What's the Difference Between Cantor Sets?

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In this article, we will look at a family of sets called the middle-α Cantor sets and we will try to show that not all of these sets are of the same "size." We will do this without computing any of the myriad kinds of fractal dimensions. Instead, we will prove three theorems about what are called the difference sets of the middle-α Cantor sets and these theorems will provide the evidence we need to conclude that some middle-α Cantor sets are "larger" than others. The proofs of these theorems will use methods that emphasize the ideas of self-similarity, rescaling and renormalization.

The middle-α Cantor sets are a straightforward generalization of the classical middle third Cantor set (see [**B**, pp. 33–36] or [**E**, pp. 1–6]). To define a middle-α Cantor set in the interval $[0,1]$, first choose $\alpha \in (0,1)$, let $\beta = (1-\alpha)/2$ and then define the affine maps

$$T_0(x) = \beta x \quad \text{and} \quad T_1(x) = \beta x + (1-\beta).$$

Let $I_0 = [0,1]$ and then, for $n \geq 1$, inductively define

$$I_n = T_0(I_{n-1}) \cup T_1(I_{n-1}).$$

Then the middle-α Cantor set in the interval $[0,1]$ is defined as

$$\Gamma_\alpha = \bigcap_{n=0}^{\infty} I_n.$$

Let's look at this definition more carefully. Notice that $T_0(I_0) = [0,\beta]$ and $T_1(I_0) = [1-\beta, 1]$, so $I_1 = [0,\beta] \cup [1-\beta, 1]$. The "hole" in I_1 has length $1-2\beta = \alpha$. So another way of describing I_1 is to say that we remove from the middle of I_0 the open interval of length α, leaving two closed intervals of length β. Now notice that $T_0(I_1) = [0,\beta^2] \cup [\beta(1-\beta),\beta]$ and that $T_1(I_1) = [1-\beta, \beta^2 + (1-\beta)] \cup [(1-\beta)+\beta(1-\beta), 1]$. So we get I_2 from I_1 by removing from the middle of each component of I_1 an open interval of length $\alpha\beta$. Each of the four components of I_2 has length β^2. In general, I_n is a disjoint union of 2^n closed intervals of length β^n and we get I_{n+1} by removing from the middle of each component of I_n an open interval of length $\alpha\beta^n$. The collection $\{I_n\}_{n=0}^{\infty}$ is a nested sequence of compact subsets of $[0,1]$ and it has the finite intersection property, so by the compactness of $[0,1]$, Γ_α is a nonempty set. Each Γ_α is a compact, nowhere dense, perfect subset of the real line and hence is a Cantor set. (How would you modify T_0 and T_1 to define a middle-α Cantor set in the interval $[a,b]$?)

Fix a choice of $\alpha \in (0,1)$ and let x be a point in Γ_α. Define the *address* in Γ_α of x to be a sequence $\{s_0, s_1, s_2, \dots\}$ where each s_i is either 0 or 1. To determine s_i, consider x as a point in I_{i+1} and if x is to the left of the nearest hole of length

$\alpha\beta^i$, then $s_i = 0$ and if x is to the right of the nearest hole of length $\alpha\beta^i$, then $s_i = 1$. The primary usefulness of the address of a point x from Γ_α is the fact, which you should check, that x has a unique representation of the form

$$x = \sum_{i=0}^{\infty} s_i\beta^i(1-\beta) \quad \text{where } s_i \in \{0,1\},$$

and the "digits" in this representation are the terms of the address. (Hint: The 2^n points of the form $\sum_{i=0}^{n-1} s_i\beta^i(1-\beta)$ are the left-hand endpoints of the 2^n components of I_n.) You should also check that when $\alpha = \beta = 1/3$, this representation is equivalent to the standard ternary representation of the points in the middle third Cantor set.

You may have noticed that the number β is more useful than α in describing Γ_α. This is because β^{-1} is a scaling factor for Γ_α. By this we mean that $\beta^{-1}(\Gamma_\alpha \cap [0,\beta]) = \Gamma_\alpha$ (where, if γ is a real number and A is a subset of the real line, then $\gamma A = \{\gamma x | x \in A\}$). In the rest of this article, when referring to a middle-α Cantor set, we will freely go back and forth between the two numbers $\beta = (1-\alpha)/2$ and $\alpha = 1 - 2\beta$.

Compare the middle-9/10 and the middle-1/10 Cantor sets. One seems "larger" than the other. But how can we demonstrate this? We could try comparing the number of points in the two sets but it's not too hard to show that all the middle-α Cantor sets have the same cardinality (in fact, any two Cantor sets are homeomorphic; see [HY, pp. 97–100]). So we can't use cardinality to compare the sizes of different middle-α Cantor sets. Another way to compare the sizes of two sets is to compare the values of their Lebesgue measure. But we'll now show that each of the middle-α Cantor sets has Lebesgue measure zero. Choose $\alpha \in (0,1)$. Each set I_n covers Γ_α. I_n contains 2^n components, each of length β^n. So the total length of I_n is $2^n\beta^n = (2\beta)^n$. So the total length of Γ_α is less than or equal to $(2\beta)^n$ for all n. Now $\alpha \in (0,1)$ implies that $\beta \in (0,1/2)$, so $2\beta < 1$ and therefore the total lengths of the I_n go to zero as n goes to infinity. And this is what it means to say that the Lebesgue measure of Γ_α is zero. So we can't use Lebesgue measure to compare the sizes of the middle-α Cantor sets. (It is not true, however, that every Cantor set has measure zero; see [**B**, pp. 63–64].)

