Two Functions Whose Powers Make Fractals

Marc Frantz

1. AN OLD FRIEND f. In [**5**] Richard Darst and Gerald Taylor investigated the differentiability of functions f^p (which for our purposes we will restrict to $(0,1)$) defined for each $p \geq 1$ by

$$f^p(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n^p & \text{if } x = m/n \text{ with } (m,n) = 1. \end{cases}$$

These functions are powers of the function $f = f^1$, which frequently appears in real analysis courses because of its interesting property of being discontinuous at rational points and continuous at irrational points. Darst and Taylor showed that if $1 \leq p \leq 2$, then f^p is nowhere differentiable, and if $p > 2$, then f^p is differentiable almost everywhere. Graphs of f^1 and its variants (Figure 1) suggest there may be fractals involved somewhere, and indeed this is the case.

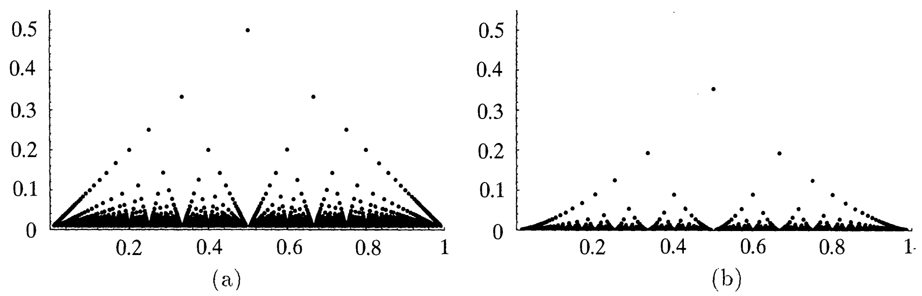

Figure 1. (a) The graph of f^1. (b) A graph of f^p for $p > 1$.

In this article, "fractal" means a subset of $\mathbb{R}^n$ for some n, whose Hausdorff dimension is not an integer. For a discussion of Hausdorff measure and dimension see [**6**, Chapter 1]. Although the graphs of the functions f^p are not themselves fractals in this sense, they do display an interesting kind of self-similarity. To be precise, if G_p is the graph of f^p for some $p \geq 1$, and the functions L_p, R_p, and T_p are defined on $[0,1] \times [0,1]$ by

$$L_p(x,y) = \left(\frac{x}{1+x}, \frac{y}{(1+x)^p}\right), \quad R_p(x,y) = \left(\frac{1}{1+x}, \frac{y}{(1+x)^p}\right),$$
$$T_p(x,y) = (1/2, 1/2^p),$$

then $L_p(G_p)$, $R_p(G_p)$, and $T_p(G_p)$ are the left half, right half, and top point, respectively, of G_p. This is left as an exercise to the reader. In fact, the functions L_p, R_p, and T_p can be used to sketch the graphs G_p by an iterative procedure essentially identical to the iterated function system method made popular by Michael Barnsley in [**1**]. For more information on graphing, see the remarks at the end of this article.

The real fractals show up when we deal with differentiability. Fortunately, we can avoid the usual difficult estimates for Hausdorff dimension by employing theorems from the theory of rational approximation of real numbers; this theory is closely related to the differentiability of the functions f^p. In the language of that theory, an irrational number x is said to be *approximable by rationals to order p* [7, p. 88] if there exists a positive constant c, depending only on x, such that the inequality

$$|x - m/n| < cn^{-p} \tag{1}$$

has infinitely many rational solutions m/n with $n > 0$ and $(m, n) = 1$. Assuming that $x, m/n \in (0, 1)$, we have $f^p(x) = 0$ and $f^p(m/n) = n^{-p}$, so condition (1) is equivalent to the existence of a positive constant c, depending only on x, such that

$$\left|\frac{f^p(x) - f^p(m/n)}{x - m/n}\right| = \frac{n^{-p}}{|x - m/n|} > \frac{1}{c} \tag{2}$$

for infinitely many rationals m/n, arbitrarily close to x. But x, being irrational, is a minimum point of f^p, so $(f^p)'(x)$ exists if and only if $(f^p)'(x) = 0$. Hence (2) implies that $(f^p)'(x)$ does not exist, and the reverse implication is clearly true as well. We have proved

Lemma 1. *An irrational number $x \in (0, 1)$ is approximable by rationals to order p if and only if f^p is not differentiable at x.*

We refer to the set of points at which f^p is nondifferentiable as Nondiff f^p, and we denote the Hausdorff dimension of a set A by dim A.

