A Golden Cantor Set

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This article shows how the golden ratio solves a simple but interesting geometric problem involving Cantor sets. We construct a special Cantor set that deserves to be called "a golden Cantor set."

Before we marry the golden ratio to a Cantor set, we should find out what these two have in common. We see that what they both bring to this relationship, what lies at the core of each of their beings, is the notion of self-similarity. We take advantage of this by using the idea of renormalization to prove several of our results. Renormalization is a technique often used in situations involving self-similarity. It has been used extensively in dynamical systems theory and mathematical physics. The renormalizations that we will use here are simple illustrations of this sophisticated technique. For some other elementary examples of how renormalization can be used, see [**2**] or [**13**, pp. 272–274 or pp. 367–369].

The golden ratio is the number $(1 + \sqrt{5})/2$, which is commonly denoted by ϕ. Let's see how it is derived. Choose a number a from the interval $(0, 1)$.

The interval $[0, 1]$ is divided by a into two subintervals, and we have several ratios of lengths that we can compare. If we choose a so that the ratio of the length of the whole interval $[0, 1]$ to the length of its left part $[0, a]$ is equal to the ratio of the length of the left part $[0, a]$ to the length of the right part $[a, 1]$, then that common ratio is the golden ratio. If a is chosen this way, then we have

$$\frac{1}{a} = \frac{a}{1 - a}, \tag{1}$$

but, by design, $\phi = 1/a$, and substituting this into the last equation gives us

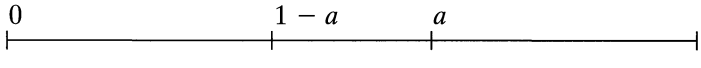
$$\phi = \frac{1}{\phi - 1}.$$

This gives us the quadratic equation

$$\phi^2 - \phi - 1 = 0,$$

whose positive root is $(1 + \sqrt{5})/2$; the negative root is $\phi' \equiv (1 - \sqrt{5})/2$. Notice that $\phi' = -\phi^{-1}$.

What is it that makes the golden ratio so interesting? Some feel that it has aesthetic value when used in works of art; see [**5**, pp. 124–125] and [**11**] for contrasting views about this. For mathematics, the golden ratio gets its interest from its amazing collection of self-reproducing properties [**13**, p. 51]. Here is a simple, but key, example. If the number a divides $[0, 1]$ according to the golden ratio, then, by symmetry, so does the number $1 - a$; by the definition of the golden ratio (1), $1 - a$ also divides the interval $[0, a]$ according to the golden ratio, since $\phi = a/(1 - a)$.



So $1 - a$ simultaneously divides two different intervals according to the golden ratio, both $[0, 1]$ and $[0, a]$. Once an interval has been divided into two subintervals according to the golden ratio, the length of the shorter of the two subintervals divides the longer subinterval according to the golden ratio. Here's another way to think about this. Suppose you use a compass and straightedge construction to divide an interval according to the golden ratio [**3**, p. 161]. Then dividing the longer of the two subintervals according to the golden ratio becomes trivial—just use the compass to copy the shorter subinterval into the longer one. The golden ratio is self-reproducing.

Other examples of the golden ratio's ability to reproduce itself are the golden rectangle, the golden triangle, the regular pentagon, the continued fraction expansion of ϕ, and the continued square root expansion of ϕ. For a wealth of information about self-similar aspects of ϕ and their applications to a wide variety of physical problems, see [**13**].

Now let's turn to the Cantor sets. We are concerned here with a family of sets called the *middle-α Cantor sets*; a member of this family will be married to the golden ratio. These Cantor sets are a generalization of the classical middle third Cantor set. Choose a number $\alpha \in (0, 1)$, let $I_0 = [0, 1]$, and let I_1 be the union of the two closed intervals that remain after the open interval of length α is removed from the middle of I_0. Each of the closed intervals in I_1 has length $(1 - \alpha)/2$; let β denote $(1 - \alpha)/2$. Notice that $\beta \in (0, 1/2)$ and $\alpha = 1 - 2\beta$. Now we do to each closed interval in I_1 what we did to I_0: we remove from the middle of each an open interval whose length is α times the length of the closed interval. This leaves us with four closed intervals, each of length β^2; call the union of these intervals I_2. Let I_n be the union of the 2^n closed intervals of length β^n that remain after the open interval of length $\alpha\beta^{n-1}$ is removed from the middle of each of the components of I_{n-1}. Figure 1 gives a picture of I_0 through I_4.

