Lambda)$ is a self-similar set in $\mathbb{R}$ with the same contraction ratio r and with digits $P_\theta(\mathbf{b}_1), \ldots, P_\theta(\mathbf{b}_m)$. (Observe that some of the digits for the projection may coincide even if all the vector digits are distinct.) However, these projected sets are often very complicated because of possible *overlap* (see Figure 3). In general, they can no longer be described via an iterative process of "removals," as was the case for the Cantor middle-half set. Figure 3 shows the projection of the *first stage* of the construction of the set K^2. When we iterate the construction, the interplay between "gaps" and "overlaps" becomes quite involved. (Note that in Figure 3 we translated the set K^2 away from the origin in order to make the picture more illuminating. This just translates the projections.)

Some projections of K^2 are easy to describe. Indeed, $P_0(K^2) = K$ is the Cantor middle-half set, hence $|P_0(K^2)| = 0$. Also, $P_{\pi/2}(K^2) = K$, so $|P_{\pi/2}(K^2)| = 0$. This observation has a remarkable consequence, in view of the following theorem of Besicovitch [2].

Theorem 2 (Besicovitch). *Let F be a compact subset of $\mathbb{R}^2$ with the property that $0 < \mathcal{H}^1(F) < \infty$. If two distinct projections of F have zero length, then almost every projection of F has zero length.*
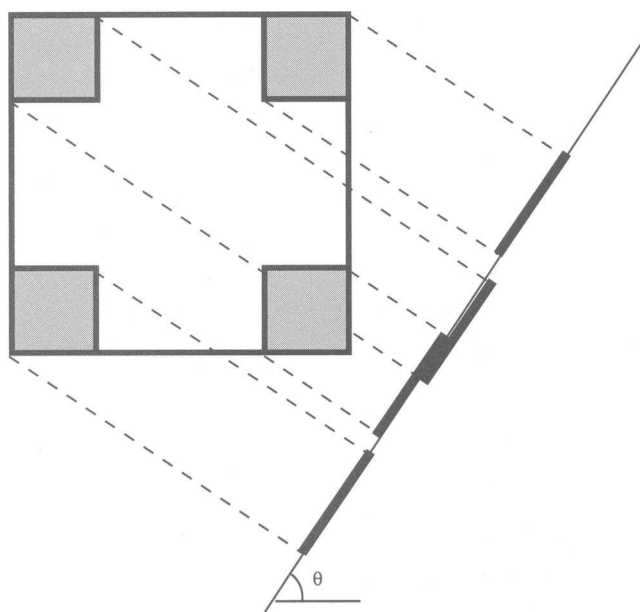
Figure 3. Projecting the "four corner" set.

The theorem clearly applies to the set K^2. "Almost every projection" means that there is a set E in $[0, \pi)$ of zero Lebesgue measure such that $|P_\theta(F)| = 0$ for all θ not belonging to E. The proof of Theorem 2 is quite complicated. Our goal is to give an elementary proof of it for a class of self-similar sets F that includes the "four corner" set K^2. One advantage of this elementary approach is that it can be refined to yield quantitative estimates (see section 3).

Proposition 3. *Suppose that* $m \geq 3$ *and that*

$$\Lambda = \left\{ \sum_{n=0}^{\infty} a_n m^{-n} : a_n \in \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\} \right\}, \tag{8}$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ *are distinct vectors in* $\mathbb{R}^2$ *such that the "pieces"* $\Lambda_i = \mathbf{b}_i + m^{-1}\Lambda$ *are pairwise disjoint. Then* $|P_\theta(\Lambda)| = 0$ *for almost every* θ *in* $[0, \pi)$.

One can check that for $m = 2$ the set Λ, defined as in (8), is a line segment, so all its projections, except one, have positive length. For $m \geq 3$ the disjointness condition precludes the set Λ from being a subset of a line; this is proved in Lemma 6.

The proof of Proposition 3 is based on several lemmas dealing with subsets of $\mathbb{R}$.