So we can't use cardinality or Lebesgue measure to distinguish between different middle-α Cantor sets. But it would be nice if we could find some way to use these two simple notions to demonstrate that the middle-1/10 Cantor set is "larger" than the middle-9/10 Cantor set. And that is exactly what we will be able to do, by applying cardinality and Lebesgue measure to the *difference set*, $\Gamma_\alpha - \Gamma_\alpha$, of a middle-$\alpha$ Cantor set. The name and notation for the difference set come from the definition,

$$\Gamma_\alpha - \Gamma_\alpha = \{x - y | x, y \in \Gamma_\alpha\}.$$

Here is a more "dynamical" way of defining the difference set,

$$\Gamma_\alpha - \Gamma_\alpha = \{t | \Gamma_\alpha \cap (\Gamma_\alpha + t) \neq \varnothing\}$$

where $\Gamma_\alpha + t = \{x + t | x \in \Gamma_\alpha\}$. In this second definition, $\Gamma_\alpha + t$ is a translate of Γ_α and $\Gamma_\alpha - \Gamma_\alpha$ tells us which of these translates intersect with Γ_α. Imagine a copy of Γ_α moving down the number line with constant velocity one such that it coincides with Γ_α when $t = 0$. The difference set describes those times when the moving copy of Γ_α intersects with Γ_α. Notice that for any choice of $\alpha \in (0,1)$, $\Gamma_\alpha - \Gamma_\alpha \subset [-1,1]$.

How can difference sets be used to show that one middle-α Cantor set is larger than another? Intuitively, the larger a set is, the more often it should intersect with a translate of itself, so the larger its difference set should be. Also, the larger a set is, the larger we would expect the intersection that set has with its translates to be. So what we will do is apply cardinality and Lebesgue measure to the difference sets $\Gamma_\alpha - \Gamma_\alpha$ and to the intersections $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ for $t \in \Gamma_\alpha - \Gamma_\alpha$. We will show that, as α decreases, the Lebesgue measure of $\Gamma_\alpha - \Gamma_\alpha$ will increase, and the minimum cardinality of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ for $t \in \Gamma_\alpha - \Gamma_\alpha$ will increase. However, the price we pay for using such simple concepts as Lebesgue measure and cardinality is that we can only demonstrate changes in the size of Γ_α for a few values of α. We will not be able to demonstrate a continuous increase in the size of the middle-α Cantor sets as α decreases, which is really the case.

Our first theorem is about the measure of the difference sets. We will state the theorem in terms of β instead of α. Notice that the sets Γ_α "grow" as β increases.

Theorem 1. *If $\beta < 1/3$, then $\Gamma_\alpha - \Gamma_\alpha$ is a Cantor set of measure zero. If $\beta \geq 1/3$, then $\Gamma_\alpha - \Gamma_\alpha = [-1, 1]$.*

In other words, if $\beta < 1/3$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is empty for almost all $t \in [-1, 1]$ and if $\beta \geq 1/3$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is nonempty for all $t \in [-1, 1]$. So there is a significant change in the way that the middle-α Cantor sets act when β (or α) crosses $1/3$. We will see latter that this change is even more dramatic than is indicated by this theorem.

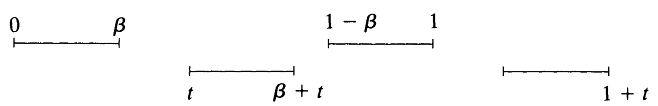
Proof: A very elegant proof by picture for the case when $\beta = 1/3$ can be found in [**B**, p. 110]. That proof can easily be seen to generalize to all $\beta \in [1/3, 1/2)$. So we will leave it to the reader to look up the proof of this part of the theorem.

We will prove the part of the theorem where $\beta < 1/3$. We will use the fact (which you should prove) that for $\alpha \in (0, 1)$,

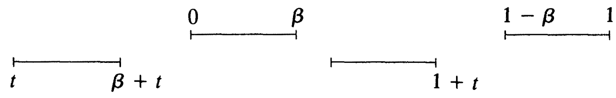
$$\Gamma_\alpha - \Gamma_\alpha = \bigcap_{n=0}^{\infty} (I_n - I_n).$$

We will show, by induction, that $I_n - I_n$ is a disjoint union of 3^n closed intervals each of which has length $2\beta^n$. This implies that $\Gamma_\alpha - \Gamma_\alpha$ has measure zero by an argument very similar to the one used to prove that Γ_α has measure zero. (Each $I_n - I_n$ is a cover of $\Gamma_\alpha - \Gamma_\alpha$ and the total length of $I_n - I_n$ is $3^n 2\beta^n = 2(3\beta)^n$ which goes to zero as n goes to infinity since $\beta < 1/3$.) The fact that $\Gamma_\alpha - \Gamma_\alpha$ is a Cantor set will follow from the way that $I_{n+1} - I_{n+1}$ is derived from $I_n - I_n$.

