# Exponential vs. Factorial

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**1. INTRODUCTION.** What is the largest number you can write with five characters, using ordinary mathematical notation?[1] Here are two tempting candidates:

$$9^{9^{9^{9^{9}}}} \qquad 9!!!!$$

But which of these two is larger? The answer will emerge in the course of our investigations, but the reader might want to take a guess before continuing. (Note that we follow the usual convention that grouping in repeated exponentials is to the right. Thus, $9^{9^9}$ means $9^{(9^9)}$, not $(9^9)^9$. Also, we use repeated exclamation points to denote repeated applications of the factorial function.)

We begin by generalizing the foregoing question. We are interested in comparing repeated applications of an exponential function, with any base, and the factorial function. In the long run, which leads to larger values? More precisely, suppose that $k$ is a positive integer, $b$ is a positive real number, and $c$ is any real number. Then we recursively define an iterated exponential sequence $\{E_n\}$ based on $b$ and $c$ and an iterated factorial sequence $\{F_n\}$ based on $k$ as follows:

$$E_0 = c, \qquad E_{n+1} = b^{E_n} \quad (n = 0, 1, \dots)$$

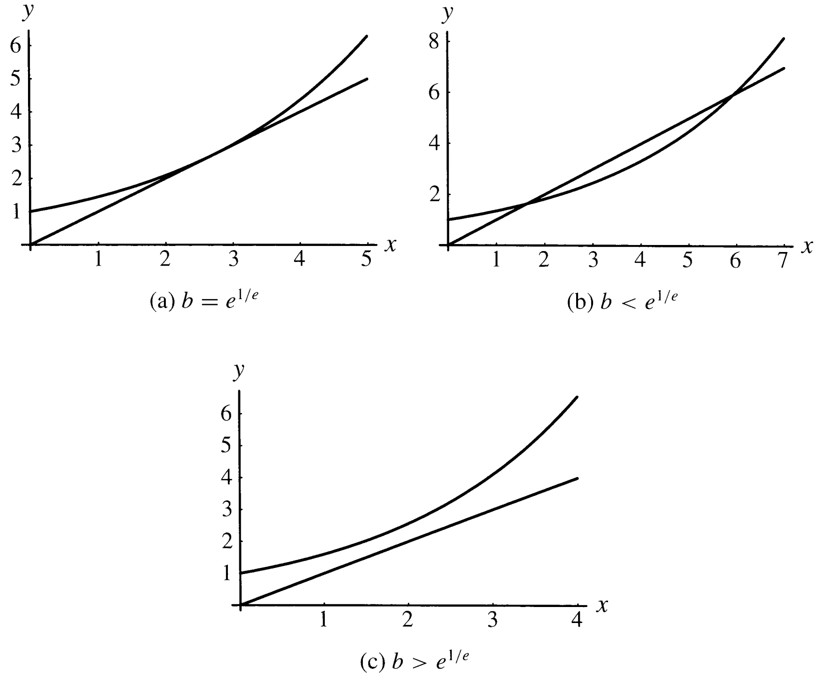
and

$$F_0 = k, \qquad F_{n+1} = F_n! \quad (n = 0, 1, \dots).$$

We wish to determine which is greater for large $n$, $E_n$ or $F_n$. If $b$, $c$, and $k$ are not clear from context, we may write $E_n(b, c)$ and $F_n(k)$ to denote the terms in the two sequences.

We are interested only in values of $b$, $c$, and $k$ that lead to sequences that grow without bound. For the iterated factorial sequence this means that we must have $k \geq 3$, but for the iterated exponential sequence the situation is a bit more complicated. Clearly we must have $b > 1$, but this does not suffice to guarantee that the sequence grows without bound. Figure 1a shows the graph of the curve $y = b^x$ for $b = e^{1/e}$. It is easy to verify that this curve is tangent to the line $y = x$ at the point $(e, e)$, and this implies that if $c \leq e$ then the sequence $\{E_n\}$ is bounded by $e$. On the other hand, if $c > e$ then the sequence $\{E_n\}$ is strictly increasing and $\lim_{n\to\infty} E_n = \infty$. When $1 < b < e^{1/e}$, the curve $y = b^x$ intersects the line $y = x$ twice, as shown in Figure 1b—in other words, the function $b^x$ has two fixed points—and it is only when $c$ is larger than the larger fixed point that the sequence $\{E_n\}$ grows without bound. If $b > e^{1/e}$, the sequence $\{E_n\}$ always increases and approaches $\infty$, as is evident from Figure 1c. Thus, we can describe the values of $b$ and $c$ that interest us as follows. For $b$ satisfying $1 < b \leq e^{1/e}$, let $f(b)$ be the largest $x$ such that $b^x = x$. Then the exponential sequence $\{E_n\}$ grows

[1] This question was suggested to me by Jim Henle of Smith College, who used it as the basis for a contest for undergraduates. My work was motivated by the difficulty that Jim had in determining the winner of his contest.




(a) $b = e^{1/e}$ (b) $b < e^{1/e}$

(c) $b > e^{1/e}$

**Figure 1.** Graphs of $y = b^x$ and $y = x$.

without bound if and only if either $b > e^{1/e}$, or $1 < b \leq e^{1/e}$ and $c > f(b)$. We define $R$ to be the set of all pairs $(b, c)$ for which the exponential sequence $\{E_n\}$ grows without bound. In other words,

$$R = \{(b, c) \in \mathbb{R}^2 : \text{either } b > e^{1/e}, \text{ or } 1 < b \leq e^{1/e} \text{ and } c > f(b)\}.$$

We can get a better understanding of the function $f$ by graphing the relation $b^x = x$ or, equivalently, $b = x^{1/x}$. It is easy to check that the function $x^{1/x}$ is continuous on $[1, \infty)$, increasing on $[1, e]$, and decreasing on $[e, \infty)$, and that $\lim_{x\to\infty} x^{1/x} = 1$, as can be seen in Figure 2. Since $f(b)$ is the largest $x$ such that $x^{1/x} = b$, the graph of $f$ is the solid line in Figure 2. It is easy to see that $f$ is a continuous, decreasing function on $(1, e^{1/e}]$, with $f(e^{1/e}) = e$ and $\lim_{b\to 1^+} f(b) = \infty$. The function $f$ is related to the Lambert $W$-function by the equation $f(b) = -W_{-1}(-\ln b)/\ln b$. For information about the Lambert $W$-function, see [**2**].

It will be convenient to summarize at the outset some simple ordering properties of iterated exponential sequences.

**Proposition 1.** *Iterated exponential sequences have the following properties:*

1. *If* $(b, c) \in R$, *then* $E_{n+1}(b, c) > E_n(b, c)$ *for all* $n$, *and* $\lim_{n\to\infty} E_n(b, c) = \infty$.
2. *If* $b > 1$ *and* $n$ *is a nonnegative integer, then* $E_n(b, c_1) > E_n(b, c_2)$ *whenever* $c_1 > c_2$, *and* $\lim_{c\to\infty} E_n(b, c) = \infty$.
3. *If* $b_1 > b_2 > 1$ *and* $c > 0$, *then* $E_n(b_1, c) > E_n(b_2, c)$ *for all* $n \geq 1$.
4. *If* $b_1 > b_2 > 1$, $n \geq 1$, *and* $E_n(b_1, c_1) \geq E_n(b_2, c_2)$, *then* $E_m(b_1, c_1) > E_m(b_2, c_2)$ *for all* $m > n$.