Figure 1

The middle-α Cantor set in the interval $[0, 1]$ is

$$C_\alpha \equiv \bigcap_{n=0}^{\infty} I_n.$$

When $\alpha = \beta = 1/3$, we get the classical middle third Cantor set.

For a middle-α Cantor set, β is more useful to us than α. One of the most important properties of a middle-α Cantor set, the property of self-similarity, is described by β; β is a scaling factor for the Cantor set. To see what we mean here by self-similarity and scaling factors, let C_α^L denote $C_\alpha \cap [0, \beta]$ and let C_α^R denote $C_\alpha \cap [1 - \beta, 1]$, so C_α^L is the left "half" of C_α and C_α^R is the right "half." Then the image of C_α under the function $T_L(x) = \beta x$ is C_α^L, and the image of C_α under the function $T_R(x) = \beta x + (1 - \beta)$ is C_α^R. This shows that the left and right "halves" of C_α are smaller scale, exact duplicates of C_α. You should convince yourself that the powers β^n, for all positive and negative integers n, are also scaling factors for C_α.

As α decreases towards zero (or, what is the same thing, as β increases towards $1/2$), the middle-α Cantor sets grow "larger." This idea, which is not by any means obvious, can be made precise by using one of the many different "fractal dimensions" (see [**4**, pp. 105–107]), but we will get a hint of this from the solution of our geometric problem stated below (see also [**8**]).

What is it that makes Cantor sets so interesting? Most mathematicians would probably say that it is their incredible versatility and usefulness for constructing examples and counterexamples in topology and analysis. However, many of these constructions have a contrived air to them; take for example Cantor's Leaky Tent in [**14**, p. 145]. This may leave the impression in some minds that Cantor sets are anything but the stuff that models for real world phenomena are made of. But lately, that is exactly what Cantor sets have been used for, particularly within the field of dynamical systems. One use of Cantor sets in dynamical systems, the study of homoclinic bifurcations, has led to many questions about how two Cantor sets can intersect [**12**, Chapter 4]. This work led to the problem, posed in [**15**], of finding all the ways that two Cantor sets can intersect in just one point. And that problem motivated the geometric problem that leads us to a golden Cantor set.

We need one more definition before we can state our geometric problem. If A is a subset of the real line and λ is a positive real number, then $\lambda A \equiv \{\lambda x | x \in A\}$. The set λA is a stretched or compressed version of A if $\lambda > 1$ or $\lambda < 1$, respectively.

Geometric problem. Given $\beta \in (0, 1/2)$, is it possible to find a $\lambda \in (0, 1)$ such that $C_\alpha \cap \lambda C_\alpha = \{0\}$?

What makes this problem interesting? Cantor sets have the property of being totally disconnected. That is, between any two points from a Cantor set there are points not in the Cantor set, i.e., a Cantor set is infinitely full of holes; in [**10**, Chapter 8], Cantor sets are aptly called "Cantor dusts." The geometric problem asks if it is possible to take two middle-α Cantor sets, C_α and λC_α, and interweave all the points from each one of them into the holes of the other one, except for their common endpoint 0. How hard is this to do? We show that there is a critical value for β below which the problem has a solution, but above which there is no solution; the sets *cannot* be interweaved. This critical value helps to demonstrate that the "size" of middle-α Cantor sets changes with β. For β below the critical value, the middle-α Cantor sets are "small enough" for the problem to have a solution; for β above the critical value, the middle-α Cantor sets are "too big" for the problem to have a solution. The critical value is $\beta = (3 - \sqrt{5})/2$, and of course this number has something to do with the golden ratio.