To make clearer the geometric structure of $\Gamma_\alpha - \Gamma_\alpha$, we will begin the induction with $n = 1$, rather than with $n = 0$. If $\beta < t < 1 - 2\beta$, then it's easy to see that $I_1 \cap (I_1 + t) = \varnothing$ (notice that if $\beta < 1/3$, then $\alpha > \beta$; now consider the following picture of I_1 and $I_1 + t$).



And if $-1 + 2\beta < t < -\beta$, then we again have $I_1 \cap (I_1 + t) = \varnothing$.



For any other choice of $t \in [-1, 1]$ we will have $I_1 \cap (I_1 + t) \neq \varnothing$. So $I_1 - I_1$ is the disjoint union of three closed intervals, and each of these intervals has length 2β, i.e.,

$$I_1 - I_1 = [-1, -1 + 2\beta] \cup [-\beta, \beta] \cup [1 - 2\beta, 1].$$

Now suppose that $I_n - I_n$ is the disjoint union of 3^n closed intervals, each of length $2\beta^n$.

For $n \geq 1$ let $I_n^L = I_n \cap [0, \beta]$ and let $I_n^R = I_n \cap [1 - \beta, 1]$, so I_n^L and I_n^R are the left and right "halves" of I_n. Because of the self-similarity of Γ_α, we have $I_n = \beta^{-1} I_{n+1}^L$. So

$$\beta^{-1} I_{n+1}^L - \beta^{-1} I_{n+1}^L = I_n - I_n$$

or

$$I_{n+1}^L - I_{n+1}^L = \beta(I_n - I_n)$$

and so the induction hypothesis implies that $I_{n+1}^L - I_{n+1}^L$ is the disjoint union of 3^n closed intervals each of length $2\beta^{n+1}$. Since $I_{n+1}^R = I_{n+1}^L + (1 - \beta)$, we have

$$I_{n+1}^L - I_{n+1}^R = \left(I_{n+1}^L - I_{n+1}^L\right) - (1 - \beta) = \beta(I_n - I_n) - (1 - \beta)$$

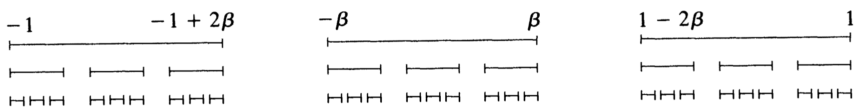
and similarly,

$$I_{n+1}^R - I_{n+1}^L = \left(I_{n+1}^L - I_{n+1}^L\right) + (1 - \beta) = \beta(I_n - I_n) + (1 - \beta)$$

so the induction hypothesis implies that both $I_{n+1}^L - I_{n+1}^R$ and $I_{n+1}^R - I_{n+1}^L$ are a disjoint union of 3^n closed intervals each of length $2\beta^{n+1}$. Now notice that

$$I_{n+1} - I_{n+1} = \left(I_{n+1}^L - I_{n+1}^R\right) \cup \left(I_{n+1}^L - I_{n+1}^L\right) \cup \left(I_{n+1}^R - I_{n+1}^L\right)$$

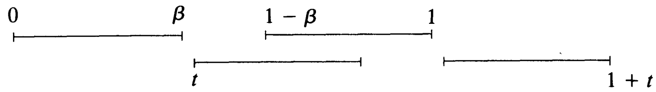
and, since $\beta < 1/3$, the unions are disjoint. So $I_{n+1} - I_{n+1}$ is a disjoint union of 3^{n+1} closed intervals each of length $2\beta^{n+1}$. The following picture illustrates $I_1 - I_1$, $I_2 - I_2$ and $I_3 - I_3$.



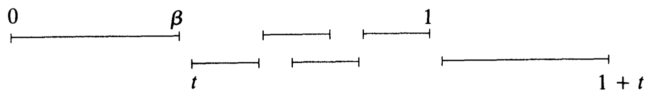
$\square$

Exercise. As β increases from 0 to $1/3$, the sets $\Gamma_\alpha - \Gamma_\alpha$ grow in much the same way that the sets Γ_α grow as β increases from 0 to $1/2$. Show that if $\beta < 1/5$, then the difference set of the difference set $(\Gamma_\alpha - \Gamma_\alpha) - (\Gamma_\alpha - \Gamma_\alpha)$ is a Cantor set of measure zero and if $\beta \geq 1/5$, then $(\Gamma_\alpha - \Gamma_\alpha) - (\Gamma_\alpha - \Gamma_\alpha) = [-2, 2]$.

When $\beta \geq 1/3$ we know that $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is nonempty for all $t \in [-1, 1]$. Now we will show that for some values of β and for some $t \in (-1, 1)$, $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ will be as small as a nonempty set can be, i.e., a single point. Consider the following picture.