*Proof.* All four parts are easily proved by induction. The first simply states formally the fact that the sequence $\{E_n(b, c)\}$ grows without bound for $(b, c)$ in $R$. Notice that in

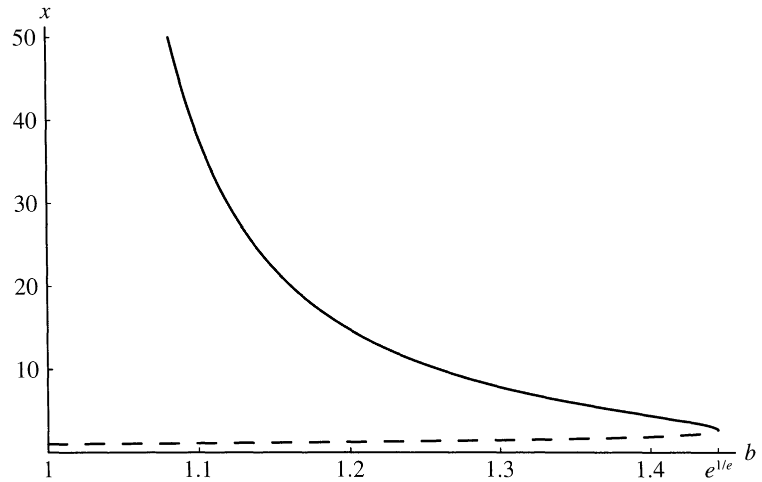



**Figure 2.** The dashed and solid lines together make up the graph of the relation $b = x^{1/x}$. The solid line is the graph of the function $f$.

part 3 the hypothesis $c > 0$ is needed to verify the base case of the induction. There is no need for a similar positivity hypothesis in part 4, because $E_n(b_2, c_2) > 0$ for every positive $n$. ■

There is a sizeable literature on iterated exponential sequences that converge, beginning with Euler [**4**]. A modern exposition is given in [**1**], and an extensive list of references can be found in [**5**]. However, iterated exponential sequences that grow without bound seem to have received less attention. Devaney et al. discuss a problem involving such sequences in [**3**]. We will have more to say about this problem later.

**2. EXPONENTIAL VS. FACTORIAL.** In comparing iterated exponential and factorial sequences it is helpful to have some bounds on the factorial function. If $n \geq 3$, then using upper rectangles to overestimate an integral gives us

$$\ln n! = \sum_{i=2}^{n} \ln i > \int_1^n \ln x \, dx = [x \ln x - x]_1^n = n \ln n - n + 1,$$

and therefore

$$n! > e^{n \ln n - n + 1} = \frac{n^n}{e^{n-1}}. \tag{1}$$

Similarly, using lower rectangles we get

$$\ln n! = \sum_{i=2}^{n-1} \ln i + \ln n < \int_2^n \ln x \, dx + \ln n = (n+1) \ln n - n - 2 \ln 2 + 2,$$

so

$$n! < \frac{n^{n+1}}{4e^{n-2}}. \tag{2}$$

Sometimes we use the simpler bounds

$$\left(\frac{n}{e}\right)^n < n! < n^n. \tag{3}$$

The lower bound in (3) is smaller than the lower bound in (1) by a factor of $1/e$, and the upper bound in (3) is larger than the upper bound in (2) by a factor of $4e^{n-2}/n$. The upper bound in (3) can also be established by observing that $n!$ is a product of $n$ integers running from 1 to $n$, whereas $n^n$ is a product of $n$ integers, all of which are equal to $n$. The following strengthening of the upper bound in (3) will also prove useful:

**Proposition 2.** *For every integer $n \geq 3$, $n! \ln n! < n^n$.*

*Proof.* By (2) and (3),

$$\frac{n! \ln n!}{n^n} < \frac{n^{n+1} \ln n^n}{4e^{n-2} n^n} = \frac{n^2 \ln n}{4e^{n-2}}.$$

Letting $g(x) = x^2 \ln x/(4e^{x-2})$, one can easily check that $g(3) \approx 0.91 < 1$ and $g'(x) < 0$ when $x \geq 3$. Therefore $n! \ln n!/n^n < 1$, and the proposition follows. ∎

We are now ready to compare an iterated exponential sequence $\{E_n(b, c)\}$ with an iterated factorial sequence $\{F_n(k)\}$. We assume that $k \geq 3$ in order to ensure that the factorial sequence grows without bound, and it will simplify our reasoning if we also assume, at least in this section, that $b \geq e$. The next two propositions are the keys to our comparison.

**Proposition 3.** *If $E_n > F_n \ln F_n/\ln b$, then $E_m > F_m \ln F_m/\ln b$ for all $m \geq n$.*

*Proof.* The proof is by induction. The base case holds by hypothesis. For the induction step, suppose that $m \geq n$ and $E_m > F_m \ln F_m/\ln b$. Then $E_m \ln b > F_m \ln F_m$, so using Proposition 2 and the fact that $b \geq e$ we have

$$\begin{aligned} E_{m+1} &= e^{E_m \ln b} > e^{F_m \ln F_m} = F_m^{F_m} > F_m! \ln F_m! \\ &= F_{m+1} \ln F_{m+1} \geq F_{m+1} \ln F_{m+1}/\ln b. \end{aligned}$$ ∎

**Proposition 4.** *If $E_n < F_n(\ln F_n - 1)/\ln b$ and $F_{n+1} \geq be$, then $E_m < F_m \leq F_m(\ln F_m - 1)/\ln b$ for all $m > n$.*

*Proof.* We have $E_n \ln b < F_n(\ln F_n - 1)$, so by (3)

$$E_{n+1} = e^{E_n \ln b} < e^{F_n(\ln F_n - 1)} = \left(\frac{F_n}{e}\right)^{F_n} < F_n! = F_{n+1}.$$

From $F_{n+1} \geq be$ we infer that $\ln F_{n+1} \geq 1 + \ln b$, and therefore $(\ln F_{n+1} - 1)/\ln b \geq 1$. Combining our conclusions, we deduce that

$$E_{n+1} < F_{n+1} \leq F_{n+1}(\ln F_{n+1} - 1)/\ln b.$$

But now we can repeat this argument with $n + 1$ in place of $n$. The proposition then follows by induction. ∎

We now consider holding $k$ and $b$ fixed but allowing $c$ to vary. Let

$$I_1 = \{c \in \mathbb{R} : \text{for some } n,\ E_n(b, c) > F_n \ln F_n / \ln b\}$$

and

$$I_2 = \{c \in \mathbb{R} : \text{for some } n,\ E_n(b, c) < F_n(\ln F_n - 1)/\ln b \text{ and } F_{n+1} \geq be\}.$$

Since $E_n(b, c)$ is continuous as a function of $c$, $I_1$ and $I_2$ are open sets, and by Propositions 3 and 4 they are disjoint. By Proposition 1, $I_1$ is closed upwards and $I_2$ is closed downwards (i.e., if $c \in I_1$ and $c' > c$, then $c' \in I_1$, while if $c \in I_2$ and $c' < c$, then $c' \in I_2$). It follows that each set is an open interval.

