"digits" are needed to make $\mathfrak{B}$ look whole. Another example: $\mathfrak{B}_9$ is disconnected, although one cannot really tell from the image—a proof is needed. The four upper right blobs can be separated from the four bottom left ones by a strip that can be physically measured and seen to be greater than $1/50$. The modulus of b is smaller than 0.64, so with the powers greater than 13 a gap wider than $0.64^{14}/(1 - 0.64)$ (which is well under $1/50$) cannot be bridged. Therefore $\mathfrak{B}_9$ is indeed disconnected.

The set of those b for which $\mathfrak{B}$ is connected is described in [**3**], where, among other things, it is proven that $\mathfrak{B}$ is connected if $|b| > \sqrt{2}/2$. Among the $\mathfrak{B}$s shown in FIGURES 1–10, only $\mathfrak{B}_9$ and $\mathfrak{B}_{10}$ are disconnected.

8. Miscellaneous. A lot of things are left to be explored. When $\mathfrak{B}$ is connected and simply connected (see FIGS. 7 and 8), is the boundary of fractal dimension strictly greater than 1 (as it seems)? Is that boundary the image of a continuous map from $[0, 1]$ in $\mathbb{R}^2$ (as it seems)? Why does $\mathfrak{B}_5$ have $\mathfrak{B}_5$-like eggs inside it (dark gray)?

I tried to replace $\mathbb{C}$ by the algebra with $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$, but got nothing worth showing. Why not?

Acknowledgement. I am grateful to the referees for many suggestions.

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A Characterization of the Cantor Function

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This note presents a characterization of the Cantor function that might be used as an alternative or addition to the development usually presented in real analysis courses. It may also be utilized to give a short program in *Mathematica* that easily generates the Cantor function and other similar functions which we call "devil's staircases" (see [**4**]).

THEOREM. *Any real-valued function $F(x)$ on $[0, 1]$ that is monotone increasing and satisfies (a) $F(0) = 0$, (b) $F(x/3) = F(x)/2$, and (c) $F(1 - x) = 1 - F(x)$, is the Cantor function.*

Before presenting the proof, recall (see [**1**]) that if we consider the closed interval $[0, 1]$ and remove the open middle third, $(1/3, 2/3)$, and next remove the open middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the two remaining intervals, and then remove the open middle thirds of the remaining four intervals and so on, indefinitely, what remains is the *Cantor set C*. Alternatively, x is in C iff x has a base-3 expansion consisting only of the digits 0 and 2.

The *Cantor function* $G(x)$ may be defined as follows. First define it on C: if $x = \Sigma_i 2 \cdot 3^{-n_i}$, then $G(x) = \Sigma_i 2^{-n_i}$. The function G is monotone and has the same values at the endpoints of each removed interval, so G extends to a continuous function on $[0, 1]$ (see FIGURE 3(a)).

On $[1/3, 2/3]$, G has value $1/2$, on $[1/9, 2/9]$ and $[7/9, 8/9]$, G has values $1/4$ and $3/4$, respectively. On the intervals $[1/27, 2/27]$, $[7/27, 8/27]$, $[19/27, 20/27]$, and $[25/27, 26/27]$, the values of G are $1/8$, $3/8$, $5/8$, and $7/8$, respectively. Since G is (locally) constant on some neighborhood of every point in $[0, 1] \setminus C$, $G'(x) = 0$ almost everywhere on $[0, 1]$. (See [**1**] or [**3**].)

Proof of the theorem. First observe that in constructing the Cantor set, the removed intervals (in base 3) are as given in TABLE 1. The endpoints of any removed interval at the nth stage are found by either multiplying those from the previous stage by .1 (base 3), or by multiplying by .1 and adding .2.

TABLE 1

Step	Values of G	Removed Intervals (closures)
1	$1/2$	[.1, .2]
2	$1/4, 3/4$	[.01, .02], [.21, .22]
3	$1/8, 3/8$	[.001, .002], [.021, .022],
	$5/8, 7/8$	[.201, .202], [.221, .222]
4	$1/16, 3/16$	[.0001, .0002], [.0021, .0022],
	$5/16, 7/16$	[.0201, .0202], [.0221, .0222]
	$9/16, 11/16$	[.2001, .2002], [.2021, .2022],
	$13/16, 15/16$	[.2201, .2202], [.2221, .2222], etc.

