AMERICAN

MATHEMATICAL MONTHLY

MATHEMATICIANS AND MUSIC.

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Presidential Address [1] delivered [2] before the Mathematical Association of America, September 6, 1923.

I.

"Mathematics and Music, the most sharply contrasted fields of intellectual activity which one can discover, and yet bound together, supporting one another as if they would demonstrate the hidden bond which draws together all activities of our mind, and which also in the revelations of artistic genius leads us to surmise unconscious expressions of a mysteriously active intelligence." In such wise wrote one [3] supremely competent to represent both musicians and mathematicians, the author of that monumental work, *On the Sensations of Tone as a physiological basis for the Theory of Music.*

"Bound together?" Yes! in regularity of vibrations, in relations of tones to one another in melodies and harmonies, in tone-color, in rhythm, in the many varieties of musical form, in Fourier's series arising in discussion of vibrating strings and development of arbitrary functions, and in modern discussions of acoustics.

This suggests that the famous affirmation of Leibniz, "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers," [4] must

[1] To the address as delivered a number of footnotes, mainly with a few references to the vast literature of the subject, have been added. A fundamental work in this connection is H. L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, and the best edition is the second English edition translated with many additions from the fourth (last) German edition by A. J. Ellis, London, 1885; the third edition was reprinted from the second in 1895, and the fourth from the third in 1912. While some mathematical discussion occurs in this work, the standard treatise on the mathematical theory is Rayleigh, *Theory of Sound*, 2 vols., second ed., London, 1894. Another work of high order is H. v. Helmholtz, *Vorlesungen über die mathematischen Akustik*, 1898, vol. 3 of *Vorlesungen über theoretische Physik*, Leipsic. H. Lamb, *The Dynamical Theory of Sound*, London, 1910, was intended as a stepping stone to the writings of Helmholtz and Raleigh. Between 1898 and 1915, 38 papers by various authors appeared at Leipsic in 8 Hefte of *Beiträge zur Akustik und Musikwissenschaft* herausgegeben von C. Stumpf. For the most part, they are reprints of articles in *Zeitschrift für Psychologie*, *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, and *6. Kongress der Gesellschaft für experimentelle Psychologie.* Another very valuable general work, discussing the writings of mathematicians on musical matters, is F. J. Fétis, *Biographie Universelle des Musiciens et Bibliographie générale de la Musique*, 8 vols., second ed., Paris, 1 (1873), 2 (1867), 3–4 (1869), 5–8 (1870). R. Eitner's *Biographisch-bibliographisches Quellen-Lexikon der Musiker der christlichen Zeitrechnung bis zur Mitte des 19. Jahrhunderts*, 10 vols., Leipsic, 1900–1904, is also sometimes useful.

[2] At a joint session of the Mathematical Association of America and of the American Mathematical Society, Vassar College, Poughkeepsie, N. Y.

[3] H. v. Helmholtz, *Vorträge und Reden*, Braunschweig, vol. 1, 1884, p. 82. See also Helmholtz, *Popular Lectures on Scientific Subjects*, London, 1873, p. 62.

[4] "Musica est exercitium arithmeticæ occultum nescientis se numerare animi," which occurs in a letter dated April 17, 1712, and addressed to Goldbach. It is letter 154 in Leibniz, *Epistolæ ad diversos*, vol. 1, Leipsic, 1734, p. 241. The quotation is in the section dealing with the question "Vnde oritur ex musica voluptas?" This is preceded and followed by two other sections on

be far from true if taken literally. But, in a very general conception of art and science, its verity may well be granted; for, in creating as in listening to music, there is no realization possible except by immediate and spontaneous appreciation of a multitude of relations of sound.

Other modes of expression and points of view were suggested by that great enthusiast to whom America owed much, him who called himself[1] "the Mathematical Adam" because of the many mathematical terms he invented; for example, mathematic—to denote the science itself in the same way as we speak of logic, rhetoric or music, while the ordinary form is reserved for the applications of the science. He referred to the cultures of mathematics and music "not merely as having arithmetic for their common parent but as similar in their habits and affections."[2] "May not Music be described," he wrote, "as the Mathematic of Sense, Mathematic as the Music of reason?[3] the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life,—each to receive its consummation from the other when the human intelligence, elevated to the perfect type, shall shine forth glorified in some future Mozart-Dirichlet, or Beethoven-Gauss —a union already not indistinctly foreshadowed in the genius and labors of a Helmholtz"![4]

But such intimacies in these cultures are not discoveries and imaginings of a later day. For two thousand years music was regarded as a mathematical science. Even in more recent times the mathematical dictionaries of Ozanam,[5] Savérien,[6] and Hutton,[7] contain long articles on music and considerable space is devoted to the subject in Montucla's revised history,[8]—which brings us to

"Quibus musicis proportionibus homines delectantur?" and "Quando rationes surdæ in musica commode locum inueniunt?"

T. W. Preyer corrupted Leibniz's sentence into "Arithmetica est exercitium musicum occultum nescientis se sonos comparare animi" (compare M. Lecat, *Pensées sur la Science, la Guerre, et sur des Sujets très variés*, Brussels, 1919, p. 438). Preyer's thought in this connection will be apparent on turning to his monograph, "Ueber den Ursprung des Zahlbegriffes aus dem Tonsinn und über das Wesen der Primzahlen," pages 1–36 of *Beiträge zur Psychologie und Physiologie der Sinnesorgane, Hermann von Helmholtz als Festgruss zu seinem siebzigsten Geburtstag.* Gesammelt und herausgegeben von A. König, Hamburg and Leipsic, 1891.

[1] J. Sylvester, in a footnote to "Note on a Proposed Addition to the Vocabulary of Ordinary Arithmetic," *Nature*, vol. 37, p. 152, 1887.

[2] J. Sylvester, British Assoc. for the Adv. of Science, *Report*, 1869, page 7 of Notices and Abstracts.

[3] Compare "Die Mathematik ist die Musik des Verstandes, die Musik die Mathematik des Gefühls" as employed by Josef Petzval, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 12, 1903, p. 327.

[4] This passage occurs in a footnote in the midst of Sylvester's memoir "Algebraical researches containing a disquisition on Newton's rule for the discovery of imaginary roots . . .," *Philosophical Transactions* for 1864, vol. 154, 1865, p. 613.

[5] J. Ozanam, *Dictionaire Mathematique*, Amsterdam, or Paris, 1691.

[6] A. Savérien, *Dictionnaire Universel de Mathematique et de Physique*, Paris, 1753, vol. 2.

[7] C. Hutton, *A Philosophical and Mathematical Dictionary*, London, 1795, vol. 2; new edition, 1815.

[8] J. F. Montucla and J. de La Lande, *Histoire des Mathématiques*, Paris, vols. 1 and 4, 1799, and 1802. Compare D. E. Smith, "The threatened loss of the second edition of Montucla's History of Mathematics" in this MONTHLY, *1921*, 207–208.

the threshold of the nineteenth century. It is, therefore, not surprising that many mathematicians wrote on musical matters. I shall presently consider these at some length. But certain other facts may first be reviewed.

The manner in which music, as an art, has played a part in the lives of some mathematicians is recorded in widely scattered sources. A few instances are as follows.

Maupertuis was a player on the flageolet and German guitar and won applause in the concert room for performance on the former.[1] At different times William Herschel served as violinist, hautboyist, organist, conductor, and composer (one of his symphonies was published) before he gave himself up wholly to astronomy.[2] Jacobi had a thorough appreciation of music.[3] Grassmann was a piano player and composer, some of his three-part arrangements of Pomeranian folk-songs having been published; he was also a good singer and conducted a men's chorus for many years.[4] János Bolyai's gifts as a violinist were exceptional and he is known to have been victorious in 13 consecutive duels where, in accordance with his stipulation, he had been allowed to play a violin solo after every two duels.[5] As a flute player De Morgan excelled.[6] The late G. B. Mathews knew music as thoroughly as most professional musicians; his copies of Gauss and Bach were placed together on the same shelf.[7] It was with good music that Poincaré best liked to occupy his periods of leisure. The famous concerts of chamber music held at the home of Emile Lemoine during half a century exerted a great influence on the musical life of Paris.[8] And in

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[1] *Basler Jahrbuch*, 1910, p. 46 in "Maupertuis" (pp. 29–53) by F. Burckhardt; see also "Maupertuis' Lebensende" by the same author in *Basler Jahrbuch*, 1886, pp. 153–159. These very interesting articles contain new material concerning the closing days of Maupertuis at the home of his friend Johann Bernoulli the second. Compare D. E. Smith, "Maupertuis and Frederick the Great" in this MONTHLY, *1921*, 430–432. The first memoir presented to the French Academy by Maupertuis was "Sur la forme des instruments de musique," *Mémoires de l'Academie Royale des Sciences, 1724*, Paris, 1726, pp. 215–226 + 1 plate; *Histoire*, pp. 90–92. This memoir is not contained in *Oeuvres de Mr. Maupertuis* published at Dresden in 1752 and at Lyons in 1756.

[2] Concerning the musical activities of Herschel, see especially *The Scientific Papers of Sir William Herschel*, vol. 1, London, 1912, pp. xiv–xxii; F. J. Fétis, *Biographie*, etc., vol. 4, *loc. cit.*, and A. Noyes, *The Torch-Bearers, Watchers of the Sky*, New York, 1922, p. 231 f. Noyes perpetuates the old error about Herschel "deserting from the army." Herschel was born in 1738 and died in 1822.

[3] S. Hensel, *Die Familie Mendelssohn, 1729–1847*, second ed., Berlin, 1880, vol. 2, pp. 364–365; English edition, London, 1881, vol. 2, p. 324.

