HISTORY OF MATHEMATICS BEFORE THE SEVENTEENTH CENTURY

A. BABYLONIAN AND EGYPTIAN MATHEMATICS AND ASTRONOMY 3100 B.C. TO 1 B.C. [13–26]

The period covered by this first lecture on the history of mathematics will be about 4700 years; for we shall begin by noting that in one of the great museums at Oxford is a royal mace [**36**] of 3100 B.C. on which there is a record of 120,000 prisoners, of 400,000 captive oxen, and of 1,422,000 captive goats. These numbers, written in Egyptian hieroglyphs, show that already in this ancient time not only was the decimal system of numeration, but also a method for writing very large numbers, thoroughly established. It is interesting to speculate on how many thousand years earlier must have been *the beginnings* when great communities, of a millennium later, with a highly developed social order, calling for the frequent use of mathematics of taxation, barter, interest, and of large numbers, were replaced by small groups of primitive people for whose simple needs such number words as "one," "two," and "heap" would wholly suffice [**4, 19**].

Any general primitive notions of this kind probably long antedated the execution of the recently discovered finest art-work of primitive man, reputed to be from 25,000 to 50,000 years old [**18**]. It is a sculpture of a white rhinoceros with a swarm of attendant tick-birds, hammered into a slab of basaltic rock, and showing extraordinary power of form, line, and perspective—elements intimately allied to things mathematical.

But there were other people besides the Egyptians who contributed notably to early mathematics. I refer to the non-Semitic Sumerians who lived just north of the Persian Gulf and south of the Semitic Akkadians, and who for many centuries prior to 2500 B.C. were generally predominant in Babylonia, but were absorbed into a larger political group by about 2000 B.C. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of rivers regulated by canals, went back to Sumerian times, as also a considerable part of their religion, their law, and their mathematical notation [**13, 17, 19**].

An extraordinary number of tablets show that the Sumerian merchant of about 2000 B.C. was familiar with such things as weights and measures, bills, receipts, notes, and accounts. Long before coins were in use (7th century B.C.) it was common custom to pay interest for the loans of produce, or of a certain weight of a precious metal. Tablets indicate that the rate of interest varied from 20 to 30 per cent, the higher rate being charged for produce.

Sumerian or Babylonian arithmetic was essentially sexagesimal, although there are influences of a decimal system. Hence the general numeral might be written in the form $a_n \cdot 60^n + a_{n-1} \cdot 60^{n-1} + \cdots + a_1 \cdot 60 + a_0 \cdot 60^0 + b_{-1} \cdot 60^{-1} + \cdots + b_{-m} \cdot 60^{-m}$ or $a_n, a_{n-1}, \cdots a_1, a_0, b_{-1}, \cdots b_{-m}$. In particular 31, 6, 15 might equal 111975. The symbol for unity was also the symbol for any one of the numbers 60^n where n is a positive or negative integer. Thus 31, 6, 15 might mean not only 111975, but also 6718500 or 1866 1/4 or 31 1/10 1/240, according as the 15 is taken as multiplied by 1, 60, or 1/60, or 1/3600. The uncertainty in this regard introduces certain difficulties in interpreting Babylonian mathematical texts. Another uncertainty was introduced through the fact that where a zero enters may be determined only by the context; so that, for example, 11, 7 or 11, , 7 might stand for 39607. But a blank space does not always mean zero in Old Babylonian tablets. So far as is at present known, no special symbol for zero was used by the Babylonians before about 400 B.C., and this symbol was used to indicate a zero both in the interior and at the end of the number.

In the field of geometry the Babylonian of 2000 to 1600 B.C. used the following results in concrete cases, from which we have to infer that they were familiar with the general rules:

1. The area of a rectangle is the product of the lengths of two adjacent sides.
2. The area of a right triangle is equal to one-half the product of the lengths of the sides about the right angle.
3. The sides about corresponding angles of two similar right triangles are proportional.
4. The area of a trapezoid with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.
5. The perpendicular from the vertex of an isosceles triangle on the base bisects the base. The area of the triangle is the product of the lengths of the altitude and half the base. Indeed the Babylonians may have thought of this result for the area of a triangle other than right or isosceles, since such a triangle may be regarded as made up of adjacent or overlapping right triangles; but there is no known example of this use of the formula.
6. The "Pythagorean" theorem; for example, for triangles with sides corresponding to the numbers 3, 4, 5; 5, 12, 13; 8, 15, 17; 20, 21, 29; and many more; see after no. 11, below.
7. The angle in a semi-circle is a right angle [**55**].
8. The length of the diameter of a circle is one-third of its circumference ($\pi=3$). The area of a circle is 1/12 of the square of its circumference (correct for $\pi=3$).
9. The volume of a rectangular parallelopiped is the product of the lengths of its three dimensions, and the volume of a right prism with a trapezoidal base is equal to the area of the base multiplied by the altitude of the prism.
10. The volume of a right circular cylinder is the area of its base multiplied by its altitude.
11. The volume of the frustum of a cone, or of a square pyramid, is equal to its altitude multiplied by one-half the sum of the areas of its bases. It has been conjectured that the Babylonians had also the equivalent of an exact

formula for the volume in the case of a square pyramid, namely

$$V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right],$$

where a and b are the lengths of the sides of the square bases [**20**]. This was known to HERON of Alexandria 1700 years later, and reduces to the extraordinary formula apparently known to the Egyptians.

The most remarkable recent discovery in connection with Babylonian mathematics was that made by Professor NEUGEBAUER, in tablet PLIMPTON 322, dated -1900 to -1600, at Columbia University, which contains a table of "Pythagorean" numbers [**16**]. Let l denote the longer, s the shorter side of a right triangle, and h its hypotenuse. The values of h and s are given in two columns of our text but the third column gives not l but h^2/l^2 with successive entries decreasing almost linearly while there is great variation in the other columns. He discusses the contents of this tablet at great length and is convincing in suggesting that the following Euclidean relations (*Elements*, book 10, prob. 28, lemma) were known more than a thousand years before Pythagoras.

$$l = 2pq, \qquad s = p^2 - q^2, \qquad h = p^2 + q^2,$$

p and q being relatively prime, and $p>q$. Also that the following relation (if $\bar{q}=1/q$ and $\bar{p}=1/p$) was not only known:

$$h/l = \tfrac{1}{2}(p\bar{q} + q\bar{p}),$$

but was the basis, with known tables of reciprocals, for the discovery of the successive entries of the tablet. In 13 of the 15 triangles the sides are relatively prime. The lengths of the sides of the largest triangle are:

650 700, 649 909, 1 080 541.

Was EUCLID indebted to Babylonians for his tenth book lemma? As yet no cuneiform mathematical tablets date from the period 1300 to 300 B.C. so that the origins of later Babylonian results are mostly unknown.

Numerous problems involving portions of a right triangle cut off by lines parallel to a side lead to systems of simultaneous equations, even as many as ten equations in ten unknowns; and also to the solution of quadratic equations. Moreover the Babylonian of 1800 B.C. evidently knew our formula for the solution of a quadratic equation with the positive sign before the radical. Many problems could be cited to prove this [**14, 16, 17, 21**]; let us consider one of them on a tablet in Strassburg dating from about 1800 B.C.

"An area [A] (consisting of) the sum of two squares [x^2+y^2] (is) 16, 40 [$=1000$]." The side of one square [y] (is) $\frac{2}{3}\left[\frac{\alpha}{\beta}\right]$ of the side of the other square [x], diminished by 10 [d]. "What are the sides of the squares [x, and y]?"

Hence

$$x^2 + y^2 = A, \qquad y = \frac{\alpha}{\beta}\, x - d.$$

If $\zeta = x/\beta$, we get a quadratic equation:

$$(\alpha^2 + \beta^2)\zeta^2 - 2d\alpha\zeta = A - d^2,$$

the solution being

$$\zeta = \frac{1}{\alpha^2 + \beta^2}\,[d\alpha \pm \sqrt{\{d^2\alpha^2 + (\alpha^2 + \beta^2)(A - d^2)\}}].$$

Bearing in mind that, in the particular case, $A = 1000$, $d = 10$, $\alpha = 40$, $\beta = 60$, let us now compare with this the working given in the text, with our numbers substituted for the Babylonian, of which a few samples are given.

"You proceed thus:	
Square 10: this gives [1, 40] 100; subtract [1, 40] 100 from [16, 40] 1000: this gives [15, 0]900.	$A - d^2 = 900.$
Square [1, 0]60: this gives [1, 0, 0] 3600; 40^2 is [25, 40] 1600: 3600+1600=5200.	$\alpha^2 + \beta^2 = 5200.$
Multiply 5200 by 900: this gives 4680000.	$(\alpha^2 + \beta^2)(A - d^2) = 4680000.$
Multiply 40 by 10: this gives 400.	$\alpha d = 400.$
Square 400: this gives 160000.	$\alpha^2 d^2 = 160000.$
Add 160000 to 4680000: this gives 4840000.	$\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2) = 4840000.$
The square root of this is 2200.	$\sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2200.$
Add 400 already found: this gives 2600.	$d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2600.$
What part of 5200 gives 2600? Answer: one-half (30 in text).	$\dfrac{d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}}}{\alpha^2 + \beta^2} [= \zeta] = \frac{1}{2}.$
$\frac{1}{2}$ multiplied by 60 gives 30 as [side of] greater square.	$\frac{1}{2}\beta = \zeta\beta = x = 30.$
Multiply $\frac{1}{2}$ by 40: this gives 20.	$\frac{1}{2}\alpha = \zeta\alpha = 20.$
Subtract 10 from 20 and this gives 10 as [side of] lesser square."	$\zeta\alpha - d = y = 10.$

Surely such work "is wonderful in itself; it is equally extraordinary that these developments in arithmetic and algebra should have remained, for the most of 1800 years at all events, unknown to, or at least without (so far as we can judge) any traceable effect upon, the Greek pioneers in the same subject" **[22]**.