Suppose that $c = k \ln k / \ln b$. Then by Proposition 2

$$E_1 = b^c = e^{k \ln k} = k^k > k! \ln k! = F_1 \ln F_1 \geq F_1 \ln F_1 / \ln b,$$

so $c$ is in $I_1$. This shows that $[k \ln k / \ln b, \infty) \subseteq I_1$. Next, we try to locate an element of $I_2$. We have $F_1(\ln F_1 - 1)/\ln b \geq 6(\ln 6 - 1)/\ln b > 0$ and $\lim_{c \to -\infty} E_1 = \lim_{c \to -\infty} b^c = 0$, which implies that, for sufficiently small $c$, $E_1 < F_1(\ln F_1 - 1)/\ln b$. If we also assume that $b \leq k!!/e$, then $F_2 \geq be$, and therefore $c$ belongs to $I_2$.

Combining everything that we know about $I_1$ and $I_2$, we conclude that if $e \leq b \leq k!!/e$, then there are numbers $c_1$ and $c_2$ such that $I_1 = (c_1, \infty)$, $I_2 = (-\infty, c_2)$, and $c_2 \leq c_1 < k \ln k / \ln b$. (Later we will see that $c_1$ and $c_2$ must be equal.) What about values of $c$ in the interval $[c_2, c_1]$? If $c_2 \leq c \leq c_1$, then for every positive integer $n$ we have

$$\frac{F_n(\ln F_n - 1)}{\ln b} \leq E_n \leq \frac{F_n \ln F_n}{\ln b},$$

and therefore

$$\frac{1}{\ln b} - \frac{1}{\ln b \ln F_n} \leq \frac{E_n}{F_n \ln F_n} \leq \frac{1}{\ln b}.$$

Letting $n \to \infty$, we see that

$$\lim_{n \to \infty} \frac{E_n}{F_n \ln F_n} = \frac{1}{\ln b}.$$

Motivated by this conclusion, we introduce the following terminology. We say that two sequences $\{a_n\}$ and $\{b_n\}$ that are eventually nonzero are *comparable* if $\lim_{n \to \infty} a_n / b_n$ is defined, finite, and nonzero. It is easy to verify that comparability is an equivalence relation. In this terminology, what we have shown is that if $e \leq b \leq k!!/e$, then there is at least one value of $c$ smaller than $k \ln k / \ln b$ such that the sequences $\{E_n\}$ and $\{F_n \ln F_n\}$ are comparable.

In particular, in the case $b = k$ we have shown that there is some $c$ less than $k$ such that $\{E_n(k, c)\}$ and $\{F_n \ln F_n\}$ are comparable. It follows that

$$\lim_{n \to \infty} \frac{E_n(k, c)}{F_n} = \lim_{n \to \infty} \frac{E_n(k, c)}{F_n \ln F_n} \cdot \ln F_n = \infty. \tag{4}$$

Thus, for large $n$, $E_n(k, c)$ is much larger than $F_n$, and therefore $E_n(k, k)$ is much larger than $F_n$. In fact, we can see that the sequence $\{E_n(k, k)\}$ takes the lead right away. By Proposition 2, $k^k > k! \ln k! > k! \ln k!/\ln k$, and therefore by Proposition 3, $E_n(k, k) > F_n \ln F_n/\ln k > F_n$ for every positive integer $n$. In particular,

$$9^{9^{9^{9^9}}} > 9!!!!,$$

which answers a question from the first paragraph of this paper.

Unfortunately, neither of these numbers is the solution to our original problem of finding the largest number that can be written with five characters. In fact, there are larger numbers that can be written with five characters using positive integers written in decimal notation, exponentiation, and the factorial function. We invite the reader to try to think of a larger number before we reveal the answer in the last section of this paper.

First, however, we turn to another issue suggested by what we have discovered so far. We have observed that for every $k \geq 3$ there are iterated exponential sequences with a range of different bases that are comparable to $\{F_n \ln F_n\}$. But comparability is an equivalence relation, so it follows that there are iterated exponential sequences with different bases that are comparable to each other! We explore this surprising phenomenon in the next section.

**3. EXPONENTIAL VS. EXPONENTIAL.** When are two iterated exponential sequences comparable to each other? Two different exponential sequences with the same base that grow without bound are never comparable, as the following proposition shows.

**Proposition 5.** *If* $c_1 > c_2$ *and* $(b, c_2) \in R$, *then*

$$\lim_{n\to\infty} \frac{E_n(b, c_1)}{E_n(b, c_2)} = \infty.$$

*Proof.* Choose a positive integer $n$ large enough that $E_n(b, c_2) \geq e^2$. Let

$$r = \frac{E_n(b, c_1)}{E_n(b, c_2)} - 1,$$

and note that, by Proposition 1, $r > 0$. We claim now that for each nonnegative $k$

$$\frac{E_{n+k}(b, c_1)}{E_{n+k}(b, c_2)} \geq 1 + 2^k r.$$

The proof is by induction. The base case is clear. For the induction step suppose that, for some $k$,

$$\frac{E_{n+k}(b, c_1)}{E_{n+k}(b, c_2)} \geq 1 + 2^k r.$$

Then

$$E_{n+k}(b, c_1) \geq (1 + 2^k r) E_{n+k}(b, c_2),$$

so

$$E_{n+k+1}(b, c_1) = b^{E_{n+k}(b,c_1)} \geq b^{(1+2^k r)E_{n+k}(b,c_2)} = (E_{n+k+1}(b, c_2))^{1+2^k r}.$$

Dividing through by $E_{n+k+1}(b, c_2)$, we find that

$$\frac{E_{n+k+1}(b, c_1)}{E_{n+k+1}(b, c_2)} \geq (E_{n+k+1}(b, c_2))^{2^k r} \geq (e^2)^{2^k r} = e^{2^{k+1} r} \geq 1 + 2^{k+1} r.$$

This proves the claim, and since $\lim_{k\to\infty}(1 + 2^k r) = \infty$, the proposition follows. ■

This proposition suggests another piece of terminology. We say that a sequence $\{a_n\}$ *dominates* a sequence $\{b_n\}$ if $\lim_{n\to\infty} a_n/b_n = \infty$. In this terminology, Proposition 5 says that if two iterated exponential sequences that grow without bound have the same base, then the one with the larger initial value dominates the other one.

Next, we consider exponential sequences with different bases. Suppose that $(b_0, c_0) \in R$, so $\{E_n(b_0, c_0)\}$ grows without bound. We wish to investigate the question: Given an arbitrary number $b$ greater than 1, is there a $c$ such that $\{E_n(b, c)\}$ and $\{E_n(b_0, c_0)\}$ are comparable? By Proposition 5, if such a $c$ exists, it is unique. We define $c_{b_0,c_0}(b)$ to be the unique $c$ such that $\{E_n(b, c)\}$ is comparable to $\{E_n(b_0, c_0)\}$, if there is such a $c$. To determine when $\{E_n(b, c)\}$ is comparable to $\{E_n(b_0, c_0)\}$, we first ascertain when one of these sequences dominates the other.