This picture represents I_1 and $I_1 + t$ for some, as yet, unspecified values of β and t. If we can choose β and t such that I_1^R and $I_1^L + t$ are positioned, relative to each other, the same way that I_0 and $I_0 + t$ are positioned, then, when we look at I_2^R and $I_2^L + t$, the above picture will reproduce itself, on a smaller scale, inside the intervals I_1^R and $I_1^L + t$. See the following picture (notice that the above picture, when it is reproduced below in the intervals I_1^R and $I_1^L + t$, has its orientation reversed).



So if β and t can be chosen so that the kind of self-similarity described above occurs, then for all n, I_n and $I_n + t$ will have only one pair of overlapping intervals. And then we will have only one point in $\Gamma_\alpha \cap (\Gamma_\alpha + t)$. The self-similarity we want can be described by looking at the proportion of $[0, 1]$ that lies to the left of $[t, 1 + t]$ and equating it to the proportion of $[1 - \beta, 1]$ that lies to the right of $[t, \beta + t]$. This can be expressed as

$$\frac{t}{1} = \frac{1 - (\beta + t)}{\beta}.$$

Solving for t we get

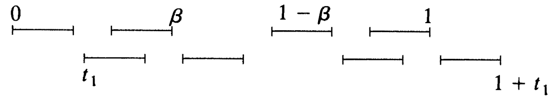
$$t = \frac{1 - \beta}{1 + \beta}. \tag{1}$$

But we also need to require that $\beta < t$. If we take the value of t given in equation (1) and put it in this last inequality, we get $\beta^2 + 2\beta - 1 < 0$ which is solved by $\beta < \sqrt{2} - 1$. So when $\beta < \sqrt{2} - 1$ and we give t the value $(1 - \beta)/(1 + \beta)$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ will contain exactly one point. (Exercise: Find the value of this point.) When $\beta = \sqrt{2} - 1$, the intersection of Γ_α and $\Gamma_\alpha + t$ will be a countable number of endpoints along with the unique point that is in the interior of all the overlapping pairs of intervals. Notice that $\sqrt{2} - 1 > 1/3$.

For any $\beta \in (0, \sqrt{2} - 1)$ and any n, if we let

$$t_n = \beta^n \left(\frac{1 - \beta}{1 + \beta} \right),$$

then the intersection of Γ_α and $\Gamma_\alpha + t_n$ will contain exactly 2^n points. For example, consider the following picture, where $n = 1$.



This picture shows I_2 and $I_2 + t_1$, I_1^L and $I_1^L + t_1$ (and also I_1^R and $I_1^R + t_1$) are positioned, relative to each other, the same way that I_0 and $I_0 + t$ were in the last example. So the intersection of Γ_α and $\Gamma_\alpha + t_1$ will contain exactly two points.

These examples leave us with two questions. First, for $\beta \in [1/3, \sqrt{2} - 1)$ how often is the intersection of Γ_α and $\Gamma_\alpha + t$ a finite number of points? Second, for $\beta \geq \sqrt{2} - 1$, can the intersection of Γ_α and $\Gamma_\alpha + t$ be finite for any choice of $t \in (-1, 1)$? Before answering these questions, let's look at a simpler one; for $\beta \geq 1/3$, how often is the intersection of Γ_α and $\Gamma_\alpha + t$ a single point?

For any $\alpha \in (0, 1)$, let Λ_α denote the middle-α Cantor set defined in the interval $[-1, 1]$ (so $\Lambda_\alpha = 2\Gamma_\alpha - 1$). Λ_α can also be defined by

$$\Lambda_\alpha = \left\{ \sum_{i=0}^{\infty} u_i \beta^i (1 - \beta) | u_i \in \{-1, 1\} \text{ for all } i \right\}.$$

(What is the geometric significance for Λ_α of the *finite* sums $\Sigma_{i=0}^{n-1} u_i \beta^i (1 - \beta)$ where $u_i \in \{-1, 1\}$?) Notice that since Λ_α is a middle-α Cantor set, it has zero measure.

The next lemma shows that Λ_α contains all the points of $\Gamma_\alpha - \Gamma_\alpha$ for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a single point. So we have an answer to our last question above. If $\beta \geq 1/3$, then for almost all $t \in [-1, 1]$, $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains more than one point.

Lemma 2. *Choose $\beta \in (0, 1/2)$. If $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a single point, then $t \in \Lambda_\alpha$.*

Proof: Let x denote the single point in $\Gamma_\alpha \cap (\Gamma_\alpha + t)$. Let $\{s_0, s_1, s_2, \dots\}$ be the address in Γ_α of x and let $\{r_0, r_1, r_2, \dots\}$ be the address in Γ_α of $x - t$. Then

$$t = x - (x - t) = \sum_{i=0}^{\infty} (s_i - r_i) \beta^i (1 - \beta).$$

To show that $t \in \Lambda_\alpha$, we must show that $s_i - r_i \in \{-1, 1\}$ for all $i \geq 0$, i.e., we need to show that $s_i \neq r_i$ for all i. Suppose there is an i such that $s_i = r_i$. Let B and B' be the components of I_i that contain x and $x - t$ respectively. Assume that $s_i = r_i = 0$. Consider the following picture of $I_{i+1} \cap B$ and $(I_{i+1} \cap B') + t$.