Another interesting method of solution of quadratic equations may be noted. On a Louvre tablet of about 300 B.C. are four problems **[23]** concerning rectangles of unit area but with the sum of adjacent sides varying $x + y = a$, $xy = 1$. The successive steps of the solution are equivalent to substitution in the formula

$$\tfrac{1}{2}(x - y) = \pm \{[\tfrac{1}{2}(x + y)]^2 - xy\}^{1/2}, \quad \text{or}$$

$$[\tfrac{1}{2}(x - y)]^2 = [\tfrac{1}{2}(x + y)]^2 - xy$$ (EUCLID'S *Elements*, II, 5 and *Data*, prop. 85).

In a Yale text of about 1700 B.C. this same method of solution is applied [24] to the equations $x-y=7$, $xy=1, 0$. By this method also the solution of somewhat more complicated rectangle problems on Strassburg and Louvre tablets, [25] of about 1800 B.C., would follow at once.

Our present knowledge of Babylonian achievements in mathematics is mainly due to the extraordinary discoveries of OTTO NEUGEBAUER, a research professor and chairman of the History of Mathematics Department of Brown University. (Dr. SACHS is also a member of this Department.) He discovered not only most of the results set forth above, but also many other things; in particular that the Babylonians discussed problems involving cubic [26] and biquadratic equations. A tablet was discovered which gives not only the squares and cubes of all integers from 1 to 30, but also the results for the sum n^3+n^2 for this same range. Pure cubic equations with integral solutions could be solved by means of the table of cubes reversed, and an example of this kind is quoted. The use of tables of cubes was not previously surmised. Another problem, of about 1800 B.C., seems to call for the solution of simultaneous equations, $xyz+xy=1\frac{1}{6}$, $y=\alpha x$, $z=\mu x$, $(\alpha=2/3, \mu=12)$, which lead to $(\mu x)^3+(\mu x)^2=252$. The solution of this equation may be found from the n^3+n^2 table. Two problems lead to such equations. Another problem, of the same period, seems to lead to the general equation $\mu x^3+(1-\mu b)x^2-bx+a=0$, being derived from $xyz+xy=a$, $z=\mu x$, $x+y=b(a=7/6, b=5/6, \mu=12)$. But in the tablet it is stated that $\frac{xyz+xy}{\mu b^3}=\frac{x}{b}\cdot\frac{y}{b}\cdot\frac{z+1}{\mu b}=\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{7}{10}$ whence it is inferred that $\frac{x}{b}=\frac{3}{5}$, $\frac{y}{b}=\frac{2}{5}$, $\frac{z+1}{\mu b}=\frac{7}{10}\cdot$ Thus, this general cubic equation is not solved by reduction to the "normal form," $n^3+n^2=c$, to which, however, NEUGEBAUER believes that the Babylonians were quite capable of reducing the general cubic equation, although as yet he has no evidence that they actually did do it. In connection with the n^3+n^2 table NEUGEBAUER noted also [27] that they may well have known the equivalent of the relation $\sum_{i=1}^{n} i^3=(\sum_{i=1}^{n} i)^2$ for various values of n.

Tablets at Yale University [16] containing hundreds of problems, without solutions, arranged in systematic order, are of great interest. Possibly these may date from 1600 B.C. In several cases of simultaneous equations for solution and leading to biquadratic equations, we find here a number of extraordinary examples of negative numbers in right-hand members. For instance, solve the equations

$$xy = 600 \quad \text{and} \quad 150(x - y) - (x + y)^2 = -1000.$$

Babylonian astronomical texts of the third century B.C. [28] make explicit use of the rules, $+ \times + = - \times - = +$;
$+ \times - = - \times + = -$.

On a Berlin tablet closely related to Yale tablets, the equations (also leading to a biquadratic equation)

$$xy = 600 \quad \text{and} \quad x^2 + \frac{1}{13}\left\{\frac{1}{19}\left[(x+y)^2 - 600\right] + 3y^2\right\} = 1000,$$

are especially interesting because of the "irregular" numbers in the fractions 1/13 and 1/19. All problems which have so far been found to call for division by such numbers as 7, 11, and 13, are arranged in such a way that the divisors will disappear in the course of the work.

Two other illustrations from the Yale tablets may be cited. A general cubic equation comes up in the discussion of volumes of frustums of a pyramid, as the result of eliminating z from equations of the type, $z(x^2+y^2)=A$, $z=(ay+b)$, $x=c$. An equation of the sixth degree (equivalent to a quadratic in x^3) results from the solution of equations of the form $xy=b$, $a_1x^2/y+a_2y^2/x+a_3=0$.

A sexagesimal number, n, is regular if its reciprocal, $\bar{n}$, is a finite sexagesimal expression, and irregular if it is not. The necessary and sufficient condition for n to be regular is that $n=2^\alpha 3^\beta 5^\gamma$, where α, β, γ are each a positive integer or zero. With the exception of a Yale tablet, tables of reciprocals contain only reciprocals of regular numbers. For an irregular number like 7 it might be thought that such approximations as 7/48 or 13/90 might be found, but this is not the case. On the unique tablet to which we have just referred are such approximations as the following:

$$\overline{59} = ;1, 1, 1 \qquad \overline{61} = ;0, 59, 0, 59 \quad \text{and} \quad \overline{78} = ;0, 56, 9, 13, 50.$$

The Babylonian method of finding the reciprocal of any regular number, however complicated, is now well known [**29**]. For such a number c the basic relation is $\bar{c}=\bar{a}(\overline{1+b\bar{a}})$, where a and b are two numbers such that $c=a+b$, and a is a number whose reciprocal may be found in a standard table.

In Babylonian mathematics tables were constantly used, and in particular, tables of reciprocals, reducing division to multiplication, but these tables rarely go beyond two sexagesimal places (3600). There is, however, a Louvre text of about 300 B.C. with scores of larger entries and including one of an n, as a seven-place number (corresponding to eleven decimal places), leading to $\bar{n}$, a seventeen-place number (corresponding to twenty-nine decimal places). Such tables were necessary in the Babylonian astronomical calculations [**30**] of the time. Thus there was mathematical development to meet astronomical needs.

On another Louvre tablet about the time of ARCHIMEDES, NEUGEBAUER found two other suggestive problems [**32**]. One states that

$$1 + 2 + 2^2 + \cdots + 2^9 = 2^9 + 2^9 - 1.$$

Is this another way of writing $2\cdot 2^9 - 1 = (2^{10}-1)/(2-1)$, indicating knowledge of EUCLID'S, or our own, formula for the sum of such a geometric series?

On the same tablet it is stated that

$$1\cdot 1 + 2\cdot 2 + 3\cdot 3 + \cdots + 10\cdot 10 = (1\cdot\tfrac{1}{3} + 10\cdot\tfrac{2}{3})\cdot 55 = 385.$$

Now $\sum_{i=1}^{n} i^2 = (1 \cdot \frac{1}{3} + n \cdot \frac{2}{3}) \cdot \sum_{i=1}^{n} i = \frac{1}{6}n(n+1)(2n+1)$, if we set $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, a result known to the PYTHAGOREANS. This sum-of-squares formula was practically equivalent to one known to ARCHIMEDES. Did contemporary Babylonians also use it?

Istanbul tablets seem originally to have had tables of c^n, for $n=1$ to 10, for $c=9$, $c=16$, $c=100$, and $c=225$. An application for such a table would be in solving exponential equations of the type $a^x=b$. Such equations arose in work of the Babylonian, even in the case where x was not integral, as one may check by a Louvre tablet of about 1700 B.C. One problem here is to find how long it would take for a certain sum of money to double itself at 20% interest. The problem is, then, to find x, the number of years, in the equation $(1;12)^x=2$. $1;12^3<2<1;12^4$, hence $x>3$ but <4. Linear interpolation gives

$$\frac{1;12^4 - 2}{1;12^4 - 1;12^3} = 0;12,46,40 \text{ years;}$$

or 2;33,20 months. That is, $x=4$ years$-$2;33,20 months.

Since the final result thus found is exactly what is given in the tablet, the method used in reaching it, may well have been that employed by the Babylonians. Both in the Berlin Museum and in Yale University are tablets with other problems in compound interest.

Babylonian approximations to the square roots of non-square numbers are of interest; for example, 1 5/12 for $\sqrt{2}$, and 17/24 for $1/\sqrt{2}$; here, and in finding $\sqrt{1700}$, an approximation formula equivalent to one employed by HERON of Alexandria [97] seems to have been used. In finding 2β as approximation to $\sqrt{2\frac{1}{2}}$, the use of the equivalent of a DIOPHANTINE equation is suggested. But the most remarkable approximation is that found in a Yale tablet of about -1600 for the diagonal of the side of a unit square $\sqrt{2} \approx 1;24,51,10$ which is 1.414213 instead of 1.414214. In explaining the possible derivation of this result NEUGEBAUER & SACHS present a strong case for the Babylonians having followed a procedure consisting of alternating approximation of $\sqrt{2}$ by arithmetic and harmonic means of approximations previously found [33].