**Proposition 6.** *For every $b > 1$ and every $c$, $\lim_{n\to\infty} E_n(b, c)/E_n(b_0, c_0) = \infty$ if and only if there is some positive integer $n$ such that the following inequalities hold:*

$$\frac{E_n(b, c)}{E_n(b_0, c_0)} > \frac{2\ln b_0}{\ln b}, \qquad E_n(b_0, c_0) > \frac{2\ln b_0}{\ln b}.$$

*Proof.* Since $\lim_{n\to\infty} E_n(b_0, c_0) = \infty$, the left-to-right direction is clear. For the other direction, suppose that the displayed inequalities hold for some $n$. By the first inequality, $E_n(b, c)\ln b > 2E_n(b_0, c_0)\ln b_0$. Applying the exponential function to both sides of this inequality yields

$$E_{n+1}(b, c) = b^{E_n(b,c)} > b_0^{2E_n(b_0,c_0)} = (E_{n+1}(b_0, c_0))^2.$$

Dividing through by $E_{n+1}(b_0, c_0)$ and invoking the second inequality, we obtain

$$\frac{E_{n+1}(b, c)}{E_{n+1}(b_0, c_0)} > E_{n+1}(b_0, c_0) > E_n(b_0, c_0) > \frac{2\ln b_0}{\ln b}.$$

But this means that we can repeat the reasoning we have just used, with $n + 1$ in place of $n$. It follows, by induction, that

$$\frac{E_m(b, c)}{E_m(b_0, c_0)} > E_m(b_0, c_0)$$

whenever $m > n$. Since $\lim_{n\to\infty} E_n(b_0, c_0) = \infty$, we infer that

$$\lim_{n\to\infty} \frac{E_n(b, c)}{E_n(b_0, c_0)} = \infty$$

as well. ■

**Corollary 7.** *For every $b > 1$ and every $c$, $\lim_{n\to\infty} E_n(b, c)/E_n(b_0, c_0) = 0$ if and only if either $(b, c) \notin R$ or there is some positive integer $n$ such that the following* *inequalities hold:*

$$\frac{E_n(b,c)}{E_n(b_0,c_0)} < \frac{\ln b_0}{2\ln b}, \quad E_n(b,c) > \frac{2\ln b}{\ln b_0}.$$

*Proof.* If $(b,c) \notin R$, then $\{E_n(b,c)\}$ is bounded. Because $\{E_n(b_0,c_0)\}$ grows without bound, this implies that $\lim_{n\to\infty} E_n(b,c)/E_n(b_0,c_0) = 0$. If $(b,c) \in R$, we appeal to Proposition 6, with the roles of $(b,c)$ and $(b_0,c_0)$ reversed and the ratios of iterated exponentials inverted. ■

Proposition 6 and Corollary 7 tell us how to determine, for any $b > 1$ and any $c$, whether $\lim_{n\to\infty} E_n(b,c)/E_n(b_0,c_0)$ is either $\infty$ or 0. It turns out that there is only one other possible value for this limit. To see why, suppose that the limit is neither $\infty$ nor 0. Then $(b,c) \in R$, so we can choose $n_0$ large enough that $E_{n_0}(b,c) > 2\ln b/\ln b_0$ and $E_{n_0}(b_0,c_0) > 2\ln b_0/\ln b$. By Proposition 6 and Corollary 7, it follows that

$$\frac{\ln b_0}{2\ln b} \le \frac{E_n(b,c)}{E_n(b_0,c_0)} \le \frac{2\ln b_0}{\ln b},$$

provided $n \ge n_0$. Applying this fact with $n+1$ in place of $n$ and taking natural logarithms, we get

$$\ln\ln b_0 - \ln\ln b - \ln 2 \le \ln E_{n+1}(b,c) - \ln E_{n+1}(b_0,c_0) \le \ln\ln b_0 - \ln\ln b + \ln 2.$$

Now, by the definition of the iterated exponential sequence,

$$\ln E_{n+1}(b,c) = \ln b^{E_n(b,c)} = E_n(b,c)\ln b,$$

and similarly $\ln E_{n+1}(b_0,c_0) = E_n(b_0,c_0)\ln b_0$. Filling in these values, we have

$$\ln\ln b_0 - \ln\ln b - \ln 2 \le E_n(b,c)\ln b - E_n(b_0,c_0)\ln b_0 \le \ln\ln b_0 - \ln\ln b + \ln 2.$$

Finally, we divide through by $E_n(b_0,c_0)\ln b$ to get

$$\frac{\ln\ln b_0 - \ln\ln b - \ln 2}{E_n(b_0,c_0)\ln b} \le \frac{E_n(b,c)}{E_n(b_0,c_0)} - \frac{\ln b_0}{\ln b} \le \frac{\ln\ln b_0 - \ln\ln b + \ln 2}{E_n(b_0,c_0)\ln b}.$$

As $n \to \infty$, the first and last fractions approach 0, so we have

$$\lim_{n\to\infty} \frac{E_n(b,c)}{E_n(b_0,c_0)} = \frac{\ln b_0}{\ln b}.$$

Thus we have proved the following proposition:

**Proposition 8.** *For all $b > 1$ and all $c$, $\lim_{n\to\infty} E_n(b,c)/E_n(b_0,c_0)$ is either $\infty$, 0, or $\ln b_0/\ln b$. Thus, either $\{E_n(b,c)\}$ and $\{E_n(b_0,c_0)\}$ are comparable, or one dominates the other.*

Imitating our reasoning in the previous section, we now hold $b_0$, $c_0$, and $b$ fixed but allow $c$ to vary. Let

$$I_\infty = \{c \in \mathbb{R} : \lim_{n\to\infty} E_n(b,c)/E_n(b_0,c_0) = \infty\}$$

and

$$I_0 = \{c \in \mathbb{R} : \lim_{n\to\infty} E_n(b, c)/E_n(b_0, c_0) = 0\}.$$

Clearly $I_\infty$ is closed upwards and $I_0$ is closed downwards, and by Propositions 5 and 8 there is at most one number that is not contained in either $I_0$ or $I_\infty$. An important consequence of Proposition 6 and Corollary 7 is that both of these sets are open.

**Proposition 9.** *The sets $I_\infty$ and $I_0$ are open.*

*Proof.* Suppose that $c \in I_\infty$. Then by Proposition 6 we can find some positive integer $n$ such that the following inequalities hold:

$$\frac{E_n(b, c)}{E_n(b_0, c_0)} > \frac{2\ln b_0}{\ln b}, \qquad E_n(b_0, c_0) > \frac{2\ln b_0}{\ln b}.$$

Since $E_n(b, c)$ is continuous as a function of $c$, there is some $\epsilon > 0$ such that if $c - \epsilon < c' < c + \epsilon$, then

$$\frac{E_n(b, c')}{E_n(b_0, c_0)} > \frac{2\ln b_0}{\ln b}.$$

But then, by Proposition 6, $(c - \epsilon, c + \epsilon) \subseteq I_\infty$, verifying that $I_\infty$ is open.