Theorem 1. *For $p > 2$,* $\dim \text{Nondiff } f^p = 2/p$.

Proof: It suffices to prove the theorem for the set N_p of irrationals in Nondiff f^p. To show that dim $N_p \geq 2/p$, we need only exhibit a subset A of N_p such that dim $A = 2/p$. The existence of such a set was established by Jarnik ([**6**, Theorem 8.16(a)], [**8**]), who proved that if $p > 2$ and A is the set of irrational numbers x such that

$$|x - m/n| \leq n^{-p}$$

for infinitely many rationals m/n, then dim $A = 2/p$. By Lemma 1, f^p is not differentiable at any such x, so $A \subset N_p$.

To show that dim $N_p \leq 2/p$, fix an arbitrary ε with $0 < \varepsilon < p - 2$. If $x \in N_p$, then there must exist a positive constant $c = c(x)$ such that for infinitely many rationals m/n,

$$|x - m/n| \leq c/n^p \leq n^{\varepsilon}/n^p,$$

the latter inequality being true for sufficiently large n. Thus for all $x \in N_p$, we have $|x - m/n| \leq n^{-(p-\varepsilon)}$ for infinitely many n, so Jarnik's theorem implies that dim $N_p \leq 2/(p - \varepsilon)$. Since ε was arbitrary, it follows that dim $N_p \leq 2/p$. ■

It is hopeless to look for second derivatives of the functions f^p, but we can come close. For sufficiently large p, the related limit

$$f^p_*(x) = \lim_{h \to 0} \frac{f^p(x+h) - 2f^p(x) + f^p(x-h)}{h^2} \tag{3}$$

exists, and fractals are again involved.

Theorem 2. *Let $N_p^* \subset (0,1)$ be the set of points x for which $f_*^p(x)$ does not exist. Then for $p \geq 2$, $N_p^* =$ Nondiff $f^{p/2}$. Thus, by Theorem 1, $\dim N_p^* = 4/p$ for $p > 4$.*

Proof: It should be clear that $f_*^p(x)$ exists if and only if x is irrational and $f_*^p(x) = 0$. Furthermore, since $f^p(x) = 0$ for every irrational x, the quotient in (3) is nonzero only if one of $x + h$ or $x - h$ is rational, and it must be exactly one of these, for if both were rational, then x would be rational, being the average of two rational numbers. Hence the quotient in (3) is nonzero if and only if exactly one of $x + h$ or $x - h$ is equal to m/n for $(m, n) = 1$, and then

$$\frac{f^p(x+h) - 2f^p(x) + f^p(x-h)}{h^2} = \frac{1/n^p}{(x - m/n)^2}$$
$$= \left(\frac{1/n^{p/2}}{x - m/n}\right)^2 = \left(\frac{f^{p/2}(x) - f^{p/2}(m/n)}{x - m/n}\right)^2.$$

Thus $f_*^p(x)$ fails to exist precisely when $x \in$ Nondiff $f^{p/2}$. ∎

2. A RELATED APPLICATION. When pondering counterexamples such as f^1 and its powers, it is natural to wonder if such things might apply to the real world. In fact, graphs like those in Figure 1 appeared some time ago in an applied setting (Figure 2).

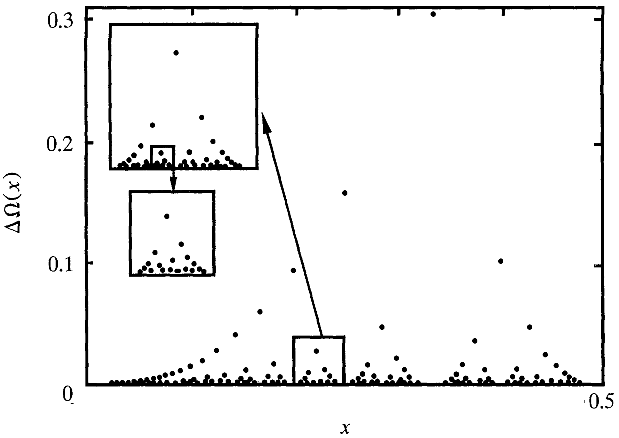

Figure 2. The function $\Delta\Omega$. After Jensen, Bak, and Bohr [**10**].