Alternatively, any permutation of $n - 1$ 0s and 2s after the ternary point followed by a 1 or a 2 gives a removed interval at the nth stage.

Recursion is now used to characterize F: By (a) and (c), $F(1) = 1$, so $F(.1) = 1/2$. By (b) and (c), $F(.2) = 1 - F(.1) = 1/2$; (b) implies $F(.01) = F(.02) = 1/4$ and (c) implies $F(.22) = F(.21) = 3/4$; (b) then yields the correct values of $1/8$, $3/8$ on the firsts two intervals at stage 3, while (c) yields the value $F(.201) = 1 - F(.022) = 5/8$. It follows by induction that F and G agree on a dense subset of $[0, 1]$. Since F is monotone increasing and has no jump discontinuities, F is continuous (because any discontinuity of a monotone function is a jump; see [**1**, p. 129]). Thus since F is continuous and agrees with the Cantor function on a dense set, F is the Cantor function, and the proof is complete.

As an alternative proof of continuity, F may be exhibited as a uniform limit of a sequence of monotone increasing step functions $\{s_n\}$ on $[0, 1]$ with the jumps converging to 0. For example, define

$$s_1(x) = \begin{cases} 0 & \text{if } x < .1 \\ 1/2 & \text{if } .1 \leqslant x \leqslant .2 \\ 1 & \text{if } .2 < x \end{cases}$$

and inductively, define

$$s_n(x) = \begin{cases} s_{n-1}(3x)/2 & \text{if } x < .1 \\ 1/2 & \text{if } .1 \leqslant x \leqslant .2 \\ 1 - s_n(1 - x) & \text{if } .2 < x \end{cases}$$

(assuming s_n as defined on the first line). The continuity of F is then a consequence of the following lemma, which may be of independent interest.

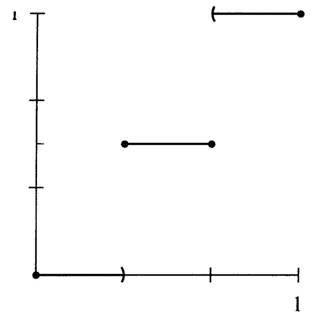


Figure 1(a) Graph of s_1.

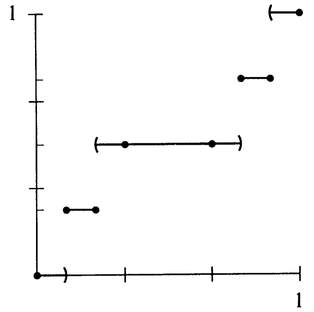


Figure 1(b) Graph of s_2.

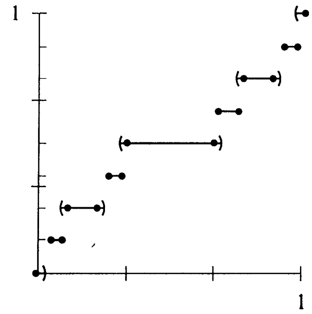


Figure 1(c) Graph of s_3.

LEMMA. *Let $\{s_n\}$ be a Cauchy sequence in uniform norm of step functions on* $[0, 1]$, *each with a finite number of jumps such that the heights of the jumps for s_n converge uniformly to 0. Then s_n converges uniformly to a continuous function F on* $[0, 1]$.

Proof. Pick x in $[0, 1]$ and $\varepsilon > 0$. There exists an integer N such that $n, m \geqslant N$ implies $|s_n(x) - s_m(x)| < \varepsilon/4$ for all x in $[0, 1]$, and so that the jumps of s_n are all less than $\varepsilon/4$. Now any s_n is uniformly continuous on each interval on which there is not a jump, so we can pick δ so that $|x - y| < \delta$ implies $|s_N(x) - s_N(y)| < 2\varepsilon/4$. Hence, $|x - y| < \delta$ implies $|F(x) - F(y)| \leqslant |F(x) - s_N(x)| + |s_N(x) - s_N(y)| + |s_N(y) - F(y)| < \varepsilon$. Thus F is continuous. QED

Since $n < m$ implies $|s_n(x) - s_m(x)| \leqslant |s_{n-1}(y) - s_{m-1}(y)|/2$ where y is one of $3x$ or $1 - 3x$, it then follows inductively that $|s_n(x) - s_m(x)| \leqslant 1/2^n$ when $n \leqslant m$. Thus $\{s_n\}$ is uniformly Cauchy and the continuity of F follows from the lemma.