Let's show that the geometric problem can be solved when $\beta < (3 - \sqrt{5})/2$. Consider the picture of I_1 and λI_1 in Figure 2.

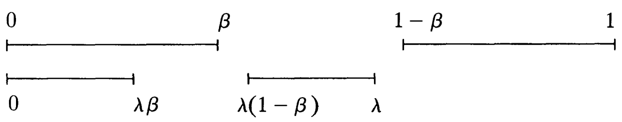


Figure 2

We need to determine those values of β and λ for which the relationship of the components of I_1 and λI_1 shown in Figure 2 is possible. If we can find such values of β and λ, then we have a solution of the geometric problem. This is because the situation shown in Figure 2 repeats itself, on a smaller scale, in the intervals $[0, \beta]$ and $[0, \lambda\beta]$ when we construct I_2 and λI_2 (in Figure 3), and, in general, in the intervals $[0, \beta^n]$ and $[0, \lambda\beta^n]$ when we construct I_{n+1} and λI_{n+1}. Notice how we are using self-similarity here.

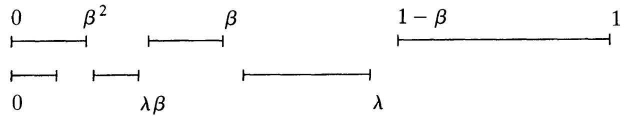


Figure 3

Then $I_n \cap \lambda I_n = [0, \lambda\beta^n]$, which implies $C_\alpha \cap \lambda C_\alpha = \{0\}$.

To get pictures such as those in Figures 2 and 3, we need $\beta < \lambda(1 - \beta)$ and $\lambda < 1 - \beta$. In other words, we need

$$\frac{\beta}{1-\beta} < \lambda < 1 - \beta,$$

so β must satisfy the inequality $0 < 1 - 3\beta + \beta^2$, which means that

$$\beta < \frac{3 - \sqrt{5}}{2}.$$

Thus, for $\beta < (3 - \sqrt{5})/2$, we can solve the geometric problem. Notice that when $\beta = (3 - \sqrt{5})/2$, if we let $\lambda = 1 - \beta$, then the intersection $C_\alpha \cap \lambda C_\alpha$ equals 0 and a countable number of common endpoints; see Figure 4.

Figure 4

Now we can see where the golden ratio comes in. When $\beta = (3 - \sqrt{5})/2$, if we look at the ratio of the length of the components of I_1 (i.e., β) to the length of the hole in I_1 (i.e., α), we get

$$\frac{\beta}{\alpha} = \frac{\beta}{1-2\beta} = \frac{(3-\sqrt{5})/2}{1-(3-\sqrt{5})} = \frac{1+\sqrt{5}}{2}.$$

This means that β divides the interval $[0, 1-\beta]$ (and also $[0,1]$) according to the golden ratio. You can check that this also means that $\beta = 1/\phi^2$. And if we let $\lambda = 1 - \beta$, then, since $1 - \beta$ divides $[0,1]$ according to the golden ratio, we have $\lambda = 1/\phi$. Later we return to the case $\beta = (3 - \sqrt{5})/2$ and look at it from a different point of view, but first let's show that there is no solution of the geometric problem when $\beta \geq (3 - \sqrt{5})/2$.

We need to show that if $\beta \in [(3 - \sqrt{5})/2, 1/2)$, then for any $\lambda \in (0,1)$, 0 is not the only point in $C_\alpha \cap \lambda C_\alpha$. To prove this, we need to consider the intersection of

C_α with a shrunken *and* translated copy of itself. If A is a subset of the real line and t is any real number, let $A + t = \{x + t | x \in A\}$, that is, $A + t$ is a translate of A.

Lemma 1. *Choose* $\beta \in [1/3, 1/2)$. *Let* $\alpha = 1 - 2\beta$. *If* $\lambda \in [\alpha, 1]$, *then for any* $t \in [-\lambda, 1]$, $C_\alpha \cap (\lambda C_\alpha + t) \neq \emptyset$.

