# TEN MISCONCEPTIONS FROM THE HISTORY OF ANALYSIS AND THEIR DEBUNKING

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ABSTRACT. The widespread idea that infinitesimals were "eliminated" by the "great triumvirate" of Cantor, Dedekind, and Weierstrass, is refuted by an uninterrupted chain of work on infinitesimal-enriched number systems. The elimination claim is an oversimplification created by triumvirate followers, who tend to view the history of analysis as a pre-ordained march toward the radiant future of Weierstrassian epsilontics. In the present text, we document distortions of the history of analysis stemming from the triumvirate ideology of ontological minimalism, which identified the continuum with a single number system. Such anachronistic distortions characterize the received interpretation of Stevin, Leibniz, d'Alembert, Cauchy, and others.

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#### 1. Introduction

Here are some common claims. The founders of infinitesimal calculus were working in a vacuum caused by an absence of a satisfactory number system. The incoherence of infinitesimals was effectively criticized by Berkeley as so much hazy metaphysical mysticism. D'Alembert's visionary anticipation of the rigorisation of analysis was ahead of his time. Cauchy took first steps toward replacing infinitesimals by rigor and epsilontics, in particular giving a modern definition of continuity. Cauchy's false 1821 version of the "sum theorem" was corrected by him in 1853 by adding the hypothesis of uniform convergence. Weierstrass finally rigorized analysis and thereby eliminated infinitesimals from mathematics. Dedekind discovered "the essence of continuity", which is captured by his cuts. One of the spectacular successes of the rigorous analysis was the mathematical justification of Dirac's "delta functions". Robinson developed a new theory of infinitesimals in the 1960s, but his approach has little to do with historical infinitesimals. Lakatos pursued an ideological agenda of Kuhnian relativism and fallibilism, inapplicable to mathematics.

Each of the above ten claims is in error, as we argue in the next ten sections (cf. Crowe [31]).

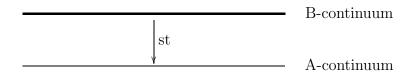


FIGURE 1. Pair of parallel conceptions of the continuum (the thickness of the top line is merely conventional). The "thick-to-thin" vertical arrow "st" represents taking the standard part (see Appendix A for details).

The historical fact of the dominance of the view of analysis as being based on the real numbers to the exclusion of infinitesimals, is beyond dispute. One could ask oneself why this historical fact is so; some authors have criticized mathematicians for adhering to an approach that others consider less appropriate. In the present text, we will *not* be concerned with either of these issues. Rather, we will be concerned with another issue, namely, why is it that traditional historical scholarship has been inadequate in indicating that alternative views have been around. We will also be concerned with documenting instances of tendentious interpretation and the attendant distortion in traditional evaluation of key figures from mathematical history.

Felix Klein clearly acknowledged the existence of a parallel, infinitesimal approach to foundations. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out in 1908 that

The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries (Klein [87, p. 214]).

Such a different conception, according to Klein, "harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of [...] infinitely small parts" (ibid.). The pair of parallel conceptions of analysis are illustrated in Figure 1.

A comprehensive re-evaluation of the history of infinitesimal calculus and analysis was initiated by Katz & Katz in [76], [77], and [78]. Briefly, a philosophical disposition characterized by a preference for a sparse ontology has dominated the historiography of mathematics for the past 140 years, resulting in a systematic distortion in the interpretation of the historical development of mathematics from Stevin

(see [78]) to Cauchy (see [77] and Borovik & Katz [19]) and beyond. Taken to its logical conclusion, such distortion can assume comical proportions. Thus, Newton's eventual successor in the Lucasian chair of mathematics, Stephen Hawking, comments that Cauchy

was particularly concerned to banish infinitesimals (Hawking [67, p. 639]),

yet on the very same page 639, Hawking quotes Cauchy's infinitesimal definition of continuity in the following terms:

the function f(x) remains continuous with respect to x between the given bounds, if, between these bounds, an infinitely small increment in the variable always produces an infinitely small increment in the function itself (ibid).

Did Cauchy banish infinitesimals, or did he exploit them to define a seminal new notion of continuity? Similarly, historian J. Gray lists continuity among concepts Cauchy allegedly defined

using careful, if not altogether unambiguous, **limiting** arguments (Gray [65, p. 62]) [emphasis added–authors],

whereas in reality *limits* appear in Cauchy's definition only in the sense of the *endpoints* of the domain of definition (see [76], [77] for a more detailed discussion). Commenting on 'Whig' re-writing of mathematical history, P. Mancosu observed that

the literature on infinity is replete with such 'Whig' history. Praise and blame are passed depending on whether or not an author might have anticipated Cantor and naturally this leads to a completely anachronistic reading of many of the medieval and later contributions [114, p. 626].

The anachronistic idea of the history of analysis as a relentless march toward the yawning heights of epsilontics is, similarly, our target in the present text. We outline some of the main distortions, focusing on the philosophical bias which led to them. The outline serves as a program for further investigation.

## 2. Were the founders of calculus working in a numerical vacuum?

Were the founders of infinitesimal calculus working in a vacuum caused by an absence of a satisfactory number system?

<sup>&</sup>lt;sup>1</sup>Related comments by Grattan-Guinness may be found in the main text at footnote 25.

2.1. Stevin, La Disme, and Arithmetique. A century before Newton and Leibniz, Simon Stevin (Stevinus) sought to break with an ancient Greek heritage of working exclusively with relations among natural numbers,<sup>2</sup> and developed an approach capable of representing both "discrete" number  $(\dot{\alpha}\rho\iota\theta\mu\dot{\rho}\varsigma)^3$  composed of units  $(\mu\nu\dot{\alpha}\delta\omega\nu)$  and continuous magnitude  $(\mu\dot{\epsilon}\gamma\varepsilon\theta\sigma\varsigma)$  of geometric origin.<sup>4</sup> According to van der Waerden, Stevin's

general notion of a real number was accepted, tacitly or explicitly, by all later scientists [151, p. 69].

## D. Fearnley-Sander wrote that

the modern concept of real number [...] was essentially achieved by Simon Stevin, around 1600, and was thoroughly assimilated into mathematics in the following two centuries [48, p. 809].

## D. Fowler points out that

Stevin [...] was a thorough-going arithmetizer: he published, in 1585, the first popularization of decimal fractions in the West [...]; in 1594, he described an algorithm for finding the decimal expansion of the root of any polynomial, the same algorithm we find later in Cauchy's proof of the Intermediate Value Theorem [54, p. 733].

Fowler [54] emphasizes that important foundational work was yet to be done by Dedekind, who proved that the field operations and other arithmetic operations extend from  $\mathbb{Q}$  to  $\mathbb{R}$  (see Section 8).<sup>5</sup> Meanwhile, Stevin's decimals stimulated the emergence of power series (see below) and other developments. We will discuss Stevin's contribution to the Intermediate Value Theorem in Subsection 2.3 below.

In 1585, Stevin defined decimals in *La Disme* as follows:

Decimal numbers are a kind of arithmetic based on the idea of the progression of tens, making use of the Arabic numerals in which any number may be written and by

<sup>&</sup>lt;sup>2</sup>The Greeks counted as numbers only 2, 3, 4, . . .; thus, 1 was not a number, nor are the fractions: ratios were relations, not numbers. Consequently, Stevin had to spend time arguing that the unit  $(\mu o \nu \acute{\alpha} \varsigma)$  was a number.

<sup>&</sup>lt;sup>3</sup>Euclid [45, Book VII, def. 2].

<sup>&</sup>lt;sup>4</sup>Euclid [45, Book V]. See also Aristotle's *Categories*, 6.4b, 20-23: "Quantity is either discrete or continuous. [...] Instances of discrete quantities are number and speech; of continuous, lines, surfaces, solids, and besides these, time and place".

<sup>&</sup>lt;sup>5</sup>Namely, there is no easy way from representation of reals by decimals, to the *field* of reals, just as there is no easy way from continuous fractions, another well-known representation of reals, to operations on such fractions.

which all computations that are met in business may be performed by integers alone without the aid of fraction. (*La Disme*, On Decimal Fractions, tr. by V. Sanford, in Smith [139, p. 23]).

By numbers "met in business" Stevin meant finite decimals,<sup>6</sup> and by "computations" he meant addition, subtraction, multiplications, division and extraction of square roots on finite decimals.<sup>7</sup> Stevin argued that numbers, like the more familiar continuous magnitudes, can be divided indefinitely, and used a water metaphor to illustrate such an analogy:

As to a continuous body of water corresponds a continuous wetness, so to a continuous magnitude corresponds a continuous number. Likewise, as the continuous body of water is subject to the same division and separation as the water, so the continuous number is subject to the same division and separation as its magnitude, in such a way that these two quantities cannot be distinguished by continuity and discontinuity (Stevin, 1585, see [140, p. 3]; quoted in A. Malet [111]).

Stevin argued for equal rights in his system for rational and irrational numbers. He was critical of the complications in Euclid [45, Book X], and was able to show that adopting the arithmetic as a way of dealing with those theorems made many of them easy to understand and easy to prove. In his *Arithmetique* [140], Stevin proposed to represent all numbers systematically in decimal notation. P. Ehrlich notes that Stevin's

viewpoint soon led to, and was implicit in, the analytic geometry of René Descartes (1596-1650), and was made explicit by John Wallis (1616-1703) and Isaac Newton (1643-1727) in their arithmetizations thereof (Ehrlich [41, p. 494]).

2.2. Decimals from Stevin to Newton and Euler. Stevin's text *La Disme* on decimal notation was translated into English in 1608 by Robert Norton (cf. Cajori [23, p. 314]). The translation contains the first occurrence of the word "decimal" in English; the word will be

<sup>&</sup>lt;sup>6</sup>But see Stevin's comments on extending the process *ad infinitum* in main text at footnote 15.

<sup>&</sup>lt;sup>7</sup>An algorithmic approach to such operations on infinite decimals was developed by Hoborski [69], and later in a very different way by Faltin et al. [47].

