HILBERT SPACE OPERATORS AND QUANTUM MECHANICS

E. W. PACKEL

1. Introduction. The beautiful theory of Hilbert space can be motivated in several ways. In one view, the Hilbert space axioms are a most natural generalization of properties of finite dimensional Euclidean space. From a historical standpoint, much of the theory of orthonormal bases and spectral decomposition grows directly out of Hilbert's work with quadratic forms and integral equations. Thirdly, and perhaps most significantly, Hilbert space provides a natural, elegant, and effective setting for one formulation of quantum mechanics. It is the purpose of this paper to elucidate in an elementary and necessarily simplified fashion this important physical role of Hilbert space.

Clearly we cannot hope to deal very generally or thoroughly with the mathematical and philosophical subtleties of the formalism of quantum mechanics without requiring more space and knowledge of physics than would be reasonable in this undertaking. For more detailed and advanced treatments see Mackey [5] and Jauch [3]. Our aim here is, rather, to use the machinery of Hilbert space theory to convey the spirit of the intellectual triumphs of the 1920's which resulted in quantum mechanics; and to do so without assuming any formal knowledge of quantum mechanics. The author hastens to point out his commitment to and qualification for this task. His very modest knowledge of quantum mechanics has resulted primarily from efforts to communicate with the intrepid and inquisitive physics students who invariably populate undergraduate courses in functional analysis. The author believes that any mathematics undergraduate or graduate who studies Hilbert space theory should have some idea of its important role in quantum mechanics. Two texts which support this belief are Packel [6] and Reed and Simon [8].

Bounded and unbounded operators on Hilbert space. With the space l^2 of square summable sequences and the space L^2 of square integrable functions as models, von Neumann proposed axioms for Hilbert space in 1929. We present below the definition, some properties, and an important example.

A **Hilbert space** H is a vector space over $\boldsymbol{C}$ on which is defined an **inner product** $\langle\, ,\rangle : H \times H \to \boldsymbol{C}$ satisfying for all f, g, and h in H and α, β in $\boldsymbol{C}$:

(i) $\langle f,g\rangle = \langle g,f\rangle^*$ (* denotes the complex conjugate),

(ii) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$,

(iii) $f \neq 0 \Rightarrow \langle f,f\rangle > 0$.

Furthermore, H must be complete with respect to the norm defined by

$$\|f\| = \langle f,f\rangle^{\frac{1}{2}}. \tag{1}$$

We shall make use of the well-known **Schwarz inequality**

$$|\langle f,g\rangle| \leqq \|f\|\ \|g\|,$$

which is used to show that

$$\|f+g\| \leqq \|f\| + \|g\|$$

so that (1) in fact does define a norm on H.

As a fundamental example of Hilbert space, and one which we shall use throughout, consider the Lebesgue integral on $\boldsymbol{R}$ and let the role of H be played by

$$L^2(\boldsymbol{R}) = \left\{f : \boldsymbol{R} \to \boldsymbol{C} : \int_{\boldsymbol{R}} |f|^2 < \infty\right\}.$$

If we define

$$\langle f, g\rangle = \int_{\boldsymbol{R}} fg^*$$

and if we play the traditional game of identifying functions which differ only on sets of Lebesgue measure zero (thereby regarding $L^2(\boldsymbol{R})$ as a space of equivalence classes), then $\langle\ ,\ \rangle$ becomes an inner product. Completeness of $L^2(\boldsymbol{R})$ is fundamentally related to the use of (and singlehandedly testifies to the importance of) the Lebesgue integral.

As with so many mathematical constructs (categories), our interest will be not so much in the Hilbert spaces themselves (the objects) as in the structure preserving mappings between spaces (the morphisms). Here, unless otherwise stated, we consider morphisms from a Hilbert space H into itself—these are the **operators** on H. Mathematicians have little trouble deciding what properties these operators should have. We define a (linear) **bounded operator** T on H as follows:

(i) $T: H \to H$ (T is defined on all of H),

(ii) $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ (T is linear),

(iii) $\|T\| \equiv \sup_{\|f\| \leqq 1} \|Tf\| < \infty$ (T is bounded).

