A Simple Example of Little Big Set

John K. Williams

Loosely speaking, the Hausdorff dimension of a set is the right place to measure that set. For s less than the Hausdorff dimension, the Hausdorff s-measure of the set is infinite and for s larger than the Hausdorff dimension the Hausdorff s-measure is zero. This note describes a simple example of a set with Hausdorff s-measure 0, little, and Hausdorff dimension s, big. Specifically we will construct a set with Hausdorff dimension 1 and Hausdorff 1-measure 0 and then show how it can be generalized to give such a set for any s.

To get a feel for Hausdorff dimension let's look at an example, the Cantor set. The Cantor set is defined as follows. Let $E_0 = [0, 1]$ and then define E_j as E_{j-1} with the open middle one third of each interval removed, i.e. $E_1 = [0, 1/3] \cup [2/3, 1]$, $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Each E_j consists of 2^j intervals of length 3^{-j}. Cantor's set is the set $E = \bigcap_{j=0}^{\infty} E_j$.

The Hausdorff 1-measure is just the same as the 1-dimensional Lebesque measure, the length of the set. From the construction, one can see that the length of the E_j is $2^j \times (1/3)^j$ and the length of $E = \text{limit}_{j\to\infty}(2/3)^j = 0$. Now if we change how we measure the length of an interval, $[a, b]$, from $|a - b|$ to $|a - b|^s$, then the length of each E_j becomes $2^j \times (1/3)^{js}$. The length of $E = \text{limit}_{j\to\infty}(2/3^s)^j$. If $s > (\ln 2/\ln 3)$, then the limit is 0; if $s < (\ln 2/\ln 3)$, the limit is infinity; and if $s = (\ln 2/\ln 3)$, the limit is 1. As we will see below, the Hausdorff dimension of this set is precisely $(\ln 2/\ln 3)$. Now a definition of Hausdorff dimension [**1**, p. 7].

Define the diameter of a non-empty subset U of $\mathfrak{R}^n$ as $|U| = \sup\{|x - y|: x, y \in U\}$. If $E \subseteq \cup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of E.

For E a subset of $\mathfrak{R}^n$, s a non-negative number, and $\delta > 0$ define:

$$\mathfrak{H}_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$

where the infinum is over all (countable) δ-covers $\{U_i\}$ of E. Then the *Hausdorff s-dimensional outer measure of E* is defined as:

$$\mathfrak{H}^s(E) = \lim_{\delta\to 0} \mathfrak{H}_\delta^s(E).$$

The limit exists since $\mathfrak{H}_\delta^s$ increases as δ decreases but may be infinite. The restriction of $\mathfrak{H}^s$ to the σ-field of $\mathfrak{H}^s$-measurable sets is called the *Hausdorff s-dimensional measure*.

For each E, $\mathfrak{H}^s(E)$ is non-increasing as s increases from 0 to ∞ (as soon as δ is less than 1, $|U_i|^s$ decreases as s goes from 0 to ∞). Also if $s < t$, then

$$\mathfrak{H}_\delta^s(E) \geq \delta^{s-t}\mathfrak{H}_\delta^t(E),$$

which implies that if $\mathfrak{H}^t(E)$ is positive, then $\mathfrak{H}^s(E)$ is infinite. There is then a unique value, dim E, called the *Hausdorff dimension* of E, such that

$$\mathfrak{H}^s(E) = \infty \text{ if } 0 \leq s < \dim E \quad\text{and}\quad \mathfrak{H}^s(E) = 0 \text{ if } \dim E < s < \infty.$$

Three observations are immediate. First, if E is a subset of F then the Hausdorff dimension of E is less than or equal to the Hausdorff dimension of F. Secondly, the Hausdorff dimension of $\mathfrak{R}^n$ is n. Putting the first two together, the Hausdorff dimension of a subset of $\mathfrak{R}^n$ is less than or equal to n. The question addressed in this note is can $\mathfrak{H}^s(E)$ be zero when s is $\dim E$?

If E is $\mathfrak{R}^1$, then $\dim E$ is 1 and $\mathfrak{H}^1(E) = \infty$. If E is a line segment of length l, then $\dim E$ is again 1 and $\mathfrak{H}^1(E)$ is l. Is there a set E where $\dim E$ is 1 while $\mathfrak{H}^1(E) = 0$? At first it does not seem possible. The idea of Hausdorff dimension is to find a place where the set should be measured, for s smaller than the dimension, the measure is infinite and for s larger, the measure is zero. A slight modification of the Cantor set will lead us to a set which has the desired properties.

