NOTES

Edited by **William Adkins**

Partial Fractions, Binomial Coefficients, and the Integral of an Odd Power of sec θ

Daniel J. Velleman

Many calculus textbooks discuss the integrals of products and powers of trigonometric functions. The trickiest of these is the integral of an odd power of $\sec\theta$, and it is usually done by using integration by parts repeatedly (or by using integration by parts to derive a reduction formula, and then using the reduction formula repeatedly). But there is another natural approach to such an integral: An odd power of $\sec\theta$ can also be thought of as an odd power of $\cos\theta$, and this can be integrated by using the substitution $x = \sin\theta$:

$$\int \sec^{2n+1}\theta\, d\theta = \int \frac{d\theta}{\cos^{2n+1}\theta} = \int \frac{\cos\theta\, d\theta}{\cos^{2n+2}\theta} = \int \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)^{n+1}} = \int \frac{dx}{(1-x^2)^{n+1}}$$

The last integral can now be evaluated by partial fractions. The purpose of this note is to give the partial fractions decomposition of $1/(1-x^2)^{n+1}$, thus allowing us to complete the calculation of $\int \sec^{2n+1}\theta\, d\theta$.

Here is our main result:

Theorem 1. *For every integer $n \geq 0$,*

$$\frac{1}{(1-x^2)^{n+1}} = \sum_{i=1}^{n+1} \frac{\binom{2n+1-i}{n}}{2^{2n+2-i}} \left[\frac{1}{(1+x)^i} + \frac{1}{(1-x)^i}\right].$$

Using Theorem 1, we can now complete the calculation of the integral of an odd power of $\sec\theta$:

$$\begin{aligned}
\int \sec^{2n+1}\theta\, d\theta &= \int \frac{dx}{(1-x^2)^{n+1}} \\
&= \int \sum_{i=1}^{n+1} \frac{\binom{2n+1-i}{n}}{2^{2n+2-i}} \left[\frac{1}{(1+x)^i} + \frac{1}{(1-x)^i}\right] dx \\
&= \frac{\binom{2n}{n}}{2^{2n+1}} [\ln(1+x) - \ln(1-x)] \\
&\quad + \sum_{i=2}^{n+1} \frac{\binom{2n+1-i}{n}}{(i-1)2^{2n+2-i}} \left[\frac{1}{(1-x)^{i-1}} - \frac{1}{(1+x)^{i-1}}\right] + C \\
&= \frac{\binom{2n}{n}}{2^{2n+1}} \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \\
&\quad + \sum_{i=1}^{n} \frac{\binom{2n-i}{n}}{i2^{2n+1-i}} \left[\frac{1}{(1-\sin\theta)^i} - \frac{1}{(1+\sin\theta)^i}\right] + C.
\end{aligned}$$

For example, in the case $n = 3$ we get:

$$\int \sec^7\theta\, d\theta = \frac{5}{32}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) + \frac{5}{32}\left[\frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta}\right] + \frac{1}{16}\left[\frac{1}{(1-\sin\theta)^2} - \frac{1}{(1+\sin\theta)^2}\right] + \frac{1}{48}\left[\frac{1}{(1-\sin\theta)^3} - \frac{1}{(1+\sin\theta)^3}\right] + C.$$

Our proof of Theorem 1 relies on the following two lemmas:

Lemma 2. *For any integers $n, m \geq 0$ and any real number r,*

$$\sum_{k=0}^{m}\binom{n+k}{n}r^k = \sum_{k=0}^{m}\binom{n+m+1}{k}r^k(1-r)^{m-k}.$$

Proof. We proceed by induction on m. It is easy to check that the lemma holds when $m = 0$. Now assume the lemma holds for some value of m. Then

$$\begin{aligned}
&\sum_{k=0}^{m+1}\binom{n+k}{n}r^k \\
&= \sum_{k=0}^{m}\binom{n+k}{n}r^k + \binom{n+m+1}{n}r^{m+1} \\
&= \sum_{k=0}^{m}\binom{n+m+1}{k}r^k(1-r)^{m-k} + \binom{n+m+1}{n}r^{m+1} \quad \text{(by inductive hypothesis)} \\
&= \sum_{k=0}^{m}\binom{n+m+1}{k}r^k(1-r)^{m-k}(r+1-r) + \binom{n+m+1}{m+1}r^{m+1} \\
&= \sum_{k=0}^{m}\binom{n+m+1}{k}\left[r^{k+1}(1-r)^{m-k} + r^k(1-r)^{m+1-k}\right] \\
&\quad + \binom{n+m+1}{m+1}r^{m+1} \\
&= \sum_{k=0}^{m}\binom{n+m+1}{k}r^{k+1}(1-r)^{m-k} + \sum_{k=0}^{m+1}\binom{n+m+1}{k}r^k(1-r)^{m+1-k} \\
&= \sum_{k=1}^{m+1}\binom{n+m+1}{k-1}r^k(1-r)^{m+1-k} + \sum_{k=0}^{m+1}\binom{n+m+1}{k}r^k(1-r)^{m+1-k} \\
&= \binom{n+m+1}{0}(1-r)^{m+1} \\
&\quad + \sum_{k=1}^{m+1}\left[\binom{n+m+1}{k-1} + \binom{n+m+1}{k}\right]r^k(1-r)^{m+1-k}
\end{aligned}$$

$$= \binom{n+m+2}{0}(1-r)^{m+1} + \sum_{k=1}^{m+1} \binom{n+m+2}{k} r^k (1-r)^{m+1-k}$$

$$= \sum_{k=0}^{m+1} \binom{n+m+2}{k} r^k (1-r)^{m+1-k},$$

as required. ∎

Lemma 3. *For any integers $n, m \geq 0$ and any real number r,*

$$(1-r)^{n+1} \sum_{k=0}^{m} \binom{n+k}{n} r^k + r^{m+1} \sum_{k=0}^{n} \binom{m+k}{m} (1-r)^k = 1. \tag{1}$$