Note. This lemma states, roughly, that if a middle-α Cantor set is sufficiently "large" (i.e., $\beta \geq 1/3$), and if it "overlaps" with a shrunken and translated copy of itself (i.e., $\lambda \geq \alpha$ and $-\lambda \leq t \leq 1$), then the middle-α Cantor set *intersects* the shrunken and translated copy of itself.

Proof: The proof uses induction and a rescaling (or renormalization) argument. We show that for every $n \geq 1$, $I_n \cap (\lambda I_n + t) \neq \emptyset$. Since the intersection of a nested sequence of nonempty compact sets is nonempty, $\cap_{n=0}^{\infty}[I_n \cap (\lambda I_n + t)] \neq \emptyset$. But $\cap_{n=0}^{\infty}[I_n \cap (\lambda I_n + t)]$ is the same as $C_\alpha \cap (\lambda C_\alpha + t)$.

Before doing the induction argument, let's reexamine the self-similar structure of a middle-α Cantor set C_α. Let J denote a component of I_n for some $n \geq 0$. We call $J \cap C_\alpha$ a *segment* of C_α. The self-similarity of C_α means that a segment of C_α is just a scaled down copy of C_α. That is, by an affine map (i.e., a map of the form $f(x) = mx + b$ with $m = \beta^{-n}$) we can map any segment of C_α onto C_α. Affinely mapping a segment of C_α onto C_α is what we mean by rescaling.

To begin the induction, we need to show that $I_1 \cap (\lambda I_1 + t) \neq \emptyset$. Since $\lambda \geq \alpha$, we cannot have $\lambda I_1 + t$ contained in the open gap in the middle of I_1, i.e., we cannot have the picture in Figure 5.

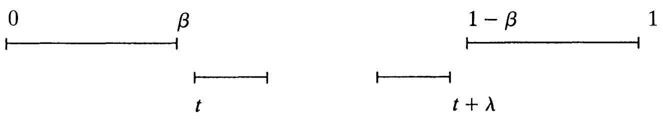


Figure 5

Since $\beta \geq 1/3$, we have $\beta \geq \alpha$, and since $\lambda \leq 1$, we have $\beta \geq \lambda\alpha$. Thus, neither component of I_1 is contained in the open gap in the middle of $\lambda I_1 + t$, i.e., we cannot have the picture in Figure 6.

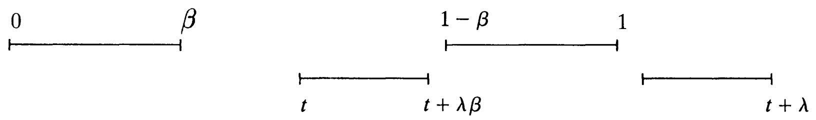


Figure 6

Therefore, $I_1 \cap (\lambda I_1 + t) \neq \emptyset$.

Now suppose that $I_n \cap (\lambda I_n + t) \neq \emptyset$. We need to show that $I_{n+1} \cap (\lambda I_{n+1} + t) \neq \emptyset$. Let J_n and J_n' denote any components of I_n and $\lambda I_n + t$, respectively, that intersect. Using an affine map, we can map the segment of C_α contained in J_n onto C_α; this same affine map takes the segment of $\lambda C_\alpha + t$ contained in J_n' onto $\lambda C_\alpha + t_n$ for some $t_n \in [-\lambda, 1]$. This shows that the first step in the proof of the lemma (the step with $n = 1$) can be applied to $J_n \cap I_{n+1}$ and $J'_n \cap (\lambda I_{n+1} + t)$ to conclude that $(J_n \cap I_{n+1}) \cap (J'_n \cap (\lambda I_{n+1} + t)) \neq \emptyset$. We have renormalized the induction step back to the original step. Therefore, $I_{n+1} \cap (\lambda I_{n+1} + t) \neq \emptyset$. ■

To show that the geometric problem does not have a solution when $\beta \in [(3 - \sqrt{5})/2, 1/2)$, we use Lemma 1 and another rescaling argument to prove the existence of a point other than 0 in $C_\alpha \cap \lambda C_\alpha$. First we prove our result with $\lambda \in [\beta, 1)$. Then we use rescaling once again to extend the result to $\lambda \in (0, 1)$.

