On a Particular One-Dimensional Random Walk
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Source: *The American Mathematical Monthly*, Vol. 69, No. 7 (Aug. - Sep., 1962), pp. 632-634
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2310829
Accessed: 30/03/2009 13:18

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$$f(x) = a\int_0^x g(t)dt.$$

Since g is continuous, it follows [in the case $f \neq 0$] that

$$f'(x) = ag(x), \qquad \text{for all } x,$$

and, since $g(0) = 1$, $a = f'(0)$. This completes the proof of (28a).

The proof of (28b) is, up to a point, similar to that of (28a). The case $f = 0$ is trivial. In the case $f \neq 0$, an argument like that above shows that

$$(**) \qquad \int_0^x f(t)dt = [1 - g(x)] \lim_{n\to\infty} \frac{x/2n}{f(x/2n)}$$

for all $x \neq 0$ for which the limit on the right exists. Now, since f is continuous, so is g and, by (28a), f is differentiable at 0. If $f'(0) = 0$ then, by (28a), f is constant—but, since $f(0) = 0$ and $f \neq 0$, this is not the case. So, $f'(0) \neq 0$ and, for all $x \neq 0$, the limit in question is $1/f'(0)$. The conclusion of (28b) now follows at once from this and (**).

MATHEMATICAL NOTES

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ON A PARTICULAR ONE-DIMENSIONAL RANDOM WALK

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Consider the familiar one-dimensional random walk in which a particle, initially at the origin, moves either one unit to the right or one unit to the left at each step. Let X_i be a random variable that assumes the values 1 and -1 depending on whether the ith step is to the right or to the left, and let $S_m = \sum_{i=1}^m X_i$. Then $S_m = 0$ is the event of a return to the origin at the mth step (which can occur, of course, only if m is even), and $\sum_{m=1}^{2n} P\{S_m = 0\}$, which we will denote by E_n, is the expected number of returns to the origin in a $2n$-step walk. The following theorem is well known ([1], p. 298):

Theorem. *If $P\{X_i = 1\} = \frac{1}{2}$, for $i = 1, 2, \cdots$, then $E_n \sim C\sqrt{n}$, where C is independent of n.*

As Feller mentions in [1], this result seems at first glance to be paradoxical. We are led to ask: what requirements must the distributions $P\{X_i = 1\}$ satisfy in order that $E_n = O(n)$? In its generality, this question appears to present many difficulties; in this note we will merely present one case in which $E_n = O(n)$.

THEOREM. *Suppose that* $P\{X_i=1\}$ *is equal to*: $p>\frac{1}{2}$ *whenever* $S_{i-1}<0$, $q=1-p$ *whenever* $S_{i-1}>0$, $\frac{1}{2}$ *whenever* $S_{i-1}=0$; *then* $E_n\sim(2-1/p)n$.

Proof. Let $P\{S_{2m}=2k\}$ be abbreviated by $P_m(k)$, and consider the expected value of $|S_{2m}|$.

$$E|S_{2m}| = \sum_{k=-m}^{m} |2k| P_m(k) = 4\sum_{k=1}^{m} kP_m(k).$$

The following recurrence relations are easily established:

$$\begin{aligned}
&P_{m+1}(0) = 2p^2P_m(1) + pP_m(0),\\
&P_{m+1}(1) = p^2P_m(2) + 2pqP_m(1) + \frac{q}{2}P_m(0),\\
(1)\qquad &P_{m+1}(k) = p^2P_m(k+1) + 2pqP_m(k) + q^2P_m(k-1) \qquad \text{for } 1 < k < m\\
&P_{m+1}(m) = 2pqP_m(m) + q^2P_m(m-1),\\
&P_{m+1}(m+1) = q^2P_m(m).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E|S_{2m+2}| &= 2\sum_{k=1}^{m+1} 2kP_{m+1}(k)\\
&= 4p^2\sum_{k=1}^{m-1} kP_m(k+1) + 8pq\sum_{k=1}^{m} kP_m(k) + 4q^2\sum_{k=2}^{m+1} kP_m(k-1) + 2qP_m(0)\\
&= 4p^2\sum_{k=1}^{m} (k-1)P_m(k) + 8pq\sum_{k=1}^{m} kP_m(k) + 4q^2\sum_{k=1}^{m} (k+1)P_m(k) + 2qP_m(0)\\
&= E|S_{2m}| + 4(q^2-p^2)\sum_{k=1}^{m} P_m(k) + 2qP_m(0)\\
&= E|S_{2m}| + 2[1-2p+pP_m(0)].
\end{aligned}$$

Since $E|S_2|=2q$, summation over $m=1, 2, \cdots, n$ results in

$$(2)\qquad E_n = \sum_{m=1}^{n} P_m(0) = \frac{2n(2p-1) + E|S_{2n+2}| - 2q}{2p}.$$

Since $E|S_{2n+2}|>0$, we have a lower bound for E_n. To find an upper bound, we show that

$$P_m(k+1) < \frac{q}{p}P_m(k), \qquad k = 0, 1, \cdots, m-1,$$

for all $m>0$. The proof, which is omitted, is by induction on m and depends on the relations (1). It follows that

$$E|S_{2n+2}| < 4\sum_{k=1}^{n+1} k\left(\frac{q}{p}\right)^k P_{n+1}(0)$$

$$< \frac{4q}{p}\sum_{k=1}^{\infty} k\left(\frac{q}{p}\right)^{k-1} = \frac{4q}{p}\cdot\frac{1}{(1-q/p)^2}.$$

Therefore, from (2),

$$\frac{n(2p-1)}{p} + C_2 > E_n > \frac{n(2p-1)}{p} + C_1$$

and this implies the theorem.

Reference

1. William Feller, An Introduction to Probability Theory and its Applications, vol. 1, 2nd ed., Wiley, New York, 1957.

THE NUMBER OF POSSIBLE AUCTIONS AT BRIDGE

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The game of bridge has given rise to a wide variety of interesting problems in combinatorial analysis, most of which have been collected by Borel and Cheron.* Many of these problems are intimately related to the strategy of bidding and play. In fact, a player cannot proceed beyond the novice stage without acquiring at least an intuitive knowledge of suit-split probabilities, etc. On the other hand, some combinatorial problems of bridge have nothing to do with the strategy of the game: It is of no value at the bridge table to know there are 635,013,559,600 possible hands and 53,644,737,765,488,792,839,237,440,000 ($\approx 5.4\times 10^{28}$) possible deals.

One interesting nonstrategic problem which, as far as I can tell, has never appeared in the literature of either bridge or mathematics, concerns the number of possible bidding auctions. At a first glance, it appears that this number could be calculated only by a laborious tallying of the possibilities, perhaps using a few obvious shortcuts. As we shall see, however, counting the auctions in a slightly unnatural way leads quite directly to the answer. We shall assume that the reader is familiar with bridge terminology, but not necessarily with bridge strategy.

We exclude from consideration the infinity of possible auctions in which improper calls are made and condoned. An auction without such irregularities consist of at most thirty-five bids, at least three *pass's*, possibly some *double's* and *redouble's*. There is one null auction, consisting entirely of four *pass's*; we omit this case from the calculation and add it in at the end.

Consider an auction in which k bids are made, where $1\leqq k\leqq 35$. As each bid must outrank its predecessor, there are $\binom{35}{k}$ possible sequences of bids leading to such an auction.

* Borel, É. and Cheron, A., La Théorie Mathématique du Bridge, Gauthier-Villars, Paris, 1955.