<sup>&</sup>lt;sup>8</sup>See also Naets [119] for an illuminating discussion of Stevin.

employed by Newton in a crucial passage 63 years later. Wallis recognized the importance of unending decimal expressions in the following terms:

Now though the Proportion cannot be accurately expressed in absolute Numbers: Yet by continued Approximation it may; so as to approach nearer to it, than any difference assignable (Wallis's Algebra, p. 317, cited in Crossley [30]).

Similarly, Newton exploits a power series expansion to calculate detailed decimal approximations to log(1+x) for  $x=\pm 0.1, \pm 0.2, \ldots$  (Newton [122]). By the time of Newton's annus mirabilis, the idea of unending decimal representation was well established. Historian V. Katz calls attention to "Newton's analogy of power series to infinite decimal expansions of numbers" (V. Katz [82, p. 245]). Newton expressed such an analogy in the following passage:

Since the operations of computing in numbers and with variables are closely similar ... I am amazed that it has occurred to no one (if you except N. Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine in decimal numbers has to common Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root-extraction may easily be learnt from the latter's provided the reader be skilled in each, both Arithmetic and Algebra, and appreciate the correspondence between decimal numbers and algebraic terms continued to infinity . . . And just as the advantage of decimals consists in this, that when all fractions and roots have been reduced to them they take on in a certain measure the nature of integers, so it is the advantage of

<sup>&</sup>lt;sup>9</sup>See footnote 12.

<sup>&</sup>lt;sup>10</sup>Newton scholar N. Guicciardini kindly provided a jpg of a page from Newton's manuscript containing such calculations, and commented as follows: "It was probably written in Autumn 1665 (see Mathematical papers 1, p. 134). White-side's dating is sometimes too precise, but in any case it is a manuscript that was certainly written in the mid 1660s when Newton began annotating Wallis's *Arithmetica Infinitorum*" [66]. The page from Newton's manuscript can be viewed at http://u.math.biu.ac.il/~katzmik/newton.html

infinite variable-sequences that classes of more complicated terms (such as fractions whose denominators are complex quantities, the roots of complex quantities and the roots of affected equations) may be reduced to the class of simple ones: that is, to infinite series of fractions having simple numerators and denominators and without the all but insuperable encumbrances which beset the others (Newton 1671, [121]).<sup>11</sup>

In this remarkable passage dating from 1671, Newton explicitly names infinite decimals as the source of inspiration for the new idea of infinite series.<sup>12</sup> The passage shows that Newton had an adequate number system for doing calculus and real analysis two centuries before the triumvirate.<sup>13</sup>

In 1742, John Marsh first used an abbreviated notation for repeating decimals (Marsh [116, p. 5], cf. Cajori [23, p. 335]). Euler exploits unending decimals in his *Elements of Algebra* in 1765, as when he sums an infinite series and concludes<sup>14</sup> that 9.999... = 10 (Euler [46, p. 170]).

2.3. Stevin's cubic and the IVT. In the context of his decimal representation, Stevin developed numerical methods for finding roots of polynomials equations. He described an algorithm equivalent to finding zeros of polynomials (see Crossley [30, p. 96]). This occurs in a corollary to problem 77 (more precisely, LXXVII) in (Stevin [141, p. 353]). Here Stevin describes such an argument in the context of finding a root of the cubic equation (which he expresses as a proportion to conform to the contemporary custom)

$$x^3 = 300x + 33915024.$$

<sup>&</sup>lt;sup>11</sup>We are grateful to V. Katz for signaling this passage.

<sup>&</sup>lt;sup>12</sup>See footnote 9.

 $<sup>^{13}</sup>$ The expression "the great triumvirate" is used by Boyer [21, p. 298] to describe Cantor, Dedekind, and Weierstrass.

<sup>&</sup>lt;sup>14</sup>This could be compared with Peirce's remarks. Over a century ago, Charles Sanders Peirce wrote with reference to 1−'0.999...': "although the difference, being infinitesimal, is less than any number [one] can express[,] the difference exists all the same, and sometimes takes a quite easily intelligible form" (Peirce [125, p. 597]; see also S. Levy [102, p. 130]). Levy mentions Peirce's proposal of an alternative notation for "equality up to an infinitesimal". The notation Peirce proposes is the usual equality sign with a dot over it, like this: "≐". See also main text at footnote 46.

Here the whimsical coefficient seems to have been chosen to emphasize the fact that the method is completely general; Stevin notes furthermore that numerous additional examples can be given. A textual discussion of the method may be found in Struik [142, p. 476].

Centuries later, Cauchy would prove the Intermediate Value Theorem (IVT) for a continuous function f on an interval I by a divide-and-conquer algorithm. Cauchy subdivided I into m equal subintervals, and recursively picked a subinterval where the values of f have opposite signs at the endpoints (Cauchy [24, Note III, p. 462]). To elaborate on Stevin's argument following [142, §10, p. 475-476], note that what Stevin similarly described a divide-and-conquer algorithm. Stevin subdivides the interval into ten equal parts, resulting in a gain of a new decimal digit of the solution at every iteration of his procedure. Stevin explicitly speaks of continuing the iteration ad infinitum:

Et procedant ainsi infiniment, l'on approche infiniment plus pres au requis (Stevin [141, p. 353, last 3 lines]).

Who needs existence proofs for the real numbers, when Stevin gives a procedure seeking to produce an explicit decimal representation of the solution? The IVT for polynomials would resurface in Lagrange before being generalized by Cauchy to the newly introduced class of continuous functions.<sup>16</sup>

One frequently hears sentiments to the effect that the pre-triumvirate mathematicians did not and could not have provided rigorous proofs, since the real number system was not even built yet. Such an attitude is anachronistic. It overemphasizes the significance of the triumvirate project in an inappropriate context. Stevin is concerned with constructing an algorithm, whereas Cantor is concerned with developing a foundational framework based upon the existence of the *totality* of the real numbers, as well as their power sets, etc. The latter project involves a number of non-constructive ingredients, including the axiom of infinity and the law of excluded middle. But none of it is needed for Stevin's procedure, because he is not seeking to re-define "number" in terms of alternative (supposedly less troublesome) mathematical objects.

Why do many historians and mathematicians of today emphasize the great triumvirate's approach to proofs of the existence of real numbers, at the expense, and almost to the exclusion, of Stevin's approach? Can this be understood in the context of the ideological foundational

 $<sup>^{15}</sup>$ Cf. footnote 6.

<sup>&</sup>lt;sup>16</sup>See further in footnote 27.

battles raging at the end of 19th and beginning of 20th century? These questions merit further scrutiny.

## 3. Was Berkeley's criticism coherent?

Was Berkeley's criticism of infinitesimals as so much hazy metaphysical mysticism, either effective or coherent?

- D. Sherry [134] dissects Berkeley's criticism of infinitesimal calculus into its metaphysical and logical components, as detailed below.
- 3.1. **Logical criticism.** The *logical criticism* is the one about the disappearing dx. Here we have a ghost:  $dx \neq 0$ , but also a departed quantity: dx = 0 (in other words eating your cake: dx = 0 and having it, too:  $dx \neq 0$ ).

Thus, Berkeley's *logical criticism* of the calculus is that the evanescent increment is first assumed to be non-zero to set up an algebraic expression, and then *treated as zero* in *discarding* the terms that contained that increment when the increment is said to vanish.<sup>17</sup>

The fact is that Berkeley's logical criticism is easily answered within a conceptual framework available to the founders of the calculus. Namely, the *rebuttal* of the logical criticism is that the evanescent increment is not *treated as zero*, but, rather, merely *discarded* through an application of Leibniz's *law of homogeneity* (see Leibniz [101, p. 380]) and Bos [20, p. 33]), which would stipulate, for instance, that

$$a + dx - a. (3.1)$$

Here we chose the sign  $\neg$  which was already used by Leibniz where we would use an equality sign today (see McClenon [117, p. 371]). The law is closely related to the earlier notion of adequality found in Fermat. Adequality is the relation of being infinitely close, or being equal "up to" an infinitesimal. Fermat exploited adequality when he sought a maximum of an expression by evaluating expression at A + E and at A, and forming the difference. In modern notation this would appear as f(A+E)-f(A) (note that Fermat did not use the function notation).

 $<sup>^{17}</sup>$ Given Berkeley's fame among historians of mathematics for allegedly spotting logical flaws in infinitesimal calculus, it is startling to spot circular logic at the root of Berkeley's own doctrine of the compensation of errors. Indeed, Berkeley's new, improved calculation of the derivative of  $x^2$  in *The Analyst* [15] relies upon the determination of the tangent to a parabola due to Apollonius of Perga [4, Book I, Theorem 33] (see Andersen [2]).

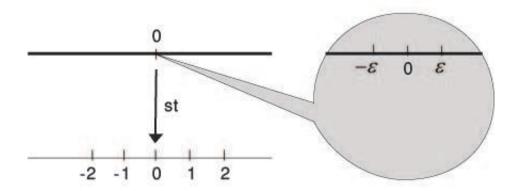


FIGURE 2. Zooming in on infinitesimal  $\varepsilon$  (see Appendix A for details)

Huygens already interpreted the "E" occurring in this expression in the method of adequality, as an infinitesimal.<sup>18</sup>

Ultimately, the heuristic concepts of adequality (Fermat) and law of homogeneity (Leibniz) were implemented in terms of the *standard part* function (see Figure 2).

In a passage typical of post-Weierstrassian scholarship, Kleiner and Movshovitz-Hadar note that

Fermat's method was severely criticized by some of his contemporaries. They objected to his introduction and subsequent suppression of the mysterious E. Dividing by E meant regarding it as not zero. Discarding E implied treating it as zero. This is inadmissible, **they rightly claimed.** In a somewhat different context, but **with equal justification**, ... Berkeley in the 18th century would refer to such E's as 'the ghosts of departed quantities' " [88, p. 970] [emphasis added–authors].