Lemma 2. *Choose* $\beta \in [(3 - \sqrt{5})/2, 1/2)$. *If* $\lambda \in [\beta, 1)$, *then* $C_\alpha \cap \lambda C_\alpha \neq \{0\}$.

Proof: First, notice that with our hypotheses, $I_1 \cap \lambda I_1$ must have at least two components, one of which contains 0. As we have shown earlier, if $\lambda \geq \beta$, then for $I_1 \cap \lambda I_1$ to have only one component (the one that contains 0) it is necessary that $\beta < (3 - \sqrt{5})/2$.

Let J and J' be intersecting components of I_1 and λI_1, respectively, such that $0 \notin J \cap J'$. There is an affine map that maps the segment of C_α contained in J onto C_α; this same affine map takes the segment of λC_α contained in J' onto $\lambda C_\alpha + t$ for some $t \in [-\lambda, 1]$. Then Lemma 1 implies that the segment of C_α contained in J has nonempty intersection with the segment of λC_α contained in J' (notice that $(3 - \sqrt{5})/2$ is greater than $1/3$). Therefore, $C_\alpha \cap \lambda C_\alpha$ contains a nonzero point. ■

Theorem 3. *Choose* $\beta \in [(3 - \sqrt{5})/2, 1/2)$. *If* $\lambda \in (0, 1)$, *then* $C_\alpha \cap \lambda C_\alpha \neq \{0\}$.

Proof: We know that the theorem is true for $\lambda \in [\beta, 1)$. Suppose that $\lambda \in [\beta^{n+1}, \beta^n)$ for some integer $n \geq 1$. Then $T(x) = \beta^{-n}x$ maps the segment of C_α contained in $[0, \beta^n]$ onto C_α, and it also maps λC_α onto $(\lambda/\beta^n)C_\alpha$. Since $\beta^{n+1} \leq \lambda < \beta^n$, we have $\lambda/\beta^n \in [\beta, 1)$, and so we can apply Lemma 2 to conclude that $(C_\alpha \cap [0, \beta^n]) \cap \lambda C_\alpha \neq \{0\}$. ■

Note. We have actually proved a little more than what is stated in Theorem 3. We have shown that if $\beta \in [(3 - \sqrt{5})/2, 1/2)$ and $\lambda \in (0, 1)$, then $C_\alpha \cap \lambda C_\alpha$ has infinite cardinality.

Now let's return to the case $\beta = (3 - \sqrt{5})/2$ and $\lambda = 1 - \beta$. Recall that we then have $\beta = 1/\phi^2$, $\lambda = 1/\phi$, and $\beta/\alpha = \phi$. Our goal now is to give $C_\alpha \cap \lambda C_\alpha$ a more vivid geometric interpretation. To do this we use the product set $C_\alpha \times C_\alpha$ and the graph of the function $f(x) = \lambda x$; the idea for our construction comes from [**1**, p. 135]. Notice that $y \in C_\alpha \cap \lambda C_\alpha$ if and only if $y \in C_\alpha$ and $y = \lambda x$ for some $x \in C_\alpha$ if and only if there is a point (x, y) contained in both $C_\alpha \times C_\alpha$ and the graph of f. This shows that there is a bijection between the points in $C_\alpha \cap \lambda C_\alpha$ and the intersection of $C_\alpha \times C_\alpha$ with the graph of f; the bijection projects $(C_\alpha \times C_\alpha) \cap \text{graph}(f)$ horizontally onto the y-axis, giving $C_\alpha \cap \lambda C_\alpha$. So let's replace our one dimensional visualization of $C_\alpha \cap \lambda C_\alpha$ with a two dimensional visualization of $(C_\alpha \times C_\alpha) \cap \text{graph}(f)$. Figure 8 is a picture of $(I_2 \times I_2) \cap \text{graph}(f)$, it "replaces" the picture of $I_2 \cap \lambda I_2$ in Figure 7.