The brief suggestions which we have given of discussions of purely numerical problems of very varied types, could be greatly magnified, and force upon us the conclusion that Babylonians during 2000 years before the Christian era laid the foundations of real algebra. Even where the foundation is apparently geometric the essence is usually strongly algebraic, illustrated by the fact that frequently operations occur which do not admit of a geometric interpretation, like the addition of areas and lengths, or multiplication of areas. The predominant problem consists in the determination of unknown quantities subject to given conditions. Of course a certain number of geometric relations were well known. Numerical calculations are everywhere carried out with the greatest facility and skill.

The criterion for scientific mathematics must be the existence of the concept of proof. Egyptian mathematics contains only two (rather trivial) general rules (RHIND papyrus [43, 44], problems 61, 66), and the much more highly developed Babylonian mathematics displays substantial illustrations of general techniques for proving its procedures [34]. But the remarkable mathematical astronomy which flourished during the last three centuries preceding the Christian era, and had as its goal the computation of ephemerides for the moon and planets, was distinctly scientific [27, 28].

Extension of our knowledge in connection with Babylonian mathematics and astronomy is likely long to continue. Let us now consider achievements of the Egyptians in pure and applied mathematics [35], where recent research has contributed little that is new.

We have referred to the mace record [36] of 3100 B.C. The erection of the great pyramid at Gizeh about 2900 B.C. must have involved many mathematical problems. Here was a huge structure covering more than 13 acres which, with its marvelous connecting roadway to the Nile, took 100,000 workmen 30 years to build. Over 2,000,000 blocks of stone, averaging $2\frac{1}{2}$ tons in weight, and fitted together with great exactness were brought from sandstone quarries on the opposite side of the Nile. For the roofs of chambers granite blocks 27 feet long and 4 feet thick, weighing 54 tons each, were transported from the quarry more than 600 miles away, and placed in their position over 200 feet above the ground level [37].

But the problems of mechanics and engineering involved in handling even the large stone blocks of the great pyramid were slight as compared with those dealt with by the Egyptians in quarrying and setting up some of their huge obelisks of pink granite. The largest existing obelisk, quarried about 1500 B.C., was no less than 105 feet long, nearly 10 feet square at the larger end, and about 430 tons in weight. It was set up in front of the Temple of the Sun at Thebes [38].

A certain amount of mathematics was used by the Ancient Egyptians in connection with their devising of timepieces, namely: (a) the sun-dial or gnomon; and (b) the clepsydra or water-clock [39]. The same may be said concerning ship-building and navigation [40] and engineering involved in the building of notable aqueducts and tunnels for providing water supplies [41].

Most of our knowledge of Egyptian mathematics is derived from two mathematical papyri, the one written about 1850 B.C., usually called the Moscow papyrus [42], containing 25 problems; and the other dating from about 1650 B.C., commonly called the RHIND mathematical papyrus [43, 44], with 85 problems. All of the 110 problems are numerical and many of them are excessively simple. Those in the RHIND papyrus are preceded by a table giving the equivalents in unit fractions of 2 divided by all odd numbers from 5 to 101; for with the exception of $\frac{2}{3}$ the Egyptian had no notation for a fraction with a numerator other than unity. Two divided by 7 is expressed as the sum of $\frac{1}{4}$ and 1/28; 2 divided by 97 is expressed as the sum of 1/56, 1/679, 1/776. Both of these results, as well as $\frac{2}{3}$ expressed as the sum of $\frac{1}{2}$ and $\frac{1}{6}$, are used in a single problem later. The table was therefore a reference list for use in solving problems.

The Egyptian carried through the multiplication of two numbers by successive multiplication of one of the numbers by twos or tens or by $\frac{2}{3}$, or by the division of the number by twos or tens. A sort of transposition was also a common operation. For example, if it had been found that $\frac{1}{5}$ of 105 was 21 the Egyptian might at once write down 1/21 of 105 is 5. Problems in division are reduced to those of multiplication. It is rather extraordinary that in order to get *one*-third of a number, the Egyptian first found *two*-thirds of the number and then took one-half of the result. This is illustrated in more than a dozen problems of the RHIND papyrus.

Nearly a score of 110 problems are such as we would now solve by algebra with equations of the first degree in one unknown quantity. For example: "A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$, added together, becomes 33. What is the quantity?" The method of solution used is generally that of trial, or false position, and in more than one case it is obvious that the idea of proportion was clearly understood.

Another score of the problems deal with such questions as the strength of bread and of different kinds of beer, the derivation of beer of great alcoholic strength from two others, and the exchange of beer for bread. An example of this type of problem is the following: "Given that 13 hekat of upper Egyptian grain is made into 18 des of besha date-substitute beer, and that 1 des of this makes $2\frac{1}{6}$ des of barley beer, what is the strength of the barley beer?"

The feed for geese, cranes, ducks, quails, doves, and also for bulls and common cattle is discussed in other problems. The following illustrates a rule-of-three problem: "A sandal maker works for 15 days receiving wages every 5 days. If he does the work in 10 days after what periods should he be paid?" Of problems in arithmetic progressions the following may be mentioned: "Divide 100 loaves among 5 men in such a way that the share received shall be in arithmetic progression and that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two." Remember that these are problems of 1650 to 1850 B.C.

Twenty-six of the 110 problems are geometric [**45**], and both volumes and areas are discussed. The area of a circle is repeatedly taken as the square of 8/9 of the diameter; this leads us to the remarkable value $256/81=3.1605\cdots$ for π, much better than the somewhat earlier Babylonian value 3. The volume of a right circular cylindric granary is taken as equal to the area of its base multiplied by the number of units in its height. The most recent discussion seems to make it clear that the Egyptian knew that in any triangle its area is equal to one-half the product of its base and altitude [**45**]. The cotangents of the angles which the faces of pyramids make with their bases are discussed. A numerical problem appears to prove the extraordinary fact that the Egyptian knew our formula for the volume of a frustum of a square pyramid $V=(h/3)(a^2+ab+b^2)$, where a and b are the lengths of the sides of the square bases and h is the number of height units of the frustum. The editor of the Moscow papyrus, which was first completely published in 1930 [**42**], believed that yet another problem gave the correct result for the area of a hemisphere, if we take the value of π to which reference was just made. But the late Professor PEET, a prominent English Egyptologist, argued that there was no proper ground for this conclusion, since the correct translation led to something quite different [45].

There is no document to prove that the Egyptian knew even a particular case of the PYTHAGOREAN theorem [43].

Our main sources of information concerning Egyptian mathematics consist of two papyri of much the same type, but all additional fragments which we possess match the same picture, which is paralleled by economic documents in which occur precisely those problems and methods which we find in the mathematical papyri. Furthermore, the Egyptian mathematical texts find their continuation in Greek papyri (on to the eighth century of our era), which again show the same pattern. It is therefore safe to say that Egyptian mathematics never rose above a very primitive level, and did not provide the most essential tools for astronomical computation of real importance [30].

Thus we conclude our consideration of the mathematics of the Babylonians and Egyptians in our first period. Before 600 B.C. there was no other mathematics than theirs worth considering.

B. GREEK MATHEMATICS 600 B.C. TO 600 A.D. [46–51]

Greek history began with the second millennium B.C. but in connection with the history of mathematics, of sculpture, of architecture, of art, of philosophy, of literature, of thought, the semi-millennium commencing about 600 B.C. is of the greatest importance. The Greeks were not confined to the Greek peninsula, as in modern times. They occupied Macedonia and Thrace, the islands of the Aegean, the northern and western seaboards of Asia Minor, Southern Italy and Sicily. Scattered settlements were also to be found as far apart as the mouth of the Rhone, the north of Africa, and the eastern end of the Black Sea. Such was the location of the people who were presently to set such marvelous everlasting beacon lights of Freedom, Truth, Reason, Beauty, Excellence, Fellowship between man and man, which were to inspire, guide, and sustain through the millenniums to follow.

What were the special aptitudes which the Greeks possessed for science? An answer by SIR THOMAS HEATH [48], long a leading authority on Greek mathematics, is given. But into this quotation two correcting interpolations are introduced.

"They had, first, a love of knowledge for its own sake, amounting, as Butcher says, to an instinct and a passion; secondly, a love of truth and a determination to see things as they are; thirdly, a remarkable capacity for accurate observation." [In "capacity for accurate observation" there seems to be no difference between Greek and other cultures. The representation of animals by Egyptians and Assyrians did not find their equal in Greece; because of their exactness they are constantly used by zoologists as source material. The astronomical observations of the Babylonians are certainly not of less importance than those of the Greeks, and also the fundamental instruments such as gnomon, sundial, water-clock. Ancient Egyptian anatomy was of the highest order.] "Fourthly, while eagerly assimilating information from all quarters, from Egypt and Babylon in particular, they had an unerring instinct for taking what was worth having and rejecting the rest. As one writer has said, 'it remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.'" ["Rejecting the rest" and "Superstitious ideas . . . rubbish on one side." These stand in contradiction to many historical facts. All forms of oriental culture found an entry into Greece (see, for example, R. REITZENSTEIN, *Die hellenistischen Mysterienreligionen nach ihren Grundgedanken und Wirkungen*, third ed., Leipzig, 1927.) Astrology in its absurd forms was first a product of Hellenism.] "Fifthly, they possessed a speculative genius unrivaled in the world's history.

"It was this unique combination of gifts which qualified the Greeks to lead the world in all the intellectual pursuits that make life worth living.