(The two holes have length $\alpha\beta^i$ and the four closed intervals have length β^{i+1}.) Since $s_i = r_i = 0$, the point x is to the left of the open intervals (if $s_i = r_i = 1$, then the point x will be to the right of the open intervals). The two "halves" of the above picture are translations of each other, so in the right half of the picture the point $y = x + \beta^i(1 - \beta)$ will also be in $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contradicting that x is the only point in this intersection. $\square$

Exercise. If $\Gamma_\alpha \cap (\Gamma_\alpha + t) = \{x\}$, then $t = 2x - 1$.

Let Λ_A denote the union of the images of Λ_α under all the affine maps of the form

$$T(x) = \beta^n x + \sum_{i=0}^{n-1} v_i \beta^i (1 - \beta)$$

where n is any integer greater than 0 and $v_i \in \{-1, 0, 1\}$ for $i = 0, \ldots, n-1$. Then

$$\Lambda_A = \left\{ \sum_{i=0}^{\infty} s_i \beta^i (1 - \beta) \middle| \text{for some } n > 0,\ s_i \in \{-1,0,1\} \text{ for } 0 \le i \le n-1 \right.$$
$$\left. \text{and } s_i \in \{-1, 1\} \text{ for } i \ge n \right\}.$$

Let's notice several things about Λ_A. First of all, $\Lambda_A \subset [-1, 1]$. In fact, $\Lambda_A \subset \Gamma_\alpha - \Gamma_\alpha$ (why?). Λ_A has measure zero since it is a countable union of sets of measure zero (the affine image of a set of measure zero has measure zero). The series representations of points in Λ_A are not unique. For example, if $\beta = \sqrt{2} - 1$ and $t = (1 - \beta)/(1 + \beta)$, then the sequences $\{0, 1, 1, 1, \ldots\}$ and $\{1, -1, 1, -1, 1, -1, \ldots\}$ can both be used to represent t.

The significance of Λ_A is that it contains all the points of $\Gamma_\alpha - \Gamma_\alpha$ for which $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a finite set of points. This will be shown in the proof of the following theorem.

Theorem 3. *If $\beta \ge 1/3$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set for almost all $t \in [-1, 1]$.*

Note. So $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ goes from being almost always empty when $\beta < 1/3$, to being almost always a Cantor set when $\beta \ge 1/3$. This shows that the change in the difference sets described by Theorem 1 is even more dramatic than was indicated by that theorem.

Proof: Let $\beta \in (0, 1/2)$. We will show that if $t \in (\Gamma_\alpha - \Gamma_\alpha) \setminus \Lambda_\alpha$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set. The intersection of any two Cantor sets is a compact, totally disconnected set. To show that $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set, we only need to show that it is a perfect set.

Suppose that $t \in \Gamma_\alpha - \Gamma_\alpha$ and $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is not a perfect set, i.e., there is a point $x \in \Gamma_\alpha \cap (\Gamma_\alpha + t)$ and an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap \Gamma_\alpha \cap (\Gamma_\alpha + t) = \{x\}$, or in other words, x is an isolated point in $\Gamma_\alpha \cap (\Gamma_\alpha + t)$. We'll show that $t \in \Lambda_A$. Choose n large enough so that $\beta^n < \varepsilon$. Let B denote the component of I_n that contains x and let B' be the component of I_n that contains $x - t$. Then $\Gamma_\alpha \cap B$ and $(\Gamma_\alpha \cap B') + t$ are two middle-α Cantor sets whose intersection contains only the point x. Let $\{s_0, \ldots, s_{n-1}, 0, 0, 0, \ldots\}$ be the address in Γ_α of the left hand endpoint of B and let $\{r_0, \ldots, r_{n-1}, 0, 0, 0, \ldots\}$ be the address in Γ_α of the left hand endpoint of B'. Let T denote the affine map

$$T(y) = \beta^{-n}\left(y - \sum_{i=0}^{n-1} s_i \beta^i (1 - \beta)\right).$$

The image of $\Gamma_\alpha \cap B$ under T is Γ_α. The image of $(\Gamma_\alpha \cap B') + t$ under T is

$\Gamma_\alpha + t'$ where

$$t' = T\left(t + \sum_{i=0}^{n-1} r_i\beta(1-\beta)\right). \tag{2}$$

Because T is an affine map, we have $\Gamma_\alpha \cap (\Gamma_\alpha + t') = T((\Gamma_\alpha \cap B) \cap ((\Gamma_\alpha \cap B') + t))$, so $\Gamma_\alpha \cap (\Gamma_\alpha + t')$ contains only one point. Then the previous lemma implies that $t' \in \Lambda_\alpha$. By the definition of Λ_α, t' has a unique series representation of the form

$$t' = \sum_{i=0}^{\infty} u_i \beta^i(1-\beta) \quad \text{where } u_i \in \{-1, 1\} \text{ for all } i. \tag{3}$$