Jensen, Bak, and Bohr [**10**] made use of the graph shown in Figure 2 (cf. [**10**, Fig. 5]) to help illustrate the behavior of an important dynamical system—the so-called critical circle map—which is used in the analysis of many physical systems, ranging from the driven, damped pendulum to phenomena in condensed-matter physics (cf. [**10**]). Figure 2 shows the left half of the graph of a function $\Delta\Omega : [0, 1] \to \mathbb{R}$ that was introduced in [**10**]. Notice that this graph bears a striking resemblance to the left half of Figure 1(b). In both cases the right half is the mirror image of the left. The interpretation of the graph is roughly as follows. The independent variable x (the winding number) describes the periodic or nonperiodic behavior of the system, and x in turn is determined by a parameter Ω (not depicted in the figure), which can represent some physical property of the system. If this parameter changes, what happens to x? That depends on $\Delta\Omega(x)$. If, say,

$\Delta\Omega(x) = 0.12$, then Ω can vary over an interval of length 0.12 without affecting x; the system is then called "mode-locked". On the other hand, if $\Delta\Omega(x) = 0$, then the slightest variation in Ω will cause x to change to a new value, and the behavior of the system changes. No exact formula is known for $\Delta\Omega(x)$, but it is known that $\Delta\Omega(x) = 0$ for each irrational $x \in [0,1]$ and $\Delta\Omega(m/n) > 0$ for each rational $m/n \in [0,1]$. Moreover, numerical evidence presented in **[10]** suggests that $0 \leq \Delta\Omega(x) \leq Af^{p_1}(x)$ for all $x \in (0,1)$, where A is a positive constant and $p_1 > 2$. If this is true, then the result of Darst and Taylor—as well as Theorem 1—imply that $\Delta\Omega$ is differentiable almost everywhere on $[0,1]$. Although it is known that $\Delta\Omega$ cannot be equal* to Af^p for any constants A and p, further numerical results in **[10]** suggest that $Af^{2.29}$, for some constant A, could serve as a rough model for $\Delta\Omega$. In view of the tractability of the functions f^p, and in view of interest in the Hausdorff dimension of sets related to $\Delta\Omega$, this could be a promising avenue of investigation; see **[2]**, **[3]**, **[9]**, **[10]**.

The function $\Delta\Omega$ is, in fact, a step-width function for a devil's staircase associated with the critical circle map (Figure 3). The function $W:[0,1] \to \mathbb{R}$,

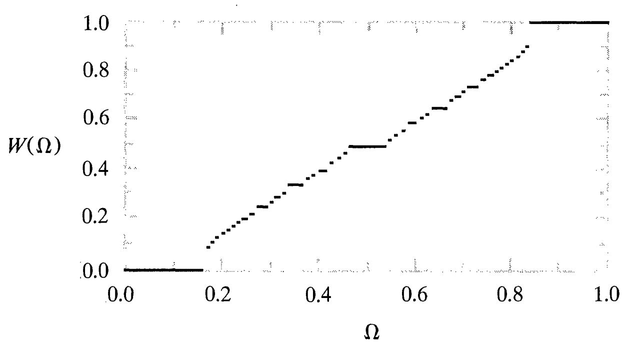

Figure 3. Mode-locking staircase. After Jensen, Bak, and Bohr **[10]**.

whose staircase graph appears in the figure, takes the previously-mentioned parameter Ω as its input; the output $W(\Omega)$ is the winding number associated with Ω. The value $\Delta\Omega(x)$ gives the width of the step on the staircase with ordinate $x = W(\Omega)$; that is, $\Delta\Omega(x) = \lambda(W^{-1}(\{x\}))$, where λ denotes Lebesgue measure.

Results in **[10]** indicate that the Hausdorff dimension of the corresponding Cantor set (the closure of the set of Ω-values with irrational winding numbers) is approximately $2/2.29$, which by Theorem 1 is the same as $\dim \text{Nondiff } f^{2.29}$. Is this a coincidence? Probably not. In the next section we use the step-width function approach with respect to a class of Cantor functions studied recently by Darst, and show that each corresponding Cantor set has the same Hausdorff dimension as the nondifferentiability set of the associated step-width function.