"Last, but not less important, the Greeks possessed the advantage over the Egyptians and Babylonians of having no priesthood which could monopolise learning as a preserve of its own, with the inevitable result of sterilising it by keeping it bound up with religious dogmas and prescribed and narrow routine."

Some of these attributes were possibly more in evidence in their great achievements in the fields of medicine, biology, and natural science, than in those of mathematics and astronomy, which now concern us for a few moments. Since hours would be necessary for any adequate description of their wonderful achievements in these fields, we must confine ourselves largely to references to a few names and results.

Greek theoretical geometry and astronomy began with **Thales** **[51]** of Miletus, on the West coast of Asia Minor, in the first half of the sixth century B.C. No wonder he was declared one of the Seven Wise Men since, apart from being a mathematician and astronomer, he was also a statesman, engineer, man of business, and philosopher. To THALES later tradition attributes the following results in elementary geometry, all of them in EUCLID'S *Elements:*

1. A circle is bisected by any diameter (I, def. 17);
2. The angles at the base of an isosceles triangle are equal (I, 5);
3. If two straight lines cut one another, the vertically opposite angles are respectively equal (I, 15);
4. If two triangles have two angles and one side in each respectively equal the triangles are equal in all respects (I, 26);
5. The angle in a semi-circle is a right angle (which we have seen, was already recognized by the Babylonians **[55]** some 1400 years earlier; III, 31).

Of practical problems he showed how to determine the distance of a ship from the shore and found the height of a pyramid by means of the shadows cast on the ground at the same moment by the pyramid and a stick; that moment was chosen when the length of the stick and its shadow were equal. There is no evidence that Thales predicted a solar eclipse which took place in 585 B.C. (See A. PANNEKOEK, "The origin of the Saros," K. Akad. van Wetens., Amsterdam, *Proc.*, 1918, v. 20, p. 955.) It may be of interest to remark that THALES was the first known individual with whom definite mathematical discoveries were associated.

About half a century after THALES came **PYTHAGORAS** [52]. Under his inspiration geometry was first pursued as a study for its own sake. A man of great ability and a most interesting and magnetic mystic, he finally settled at Crotona on the southeastern coast of Italy. Here among the young men of well-to-do families he established a secret society or brotherhood most of whose mathematical discoveries were pooled. According to HEATH [22], it is to the PYTHAGOREANS, that is about 500–350 B.C., that the following geometric results are due:

1. The properties of parallels and their application to prove that the sum of the angles of a triangle is equal to two right angles. From this were deduced the familiar results concerning the sums of (*a*) exterior and (*b*) interior angles of a polygon.
2. The transformation of areas of rectilinear figures, and the sums and differences of such areas, into equivalent areas of different shapes. To this end they invented the powerful method of application of areas, the main constituent of the geometric algebra by which they effected the geometric equivalent of addition, subtraction, division, extraction of the square root, and the complete solution of the general quadratic equation $x^2 \pm px \pm q = 0$, so far as it has real roots.

At a considerably later day APOLLONIUS of Perga named the conic sections, parabola, ellipse, and hyperbola, because he had shown that these curves were respectively defined by the application of an area, the application of an area falling short, the application of an area exceeding; that is,

$$y^2 = px, \quad y^2 = px - \frac{p}{d}x^2, \quad \text{and} \quad y^2 = px + \frac{p}{d}x^2.$$

3. The PYTHAGOREANS had a theory of proportion pretty fully developed, though it was only applicable to commensurable magnitudes, being presumably a numerical theory. They knew properties of similar figures such as similar rectilinear figures being in the duplicate ratio of corresponding sides.
4. They had discovered, or were aware of the existence of, at least three of the regular solids—the tetrahedron, cube, and dodecahedron.
5. They discovered the existence of the incommensurable in at least one case, that of the diagonal of a square in relation to its side, and they devised a method of obtaining closer and closer approximations to the value of $\sqrt{2}$ in the form of numerical fractions, x/y whose elements

are the successive solutions of the indeterminate equations $x^2-2y^2=\pm 1$; [$y=2, 5, 12, 29, \cdots; x=3, 7, 17, 41, \cdots$].

Arithmetic in the sense of the theory of numbers began in discussions connected with problems of the PYTHAGOREANS. With the Greeks, Arithmetic dealing with absolute numbers, or numbers in the abstract, was distinguished from Logistic [**50**], the science dealing with ordinary arithmetical operations, and certain problems of elementary algebra.

Two numbers are called amicable or friendly if each equals the sum of the aliquot divisors of the other. PYTHAGORAS gave the first pair 220, 284 [**56**]. The second and third pairs were given in the seventeenth century by FERMAT and DESCARTES respectively, and the next 61 pairs were obtained by Euler in the eighteenth century. PYTHAGOREANS discussed also various forms of figured numbers—triangular, square, pentagonal, *etc.*, the number of dots in a figure corresponding to the number. Thus $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$ is any triangular number; $1+3+5+7+\cdots+(2n-1)=n^2$ is any square number and so on. To them is also attributed the formula $m^2+[\frac{1}{2}(m^2-1)]^2=[\frac{1}{2}(m^2+1)]^2$, where m was any odd number. But their most wonderful discovery about numbers was in showing the dependence of musical intervals upon numerical ratios. They found that for strings of the same tension, the different lengths of strings would be in the ratio of 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth. They went on to form a diatonic scale and thus initiated a study which was to be extensively elaborated by later Greeks [**57**]. The PYTHAGOREANS showed interest in astronomy, associating musical notes with the heavenly bodies (the highest with the fixed stars, the next highest with Saturn, the lowest with the Moon) and also various numbers [**58**]. The planets were thought as moving in independent circles. There is no ground for thinking that they regarded the earth as spherical [**59**].

Thus geometry, arithmetic, music, and astronomy came to be grouped together by the PYTHAGOREANS as fundamental liberal arts for study and to form what was, in the middle ages, called the *quadrivium.* The word "mathematics" is derived from a Greek word μάθημα meaning simply a "subject of instruction"; but by the time of ARISTOTLE [**60**] (−340) the term was definitely restricted to subjects of the *quadrivium.*

PLATO [**61, 62**] regarded mathematics in its four branches, arithmetic, geometry, stereometry, and astronomy, as the first essential in the training of philosophers and of those who should rule his ideal State; "let no one destitute of geometry enter my doors," said the inscription over the door of his school near Athens. He emphasized that mathematics is of value for the training of the mind, and that by comparison, its practical value is of no account. It is not known that PLATO made any mathematical discovery. **THEAETETUS**, pupil of SOCRATES and friend of PLATO, one of whose dialogues, *Theaetetus*, is a commemorative tribute, advanced the theory of irrationals and was probably the first to construct all five of the regular solids [**62**] theoretically, and to investigate fully their relations to one another, and to the circumscribed sphere as elaborated in EUCLID'S *Elements*, XIII; indeed THEAETETUS was the discoverer of the octahedron and icosahedron. In the fifth century B.C. flourished **ZENO** of Elea [63] who also does not seem to have been a mathematician, yet his paradoxes of motion had a profound influence on the course which geometry thereafter took.

The PYTHAGOREAN difficulties in connection with geometric proportion and incommensurables were completely solved in the fourth century B.C. by a pupil of ARCHYTAS and PLATO named **EUDOXUS** [64], born at Cnidos on the west coast of Asia Minor. He was an original genius second only to ARCHIMEDES. His masterly setting forth of incommensurables, as in the fifth book of EUCLID'S *Elements*, is practically identical with the modern formulation of DEDEKIND. EUDOXUS discovered also the so-called "method of exhaustion" by means of which he gave the first rigorous proof of the results for the volume of a cone, and of a pyramid, and probably showed that: (1) the areas of two circles are to one another as the squares of their diameters; and (2) the volumes of spheres are to one another as the cubes of the lengths of their respective diameters.

It seems certain that **MENAECHMUS**, a pupil of EUDOXUS, was the discoverer of the conic sections—parabola, ellipse, hyperbola, which were originally thought of as sections perpendicular to generators of right-angled, acute-angled, and obtuse-angled cones; see NEUGEBAUER, 1948, [86]. He showed that by means of the intersection of two parabolas, or of a parabola and a rectangular hyperbola so obtained, we could find two mean proportionals between two lines of lengths a and b. This implies the recognition of a geometric relation, in the case of the parabola, equivalent to our ordinary Cartesian equation, and of a similar relation for a rectangular hyperbola referred to its asymptotes, as axes.

This discovery of MENAECHMUS was of particular interest because it gave a new solution of the problem of the duplication of the cube, one of the three famous problems which had been formulated in the fifth century B.C. These problems are [65–69].

(*a*) To find a line which shall be the edge of a cube whose volume is double that of a given cube, the problem of the duplication of the cube.
(*b*) To trisect any given angle.
(*c*) To find a line which shall be the side of a square whose area shall be exactly equal to that of a given circle, the problem of squaring the circle [67].

All of these problems were solved by the Greeks within a century, but more than 22 centuries were to pass before it was finally proved that no one of them could be solved with ruler and compasses alone. As early as the fifth century B.C., by means of a curve called the quadratrix [69], the problem of the trisection of an angle was solved by **HIPPIAS** of Elis, and **HIPPOCRATES** of Chios had shown that the problem of duplicating the cube could be reduced to that of finding two mean proportionals between such lengths as a and $2a$. Such mean proportionals were found by **ARCHYTAS**, a PYTHAGOREAN, a friend of PLATO, a statesman and philosopher, in a marvellous construction by means of the intersection of a right cone, a cylinder, and an anchor ring with inner diameter zero. **DINOSTRATUS**, a brother of MENAECHMUS, showed that the quadratrix of HIPPIAS could also be used to solve the problem of squaring the circle. In the third century this problem was also solved, in effect, by a spiral invented by ARCHIMEDES [69]. In the third century a quartic curve, the conchoid of **NICOMEDES** [69], was used to solve the problems of trisecting an angle, and duplicating a cube, and the cissoid of **DIOCLES** [69], a cubic curve, was also employed for solving the latter problem.