Now suppose that $c \in I_0$. If $(b, c) \in R$, then we can imitate the preceding argument, using Corollary 7 instead of Proposition 6, to find a neighborhood of $c$ that is contained in $I_0$. Now assume that $(b, c) \notin R$. This means that $1 < b \le e^{1/e}$ and $c \le f(b)$. Choose a number $x$ large enough that $x > 2\ln b/\ln b_0$ and $x > f(b)$, and choose a positive integer $n$ large enough that

$$\frac{x}{E_n(b_0, c_0)} < \frac{\ln b_0}{2\ln b}.$$

Since $E_n(b, f(b)) = f(b) < x$, $\lim_{c\to\infty} E_n(b, c) = \infty$, and $E_n(b, c)$ is continuous as a function of $c$, we can now find some $c'$ such that $c' > f(b)$ and $E_n(b, c') = x$. Then

$$\frac{E_n(b, c')}{E_n(b_0, c_0)} = \frac{x}{E_n(b_0, c_0)} < \frac{\ln b_0}{2\ln b}$$

and

$$E_n(b, c') = x > \frac{2\ln b}{\ln b_0},$$

so Corollary 7 implies that $\lim_{n\to\infty} E_n(b, c')/E_n(b_0, c_0) = 0$. In other words, $c' \in I_0$. But $I_0$ is closed downwards, so we have $c \in (-\infty, c') \subseteq I_0$. Thus, we have found a neighborhood of $c$ that is contained in $I_0$, as required. ∎

We now know that either $I_0 = \mathbb{R}$ and $I_\infty = \varnothing$, or $I_\infty = \mathbb{R}$ and $I_0 = \varnothing$, or there is some number $c$ such that $I_0 = (-\infty, c)$ and $I_\infty = (c, \infty)$. However, the first of these possibilities can be ruled out. To see why, choose a positive integer $n$ large enough that $E_n(b_0, c_0) > 2\ln b_0/\ln b$. Now since $\lim_{c\to\infty} E_n(b, c) = \infty$, there is some $c$ such that $E_n(b, c)/E_n(b_0, c_0) > 2\ln b_0/\ln b$. By Proposition 6, $c \in I_\infty$, so $I_\infty \neq \varnothing$. Thus, either

$I_\infty = \mathbb{R}$ and $I_0 = \varnothing$, or there is some number $c$ such that $I_0 = (-\infty, c)$ and $I_\infty = (c, \infty)$. In the first case, every iterated exponential sequence with base $b$ dominates $\{E_n(b_0, c_0)\}$, and $c_{b_0,c_0}(b)$ is undefined. In the second case, $c_{b_0,c_0}(b) = c$. Thus, we have established the following proposition:

**Proposition 10.** *For all $(b_0, c_0) \in R$ and $b > 1$, $c_{b_0,c_0}(b)$ is defined if and only if there is some $c$ such that $\{E_n(b, c)\}$ does not dominate $\{E_n(b_0, c_0)\}$.*

Notice that if $c_{b_0,c_0}(b)$ is defined, then for all $c$,

$$\lim_{n\to\infty} \frac{E_n(b,c)}{E_n(b_0,c_0)} = \begin{cases} \ln b_0/\ln b & \text{if } c = c_{b_0,c_0}(b), \\ \infty & \text{if } c > c_{b_0,c_0}(b), \\ 0 & \text{if } c < c_{b_0,c_0}(b). \end{cases}$$

This means that we can use Proposition 6 and Corollary 7 to compute $c_{b_0,c_0}(b)$ to any desired degree of accuracy. For example, consider the case $b_0 = c_0 = 2$ and $b = 3$. Here are the first few terms of the sequence $\{E_n(2, 2)\}$:

$$2,\ 2^2 = 4,\ 2^4 = 16,\ 2^{16} = 65536,\ 2^{65536} \approx 2.00353 \times 10^{19728},\ \ldots.$$

For $c = c_{2,2}(3)$ we expect to have $E_4(3, c)/E_4(2, 2) \approx \ln 2/\ln 3 \approx 0.63093$, and solving for $c$ leads to the approximation $c \approx 0.660434$. If we let $c = 0.66043$, then the first few terms of the sequence $\{E_n(3, c)\}$ are approximately

$$0.66043,\ 2.06588,\ 9.67555,\ 4.13436 \times 10^4,\ 8.44817 \times 10^{19725},\ \ldots.$$

Thus, we have

$$\frac{E_4(3,c)}{E_4(2,2)} \approx 0.0042 < \frac{\ln 2}{2\ln 3}, \qquad E_4(3,c) \approx 8.44817 \times 10^{19725} > \frac{2\ln 3}{\ln 2}.$$

It follows by Corollary 7 that $\lim_{n\to\infty} E_n(3, c)/E_n(2, 2) = 0$. On the other hand, if $c = 0.66044$, then the first few terms of the sequence $\{E_n(3, c)\}$ are approximately

$$0.66044,\ 2.06590,\ 9.67579,\ 4.13546 \times 10^4,\ 1.43133 \times 10^{19731},\ \ldots.$$

This time we can invoke Proposition 6 to conclude that

$$\lim_{n\to\infty} \frac{E_n(3,c)}{E_n(2,2)} = \infty.$$

Therefore

$$0.66043 < c_{2,2}(3) < 0.66044.$$

We can construct the graph of the function $c_{2,2}$ by imitating the foregoing calculations to compute values of $c_{2,2}(b)$ for many values of $b$. The resulting graph is shown in Figure 3; Figure 4 shows the graph for $b \le 20$. The points $(b, c)$ on this graph represent iterated exponential sequences $\{E_n(b, c)\}$ that belong to a single equivalence class under the comparability relation.

Figure 5 shows the graphs of $c_{2,c_0}$ for several values of $c_0$. Notice that we have used a logarithmic scale on the $b$-axis in this figure because the domains of some of the functions extend quite far. Each curve in the figure represents an equivalence class under comparability. One of the most striking features of these graphs is that some of

