Continuous Nowhere-Differentiable Functions—an Application of Contraction Mappings

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HIDEFUMI KATSUURA was born and raised in Wakayama about 300 miles south of Tokyo, Japan. He came to the United States to study at the College of Charleston, where he received his B.A. He earned his Ph.D. at the University of Delaware under Professor David P. Bellamy. Currently, he is teaching and doing research in point set topology and in analysis.

1. Introduction. The existence of a continuous nowhere-differentiable function is well known, and many such functions are known. Most of them are variations of the following examples.

Examples of nowhere differentiable functions:

$$\text{If } f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(3^n x), \tag{1}$$

then f is a continuous nowhere-differentiable function. (This is due to Weierstrass. See [**2**, p. 195].)

(2) Let $g(x)$ be the distance from x to the nearest integer. If

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} g(2^n x),$$

then f is also a continuous nowhere-differentiable function. (See [**3**, p. 115].)

The above examples have concise definitions and establish the existence of continuous nowhere-differentiable functions. However, it is not easy to visualize or guess what their graphs look like, let alone to see intuitively why they work. Our continuous nowhere-differentiable function $\mathbf{f}$: $[0,1] \to \mathbf{R}$ is the uniform limit of a sequence of piecewise linear continuous functions f_n: $[0,1] \to \mathbf{R}$ with steep slopes. In constructing the sequence $\langle f_n \rangle$, we will be using a contraction mapping w from the family of all closed subsets of $X = [0,1] \times [0,1]$ into itself with respect to the Hausdorff metric induced by the Euclidean metric. (A contraction mapping on a metric space (Y, d) is a function g: $Y \to Y$ for which there is a positive constant $k < 1$ such that $d(g(x), g(y)) \leq kd(x, y)$ for every $x, y \in Y$.) It is interesting to note that, in our example, if A is *any non-empty closed subset* of the square X, even if it is a singleton set, the iterated sequence $\langle w^n(A) \rangle$ of sets converges to the graph of $\mathbf{f}$ in the Hausdorff metric. If A is the diagonal of slope 1 in the square X, then $w^n(A)$ is the graph of f_n, and hence, a reader can obtain an intuitive idea of the graph of $\mathbf{f}$. This idea first occurred to me when I attended Mr. Gary Church's master's thesis defense at San Jose State University at which he talked about attractors of contraction mappings (see [**4**]).

II. The construction of a continuous nowhere-differentiable function. Let $X = [0,1] \times [0,1]$ be the closed unit square in the plane with the usual Euclidean metric. Let w_i: $X \to X$, $i = 1, 2, 3$ be contraction mappings defined by

$$w_1(x, y) = \left(\frac{x}{3}, \frac{2y}{3}\right),$$

$$w_2(x, y) = \left(\frac{2-x}{3}, \frac{1+y}{3}\right),$$

and

$$w_3(x, y) = \left(\frac{2+x}{3}, \frac{1+2y}{3}\right).$$

(See FIGURE 1.)

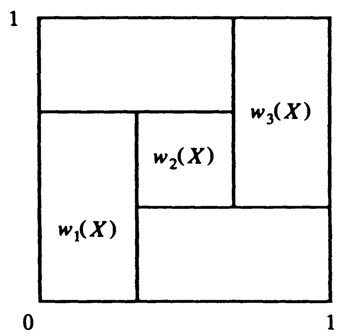

FIG. 1

In FIGURE 1, w_1 shrinks X onto $w_1(X)$, with the fixed point $(0, 0)$, w_2 flips X about the line $x = 1/2$ and shrinks it onto $w_2(X)$, with the fixed point $(1/2, 1/2)$, and w_3 shrinks X onto $w_3(X)$, with the fixed point $(1, 1)$.