However, Fermat scholar P. Strømholm already pointed out in 1968 that in Fermat's main method of adequality,

there was never [...] any question of the variation E being put equal to zero. The words Fermat used to express the process of suppressing terms containing E was "elido", "deleo", and "expungo", and in French "i'efface" and "i'ôte". We can hardly believe that a

 $<sup>^{18}</sup>$ Huygens explained Fermat's method of adequality in a presentation at the *Académie des Sciences* in 1667. Huygens noted that "E" is an "infinitely small quantity" (see Huygens [73]). See also André Weil [152, p. 1146], [153, p. 28].

sane man wishing to express his meaning and searching for words, would constantly hit upon such tortuous ways of imparting the simple fact that the terms vanished because E was zero (Strømholm [143, p. 51]).

Thus, Fermat planted the seeds of the answer to the logical criticism of the infinitesimal, a century before George Berkeley ever lifted up his pen to write *The Analyst*.

The existence of two separate binary relations, one "equality" and the other, "equality up to an infinitesimal", was already known to Leibniz (see E. Knobloch [89, p. 63] and Katz and Sherry [80] for more details).

3.2. **Metaphysical criticism.** Berkeley's *metaphysical criticism* targets the absence of any empirical referent for "infinitesimal". The metaphysical criticism has its roots in empiricist dogma that every meaningful expression or symbol must correspond to an empirical entity. Ironically, Berkeley accepts many expressions lacking an empirical referent, such as 'force', 'number', or 'grace', on the grounds that they have pragmatic value. It is a glaring inconsistency on Berkeley's part not to have accepted "infinitesimal" on the same grounds (see Sherry [135]).

It is even more ironic that over the centuries, mathematicians were mainly unhappy with the logical aspect, but their criticisms mainly targeted what they perceived as the metaphysical/mystical aspect. Thus, Cantor attacked infinitesimals as being "abominations" (see Ehrlich [42]); R. Courant described them as "mystical", "hazy fog", etc. E. T. Bell went as far as describing infinitesimals as having been

- slain [10, p. 246],
- scalped [10, p. 247], and
- disposed of [10, p. 290]

by the cleric of Cloyne (see Figure 3). Generally speaking, one does not slay either scientific concepts or scientific entities. Bellicose language of this sort is a sign of commitments that are both emotional and ideological.

#### 4. Were d'Alembert's anticipations ahead of his time?

Were d'Alembert's mathematical anticipations ahead or behind his time?

<sup>&</sup>lt;sup>19</sup>The interplay of empiricism and nominalism in Berkeley's thought is touched upon by D. Sepkoski [133, p. 50].

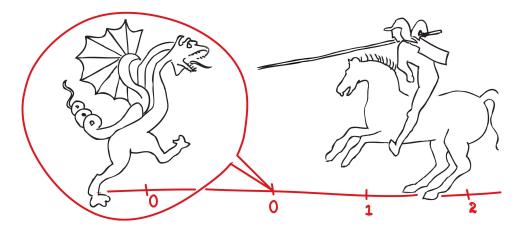


FIGURE 3. George's attempted slaying of the infinitesimal, following E. T. Bell and P. Uccello. Uccello's creature is shown as inhabiting an infinitesimal neighborhood of 0, cf. Figure 2.

One aspect of d'Alembert's vision as expressed in his article for the *Encyclopedie* on *irrational* numbers, is that irrational numbers *do not exist*. Here d'Alembert uses terms such as "surds" which had already been rejected by Simon Stevin two centuries earlier (see Section 2). From this point of view, d'Alembert is not a pioneer of the rigorisation of analysis in the 19th century, but on the contrary, represents a throwback to the 16th century. D'Alembert's attitude toward irrational numbers sheds light on the errors in his proof of the fundamental theorem of algebra;<sup>20</sup> indeed, in the anemic number system envisioned by d'Alembert, numerous polynomials surely fail to have roots.

D'Alembert used the term "charlatanerie" to describe infinitesimals in his article *Différentiel* [33]. D'Alembert's anti-infinitesimal vitriol is what endears him to triumvirate scholars, for his allegedly visionary remarks concerning the centrality of the limit concept fall short of what is already found in Newton.<sup>21</sup> He never went beyond a kinetic notion of limit, so as to obtain the epsilontic version popularized in the 1870s.

<sup>&</sup>lt;sup>20</sup>D'Alembert's error was first noticed by Gauss, who gave some correct proofs, though S. Smale argues his first proof was incomplete [138]. See Baltus [8] for a detailed study of d'Alembert's proof.

<sup>&</sup>lt;sup>21</sup>See Pourciau [126] who argues that Newton possessed a clear kinetic conception of limit (Sinaceur [136] and Barany [9] argue that Cauchy's notion of limit was kinetic, rather than a precursor of a Weierstrassian notion of limit). Pourciau cites Newton's lucid statement to the effect that "Those ultimate ratios ... are not actually ratios of ultimate quantities, but limits ... which they can approach so closely that their difference is less than any given quantity..." (Newton, 1946 [123,

D'Alembert was particularly bothered by the characterisation of infinitesimals he found in "the geometers". He does not explain who these geometers are, but the characterisation he is objecting to can be already found in Leibniz and Newton. Namely, the geometers used to describe infinitesimals as what remains

"not before you pass to the limit, nor after, but at the very moment of passage to the limit".

In the context of a modern theory of infinitesimals such as the hyperreals (see Appendix A), one could explain the matter in the following terms. We decompose the procedure of taking the limit of, say, a sequence  $(r_n)$  into two stages:

- (i) evaluating the sequence at an infinite hypernatural<sup>22</sup> H, to obtain the hyperreal  $r_H$ ; and
- (ii) taking its standard part  $L = st(r_H)$ .

Thus  $r_H$  is adequal to L, or  $r_H \cap L$  (see formula (3.1)), so that we have  $\lim_{n\to\infty} r_n = L$ . In this sense, the infinitesimals exist "at the moment" of taking the limit, namely *between* the stages (i) and (ii).

Felscher [51] describes d'Alembert as "one of the mathematicians representing the heroic age of calculus" [51, p. 845]. Felscher buttresses his claim by a lengthy quotation concerning the definition of the limit concept, from the article Limite from the Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers:

On dit qu'une grandeur est la limite d'une autre grandeur, quand la seconde peut approcher de la première plus près que d'une grandeur donnée, si petite qu'on la puisse supposer, sans pourtant que la grandeur qui approche, puisse jamais surpasser la grandeur dont elle approche; ensorte que la différence d'une pareille quantité à sa limite est absolument inassignable (Encyclopédie, volume 9 from 1765, page 542).

One recognizes here a kinetic definition of limit already exploited by Newton.<sup>23</sup> Even if we do attribute visionary status to this passage as many historians seem to, the fact remains that d'Alembert didn't write it. Felscher overlooked the fact that the article *Limite* was written by two authors. In reality, the above passage defining the concept of "limit" (as well as the two propositions on limits) did not originate

p. 39] and 1999 [124, p. 442]). The same point, and the same passage from Newton, appeared a century earlier in Russell [130, item 316, p. 338-339].

<sup>&</sup>lt;sup>22</sup>See main text at footnote 39.

<sup>&</sup>lt;sup>23</sup>See footnote 21 on Pourciau's analysis.

with d'Alembert, but rather with the encyclopedist Jean-Baptiste de La Chapelle. De la Chapelle was recruited by d'Alembert to write 270 articles for the *Encyclopédie*. The section of the article containing these items is signed (E) (at bottom of first column of page 542), known to be de La Chapelle's "signature" in the *Encyclopedie*. Felscher had already committed a similar error of attributing de la Chapelle's work to d'Alembert, in his 1979 work [50].<sup>24</sup> Note that Robinson [128, p. 267] similarly misattributes this passage to d'Alembert.

#### 5. DID CAUCHY REPLACE INFINITESIMALS BY RIGOR?

Did Cauchy take first steps toward replacing infinitesimals by rigor, and did he give an epsilontic definition of continuity?

A claim to the effect that Cauchy was a fore-runner of the epsilontisation of analysis is routinely recycled in history books and textbooks. To put such claims in historical perspective, it may be useful to recall Grattan-Guinness's articulation of a historical reconstruction project in the name of H. Freudenthal [56], in the following terms:

it is mere feedback-style ahistory to read Cauchy (and contemporaries such as Bernard Bolzano) as if they had read Weierstrass already. On the contrary, their own pre-Weierstrassian muddles<sup>25</sup> need historical reconstruction [64, p. 176].

5.1. Cauchy's definition of continuity. It is often claimed that Cauchy gave an allegedly "modern", meaning epsilon-delta, definition of continuity. Such claims are anachronistic. In reality, Cauchy's definition is an infinitesimal one. His definition of the continuity of y = f(x) takes the following form: an infinitesimal x-increment gives rise to an infinitesimal y-increment (see [24, p. 34]). The widespead misconception that Cauchy gave an epsilontic definition of continuity is analyzed in detail in [76].

Cauchy's primary notion is that of a variable quantity. The meaning he attached to the latter term in his Cours d'Analyse in 1821 is generally agreed to have been a sequence of discrete values. He defines infinitesimals in terms of variable quantities, by specifying that a variable quantity tending to zero becomes an infinitesimal. He similarly defines limits in terms of variable quantities in the following terms:

 $<sup>^{24}</sup>$ We are grateful to D. Spalt for the historical clarification concerning the authorship of the *Limite* article in the *Encyclopedie*.

<sup>&</sup>lt;sup>25</sup>Grattan-Guinness's term "muddle" refers to an irreducible ambiguity of historical mathematics such as Cauchy's sum theorem of 1821. See footnote 1 for a related comment by Mancosu.

lorsque les valeurs successivement attribuées à une même variable s'approche indéfiniment d'une valeur fixe de manière à finir par en différer aussi peu que l'on voudra cette dernière est appelée limite de toutes les autres (Cauchy [24, p. 4]).