3. A RELATED FUNCTION g WHOSE POWERS CORRESPOND TO CANTOR SETS. In **[4]** Darst describes a class of Cantor sets C_a as follows. For each $a \in (0, 1/2)$, put $b = 1 - 2a$ and delete an open interval of length b from the center of $[0,1]$, leaving two closed intervals of length a. From the center of each of

*For fixed n, the values $Af^p(m/n)$ are the same whenever $(m,n) = 1$. The corresponding values $\Delta\Omega(m/n)$ are not all equal; it is their *average* that is well-approximated by $Af^{2.29}(m/n)$. See **[10]** for details.

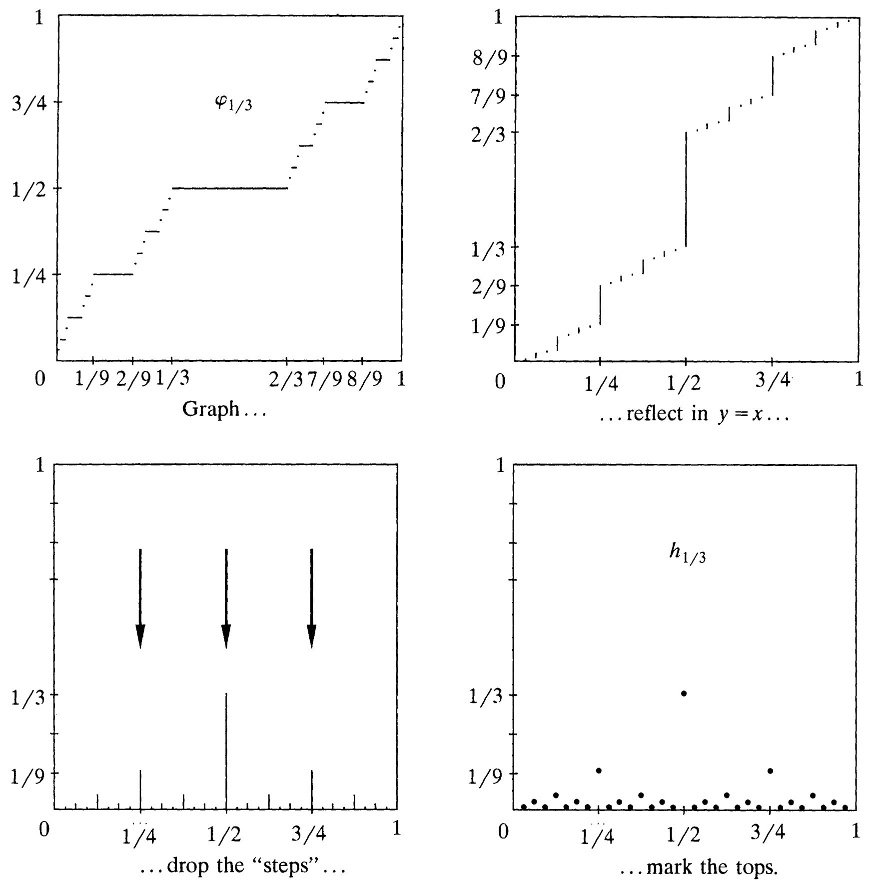

Figure 4. Obtaining the step-width function $h_{1/3}$ from the classical Cantor function $\varphi_{1/3}$. For any $a \in (0, 1/2)$, the corresponding Cantor set C_a satisfies $\dim C_a = \dim \text{Nondiff } h_a$.

these intervals, delete a scaled version of the first open interval; that is, delete an interval of length ab, leaving four closed intervals of length a^2. Repeating the process indefinitely leaves us with the set C_a; in particular, $C_{1/3}$ is the classical Cantor middle-thirds set. Darst then defines corresponding Cantor functions $\varphi_a : [0,1] \to [0,1]$ in the obvious way. The graphs of these functions, including that of the classical Cantor function $\varphi_{1/3}$, are staircases with steps of finite width at each ordinate $m/2^n \in (0,1)$ where m is an odd integer, and steps of zero width elsewhere. For our purposes, let us ignore the "steps" at ordinates 0 and 1, and for each $a \in (0, 1/2)$ define a step-width function $h_a : (0,1) \to \mathbb{R}$ by $h_a(x) = \lambda(\varphi_a^{-1}(\{x\}))$ (see Figure 4). Where the steps have finite widths, they are the lengths $a^n b$ of deleted intervals in the construction of C_a. A little thought shows that $h_a(x) = a^{n-1}b$ if $x = m/2^n$ with m odd, and $h_a(x) = 0$ otherwise. If we set $p_a = \ln(1/a)/\ln 2 = \log_2(1/a)$, we can write