It is readily shown that the operator norm defined in (iii) also satisfies for bounded operators S and T in H:

$$\|S + T\| \leqq \|S\| + \|T\| \tag{2}$$

and

$$\|ST\| \leqq \|S\| \, \|T\|. \tag{3}$$

The boundedness property ($\|T\| < \infty$) turns out to be equivalent to the continuity of T and hence seems a very natural condition to impose.

As some very special examples consider the operators I, U, and V on $L^2(\boldsymbol{R})$ defined by

$If = f$ (identity operator),

$(Uf)(x) = f(x+1)$ (shift one unit to the left),

$(Vf)(x) = e^{ix}f(x)$ (multiplication by e^{ix}).

It is clear that each of these operators is bounded and in fact has norm 1. It may also be helpful to observe that, relative to a choice of basis, an n by n matrix with complex entries can be regarded as a bounded operator on the Hilbert space $\boldsymbol{C}^n$.

Despite the soundest of mathematical reasons for studying these bounded operators, we shall see that many operators required in the Hilbert space formulation of quantum mechanics (and hence of interest to physicists) are of necessity **unbounded operators**. We define T to be such an operator if:

(i) $T:\Omega \to H$ where Ω is a dense subspace of H,

(ii) T is linear on Ω,

(iii) $\|T\| \equiv \sup\limits_{\substack{\|f\| \leq 1 \\ f \in \Omega}} \|Tf\| = \infty$.

For the sake of accuracy (though we do not invoke it in what follows) we also require that T be **closed**, by which we mean that

$$\operatorname{graph}(T) = \{(f, Tf) \in H \times H : f \in \Omega\}$$

is a closed subspace of $H \times H$. In contrast to the bounded situation, it turns out that an unbounded operator is continuous at no point of its domain. For a thorough study of unbounded operators see Goldberg [**2**].

As examples of unbounded operators, consider p and q defined by

$$(pf)(x) = -if'(x) \qquad \text{(the ``momentum'' operator)},$$

$$(qf)(x) = xf(x) \qquad \text{(the ``position'' operator)},$$

on suitable domains in $L^2(\boldsymbol{R})$. Both p and q can readily be shown to have infinite norms (for q, look at $\|q(f_k)\|$ with $k = 1, 2, \cdots$, where

$$f_k(x) = \begin{cases} 1 & x \in [k-1, k] \\ 0 & \text{otherwise} \end{cases}$$

and note that $\|f_k\| = 1$ for all k). It is more demanding to show that these operators are closed and that their domains are dense. Indeed it requires more development than is appropriate here even to make precise what these domains are, though they are nicely related by the Fourier transform. The operators p and q play a crucial role in our glimpse of quantum mechanics.

We conclude this section by defining two additional subclasses of operators, the first of which we generalize to allow mappings between distinct Hilbert spaces.

Given Hilbert spaces H and K, a linear transformation $U: H \to K$ is called **unitary** if U is surjective and $\|Uf\| = \|f\|$ for all f in H. It is a standard result that U must also preserve inner products ($\langle Uf, Ug\rangle = \langle f, g\rangle$) and that U, preserving all existing structure, provides the notion of the "essential sameness" of H and K. In this case we call H and K **unitarily equivalent**. It is easily checked that each of our examples I, U, and V is a unitary operator on $L^2(\boldsymbol{R})$.

A (bounded or unbounded) operator T on H is called **self-adjoint** if

(i) $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in$ domain (T),

(ii) range $(T + iI) =$ range$(T - iI) = H$.

In the bounded case, (i) alone provides a complete definition of self-adjoint. In the unbounded case the above is not the standard definition, but is equivalent to it. The operators p and q serve as examples of unbounded self-adjoint operators on $L^2(\boldsymbol{R})$. Property (i) can be verified directly (using integration by parts for p), while (ii) requires more work.