Cauchy's definition is patently a kinetic, not an epsilontic, definition of limit, similar to Newton's.<sup>26</sup> While epsilontic-style formulations do occasionally appear in Cauchy (though without Bolzano's proper attention to the order of the quantifiers), they are not presented as definitions but rather as consequences, and at any rate never appear in the context of the property of the continuity of functions.

Thus, Grabiner's famous essay Who gave you the epsilon? Cauchy and the origins of rigorous calculus cites a form of an epsilon-delta quantifier technique used by Cauchy in a proof:

Let  $\epsilon, \delta$  be two very small numbers; the first is chosen so that for all numerical [i.e., absolute] values of h less than  $\delta$ , and for any value of x included [in the interval of definition], the ratio (f(x+h)-f(x))/h will always be greater than  $f'(x)-\epsilon$  and less than  $f'(x)+\epsilon$  (Grabiner [63, p. 185] citing Cauchy).

Grabiner describes such an epsilon-delta technique as "the algebra of inequalities". The thrust of her argument is that Cauchy sought to establish a foundation for analysis based on the algebra of inequalities. Is this borne out by the evidence she presents? Let us consider Grabiner's evidence:

Cauchy gave essentially the modern definition of continuous function, saying that the function f(x) is continuous on a given interval if for each x in that interval "the numerical [i.e., absolute] value of the difference  $f(x + \alpha) - f(x)$  decreases indefinitely with  $\alpha$  [63, p. 190].

Is this "essentially the modern definition of continuity", as Grabiner claims? Hardly so. Cauchy's definition is a blend of a kinetic (rather than epsilontic) and an infinitesimal approach. Grabiner fails to mention three essential items:

• Cauchy prefaces the definition she cited, by describing his  $\alpha$  as an *infinitely small increment*:

<sup>&</sup>lt;sup>26</sup>See footnote 21.

Si, en partant d'une valeur de x ... on attribue à la variable x un accroissement infiniment petit  $\alpha$  ..." (Cauchy [24, p. 34]) [emphasis added—the authors];

- Cauchy follows this definition by another, *italicized*, definition, where both  $\alpha$  and the difference  $f(x + \alpha) f(x)$  are described as being *infinitesimal*: if the former is infinitesimal, then so is the latter;
- Infinitesimals provide a method for calculating limits, whereas epsilon, delta methods require the answer in advance (see Madison and Stroyan [109, p. 497]).

The advantage of infinitesimal definitions, such as those found in Cauchy, is their covariant nature (cf. Lutz et al. [105]). Whereas in the epsilontic approach one needs to work one's way backwards from the value of the limit, in the infinitesimal approach one can proceed from the original expression, simplify it, and eventually arrive at the value of the limit. This indicates that the two approaches work in opposite directions. The infinitesimal calculation goes with the natural flow of our reasoning, whereas the epsilontic one goes in the opposite direction. Notice, for example, that delta corresponds to the independent variable even though the value of delta depends on our choice of epsilon, which corresponds to the dependent variable. The infinitesimal calculation, in contrast, begins with the the independent variable and computes from that the value of the dependent variable.

5.2. Cauchy's intermediate value theorem. Did Cauchy exploit epsilon-delta techniques in building foundations for analysis? Let us examine Grabiner's evidence. She claims that, in Cauchy's proof of the intermediate value theorem (IVT),

we have the algebra of inequalities providing a technique which Cauchy transformed from a tool of approximation to a tool of rigor (Grabiner [63, p. 191]).

Yet Grabiner's treatment of Cauchy's proof of the IVT in [63, p. 190] page offers no evidence that Cauchy employed an epsilon-delta technique.<sup>27</sup>

An examination of Cauchy's proof in his *Note III* reveals that, on the contrary, it is closely tied to Cauchy's infinitesimal definition of continuity. Thus, Cauchy constructs an increasing sequence and a decreasing sequence, denoted respectively  $x_n$  and  $X^{(n)}$  (Cauchy [24, p. 462]) with

<sup>&</sup>lt;sup>27</sup>Grabiner further attributes to Lagrange the polynomial case of Cauchy's divide-and-conquer argument in the proof of the IVT, whereas we saw in Subsection 2.3 that Stevin did this two centuries before Lagrange (see main text at footnote 16).

a common limit a, such that f has opposite sign at the corresponding pairs of points. Cauchy concludes that the values of f at the respective sequences converge to a common limit f(a). Being both nonpositive and nonnegative, the common limit f(a) must vanish.

Koetsier [90] speculates that Cauchy may have hit upon his concept of continuity by analyzing his proof of the IVT (perhaps in the case of polynomials). The evidence is compelling: even though Cauchy does not mention infinitesimals in his Note III,  $(x_n)$  and  $(X^{(n)})$  are recognizably variable quantities differing by an infinitesimal from the constant quantity a. By Cauchy's definition of continuity,  $(f(x_n))$  and  $(f(X^{(n)}))$  must similarly differ from f(a) by an infinitesimal. Contrary to Grabiner's claim, a close examination of Cauchy's proof of the IVT reveals no trace of epsilon-delta. Following Koetsier's hypothesis, it is reasonable to place it, rather, in the infinitesimal strand of the development of analysis, rather than the epsilontic strand.

After constructing the lower and upper sequences, Cauchy does write that the values of the latter "finiront par differer de ces premiers valeurs aussi peu que l'on voudra". That may sound a little bit epsilon/delta. Meanwhile, Leibniz uses language similar to Cauchy's:

Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like ["ut fiat tam parvus quam velimus"] (Leibniz [99, p. 99]).

Cauchy used epsilontics if and only if Leibniz did, over a century before him.

5.3. Cauchy's influence. The exaggerated claims of a Cauchy provenance for epsilontics found in triumvirate literature go hand-in-hand with neglect of his visionary role in the development of infinitesimals at the end of the 19th century. In 1902, E. Borel [17, p. 35-36] elaborated on Paul du Bois-Reymond's theory of rates of growth, and outlined a general "theory of increase" of functions, as a way of implementing an infinitesimal-enriched continuum. In this text, Borel specifically traced the lineage of such ideas to a 1829 text of Cauchy's [27] on the rates of growth of functions (see Fisher [52, p. 144] for details). In 1966, A. Robinson pointed out that

Following Cauchy's idea that an infinitely small or infinitely large quantity is associated with the behavior

of a function f(x), as x tends to a finite value or to infinity, du Bois-Reymond produced an elaborate theory of orders of magnitude for the asymptotic behavior of functions ... Stolz tried to develop also a theory of arithmetical operations for such entities [128, p. 277-278].

Robinson traces the chain of influences further, in the following terms:

It seems likely that Skolem's idea to represent infinitely large natural numbers by number-theoretic functions which tend to infinity (Skolem [1934]),<sup>28</sup> also is related to the earlier ideas of Cauchy and du Bois-Reymond [128, p. 278].

One of Cantor's bêtes noires was the neo-Kantian philosopher Hermann Cohen (1842–1918) (see also Mormann [118]), whose fascination with infinitesimals elicited fierce criticism by both Cantor and B. Russell. Yet at the end of the day, A. Fraenkel (of Zermelo–Fraenkel fame) wrote:

my former student Abraham Robinson had succeeded in saving the honour of infinitesimals - although in quite a different way than Cohen and his school had imagined (Fraenkel [55, p. 107]).

## 6. Was Cauchy's 1821 "sum theorem" false?

Was Cauchy's 1821 "sum theorem" false, and what did he add in 1853?

As discussed in Section 5, Cauchy's definition of continuity is explicitly stated in terms of infinitesimals: "an infinitesimal x-increment gives rise to an infinitesimal y-increment". Boyer [21, p. 282] declares that Cauchy's 1821 definition is "to be interpreted" in the framework of the usual "limits", at a point of an Archimedean continuum. Traditional historians typically follow Boyer's lead.

But when it comes to Cauchy's 1853 modification of the hypothesis of the "sum theorem" <sup>29</sup> in (Cauchy [28]), some historians declare that it

<sup>&</sup>lt;sup>28</sup>The reference is to Skolem's 1934 work [137]. The evolution of modern infinitesimals is traced in more detail in Table 1 and in Borovik et al. [19].

<sup>&</sup>lt;sup>29</sup>The assertion of the theorem is the continuity of the sum of a *convergent* series of continuous functions, with the italicized term requiring clarification. Modern versions of the theorem require a hypothesis *uniform* convergence. The nature of the hypothesis Cauchy himself had in mind is hotly disputed (see Borovik & Katz [19] as well as [77]).

is to be interpreted as adding the hypothesis of "uniform convergence" (see e.g., Lützen [107, p. 183-184]). Are Boyer and Lützen compatible?

Note that an epsilontic definition (in the context of an Archimedean continuum) of the uniform convergence of a sequence  $\langle f_n : n \in \mathbb{N} \rangle$  to f necessarily involves a pair of variables x, n (where x ranges through the domain of f and n ranges through  $\mathbb{N}$ ), rather than a single variable: we need a formula of the sort

$$\forall n \in \mathbb{N} \ \forall x \ (n > n_0 \implies |f_n(x) - f(x)| < \epsilon) \tag{6.1}$$

(prefaced by the usual clause " $(\forall \epsilon > 0)$  ( $\exists n_0 \in \mathbb{N}$ )"). Now Cauchy's 1853 modification of the hypothesis is stated in terms of a *single* variable x, rather than a pair of variables x, n. Namely, Cauchy specified that the condition of convergence should hold "always". The meaning of the term "always" becomes apparent only in the course of the proof, when Cauchy gives an explicit example of evaluating at an infinitesimal generated by the sequence x = 1/n. Thus the term "always" involves adding extra values of x at which the convergence condition must be satisfied (see Bråting [22] and Katz & Katz [77]).

Cauchy's approach is based on two assumptions which can be stated in modern terminology as follows:

- (1) when you have a closed expression for a function, then its values at "variable quantities" (such as  $x = \frac{1}{n}$ ) are calculated by using the same closed expression as at real values;
- (2) to evaluate a function at a variable quantity generated by a sequence, one evaluates term-by-term.