Let $\mathbf{F}(X)$ be the collection of all non-empty closed subsets of X. We define a function w from $\mathbf{F}(X)$ into itself by

$$w(A) = w_1(A) \cup w_2(A) \cup w_3(A)$$

for every A in $\mathbf{F}(X)$. For every A, B in $\mathbf{F}(X)$, let

$$d_H(A, B) = \inf\{\varepsilon > 0: N_\varepsilon(A) \supset B \text{ and } N_\varepsilon(B) \supset A\},$$

where $N_\varepsilon(A)$ and $N_\varepsilon(B)$ are ε-neighborhoods of A and B, respectively. Then d_H is a complete metric on $\mathbf{F}(X)$, and is called a *Hausdorff metric*. It is interesting to note that the function w is a contraction mapping on $\mathbf{F}(X)$ under this metric d_H (see [5]).

Let $D_0 = \{(x, x) \in X\}$ be the diagonal in X. For each $n = 1, 2, 3, \ldots$, let $D_n = w(D_{n-1})$. Then for every $n = 0, 1, 2, \ldots$, D_n is the graph of a continuous function, call it f_n, from $[0, 1]$ onto itself.

Note that if $m \leq n$, then $w^n(X) \supseteq D_m$, and $w^n(X)$ is the union of 3^n rectangles of heights $\leq (2/3)^n$. Hence, we have

$$\sup\{|f_m(t) - f_n(t)|: t \in [0, 1]\} \leq (2/3)^n.$$

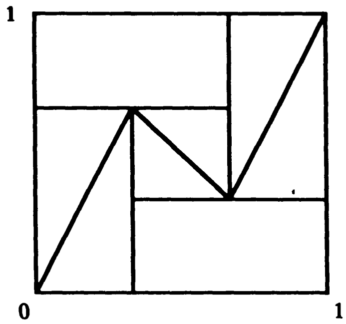

FIG. 2. $w(X)$ and D_1.

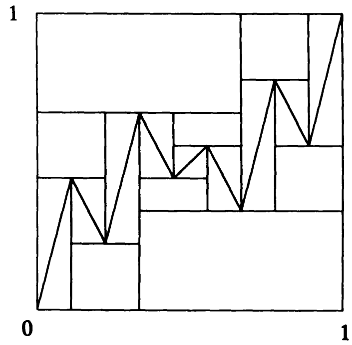

FIG. 3. $w^2(X)$ and D_2.

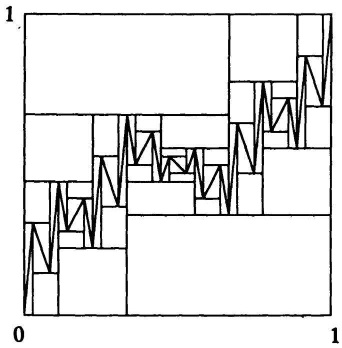

FIG. 4. $w^3(X)$ and D_3.

This implies that the sequence of continuous functions $\langle f_n \rangle$ is uniformly Cauchy. Thus there is a continuous function **f** from [0, 1] onto itself such that f_n converges to **f** uniformly on [0, 1]. We will prove that this function **f** is nowhere-differentiable on the interval (0, 1).

Since w is a contraction mapping on $\mathbf{F}(X)$, it has a unique fixed point D in $\mathbf{F}(X)$ (see Theorem 4.48 on page 92, [1]). Note that if A is an arbitrary element of $\mathbf{F}(X)$, then the sequence $\langle w^n(A) \rangle$ converges to D with respect to the metric d_H. Hence, D_n converges to D and D is the graph of **f**. If we let $B = \{(0,0,(1/2,1/2),(1,1)\}$ be the set of fixed points of w_1, w_2, and w_3, then the sequence $\langle w^n(B) \rangle$ in $\mathbf{F}(X)$ also converges to D. Moreover, $D \supseteq w^n(B)$. Hence, if one plots the points of $w^n(B)$ for a large n, then not only does it resemble the graph of **f**, but actually it is a part of the graph of **f**.

III. Proof of the nowhere-differentiability of f. Let T be the set of all ternary rationals in the interval (0, 1), and for each $n = 1, 2, 3, \ldots$, let T_n be the set of n-digit ternary rationals.