Now solve for t in Equation (2) using the series (3) to represent t'. We get

$$\begin{aligned}
t &= T^{-1}(t') - \sum_{i=0}^{n-1} r_i \beta^i(1-\beta) \\
&= \beta^n t' + \sum_{i=0}^{n-1} (s_i - r_i)\beta^i(1-\beta) \\
&= \beta^n \sum_{i=0}^{\infty} u_i \beta^i(1-\beta) + \sum_{i=0}^{n-1} (s_i - r_i)\beta^i(1-\beta) \\
&= \sum_{i=0}^{n-1} (s_i - r_i)\beta^i(1-\beta) + \sum_{i=n}^{\infty} u_{i-n}\beta^i(1-\beta)
\end{aligned}$$

which implies that $t \in \Lambda_A$.

So far, we have only assumed that $\beta \in (0, 1/2)$ and we have shown that whenever $t \in (\Gamma_\alpha - \Gamma_\alpha) \setminus \Lambda_A$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set. When $\beta \geq 1/3$, $\Gamma_\alpha - \Gamma_\alpha = [-1, 1]$. So when $\beta \geq 1/3$ and $t \in [-1, 1] \setminus \Lambda_A$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set. The fact that Λ_A has measure zero completes the proof. □

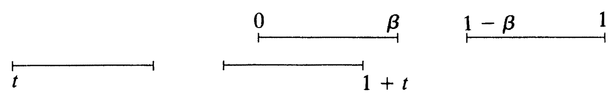
We have now answered one of our earlier questions. For $\beta \in [1/3, \sqrt{2} - 1]$, the intersection of Γ_α and $\Gamma_\alpha + t$ contains an infinite number of points for almost all t in $[-1, 1]$. The next theorem answers the other question.

Theorem 4. *If $\beta \in (\sqrt{2} - 1, 1/2)$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains a Cantor set for all $t \in (-1, 1)$.*

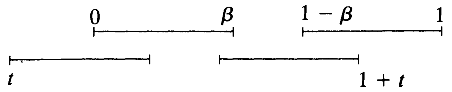
Proof: For $\beta \in (\sqrt{2} - 1, 1/2)$, define a function $f_\beta\colon [-1, 1] \to [-1, 1]$ by

$$f_\beta(t) = \begin{cases} \beta^{-1}t + (\beta^{-1} - 1) & \text{for } -1 \leq t \leq -\beta \\ \beta^{-1}t & \text{for } -\beta < t < \beta \\ \beta^{-1}t - (\beta^{-1} - 1) & \text{for } \beta \leq t \leq 1. \end{cases}$$

Let's see how f_β was derived. If $t \in [-1, -\beta]$, then I_1^L and $I_1^R + t$ overlap but $I_1^R + t$ and I_1^R do not overlap (except when $t = -\beta$ and they have a common endpoint).



If we concentrate only on the intervals I_1^L and $I_1^R + t$, then we can *renormalize* (or rescale) this pair of intervals by mapping I_1^L onto $[0,1]$ and mapping $I_1^R + t$ onto an interval $[t', 1 + t']$ in such a way that the relation of $[0,1]$ to $[t', 1 + t']$ is the same as the relation of I_1^L to $I_1^R + t$. The required value of t' is given by $f_\beta(t)$. If $t \in (-\beta, \beta)$, then I_1^L and $I_1^L + t$ overlap and of course I_1^R and $I_1^R + t$ also overlap (it may be that the pair I_1^L and $I_1^R + t$ or the pair I_1^R and $I_1^L + t$ overlap, but we will ignore these pairs in this case).



The value of $f_\beta(t)$ gives us, in this case, the renormalization of the pair of intervals I_1^L and $I_1^L + t$ (and simultaneously the renormalization of I_1^R and $I_1^R + t$). If $t \in [\beta, 1]$, then only I_1^R and $I_1^L + t$ overlap, and $f_\beta(t)$ renormalizes this pair of intervals.

The orbit $\{f_\beta^n(t)\}_{n=0}^\infty$ of the point t gives us information about $\Gamma_\alpha \cap (\Gamma_\alpha + t)$. For example, if the nth iterate of t is in the interval $(-\beta, \beta)$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains at least two points (why?). If $f_\beta^{n_i}(t) \in (-\beta, \beta)$ for some finite set of integers $\{n_i\}_{i=1}^k$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains at least 2^k points. What we will prove below is that if $f_\beta^{n_i}(t) \in (-\beta, \beta)$ for some infinite sequence of integers $\{n_i\}_{i=1}^\infty$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains a Cantor set. But first let's show that for any $\beta \in (\sqrt{2} - 1, 1/2)$ and for any $t \in (-1, 1)$, there is a sequence $\{n_i\}_{i=1}^\infty$ of integers such that $f_\beta^{n_i}(t) \in (-\beta, \beta)$ for all $i \geq 1$.