$$h_a = \begin{cases} (b/a)(1/2^n)^{p_a} & \text{if } x = m/2^n \text{ with } m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Now the rationals $m/2^n$ are sometimes referred to as the dyadic rationals. If, in our definition of the functions f^p, we replace the rationals with the dyadic rationals, we get analogous functions $g^p : (0,1) \to \mathbb{R}$ defined by

$$g^p(x) = \begin{cases} (1/2^n)^p & \text{if } x = m/2^n \text{ with } m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

for each $p \geq 1$. Analogously, the function $g = g^1$ and its powers are discontinuous at and only at each dyadic rational point of $(0,1)$. Notice that $h_a = (b/a)g^{p_a}$, and as a runs over $(0,1/2)$, the values p_a run over $(1,\infty)$. Thus the functions g^p for $p > 1$ correspond in a precise one-to-one way to the Cantor sets C_a.

In [**4**] Darst addresses the nondifferentiability of the Cantor functions φ_a. He defines their nondifferentiability points as those points where there is no derivative, finite or infinite, and shows that the corresponding set has Hausdorff dimension $[\ln(2)/\ln(1/a)]^2$. He also shows that for each $a \in (0,1/2)$, $\dim C_a = \ln(2)/\ln(1/a)$. The next result fits nicely with these.

Theorem 3. *For each* $a \in (0,1/2)$, $\dim \operatorname{Nondiff} h_a = \ln(2)/\ln(1/a)$. *Equivalently, for each* $p > 1$, $\dim \operatorname{Nondiff} g^p = 1/p$.

Proof: Clearly $\operatorname{Nondiff} h_a = \operatorname{Nondiff} g^{p_a}$, and $1/p_a = \ln(2)/\ln(1/a)$. Thus it suffices to show that $\dim \operatorname{Nondiff} g^p = 1/p$ for each $p > 1$. Moreover, following the proof of Theorem 1, it suffices to prove this result for the subset M_p of points in $\operatorname{Nondiff} g^p$ that are not dyadic rationals. To show that $\dim M_p \geq 1/p$, we need only exhibit a subset B of M_p such that $\dim B = 1/p$. The existence of such a set follows from a result by Falconer [**6**, Theorem 8.15], which implies that if $p > 1$ and B is the set of numbers x that are not dyadic rationals and satisfy

$$|x - m/2^n| \leq (2^n)^{-p} \tag{4}$$

for infinitely many dyadic rationals $m/2^n$, then $\dim B = 1/p$. To see how this applies, observe that x is a minimum point of g^p, so $(g^p)'(x)$ can exist only if it is zero. But inequality (4) implies

$$\left| \frac{g^p(x) - g^p(m/2^n)}{x - m/2^n} \right| = \frac{(2^n)^{-p}}{|x - m/2^n|} \geq 1$$

for infinitely many dyadic rationals $m/2^n$. Thus g^p is not differentiable at any $x \in B$, and $B \subset M_p$.

To show that $\dim M_p \leq 1/p$, fix an arbitrary ε with $0 < \varepsilon < p - 1$. If $x \in M_p$, then there must exist a positive constant $c = c(x)$ such that for infinitely many dyadic rationals $m/2^n$,

$$|x - m/2^n| \leq c/(2^n)^p \leq (2^n)^\varepsilon/(2^n)^p,$$

the latter inequality being true for sufficiently large n. Thus for all $x \in M_p$, we have $|x - m/2^n| \leq (2^n)^{-(p-\varepsilon)}$ for infinitely many n, so Falconer's result implies that $\dim M_p \leq 1/(p - \varepsilon)$. Since ε was arbitrary, it follows that $\dim M_p \leq 1/p$. ■

The functions g^p make fractals, except for g^1: $\operatorname{Nondiff} g^1$ is all of $(0,1)$. However, the graph of g^1 is very fractal-like (Figure 5). If you think you see a right isosceles Sierpiński triangle there, it's because the points of the graph above the x-axis are precisely the apexes of those subtriangles of the Sierpiński triangle whose hypotenuses lie on the x-axis. The picture can also be used to give an easy proof of the nondifferentiability of g^1; one can actually see its nondifferentiability.