Cauchy's strengthened condition amounts to requiring the error  $r_n(x) = f(x) - f_n(x)$  to become infinitesimal:

if 
$$x$$
 infinitesimal then  $r_n(x)$  infinitesimal, (6.2)

which in the case of x given by (1/n) translates into the requirement that  $r_n(1/n)$  tends to zero.

An epsilontic interpretation (in the context of an Archimedean continuum) of Cauchy's 1821 and 1853 texts is untenable, as it necessitates a pair of variables as in (6.1), where Cauchy only used a single one, namely x, but one drawn from a "thicker" continuum including infinitesimals. Namely, Cauchy draws the points to be evaluated at from an infinitesimal-enriched continuum.

We will refer to an infinitesimal-enriched continuum as a Bernoullian continuum, or a "B-continuum" for short, in an allusion to Johann Bernouilli.<sup>30</sup>

<sup>&</sup>lt;sup>30</sup>Bernoulli was the first to use infinitesimals in a systematic fashion as a foundational concept, Leibniz himself having employed both a syncategorematic and a

A null sequence such as 1/n "becomes" an infinitesimal, in Cauchy's terminology. Evaluating at points of a Bernoullian continuum makes it possible to express uniform convergence in terms of a single variable x rather than a pair (x, n).

Once one acknowledges that there are *two* variables in the traditional epsilontic definition of uniform continuity and uniform convergence, it becomes untenable to argue that the condition Cauchy introduced was epsilontic uniform convergence. A historian who describes Cauchy's condition as uniform convergence, must acknowledge that the definition involves an infinitesimal-enriched continuum, at variance with Boyer's interpretation.

In Appendix A, Subsection A.2 we present a parallel distinction between continuity and uniform continuity, where a similar distinction in terms of the number of variables is made.

### 7. DID WEIERSTRASS SUCCEED IN ELIMINATING INFINITESIMALS?

Did Weierstrass succeed in eliminating infinitesimals from mathematics?

The persistent idea that infinitesimals have been "eliminated" by the great triumvirate of Cantor, Dedekind, and Weierstrass was soundly refuted by Ehrlich [42]. Ehrlich documents a rich and uninterrupted chain of work on non-Archimedean systems, or what we would call a Bernoullian continuum. Some key developments in this chain are listed in Table 1 (see [19] for more details).

The elimination claim can only be understood as an oversimplification by Weierstrass's followers, who wish to view the history of analysis as a triumphant march toward the radiant future of Weierstrassian epsilontics. Such a view of history is rejected by H. Putnam who comments on the introduction of the methods of the differential and integral calculus by Newton and Leibniz in the following terms:

If the epsilon-delta methods had not been discovered, then infinitesimals would have been postulated entities (just as 'imaginary' numbers were for a long time). Indeed, this approach to the calculus—enlarging the real number system—is just as consistent as the standard approach, as we know today from the work of Abraham Robinson [...] If the calculus had not been 'justified' Weierstrass style, it would have been 'justified' anyway (Putnam [127]).

true infinitesimal approach. The pair of approaches in Leibniz are discussed by Bos [20, item 4.2, p. 55]; see also Appendix A, footnote 38.

years	author	contribution	
1821	Cauchy	Infinitesimal definition of continuity	
1827	Cauchy	Infinitesimal delta function	
1829	Cauchy	Defined "order of infinitesimal" in terms of rates of growth of functions	
1852	Björling	Dealt with convergence at points "indefinitely close" to the limit	
1853	Cauchy	Clarified hypothesis of "sum theorem" by requiring convergence at infinitesimal points	
1870- 1900	Stolz, du Bois- Reymond, Veronese, and others	Infinitesimal-enriched number systems defined in terms of rates of growth of functions	
1902	Emile Borel	Elaboration of du Bois-Reymond's system	
1910	G. H. Hardy	Provided a firm foundation for du Bois-Reymond's orders of infinity	
1926	Artin–Schreier	Theory of real closed fields	
1930	Tarski	Existence of ultrafilters	
1934	Skolem	Nonstandard model of arithmetic	
1948	Edwin Hewitt	Ultrapower construction of hyperreals	
1955	Łoś	Proved Łoś's theorem forshadowing the transfer principle	
1961, 1966	Abraham Robinson	Non-Standard Analysis	
1977	Edward Nelson	Internal Set Theory	

 $\ensuremath{\mathrm{TABLE}}$  1. Timeline of modern infinitesimals from Cauchy to Nelson.

In short, there is a cognitive bias inherent in a postulation in an inevitable outcome in the evolution of a scientific discipline.

The study of cognitive bias has its historical roots in Francis Bacon's proposed classification of what he called *idola* (a Latin plural) of several kinds. He described these as things which obstructed the

path of correct scientific reasoning. Of particular interest to us are his *Idola fori* ("Illusions of the Marketplace": due to confusions in the use of language and taking some words in science to have meaning different from their common usage); and *Idola theatri* ("Illusions of the Theater": the following of academic dogma and not asking questions about the world), see Bacon [7].

Completeness, continuity, continuum, Dedekind "gaps": these are terms whose common meaning is frequently conflated with their technical meaning. In our experience, explaining infinitesimal-enriched extensions of the reals to an epsilontically trained mathematician typically elicits a puzzled reaction on her part: "But aren't the real numbers already *complete* by virtue of having filled in all the *gaps* already?"

This question presupposes an academic dogma, viz., that there is a single coherent conception of the continuum, and it is a complete, Archimedean ordered field. This dogma has recently been challenged. Numerous possible conceptions of the continuum range from S. Feferman's predicative conception of the continuum [49], to F. William Lawvere's [98] and J. Bell's conception in terms of an intuitionistic topos [11], [12], [13].

To illustrate the variety of possible conceptions of the continuum, note that traditionally, mathematicians have considered at least two different types of continua. These are Archimedean continua, or Acontinua for short, and infinitesimal-enriched (Bernoulli) continua, or B-continua for short. Neither an A-continuum nor a B-continuum corresponds to a unique mathematical structure (see Table 2). Thus, we have two distinct implementations of an A-continuum:

- the real numbers (or Stevin numbers),<sup>31</sup> in the context of classical logic (incorporating the law of excluded middle);
- Brouwer's continuum built from "free-choice sequences", in the context of intuitionistic logic.

John L. Bell describes a distinction within the class of an infinitesimal-enriched B-continuum, in the following terms. Historically, there were two main approaches to such an enriched continuum, one by Leibniz, and one by B. Nieuwentijt, who favored nilpotent (nilsquare) infinitesimals whose squares are zero. Mancosu's discussion of Nieuwentijt in [112, chapter 6] is the only one to date to provide a contextual understanding of Nieuwentijt's thought (see also Mancosu and Vailati [115]). J. Bell notes:

 $<sup>^{31}</sup>$ See Section 2.

	Archimedean	Bernoullian
classical	Stevin's continuum <sup>31</sup>	Robinson's continuum <sup>32</sup>
intuitionistic	Brouwer's continuum	Lawvere's continuum <sup>33</sup>

 $\rm TABLE~2.$  Varieties of continua, mapped out according to a pair of binary parameters: classical/intuitionistic and Archimedean/Bernoullian.

Leibnizian infinitesimals (differentials) are realized in [A. Robinson's] nonstandard analysis,<sup>32</sup> and nilsquare infinitesimals in [Lawvere's] smooth infinitesimal analysis (Bell [11, 12]).

The latter theory relies on intuitionistic logic.<sup>33</sup> An implementation of an infinitesimal-enriched continuum was developed by P. Giordano (see [59, 60]), combining elements of both a classical and an intuitionistic continuum. The Weirstrassian continuum is but a single member of a diverse family of concepts.

## 8. Did Dedekind discover the essence of continuity?

Did Dedekind discover "the essence of continuity", and is such essence captured by his cuts?

In Dedekind's approach, the "essence" of continuity amounts to the numerical assertion that two non-rational numbers should be equal if and only if they define the same Dedekind cut<sup>34</sup> on the rationals. Dedekind formulated his "essence of continuity" in the context of the geometric line in the following terms:

If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into

<sup>&</sup>lt;sup>32</sup>More precisely, the Hewitt-Łoś-Robinson continuum; see Appendix A.

<sup>&</sup>lt;sup>33</sup>Lawvere's infinitesimals rely on a category-theoretic framework grounded in intuitionistic logic (see J. Bell [12]).

<sup>&</sup>lt;sup>34</sup>We will ignore the slight technical complication arising from the fact that there are two ways of defining the Dedekind cut associated with a *rational* number.

two classes, this severing of the straight line into two portions (Dedekind [38, p. 11]).

We will refer to this essence as the *geometric essence of continuity*.<sup>35</sup> Dedekind goes on to comment on the epistemological status of this statement of the essence of continuity:

[...] I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power (ibid.).

Having enriched the domain of rationals by adding irrationals, numbers defined completely by cuts not produced by a rational, Dedekind observes:

From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal always and only when they correspond to essentially different cuts (Dedekind [38, p. 15]).

By now, Dedekind postulates that two numbers should be equal "always and only" [i.e., if and only if] they define identical cuts on the rational numbers. Thus, Dedekind postulates that there should be "one and only one" number which produces such a division. Dedekind clearly presents this as an exact arithmetic analogue to the geometric essence of continuity. We will refer to such a postulate as the rational essence of continuity.

Dedekind's postulation of rational essence is not accompanied by epistemological worries as was his geometric essence a few pages earlier. Yet, rational essence entails a suppression of infinitesimals: a pair of distinct non-rational numbers can define the same Dedekind cut on  $\mathbb{Q}$ , such as  $\pi$  and  $\pi + h$  with h infinitesimal; but one cannot have such a pair if one postulates the rational essence of continuity, as Dedekind does.