To prove this fact, it suffices to show that if $t \in (-1, -\beta] \cup [\beta, 1)$, then for some n, $f_\beta^n(t) \in (-\beta, \beta)$ (so the orbit of t will always return to $(-\beta, \beta)$ after it leaves it). Notice that the points -1 and 1 are both fixed points for f_β. Suppose that $t \in [\beta, 1)$ (the proof for $t \in (-1, -\beta]$ is similar). The image under f_β of $[\beta, 1)$ is the interval $[2 - \beta^{-1}, 1)$ which, for $\beta > \sqrt{2} - 1$, is a subset of $(-\beta, 1)$. So the point $f_\beta(t)$ is contained in either $(-\beta, \beta)$ (in which case $n = 1$ and we are done) or it's still in $[\beta, 1)$. Notice that the distance of $f_\beta(t)$ from 1, i.e., $1 - f_\beta(t) = 1 - (\beta^{-1}t - (\beta^{-1} - 1)) = \beta^{-1}(1 - t)$, is strictly greater than the distance that t was from 1, i.e., $1 - t$ (since $\beta^{-1} > 1$). So as long as $f_\beta^n(t)$ stays in $[\beta, 1)$, its distance from 1 is $\beta^{-n}(1 - t)$. But eventually $\beta^{-n}(1 - t)$ will be strictly greater than $1 - \beta$ (the length of $[\beta, 1)$) and $f_\beta^n(t)$ will be in $(-\beta, \beta)$.

Now let's show that if $\beta \in (\sqrt{2} - 1, 1/2)$ and $t \in (-1, 1)$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains a Cantor set. If B is a component of I_n, then B contains two components of I_{n+1}. Let $(B \cap I_{n+1})^L$ denote the left hand component of I_{n+1} contained in B and let $(B \cap I_{n+1})^R$ denote the right hand component. Let $B_0 = B_0' = [0, 1]$. For $n \geq 0$, define inductively the following sequences of nested closed intervals

$$B_{n+1} = \begin{cases} (B_n \cap I_{n+1})^L & \text{if } f_\beta^n(t) \in [-1, \beta) \\ (B_n \cap I_{n+1})^R & \text{if } f_\beta^n(t) \in [\beta, 1] \end{cases}$$

and

$$B_{n+1}' = \begin{cases} (B_n' \cap I_{n+1})^R & \text{if } f_\beta^n(t) \in [-1, -\beta] \\ (B_n' \cap I_{n+1})^L & \text{if } f_\beta^n(t) \in (-\beta, 1]. \end{cases}$$

The lengths of the B_n go to zero, so $\bigcap_{n=0}^{\infty} B_n$ is a single point in Γ_α. Let x denote this point. Similarly, let $y = \bigcap_{n=0}^{\infty} B'_n$. B_n and B'_n have been defined so that B_n and $B'_n + t$ are the overlapping pair of components from I_n and $I_n + t$ that are renormalized by applying f_β to $f_\beta^{n-1}(t)$. In the case where $f_\beta^{n-1}(t) \in (-\beta, \beta)$, so f_β is simultaneously renormalizing two pairs of overlapping intervals, B_n and $B'_n + t$ are defined to be the left-hand pair. It follows that $\bigcap_{n=0}^{\infty} B_n = \bigcap_{n=0}^{\infty}(B'_n + t)$, i.e., that $x = y + t$. Let $\{s_0, s_1, s_2, \dots\}$ and $\{r_0, r_1, r_2, \dots\}$ denote the addresses in Γ_α of x and y respectively. So

$$s_n = \begin{cases} 0 & \text{if } f_\beta^n(t) \in [-1, \beta) \\ 1 & \text{if } f_\beta^n(t) \in [\beta, 1] \end{cases} \quad \text{and} \quad r_n = \begin{cases} 1 & \text{if } f_\beta^n(t) \in [-1, -\beta] \\ 0 & \text{if } f_\beta^n(t) \in (-\beta, 1]. \end{cases}$$

Notice that $s_n = r_n$ if and only if $f_\beta^n(t) \in (-\beta, \beta)$ and $s_n = r_n = 0$.

Let $\{n_i\}_{i=1}^{\infty}$ be the set of integers for which $f_\beta^{n_i}(t) \in (-\beta, \beta)$. Let $\mathscr{A}$ denote the set of all sequences $\{a_0, a_1, a_2, \dots\} \in \{0,1\}^N$ such that $a_n = 0$ if $n \notin \{n_i\}_{i=1}^{\infty}$. If $a \in \mathscr{A}$, let

$$x_a = \sum_{n=0}^{\infty} (s_n + a_n)\beta^n(1-\beta) \quad \text{and} \quad y_a = \sum_{n=0}^{\infty} (r_n + a_n)\beta^n(1-\beta).$$