[4] Compare F. Engel, *Grassmann's Leben* in *Hermann Grassmann's Gesammelte Mathematische und Physikalische Werke*, vol. 3, part 2, Leipsic, 1911, pp. 250–253, 371–372. Grassmann was born in 1809 and died in 1877.

[5] F. Schmidt, "Lebensgeschichte des ungarischen Mathematikers Johann Bolyai," *Abhandlungen zur Geschichte der Mathematik*, Heft 8, 1898, p. 141. Bolyai was born in 1802 and died in 1860; sketches by G. B. Halsted appeared in this MONTHLY, *1896*, 1–5, and *1898*, 35–38.

[6] Sophia E. De Morgan, *Memoir of Augustus De Morgan*, London, 1882, p. 16. See also A. M. Stirling, *William De Morgan and his Wife*, London, 1922, p. 61. De Morgan was born in 1806 and died in 1871.

[7] *Proceedings of the Royal Society*, London, vol. 101 A, 1922, p. xiv; also in *Proceedings of the London Mathematical Society*, 2 series, vol. 21, 1923, p. l. Mathews was born in 1861 and died in 1922.

[8] L. Augé de Lassus, *La Trompette. Un demi-siècle de Musique de Chambre*, Paris, 1911. See also D. E. Smith, "Emile Michel Hyacinthe Lemoine," in this MONTHLY, *1896*, 29–33. Lemoine was born in 1840 and died in 1912.

America we have only to recall colleagues in the mathematics departments of the Universities of California, Chicago and Iowa, and of Cornell University, who are, to use Shakespeare's phrase, "cunning in music and mathematics."

While Friedrich T. Schubert, the Russian astronomer and mathematician, played the piano, flute, and violin in an equally masterly fashion,[1] his great-grand-daughter Sophie Kovalevsky was devoid of musical talent; but she is said to have expressed her willingness to part with her talent for mathematics could she thereby become able to sing.[2] Abel had no interest in music as such, but only for the mathematical problems it suggested. His close attention to a performer at a piano was once explained by the fact that he sought to find a relation between the number of times that each key was struck by each finger of the player.[3] Lagrange welcomed music at a reception because he could by the fourth measure become oblivious to his surroundings and thus work out mathematical problems; for him the most beautiful musical work was that to which he owed the happiest mathematical inspirations.[4] Dirichlet seemed to be sensible to the charms of music in a similar manner.[5]

Such are a few instances, which could be considerably multiplied, of the relation of mathematicians to the art of music

"that gentlier on the spirit lies
Than tir'd eyelids upon tir'd eyes."

They suggest the accuracy of at least a part of the following observations of Möbius in his book on mathematical abilities:[6] "Musical mathematicians are frequent . . . but there are wholly unmusical mathematicians and many more musicians without any mathematical capability." That there are musicians with some mathematical ability will be granted when we recall, not only that Henderson, the prominent New York music critic and the author of many works on musical topics, has written a little book on navigation,[7] but also that the late Sergei Tanaïeff, pupil of Rubenstein and Tchaikovsky, successor of the latter as professor of composition and instrumentation at the Moscow Conservatory of Music, and one of the most prominent of modern Russian composers, found algebraic symbolism and formulæ of fundamental importance in his lectures and work on counterpoint.[8]

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[1] *Allgemeine deutsche Biographie.* Schubert was born in 1758 and died in 1825.

[2] Sophie von Adelung, *Deutsche Rundschau,* vol. 89, 1896, p. 405. Sophie Kovalevsky was born in 1850 and died in 1891.

[3] *Niels Henrik Abel Memorial publié à l'occasion du centenaire de sa naissance,* Christiania, 1902, pp. 57–58. Abel was born in 1802 and died in 1829.

[4] *Œuvres de Lagrange,* Paris, vol. 1, 1867, p. xlviii. Lagrange was born in 1736 and died in 1813.

[5] *G. Lejeune Dirichlets Werke,* vol. 2, Berlin, 1897, p. 343. Dirichlet was born in 1805 and died in 1859.

[6] P. J. Möbius, *Ueber die Anlage zur Mathematik,* zweite vermehrte und veränderte Auflage, Leipsic, 1907, p. 124.

[7] W. J. Henderson, *The Elements of Navigation,* New York, 1895; new and enlarged edition, 1917. He was formerly lieutenant in the first battalion, naval militia of New York.

[8] A. E. Hull, "Music and Mathematics," *The Monthly Musical Record,* London, May 1, 1916, vol. 46, p. 133 f. Tanaïeff was born in 1856 and died in 1915. See also "Russian Music, A Taneyef Souvenir," *The Monthly Musical Record,* vol. 46, pp. 313–314; and *Life and Letters*

A question which has interested more than one group of inquirers is: Can one establish any relationship between mathematical and musical abilities? Within the past year two Jena professors, Haecker and Ziehen, published the results of an elaborate inquiry as to the inheritance of musical abilities in *musical* families.[1] As a by-product of the inquiry they arrived at the result that in only about 2 per cent. of the cases considered was there any appreciable correlation between talent for music and talent for mathematics; they found also that the percentage of males lacking in talent for music but showing a talent for mathematics was comparatively high, about 13 per cent. At the Eugenics Record Office of Cold Spring Harbor, Long Island, there has been collected a considerable body of data upon which a study of the correlation of mathematical and musical abilities could be based. It will be interesting to see if the conclusions of Haecker and Ziehen are here checked, and also if some results are found as to the extent to which musical abilities are present in a group of mathematicians.[2]

II

Turning now to the *theory* of music, it is natural to inquire: What are the relations of mathematics to music? What have mathematicians written about music or its theory? Even on the part of one fully informed and competent, to answer these questions with any degree of completeness would require not one hour only, but many hours. I shall therefore limit myself to brief statements, with references to only a score or so of the better known mathematicians.

In any consideration of the history of music and its relation to mathematics it is important to have in mind the general character of music of different periods. With Helmholtz [3] these may be stated as follows:

(*a*) The Homophonic or Unison Music of the ancients, including the music of the Christian era up to the eleventh century, to which also belongs the existing music of Oriental and Asiatic nations.

(*b*) Polyphonic Music of the middle ages, with several parts, but without regard to independent musical significance of the harmonies, extending from the tenth to the seventeenth century, when it passes into

(*c*) Harmonic or Modern Music, characterized by the independent significance attributed to the harmonies as such.

of Peter Ilich Tchaikovsky, by M. Tchaikovsky, edited from the Russian by R. Newmarsh, London, 1916, with many references.

[1] "Ueber die Erblichkeit der musikalischen Begabung, nebst allgemeinen methodologischen Bemerkungen uber die psychische Vererbung," *Zeitschrift für Psychologie*, 1922, vol. 88, pp. 265–307; vol. 89, pp. 273–312; vol. 90, pp. 204–306; see particularly pp. 290–298. This was reprinted in book form with the title, *Zur Vererbung und Entwicklung der musikalischen Begabung*, Leipsic, 1922, 3 + 186 pages. Haecker is also the author of *Der Gesang der Vögel, seine anatomischen und biologischen Grundlagen*, Jena, 1900, 6 + 102 pp.

[2] The correlations between arithmetic and singing, space intuition and singing and other subjects studied by a group of 42 boys in the sixth year of a grammar school at Kiel, Germany have been set forth in M. Lobsien, "Korrelationem zwischen den unterrichtlichen Leistungen einer Schülergruppe," *Zeitschrift für experimentelle Pädagogik*, Leipsic, vol. 11, 1910, pp. 146–164

[3] *Loc. cit.*, p. 236.

Our first consideration is therefore to be given to the homophonic music of the *Greeks:* for in music as in mathematics the period of real development began in the sixth century B.C. with Pythagoras. Before his time tones an octave or a fifth apart, above and below, were regarded as consonant and as the basis of ordinary needs in declamation. If the c be taken as a point of departure, its fifth is g, and its fifth below is f. If this last note f be raised an octave so as to bring it nearer to the other notes, and if the octave of c be also added, the following four notes are obtained: c, f, g, c. Tradition affirms that these four notes constituted the range of the lyre of Orpheus. As Blaserna remarks,[1] "Musically speaking it is certainly poor, but the observation is interesting that it contains the most important musical intervals of declamation. In fact, when an interrogation is made, the voice rises a fourth. To emphasize a word, it rises another tone and goes to the fifth. In ending a story, it falls a fifth, etc. Thus it may be understood that Orpheus' lyre, notwithstanding its poverty, was well suited to a sort of musical declamation."

The notable contribution of Pythagoras was his enunciation of the law governing such sounds which are found in all the musical scales known. "He proclaimed the remarkable fact, of which the proof existed in his famous experiments with stretched strings of different lengths, that the ratios of the intervals perceived as consonant could all be expressed by the numbers 1, 2, 3, 4. His method of demonstration was afterward improved and rendered more exact by the invention of the monochord, and his law may now be stated as follows:[2]

"If a string be divided into two parts by a bridge, in such a manner as to give two consonant sounds when struck, the lengths of those parts will be in the ratio of two of the first four positive integers. If the bridge be so placed that two thirds of the string lie to the right and one third to the left, so that the two lengths are in the ratio of $1 : 2$, they produce the interval of the octave, the greater length being given to the deeper note. If the bridge be so placed that three fifths of the string lie to the right and two fifths to the left, the ratio of the two lengths is $2 : 3$ and the interval produced is the fifth. If the bridge be again shifted to a position which gives four sevenths on the right and three sevenths on the left, the ratio is $3 : 4$ and the interval is the fourth." Thus corresponding to the successively higher notes c, f, g, and c we have the numbers 1, 3/4, 2/3, and 1/2 for the relative lengths of the strings corresponding to the different notes.