Dedekind's technical work on the foundations of analysis has been justly celebrated (see D. Fowler [54]). Whereas everyone before Dedekind had assumed that operations such as powers, roots, and logarithms can

 $<sup>^{35}</sup>$ Note that the geometric essence of continuity necessarily fails over an ordered non-Archimedean field F. Thus, defining infinitesimals as elements of F violating the traditional Archimedean property, we can start with the cut of F into positive and negative elements, and then modify this cut by assigning all infinitesimals to, say, the negative side. Such a cut does not correspond to an element of F.

be performed, he was the first to show how these operations can be defined, and shown to be coherent, in the realm of the real numbers (see Dedekind [37, §6]).

Meanwhile, the nature of his interpretive speculations about what does or does not constitute the "essence" of continuity, is a separate issue. For over a hundred years now, many mathematicians have been making the assumption that space conforms to Dedekind's idea of "the essence of continuity", which in arithmetic translates into the numerical assertion that two numbers should be equal if and only if they define the same Dedekind cut on the rationals. Such an assumption rules out infinitesimals. In the context of the hyperreal number system, it amounts to an application of the standard part function (see Appendix A), which forces the collapse of the entire halo (cluster of infinitely close, or adequal, points) to a single point.

The formal/axiomatic transformation of mathematics accomplished at the end of the 19th century created a specific foundational framework for analysis. Weierstrass's followers raised a philosophical prejudice against infinitesimals to the status of an axiom. Dedekind's "essence of continuity" was, in essence, a way of steamrolling infinitesimals out of existence.

In 1977, E. Nelson [120] created a set-theoretic framework (enriching ZFC) where the usual construction of the reals produces a number system containing entities that behave like infinitesimals. Thus, the elimination thereof was not the only way to achieve rigor in analysis as advertized at the time, but rather a decision to develop analysis in just one specific way.

### 9. Who invented Dirac's delta function?

A prevailing sentiment today is that one of the spectacular successes of the rigorous analysis was the justification of delta functions, originally introduced informally by to P. Dirac (1902–1984), in terms of distribution theory. But was it originally introduced informally by Dirac?

In fact, Fourier [53] and Cauchy exploited the "Dirac" delta function over a century earlier. Cauchy defined such functions in terms of infinitesimals (see Lützen [106] and Laugwitz [96]). A function of the type generally attributed to Dirac was specifically described by Cauchy in 1827 in terms of infinitesimals. More specifically, Cauchy uses a unit-impulse, infinitely tall, infinitely narrow delta function, as an integral kernel. Thus, in 1827, Cauchy used infinitesimals in his definition of a "Dirac" delta function [26, p. 188]. Here Cauchy uses

infinitesimals  $\alpha$  and  $\epsilon$ , where  $\alpha$  is, in modern terms, the "scale parameter" of the "Cauchy distribution", whereas  $\epsilon$  gives the size of the interval of integration. Cauchy wrote:

Moreover one finds, denoting by  $\alpha$ ,  $\epsilon$  two infinitely small numbers,

$$\frac{1}{2} \int_{a-\epsilon}^{a+\epsilon} F(\mu) \frac{\alpha \ d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a) \tag{9.1}$$

(see Cauchy [26, p. 188]). Such a formula extracts the value of a function F at a point a by integrating F against a delta function defined in terms of an infinitesimal parameter  $\alpha$  (see and Laugwitz [95, p. 230]). The expression

$$\frac{\alpha}{\alpha^2 + (\mu - a)^2}$$

(for real  $\alpha$ ) is known as the *Cauchy distribution* in probability theory. The function is called the probability density function. A Cauchy distribution with an infinitesimal scale parameter produces a function with Dirac-delta function behavior, exploited by Cauchy already in 1827 in work on Fourier series and evaluation of singular integrals.

#### 10. IS THERE CONTINUITY BETWEEN LEIBNIZ AND ROBINSON?

Is there continuity between historical infinitesimals and Robinson's theory?

Historically, infinitesimals have often been represented by null sequences. Thus, Cauchy speaks of a variable quantity as becoming an infinitesimal in 1821, and his variable quantities from that year are generally understood to be sequences of discrete values (on the other hand, in his 1823 he used continuous variable quantities). Infinitesimal-enriched fields can in fact be obtained from sequences, by means of an ultrapower construction, where a null sequence generates an infinitesimal. Such an approach was popularized by Luxemburg [108] in 1962, and is based on the work by E. Hewitt [68] from 1948. Even in Robinson's approach [128] based on the compactness theorem, a null sequence is present, though well-hidden, namely in the countable collection of axioms  $\epsilon < \frac{1}{n}$ . Thus, null sequences provide both a cognitive and a technical link between historical infinitesimals thought of as variable quantities taking discrete values, on the one hand, and modern infinitesimals, on the other (see Katz & Tall [81]).

Leibniz's heuristic *law of continuity* was implemented mathematically as Łoś's theorem and later as the *transfer principle* over the hyperreals (see Appendix A), while Leibniz's heuristic law of homogeneity

(see Leibniz [101, p. 380]) and Bos [20, p. 33]) was implemented mathematically as the standard part function (see Katz and Sherry [80]).

# 11. Is Lakatos' take on Cauchy tainted by Kuhnian Relativism?

Does Lakatos's defense of infinitesimalist tradition rest upon an ideological agenda of Kuhnian relativism and fallibilism, inapplicable to mathematics?

### G. Schubring summarizes fallibilism as an

enthusiasm for revising traditional beliefs in the history of science and reinterpreting the discipline from a theoretical, epistemological perspective generated by Thomas Kuhn's (1962) work on the structure of scientific revolutions. Applying Popper's favorite keyword of fallibilism, the statements of earlier scientists that historiography had declared to be false were particularly attractive objects for such an epistemologically guided revision (Schubring [132, p. 431–432]).

Schubring then takes on Lakatos in the following terms:

The philosopher Imre Lakatos (1922-1972)<sup>36</sup> was responsible for introducing these new approaches into the history of mathematics. One of the examples he analyzed and published in 1966 received a great deal of attention: Cauchy's theorem and the problem of uniform convergence. Lakatos refines Robinson's approach by claiming that Cauchy's theorem had also been correct at the time, because he had been working with infinitesimals (ibid.).

However, Schubring's summary of the philosophical underpinnings of Lakatos' interpretation of Cauchy's sum theorem is not followed up by an analysis of Lakatos's position (see [92, 93]). It is as if Schubring felt that labels of "Kuhnianism" and "fallibilism" are sufficient grounds for dismissing a scholar. Schubring proceeds similarly to dismiss Laugwitz's reading of Cauchy as "solipsistic" [132, p. 434]. Schubring accuses Laugwitz of interpreting Cauchy's conceptions as

some hermetic closure of a *private* mathematics (Schubring [132, p. 435]) [emphasis in the original—the authors];

as well as being "highly anomalous or isolated" [132, p. 441].

The fact is that Laugwitz is interpreting Cauchy's words according to their plain meaning (see [94, 97]), as revealed by looking, as

<sup>&</sup>lt;sup>36</sup>The dates given by Schubring are incorrect. The correct dates are 1922-1974.

Kuhn would suggest, at the context in which they occur. The context strongly recommends taking Cauchy's infinitesimals at face value, rather than treating them as a sop to the management. The burden of proof falls upon Schubring to explain why the triumvirate interpretation of Cauchy is not "solipsistic", "hermetic", or "anomalous". The latter three modifiers could be more appropriately applied to Schubring's own interpretation of Cauchy's infinitesimals as allegedly involving a compromise with rigor, allegedly due to tensions with the management of the Ecole polytechnique. Schubring's interpretation is based on Cauchy's use of the term concilier in Cauchy's comment on the first page of his Avertissement:

Mon but principal a été de concilier la rigueur, dont je m'étais fait une lois dans mon *Cours d'Analyse*, avec la simplicité qui résulte de la considération directe des quantités infiniment petites (Cauchy [25, p. 10]).

Let us examine Schubring's logic of conciliation. A careful reading of Cauchy's Avertissement in its entirety reveals that Cauchy is referring to an altogether different source of tension, namely his rejection of some of the procedures in Lagrange's Mécanique analytique [91] as unrigorous, such as Lagrange's principle of the "generality of algebra". While rejecting the "generality of algebra" and Lagrange's flawed method of power series, Cauchy was able, as it were, to sift the chaff from the grain, and retain the infinitesimals endorsed in the 1811 edition of the Mécanique analytique. Indeed, Lagrange opens his treatise with an unequivocal endorsement of infinitesimals. Referring to the system of infinitesimal calculus, Lagrange writes:

Lorsqu'on a bien conçu l'esprit de ce système, et qu'on s'est convaincu de l'exactitude de ses résultats par la méthode géométrique des premières et dernières raisons, ou par la méthode analytique des fonctions dérivées, on peut employer les infiniment petits comme un instrument sûr et commode pour abréger et simplifier les démonstrations<sup>37</sup> (Lagrange [91, p. iv]).

Lagrange describes infinitesimals as dear to a scientist, being reliable and convenient. In his *Avertissement*, Cauchy retains the infinitesimals

<sup>&</sup>lt;sup>37</sup>"Once one has duly captured the spirit of this system [i.e., infinitesimal calculus], and has convinced oneself of the correctness of its results by means of the geometric method of the prime and ultimate ratios, or by means of the analytic method of derivatives, one can then exploit the infinitely small as a reliable and convenient tool so as to shorten and simplify proofs" (Lagrange).

that were also dear to Lagrange, while criticizing Lagrange's "generality of algebra" (see [76] for details).

It's useful here to evoke the use of the term "concilier" by Cauchy's teacher Lacroix. Gilain quotes Lacroix in 1797 to the effect that

"lorsqu'on veut concilier la rapidité de l'exposition avec l'exactitude dans le langage, la clarté dans les principes, [...], je pense qu'il convient d'employer la méthode des limites" (p.XXIV)." [58, footnote 20].

Here Lacroix, like Cauchy, employs "concilier", but in the context of discussing the *limit* notion. Would Schubring's logic of conciliation dictate that Lacroix developed a compromise notion of limit, similarly with the sole purpose of accommodating the management of the *Ecole*?