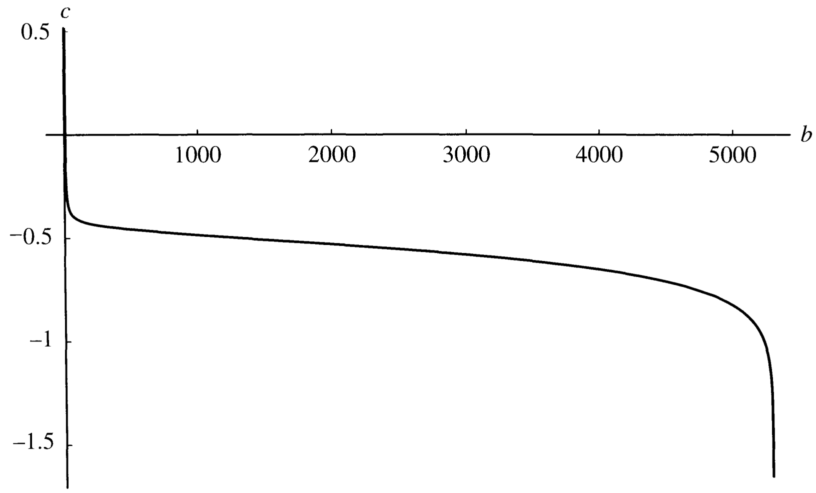



**Figure 3.** The graph of $c = c_{2,2}(b)$.

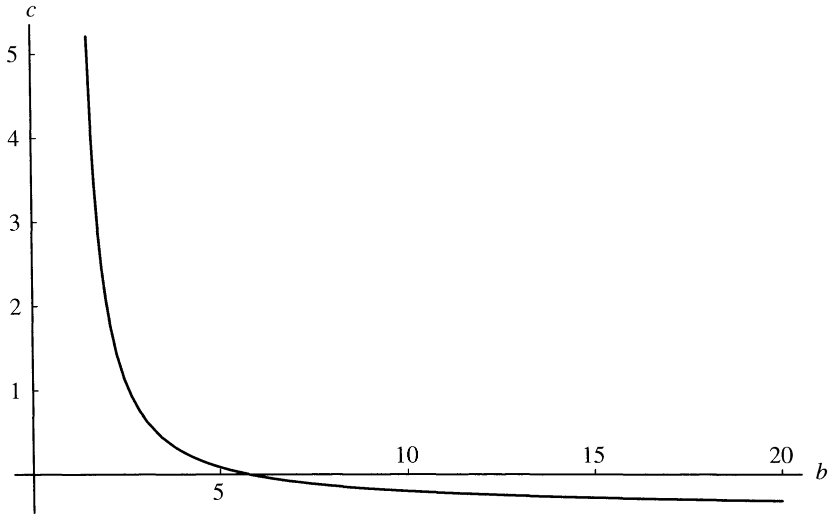



**Figure 4.** The graph of $c = c_{2,2}(b)$ for $b \leq 20$.

the functions are not monotone, and therefore there are pairs of comparable iterated exponential sequences with the same $c$ value but different $b$ values. For example, the sequences $\{E_n(100, -0.1)\}$ and $\{E_n(b, -0.1)\}$ are comparable for $b \approx 3.913 \times 10^{17}$. Here are the first few terms of both sequences:

$$\{E_n(100, -0.1)\}: \; -0.1, 0.63096, 18.2774, 3.58776 \times 10^{36}, \ldots,$$

$$\{E_n(b, -0.1)\}: \; -0.1, 0.017408, 2.02419, 4.07868 \times 10^{35}, \ldots.$$

Taking the ratios of corresponding terms, we find that

$$\frac{E_3(100, -0.1)}{E_3(b, -0.1)} \approx 8.796 \approx \ln b / \ln 100,$$

exactly as expected.

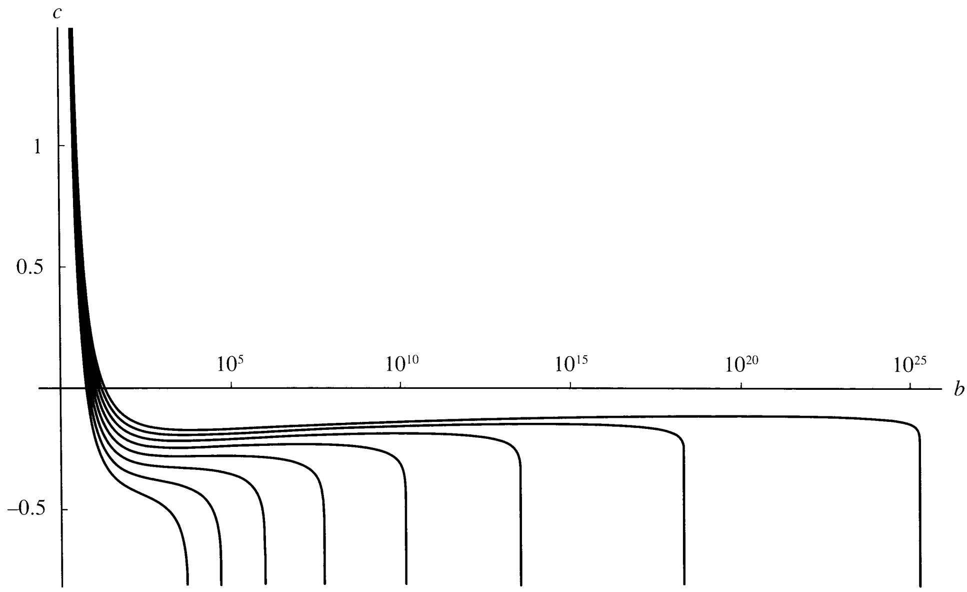



**Figure 5.** The graphs of $c = c_{2,c_0}(b)$ for $c_0 = 2, 2.1, 2.2, \ldots, 2.7$.

Our previous examples suggest a number of properties of the function $c_{b_0,c_0}$. These are summarized in the following theorem:

**Proposition 11.** *There is a number $\bar{b}$ such that the domain of $c_{b_0,c_0}$ is the open interval $(1, \bar{b})$, $c_{b_0,c_0}$ is continuous on this interval, $\lim_{b\to 1^+} c_{b_0,c_0}(b) = \infty$, and $\lim_{b\to \bar{b}^-} c_{b_0,c_0}(b) = -\infty$. Also, there is a number $b^*$ such that $1 < b^* < \bar{b}$, $c_{b_0,c_0}(b^*) = 0$, $c_{b_0,c_0}$ is strictly decreasing on the interval $(1, b^*]$, and $c_{b_0,c_0}(b) < 0$ for all $b$ in the interval $(b^*, \bar{b})$.*

*Proof.* Suppose that $1 < b_1 < b_2$ and $b_2$ is in the domain of $c_{b_0,c_0}$. Let $c_2 = c_{b_0,c_0}(b_2)$, and choose $c_1$ small enough that $b_1^{c_1} < b_2^{c_2}$. Then $E_n(b_1, c_1) < E_n(b_2, c_2)$ for every positive integer $n$. It follows that $\{E_n(b_1, c_1)\}$ does not dominate $\{E_n(b_2, c_2)\}$, and since $\{E_n(b_2, c_2)\}$ is comparable to $\{E_n(b_0, c_0)\}$, $\{E_n(b_1, c_1)\}$ does not dominate $\{E_n(b_0, c_0)\}$ either. Therefore, by Proposition 10, $b_1$ is in the domain of $c_{b_0,c_0}$. This shows that the domain is an interval, with left endpoint 1.