One can view the Cantor set as the invariant set for a pair of linear contractions applied to the unit interval. If we define:

$$f_1(x) = (1/3)x \quad \text{and} \quad f_2(x) = (1/3)x + 2/3,$$

Then each E_j above may defined inductively as $E_0 = [0,1]$ and $E_j = f_1(E_{j-1}) \cup f_2(E_{j-1})$.

The advantage of viewing the Cantor set in this manner is that we can apply a theorem of Moran [**3**, Thm. II] which in this context can be stated as follows:

Theorem. *Let $\{f_1, \dots, f_n\}$ be a set of linear contractions, each of which contracts by a factor of w_n. Let E_0 be a set where $f_j(E_0)$ and $f_k(E_0)$ are disjoint for $j \neq k$, $E_j = \bigcup_{i=1}^n f_i(E_{j-1})$, and $E = \bigcap_{j=0}^\infty E_j$. Then the Hausdorff dimension of E is s where s is defined by the equation*:

$$\sum_{i=1}^n w_i^s = 1$$

and the Hausdorff s-measure is finite and positive.

Applying this to the Cantor set, we see that the Hausdorff dimension is $\log 2/\log 3$ and has positive Hausdorff $(\log 2/\log 3)$-measure.

We can modify the construction of the Cantor set slightly by taking out the middle $1/m$th section at each step. This means modifying the linear contractions to be:

$$f_1(x) = \left(\frac{m-1}{2m}\right)x \quad \text{and} \quad f_2(x) = \left(\frac{m-1}{2m}\right)x + \frac{m+1}{2m}.$$

Applying Moran theorem we have the following theorem:

Theorem. *The Hausdorff dimension of $\mathscr{C}_m$ is*

$$s = \log 2/\log\left(\frac{2m}{m-1}\right)$$

and $\mathfrak{H}^s(\mathscr{C}_m)$ is finite and positive.

Now the set we are looking for is just the union of the $\mathscr{C}_m$. Set $\mathscr{C} = \bigcup_{j=3}^\infty \mathscr{C}_m$.

Theorem. *The Hausdorff dimension of $\mathscr{C}$ is* 1 *and* $\mathfrak{H}^1(\mathscr{C}) = 0$.

Since the dimension of each of the $\mathscr{C}_m$ is less than one, $\mathfrak{H}^1(\mathscr{C}_m) = 0$. Therefore

$$\mathfrak{H}^1(\mathscr{C}) = \mathfrak{H}^1\left(\bigcup_{m=3}^\infty \mathscr{C}_m\right) \leq \sum_{m=3}^\infty \mathfrak{H}^1(\mathscr{C}_m) = 0.$$

But the dimension of $\mathscr{C}$ must be greater than or equal to the dimension of each $\mathscr{C}_m$. Since the dimension of $\mathscr{C}_m$ tends to 1 as m tends to ∞, the dimension of $\mathscr{C}$ must be one. □

This construction can be generalized in two directions. First we can construct a Cantor like set of any dimension s between 0 and 1 by pulling out the middle α at each stage where s and α are related by:

$$\alpha = 1 - 2^{1-1/s}.$$

To find a set with dimension s and Hausdorff s-measure 0, find a sequence which converges to s, $\{s_i\}$; construct a Cantor like set for each s_i; and form the union of these sets.

Secondly we can reach dimensions higher than 1 by pulling squares out the unit square {a Sierpinski Gasket} or pulling cubes out the unit cube in dimension three and above.

Finally, there are other sets with this property. Most notably, a Besicovitch set is a set with zero planar measure, lines in all directions and Hausdorff dimension 2. The exact description and proof of its properties is a little more difficult [**1**, ch. 7].

ACKNOWLEDGMENT. My thanks to the referee who pointed out that Federer [**2**] has a similar construction, in greater generality of course.

REFERENCES

1. K. J. Falconer, *The Geometry of Fractal Sets*, New York: Cambridge University Press, 1985.
2. H. Federer, *Geometric Measure Theory*, New York, Springer, 1969.
3. P. A. P. Moran, Additive functions of intervals and Hausdorff Measure' *Proceedings of the Cambridge Philosophical Society*, 42 (1946) 15–23.

Department of Mathematics, Physics and Computer Science
University of Hartford
West Hartford, CT 06117

E 402 [1940, 48]. *Proposed by Irving Kaplansky, Harvard University.*

If n, r, and a are positive integers, the congruence $n^2 \equiv n \pmod{10^a}$ obviously implies $n^r \equiv n \pmod{10^a}$. (When such a number n has only a digits, it is called an automorphic number.) For what values of r does $n^r \equiv n \pmod{10^a}$ imply $n^2 \equiv n \pmod{10^a}$?

—*American Mathematical Monthly* 47, (1940) p. 572.