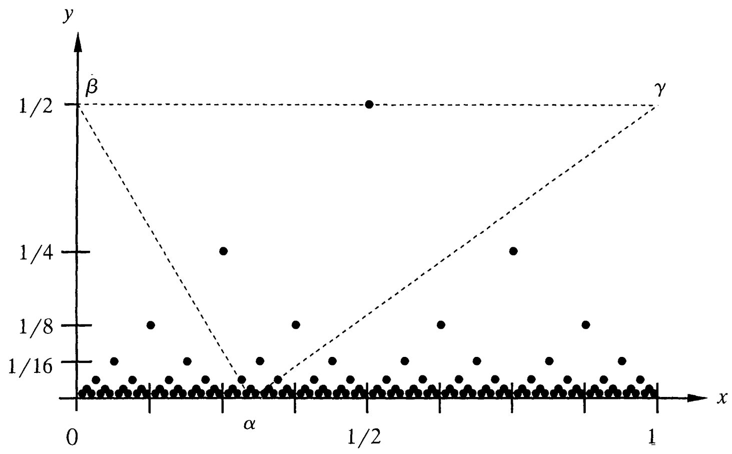

Figure 5. The function g^1.

Theorem 4. *The function* g^1 *is not differentiable at any point of* $(0,1)$.

Proof: Since g^1 is not continuous at any dyadic rational, let us focus on the other points. Choose any point $x \in (0,1)$ that is not a dyadic rational. As in the previous proof, $(g^1)'(x)$ exists only if $(g^1)'(x) = 0$; we show that this cannot happen. Draw the triangle $\triangle\alpha\beta\gamma$, where $\alpha = (x,0)$, $\beta = (0,1/2)$, and $\gamma = (1,1/2)$. Thinking of the top as the base, $\triangle\alpha\beta\gamma$ has base 1 and height $1/2$, so at any ordinate $1/2^n$, the triangle has width $2/2^n$. Additionally, at any ordinate $1/2^n$, the graph of g^1 has points with abscissas $1/2^n, 3/2^n, \ldots, (2^n - 1)/2^n$, which are spaced $2/2^n$ units apart. Thus at every ordinate $1/2^n$, triangle $\triangle\alpha\beta\gamma$ contains a point of the graph. Since its slanted sides have slope at least $1/2$, it follows that $(g^1)'(x)$ cannot be zero. ∎

Remarks. An avenue for future work is to think of the functions f^p for $p > 2$ as step-width functions, construct corresponding Cantor sets K_p and Cantor functions ψ_p, and show (presumably) that $\dim K_p = 2/p$. Staircase graphs of the functions ψ_p would of course have steps of finite width at every rational ordinate, making them similar to the mode-locking staircase in Figure 3. Put differently, we can start with f^p for $p > 2$, and in effect reverse the process in Figure 4. This is possible because $f^p(m/n) = 1/n^p$ for precisely $\phi(n)$ rationals $m/n \in (0,1)$, where ϕ denotes the Euler totient function. Hence the total length $c_p = \sum_{n=1}^{\infty} \phi(n)/n^p$ of the steps is finite, since $p > 2$ and $\phi(n) < n$ for each n. If desired, the total length can be scaled to 1 by using the functions $(1/c_p)f^p$ as step-width functions, just as the functions h_a were defined as suitable multiples of the functions g^{p_a}.

Since the sets Nondiff f^p and Nondiff g^p, for sufficiently large p, are fractals by the definition used here, it is natural to wonder whether they might be indicated graphically in some way, as is often done with the Cantor middle-thirds set. Unfortunately, even without the rationals, both sets are dense in $(0,1)$, so no helpful illustrations are possible. To prove this for Nondiff g^p, one can start with an arbitrary open interval in $(0, 1)$, and construct in it a nondifferentiable irrational point x by requiring x to have a binary "decimal" expansion which ultimately consists of longer and longer strings of 0's separated by single 1's. The growth of the strings must depend appropriately on p; the details are left to the interested reader.