Lemma 4. *Let* F *be a compact subset of* $\mathbb{R}$ *with positive Lebesgue measure. Then for any* $\delta > 0$ *there is an interval* J *such that* $|F \cap J| \geq (1 - \delta)|J|$.

Proof. (This follows immediately from Lebesgue's theorem on points of density, but is actually more elementary.) By (1) and the compactness assumption, F can be covered by finitely many open intervals $\{I_i\}$ such that

$$\sum_i |I_i| \leq (1 - \delta)^{-1}|F|.$$

If $|F \cap I_i| < (1 - \delta)|I_i|$ held for all i, then summing over i would yield a contradiction. ∎

In the following three lemmas we consider the set

$$K = K(\{d_1, \ldots, d_m\}) = \left\{ \sum_{n=0}^{\infty} a_n m^{-n} : a_n \in \{d_1, \ldots, d_m\} \right\}, \tag{9}$$

where $d_1, d_2, \ldots, d_m$ are real numbers (not necessarily distinct). Thus, K is a self-similar subset of $\mathbb{R}$ given by (5), but of the special form in which the contraction ratio r is equal to m^{-1}, the reciprocal of the number of digits. In the proof of Proposition 3 we will have $K = P_\theta(\Lambda)$ for a fixed θ; then $d_i = P_\theta(\mathbf{b}_i)$.

Lemma 5. *For K as in (9), let $K_i = d_i + m^{-1}K$. Then $K = \bigcup_{i=1}^{m} K_i$ and $|K_i \cap K_j| = 0$ for $i \neq j$.*

Proof. The first statement follows immediately from the definition of K. The second statement is an easy consequence of self-similarity. The set K is a union of m pieces, each of which is a translate of $m^{-1}K$. Since $|K| = m \cdot |m^{-1}K|$, the pieces have to be pairwise disjoint in measure. The formal proof goes as follows: Fix $1 \leq i < j \leq m$. We have

$$|K| = \left| \bigcup_{k=1}^{m} K_k \right| \leq |K_i \cup K_j| + \sum_{k \leq m;\ k \neq i,j} |K_k| = \sum_{k=1}^{m} |K_k| - |K_i \cap K_j|.$$

But $|K_k| = |d_k + m^{-1}K| = m^{-1}|K|$, hence $|K_i \cap K_j| \leq 0$. ∎

Observe that the lemma covers the case $d_i = d_j$ as well. Then $K_i \cap K_j = K_i$, but $|K| = |K_i| = 0$, so there are no surprises here.

The previous lemma says that the pairwise intersections $K_i \cap K_j$ cannot be "large." However, at least one of them must be nonempty.

Lemma 6. *Let K and K_i be as in Lemma 5. There exist indices i and j, $i \neq j$, such that $K_i \cap K_j \neq \emptyset$.*

Proof. Let $N_\varepsilon(F) = \{x : \text{dist}(x, F) < \varepsilon\}$ denote the neighborhood of radius ε of a set F. Suppose that the sets K_k are pairwise disjoint. Since they are compact, the distance between any two of them is positive. Thus we can find $\varepsilon > 0$ so that $N_\varepsilon(K_i) \cap N_\varepsilon(K_j) = \emptyset$ whenever $i \neq j$. Then we have

$$|N_\varepsilon(K)| = \left| \bigcup_{k=1}^{m} N_\varepsilon(K_k) \right| = \sum_{k=1}^{m} |N_\varepsilon(K_k)|.$$

On the other hand,

$$|N_\varepsilon(K_k)| = |N_\varepsilon(d_k + m^{-1}K)| = |N_\varepsilon(m^{-1}K)| = m^{-1}|N_{m\varepsilon}(K)|,$$

since $mN_\varepsilon(m^{-1}K) = N_{m\varepsilon}(K)$. We conclude that $|N_\varepsilon(K)| = |N_{m\varepsilon}(K)|$. This is a contradiction, for both $N_\varepsilon(K)$ and $N_{m\varepsilon}(K)$ are bounded open sets, and $N_\varepsilon(K)$ is obviously a proper subset of $N_{m\varepsilon}(K)$. ∎