The fourth and fifth gave the means of fixing a much smaller interval, called a tone, corresponding to which is the number $8/9 = 2/3 \div 3/4$. Starting with a fundamental c and inserting two tones between it and its fourth, two more between its fifth and its octave, the corresponding numbers for the succession $c\ d\ e\ f\ g\ a\ b\ c$ would be 1, 8/9, 64/81, 3/4, 2/3, 16/27, 128/243, 1/2. The numbers corresponding to successive *pairs* of notes would be 8/9, 8/9, 243/256, 8/9, 8/9,

[1] P. Blaserna, *The Theory of Sound in its relation to Music*, London, 1875, p. 117. Compare Helmholtz, *loc. cit.*, p. 255, and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 113 f.

[2] H. E. Wooldridge, *The Oxford History of Music*, vol. 1, Oxford, 1901, p. 11.

8/9, 243/256, the 243/256 being that number by which it is necessary to multiply into $8/9 \times 8/9$ in order to give 3/4.

Pythagoras looked upon the diatonic scale to which we have just referred in quite a different manner, namely, as derived from a succession of fifths. Thus starting from a prime *c* we have

$$c\ g\ d\ a\ e\ b.$$

Reducing *d* an octave, *a* an octave, *e* two octaves, and *b* two octaves, we have the series

$$c\ d\ e\quad g\ a\ b.$$

To obtain the *f* missing in this series and to fill up the wide interval between *e* and *g* it appears that *c* as a fifth below the prime was raised an octave. It may be readily verified that we are thus led to the same results as before; for example, *d*, the second fifth above the prime, is given by $2/3 \times 2/3$; to the *d* an octave lower corresponds $2 \times 2/3 \times 2/3 = 8/9$.

Pythagoras proposed to find in the order of the universe, where whole numbers and simple ratios prevail, an answer to the question: Why is consonance (the beautiful in sound) determined by the ratio of small whole numbers? The correct numerical ratios existing between the seven tones of the diatonic scale corresponded, according to Pythagoras, to the sun, moon and five planets, and the distances of the celestial bodies from the central fire, etc.

"It was the elaboration of these figments of philosophy, and because the fifth as the central tone of the octave corresponded to the astronomical order in which the Samian sage ranged the sun and planets, that he laid such a deep stress upon the *c* scale obtained from fifths only." [1]

Pythagoras limited himself to the insertion of seven notes within the octave. But from the primal scale he evolved six others. This was done not by setting up a new succession of fifths on the several notes of the primal scale but by making the second note of his first scale the prime of his second and so for each of five remaining notes. In this way, for example, we get the scale *d, e, f, g, a, b, c, d* with the corresponding numbers 1, 8/9, 27/32, 3/4, 2/3, 16/27, 9/16, 1/2. To the succession *b, c, d, e, f, g, a, b* corresponds 1, 243/256, 27/32, 3/4, 729/1024, 81/128, 9/16, 1/2.

It is not apparent in this latter scale that the method of Pythagoras can be said to illustrate the principle that the beautiful in sound must depend upon a succession of notes related to each other and a prime, by the simplest possible ratios.

The most noted of all the musical theorists of antiquity was Aristoxenus of Tarantum, a contemporary and pupil of Aristotle. To him as author have been assigned no less than 453 works but of these none now remain except the Harmonics,[2] portions of a treatise on rhythm, and some fragments recently found in

[1] H. Wylde, *The Evolution of the Beautiful in Sound*, London, 1888. Compare "The music of the spheres," *Harper's Magazine*, vol. 63, 1881, pp. 286–288. But best of all in this connection see chapter 12 of T. L. Heath, *Aristarchus of Samos, the Ancient Copernicus*, Oxford, 1913, with references to still more elaborate discussions.

[2] ΑΡΙΣΤΟΞΕΝΟΥ ΑΡΜΟΝΙΚΑ ΣΤΟΙΧΕΙΑ. *The Harmonics of Aristoxenus, Edited with*

Egypt. According to Macran (page 87), his great service was rendered "firstly, in the accurate determination of the scope of musical science, lest on the one hand it should degenerate into empiricism, or on the other hand lose itself in mathematical physics; and secondly, in the application to all questions and problems of music of a deeper and truer conception of the ultimate nature of music itself."

Of two treatises on music attributed to Euclid,[1] only the Theory of Intervals or Section of the Canon, as it is sometimes called, may be regarded as genuine.[2] It is based on the Pythagorean theory of music, "is mathematical, and clearly and well written, the style and form of the propositions agreeing well with what we find in the Elements."

The way in which the work starts out seems somewhat remarkable when we remember that it was written about three hundred years before Christ. It commences as follows:[3] "If all things were at rest, and nothing moved, there must be perfect silence in the world; in such a state of absolute quiescence nothing could be heard. For motion and percussion must precede sound; so that as the immediate cause of sound is some percussion, the immediate cause of all percussion must be motion. And whereas of vibratory impulses or motions causing a percussion on the ear, some there be returning with a greater quickness which consequently have a greater number of vibrations in a given time, whilst others are repeated slowly and of consequence are fewer in an assigned time, the quick returns and greater number of such impulses produce the higher sounds, whilst the slower which have fewer courses and returns, produce the lower. Hence it follows, that if sounds are too high they may be rendered lower by a diminution of the number of such impulses in a given time, and that sounds which are too low, by adding to the number of their impulses in a given time, may be made as high as we choose. The notes of music may be said then to

translation, notes, introduction, and index of words by H. S. Macran, Oxford, 1902. See also L. Laloy, *Aristoxène de Tarente et la musique de l'Antiquité*, Paris, 1904; C. F. A. Williams, *The Aristoxenian Theory of Musical Rhythm*, Cambridge, 1911; and F. A. Wright, *The Arts in Greece*, London, 1923, pp. 52–55.

[1] The best text with Latin translation, for the musical works attributed to Euclid, is that edited by H. Menge in *Euclidis Opera Omnia* edited by Heiberg and Menge, Leipsic, vol. 8, 1916. There is a critical introduction, "De scriptis musicis," pages xxxvii–liv. These same works, in Greek and Latin, are to be found in David Gregory's edition of Euclid's works (Oxford, 1703, pp. 531–536). A Latin-French edition by P. Herigone appears in his *Cursus Mathematicus*, vol. 5, Paris (1637), 1644, pp. 802–856. There is a French translation by P. Forcadel, *Le livre de la musique d'Euclide*, Paris, 1566; this is also in L. Lucas, *Une révolution dans la musique . . .*, Paris, 1849, and in L. Lucas, *L'Acoustique nouvelle . . .*, Paris, 1854. Another French translation is by C. E. Ruelle, *L'introduction harmonique de Cléonide, La division du canon d'Euclide . . .*, Paris, 1884. An English translation appears in C. Davey, *Letters addressed chiefly to a young gentleman upon subjects of Literature: including a translation of Euclid's section of the Canon; and his treatise on Harmonic; with an explanation of the Greek musical modes according to the Doctrine of Ptolemy*, Bury St. Edmunds, 1787, vol. 2, pp. 264–410. The Theory of Intervals, in Greek, is also given in K. v. Jan, *Musici Scriptores Græci*, Leipsic, 1895–1898, vol. 1, pp. 148–166; there is a "prolegomena" (in Latin), by Jan, pp. 115–147.

[2] Compare Heath, *The Thirteen Books of Euclid's Elements*, Cambridge, 1908, vol. 1, p. 17, and vol. 2, p. 295.

[3] C. Davey, *loc. cit.*, pp. 264–265; the punctuation and slight changes in the wording have been made in the quotation.

consist of parts, inasmuch as they are capable of being rendered precisely and exactly tunable, either by increasing or diminishing the number of the vibratory motions which excite them. But all things which consist of numerical parts when compared together, are subject to the ratios of numbers, so that musical sounds or notes compared together, must consequently be in some numerical ratio to each other."

Nearly two thousand years passed before Galileo went one step further,[1] and proved that the lengths of strings of the same size and tension were in the inverse ratios of the numbers of the vibrations of the tones they produced.[2] It was not for another seventy years that the actual number of vibrations corresponding to a given tone was determined;[3] but we shall return to this a little later.

Euclid's work contains 19 theorems. They are mostly concerned with results which may be obtained by the division of a monochord, or string to be experimented upon, which Euclid calls Proslambanomenos. Let this be named A.[4]

This string A was first divided into four parts; three parts were taken and the perfect fourth established with the ratio 3 : 4; two parts were taken and the sound of the octave established; one part was taken and the sound of the double octave A was given.

The next experiment was to divide the length which produced the fourth of the prime into two equal parts, when the sound, the octave of the fourth, was established.

Proslambanomenos was then divided into two equal parts, and one of these being again divided into three parts, two parts were taken and the octave of the fifth was established.

And so till all the tones in two octaves were determined. By beginning with different letters in the series thus determined, Euclid got the seven Pythagorean scales covering two octaves instead of one. Euclid arrived at these sounds by the division of the monochord instead of by successions of fifths employed by Pythagoras.

[1] Galileo Galilei, *Discorsi e Dimostrazioni Matematiche*, Leyden, 1638, at the end of the "first day." A new English translation by H. Crew and A. de Salvio appeared at New York in 1914 with the title: *Dialogues concerning Two New Sciences*. The manuscript of the original work was sent to the printer in 1636 and the printing was completed in 1637; but many of the results were given by Galileo in lectures long before.

[2] Credit for this result is given to Galileo with full knowledge of the claims of Mersenne. Compare F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 35–36.