LEMMA 1. *Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences in T such that, for each $n = 1, 2, \ldots$,*
(1) *x_n and y_n are in T_n,*
(2) *$y_n - x_n = 1/3^n$, and*
(3) *either $x_n = x_{n+1}$ or $y_n = y_{n+1}$.*

Then

$$\lim_{n\to\infty}\left|\frac{\mathbf{f}(y_n)-\mathbf{f}(x_n)}{y_n-x_n}\right|=\infty.$$

Proof. We claim that

$$\left|\frac{\mathbf{f}(y_n)-\mathbf{f}(x_n)}{y_n-x_n}\right|\geq 2^{n-1}\qquad(n=1,2,3,\ldots).$$

First, note that if $x\in T_n$, then $\mathbf{f}(x)=f_n(x)$. If $n=1$, then

$$\frac{\mathbf{f}(y_1)-\mathbf{f}(x_1)}{y_1-x_1}=\begin{cases}\dfrac{\frac{2}{3}}{\frac{1}{3}}=2 & \text{if } (x_1=0, \text{ and } y_1=\frac{1}{3}) \text{ or } (x_1=\frac{2}{3} \text{ and } y_1=1),\\[2ex] \dfrac{-\frac{1}{3}}{\frac{1}{3}}=-1 & \text{if } x_1=\frac{1}{3} \text{ and } y_1=\frac{2}{3}.\end{cases}$$

Hence, if $n=1$, we have proved the claim.

Suppose, for some integer $k\geq 1$, the claim is true, i.e.,

$$|\mathbf{f}(y_k)-\mathbf{f}(x_k)|\geq 2^{k-1}\cdot|y_k-x_k|.$$

Then

$$\left|\frac{\mathbf{f}(y_{k+1})-\mathbf{f}(x_{k+1})}{y_{k+1}-x_{k+1}}\right|=\frac{\frac{2}{3}|\mathbf{f}(y_k)-\mathbf{f}(x_k)|}{\frac{1}{3}|y_k-x_k|}$$

(see FIGURE 5 for the case $x_k=x_{k+1}$)

$$\geq 2\cdot\frac{2^{k-1}|y_k-x_k|}{|y_k-x_k|}$$

by the induction hypothesis

$$=2^{(k+1)-1}$$

which proves the lemma, by induction. ♦

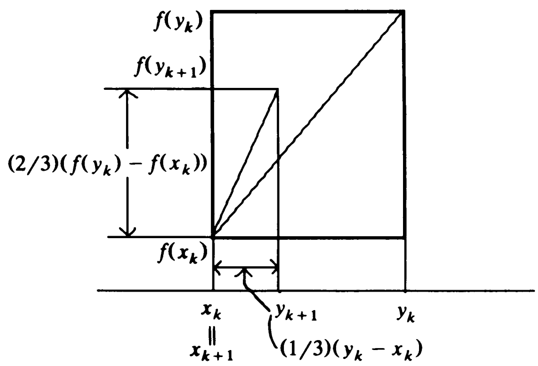

FIG. 5

LEMMA 2. *Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences in T such that for each $n = 1, 2, \ldots, x_n, y_n \in T_n$, and $y_n - x_n = 1/3^n$. Suppose $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for infinitely many $n = 1, 2, \ldots$. Then*

$$\lim_{n \to \infty} \frac{\mathbf{f}(y_n) - \mathbf{f}(x_n)}{y_n - x_n}$$

does not exist.

Proof. First, we claim that

$$\left| \frac{\mathbf{f}(y_n) - \mathbf{f}(x_n)}{y_n - x_n} \right| \geq 1 \quad \text{for each } n = 1, 2, 3, \ldots$$

From the proof of the previous lemma, this is surely true if $n = 1$. First, if $x_k = x_{k+1}$ or $y_k = y_{k+1}$, then as in the previous lemma, we have

$$\left| \frac{\mathbf{f}(y_{k+1}) - \mathbf{f}(x_{k+1})}{y_{k+1} - x_{k+1}} \right| = \frac{\frac{2}{3}|\mathbf{f}(y_k) - \mathbf{f}(x_k)|}{\frac{1}{3}|y_k - x_k|}$$