Before stating the next lemma we need to introduce some notation. Recall that the self-similar set K has a representation

$$K = \bigcup_{i=1}^{m} K_i = \bigcup_{i=1}^{m} (d_i + m^{-1}K).$$

Substituting this formula into each term in its right-hand side, we obtain

$$K = \bigcup_{i,j=1}^{m} K_{ij}, \qquad K_{ij} = d_i + m^{-1}d_j + m^{-2}K.$$

The sets K_i and K_{ij} are called the *cylinder sets* of K of orders 1 and 2, respectively. This operation can be iterated. For each positive integer ℓ the set K is the union of m^ℓ pieces, called *cylinders of order* ℓ, each of which is a translate of $m^{-\ell}K$. Let $\mathcal{A} = \{1, 2, \ldots, m\}$ and $\mathcal{A}^\ell = \{u = u_1 \ldots u_\ell : u_i \in \mathcal{A}\}$. Then

$$K = \bigcup_{u \in \mathcal{A}^\ell} K_u, \qquad K_u = K_{u_1 \ldots u_m} = \sum_{n=1}^{\ell} d_{u_n} m^{-n+1} + m^{-\ell}K.$$

Repeating the proof of Lemma 5 for this decomposition demonstrates that

$$|K_u \cap K_v| = 0 \tag{10}$$

for different u and v in $\mathcal{A}^\ell$.

We want to understand when $|K| = 0$. There is an easy sufficient condition: $|K| = 0$ if two cylinders of K coincide; i.e., if $K_u = K_v$ for some distinct u and v in $\mathcal{A}^\ell$. This can be seen in many ways; for instance, $|K| = m^\ell |K_u| = m^\ell |K_u \cap K_v| = 0$ by (10). This condition is too strong to be necessary, however: for the planar self-similar set Λ in (8), there are just countably many θ in $[0, \pi)$ for which $\Lambda^\theta = P_\theta(\Lambda)$ has two coinciding cylinders. Thus we would like to know what happens if some cylinders "almost" coincide.

Definition 7. Two cylinders K_u and K_v are *ε-relatively close* if u and v belong to $\mathcal{A}^\ell$ for some ℓ and $K_u = K_v + x$ for some x with $|x| \le \varepsilon \cdot \mathrm{diam}(K_u)$.

Note that $\mathrm{diam}(K_u) = \mathrm{diam}(K_v) = m^{-\ell} \mathrm{diam}(K)$ for all u and v in $\mathcal{A}^\ell$. Let

$$d_u = \sum_{n=1}^{\ell} d_{u_n} m^{-n+1},$$

so that $K_u = d_u + m^{-\ell}K$. Then K_u and K_v are ε-relatively close whenever

$$|d_u - d_v| \le \varepsilon m^{-\ell} \mathrm{diam}(K). \tag{11}$$

Lemma 8. *If for every $\varepsilon > 0$ there exist an index ℓ and distinct u and v in $\mathcal{A}^\ell$ such that K_u and K_v are ε-relatively close, then $|K| = 0$.*

This lemma (and its converse, which holds as well) is a very special case of a theorem by Bandt and Graf [1].