In J. C. Poggendorff, *Histoire de la Physique*, translated by E. Bibart and G. de la Quesnerie, Paris, 1883, the following sentence occurs on page 487: "Deux siècles après *Pythagore*, *Aristote* écrivit sur les sons, et fit preuve d'une connaissance exact des faits, donts il est peut-être redevable aux Pythagoriciens. Il savait, par example, que dans les cordes de tension égale et dans les tuyaux, le nombres des vibrations est en raison inverse des longueurs, et que les sons sont produits par des vibrations qui passent des corps sonores à l'air qui les transmet à notre oreille." I can find no verification of this statement as to Aristotle's knowledge.

[3] Compare Newton, *Philosophiæ Naturalis Principia Mathematica*, London, 1687, book 2, section 8, prop. 50, pp. 369–372; second ed., Cambridge, 1713, pp. 342–344.

[4] Compare "Euclid's mathematical divisions of a string, and resulting series of sounds" in H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, pp. 84–93. See also T. P. Thompson, *Theory and Practice of Just Intonation*, London, 1850, pp. 79–80; fourth ed., 1860, pp. 104–108. The fourth edition has the title *On the Principles and Practice of Just Intonation*.

Two of Euclid's theorems prove that an octave is less than six tones, the ratio of the interval being $(8/9)^6 \div (1/2) = 524288/531441$, or nearly 80 : 81.0915. This same ratio is got from $(2/3)^{12} \div (1/2)^7$. In other words it is the ratio determined by the difference of tones derived by counting 12 fifths and 7 octaves from a fundamental. This interval, between notes theoretically the same, was noted by Pythagoras and is called a Pythagorean comma.[1]

The scales of Pythagoras and Euclid differ in two important respects from our major scales, namely, in the ratios for the intervals of a third and a sixth. In the scale of *c*, the interval of a major third from the tonic is now $4/5 = 64/80$ instead of the Pythagorean $64/81 = (8/9)(8/9)$. This substitution of 4/5, even though not mentioned by Euclid, is not modern, but was already suggested in the late Pythagorean school.[2] The second substitution of 3/5 for the major sixth interval from the tonic naturally followed from this, since it is the octave of the fifth below the third. In this way the ratios of the intervals of the major scale became[3] 1, 8/9, 4/5, 3/4, 2/3, 3/5, 8/15, 1/2, while the intervals between successive pairs of notes became 8/9, 9/10, 15/16, 8/9, 9/10, 8/9, 15/16.

In such a scale if we tune up four perfect fifths on the one hand and two octaves and a major third on the other, we ought to arrive at the same note. The resulting comma here is 80/81 instead of the 80/81.0915 already referred to. It is the distribution of this comma which is ordinarily carried through in our equal-tempered scale. This temperament is said to have been proposed by Aristoxenus.

And last among the Greeks to whom we shall refer is the celebrated mathematician, astronomer and geographer, Claudius Ptolemy, who flourished in the second century of the Christian era. Apart from the Almagest, works on optics and mechanics, a book on stereographic projection, a book in which he tried to show that the possible number of dimensions is limited to three, and other works, Ptolemy wrote a remarkable treatise on music.[4] In it he discusses critically the earlier Pythagorean and Aristoxenean modes and tonalities and presents new developments. But the restrictions made in connection with the music seem to indicate the beginning of a decline.

Some interesting suggestions have been made by Paul Tannery as to the

[1] Compare Helmholtz, *On Sensations . . ., loc. cit.*, p. 432.

[2] D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 123.

[3] If in the series of ratios for the major scale we substitute 5/6 for a minor third, instead of 4/5 as for the major, and 9/16 for the 8/15, we have the succession at which Newton arrived, in an experiment with the prismatic colors of pure light published in his *Optiks*, London, 1704, p. 92. Measuring from an origin to the left to determine the points 1, 8/9, 5/6, 3/4, 2/3, 3/5, 9/16, 1/2, and erecting cross lines he found, as he states, "the said cross lines divided after the manner of musical chord . . . to represent the Chords of that Key, and of a Tone, a third Minor, a fourth, a fifth, a sixth Major, a seventh, and an eighth above the Key: And the intervals . . . will be spaces which the several Colours (red, orange, yellow, green, blue, indiao, violet) take up."

[4] A somewhat defective text of this work, together with a Latin translation by John Wallis, was published at Oxford in 1680. Compare Fétis, *Biographie . . .*, vol. 7. See also *The Oxford History of Music*, vol. 1, by H. E. Wooldridge, Oxford, 1901, pp. 15–22; H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, chapter XI, etc.; and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, pp. 108–112.

possible rôle of Greek music in the development of pure mathematics.[1] One of these is to the effect that the idea of logarithms may have been suggested by such mathematical relations as the following going back to Pythagoras:

$$\frac{1}{2}=\frac{2}{3}\times\frac{3}{4} \qquad \frac{1}{2}=\left(\frac{3}{4}\right)^2\times\frac{8}{9}$$

being immediately interpreted in music by: The octave is composed of a fifth and a fourth; the octave is composed of two fourths and of a major tone. Thus mathematical multiplication is changed into musical addition.

Another of Tannery's suggestions involves finding solutions of a Diophantine equation in three variables. In the first four notes of the major scale we had the relation

$$\frac{8}{9}\times\frac{9}{10}\times\frac{15}{16}=\frac{3}{4}.$$

Ptolemy derived many scales[2] in which the relations were similar; for example,

$$\frac{7}{8}\times\frac{9}{10}\times\frac{20}{21}=\frac{8}{9}\times\frac{7}{8}\times\frac{27}{28}=\frac{9}{10}\times\frac{10}{11}\times\frac{11}{12}=\frac{3}{4}.$$

In other words the question of the composition of the tetrachord reduces to the following mathematical problem: "Determine all possible ways of decomposing the ratio 3/4 into a product of three ratios of the form $n/(n+1)$." From these results, those were finally selected which seemed practicable after trial with the monochord.

In my brief sketch of the work done by the Greeks, I have not intended to give you any idea of their music, but merely to select a few illustrations of the manner in which their music is connected with mathematics. On the varieties of their scales and their coloring through chromatics (as the name implies) and quarter tones, I have not touched. Nor have I commented on the great beauties of the music even though it was homophonic. Authorities agree with the following summing up of Helmholtz:[3] "Of course where delicacy in any artistic observations made with the senses come into consideration, moderns must look upon the Greeks in general as unsurpassed masters. And in this particular case they had very good reason and abundance of opportunity for cultivating their ears better than ours. From youth upwards we are accustomed to accommodate our ears to the inaccuracies of equal temperament, and the whole of the former variety of tonal modes, with their different expression, has reduced itself to such an easily apprehended difference as that between major and minor. But the varied gradations of expression, which moderns attain by harmony and modulation, had to be effected by the Greeks and other nations that used homo-

[1] P. Tannery, "Du rôle de la musique grecque dans le developpement de la mathématique pure," *Bibliotheca Mathematica*, series 3, vol. 3, 1902, pp. 161–175; also in P. Tannery, *Mémoires Scientifiques*, vol. 3, 1915, pp. 68–89.

[2] Compare "Examen des series d'Archytas" in L. Laloy, *Aristoxène de Tarente et la musique de l'antiquité*, Paris, 1904, pp. 364–365.

[3] *On Sensations . . .*, *loc. cit.*, p. 266.

phonic music by a more delicate and varied gradation of tonal modes. Can we be surprised, then, if their ear became much more finely cultivated for differences of this kind than it is possible for ours to be?"

The next outstanding figure in our survey is Boetius who flourished in the early part of the sixth century of our Christian era. He was a Roman senator and a philosopher,—"the last of the Romans whom Cato or Tully could have acknowledged for their countryman," as Gibbon expresses it. Not only did Boetius exert great influence in his own time through his summaries of logical and scientific works of the ancients, but for six centuries after his death they were the leading authorities. He wrote works on arithmetic, geometry and music. While the first printed edition of the arithmetic appeared at Venice in 1488, all three united seem to have first been published in 1492. Details of the mathematical works have been given by Cantor.[1] His extensive treatise on music[2] is a valuable repertory of the knowledge of the ancients in this art. It was long used as a text at the Universities of Oxford and Cambridge.[3] Boetius sets forth the details of the accomplishments of the Pythagoreans and the teachings of such writers as Aristoxenus which were opposed to those of Pythagoras. He also surveys the Ptolemaic musical scheme in connection with those of Pythagoras and Aristoxenus. Since the doctrine of Boetius was mainly Pythagorean, this was the system which prevailed for centuries later.

As the Roman absorbed the Greek, so the Christians accepted the Roman organization of learning. In the medieval curriculum the scope of this learning on the secular side was comprised within the seven liberal arts[4] and philosophy. The seven liberal arts, divided into the *Trivium* (grammar, dialectic, rhetoric), and the more advanced *Quadrivium* (geometry, arithmetic, music and astronomy), were an inheritance from a period at least as early as the second century before Christ; indeed the Quadrivium division of mathematical studies is Pythagorean.[5] Some explanation of the nature of the subjects of the Trivium is necessary in order to make their scope clear; but we are only concerned with the mathematical sciences of the Quadrivium in which, early in the middle ages, the course in geometry was more a course in geography and surveying than in the subject matter of Euclid's *Elements* which later became a text. The study of music consisted mainly in becoming acquainted with the mathematics of the subject, and with the mystic properties of its numbers,—much as taught by the Pythagoreans. As a liberal art it concerned itself neither with singing (apart from its rules), nor with playing on an instrument. Astronomy with its practical applications to the calendar and sun dial was the most popular of the Quadrivium subjects but there was probably more of astrology in it than astronomy as we now understand the term.