(see FIGURE 5 for the case $x_k = x_{k+1}$)

$$\geq 2 \cdot 1$$

by the induction hypothesis

$$> 1.$$

On the other hand, if $x_k \neq x_{k+1}$ and $y_k \neq y_{k+1}$, then

$$\left| \frac{\mathbf{f}(y_{k+1}) - \mathbf{f}(x_{k+1})}{y_{k+1} - x_{k+1}} \right| = \frac{\frac{1}{3}|\mathbf{f}(y_k) - \mathbf{f}(x_k)|}{\frac{1}{3}|y_k - x_k|}$$

(see FIGURE 6)

$$\geq 1$$

by the induction hypothesis, which proves the claim.

Now, if n is an integer such that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$, then

$$\frac{\mathbf{f}(y_{n+1}) - \mathbf{f}(x_{n+1})}{y_{n+1} - x_{n+1}} = \frac{-\frac{1}{3}(\mathbf{f}(y_n) - \mathbf{f}(x_n))}{\frac{1}{3}(y_n - x_n)}$$

(see FIGURE 6)

$$= -\left(\frac{\mathbf{f}(y_n) - \mathbf{f}(x_n)}{y_n - x_n} \right).$$

So surely if the limit exists it must be 0. But by the claim proved above, it can't be 0 either. This means that the limit does not exist. ♦

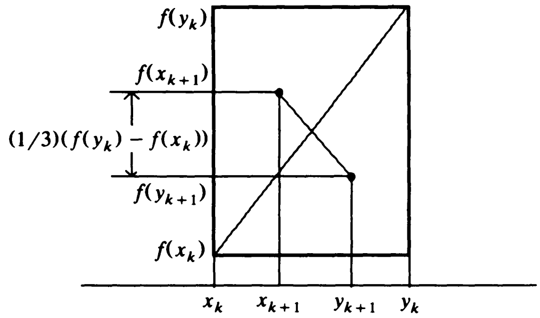

FIG. 6

The following is well known.

LEMMA 3. *Let g be a real valued function on* $[0,1]$ *differentiable at* t, $0 < t < 1$. *If, for every* $n = 1,2,3,\ldots$, $0 < x_n < t < y_n < 1$, *and if* $x_n \to t$ *and* $y_n \to t$ *then,*

$$\lim_{n\to\infty} \frac{g(y_n) - g(x_n)}{y_n - x_n} = g'(t).$$

THEOREM. *The function* $\mathbf{f}\colon [0,1] \to [0,1]$ *is not differentiable at any* $x \in (0,1)$.

Proof. If $x \in T_n$ for some integer n, then let $y_k = x + 1/3^{n+k}$ for every $k = 1,2,\ldots$. Then $\lim_{k\to\infty} y_k = x$ and

$$\lim_{k\to\infty} \left| \frac{\mathbf{f}(y_k) - \mathbf{f}(x_k)}{y_k - x_k} \right| = \infty$$

by Lemma 1. Hence, $\mathbf{f}$ is not differentiable at x.

On the other hand, if $x \notin T$, then there are sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ such that, for each $n = 1,2,\ldots$,

(1) x_n and y_n are both in T_n,

(2) $y_n - x_n = 1/3^n$, and

(3) $a_n < x < b_n$.

Then by Lemmas 1 and 2, we know that

$$\lim_{n\to\infty} \frac{\mathbf{f}(y_n) - \mathbf{f}(x_n)}{y_n - x_n}$$

does not exist. Hence, by Lemma 3, $\mathbf{f}$ is not differentiable at x. ◆

REFERENCES

1. Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, 1974.
2. Robert G. Bartle, The Elements of Real Analysis, John Wiley and Sons, Inc., 1976.
3. Ralph P. Boas, Jr., A Primer of Real Functions, The Math. Asso. of Amer., 1960.
4. Gary Church, Computer Approximation of Fractal Sets using Iterated Function Systems, Master's Thesis, San Jose State University, San Jose, California, 1988.
5. J. Hutchinson, Fractals and Self Similarity, *Indiana Univ. Math. J.*, 30 (1981) 713–747.