3. Interplay with quantum mechanics. Physics in the early 20th century was in a state of great excitement and confusion. Einstein's theories of special and general relativity had initiated a break with classical Newtonian physics and had "proven" themselves by resolving numerous previously unexplained results and by successfully predicting new results. The determinism of Newtonian physics still remained as did many unresolved and seemingly contradictory experimental facts. A fundamental tenet of the emerging quantum theory was that a deterministic view of the universe must give way to a description of particle behavior by means of a probability distribution. Despite Einstein's belief that "God does not play dice with the world," this view of a universe behaving and evolving according to the laws of chance continues to provide the most satisfactory model to date of "the way things are." In what follows we outline (in a one-dimensional nonrelativistic setting) the way in which Hilbert space theory helps to formalize this view. We begin with a few basic notions from probability theory.

Given a variable quantity which takes on real values in some probabilistic fashion (more formally, a **random variable**), its **probability density function** is a function $g: \boldsymbol{R} \to \boldsymbol{R}$ such that for any subset S of $\boldsymbol{R}$

$$\int_S g = \text{the probability that the variable lies in } S.$$

With such a density function g arising from a random variable one associates an **expectation** E_g defined by

$$E_g = \int_{\boldsymbol{R}} xg(x)\,dx,$$

which can be thought of as an average value of the variable. The **variance** D_g of g (or of its generating random variable) is defined by

$$D_g = \int_{\boldsymbol{R}} (x - E_g)^2 g(x)\,dx.$$

The variance D_g provides a measure of how much the random variable deviates from its expectation value, and $(D_g)^{\frac{1}{2}}$ is called the **standard deviation** of the random variable. All these formulas are natural continuous analogues of the more familiar summation formulas arising from discrete probabilistic situations.

We now consider a simplified but fundamental situation with which quantum mechanics deals — that of a particle constrained to one dimension (i.e., $\boldsymbol{R}$) in a physical system. Whereas the instantaneous state of the system is described in classical mechanics by specifying the position and velocity of the particle, the instantaneous state of the system is described in quantum mechanics by specifying a unit vector ψ in $L^2(\boldsymbol{R})$. The most direct interpretation of this vector is that

$$\int_S |\psi|^2$$

represents the probability that the particle in state ψ is in the region S. Thus $|\psi|^2$ provides a probability density function for the random variable defined by the position of the particle in state ψ. Since the particle must be somewhere on $\boldsymbol{R}$, we are compelled to require that

$$\|\psi\|^2 = \langle\psi,\psi\rangle = \int_{\boldsymbol{R}} \psi\psi^* = \int_{\boldsymbol{R}} |\psi|^2 = 1.$$

This approach results in an identification between states of the system and rays (one-dimensional subspaces) in $L^2(\boldsymbol{R})$; and any norm 1 representative from a ray provides a probability density function for the state to which it corresponds.

If the use of Hilbert space stopped here, we should have achieved very little for all our effort. Fortunately there is very much more to be said. In studying a physical system, one generally considers various **observables** of the system such as position, momentum, and energy. In classical mechanics an observable is represented mathematically by a (real-valued) function of position and momentum, and in quantum mechanics the mathematical entity corresponding to the notion of observable for our system is a self-adjoint operator on $L^2(\boldsymbol{R})$. If T is an operator corresponding to some observable, then

$$E_\psi(T) \equiv \langle T\psi,\psi\rangle$$

represents the expectation value of the observable, given an initial state ψ. Since the expectation $\langle T\psi,\psi\rangle$ must be real for any state ψ, which can be shown to imply (i) in the definition of self-adjoint, we have considerable motivation for associating an observable with a *self-adjoint* operator. As expected, two most important examples of this observable-operator correspondence are the position and momentum operators.