[1] *Vorlesungen über Geschichte der Mathematik*, vol. 1, third ed., Leipsic, 1907. See also D. E. Smith, *Rara Arithmetica*, Boston, 1908.

[2] See Fétis, *Biographie . . .*, vol. 1, *loc. cit.*, and Eitner, *Quellen-Lexikon*, vol. 2, *loc. cit. Arithmetica et Musica* of Boetius, ed. by Friedlein, Leipsic, 1867, is the standard edition.

[3] See article "Boetius" in *Encyclopædia Britannica*, eleventh edition.

[4] Compare P. Abelson, *The Seven Liberal Arts*, New York, 1906.

[5] T. L. Heath, *A History of Greek Mathematics*, Oxford, 1921, vol. 1, pp. 11–12.

Some four hundred years after the time of Boetius, the polyphonic period in the development of music had its inception in the composition of certain two-part song-forms. During the six hundred years which followed, that is, till towards the close of the sixteenth century, polyphonic music adorned with canon, fugue, and counterpoint was developed to a notable degree.

Of mathematicians who flourished in this period I shall refer to only one, Girolamo Cardano.[1] Among those of the sixteenth century achieving a reputation in mathematics and medicine none was better known than he, whose greatest mathematical work, *Ars Magna* (1545), contains the first solution of the general cubic equation in print.

Cardano was an ardent lover of music and while living in Milan his house was constantly filled with men and boys of somewhat sinister reputation but capable of joining with him in part-singing so popular in the polyphonic period. During the last twenty-five years of his life he spent considerable time in writing a work on music,[2] which was in many respects original and must have been welcomed by all musical students as a valuable contribution to the literature of the subject. This work begins by laying down at length the general rules and principles of the art, and then goes on to treat of ancient music in all its forms; of music as Cardano knew and enjoyed it; of the system of counterpoint and composition, and of the construction of musical instruments.[3]

An interesting glimpse of Cardano's personality may be gleaned in another place from his listing of the joys of home and children. Incidentally he suggests: "Let the young child . . . be shut out from the sight or hearing of all ill. When he is about seven years old let him be taught elements of geometry to cultivate his memory and imagination. With syllogisms cultivate his reason. Let him be taught music, and especially to play upon stringed instruments; let him be instructed in arithmetic and painting, so that he may acquire taste for them, but not be led to immerse himself in such pursuits. He should be taught also a good hand-writing, astrology, and when he is older, Greek and Latin."

In the early part of the *third* period in the development of music, namely, the period of Harmonic or Modern Music, we have the first opera and the first oratorio, and, as I have already said, the discovery by Galileo that the simple ratios of the lengths of strings existed also for the pitch numbers of the tones they produced, an observation later generalized by Newton. By the time of Rameau, the most eminent French composer and writer on the theory of music in the eighteenth century, the harmonics or upper partial tones of the human voice had been recognized and made the basis for more satisfying harmonic development. A string, for example, vibrates not only as a whole but also, at

[1] 1501–1576.

[2] G. Cardano, *Opera Omnia*, Leyden, 1663, vol. 4. See also fragment no. 6 in vol. 10. Volume 4, no. 10 is *Opus novum de proportionibus numerorum, motuum, ponderum, sonorum aliarumque rerum mensurandarum* . . . first published at Basle in 1570.

[3] W. G. Waters, *Jerome Cardan, a biographical Study*, London, 1898, p. 256; see also pp. 163, 235. In the more extensive biography by H. Morley, *The Life of Girolamo Cardano of Milan, Physician*, 2 vols., London, 1854, there are references to music on the following pages: vol. 1, pp. 41, 45, 202, 295; vol. 2, pp. 19, 43, 53.

the same time, in each of its aliquot parts 1/2, 1/3, 1/4, 1/5, 1/6, and so on. Thus the first upper partial tone is the upper octave of the prime tone, the second is the fifth of this octave, the third upper partial is the second higher octave, the fourth is the major third of this second higher octave, the fifth is the fifth of the second higher octave, making six times as many vibrations as the prime in the same time; and so on, each successive upper partial tone being fainter than the preceding. It may be shown that beginning with the twenty-fourth upper partial all the notes of a major scale may be obtained from the dominant, that is, the fifth.[1] The dominant and not the tonic is thus the root, of the whole scale. In the bugle, trumpet, French horn, and other instruments only the fundamental tone of the instrument and some of its harmonics can be sounded. On a horn about four feet long the notes are c, c', g', c'', e'', and g'', —the primes denoting tones in higher octaves.

Not all upper partials need exist in connection with a fundamental musical tone. Certain tuning forks have no upper partials.[2] In 1800, the noted physicist, Thomas Young, who first furnished the key to decipher Egyptian hieroglyphics, was also the first to show that "when a string is plucked or struck, or, as we may add 'bowed' at any point in its length which is the node of any of its so-called harmonics, those simple vibrational forms of the string which have a node in that point are not contained in the compound vibrational form. Hence if we attack at its middle point, all the simple vibrations due to the even numbered partials, each of which has a node at that point, will be absent. This gives the sound of the string a peculiarly hollow or nasal twang."[3] Because of this law piano makers eliminate[4] certain undesirable upper partials by striking the middle strings of their instruments at a point 1/7 to 1/9 of their lengths from their extremities. So too in making other instruments it is possible to eliminate, or reinforce, certain partials.

But we have got ahead of our story. Returning to the beginning of the Harmonic period let us consider the musical writings which were issued in the seventeenth century by such mathematicians as Kepler, Wallis, Mersenne, Desargues, Descartes and Christian Huygens.

Pythagorean ideas on the ratios of numbers and of proportions applied to the constitution of the universe seem to have been the point of departure of Kepler in his famous work *Harmonices Mundi* published in 1619.[5] It is in the fifth book of this work that one first finds the third fundamental law of modern astronomy, "The squares of the periodic times of the several planets are proportional to the cubes of their mean distances from the sun," demonstration of which furnished Newton with the basis for his theory of gravitation. The third book of the work is especially devoted to music and it may be characterized

[1] The scale is made from the following partials: 24, 27, 30, 32, 36, 40, 45, 48.

[2] Helmholtz, *On Sensations* . . ., *loc. cit.*, pp. 54, 528.

[3] Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 52.

[4] Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 77; but compare pp. 545–546.

[5] *Joannis Kepleri astronomi opera omnia*, ed. C. Frisch, vol. 5, Frankfurt, 1864. Compare Fétis, *Biographie* . . ., *loc. cit.*, vol. 5. Kepler was born in 1571 and died in 1630.

as mainly a work on the philosophy of music. The fifth book to which I have referred is somewhat allied to the third, since in it the author endeavored to establish curious analogies between the harmonic proportions of music and astronomy.

Markedly contrasted to Kepler in abilities and habits of thought was John Wallis, the notably able Savilian professor at Oxford University, where a brilliant mathematical school was developed under his direction. He is well known as mathematician and cryptographer,[1] but few have observed his extensive writings on musical matters [2] filling more than 500 folio pages in the third volume of his collected works. The first of these is a Greek and Latin edition of Ptolemy's Harmony, and Porphyry's third century commentary [3] on the same, with an extensive appendix by Wallis on ancient and modern music. Then comes the only published text, with Latin translation, of a musical work by Manuel Bryenne, a fourteenth century Greek, four manuscripts of whose work are to be found at the Bodleian. Among other writings of Wallis on acoustics and music may be mentioned four memoirs published in the *Philosophical Transactions*,[4] and bearing the following titles: "On the trembling of consonant strings," "On the division of the monochord, or section of the musical canon," "On the imperfections of an organ," and "On the strange effects of music in former times."

The Franciscan friar Marin Mersenne, Wallis's senior by nearly 30 years, is known to the general run of mathematicians through the numbers with which his name is associated and which arise in discussion of perfect numbers. He was widely acquainted with French and foreign contemporary mathematicians and actively corresponded with them. His work in physics [5] dealt chiefly with questions in acoustics. He determined ratios of the vibration numbers of strings varying in thickness and tension, results included in those of Brook Taylor derived mathematically about 70 years later. I have not been able to verify the statement [6] that Mersenne noticed, but attached no importance to the observation, that a vibrating string gave forth not only the fundamental tone but also higher sounds. We have already remarked that Rameau made much of the fact in the following century. Mersenne wrote half a dozen works on harmony and musical instruments [7] but his most notable one is *L'Harmonie Universelle*, a great work of 1500 pages with an immense quantity of engraved plates and musical examples. This was published in 1636–7. It is really a combination of several treatises, for example, On the Nature of Sounds and

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[1] Compare D. E. Smith, "John Wallis as a Cryptographer," *Bulletin of the American Mathematical Society*, vol. 24, pp. 82–96, 1917. Wallis was born in 1616 and died in 1703.

[2] Compare Fétis, *Biographie* . . ., vol. 8, *loc. cit.* Also H. Mendel, *Musikalisches Conversations-Lexikon*, vol. 11, 1878.

[3] Jan believes that this was probably mostly compiled by Pappus or some other competent mathematician; see K. v. Jan, *Musici Græci Scriptores Græci*, Leipsic, 1895, p. 116.

[4] 1677–1698.

[5] F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 93–95, etc.; also J. C. Poggendorff, *Histoire de la Physique*, Paris, 1883, pp. 488–489.

[6] J. K. Fischer, *Geschichte der Physik*, vol. 1, Göttingen, 1801, pp. 468, 470.