Next, we verify that the domain is open and that $c_{b_0,c_0}$ is continuous. Suppose that $b$ is in the domain of $c_{b_0,c_0}$, and let $c = c_{b_0,c_0}(b)$. Let $\epsilon > 0$ be arbitrary. Since $(b, c) \in R$ and $R$ is open, we can choose $c'$ such that $c - \epsilon \le c' < c$ and $(b, c') \in R$. Because $c' < c_{b_0,c_0}(b)$, we have $\lim_{n\to\infty} E_n(b, c')/E_n(b_0, c_0) = 0$, and by Corollary 7 it follows that there is some $n$ such that

$$\frac{E_n(b, c')}{E_n(b_0, c_0)} < \frac{\ln b_0}{2 \ln b}, \quad E_n(b, c') > \frac{2 \ln b}{\ln b_0}.$$

But now since $E_n(b, c')$ and $\ln b$ are continuous functions of $b$, there is a $\delta > 0$ such that if $b - \delta < b' < b + \delta$, then

$$\frac{E_n(b', c')}{E_n(b_0, c_0)} < \frac{\ln b_0}{2 \ln b'}, \quad E_n(b', c') > \frac{2 \ln b'}{\ln b_0}.$$

It follows that, for all $b'$ in $(b-\delta, b+\delta)$, $\lim_{n\to\infty} E_n(b', c')/E_n(b_0, c_0) = 0$, and therefore $c_{b_0,c_0}(b')$ is defined and $c_{b_0,c_0}(b') > c' \geq c - \epsilon$. Similar reasoning, using Proposition 6 instead of Corollary 7, shows that we can also ensure (by shrinking $\delta$ if necessary) that $c_{b_0,c_0}(b') < c + \epsilon$ whenever $b'$ is in $(b-\delta, b+\delta)$. This implies that the domain of $c_{b_0,c_0}$ is an open interval and that $c_{b_0,c_0}$ is continuous on this interval.

To see that the domain is bounded, first note that for any $b$ and $c$ with $b > 1$

$$E_3(b, c) = b^{b^{b^c}} > b^{b^0} = b.$$

Now let $b = E_3(b_0, c_0) > b_0$. Suppose that $b$ is in the domain of $c_{b_0,c_0}$, and let $c = c_{b_0,c_0}(b)$. Then $E_3(b, c) > b = E_3(b_0, c_0)$, and it follows that $E_n(b, c) > E_n(b_0, c_0)$ for all $n \geq 3$. But this contradicts the fact that

$$\lim_{n\to\infty} \frac{E_n(b, c)}{E_n(b_0, c_0)} = \frac{\ln b_0}{\ln b} < 1.$$

Therefore $E_3(b_0, c_0)$ is not in the domain of $c_{b_0,c_0}$, so there is a number $\bar{b} \leq E_3(b_0, c_0)$ such that the domain is the interval $(1, \bar{b})$.

If $1 < b \leq e^{1/e}$, then $\{E_n(b_0, c_0)\}$ dominates $\{E_n(b, f(b))\}$, and it follows that $c_{b_0,c_0}(b) > f(b)$. Since $\lim_{b\to 1^+} f(b) = \infty$, we infer that $\lim_{b\to 1^+} c_{b_0,c_0}(b) = \infty$ as well. To evaluate $\lim_{b\to \bar{b}^-} c_{b_0,c_0}(b)$, consider an arbitrary real number $c$. Since $c_{b_0,c_0}(\bar{b})$ is undefined, Proposition 10 implies that $\{E_n(\bar{b}, c)\}$ dominates $\{E_n(b_0, c_0)\}$. Using Proposition 6 we can then find some $\delta > 0$ such that, for all $b$ in $(\bar{b}-\delta, \bar{b})$, $\{E_n(b, c)\}$ dominates $\{E_n(b_0, c_0)\}$. Therefore $c_{b_0,c_0}(b) < c$, and since $c$ was arbitrary, it follows that $\lim_{b\to \bar{b}^-} c_{b_0,c_0}(b) = -\infty$.

It is now clear that there is some number $b^*$ such that $c_{b_0,c_0}(b^*) = 0$. To establish the remaining claims about $b^*$, suppose that $1 < b_1 < b_2$, $c_2 = c_{b_0,c_0}(b_2) \geq 0$, and $c_1 = c_{b_0,c_0}(b_1) \leq c_2$. Then

$$E_1(b_1, c_1) \leq E_1(b_1, c_2) \leq E_1(b_2, c_2),$$

where the second inequality holds because $c_2 \geq 0$. Therefore, by Proposition 1, $E_n(b_1, c_1) < E_n(b_2, c_2)$ when $n > 1$. But $\{E_n(b_1, c_1)\}$ and $\{E_n(b_2, c_2)\}$ must be comparable, since they are both comparable to $\{E_n(b_0, c_0)\}$, and therefore

$$\lim_{n\to\infty} \frac{E_n(b_1, c_1)}{E_n(b_2, c_2)} = \frac{\ln b_2}{\ln b_1} > 1,$$

so we have reached a contradiction. ■

One consequence of our analysis of the graph of $c_{b_0,c_0}$ is that there is a unique number $b$ such that $c_{b_0,c_0}(b) = b$. In other words, there is a unique $b$ such that $\{E_n(b_0, c_0)\}$ is comparable to $\{E_n(b, b)\}$. Thus, among iterated exponential sequences that grow without bound, each equivalence class under the comparability relation contains exactly one sequence of the form $b, b^b, b^{b^b}, \dots$.

These observations can be used to give another solution to the problem posed in [**3**]. In our notation, the problem involves comparing $E_n(b, b)$ with $E_{n+1}(a, a) = E_n(a, a^a)$ when $e^{1/e} < a < b$. We now know that for each $a$ satisfying $a > e^{1/e}$ there is a unique $b$ such that $\{E_n(b, b)\}$ is comparable to $\{E_n(a, a^a)\}$. It is not hard to see that $a < b < a^a$ and $E_n(a, a^a) > E_n(b, b)$ for all $n$, and of course we know that

$$\lim_{n\to\infty} \frac{E_n(a, a^a)}{E_n(b, b)} = \frac{\ln b}{\ln a} > 1.$$

If $1 < b' < b$, then $E_n(a, a^a) > E_n(b', b')$ for all $n$. On the other hand, if $b' > b$, then $\lim_{n\to\infty} E_n(a, a^a)/E_n(b', b') = 0$, and therefore $E_n(a, a^a) < E_n(b', b')$ for sufficiently large $n$. The number $b$ here is what is called $\gamma(a)$ in [**3**].

We can also use what we have discovered about comparability of iterated exponential sequences to learn more about how iterated exponential and factorial sequences compare. We showed in section 2 that for every integer $k \geq 3$ there is some $c < k$ such that $\{E_n(k, c)\}$ and $\{F_n(k) \ln F_n(k)\}$ are comparable. By Proposition 5, we now know that this value of $c$ is unique. In fact, we know that for every $b$ in some interval containing $(1, k]$ there is a unique $c$ such that $\{E_n(b, c)\}$ is comparable to $\{F_n \ln F_n\}$. Also, there is a unique $b$ such that $\{E_n(b, b)\}$ is comparable to $\{F_n \ln F_n\}$.