Proof. Suppose, to the contrary, that $|K| > 0$. Then we can find an interval J such that $|J \cap K| \geq 0.9|J|$ (Lemma 4). Let $\varepsilon = |J|/(2\,\mathrm{diam}(K))$. By assumption, there exist an index ℓ in $\mathbb{N}$ and distinct u and v in $\mathcal{A}^\ell$ such that the cylinders K_u and K_v are ε-relatively close. If $J_u = d_u + m^{-\ell}J$ and $J_v = d_v + m^{-\ell}J$, then

$$J_u = J_v + (d_u - d_v)$$

and

$$|d_u - d_v| \leq \varepsilon m^{-\ell}\,\mathrm{diam}(K) = 0.5|J_u|.$$

This means that J_u and J_v have a large overlap—at least half of J_u lies in J_v. Since J was chosen to ensure that at least 90 percent of its length belongs to the set K, this property carries over to J_u and K_u. To be more precise,

$$|J_u \cap K_u| = |(d_u + m^{-\ell}J) \cap (d_u + m^{-\ell}K)| = m^{-\ell}|J \cap K| \geq 0.9m^{-\ell}|J| = 0.9|J_u|.$$

Similarly,

$$|J_v \cap K_v| \geq 0.9|J_v|.$$

Since at least 90 percent of J_v is in K_v and at least 50 percent of J_u is in J_v, we find that at least 40 percent of J_u is in K_v. But at least 90 percent of J_u is in K_u, so at least 30 percent of J_u is in $K_u \cap K_v$. This is in contradiction with (10), and the lemma is proved. ∎

Proof of Proposition 3. Recall that Λ is a planar Cantor set of description (8). If we again write $\Lambda^\theta = P_\theta(\Lambda)$, then

$$\Lambda^\theta = \bigcup_{i=1}^{m}(P_\theta(\mathbf{b}_i) + m^{-1}\Lambda^\theta),$$

so all the foregoing discussion (in particular, Lemma 8) applies to Λ^θ. Let

$$\mathcal{V}_\varepsilon := \bigcup_{\ell \geq 1}\{\theta \in [0, \pi) : \text{there exist distinct } u \text{ and } v \text{ in } \mathcal{A}^\ell$$

with Λ_u^θ and Λ_v^θ ε-relatively close$\}$.

Note that if θ lies in $\bigcap_{\varepsilon > 0}\mathcal{V}_\varepsilon$, then $|\Lambda^\theta| = 0$ by Lemma 8. The proposition will be proved if we are able to show that $|[0, \pi) \setminus \bigcap_{\varepsilon > 0}\mathcal{V}_\varepsilon| = 0$. But $\bigcap_{\varepsilon > 0}\mathcal{V}_\varepsilon = \bigcap_{n=1}^{\infty}\mathcal{V}_{1/n}$, and by DeMorgan's law,

$$[0, \pi) \setminus \bigcap_{n=1}^{\infty}\mathcal{V}_{1/n} = \bigcup_{n=1}^{\infty}([0, \pi) \setminus \mathcal{V}_{1/n}).$$

Thus, it remains to verify that

$$|[0, \pi) \setminus \mathcal{V}_\varepsilon| = 0 \tag{12}$$

for a *fixed* ε.

We would like to show that any subinterval of $[0, \pi)$ has at least a fixed percentage of its length (depending on ε but not on the subinterval's size) lying in $\mathcal{V}_\varepsilon$. In view of Lemma 4, this will imply (12). The desired conclusion will follow if we can find positive constants C_1 and C_2 such that for any θ in $(0, \pi)$ and any ℓ in $\mathbb{N}$ there is a θ_0 satisfying

$$|\theta - \theta_0| \leq C_1 m^{-\ell}, \qquad (\theta_0 - C_2 \varepsilon m^{-\ell}, \theta_0 + C_2 \varepsilon m^{-\ell}) \subset \mathcal{V}_\varepsilon. \tag{13}$$

(There is a minor technical issue when the interval in (13) is not contained in $[0, \pi)$, but it is easy to handle.)