It follows from the definition that the expectation value of the position of a particle in state ψ is

$$\int_{\boldsymbol{R}} x|\psi(x)|^2 dx = \int_{\boldsymbol{R}} x\psi(x)\psi^*(x)\, dx.$$

If we let q denote the position observable, then

$$E_\psi(q) = \langle q\psi,\psi\rangle = \int_{\boldsymbol{R}} (q\psi)\,(x)\psi^*(x)\; dx,$$

and we see a compelling reason for defining the position operator q on $L^2(\boldsymbol{R})$ by $(q\psi)(x) = x\psi(x)$. A more involved motivational argument (using Fourier transforms) which we omit suggests defining the momentum operator p on $L^2(\boldsymbol{R})$ as we have defined it; i.e., $p\psi = -i\psi'$. We have deliberately suppressed a factor $h/(2\pi)$ (h = Planck's constant) in the definition of p and we shall reinstate it after we have done some computation.

Having defined the expectation $E_\psi(T)$ for T self-adjoint and any state ψ, we approach a notion of variance by defining

$$D_\psi(T) \equiv \| (T - E_\psi(T)I)\psi \|^2 = \langle (T - E_\psi(T)I)\psi,\ (T - E_\psi(T)I)\psi \rangle.$$

We note that the idea of a state vector and the definitions of $E_\psi(T)$ and $D_\psi(T)$ are applicable in any Hilbert space H. If, in particular, T is self-adjoint on $L^2(\boldsymbol{R})$, then we may write

$$D_\psi(T) = \int_{\boldsymbol{R}} (T - E_\psi(T)I)^2 \psi(x) \cdot \psi^*(x)\, dx. \tag{4}$$

A comparison of (4) with the definition of variance D_g suggests the idea that $D_\psi(T)$ is a variance in the sense that it provides a measure of how much the observable deviates from its expectation value.

It is possible to establish further connections between physical ideas and Hilbert space operators. In particular, measuring an observable for a particle in an initial state ψ can be related to operating on ψ (perhaps changing the state) with the corresponding self-adjoint operator. Also, allowable values of an observable (energy levels for example) correspond (in the prophetic terminology of Hilbert) to the **spectrum** or generalized eigenvalues of the operator. For an elementary exposition of these additional details see Gillespie [**1**].

We now prove a result in a general Hilbert space H from which the famous uncertainty relation of Heisenberg will swiftly emerge.

THEOREM 1. *If A and B are self-adjoint operators on a Hilbert space H and if ψ is in domain $(AB) \cap$ domain (BA), then*

$$D_\psi(A) D_\psi(B) \geqq \tfrac{1}{4} | E_\psi(AB - BA) |^2.$$

Proof. Using properties of inner products and of self-adjointness, we have

$$\begin{aligned} | E_\psi(AB - BA) |^2 &= | \langle (AB - BA)\psi, \psi \rangle |^2 = | \langle AB\psi, \psi \rangle - \langle BA\psi, \psi \rangle |^2 \\ &= | \langle AB\psi, \psi \rangle - \langle \psi, AB\psi \rangle |^2 = | \langle AB\psi, \psi \rangle - \langle AB\psi, \psi \rangle^* |^2 \\ &= (2\, \mathrm{Im} \langle AB\psi, \psi \rangle)^2 \quad \text{(Im denotes imaginary part).} \end{aligned}$$

Noting that for any a and b in $\boldsymbol{R}$

$$AB - BA = (A - aI)(B - bI) - (B - bI)(A - aI),$$

letting $E_\psi(A) = a$ and $E_\psi(B) = b$, and applying the identity developed at the beginning of the proof to the self-adjoint operators $A - aI$ and $B - bI$, we obtain

$$\begin{aligned}\tfrac{1}{4}|E_\psi(AB - BA)|^2 &= \tfrac{1}{4}|E_\psi[(A - aI)(B - bI) - (B - bI)(A - aI)]|^2\\ &= (\operatorname{Im}\langle(A - aI)(B - bI)\psi, \psi\rangle)^2\\ &= (\operatorname{Im}\langle(A - aI)\psi, (B - bI)\psi\rangle)^2\\ &\leqq \|(A - aI)\psi\|^2 \, \|(B - bI)\psi\|^2 \quad \text{(Schwarz)}\\ &= D_\psi(A)D_\psi(B).\end{aligned}$$

This completes the proof.