Note also that F may be generated by a geometric algorithm, where we begin by defining $F(x) = 1/2$ on the interval $[.1, .2]$ and $F(0) = 0$, $F(1) = 1$, and where at each stage of the algorithm we shrink the x axis by a factor of $1/3$ and the y axis by a factor of $1/2$ and then flip the resultant graph across the line $x = 1/2$, and then again across the line $y = 1/2$. We see F as coming more into focus at each stage.

The preceding characterization was used (together with the recursive feature of *Mathematica*) to generate FIGURES 3(a)–(b) (programmed by Gerald Harnett). Other "devil's staircases" were obtained by taking any $p > 0$ and changing condition (c) to (c'): $F(1 - x) = 1 - pF(x)$, for $x \leqslant 1/3$ and condition (b) to (b'): $F(x/3) = F(x)/(p + 1)$. These generalizations arise as the cumulative distribution functions of some of the probability measures invariant under the "inverted

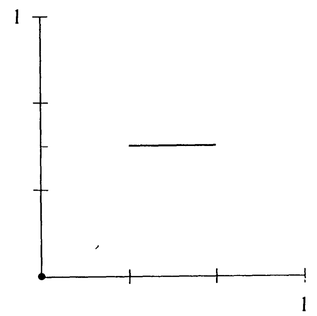


Figure 2(a).

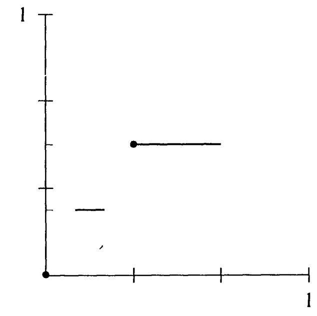


Figure 2(b)
Shrink 2(a) and superimpose on 2(a).

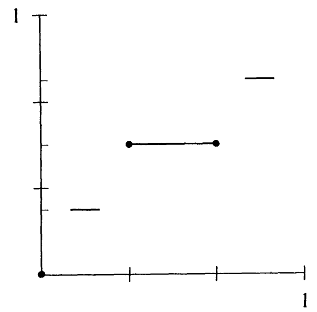


Figure 2(c)
Double flip 2(b) and superimpose on 2(b).

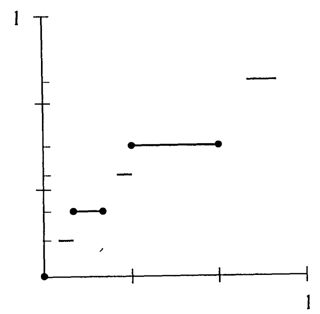
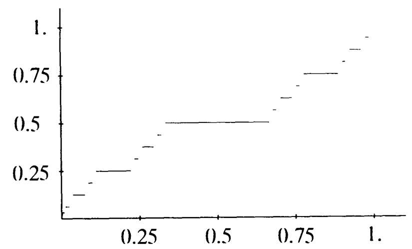
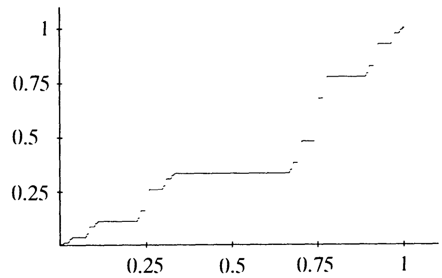


Figure 2(d)
Shrink 2(c) and superimpose on 2(c).



The Cantor function ($p = 1$).
FIG. 3a



The Cantor function ($p = 2$).
FIG. 3b

V" transformation discussed by Mandelbrot (see [**2, 4**]). Finally, by letting p take on, for example, the values .01, .1, .3, .5, 1, 4, 10, and 100, one can create a movie in *Mathematica* that shows the Cantor function "stressed" by the varying of the parameter p (see FIGURE 3).

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