We fix θ in $(0, \pi)$ and ℓ in $\mathbb{N}$. Appealing to Lemma 6, we choose i and j with $i \neq j$ for which $\Lambda_i^\theta \cap \Lambda_j^\theta \neq \emptyset$. Since

$$\Lambda_i^\theta = \bigcup_{u \in \mathcal{A}^\ell, \; u_1 = i} \Lambda_u^\theta, \qquad \Lambda_j^\theta = \bigcup_{v \in \mathcal{A}^\ell, \; v_1 = j} \Lambda_v^\theta,$$

there exist u and v in $\mathcal{A}^\ell$, with $u_1 = i$ and $v_1 = j$, such that $\Lambda_u^\theta \cap \Lambda_v^\theta \neq \emptyset$. This means that there are points y_u in Λ_u and z_v in Λ_v such that $P_\theta(y_u) = P_\theta(z_v)$. Denote by z_u the point in Λ_u corresponding to z_v (i.e., $\Lambda_v - z_v = \Lambda_u - z_u$), and let θ_0 be the angle such that $P_{\theta_0}(z_u) = P_{\theta_0}(z_v)$, whence $\Lambda_u^{\theta_0} = \Lambda_v^{\theta_0}$. Then $|\theta - \theta_0|$ is the angle at z_v for the triangle with vertices z_u, z_v, and y_u (see Figure 4), and therefore $|z_u - y_u| \geq |y_u - z_v| \sin|\theta - \theta_0|$. This implies that

$$\sin|\theta - \theta_0| \leq \frac{\mathrm{diam}(\Lambda_u)}{\mathrm{dist}(\Lambda_u, \Lambda_v)}.$$
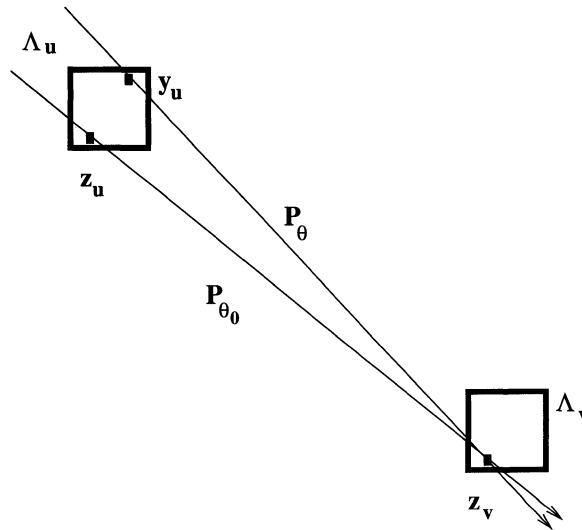

Figure 4. Finding θ_0.

Note that $\mathrm{diam}(\Lambda_u) = m^{-\ell} \mathrm{diam}(\Lambda)$ and $\mathrm{dist}(\Lambda_u, \Lambda_v) \geq \mathrm{dist}(\Lambda_i, \Lambda_j) \geq \delta > 0$ for some $\delta = \delta(\Lambda) > 0$ by the hypothesis of pairwise disjointness in Proposition 3. Thus the first condition in (13) holds with the constant $C_1 = \pi \, \mathrm{diam}(\Lambda)/(2\delta)$. By the choice

of θ_0 we have θ_0 in $\mathcal{V}_0$, which is a subset of $\mathcal{V}_\varepsilon$. Using (7), we conclude that

$$|(P_\alpha - P_{\theta_0})(\mathbf{z} - \mathbf{w})| \leq |\alpha - \theta_0| \cdot |\mathbf{z} - \mathbf{w}|$$

for any two vectors $\mathbf{z}$ and $\mathbf{w}$ and any $\alpha \in [0, \pi)$. Consequently, the projected set Λ_u^α can be obtained from Λ_v^α by a translation of at most $|\alpha - \theta_0| \cdot \text{diam}(\Lambda)$. On the other hand, the diameter of Λ_u^α is at least $m^{-\ell} \cdot \text{width}(\Lambda)$, where $\text{width}(\Lambda)$ signifies the minimal width of a strip that contains Λ; it is nonzero because the assumption that the sets Λ_i are disjoint prevents Λ from being contained in a straight line (recall Lemma 6). Set $C_2 = \text{width}(\Lambda)/\text{diam}(\Lambda)$. Then by Definition 7 the cylinders Λ_u^α and Λ_v^α are ε-relatively close for all α in the $C_2\varepsilon m^{-\ell}$-neighborhood of θ_0. It follows that this neighborhood lies in $\mathcal{V}_\varepsilon$, so the second condition in (13) is verified. This completes the proof. ∎