We come now to the consideration of the golden period of Greek mathematics and of the greatest mathematical school of ancient times. This was at the magnificently endowed university of Alexandria which had been founded by ALEXANDER the Great in 332 B.C. [70]. The university was opened about 300 B.C. and within the first 40 years of its existence over 600,000 rolls had been collected in its great library. **EUCLID** [71] was a professor of mathematics in the university. Practically nothing is known about his life except that he was the author of at least ten treatises [72] of which approximately complete texts of five are available. These include three on applied mathematics, namely on phaenomena [75], on optics [76], and on music [77]. But by far the most famous one is his treatise, in 13 books, called the Elements [73, 74]. More than a thousand editions have appeared since the first one printed in 1482, and for 1800 years before that, manuscript copies dominated all teaching of geometry. Though a large portion of the subject matter had been investigated by predecessors the whole arrangement was due to the great genius of EUCLID who supplied innumerable details. It is quite impossible to give in a few sentences any adequate idea of the contents of the 465 propositions in this monumental work. Practically all of the geometric material of American school texts in plane and solid geometry is contained in parts of six of the books (1, 3, 4, 6, 11 and 12) of the Elements. Most books are prefaced by definitions, but before the first book are certain postulates the fifth of which is the famous one which differentiates Euclidean from noneuclidean geometry [78]. The statement is as follows: "Let it be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles."

In Book II of the Elements are a number of propositions giving geometric proofs of algebraic identities, such as $(a+b)^2=a^2+2ab+b^2$, and in Books II and VI are the propositions giving, among many other things, geometric solutions of quadratic equations [98], which were treated algebraically by the Babylonians. In Book V is expounded the remarkable EUDOXUS theory of proportion, alike applicable to incommensurable and commensurable magnitudes of every kind. There are many masterly Euclidean developments. Book VI applies to plane geometry the general theory of proportion set out in Book V. The 102 propositions in Books VII, VIII, IX deal with questions in the theory of numbers. Special reference may be made to three propositions in Book IX. It is elegantly proved in the 20th that there are an infinite number of prime numbers; in the 35th there is a beautiful geometric derivation of what is practically equivalent to our algebraic formula for the sum of the first n terms of a geometric progression; and in the 36th it is proved that if $S_n = 2^n - 1$ is prime, then $P = 2^{n-1}S_n$ is a perfect number [**79**], that is, a number which is equal to the sum of its divisors smaller than itself. The first four perfect numbers are 6, 28, 496, and 8128; only 12 perfect numbers are known, and they are all of the EUCLIDEAN type.

Book X contains 115 propositions on incommensurables, and is perhaps the most remarkable and most finished of all the books of the Elements. Books XI, XII, XIII deal with geometry of three dimensions. In propositions 16 and 17 of the last book all the details are carried out for the actual construction of an icosahedron, and of a dodecahedron, inscribed in a sphere.

The logical form of the presentation of the propositions in the Elements is especially notable. There is first of all the enunciation of the proposition; secondly, the statement of the precise data; thirdly, the statement of what we are required to do with reference to the precise data mentioned; fourthly, the construction, the addition when necessary of more lines to the figure; fifthly, the proof; and sixthly, the conclusion, stating what has actually been done, which generally follows the wording of the original enunciation.

In the period between EUCLID and ARCHIMEDES came **ARISTARCHUS** of Samos [**58**], whose great achievement lies in the fact that he was the first to assert that the earth and the other planets (Venus, Mercury, Mars, Jupiter, and Saturn) revolved about the sun, thus anticipating COPERNICUS by seventeen centuries.

ARCHIMEDES [**80–82**] was born at Syracuse, in Sicily, about 287 B.C. and was killed in the sack of that city by the Romans in 212. He studied with successors of EUCLID at Alexandria and it was probably in Egypt that he invented the water screw, known by his name, for drawing water to irrigate fields. On returning to Syracuse he devoted his time wholly to mathematical research. Up to the time of NEWTON at least he was the greatest mathematical genius that the world had seen. The following summary of his work is mainly due to HEATH [**22**].

In geometry his works consisted chiefly of original researches in the quadrature of plane curvilinear figures and in the quadrature of curved surfaces and the cubature of solids bounded by such surfaces. By methods equivalent to integration [**82**], ARCHIMEDES found the area of a parabolic segment, the area of a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of the second degree.

In arithmetic he calculated approximations to the value of π and in the course of these calculations showed that he could find approximate values for the square roots of large or small non-square numbers; he found $3\frac{1}{7} > \pi > 3\frac{10}{71}$, and also, apparently, $3.141697 \cdots > \pi > 3.141495 \cdots$, the mean of which is 3.141596. ARCHIMEDES also invented a system of arithmetic nomenclature by which he could express in language enormously large numbers, in fact all numbers up to that which we would write as 1 followed by 80,000 million million ciphers.

In mechanics he laid down certain postulates and, on the basis of these postulates, established certain fundamental theorems on magnitudes balancing about a point and on centers of gravity [82], going so far as to find the center of gravity of any segment of a parabola, a semi-circle, a cone, a hemisphere, a segment of a sphere, and a right segment of a paraboloid of revolution. As we learn from his *Method* [81], discovered as recently as 1906, ARCHIMEDES made most ingenious use of mechanics as an aid for suggesting probable results in geometry; the first theorem which he found in this way was in connection with the quadrature of the parabola.

He invented the whole science of hydrostatics, which he carried so far as to give a complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid with the base either upwards or downwards, but so that the base is entirely above or entirely below the surface of the fluid. HEATH then sums up as follows:

"The treatises themselves are, without exception, models of mathematical exposition; the gradual unfolding of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant, the perfect finish of the whole, combine to produce a deep impression, almost a feeling of awe in the mind of the reader."

In accordance with the desire of ARCHIMEDES the figure of a sphere inscribed in a cylinder was engraved on his tomb since he seemed to regard the discovery that the volume of the sphere was two-thirds that of the circumscribed cylinder as his greatest achievement. CICERO found the tomb with this inscription.

Other results due to ARCHIMEDES were the following: (a) our familiar formula for the area of a triangle in terms of its sides: (b) the construction of 14 inscribable semi-regular polyhedra [83]; (c) the solution of a three-term cubic equation, the term of the first degree lacking [80], by means of the intersection of a parabola and a rectangular hyperbola; this occurs in his work on the sphere and cylinder.

The tradition that ARCHIMEDES destroyed Roman ships, by concentrating sun's rays by means of a series of mirrors, was proved by experiment to be entirely feasible [84]. And finally, ARCHIMEDES was much occupied with astronomy and wrote a book on the construction of a sphere so as to imitate the motion in the heavens of the sun, the moon, and the five planets. When CICERO was quaestor in Sicily in 75 B.C., he actually saw the contrivance and gives a description of it.

Almost contemporary with ARCHIMEDES was his friend **ERATOSTHENES**, who taught in Alexandria and was librarian at the University. To him ARCHIMEDES dedicated his treatise on *Method*. The most famous scientific achievement of ERATOSTHENES was his apparently very accurate determination of the polar circumference of the earth. This was done by observing that when the sun was in the zenith at Syene, it was exactly 7° 12′ south of the zenith at Alexandria, known to be 5000 stadia distant. According to a recent interpretation [**85**], an ERATOSTHENES' device, called a sieve, for determining prime numbers, is of considerable interest.

The third great name in connection with the Alexandrian School was **APOLLONIUS** of Perga [**86**] who was about 25 years younger than ARCHIMEDES. He was called by his contemporaries the "Great Geometer," because of his extraordinary treatise on Conics, and he was also mentioned as a famous astronomer. He studied at Alexandria. Only seven of the eight books of his Conics have been preserved. The first three seem to correspond to a treatise which EUCLID wrote, but books V, VI, VII contain the discoveries which APOLLONIUS himself had made. To him are due the terms parabola, ellipse, hyperbola, for reasons which we have already explained, and his lines of reference are any diameter and a tangent at its extremity, the cases of these when at right angles being considered only as special cases. Book V treats of various questions concerning normals to conics, and from certain propositions we can readily deduce the CARTESIAN equations to the evolutes of the three conics. Book VII deals mainly with various results in connection with conjugate diameters. There are nearly 400 propositions. The suggestions that I have given are sufficient to indicate that in this work of APOLLONIUS there is far more than is contained in any of our American textbooks on analytical geometry, so far as conic sections are concerned. Foci of central conics are constructed, and that the sum or difference of focal radii of points of such curves is constant, is proved. There is no reference to the focus of the parabola. It seems likely that the focus-directrix property of all three conics was given in EUCLID's Conics, or well known in his time, although it is not explicitly mentioned before PAPPUS, hundreds of years later [**86a**].

Among a number of other works of APOLLONIUS we shall refer to only three. In his Plane Loci the following familiar results were given: (*a*) If A, B be fixed points, P any other point in the plane, and AP and BP are in a given ratio, the locus of P is a straight line or a circle (circle locus given earlier by ARISTOTLE) according as the given ratio is or is not one of equality; (*b*) If A, B, C, $\cdots$ be any number of fixed points and α, β, γ, $\cdots$ any constants, the locus of point P, such that $\alpha\cdot AP^2+\beta\cdot BP^2+\gamma\cdot CP^2+\cdots=$a constant, is a circle.