The quantity $AB - BA$ arising above is called the commutator of A and B, and is of considerable importance in quantum mechanics. A simple calculation using the differentiation product rule shows that the commutator for p and q on $L^2(\boldsymbol{R})$ is given by

$$pq - qp = -iI.$$

If we apply Theorem 1 to the self-adjoint operators p (momentum) and q (position), then we obtain for any appropriate state ψ (recall $\|\psi\| = 1$)

$$D_\psi(p)D_\psi(q) \geqq \tfrac{1}{4}E_\psi(I) = \tfrac{1}{4}. \tag{5}$$

In physics $D_\psi(p)$ and $D_\psi(q)$ are frequently denoted by $(\Delta p)^2$ and $(\Delta q)^2$, so Δp and Δq can be thought of as standard deviations. Making this replacement and taking square roots in (5), we obtain $\Delta p \cdot \Delta q \geqq \frac{1}{2}$. We now confess that in traditional units the momentum operator includes a factor $h/(2\pi)$, where $h = 6.625 \cdot 10^{-34}$ joule $\cdot$ sec is Planck's constant. If we allow for this, the inequality becomes

$$\Delta p \cdot \Delta q \geqq \frac{h}{4\pi},$$

and we have quantified the famous **Heisenberg uncertainty principle** that the position and complementary momentum of a particle cannot simultaneously be determined with complete precision. Specifically, the product of "uncertainties" associated with determining momentum and position must always exceed a magnitude on the order of Planck's constant, which fortunately for us all is rather small.

The above development can be carried out in a general Hilbert space H for any pair P and Q of self-adjoint operators whose commutator is a nonzero multiple of the identity. If (as indicated by the importance of p and q) we accept the importance to physicists of operators whose commutator has this property, the following problem suggests itself as one of fundamental significance.

General problem. Find all pairs of self-adjoint operators P and Q on H which satisfy

$$PQ - QP = -iI \tag{c}$$

on some "sufficiently large" domain Ω.

Any tendency on the part of mathematicians to confine their attention to the bounded (continuous) operators on H would appear to be challenged by the following elegant result.

THEOREM 2 (Wielandt, 1949). *There do not exist bounded operators P and Q on H satisfying* (c) *on all of H.*

Proof. There is no loss of generality if we replace (c) by

$$I = PQ - QP, \tag{6}$$

since this could be arranged by using iP instead of P. Suppose, as the basis of an indirect proof, that (6) holds everywhere on H for bounded operators P and Q. Then an induction argument shows that for every $n = 1, 2, \cdots$

$$nQ^{n-1} = PQ^n - Q^nP. \tag{7}$$

Indeed the $n = 1$ case is simply the assumed result (6); and, assuming (7) holds for a general n, we have

$$(n+1)Q^n = nQ^{n-1}Q + Q^nI = (PQ^n - Q^nP)Q + Q^n(PQ - QP) = PQ^{n+1} - Q^{n+1}P.$$

This establishes the induction step. Applying (2) and (3) to (7), we obtain for $n = 1, 2, \cdots$

$$n\| Q^{n-1} \| \leqq 2\| P \| \; \| Q \| \; \| Q^{n-1} \|.$$

This result requires that $\| Q^n \| = 0$ for *some* n since otherwise we would have $n \leqq 2\| P \| \; \| Q \|$ for every n, which is impossible under the assumption that P and Q are bounded. Finally, using (7) repeatedly, we obtain

$$\| Q^n \| = 0 \Rightarrow Q^n = 0 \Rightarrow Q^{n-1} = 0 \Rightarrow \cdots \Rightarrow Q = 0 \Rightarrow I = 0,$$

which is untenable (except in 0 dimensional Hilbert space). This establishes our contradiction and shows that such bounded operators cannot exist.