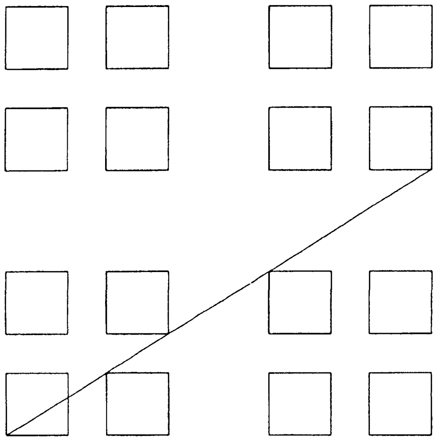


FIGURE 8

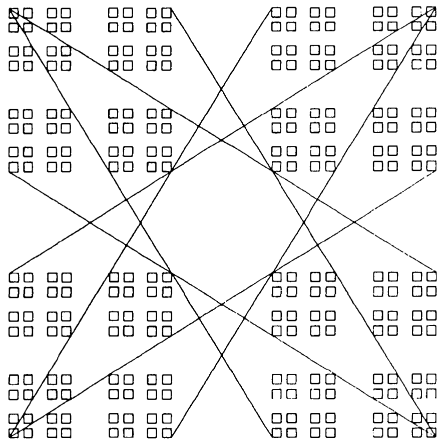


FIGURE 9

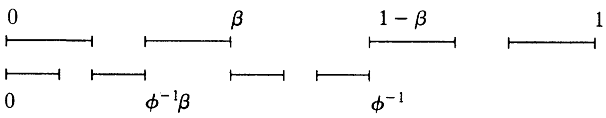


FIGURE 7

Because of the symmetry of $C_\alpha \times C_\alpha$, there are seven other lines that play the same role with respect to $C_\alpha \times C_\alpha$ as does the graph of $f(x) = \lambda x$. In Figure 9 these lines are added to a picture of $I_4 \times I_4$. Each of the eight lines in Figure 9 has slope equal to either $\pm\phi$ or $\pm\phi'$. Any rectangle with a diagonal along one of these lines is a golden rectangle, i.e., a rectangle where the ratio of the lengths of its sides is ϕ. Thus, Figure 9 is full of golden rectangles. Interestingly, Figure 9 is in Plate LVI of [**5**, p. 137] as an example of a "harmonic decomposition of the square in the theme of the golden ratio." It also appears in Plate LXXVII of [**5**, p. 166].

Three final notes. First, we have seen that at our critical value of β, the middle-α Cantor set has ϕ for the value of the ratio β/α. This ratio is called the *thickness* of the middle-α Cantor set. The idea of thickness can be generalized to any Cantor set embedded in the real line, and it is a way of measuring the size of a Cantor set that has proved to be useful in the study of dynamical systems [**6**, pp. 332–336]. The critical value of thickness is ϕ. What makes this result especially appealing is that it does not really depend on the special structure of the middle-α Cantor sets; in fact, ϕ is a "universal" critical value for (arbitrary) Cantor sets embedded in the real line. If we are given a real number τ, with $\tau < \phi$, then there exist Cantor sets C_1 and C_2, defined in intervals $[0, b_1]$ and $[0, b_2]$, respectively, such that C_1 and C_2 both have thickness τ and $C_1 \cap C_2 = \{0\}$. If $\tau > \phi$, then no such Cantor sets C_1 and C_2 with thickness τ can exist. The proof of this, and a generalization to pairs of Cantor sets with different thicknesses, can be found in [**7**, Chapter 4].

Second, Lemma 1 is a special case of a useful result in dynamical systems that is often referred to as the Gap Lemma. The Gap Lemma gives conditions under which two sufficiently thick Cantor sets must have nonempty intersection, see [**6**, pp. 333–334].

Third, the middle-α Cantor set with $\beta = (3 - \sqrt{5})/2$ and thickness equal to ϕ is not the only candidate for the title "a golden Cantor set." Another contender would be the golden subshift of the full 2-shift in symbolic dynamics; see [**9**, pp. 100–102].

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