[7] Compare Fétis, *Biographie* . . ., vol. 6, *loc. cit.*; compare Eitner, *Quellen-Lexikon* . . ., vol. 6, *loc. cit.*

Movements of All Sorts of Bodies, On Voice and Songs, and On Instruments. There is also a treatise on mechanics, by Roberval, which no one but a Mersenne could regard as appropriately placed in his work on harmony. While no sections of the work are of transcendent merit, one finds a great amount of information, especially regarding Frenchmen, which is no longer to be found elsewhere. It is only here, for example, that we learn that the geometer Desargues was the author of a method of singing.

Among Mersenne's friends was one, some eight years his junior, René Descartes. That he was interested in music [1] is attested by the fact that a score of his published letters treat of motions of vibrating strings and various musical topics. Moreover in 1618, when 22 years of age, he wrote a *Compendium Musicæ*, but this was first published as a little tract of 58 pages [2] in 1650, the year of his death. The material is arranged under about a dozen headings such as: the object of music is the sound; number and time that one should observe in the sounds; concerning the diversity of sounds; consonances; the octave; the fifth; the fourth; the second, minor third, and sixth; the degrees or tones of music; dissonances; and the manner of composing—in connection with which five principles are laid down in an interesting manner.

A copy of the manuscript of the *Compendium* found its way to one afterwards to become a particular friend of Descartes.[3] This was Constantin Huygens, a many-sided genius possibly best known as a poet and a musician; he was a competent performer on several instruments and author of several musical works. His second son was Christian the great Dutch mathematician, mechanician, astronomer and physicist. Two publications dealing with musical matters were written by Christian Huygens. The first of these is a brief sketch of 1691, entitled "Novus Cyclus Harmonicus," and occupying only 8 quarto pages.[4] In them he suggests another solution of the problem of how suitably to arrive at a tempered scale. If we divide the octave into twelve equal parts or degrees, we have a cycle in which a fifth of 7 and a major third of 4 degrees approximates to Pythagorean intonation. A cycle of 53, with a fifth of 31 and a major third of 18, had also been proposed, and led to similar results. The new harmonic cycle of Huygens contained 31 degrees, with a fifth of 23 and a major third of 10, and closely imitates mean tone temperament.[5] He refers to the writings of

[1] Compare Fétis, *Biographie* . . ., vol. 3, *loc. cit.*; and Eitner, *Quellen-Lexikon* . . ., vol. 3, *loc. cit.* Descartes was born in 1596 and died in 1650.

[2] See also *Œuvres de Descartes* publiées par Charles Adam et Paul Tannery, Paris, vol. 10, 1908, pp. 79–150. An English Translation was published at London in 1653. There were four other French editions.

[3] See many references in *Œuvres de Descartes*, vol. 12, 1910, *Vie & Œuvres de Descartes* by C. Adam. See also in this MONTHLY, *1921*, 167, where in the course of an article by D. E. Smith on "Descartes's appreciation of Huygens the elder" a letter from Descartes dated May 23, 1632, contains the following clause: "I do not know how to respond to the courtesy of Monsieur Huygens, except that I cherish the honor of his acquaintance as one of the greatest pieces of fortune that has come to me."

[4] In *L'Histoire des Ouvrages des Sçavans*, Rotterdam, October, 1691, p. 78; also C. Huygens, *Opera Varia*, Amsterdam, 1724, vol. 3, pp. 747–754. Christian Huygens was born in 1629 and died in 1695.

[5] Compare Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 436. See also *Proc. Roy. Soc. of London*, vol. 13, 1864, p. 412.

Mersenne, and of Zarlino, "one of the most learned and enlightened music theorists of the sixteenth century." I have already drawn attention to the natural way in which logarithms enter into the discussion of musical intervals. So far as I have been able to determine, this little publication of Huygens is the first to illustrate this fact.

The second work of Huygens containing musical material was finished for the press just before his death. Three years later it appeared simultaneously in Latin and English and is an exceedingly entertaining work. It is entitled *The Celestial Worlds discover'd: or Conjectures concerning the Inhabitants Plants and Productions of the Worlds in the Planets.*[1] In order adequately to present an idea of a section on mathematics and music I shall quote somewhat extensively.

The author surmises that if the surfaces of Jupiter and Saturn are divided like ours into sea and land it is reasonable to suppose that the inhabitants must know of the art of navigation. He then infers that they must have the "Mechanical Arts and Astronomy, without which Navigation can no more subsist, than they can without Geometry." Huygens then continues (page 84): "But Geometry stands in no need of being prov'd after this manner. Nor doth it want assistance from other Arts which depend upon it, but we may have a nearer and shorter assurance of their not being without it in those Earths. For that Science is of such singular Worth and Dignity, so peculiarly imploys the Understanding, and gives it such a full Comprehension, and infallible certainty of Truth, as no other Knowledge can pretend to: it is moreover of such a Nature, that its Principles and Foundations must be so immutably the same in all Times and Places, that we cannot without Injustice pretend to monopolize it and rob the rest of the Universe of such an incomparable Study. Nay Nature itself invites us to be Geometricians: it presents us with Geometrical Figures, with Circles and Squares, with Triangles, Polygons, and Spheres, and proposes them as it were to our Consideration and Study which abstracting from its usefulness is most delightful and ravishing. Who can read *Euclid* or *Apollonius*, about the Circle, without Admiration? Or *Archimedes* of the Surface of the Sphere, and Quadrature of the Parabola without Amazement? Or consider the late ingenious Discoveries of the Moderns, with Boldness and Unconcernedness? And all these Truths are as naked and open, and depend upon the same plain Principles and Axioms in *Jupiter* and *Saturn* as here, which makes it not improbable that there are in the Planets some who partake with us in these delightful and pleasant studies." Then a little later the author continues (page 86): "It's the same with Musick as with Geometry, it's everywhere immutably the same, and always will be so. For all Harmony consists in Concord, and Concord is all the World over fix'd according to the same invariable Measure and Proportion so that in all Nations the Difference and Distance of Notes is the same, whether they be in a continued gradual Progression, or the Voice skips over one to the next. Nay, very credible Authors report, that there's a sort of Bird in *America*, that can plainly sing in

[1] London, 1698; "second edition corrected and enlarged" in 1722; "new edition corrected," Glasgow, 1762. The Latin edition published at The Hague is entitled Κοσμοθεωρός *sive de terres cœlestibus, earumque ornatu, conjecturae*; second edition, 1699.

order six musical Notes: Whence it follows, that the Laws of Musick are unchangeably fix'd by Nature, and therefore the same Reason holds for their Musick, as we e'en now shewed for their Geometry."

Discussing the probability of other planets' being inhabited and of the inhabitants' possible interest in music and invention of musical instruments, he continues (page 88): "What if they should excell us in the Theory and practick part of Musick, and outdo us in consorts of vocal and instrumental Musick, so artificially compos'd, that they shew their skill by the Mixtures of Discords and Concords and of this last sort 'tis very likely the 5th and 3d are in use with them.

"This is a very bold Assertion, but it may be true for aught we know, and the Inhabitants of the Planets may possibly have a greater insight into the Theory of Musick than has yet been discover'd among us. For if you ask any of our Musicians why two or more perfect Fifths cannot be used regularly in Composition; some say 'tis to avoid that Sweetness and Lushiousness which arises from the repetition of this pleasing Chord. Others say, this must be avoided for the sake of that Variety of Chords that are requisite to make a good Composition; and these Reasons are brought by Descartes [1] and others. But an Inhabitant of *Jupiter* or *Venus* will perhaps give you a better Reason for this, *viz.* because when you pass from one perfect Fifth to another, there is such a Change made as immediately alters your Key, you are got into a new key before the Ear is prepared for it, and the more perfect Chords you use of the same kind in Consecution, by so much the more you offend the Ear by these abrupt Changes."

It may interest harmony students of our day to learn that the prohibition of consecutive fifths [2] was not something recently invented for their undoing, but was a matter of fundamental importance adequately explained over two hundred years ago. And this is not the only passage of interest for such students in the last work of Christian Huygens.

I have already referred to Thomas Young's memoir of 1800 and his explanations of varied qualities of tone through agitation of a string at different points. The mere fact of such differences of quality had been already noted by Huygens [3] in connection with harpsichords—those precursors of pianos in the sixteenth, seventeenth, and eighteenth centuries.

III.

In the eighteenth century when calculus had become a tool, there was a notable series of theoretical discussions of vibrating strings. But before considering these I wish to draw special attention to the first English scientific treatment of harmony, a work of high order, by Robert Smith.[4] It was entitled *Harmonics or the Philosophy of Musical Sounds* and was first published [5] in 1749.

[1] This was simply "*Cartes*" in the original.

[2] Compare "Modern music and 'fifths,' " *Monthly Musical Record*, vol. 46, 1916, pp. 43–44.

[3] Helmholtz, *On Sensations . . .*, *loc. cit.*, p. 77.

[4] Born 1689, died 1768. See *Dictionary of National Biography*, Oxford. Eitner, *Quellen-Lexikon . . .*, *loc. cit.*, vol. 9, confuses this Robert Smith with an earlier musician and composer; see *Musical Antiquary*, Oxford, vol. 2, pp. 171–173, 1911.

[5] "Second edition much improved and augmented," London, 1759, 20 + 293 pp. + 28 plates.

The theory of intervals and various systems of temperament are discussed in a manner very attractive even for a reader in the present day. Smith held the Plumian chair at Cambridge, the one of which A. S. Eddington is the present incumbent, and his work on harmonics contained the substance of lectures he had delivered for many years. It was he who was the author of the notable work on *Optics* which has been translated into several languages. He was also the founder of the well-known Smith's Prizes "annually awarded to those candidates who present the essays of greatest merit on any subject in mathematics or natural philosophy." [1]

First in the series of theoretical discussions to which I have referred are those of Brook Taylor, who, according to his biographer,[2] "possessed considerable ability as a musician and an artist." His discussions appeared in the *Philosophical Transactions* [3] for 1713 and 1715 and in his book *Methodus Incrementa Directa et Inversa*,[4] the first treatise dealing with finite differences, and the one which contains the celebrated theorem regarding expansions, now connected with Taylor's name. He solved the following problem which he believed to be entirely new: "To find the number of vibrations that a string will make in a certain time having given its length, its weight, and the weight that stretches it." In discussing the *form* of the vibrating string, his suppositions regarding initial conditions, including that it vibrated only as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier.