What about the original iterated factorial sequence $\{F_n\}$? Let $c_0$ be the unique real number such that $\{E_n(k, c_0)\}$ is comparable to $\{F_n \ln F_n\}$. As we observed in equation (4), it follows that $\{E_n(k, c_0)\}$ dominates $\{F_n\}$. Now consider any $(b, c)$ in $R$. We have demonstrated that either $\{E_n(b, c)\}$ and $\{E_n(k, c_0)\}$ are comparable, or one dominates the other. If $\{E_n(b, c)\}$ either dominates $\{E_n(k, c_0)\}$ or is comparable to it, then clearly $\{E_n(b, c)\}$ also dominates $\{F_n\}$. Now suppose that $\{E_n(k, c_0)\}$ dominates $\{E_n(b, c)\}$. Applying Propositions 5 and 9, we can find $c'$ with $c' < c_0$ such that $\{E_n(k, c_0)\}$ dominates $\{E_n(k, c')\}$ and $\{E_n(k, c')\}$ dominates $\{E_n(b, c)\}$. It is easy to see that $\{F_n(\ln F_n - 1)\}$ is comparable to $\{F_n \ln F_n\}$, which is comparable to $\{E_n(k, c_0)\}$, so $\{F_n(\ln F_n - 1)\}$ dominates $\{E_n(k, c')\}$. Thus, for sufficiently large $n$, $E_n(k, c') < F_n(\ln F_n - 1)/\ln k$. By Proposition 4, it follows that $E_n(k, c') < F_n$ for large $n$. Since $\{E_n(k, c')\}$ dominates $\{E_n(b, c)\}$, we can conclude that $\{F_n\}$ does as well.

In summary, the iterated factorial sequence $\{F_n\}$ is not comparable to any iterated exponential sequence. Rather, it occupies a position between the iterated exponential sequences that are dominated by $\{E_n(k, c_0)\}$ and those that either dominate or are comparable to $\{E_n(k, c_0)\}$.

For example, consider the case $k = 9$. The first few terms of the sequence $\{F_n(9)\}$ are

$$9,\ 9! = 362880,\ 9!! \approx 1.60971 \times 10^{1859933},\ \ldots.$$

Using Propositions 3 and 4, we can show that $\{F_n \ln F_n\}$ is comparable to $\{E_n(9, c_0)\}$ for $c_0 \approx 6.59145$. The first few terms of the sequence $\{E_n(9, c_0)\}$ are

$$c_0 \approx 6.59145,\ 9^{c_0} \approx 1.94913 \times 10^6,\ 9^{9^{c_0}} \approx 3.13753 \times 10^{1859939},\ \ldots.$$

The sequence $\{E_n(9, c_0)\}$ dominates $\{F_n\}$, but $\{F_n\}$ dominates $\{E_n(9, c)\}$ whenever $c < c_0$. There are many other iterated exponential sequences comparable to $\{F_n \ln F_n\}$, but perhaps the most interesting is the sequence $\{E_n(b, b)\}$ for $b \approx 7.32339$. The first few terms of this sequence are

$$b \approx 7.32339,\ b^b \approx 2.15093 \times 10^6,\ b^{b^b} \approx 3.46238 \times 10^{1859939},\ \ldots.$$

**4. INTRODUCTION REVISITED.** We now return to the problem of finding the largest number that can be written with five characters using decimal notation for positive integers, exponentiation, and the factorial function. For each $n$ let $I_n$ be the set of all $n$-digit positive integers. We define sets $T_n$ as follows:

$$T_1 = I_1 = \{1, 2, 3, \ldots, 9\},$$

and for $n > 1$,

$$T_n = I_n \cup \{p^q : \text{for some } i \text{ and } j,\ p \in T_i,\ q \in T_j, \text{ and } i + j = n\} \cup \{p! : p \in T_{n-1}\}.$$

Clearly every number that can be written with $n$ characters using positive integers, exponentiation, and the factorial function is an element of $T_n$. (However, some elements of $T_n$ may require more than $n$ characters, because parentheses may be needed.) We now find the largest element of $T_n$.

Let $m_n$ be the largest element of $T_n$. Clearly $m_1 = 9$, and by listing the elements of $T_2$ it is not hard to check that $m_2 = 9^9 = 387{,}420{,}489$. The rest of the values of $m_n$ are determined by the following proposition:

**Proposition 12.** *For $n \geq 2$, $m_{n+1} = m_n!$.*

*Proof.* Clearly $m_{n+1}$ is either the largest element of $I_{n+1}$, which is $10^{n+1} - 1$, or a number of the form $m_i^{m_j}$, where $i + j = n + 1$, or the number $m_n!$. Thus, we must show that $m_n!$ is the largest of these numbers.

Consider first the number $10^{n+1} - 1$. We know that $m_n \geq m_2 > 10e$, so $m_n/e > 10$, and also $m_n \geq 10^n - 1 > n + 1$. Therefore by (3),

$$10^{n+1} - 1 < 10^{n+1} < \left(\frac{m_n}{e}\right)^{m_n} < m_n!.$$

Next, we consider numbers of the form $m_i^{m_j}$, where $i + j = n + 1$. If $i < n$, then notice that $m_n \geq m_{i+1} \geq m_i! > m_i e$, implying that $m_n/e > m_i$ and thus that

$$m_i^{m_j} < \left(\frac{m_n}{e}\right)^{m_n} < m_n!.$$

Now suppose that $i = n$, in which event $j = 1$. In this case we use the fact that $e^{10} \approx 22026.5 < m_2 \leq m_n$, which means that $m_n/e > e^9$, and compute:

$$m_i^{m_j} = m_n^9 = (e^9)^{\ln m_n} < \left(\frac{m_n}{e}\right)^{m_n} < m_n!.$$

We conclude that among all the candidates for $m_{n+1}$, the largest is $m_n!$. ■

In particular, Proposition 12 tells us that $m_5 = m_2!!!$. Thus, the largest number that can be written with five characters, using positive integers, exponentiation, and the factorial function, is $9^9!!!$.

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**Hotel Infinity**

On a dark desert highway—not much scenery
Except this long hotel stretchin' far as I could see.
Neon sign in front read "No Vacancy,"
But it was late and I was tired, so I went inside to plea.

The clerk said, "No problem. Here's what can be done—
We'll move those in a room to the next higher one.
That will free up the first room and that's where you can stay."
I tried understanding this as I heard him say:

*CHORUS*: "Welcome to the HOTEL INFINITY—
Where every room is full (every room is full)
Yet there's room for more.
Yeah, plenty of room at the HOTEL INFINITY—
Move 'em down the floor (move 'em down the floor)
To make room for more."

I'd just gotten settled, I'd finally unpacked
When I saw eight more cars pull into the back.
I had to move to room 9; others moved up eight rooms as well.
Never more will I confuse a Hilton with a Hilbert Hotel!

My mind got more twisted when I saw a bus without end
With an infinite number of riders coming up to check in.
"Relax," said the nightman. "Here's what we'll do:
Move to the double of your room number:
That frees the odd-numbered rooms." (Repeat Chorus)

Last thing I remember at the end of my stay—
It was time to square the bill but I had no means to pay.
The man in 19 smiled, "Your bill is on me.
20 pays mine, and so on, so you get yours for free!"

—Submitted by Lawrence Lesser, University of Texas at El Paso;
lyrics copyright Lawrence Mark Lesser, all rights reserved;
(sung to the tune of the Eagles' "Hotel California").