Theorem 2 provides a partial answer to our "general problem" in the sense that we need not look among the bounded operators for solutions to (c). It also seems to create a new philosophical problem. In addition to the various well-known and often serious differences of viewpoint, the beloved continuity of mathematicians now appears quite incompatible with the "unbounded" requirements of physicists as regards one important aspect of quantum mechanics. In the final section we shall achieve a reconciliation by establishing a correspondence between special kinds of bounded and unbounded operators. At the same time a very pleasing solution to our general problem will emerge.

4. Unitary groups and Schrödinger couples. There is a useful analogy which relates self-adjoint operators to real numbers and unitary operators to complex numbers of modulus 1. Since the transformation

$$\tau \mapsto e^{it\tau} \quad (t \in \boldsymbol{R})$$

maps a real number τ to a one-parameter multiplicative group of complex numbers with modulus 1, analogy suggests such a transformation might convert self-adjoint operators (even unbounded ones) into one-parameter groups of unitary operators (which are of necessity bounded). All this is formalized in the following famous theorem. See Reed and Simon [**8**, p. 265] for a proof and for a definition of "strongly continuous".

THEOREM (Stone, 1932). *Every self-adjoint operator T on a Hilbert space H generates a strongly continuous one-parameter group of unitary operators e^{itT} on H. Conversely, every such one-parameter group is generated by a unique self-adjoint operator.*

The procedure for "exponentiation" of operators is of considerable interest and complexity (thus witness the 808 pages of [**4**]). Here we motivate plausibility by noting that if T is bounded, then the familiar power series expansion

$$e^{it\tau} = \sum_{k=0}^{\infty} \frac{(it\tau)^k}{k!}$$

carries over directly; and if T is unbounded, then the identity

$$e^{it\tau} = \lim_{k\to\infty} \left(1 - \frac{it\tau}{k}\right)^{-k}$$

can be generalized to obtain bounded operators. More precisely, it can be shown (with T self-adjoint) that for $t \in \boldsymbol{R}$

$$\left(I - \frac{itT}{k}\right)^{-1}$$

always exists and is bounded; and that for every $f \in H$

$$e^{itT}f = \lim_{k\to\infty} \left(I - \frac{itT}{k}\right)^{-k} f.$$

As an example of Stone's theorem we extend the unitary operators U and V on $L^2(\boldsymbol{R})$ which we defined earlier. For $t \in \boldsymbol{R}$ define operators $U(t)$ and $V(t)$ on $L^2(\boldsymbol{R})$ by

$$(U(t)f)(x) = f(x+t) \text{ and } (V(t)f)(x) = e^{itx}f(x).$$

Then $U(t)$ and $V(t)$ are strongly continuous one-parameter groups of unitary operators, and their generators are, respectively, the self-adjoint operators p and q. Thus $U(t) = e^{itp}$ and $V(t) = e^{itq}$. To motivate these claims, we show that p can be recaptured from $U(t)$ in much the same way that τ can be recaptured from $u(t)$

$= e^{it\tau}$. Indeed $u'(0) = i\tau$, and similarly

$$(U'(0)f)\,(x) = \lim_{h\to 0} \left(\left(\frac{U(h) - U(0)}{h}\right)f\right)(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Hence $U'(0) = ip$, suggesting that p is the generator of $U(t)$. A similar argument exists for q and $V(t)$.