3. CONCLUDING REMARKS.

First we explain what we meant by "quantitative estimates" alluded to after the statement of Theorem 2. Recall that the Favard length of a planar set F is defined by $\text{Fav}(F) = \int_0^\pi |P_\theta(F)| \, d\theta$. Denote by K the Cantor middle-half set. By Proposition 3, $\text{Fav}(K^2) = 0$. Now consider the nth stage of the Cantor set construction for K:

$$K_n = \left\{ \sum_{k=1}^\infty a_k 4^{-k} : a_k \in \{0, 3\} \text{ for } 1 \leq k \leq n \text{ and } a_k \in \{0, 1, 2, 3\} \text{ for } k > n \right\}.$$

Then K_n^2 is a union of 4^n squares of side-length 4^{-n}. Clearly, $\text{Fav}(K^2) = 0$ implies that $\lim_{n \to \infty} \text{Fav}(K_n^2) = 0$, but it does not yield an estimate for the rate of convergence. A lower bound $\text{Fav}(K_n^2) \geq c/n$ for some $c > 0$ follows from Theorem 4.1 in [14]. The elementary method described in the present article can be refined to yield a quantitative upper bound [17]. For $y \geq 1$ we employ the notation

$$\log_* y = \min \left\{ n \geq 0 : \underbrace{\log \log \ldots \log}_{n} y \leq 1 \right\}. \tag{14}$$

Theorem 9 (Peres and Solomyak). *There exist positive constants C and a such that*

$$\text{Fav}(K_n^2) \leq C \exp[-a \log_* n]$$

for all n in $\mathbb{N}$.

The convergence of the upper bound in Theorem 9 to zero is extremely slow, but it is the best estimate currently known. It is a challenging unsolved problem to determine the correct asymptotics. We would guess that the lower bound c/n is closer to the truth.

We make a few additional comments.

- It follows from results of Kenyon [12] and Lagarias and Wang [13] that $|P_\theta(K^2)| = 0$ for all θ such that $\tan \theta$ is irrational. (This is, of course, much stronger than Proposition 3!) However, this information does not seem helpful in obtaining an upper bound for $\text{Fav}(K_n^2)$.

- Using polar duality, one can infer from the properties of the four-corner set K^2 the existence of *Kakeya sets*, i.e., compact planar sets of zero area that contain line segments in every direction. This connection was discovered by Besicovitch [4]; see

also Besicovitch [2], [3] and Kahane [11]. Note that the latter paper does not really prove that almost every projection of K^2 has zero length.

• The approach of this paper was originally developed in [16] to study the Hausdorff measure of certain parameterized families of fractals.

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Prof. [Manindra] Agrawal is a 36-year old theoretical computer scientist at the Indian Institute of Technology in Kanpur, India. In August, he solved a problem that had eluded millennia of mathematicians: developing a method to determine with complete certainty if a number is prime. . . . Besides being a show stopping bit of mathematics, the work was big news for the Internet. Very large prime numbers are the bedrock of Internet encryption, the sort your browser uses when you are shopping online.

[H]e started his work three years ago. He was dealing with a different problem, called identity testing, when he noticed the solution hinted at a potential fresh assault on prime-number testing. It was a long three years. While no slouch in math, Prof. Agrawal said he sometimes had to use Google to find information on the more recondite aspects of number theory. His Eureka! moment came in July. As he was driving his daughter to school on his motor scooter, a particularly complicated mathematical set suddenly fell into place.

———Lee Gomes, "One Beautiful Mind From India
Is Putting The Internet on Alert,"
The Wall Street Journal, 4 November 2002, p. B1