The density of the irrational subsets N_p of the sets Nondiff f^p has an interesting reason behind it. Recall that the Liouville numbers are irrational numbers x such that for each positive integer k, there exist integers m and n with $n > 1$ such that

$$|x - m/n| < n^{-k}.$$

Since we may assume that $(m, n) = 1$, it is easy to see that x is not a differentiable point of f^p for any p. Consequently, the Liouville numbers form a subset of $\bigcap_{p>2} N_p$. Conversely, if $x \in \bigcap_{p>2} N_p$, then for each positive integer k we have $x \in N_{2+k}$, so there must exist a positive constant $c = c(x, k)$ such that

$$|x - m/n| < cn^{-(2+k)} = (cn^{-2})n^{-k}$$

for infinitely many rationals m/n. Thus $|x - m/n| < n^{-k}$ for some rational m/n, and hence x is a Liouville number. It follows that $\bigcap_{p>2} N_p$ is the set of Liouville numbers in $(0, 1)$. The Liouville numbers are transcendental numbers that form a first category subset of $\mathbb{R}$, and there are uncountably many of them in every open interval; thus each set N_p is also dense in $(0, 1)$. These facts are proved in [**11**, Chapter 2], where it is also shown that the set of Liouville numbers has Hausdorff dimension zero. This last fact is easily deduced from Theorem 1, since $\dim \text{Nondiff} f^p \downarrow 0$ as $p \uparrow \infty$.

Finally, a word is in order concerning the graphs of the functions f^p and g^p. Many texts include f^1 as an example, but the vast majority do not provide a detailed graph (an exception being [**12**, p. 129]). Perhaps it is thought that such graphs would exhibit no interesting structure, or that they would be hard to generate. Neither of these assumptions is true. The graphs of the functions f^p and g^p can be used to link important basic concepts of real analysis with notions of rational approximation, self-similarity, fractal geometry, and dynamical systems, and hence can serve to motivate students in the classroom. As for generating the graphs, there are several easy ways to do it. I would be happy to send the appropriate Maple or Mathematica code in response to email requests.

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REFERENCES

1. Barnsley, M., *Fractals Everywhere*, Academic Press, San Diego, 1988.
2. Cvitanović, P., Circle Maps: Irrationally Winding. In Walschmidt, M., Moussa, P., Luck, J.-M., and Itzykson, C. (eds): *From Number Theory to Physics*, Springer-Verlag, New York, 1992, 631–658.
3. Cvitanović, P., Jensen, M. H., Kadanoff, L. P., and Procaccia, I., Renormalization, Unstable Manifolds, and the Fractal Structure of Mode Locking, *Phys. Rev. Lett.* 55 (1985) 343–346.
4. Darst, R. B., Hausdorff dimension of sets of nondifferentiability points of Cantor functions, *Math. Proc. Camb. Phil. Soc.* 117 (1995) 185–191.
5. Darst, R. B. and Taylor, G. D., Differentiating powers of an old friend, *Amer. Math. Monthly* 103 (1996) 415–416.
6. Falconer, K. J., *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

7. Niven, I., *Irrational Numbers*, The Mathematical Association of America, Rahway, N.J., 1956.
8. Jarnik, V., Über die simultanen diophantischen Approximationen, *Math. Zeit.* 33 (1931) 505–543.
9. Jensen, M. H., Bak, P., and Bohr, T., Complete Devil's Staircase, Fractal Dimension, and Universality of Mode-Locking Structure in the Circle Map, *Phys. Rev. Lett.* 50 (1983) 1637–1639.
10. Jensen, M. H., Bak, P., and Bohr, T., Transition to chaos by interaction of resonances in dissipative systems, *Phys. Rev.* A30 (1984) 1960–1969.
11. Oxtoby, J. C., *Measure and Category*, Springer-Verlag, New York, 1971.
12. Walter, W., *Analysis I*, Springer-Verlag, Berlin, 1985.

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Standard mathematics has recently been rendered obsolete by the discovery that for years we have been writing the numeral five backward. This has led to reevaluation of counting as a method of getting from one to ten. Students are taught advanced concepts of Boolean algebra, and formerly unsolvable equations are dealt with by threats of reprisals.

Woody Allen, in Howard Eves' *Return to Mathematical Circles*,
Boston: Prindle, Weber, and Schmidt, 1988.