Proof. We apply Lemma 2 to both summations, with the roles of m and n and the roles of r and $1-r$ reversed for the second summation. Then we reindex the second summation by $j = n+m+1-k$ and apply the binomial theorem:

$$(1-r)^{n+1} \sum_{k=0}^{m} \binom{n+k}{n} r^k + r^{m+1} \sum_{k=0}^{n} \binom{m+k}{m} (1-r)^k$$

$$= (1-r)^{n+1} \sum_{k=0}^{m} \binom{n+m+1}{k} r^k (1-r)^{m-k}$$

$$+ r^{m+1} \sum_{k=0}^{n} \binom{n+m+1}{k} (1-r)^k r^{n-k}$$

$$= \sum_{k=0}^{m} \binom{n+m+1}{k} r^k (1-r)^{n+m+1-k}$$

$$+ \sum_{k=0}^{n} \binom{n+m+1}{n+m+1-k} r^{n+m+1-k} (1-r)^k$$

$$= \sum_{k=0}^{m} \binom{n+m+1}{k} r^k (1-r)^{n+m+1-k} + \sum_{j=m+1}^{n+m+1} \binom{n+m+1}{j} r^j (1-r)^{n+m+1-j}$$

$$= \sum_{k=0}^{n+m+1} \binom{n+m+1}{k} r^k (1-r)^{n+m+1-k}$$

$$= (r+1-r)^{n+m+1} = 1^{n+m+1} = 1.$$ ∎

If $0 \leq r \leq 1$, then Lemma 3 has an interesting probabilistic interpretation. Consider a sequence of $n+m+1$ Bernoulli trials in which the probability of success on each trial is r. If there are at least $n+1$ failures, then the $(n+1)$th failure must occur on trial $n+1+k$, for some k between 0 and m. But for the $(n+1)$th failure to occur on trial $n+1+k$, there must be exactly k successes and n failures among the first $n+k$ trials, followed by a failure on trial number $n+1+k$, and thus the probability of trial $n+1+k$ being the $(n+1)$th failure is $\binom{n+k}{n} r^k (1-r)^{n+1}$. Thus, the first sum on the left side of (1) gives the probability of having at least $n+1$ failures. Similar reasoning shows that the second sum on the left side of (1) gives the probability of having at least $m+1$ successes. Since exactly one of these two events must occur, the two probabilities must sum to 1, as claimed in Lemma 3.

We are now ready to prove our main theorem.

Proof of Theorem 1. We begin by applying Lemma 3 with $m = n$ and $r = (1+x)/2$, and therefore $1 - r = (1-x)/2$.

$$\begin{aligned}1 &= (1-r)^{n+1}\sum_{k=0}^{m}\binom{n+k}{n}r^k + r^{m+1}\sum_{k=0}^{n}\binom{m+k}{m}(1-r)^k \\ &= \left(\frac{1-x}{2}\right)^{n+1}\sum_{k=0}^{n}\binom{n+k}{n}\left(\frac{1+x}{2}\right)^k + \left(\frac{1+x}{2}\right)^{n+1}\sum_{k=0}^{n}\binom{n+k}{n}\left(\frac{1-x}{2}\right)^k \\ &= \sum_{k=0}^{n}\frac{\binom{n+k}{n}}{2^{n+1+k}}\left[(1+x)^k(1-x)^{n+1} + (1+x)^{n+1}(1-x)^k\right]. \qquad (2)\end{aligned}$$

Dividing both sides of (2) by $(1-x^2)^{n+1}$ we get:

$$\frac{1}{(1-x^2)^{n+1}} = \sum_{k=0}^{n}\frac{\binom{n+k}{n}}{2^{n+1+k}}\left[\frac{1}{(1+x)^{n+1-k}} + \frac{1}{(1-x)^{n+1-k}}\right].$$

The theorem now follows by reindexing the last summation by $i = n+1-k$. ∎

We close with one more application of Lemma 3.

Theorem 4. *For any integer $n \geq 0$ and any real number r in $(-1, 1)$,*

$$\sum_{k=0}^{\infty}\binom{n+k}{n}r^k = \frac{1}{(1-r)^{n+1}}.$$

Proof. Fix $n \geq 0$ and r in $(-1, 1)$. Solving equation (1) for $\sum_{k=0}^{m}\binom{n+k}{n}r^k$ we get:

$$\begin{aligned}\sum_{k=0}^{m}\binom{n+k}{n}r^k &= \frac{1}{(1-r)^{n+1}}\left[1 - r^{m+1}\sum_{k=0}^{n}\binom{m+k}{m}(1-r)^k\right] \\ &= \frac{1}{(1-r)^{n+1}}\left[1 - r^{m+1}p(m)\right], \qquad (3)\end{aligned}$$

where $p(m)$ is a polynomial in m of degree n. (For example, in the case $n = 0$ we have $p(m) = 1$, and (3) becomes the familiar $\sum_{k=0}^{m} r^k = (1 - r^{m+1})/(1-r)$.) Since $-1 < r < 1$, $\lim_{m\to\infty} r^{m+1}p(m) = 0$, and the theorem follows. ∎

Another way to prove Theorem 4 is to begin with the formula $\sum_{k=0}^{\infty} r^k = 1/(1-r)$, differentiate n times with respect to r, and then divide by $n!$. However, the proof by way of Lemma 3 shows explicitly how close the mth partial sum is to the sum of the series.

Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002
djvelleman@amherst.edu