Armed with this relationship between self-adjoint operators and one-parameter groups of unitary operators, we return to our general problem of determining all pairs of self-adjoint operators satisfying the identity (c). One would expect the existence of a corresponding identity involving one-parameter unitary groups. Such expectations are confirmed by the unitary operator identity

$$\text{(C)} \qquad e^{itP}e^{isQ} = e^{its}e^{isQ}e^{itP} \quad (s,\, t \in \boldsymbol{R}),$$

which corresponds to (c) in the following way:

THEOREM 3 (Fojas, Geher, Nagy, 1960). *Given self-adjoint operators P and Q satisfying* (c) *on Ω such that $(P + iI)\,(Q + iI)\Omega$ or $(Q + iI)\,(P + iI)\Omega$ is dense in H, then P and Q satisfy* (C).

Proof. See Putnam [7, p. 76].

In attempting to obtain pairs of self-adjoint operators satisfying (c) or (C), a direct procedure is to start with p and q and any unitary transformation $U: L^2(\boldsymbol{R}) \to H$. Then the operators P and Q on H defined by $P = UpU^{-1}$ and $Q = UqU^{-1}$ are readily seen to satisfy (c) and (C) since p and q do. Accordingly, we define a **Schrödinger couple** as a pair (P, Q) of self-adjoint operators on H such that $P = UpU^{-1}$ and $Q = UqU^{-1}$ for some unitary operator $U: L^2(R) \to H$.

A further procedure for generating pairs satisfying (c) and (C) is to start with two or more Schrödinger couples and to "splice" together the various P components and also the Q components in a natural fashion, thereby obtaining a new pair satisfying (c) or (C) on the direct sum of the underlying Hilbert spaces. We refer to such a pair as a **direct sum** of Schrödinger couples.

The following theorem takes the bounded operator condition (C) and states most pleasingly that the simple procedures described in the above two paragraphs account for all self-adjoint pairs satisfying (C).

THEOREM 4 (von Neumann, 1931). *Given self-adjoint operators P and Q satisfying* (C), *then (P, Q) is a Schrödinger couple or a direct sum of Schrödinger couples.*

Proof. See Putman [7, p. 65].

It is worth noting that von Neumann's beautiful and rather complicated proof seems to leave little room for improvement or simplification, despite considerable advances in the theory of groups and semigroups of operators.

The astute reader may now see that the end is in sight. Indeed, a careful look at Theorems 3 and 4 with use of the fact that p and q satisfy (c) and (C) shows that we have a very satisfactory answer to our general problem: the self-adjoint pairs P and Q satisfying (c) on a "sufficiently large" domain Ω are precisely the Schrödinger couples and their direct sums; and "sufficiently large" is given meaning in terms of the density of $(P + iI)(Q + iI)\Omega$ or $(Q + iI)(P + iI)\Omega$.

We conclude by observing that a very satisfactory rapprochement has been achieved. The theory of Hilbert space and its operators provides a most effective and elegant setting for formulating ideas of quantum mechanics, and the mathematically natural bounded operators prove to be most valuable in studying the physically indispensable unbounded ones.

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DEPARTMENT OF MATHEMATICS, LAKE FOREST COLLEGE, LAKE FOREST, IL 60045.

R. W. BRINK — AN OBITUARY

J. M. H. OLMSTED

Raymond Woodard Brink was born in Newark, New Jersey, on January 4, 1890, and died in La Jolla, California, on December 27, 1973. He was an undergraduate student at Kansas State University, where he received a B. S. degree in 1908 and a B. S. E. E. degree in 1909. After a brief period of teaching, Brink entered the graduate school of Harvard University, where he earned the Ph. D. degree in 1916. His dissertation, entitled "Some Integral Tests for the Convergence and Divergence of Infinite Series," was published under a slightly different title in the *Transactions of the American Mathematical Society*. He retained an active interest in integral tests for infinite series, and published two more papers on the subject, one in the *Annals of Mathematics* and one in this MONTHLY.

Awarded a Sheldon Traveling Fellowship for postdoctoral study, Brink spent the year 1916–17 studying at the Collège de France and the Sorbonne in Paris, where he began a lifelong devotion to France and to the French language and culture. On two