The second work, on Contacts or Tangencies, has not come down to us, but its principal problem is to describe a circle tangent to three given circles, which is usually known as the problem of APOLLONIUS. Many celebrated mathematicians, such as VIETA, EULER and NEWTON, have worked on this problem, and what is known of the work as a whole has made it the basis of numerous restorations and discussions [**87**]. The analogous form for spheres was treated synthetically by FERMAT. From the Plane Loci, and the account of Tangencies by PAPPUS, it has been deduced that APOLLONIUS made the following result, in effect, the basis of his solution of the general tangencies problem: The six centers of similitude of three coplanar circles lie by threes on four straight lines [**88**].

The third work, on Vergings, is also lost, but some of its problems and solutions have been preserved. One of these problems is: Between the side of a given rhombus and its adjacent side produced, insert a straight line of given length and verging to the opposite corner. HUYGENS gave many solutions and NEWTON used the particular case of a square in illustration of discussion [89]. The work of APOLLONIUS in Astronomy was also of importance [89].

We have already seen the marvellous development of geometry by the Greeks from the beginning up to difficult problems of the integral calculus, and all of this within the short space of 350 years.

Possibly contemporary with scientific activity among the Greeks of the fourth and fifth centuries B.C. are two notable Babylonian astronomers [90], **NABURIANU** and **CIDENAS** or **KIDINNU**, who had at their disposal an extraordinary number of Babylonian observations of eclipses for more than three centuries. The next name of mathematical interest among the Greeks is that of their most eminent astronomer, **HIPPARCHUS** [91], who flourished about 140 B.C. From the great mass of data available from about 750 B.C., it was almost inevitable that he should be the first to note the precession of the equinoxes [92]; but it was not till the time of NEWTON that the explanation of this precession made it a matter of real importance. In masterly fashion HIPPARCHUS improved on ARISTARCHUS's calculations of the sizes and distances of the sun and moon, and developed a systematic theory with regard to them. He also compiled a catalogue of 850 stars [93], stating their places and apparent sizes. Trigonometry as a science seems to have begun with him, and also the introduction into Greece of the division of the circle [94] into 360°.

About two centuries after HIPPARCHUS we come to **HERON** [95] of Alexandria whose date, until comparatively recently, has been a matter of uncertainty, even to the extent of 400 years [96]. He was an almost encyclopedic writer on mathematical and physical subjects, and aimed at practical utility rather than theoretical completeness. In his *Pneumatica* are many mechanical devices such as a siphon, a fire engine, a device whereby temple doors are opened by fire on an altar, an altar organ blown by the agency of hand labor or by a windmill, and a jet of steam supporting a sphere [314].

He gave an elegant geometric proof of the formula for the area of a triangle in terms of the sides, now attributed to ARCHIMEDES. His method of approximating to the square root of non-square numbers [97] seems to have been used by early Babylonians. His formula for the volume of the frustum of a square pyramid can be readily reduced to the one used 2000 years earlier in the Moscow papyrus [42].

Quadratic equations are solved in a manner very similar to that of the Babylonians 1900 years before [98]. HERON showed how he obtained the cube root of a non-cube number, 100. He found also the volume of the five regular solids, and of an anchor ring (as given by DIONYSODORUS, a contemporary of APOLLONIUS of Perga). His works on surveying, and the *Dioptra* are of interest, and it is by means of the latter that his date has been narrowed down.

Modern research seems to indicate that **DIOPHANTUS** [99] of Alexandria, one of the great mathematicians of Greek civilization, should be assigned to the first century of our era [100], rather than to the third century. He was the first to make systematic use of symbols in algebraic work, a sign for unknown, a sign for minus, and signs for the various powers, that is, square, cube, and so on. Of his great work called Arithmetic only 6 of its thirteen books survive and they are mostly taken up with problems in indeterminate analysis of the second degree. Hence the term Diophantine Analysis [101]. The answers are always in positive rational numbers. The collection is extraordinarily varied and the devices resorted to are highly ingenious. In Book V, 9, 11, not only does he find particular solutions of $x^2-26y^2=1$ and $x^2-30y^2=1$, but also shows (VI, 15, lemma) that they have an infinite number of solutions. DIOPHANTUS may have discussed the equation $x^2-Ay^2=1$ more fully in the part of the Arithmetic which has been lost. See later under BHĀSKARA. It has been surmised that the last seven books of the Arithmetic may have been filled with material between the most difficult of DIOPHANTUS' problems and the famous cattle problem of ARCHIMEDES.

A possible contemporary of HERON was **MENELAUS** of Alexandria [102], who wrote a variety of treatises among which his *Sphaerica* [103] contained for the first time the conception and definition of a spherical triangle; here also are the MENELAUS theorem for the sphere, and deductions from it, furnishing the equivalent of formulae in spherical trigonometry.

Much of our knowledge of the achievements of HIPPARCHUS is derived from a work of **CLAUDIUS PTOLEMY** [104] of Alexandria, who flourished in the second century of our era. This great work, of extraordinary compactness and elegance, was called the *Almagest*, which overshadowed all previous works of the kind. In the trigonometry of the early part is derived a table of chords equivalent to a table of $\sin A$ for each $15'$ of the quadrant, the values being expressed in parts, minutes and seconds.

In the first book of the *Almagest* is PTOLEMY's theorem regarding a quadrilateral inscribed in a circle, and from this $\sin^2 A+\cos^2 A=1$, and the well-known formulae for $\sin (A\pm B)$ and $\cos (A\pm B)$, among others, are readily derived [105]. In the fourth book occurs a solution of the so-called "Problem of SNELL" (1617), or of POTHENOT 1692 (publ. 1730): to determine the point from which pairs of three given points are seen under given angles [105].

In this same work results of much of PTOLEMY's observational work are contained in his catalogue of 1028 stars, with the latitude and longitude of each [106]. But his outstanding achievement was to develop a lunar theory and a planetary theory, close to KEPLER's, with planetary movements in eccentric orbits almost elliptical.

PTOLEMY used various projections including stereographic. The essentials of his argument in an attempted proof of EUCLID's parallel postulate have been preserved. It is recorded that he wrote also a work on dimension in which he attempted to prove that the possible number of dimensions is only three.

PTOLEMY wrote also a remarkable geographical treatise with maps, various later printed editions of which are treasured by libraries [**106a**]. His work on Optics is especially interesting because it contains the first attempt at a theory of refraction [**106b**]. He also wrote a notable treatise on music [**106c**].

And finally we come to **PAPPUS** [**107**] of Alexandria who lived at the end of the third century. His great work, entitled *Mathematical Collection,* [**108**] covers the whole range of Greek geometry and was intended "to be read with the original works (where extant) rather than independently. Where, however, the history of a subject is given, *e.g.*, that of the duplication of the cube, or the finding of the two mean proportionals, the solutions themselves are reproduced, presumably because they were not readily accessible but had to be collected from scattered sources. Even when some accessible classic is being described, the opportunity is taken to give alternative methods or to make improvements in proofs, extensions and so on. Without pretending to great originality, the whole work shows, on the part of the author, a complete grasp over the subjects treated, independence of judgment, and mastery of technique; the style is concise and clear; in short PAPPUS stands out as an accomplished and versatile mathematician and a worthy representative of the classical Greek geometry" [**22**]. Among the original contributions of PAPPUS are—(*a*) a generalization of the "Pythagorean" theorem to any triangle with certain parallelograms on its sides; (*b*) generalization of the four-line locus to five- or six- or *n*-line loci [**86**]; (*c*) proof of the invariance of the cross-ratio under a projective transformation; and (*d*) a result often attributed to GULDIN [**109**], a Swiss mathematician of the seventeenth century: The volume of a solid of revolution is equal to the area of the generating figure, multiplied by the circumference described by the center of gravity of the figure. PAPPUS was also the first to give results possibly known earlier: (*e*) the construction of a conic through five points [**86**]; and (*f*) the presentation of the focus-directrix property of the three conics [**86a**].

The fifth book is mainly devoted to the subject of isoperimetry. There is a very interesting passage concerning bees, their orderliness, and the hexagonal form of their cells exactly filling up space about a point [**110**]. It is later shown that (*a*) a circle is greater than any regular polygon of equal contour; (*b*) a sphere is greater than any of the five regular solids with equal area. The sixth book is mostly astronomical [**111**], dealing with treatises studied as an introduction to the great *Almagest* of PTOLEMY. There are, however, propositions of mathematical interest including some related to EUCLID's *Optics* [**76**]. The seventh book, *On the Treasury of Analysis,* is historically the most important of all the books, since it gives an account of a collection of treatises which, after the Elements of EUCLID, provided the body of doctrine necessary for the professional mathematician to know if he was to be regarded as fully equipped for the solution of problems arising in geometry. Of the 12 works listed seven are by APOLLONIUS and three by EUCLID.

THEON [**111a**] of Alexandria, a fourth century author of an edition of Euclid's Elements, and a commentator on works of EUCLID and PTOLEMY, is a name of importance in the history of mathematics. In the fifth century, **PRO-**

CLUS, more of a philosopher than a mathematician, placed the historian under great obligation by his commentary on the first book of EUCLID's elements, which is one of our main sources of information on the history of elementary geometry. [**313**].

In the early part of the sixth century JOHN PHILOPONUS of Alexandria wrote the earliest extant Greek treatise on the Astrolabe [**111b**], which played an important role in the history of astronomy and of a mathematical projection.