Why are Lakatos and Laugwitz demonized, rather than analyzed, by Schubring? We suggest that the act of contemplating for a moment the idea that Cauchy's infinitesimals can be taken at face value is unthinkable to a triumvirate historian, as it would undermine the epsilontic Cauchy-Weierstrass tale that the received historiography is erected upon. The failure to appreciate the Robinson-Lakatos-Laugwitz interpretation, according to which infinitesimals are mainstream analysis from Cauchy onwards, is symptomatic of a narrow Archimedean-continuum vision.

### Appendix A. Rival continua

This appendix summarizes a 20th century implementation of an alternative to an Archimedean continuum, namely an infinitesimal-enriched continuum. Such a continuum is not to be confused with incipient notions of such a continuum found in earlier centuries in the work of Fermat, Leibniz, Euler, Cauchy, and others.

Johann Bernoulli was one of the first to exploit infinitesimals in a systematic fashion as a foundational tool in the calculus.<sup>38</sup> We will therefore refer to such a continuum as a Bernoullian continuum, or B-continuum for short.

A.1. Constructing the hyperreals. Let us start with some basic facts and definitions. Let

$$(\mathbb{R}, +, \cdot, 0, 1, <)$$

be the field of real numbers, let  $\mathcal{F}$  be a fixed nonprincipal ultrafilter on  $\mathbb{N}$  (the existence of such was established by Tarski [149]). The

<sup>&</sup>lt;sup>38</sup>See footnote 30 for a comparison with Leibniz.

relation  $\equiv$  defined by

$$(r_n) \equiv (s_n) \leftrightarrow_{\mathrm{def}} \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

is an equivalence relation on the set  $\mathbb{R}^{\mathbb{N}}$ . The set of hyperreals  $\mathbb{IR}$ , or the B-continuum for short, is the quotient set

$$\mathbb{IR} =_{\mathrm{def}} \mathbb{R}^{\mathbb{N}}/_{=}$$
.

Addition, multiplication and order of hyperreals are defined by

$$[(r_n)] + [(s_n)] =_{\text{def}} [(r_n + s_n)], \qquad [(r_n)] \cdot [(s_n)] =_{\text{def}} [(r_n \cdot s_n)],$$
  
 $[(r_n)] \prec [(s_n)] \leftrightarrow_{\text{def}} \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}.$ 

The standard real number r is identified with equivalence class  $r^*$  of the constant sequence  $(r, r, \dots)$ , i.e.  $r^* =_{\text{def}} [(r, r, \dots)]$ .

The set  $\mathbb{I}\mathbb{N}$  of hypernaturals (mentioned in Section 4)<sup>39</sup> is the subset of  $\mathbb{I}\mathbb{R}$  defined by

$$[(r_n)] \in \mathbb{IN} \leftrightarrow_{\mathrm{def}} \{n \in \mathbb{N} : r_n \in \mathbb{N}\} \in \mathcal{F}.$$

In particular, each sequence of natural numbers  $(n_j)$  represents a hypernatural number, i.e.  $[(n_j)] \in \mathbb{I}\mathbb{N}$ .

The set of hypernaturals can be represented as a disjoint union

$$\mathbb{IN} = \{n^* : n \in \mathbb{N}\} \cup \mathbb{IN}_{\infty},$$

where the set  $\{n^* : n \in \mathbb{N}\}$  is just a copy of the usual natural numbers, and  $\mathbb{IN}_{\infty}$  consists of infinite (sometimes called "unlimited") hypernaturals. Each element of  $\mathbb{IN}_{\infty}$  is greater than every usual natural number, i.e.

$$(\forall H \in \mathbb{IN}_{\infty})(\forall n \in \mathbb{N}) [H \succ n^*].$$

**Theorem A.1.**  $(\mathbb{R}^*, +, \cdot, 0^*, 1^*, \prec)$  is a non-Archimedean, real closed field.

The set of infinitesimal hyperreals  $\Omega$  is defined by

$$x \in \Omega$$
 if and only if  $(\forall \theta \in \mathbb{R}_+) [|x| \prec \theta^*]$ ,

where |x| stands for the absolute value of x, which is defined as in any ordered field. We say that x is infinitely close to y, and write  $x \approx y$ , if and only if  $x - y \in \Omega$ .

To give some examples, the sequence  $(\frac{1}{n})$  represents a positive infinitesimal  $[(\frac{1}{n})]$ . Next, let  $(r_n) \in \mathbb{R}^{\mathbb{N}}$  be a sequence of reals such that

<sup>&</sup>lt;sup>39</sup>See footnote 22.

 $\lim_{n\to\infty} r_n = 0$ , then  $(r_n)$  represents an infinitesimal,<sup>40</sup> i.e.  $[(r_n)] \in \Omega$ . And finally, sequence  $\left(\frac{(-1)^n}{n}\right)$  represents a nonzero infinitesimal  $\left[\left(\frac{(-1)^n}{n}\right)\right]$ , whose sign depends on whether or not the set  $2\mathbb{N}$  is a member of the ultrafilter.

The set of limited hyperreals  $\mathbb{I}\mathbb{R}_{<\infty}$  is defined by

$$x \in \mathbb{IR}_{<\infty} \leftrightarrow_{\text{def}} \exists \theta \in \mathbb{R}_{+}[\ |x| \prec \theta^*\ ],$$

so that we have a disjoint union

$$\mathbb{IR} = \mathbb{IR}_{\leq \infty} \cup \mathbb{IR}_{\infty}, \tag{A.1}$$

where  $\mathbb{IR}_{\infty}$  consists of unlimited hyperreals (i.e., inverses of nonzero infinitesimals).

**Theorem A.2** (Standard Part Theorem).

$$(\forall x \in \mathbb{I}\mathbb{R}_{<\infty})(\exists ! r \in \mathbb{R}) [r^* \approx x].$$

The unique real r such that  $r^* \approx x$  is called the standard part of x, and we write  $\operatorname{st}(x) = r$ .

Note that if a sequence  $(r_n : n \in \mathbb{N})$  happens to be Cauchy, one can relate standard part and limit as follows:<sup>41</sup>

$$\operatorname{st}([(r_n)]) = \lim_{n \to \infty} r_n. \tag{A.2}$$

**Theorem A.3.**  $\Omega$  is a maximal ideal of the ring ( $\mathbb{IR}_{<\infty}$ , +, ·, 0\*, 1\*), and the quotient ring is isomorphic to the field of standard real numbers ( $\mathbb{R}$ , +, ·, 0, 1).

Since the map

$$\mathbb{R} \ni r \mapsto r^* \in \mathbb{R}$$

is an order preserving morphism, we can treat the field of hyperreals as an extension of standard reals and use the usual notation ( $\mathbb{IR}, +, \cdot, 0, 1, <$ ). Now, the map "st" sends each finite point  $x \in \mathbb{IR}_{<\infty}$ ,

 $<sup>^{40}</sup>$ In this construction, every null sequence defines an infinitesimal, but the converse is not necessarily true. Modulo suitable foundational material, one can ensure that every infinitesimal is represented by a null sequence; an appropriate ultrafilter (called a P-point) will exist if one assumes the continuum hypothesis, or even the weaker Martin's axiom. See Cutland  $et\ al\ [32]$  for details.

<sup>&</sup>lt;sup>41</sup>This theme is pursued further by Giordano et al. [61].

to the real point  $st(x) \in \mathbb{R}$  infinitely close to x, as follows:<sup>42</sup>

$$\begin{array}{c}
\mathbb{I}\mathbb{R}_{<\infty} \\
\downarrow^{\text{st}} \\
\mathbb{R}
\end{array}$$

Robinson's answer to Berkeley's *logical criticism* (see D. Sherry [134]) is to define the basic concept of the calculus as

st 
$$\left(\frac{\Delta y}{\Delta x}\right)$$
,

rather than the differential ratio  $\Delta y/\Delta x$  itself, as in Leibniz. Robinson comments as follows: "However, this is a small price to pay for the removal of an inconsistency" (Robinson [128, p 266]).<sup>43</sup>

A sequence  $(r_n : n \in \mathbb{N})$  of real numbers can be extended to a hypersequence  $(r_K : K \in \mathbb{IN})$  of hyperreals, indexed by *all* the hypernaturals, by setting

$$r_K = [(r_{k_1}, r_{k_2}, \dots, r_{k_i}, \dots)], \text{ where } K = [(k_1, k_2, \dots, k_i, \dots)].$$

**Theorem A.4.** Let  $(r_n)$  be a sequence of real numbers, and let  $L \in \mathbb{R}$ . Then

$$\lim_{n \to \infty} r_n = L \leftrightarrow (\forall H \in \mathbb{IN}_{\infty}) [r_H \approx L^*].$$

A.2. **Uniform continuity.** We present a discussion of uniform continuity so as to supplement and clarify the discussion of uniform *convergence* in Section 6. The idea of "one variable *versus* two variables" is a little easier to explain in the context of uniform continuity.

Thus, the traditional definition of ordinary continuity on an interval can be expressed in terms of a single variable x running through the domain  $D_f$  of the function f: namely,

for each 
$$x \in D_f$$
,  $\lim f(x') = f(x)$  as  $x'$  tends to  $x$ .

Meanwhile, uniform continuity cannot be expressed in a similar way in the traditional framework. Namely, one needs a pair of variables to run through the domain of f:

$$\forall x, y \in D_f$$
, if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$ 

<sup>&</sup>lt;sup>42</sup>This is the Fermat-Robinson standard part whose seeds are found in Fermat's adequality, as well as in Leibniz's treanscendental principle of homogeneity.

<sup>&</sup>lt;sup>43</sup>However, as argued in [80], an alleged inconsistency was not there in the first place. Briefly, Leibniz is able to employ his transcendental law of homogeneity to the same effect as Robinson's standard part function (see Bos 1974, [20, p. 33]).