I have already referred to the discovery of upper partial tones by Rameau and how he made this the basis of a system of harmony; his first work on this subject was published in 1726, but the first mathematician who seemed to take account of the fact was Daniel Bernoulli in a memoir of 1741–43 though not published [5] till 1751. About this time D'Alembert's thorough acquaintance with Rameau's theories was shown by his publication in 1752 of a volume entitled "Elements of theoretical and practical music according to the principles of Monsieur Rameau, clarified, developed, and simplified." Of this work six French editions [6] and one in German were published.[7] Helmholtz remarks [8] that D'Alembert's book "is an extremely clear and masterly performance, such

A *Postscript* (12 pp. + 1 plate) was published in 1762. A German edition was published at Berlin in 1771.

[1] *Cambridge University Calendar*, 1921–22.

[2] E. I. Carlyle in *Dictionary of National Biography*, Oxford. Taylor was born in 1685 and died in 1731.

[3] Vol. 28, London, 1714, "De motu tensi," pp. 26–32; vol. 29, 1715, "An account of a book entitled Methodus Incrementorum by the author," pp. 339–350.

[4] London, 1715; other title pages are dated 1717.

[5] D. Bernoulli, *Comment. acad. sc. Petrop.*, vol. 13 (1741/3), 1751, p. 173, § 8. Bernoulli was born in 1700 and died in 1782.

[6] *Elémens de musique théorique et pratique, suivant les principes de M. Rameau, éclaircis, développés et simplifiés*, Paris, 1752; second edition, 1759; third and fourth editions, Lyons, 1762, 1766, 1772, and 1779. D'Alembert was born in 1717 and died in 1783. Compare Eitner, *Quellen-Lexikon* . . ., vol. 1, *loc. cit.*

[7] The German edition by F. W. Marpurg was published at Leipsic in 1757.

[8] *On Sensations* . . ., *loc. cit.*, p. 232.

as was to be expected from a sharp and exact thinker, who was at the same time one of the greatest physicists and mathematicians of his time. Rameau and D'Alembert lay down two facts as the foundation of their system. The first is that every resonant body audibly produces at the same time as the prime its twelfth and next higher third as upper partials. The second is that the resemblance between any tone and its octave is generally apparent. The first fact is used to show that the major chord is the most natural of all chords, and the second to establish the possibility of lowering the fifth and the third by one or two octaves without altering the nature of the chord, and hence to obtain the major triad in all its different inversions and positions."

D'Alembert wrote also a long essay on the liberty of music [1] and articles of musical interest in the great *Encyclopédie Methodique*. But from a mathematical point of view, his memoir of 1747 dealing [2] with Taylor's problem of the vibrating string, and taking account of matters previously overlooked, is very notable. He was led to the differential equation (with a, a constant, equal to unity)

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2},$$

where the origin of coördinates was at the end of the chord whose length is l, the axis of x in the direction of the chord, and y the displacement at any time t. Of this equation he found the solution

$$y = f(at + x) - f(at - x),$$

where f represents any function such that $f(z) = f(z + 2l)$. He then found certain equations for determining the functions satisfying this relation of periodicity.

Euler immediately raised the question of the generality of the solution and set forth *his* interpretation. D'Alembert had supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions. Lagrange [3] joined in the discussion, to which Daniel Bernoulli contributed chiefly from physical rather than mathematical considerations. He started with Taylor's particular solution and found, in effect, that the function for determining the position of the string after starting from rest could n aturally be expressed in a form later called a Fourier series.[4] Thus

[1] "*De la liberté de la musique*" in his *Mélanges de Littérature et de Philosophie*, Amsterdam, 1767–1773, and reprinted in d'Alembert, *Œuvres Philosophiques, historiques et littéraires*, Paris, 1805, vol. 3, pp. 335–409. The first sentence gives a clue as to the meaning of the title. It is: "Il y a chez toutes les nations, deux choses qu'on doit respecter, la *religion* et le *gouvernement;* en France on y en ajoute une troisième, la musique du pays."

[2] "Recherches sur la courbe que forme une corde tendue mis en vibration," *Memoires de l'Acad. Royale des Sciences et belles Lettres, 1747*, 1749, pp. 214–219; "Suite des Recherches sur . . .," pp. 220–249.

[3] Euler (1707–1783); Lagrange (1736–1813).

[4] The series is not stated explicitly by Bernoulli in the memoir, "Sur le mêlange de plusieurs espaces de vibrations simples isochrones, qui peuvent coexister dans un même système de corps," *Histoire de l'Acad. Royale des Science, 1753*, Berlin, 1755; but it can be put together from different places, no. 17, p. 160, and no. 23, p. 165.

were such series first introduced into mathematical physics. Bernoulli remarked that since his solution was perfectly general it should include those of Euler and D'Alembert. In this way mathematicians were led to consideration of the famous problem of expanding an arbitrary function as a trigonometric series. No mathematician would admit even the possibility of its solution till this was thoroughly demonstrated, in connection with certain problems in the flow of heat, by Fourier who gives due credit to the suggestiveness of the work of those in the previous century to whom I have referred. Fourier's results were contained in a memoir crowned by the French Academy in 1812 but not printed till more than a decade later. It is sometimes asserted that the first mathematical proof of Fourier's results, with the limits of arbitrariness of the function carefully stated, was given by Dirichlet in his classic memoirs of 1829 and 1837. So far as the limits of arbitrariness are concerned this is correct; but that Fourier rigorously established his expansion of an arbitrary function seems to admit of no denial or qualification.[1]

One of Euler's most notable papers connected with the history of Fourier's series did not appear in print till 1793, ten years after his death. Thus for eighty years, from Taylor to Euler and Lagrange, mathematicians were occupied with the problem of the vibrating string[2] and allied problems including the vibration of a column of air and of an elastic rod. Then thirty years of silence and the great advance by Fourier.

I have indicated only a few bald facts since details in this regard are readily available elsewhere.[3]

Although more than twenty years Fourier's senior, Gaspard Monge,[4] so well known as an expounder of the applications of analysis to geometry,[5] and of descriptive geometry, was associated with him in more than one undertaking. They were professors at the École Polytechnique in Paris, which Monge was largely instrumental in founding. They both accompanied Napoleon to Egypt where Monge was the first president of the Institute of Egypt and Fourier its secretary. Monge was a passionate devotee of music and made a journey to Italy in order to procure copies of all of the musical works in the chapel of St. Mark's, Venice. He was also an ardent republican and, according to Arago,[6] an enthusiast for the "Marseillaise" which he sang every day at the top of his

[1] Compare Darboux, in *Œuvres de Fourier*, vol. 1, Paris, 1888, p. 512.

[2] D'Alembert's own account of this is interesting. See his article "Cordes, (*vibrations des*)" in his *Encyclopédie Methodique, Mathématiques*, vol. 1, Paris, 1784.

[3] I have found Burkhardt's monumental report to be the most complete and most reliable source of information. It is entitled "Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 10, Heft 2, 1901–1908.

[4] Compare Fétis, *Biographie* . . ., vol. 6, *loc. cit.* See also D. E. Smith, "Monge and the American colonies," in this MONTHLY, *1921*, 166. Monge was borne in 1746 and died in 1818.

[5] When 25 years of age Monge published his "Recherche des équations des surfaces d'apres leurs mode de génération" in the memoirs of the academy of Turin which had been founded through Lagrange's influence. On reading this paper Lagrange exclaimed, "Avec son application de l'analyse à la représentation des surfaces, ce diable d'homme sera immortel." (D. F. J. Arago, *Œuvres Complètes*, vol. 2, Paris, 1854, pp. 447–448.)

[6] Arago, *Œuvres*, vol. 2, p. 516.

voice before seating himself at the table. He, too, occupied himself with the problem of the vibrating string[1] and constructed a model of a surface, certain parallel sections of which give the form of the curve of the vibrating string at any time under conditions which Monge states. This model which was made in 1794 is still preserved in the École Polytechnique.

And finally in connection with great mathematicians of the eighteenth century, the extent of Euler's contributions to the theory of vibrating bodies, acoustics, and music, may be indicated somewhat further.[2] About 30 of his published memoirs, and a treatise, *Tentamen novæ theoriæ musicæ*,[3] not to speak of letters in his Letters to a German Princess, deal with such subjects. They appeared during about 60 years[4] from the first, a dissertation on sound, published in 1727, when he was 20 years old. Among the topics of memoirs not already referred to are: On the sound of bells, Conjectures as to the reason of some dissonances generally accepted in music, The true character of modern music, and On the vibratory motion of drums. It is in this last mentioned memoir of 1766 that the general so-called Bessel's functions of integral order first occur.