For any $a \in \mathscr{A}$, $x_a, y_a \in \Gamma_\alpha$ (recall that $s_n = r_n = 0$ if $n \in \{n_i\}_{i=1}^{\infty}$), $x_a - y_a = x - y = t$ and so $x_a \in \Gamma_\alpha \cap (\Gamma_\alpha + t)$. Let $\mathscr{X}_{\mathscr{A}} = \{x_a | a \in \mathscr{A}\}$. So $\mathscr{X}_{\mathscr{A}} \subset \Gamma_\alpha \cap (\Gamma_\alpha + t)$. Let's show that $\mathscr{X}_{\mathscr{A}}$ is a Cantor set. Since $\mathscr{X}_{\mathscr{A}} \subset \Gamma_\alpha$, it is totally disconnected. We need to show that it is closed and perfect. If $\{z_n\}$ is a sequence of points from $\mathscr{X}_{\mathscr{A}}$ that converge to z, then $z \in \Gamma_\alpha$ (since Γ_α is closed). It's not hard to show that for any M there is an N such that for $n > N$ the addresses of z_n and z will agree in the first M places. This implies that $z \in \mathscr{X}_{\mathscr{A}}$, so $\mathscr{X}_{\mathscr{A}}$ is closed. To show that $\mathscr{X}_{\mathscr{A}}$ is perfect, choose $a \in \mathscr{A}$ and define $a^k = \{a_0^k, a_1^k, a_2^k, \dots\} \in \mathscr{A}$ by $a_n^k = a_n$ if $n \le k$ and $a_n^k = 0$ if $n > k$. Then for all k, $x_{a^k} \in \mathscr{X}_{\mathscr{A}}$ and the sequence $\{x_{a^k}\}_{k=0}^{\infty}$ converges to x_a. So $\mathscr{X}_{\mathscr{A}}$ is a perfect set. □

Exercise. When $\beta < 1/3$, show that every $t \in \Gamma_\alpha - \Gamma_\alpha$ has a unique series representation

$$t = \sum_{i=0}^{\infty} s_i \beta^i (1-\beta) \quad \text{where } s_i \in \{-1, 0, 1\}.$$

Call $\{s_0, s_1, s_2, \dots\}$ the address of t. Show that if $\{s_0, s_1, s_2, \dots\}$ is the address of a point t in $\Gamma_\alpha - \Gamma_\alpha$ and a is the cardinality of $\{n | s_n = 0\}$, then the cardinality of $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is equal to 2^a. (What about the case $\beta = 1/3$? See also **[BM]**.)

Let's summarize what we have proven. If $\beta < 1/3$, then $\Gamma_\alpha - \Gamma_\alpha$ is a Cantor set with Lebesgue measure zero. If $\beta \in [1/3, 1/2)$, then $\Gamma_\alpha - \Gamma_\alpha = [-1, 1]$. In addition, if $\beta \in [1/3, \sqrt{2} - 1)$, then for almost all $t \in [-1, 1]$, $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set, but for some $t \in (-1, 1)$, $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is as small as one point. If $\beta = \sqrt{2} - 1$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ is a Cantor set for all but a countably infinite subset of $[-1, 1]$, and for $t \in (-1, 1)$, the smallest cardinality that $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ can have is countably infinite. And if $\beta \in (\sqrt{2} - 1, 1/2)$, then $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ contains a Cantor set for all $t \in (-1, 1)$. (The case where $\beta = \sqrt{2} - 1$ is actually an exercise for the reader.) These results can be generalized to arbitrary Cantor sets embedded in the real line if we replace the parameter β with something called the thickness of a Cantor set (see **[K]**).

In this article we have used the difference sets of middle-α Cantor sets as a way to make concrete the intuitive idea that a middle-α Cantor set with α close to zero is larger than a middle-α Cantor set with α close to one. We did this using just cardinality and Lebesgue measure. Of course, what we lose by using such simple concepts of size is the ability to distinguish between, say, Γ_{α_1} and Γ_{α_2} when both α_1 and α_2 are in $(1/3,\ 1)$. For this, more sophisticated concepts such as Hausdorff or box-counting dimension are needed (see [E]).

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Excerpt from "'Billiards Is a Good Game': Gamesmanship and America's First Nobel Prize Scientist" by Norman Maclean, *The University of Chicago Magazine* 67 (Summer 1975, pp. 19–23).

For instance, Leonard Eugene Dickson, the outstanding mathematician, who at the time was writing his classic works on the theory of numbers, was sometimes a poor card player. Anton J. Carlson was also not a good bridge player, although he was nationally famous as an exponent of the scientific method of biological sciences.

. . .

Dickson, the master of numbers, was sometimes expectedly brilliant in a game where only 13×4 numbers were involved; his habitual troubles were at least partly environmental—he had come here by way of Texas. He almost consistently overbid and, when he lost three of four hands in a row, he would slam his cards down on the table and leave the room in a rage, always denouncing Carlson on the way out. No matter who had misplayed—Carlson, Michelson, or himself—he always denounced Carlson. While the cards were still shivering on the table he would shout, "Why the hell, Carlson, don't you go back to your lab and feed your dogs? And don't let Irene Castle catch you killing any of them."

Overbidding three or four hands in a row and then blaming the great biologist seemed to put the great mathematician in the right state of mind to race back to his office and resume his classic studies on the theory of numbers.

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