(of course, this has to be prefaced by the traditional epsilon-delta yoga). Now the crucial observation is that in the context of a B-continuum, one no longer needs a *pair* of variables to define uniform continuity. Namely, it can be defined using a single variable, by exploiting the notion of *microcontinuity* at a point (see Gordon et al. [62]). We will use Leibniz's symbol  $\neg$  for the relation of being infinitely close. Thus, f is called microcontinuous at x if

whenever 
$$y \neg x$$
, also  $f(y) \neg f(x)$ .

In terms of this notion, the uniform continuity of a real function f is defined in terms of its natural extension  $f^*$  to the hyperreals as follows:

for all 
$$x \in D_{f^*}$$
,  $f^*$  is microcontinuous at  $x$ .

This sounds startlingly similar to the definition of continuity itself, but the point is that microcontinuity is now required at every point of the B-continuum, i.e., in the domain of  $f^*$  which is the natural extension of the (real) domain of f.

To give an example, the function  $f(x) = x^2$  fails to be uniformly continuous on  $\mathbb{R}$  because of the failure of microcontinuity of its natural extension  $f^*$  at a single infinite hyperreal H. The failure of microcontinuity at H is checked as follows. Consider the infinitesimal  $e = \frac{1}{H}$ , and the point H + e infinitely close to H. To show that  $f^*$  is not microcontinuous at H, we calculate

$$f^*(H+e) = (H+e)^2 = H^2 + 2He + e^2 = H^2 + 2 + e^2 - H^2 + 2$$
.

This value is not infinitely close to  $f^*(H) = H^2$ , hence microcontinuity fails at H. Thus the squaring function  $f(x) = x^2$  is not uniformly continuous on  $\mathbb{R}$ .

We introduced the term "microcontinuity" (cf. Gordon et al [62]) since there are two definitions of continuity, one using infinitesimals, and one using epsilons. The former is what we refer to as microcontinuity. It is given a special name to distinguish it from the traditional definition of continuity. Note that microcontinuity at a non-standard hyperreal does not correspond to any notion available in the epsilontic framework. To give another example, if we consider the function f given by  $f(x) = \frac{1}{x}$  on the open interval (0,1) as well as its natural extension  $f^*$ , then  $f^*$  fails to be microcontinuous at a positive infinitesimal e > 0. It follows that f is not uniformly continuous on (0,1).

A.3. **Pedagogical advantage of microcontinuity.** The expressibility of uniform continuity in terms of a condition on a *single* variable, as explained above, is a pedagogical advantage of the microcontinuous

approach. The pedagogical difficulty of the traditional two-variable definition in an epsilontic framework is compounded by its multiple alternations of quantifiers, while the hyperreal approach reduces the logical complexity of the definition by two quantifiers (see, e.g., Keisler [86]). The natural hyperreal extension  $f^*$  of a real function f is, of course, necessarily continuous at non-standard points, as well, by the transfer principle; on the other hand, this type of continuity at a non-standard point is of merely theoretical relevance in a calculus classroom. The relevant notion is that of microcontinuity, which allows one to distinguish between the classical notions of continuity and uniform continuity in a lucid way available only in the hyperreal framework. Similarly, the failure of uniform continuity can be checked by a "covariant" (direct) calculation at a single non-standard point, whereas the argument in an epsilontic framework is a bit of a "contravariant" multiple-quantifier tongue twister.

We are therefore puzzled by Hrbacek's dubious laments [71, 72] of alleged "pedagogical difficulties" related to behavior at non-standard points, directed at the framework developed by Robinson and Keisler. On the contrary, such a framework bestows a distinct pedagogical advantage.<sup>45</sup>

A.4. **Historical remarks.** Both the term "hyper-real field", and an ultrapower construction thereof, are due to E. Hewitt in 1948 (see [68, p. 74]). In 1966, Robinson referred to the

theory of hyperreal fields (Hewitt [1948]) which ... can serve as non-standard models of analysis [128, p. 278].

<sup>&</sup>lt;sup>44</sup>See discussion at the end of Subsection 5.1.

<sup>&</sup>lt;sup>45</sup>Another critic of Robinson's framework is A. Connes. He criticizes Robinson's infinitesimals for being dependent on non-constructive foundational material. He further claims it to be a weakness of Robinson's infinitesimals that the results of calculations that employ them, do not depend on the choice of the infinitesimal. Yet, Connes himself develops a theory of infinitesimals bearing a similarity to the ultrapower construction of the hyperreals in that it also relies on sequences (more precisely, spectra of compact operators). Furthermore, he freely relies on such results as the existence of the Dixmier trace, and the Hahn-Banach theorem. The latter results rely on similarly nonconstructive foundational material. Connes claims the independence of the choice of Dixmier trace to be a strength of his theory of infinitesimals in [29, p. 6213]. Thus, both of Connes' criticisms apply to his own theory of infinitesimals. The mathematical novelty of Connes' theory of infinitesimals resides in the exploitation of Dixmier's trace, relying as it does on non-constructive foundational material, thus of similar foundational status to, for instance, the ultrapower construction of a non-Archimedean extension of the reals (see also [79]).

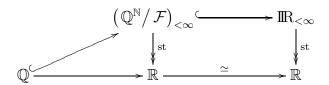


FIGURE 4. An intermediate field  $\mathbb{Q}^{\mathbb{N}}/\mathcal{F}$  is built directly out of  $\mathbb{Q}$ 

The transfer principle is a mathematical implementation of Leibniz's heuristic law of continuity: "what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa" (see Robinson [128, p. 262, 266] citing Leibniz 1701, [100]). The transfer principle, allowing an extension of every first-order real statement to the hyperreals, is a consequence of the theorem of J. Łoś in 1955, see [104], and can therefore be referred to as a Leibniz-Łoś transfer principle. A Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle.

A.5. Comparison with Cantor's construction. To indicate some similarities between the ultrapower construction and the so called Cantor's construction of the real numbers, let us start with the field of rational numbers  $\mathbb{Q}$ . Let  $\mathbb{Q}^{\mathbb{N}}$  be the ring of sequences of rational numbers. Denote by

$$(\mathbb{Q}^{\mathbb{N}})_{C}$$

the subspace consisting of Cauchy sequences, and let  $\mathcal{F}_{ndl} \subset (\mathbb{Q}^{\mathbb{N}})_{C}$  be the subspace of all null sequences. The reals are by definition the quotient field

$$\mathbb{R} = \left(\mathbb{Q}^{\mathbb{N}}\right)_{C} / \mathcal{F}_{mil}. \tag{A.3}$$

Meanwhile, an infinitesimal-enriched field extension of  $\mathbb{Q}$  may be obtained by forming the quotient

$$\mathbb{Q}^{\mathbb{N}}/\mathcal{F}$$
.

Here  $[(q_n)]$  maps to zero in the quotient if and only if one has

$${n \in \mathbb{N} : q_n = 0} \in \mathcal{F},$$

where  $\mathcal{F}$ , as above, is a fixed nonprincipal ultrafilter on  $\mathbb{N}$ .

To obtain a full hyperreal field, we replace  $\mathbb Q$  by  $\mathbb R$  in the construction, and form a similar quotient

$$IR = \mathbb{R}^{\mathbb{N}} / \mathcal{F}. \tag{A.4}$$

We wish to emphasize the analogy with formula (A.3) defining the A-continuum. We can treat both  $\mathbb{R}$  and  $\mathbb{Q}^{\mathbb{N}}/\mathcal{F}$  as subsets of  $\mathbb{R}$ . Note that, while the leftmost vertical arrow in Figure 4 is surjective, we have

$$(\mathbb{Q}^{\mathbb{N}}/\mathcal{F}) \cap \mathbb{R} = \mathbb{Q}.$$

A.6. Applications. A more detailed discussion of the ultrapower construction can be found in M. Davis [36] and Gordon, Kusraev, & Kutateladze [62]. See also Błaszczyk [16] for some philosophical implications. More advanced properties of the hyperreals such as saturation were proved later (see Keisler [84] for a historical outline). A helpful "semicolon" notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [103]. See also P. Roquette [129] for infinitesimal reminiscences. A discussion of infinitesimal optics is in K. Stroyan [144], J. Keisler [83], D. Tall [145], and L. Magnani and R. Dossena [110, 40].

Edward Nelson [120] in 1977 proposed an axiomatic theory parallel to Robinson's theory. A related theory was proposed by Hrbáček [70] (who submitted a few months earlier and published a few months later than Nelson). Another axiomatic approach was proposed by Benci and Di Nasso [14]. As Ehrlich [43, Theorem 20] showed, the ordered field underlying a maximal (i.e., On-saturated) hyperreal field is isomorphic to J. H. Conway's ordered field No, an ordered field Ehrlich describes as the absolute arithmetic continuum.

Infinitesimals can be constructed out of integers (see Borovik, Jin, and Katz [18]). They can also be constructed by refining Cantor's equivalence relation among Cauchy sequences (see Giordano & Katz [61]). A recent book by Terence Tao contains a discussion of the hyperreals [148, p. 209-229].

The use of the B-continuum as an aid in teaching calculus has been examined by Tall [146], [147]; Ely [44]; Katz and Tall [81] (see also [74, 75]). These texts deal with a "naturally occurring", or "heuristic", infinitesimal entity 1 - 0.999...' and its role in calculus pedagogy. Applications of the B-continuum range from the Bolzmann equation (see L. Arkeryd [5, 6]); to modeling of timed systems in computer science (see H. Rust [131]); Brownian motion, economics (see R. Anderson [3]); mathematical physics (see Albeverio et al. [1]); etc.

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<sup>&</sup>lt;sup>46</sup>See footnote 14 for Peirce's take on 1 - 0.999....

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A joint study with David Sherry entitled "Leibniz's infinitesimals: Their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond" is due to appear in *Erkenntnis*.

A joint study with David Tall, entitled "The tension between intuitive infinitesimals and formal mathematical analysis", appeared as a chapter in a book edited by Bharath Sriraman, see

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