Euler's treatise on music was first published in 1739, but we learn from a letter Euler wrote to Daniel Bernoulli in May, 1731, that he had already almost completed the manuscript of the work. This letter describing the ideals of the work in some detail, as well as Bernoulli's reply in the following August, are readily accessible.[5] I shall therefore make but brief extracts from the early parts of the letters. Euler explains, "My main purpose was that I should study music as a part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing. In the whole discussion I have necessarily had a metaphysical basis, wherein the cause is contained why a piece of music can give one pleasure and the basis for it is to be located, and why a thing to us pleasing is to another displeasing." To this Bernoulli replied, "I cannot readily divine wherein that principle should exist, however metaphysical it may be, whereby the reason could be given why one could take pleasure in a piece of music, and why a thing pleasant for us, may for another be unpleasant. One has indeed a general idea of harmony that it is charming if it is well arranged and the consonances are well managed; but, as it is well known, dissonances in music also have their use since by means of them the charm of the immediately following consonances is brought out the better, according to the common saying *opposita juxta se posita magis elucescunt* [opposites placed together shine brighter]; also in the art of painting, shadows must be relieved by light."

[1] Monge, "Construction de l'équation des cordes vibrantes," *Journal de l'Ecole Polytechnique*, cahier 15, tome 8, Paris, 1809, pp. 118–145.

[2] Compare Fétis, *Biographie* . . ., vol. 3, *loc. cit.*

[3] First published at St. Petersburg in 1739; there were French editions in 1839 and 1865.

[4] These are all listed on pages 319–321, 332–333, of G. Eneström's remarkable *Verzeichnis der Schriften Leonard Eulers*, Leipsic, 1910, 1913.

[5] *Bibliotheca Mathematica*, third series, vol. 4, 1903, pp. 383–388. Bernoulli's letter was published earlier in *Journal für Mathematik*, vol. 23, 1842, pp. 199–200, and in *Correspondance mathématique et physique* . . ., ed. by Fuss, St. Petersburg, 1843, vol. 2, pp. 8–11.

Euler's treatise does not seem to have met with unqualified favor. Brewster reports Fuss to have said [1] "it had no great success as it contained too much geometry for musicians, and too much music for geometers." Helmholtz gives a good deal of space [2] to setting forth the psychological considerations which Euler explains had influenced him to found his relations of consonances to whole numbers.

But here we must leave this "myriad-minded" eighteenth century genius.

And now there is time for but the briefest references to mathematics and music during the past one hundred years—the century in which niceties of mathematical calculation were surely contributory to the improvement of such instruments as the flute and organ, to the wonders of phonograph-record manufacture, of broadcasted concerts, and of sound-wave photography—the century in which Helmholtz and Rayleigh lived and worked.

Helmholtz's epoch-making work, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*, first appeared in 1862. The great feature of this work is the formulation and proof of the laws by which the ear hears musical sounds from one or more distinct sources; how the theory of combined musical sounds is reduced to the theory of combined simple sounds. The starting point of these discoveries was the fact, recognized by Rameau just two hundred years ago, that upper partials were associated with fundamental tones. From these laws we learn the *nature* of consonance and dissonance, knowledge so necessary for building up a system of harmony; we learn the principles which determined those degrees of musical sound selected by various nations at various times; we understand the reasons for the simple ratios of the lengths of strings producing consonant tones and the limitation of the numbers of these ratios; and we appreciate the value of temperaments for different instruments.[3]

In his *Tonempfindung* Helmholtz relegated to appendices the purely mathematical discussions. For example, the third appendix is "On the motion of plucked strings;" the fifth is "On the vibrational forms of pianoforte strings"; the sixth is an "Analysis of the motion of violin strings"; and the seventh is "On the theory of pipes." [4] He goes into such matters more extensively in the volume of his lectures on the mathematical principles of acoustics.[5]

Such subjects are also treated in masterly fashion by Rayleigh in his *Theory of Sound* [6] and in his papers. Among other works will be mentioned only the mathematical elements of music, as presented some twenty-five years earlier by Airy,[7] senior wrangler and astronomer.

[1] *Letters of Euler on Different Subjects in Natural Philosophy to a German Princess*, ed. by D. Brewster, third ed., vol. 1, Edinburgh, 1823, p. xxv.

[2] Helmholtz, *On Sensations . . .*, *loc. cit.*, pp. 229–231. Compare H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, pp. 171–172.

[3] Compare Helmholtz, *On Sensations . . .*, first English edition, 1875, p. vi.

[4] Helmholtz, *On Sensations . . .*, *loc. cit.*, III, pp. 374–477; V, pp. 380–384; VI, 384–387; VII, pp. 388–396.

[5] Helmholtz, *Vorlesungen . . .*, *loc. cit.;* see, for example, discussion regarding the violin on pp. 121–139.

[6] *Loc. cit.*

[7] G. B. Airy, *On Sound and Atmospheric Vibrations with the Mathematical Elements of Music*, London and Cambridge, 1868; second edition, 1871.

In such works, in the comparatively recent notable paper in this country by Harvey Davis,[1] on vibrations of a rubbed string, and, of course, in other mathematical treatments of similar material, Fourier series must enter in a fundamental manner. With specified conditions the series and its coefficients for a given tone or combination of tones may be determined. Or, if we have a graph of the vibrations corresponding to such tones, the series may also be calculated, various terms in the series corresponding to simple elements compounded in the tone or tones.

During the past twenty years photography has contributed in a remarkable manner to the analysis of musical sounds. In England, from 1905 to 1912, E. H. Barton and his associates published[2] a series of papers illustrated by photographs of vibration curves particularly as issuing from the violin strings, bridge, and belly.

In India, five years ago, R. C. V. Raman published an extensive bulletin "On the mechanical theory of the vibration of bowed strings and of musical instruments of the violin family, with experimental verification of the results."[3] It is illustrated by reproductions of many photographs, those of the wolf-notes, so well known to stringed-instrument players, having especial interest. The more recent publications of S. Garten and F. Kleinknecht contain a discussion of tones produced by the voice.[4] And with us the work that D. C. Miller, of the Case School of Applied Science, has done in this connection is known to many, not only through his volume on *The Science of Musical Sounds*,[5] but also through his remarkably interesting public lectures where his extraordinary instrument called the phonodeik, which photographically records sound waves, may also be used for projecting traces of the waves, as generated, on the screen of a lecture platform.

For the mathematician a great advantage of a photograph is that he can, after much labor, from it calculate the corresponding Fourier series. But in the laboratory, work of this kind is often saved by the employment of a machine called the harmonic analyzer.[6] The first instrument of this kind was made by Lord Kelvin in 1878; two were put forth by Henrici in 1894, and among others is that of Michelson and Stratton, constructed in 1898. By means of a Henrici machine, when the stylus of the instrument is moved along the curve of the

[1] H. N. Davis, "The Longitudinal Vibrations of a Rubbed String," *Proceedings of the American Academy of Arts and Sciences*, vol. 41, 1906, pp. 691–727 + 3 plates.

[2] *Philosophical Magazine*, 1905–1907, 1909, 1910, 1912.

[3] *Indian Association for the Cultivation of Science, Bulletin no. 15*, Calcutta, 1918, part 1, 158 pp.

[4] "Beiträge zur Vokallehre," I–III, *Abhandlungen der mathematischen-physikalischen Klasse der Sächsischen Akademie der Wissenschaften*, vol. 38, 1921–22; the sub-title for part III is: "Die automatische Analyse der gesungenen Vokale."

[5] New York, 1916.

[6] Compare G. A. Carse and J. Urquhart, "Harmonic Analysis" in *Modern Instruments and Methods of Calculation*, edited by E. M. Horsburgh, London, 1914, pp. 220–247. H. de Morin, *Les Appareils d'Intégration*, Paris, 1913, pp. 147–190. W. Dyck, *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, Munich, 1892, pp. 212–222, and 1893, pp. 34–36.

photograph the numerical values of the coefficients in the corresponding Fourier series may be read off. In 1910 Miller reconstructed a Henrici analyzer so as to care for thirty components with precision.[1] That is, a tone made up of 30 simple tones can be analyzed and the coefficients of the corresponding number of terms in the Fourier series written down. Regarding this kind of work I must not pause to do more than suggest that it has applications of high importance for tone generation and for perfecting musical instruments.

In concluding references to activities of the past one hundred years, I should, however, take time to recall that when, in these latest days, there arose a question as to the manner in which our present musical notation for equal temperament scales could best be simplified, it was a former president of this Association who brought forward a scheme[2] so beautifully simple that further advance in this regard cannot be imagined.

Speculation as to music of the future furnishes tempting themes for discussion. I shall merely mention some of these in conclusion.

The possibilities of melody and harmony in the trinity of musical fundamentals have, within the limits of our hampering scale systems, been largely explored. But what is to be the future of the almost untried vast *rhythmic* possibilities so intimately bound up with mathematical relations? Practically all of our music is modulo 2, 3, 4, 6, 8, 9, 12; but why not have modulo 5, 7, 10, 11, 13, for example, or combinations of these moduli in the same measures?

Again, is it not within the realms of possibility that some day the inadequacies of the present vehicle of musical expression may lead us to revive some of the ideals of Greek music during the golden period of Aristoxenus?

And yet again, when we recall the many results in connection with musical tones found empirically by makers of musical instruments but for which no satisfactory explanations have been furnished by the mathematician or physicist, may we not conclude that when such explanations are forthcoming, a new era shall have dawned in the evolution of musical instruments?

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[1] D. C. Miller, *loc. cit.*, p. 100.

[2] E. V. Huntington, "A simplified musical notation," *The Scientific Monthly*, vol. 11, 1920, pp. 276–283. The following footnote occurs on p. 282: "By the addition of an occasional single letter (Ellis's 'duodenal'), the new notation can even be made to indicate the note required for 'just intonation' with complete accuracy. A discussion of this phase of the subject is reserved for another occasion." In *Science*, 1921, T. P. Hall endorses the scheme (January 28, pp. 91–92) and R. P. Baker raises several pertinent considerations (March 11, pp. 235–236).