Alexandria was razed by the Arabs in the seventh century. Contrary to the imaginings of some writers, no library of importance was then destroyed, or even in existence there [**112**].

For the period of history which we have so far been considering, it is to be noted that, in the case of the Babylonians and Egyptians, the actual source material written two to four thousand years ago is available for study. On the other hand, in the case of the Greeks practically not a single source manuscript is dated earlier than a thousand years after the writer flourished. In spite of this great handicap, scholars feel that some reliable complete or partial original texts of works of such writers as EUCLID, ARCHIMEDES, and APOLLONIUS are now available [**313**].

C. HINDU [113–116], ARABIC, PERSIAN MATHEMATICS 600 to 1200

With PAPPUS creative Greek mathematicians came to an end, and very soon all the great traditions of Greek learning had died out; then followed nearly a thousand years in which slight additions to the sum of mathematical knowledge were made in Europe. Significant contributions came from India and the Arabs.

In this period the Arabian Empire stretched from India to Spain, Bagdad and Cordova being the centers for the reigning caliphs [**117**]. In extraordinary fashion Arabs at Bagdad developed profound interest in translated Greek mathematical and astronomical works, and also in similar Hindu material. The ninth and tenth centuries may be regarded as the golden age of Arabian mathematicians [**118**], to whom the world owes a great debt for preserving, and thus making possible the transmission to Europe, of classics in Greek mathematics otherwise lost. A vast field for further investigation in this connection awaits the attention of scholars.

The formula for the area of an inscribed quadrilateral, similar in form to that for a triangle, ascribed to ARCHIMEDES, was first given in the early part of the seventh century by an outstanding Hindu mathematician, named **BRAHMAGUPTA** [**120**], but it was not recognized as true for a cyclic figure only. If a, b, c, and d are the lengths of the sides of the quadrilateral, $s=\frac{1}{2}(a+b+c+d)$, and m and n are the lengths of the diagonals, the area is equal to $\sqrt{[(s-a)(s-b)(s-c)(s-d)]}$, $m=\sqrt{[(ab+cd)(ac+bd)/(ad+bc)]}$, and $n=\sqrt{[(ac+bd)(ad+bc)/(ab+cd)]}$. These formulae were considered with a view to determining quadrilaterals whose sides, diagonals, and areas were all rational quantities. In particular BRAHMAGUPTA gave the rule: If $a^2+b^2=c^2$ and $\alpha^2+\beta^2=\gamma^2$, then the quadrilateral ($a\gamma$, $c\beta$, $b\gamma$, $c\alpha$) is cyclic, the area is rational, and its rational diagonals are at right angles. BHĀSKARA pointed out that similarly rational is such an inscribed quadrilateral, if a pair of adjacent sides are interchanged; he also indicated how to determine the length of the third diagonal, no longer orthogonal to the other diagonal of the new quadrilateral. BRAHMAGUPTA also noted, in terms of any three rational numbers, the formulae for the lengths of sides of an oblique triangle whose altitudes and areas are rational [121].

We have already seen that the PYTHAGOREANS were led to solutions of the equations $x^2-2y^2=\pm 1$, in getting approximations to $\sqrt{2}$. BRAHMAGUPTA and BHĀSKARA [120, 122] (of the twelfth century) gave the method for finding remarkable particular solutions of the equations $x^2-Ay^2=1$, $A=8$, 61, 67, and 92; and the latter found general solutions [123]. BRAHMAGUPTA's rules lead, in effect, to the general solution of $ax+by=c$, a, b, c integers, and a and b relatively prime, as $x=\pm(cq-bt)$, $y=\pm(-cp+at)$, where t is zero or any integer, and p/q is (in modern phraseology) the second last convergent of a/b, expressed as a continued fraction. These are the most important developments in Hindu mathematics of this period.

Let us now outline the origin of our common numerals [124]. So far as we know the largest number of our numeral forms were first used in India [125]. The 1, 4, and 6 are found in inscriptions of the third century B.C.; the 2, 4, 6, 7, and 9 appear in another inscription about a century later; and the 2, 3, 4, 5, 6, 7, and 9 in caves of the first and second centuries of our era, all in forms which have a considerable resemblance to our own.

The first definite external reference to Hindu numerals is in a note by a bishop who lived in Mesopotamia about 650 A.D. Early in the ninth century the numerals became known to Arabic scholars. Indeed, one of the most prominent of these, MOHAMMED IBN MÛSÂ AL-KHOWÂRIZMÎ [126] (that is, of Khowârizmî, now Khiva, in Russian Turkestan) worked at Bagdad, and wrote a treatise on Hindu numeration and arithmetic, which became known to Europeans through a twelfth century Latin translation. The earliest undoubted occurrence of zero in India [127] was in an inscription of 876, in connection with the numbers 50 and 270. But it was used much earlier, indeed by HYPSICLES (-180) and later Greek astronomers, from whom Hindus probably borrowed the sign. A somewhat similar sign is also in astronomical records of the Mayas of Central America [128], which may date back to the beginning of our era; they had a vigesimal system with the principle of local value. About -400 the Babylonians possessed both the principle of relative local value, and a zero symbol which was used systematically in astronomical texts and computations. In Europe the complete system of numerals with the zero was derived from the Arabs in the twelfth century.

We have referred to the astronomer and mathematician AL-KHOWÂRIZMÎ (the first of several mathematicians often referred to by their place of birth). He wrote a work on algebra with the title "ḥisāb alǧabr walmuḳābalah," which has been translated "the science of reduction and cancellation." The Arabic word for reduction, "alḡabr," thus became our word alḡebra. The English word algorism is simply a corruption of AL-KHOWÂRIZMÎ, as in the Latinized title form *Algoritmi di numero indorum*, of AL-KHOWÂRIZMÎ's arithmetic. The algebra contained sections on geometry, quadratic equations solved geometrically [**129**], and problems of inheritance. But most of AL-KHOWÂRIZMÎ's work was in astronomy, and among 100 of his tables in this field are tables of sines and cotangents [**130**], the former doubtless indicating Hindu influence. For already in the fifth century Greek trigonometry operating with chords was transformed by the Hindus into consideration of half-chords or sines. Thus we there have a table [**131**] of 3438 sin A, where the radius $=3438$.

From the time of the PYTHAGOREANS, constructions of elementary geometry were imagined as carried through with a ruler and compasses with variable opening. A prominent Arabian mathematician of the tenth century, **ABÛ'L WEFÂ** [**132**], lectured on geometric constructions with a ruler and compasses with a fixed opening [**133**]. It was not until centuries later that it was recognized that there was no real limitation here, since it was shown by PONCELET and later by STEINER, in the early part of the nineteenth century, that if a single circle and its center are once drawn in a plane, every construction with ruler and compasses can be carried through with ruler alone.

ABÛ'L WEFÂ contributed notably to the development of trigonometry. Already in his time all six of the trigonometric functions were in use, and he employed our formulae for versed sine, for the tangent and cotangent in terms of sine and cosine, and for the sin of an angle in terms of the sine and cosine of the half angle. PTOLEMY's table of sines for each $\frac{1}{2}°$ was extended by him to each $\frac{1}{4}°$, and he made a similar table for tangents. He found the value of sin $30'$ correct to the equivalent of eight places of decimals [**134**], but his great contributions were in spherical trigonometry where he first used our formulae for a right spherical triangle and also our law of sines formula for an oblique spherical triangle.

In concluding the period under consideration we should make brief reference to a remarkable work of the Persian mathematician, astronomer, poet, **OMAR KHAYYÂM** (that is, OMAR the Tentmaker) [**135**], known to the western world as the author of the Rubaiyat [**136**]. He wrote a treatise on algebra in which his geometric treatment of systematically classified cubic equations is central [**137**]. He obtains a root as the abscissa of a point of intersection of a conic and a circle, or of two conics. Various forms of cubic equations are considered; negative roots are rejected, and not all positive roots are discovered. While we have already noted the solution of cubic equations by MENAECHMUS and ARCHIMEDES it is to be remarked that here the point of view is different; the problem is: How can we solve cubic equations with numerical coefficients? Another notable achievement of OMAR was his correction of the calendar by the introduction of cycles of 33 years. This calendar was more accurate than the one we use today.

In order to round out our survey of Islamic contributions to the development of trigonometry we append to the treatment of this period some remarks on contributions of two men of later date. The first is **Nasîr ed-Dîn al-Tûsî** [138], exceptionally able astronomer, mathematician and politician at Bagdad in the thirteenth century. Displaying remarkable scientific thoroughness, and power to integrate and extend earlier discoveries, he wrote the first complete plane and spherical trigonometry independent of any astronomical application [139]. In his discussion the theorem of Menelaus is basic. All six trigonometric functions are used, and necessary formulae for solving right, and all cases of oblique, spherical triangles are given. Such was the status of trigonometry at the close of the thirteenth century. A recently discovered fourteenth century Arabic commentary by Nasîr ed-Dîn suggests that it contains the first part of Omar Khayyàm's discussion of difficulties of Euclid [140]. This part treats material similar to introductory propositions in the noneuclidean geometry of Saccheri, hundreds of years later.

The second supplementary name is that of **Ulugh Beg** [141], a fifteenth century Persian prince and astronomer, who compiled extraordinary tables of sines and tangents for every minute and correct to the equivalent of eight or ten places of decimals [142]. In calculating the sine table he was led approximately to solve the cubic equation of the angle-trisection problem [143]: given sin 3°, to find sin 1°.

D. EUROPEAN MATHEMATICS 1200 TO 1600

During the period 500 to 1200 the student went to the teacher in the monastery and heard his lectures. But in the thirteenth century universities commenced to spring up at such places as Bologna, Padua, Naples, Paris, Oxford and Cambridge. Scribes making copies of treatises were thus kept busily employed by the universities. By the middle of the fifteenth century, however, their products were being sold as books are today. But such methods of disseminating knowledge were crude when compared with that of the distribution of the printed work. The publication of these with movable type commenced about 1450. More than two hundred mathematical works were printed, in Italy alone, before 1500; but this number was increased to 1527 in the next century.

During three and a half of the four centuries now under consideration Italy made the chief contributions to mathematics, and by far the most outstanding mathematician was one who flourished at the beginning of this period, **Leonardo** of Pisa [144], often called Fibonacci, that is, son of Bonaccio. During early life he travelled extensively about the Mediterranean, visiting Egypt, Syria, Greece, Sicily, and southern France, and knowledge thus gleaned regarding arithmetic systems used by merchants of different countries was the basis of a notable work, entitled *Liber Abaci* [145], which he wrote in 1202. This is a storehouse from which for centuries authors got material for works on arithmetic and algebra. The Hindu-Arabic system of numerals was here strongly advocated and illustrated, and the work did much to introduce it into Europe. Leonardo discussed problems in arithmetic processes, barter, alligation, false position, square and cube roots. It is in this work that we find the problem: "7 old women went to Rome; each woman had 7 mules; each mule carried 7 sacks; each sack contained 7 loaves; and with each loaf were 7 knives; each knife was put up in 7 sheaths. What is the sum total of all named?" It was this problem which gave the clue to the interpretation of a problem in the RHIND papyrus 2800 years earlier, "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, and each ear of spelt will produce 7 hekat of grain; how much grain is thereby saved?" The modern conundrum starting out: "As I was going up to St. Ives I met a man with 7 wives, each wife had 7 sacks," etc., further illustrates the perpetuation of this number succession through the centuries. An interesting chapter on similar perpetuation of other problems has been written [**146**].

In *Liber Abaci* is also the following: How many pairs of rabbits can be produced from a single pair in a year if it is supposed: (a) that every month each pair begets a new pair which from the second month on becomes productive; and (b) that deaths do not occur? In this way we are led to the famous recurrent FIBONACCI series: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$, in which each term after the second is the sum of the two preceding [**147**].

Among many other things the work contains also a proof of the well-known algebraic identity expressing the product of the sums of two squares as the sum of two squares: $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2$.

LEONARDO wrote two other important works: *Liber Quadratorum* [**148**] in 1220 and *Practica Geometriae* in 1225. The first of these is a brilliantly written, original, and able work on indeterminate analysis, stamping the author as the outstanding mathematician in the field from the time of DIOPHANTUS to the time of FERMAT over 400 years later. The great *Practica Geometriae* brings together a vast amount of material in geometry and trigonometry, and it would seem as if some works of the ancients now lost had been still available to LEONARDO. In particular this seems to have been true of EUCLID's work on the Division of Figures [**72**].

LEONARDO's great reputation led to a sort of mathematical tournament when JOHN of Palermo presented three problems which LEONARDO solved. The second of these was to find the solution of the equation $x^3+2x^2+10x=20$, the value for which, given without discussion in his *Flos*, correct to 10 places of decimals, has excited much wonder. Various surmises have been made as to his method of arriving at this result evidently based on Arabic methods [**149**].

Since we have already referred to NASÎR ED-DÎN [**138**] in the latter part of the thirteenth century, and ULUGH BEG [**141**] in the fifteenth century, we shall skip over the period of about 250 otherwise barren years and come to a German named **JOHANN MÜLLER**, born near Königsberg, Lower Franconia, and as a result known in the history of mathematics as **REGIOMONTANUS** [**150**]. He was the most influential and best-known German mathematician of the fifteenth century. We shall simply note that his work on trigonometry, written about 1464, but posthumously published in 1533, was the first European systematic exposition of plane and spherical trigonometry; the only functions here introduced are sines and cosines [**151**]. The work had great influence in establishing trigonometry as a science independent of astronomy. The period of REGIOMONTANUS is also that of the Italian, **LUCA PACIOLI**, as well as that in which two mathematical works were printed, namely the first dated arithmetic [**152**] (Treviso, 1478), an anonymous commercial work, and the first edition of EUCLID's *Elements*, a Latin translation by CAMPANUS (Venice, Ratdolt, 1482) [**153**]. PACIOLI [**154**] published two notable works, the first, usually referred to as *Sūma* (Venice, 1494), a great mathematical compilation drawn almost wholly from a variety of sources, and containing the first account of double-entry bookkeeping [**155**]; the second, *De divina proportione* (Venice, 1509), of special interest from the geometrical point of view, since the "divine proportion" here discussed was called "golden section" in the nineteenth century.

In the sixteenth century the chief Italian achievement was the solution of equations of the third and fourth degrees. The facts with regard to the general solution of the cubic equation seem to be as follows:

SCIPIONE DEL FERRO, a professor of mathematics at the University of Bologna, solved the equation $x^3+mx=n$ in 1515, imparting the result to his pupil **ANTONIO FIOR**, without publication. About 1535 **NICOLO** of Brescia who stammered badly because of an injury received as a child, and was therefore called **TARTAGLIA**, the stammerer, discovered the solution of the cubic $x^3+px^2=n$ as well as that of FERRO's form. In a public contest provoked by FIOR who had been suspicious of TARTAGLIA's achievements, TARTAGLIA triumphed completely [**156**]. Under a pledge of secrecy TARTAGLIA confided his method of solution to **GIROLAMO CARDANO** [**157**], a genius of great ability and the foremost Italian mathematician of his time, who taught mathematics and practised medicine in Milan. By far the most notable of the numerous works which CARDANO wrote was his *Ars Magna* published at Nürnberg, Germany, in 1545. This was the first great Latin treatise devoted solely to algebra. It was here that TARTAGLIA's general solution of the cubic equation was first published without TARTAGLIA's consent. Here, too, appeared the first solution of the general biquadratic equation [**158**]; this had been found by **FERRARI**, a pupil of CARDANO.

Already at this time the astronomers were feeling the need of trigonometric tables. After twelve years of incessant labor with computers, two works of extraordinary merit, and of value even to the present day, were finally prepared by **RHETICUS** or GEORG JOACHIM of Rhaetia, but not published until (1596, 1613) after his death (1576). One of the works was a ten-place table, with differences, of all six of the trigonometric functions, at interval 10″; the other was a fifteen-place table of natural sines, to every ten seconds of arc, with first, second, and third differences [**159**]. RHETICUS was the first: (i) to define trigonometric functions in connection with the ratios of sides of a right triangle; (ii) to employ the semiquadrantal arrangement of trigonometric tables. His table of secants was also the first table of the kind. RHETICUS was the leading mathematical astronomer in Teutonic countries in the sixteenth century. From 1539–

1542 he was a disciple of the Polish astronomer **Copernicus** [**160**] who after thirty-six years of labor had completed his great work *De Revolutionibus Orbium Coelestium* [**161**] whose publication in 1543 was due to importunities of Rheticus, like the *Principia* of Newton because of Halley. The Copernican Theory thus formulated had a great effect on thought of the time. We have seen that it had been put forward by **Aristarchus** about 1800 years earlier. A very influential sixteenth century English mathematician, **Robert Recorde** [**162**], was one of the first to bring the Copernican system to the attention of English readers.

The most influential of all the mathematicians in the Netherlands in the sixteenth century was **Simon Stevin** [**163**], famed in his time for his contributions to statics and hydrostatics. But he published the first compound interest tables [**164**] (1582), and in a tiny publication of 1585 he was the first to give a systematic explanation of decimal fractions. Here also he advocated the decimal division of the degree [**164**].

And finally we come to the greatest of all French mathematicians of the sixteenth century, **François Vieta** [**165**], a lawyer and Royal Commissioner who devoted most of his leisure to mathematics. His collected mathematical works form a considerable volume. He contributed extensively to the development of algebra and trigonometry [**166**]. He was among the first to employ letters to represent numbers in algebra, using vowels for the unknowns, and consonants for the knowns. He found the formula for $\cos n\phi$ in terms of $\cos \phi$ for any natural number n and wrote it down for values of n up to $n=9$; made an advance towards proving that a polynomial of the nth degree is made up of n linear factors; showed how to increase, decrease, multiply, or divide the roots of the equation $f(x)=0$, by k; gave the earliest evaluation of π as an infinite product [**167**]; applied algebra to geometry in such a way as to lay a foundation for analytic trigonometry; indicated powers more simply than his predecessors had done, using *A quadratum* or *A quad* for the square of the unknown, *A cubum* or *A cub* for its cube, *A quad quad* for its fourth power, and so on; and showed clearly the relation between the problems of the trisection of an angle and the solution of a cubic equation [**65, 143**].

In concluding our notes on mathematical developments during the 4700 years ending with the sixteenth century, we may draw attention to the fact that the first work on mathematics printed in the New World appeared at Mexico City in 1556, within 64 years of the discovery of America. It was the *Sumario Compendioso* by **Juan Diez** [**168**], and contained (*a*) tables intended to assist merchants in buying gold and silver, (*b*) an arithmetic suited to needs of apprentices in counting-houses, and (*c*) a few pages of algebraic problems, chiefly relating to quadratic equations.