The Mandelbrot Set, the Farey Tree, and the Fibonacci Sequence

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1. INTRODUCTION. Our goal is to explain and to make precise several "folk theorems" involving the Mandelbrot set and the Farey tree [**4**].

The *Mandelbrot set* is a subset of the parameter plane for iteration of the complex quadratic function $Q_c(z) = z^2 + c$. Here the parameter c is complex. The Mandelbrot set $\mathscr{M}$ consists of those c values for which the *orbit* of 0—the sequence $0, Q_c(0), Q_c(Q_c(0)) = Q_c^2(0), Q_c^3(0), \ldots$—is bounded.

One reason for singling out the orbit of 0 is the following important fact from complex dynamics: If Q_c possesses an attracting cycle, then the orbit of 0, the critical point, must converge to that cycle [**3**]; a *cycle* is an orbit $z_0, Q_c(z_0), \ldots, Q_c^n(z_0) = z_0$ that returns to itself after n iterations. A cycle is called *attracting* if all sufficiently nearby points have orbits that tend to the cycle.

Since the orbit of 0 tends to any attracting cycle of Q_c, it follows that Q_c admits at most one attracting cycle. Also, a c-value for which Q_c has an attracting cycle must lie in $\mathscr{M}$ since the orbit of 0 is bounded. In fact, the c-values for which Q_c has an attracting cycle comprise all of the visible interior of the Mandelbrot set. By visible, we mean that nobody has ever found experimentally or otherwise a component of the interior that does not have this property. One of the main conjectures concerning $\mathscr{M}$ is that its interior consists *only* of c-values for which there is an attracting cycle.

The Mandelbrot set features a basic cardioid shape from which hang numerous "bulbs" or "decorations"; see Figure 1. Each of these bulbs is a large disk that is directly attached to the cardioid, together with numerous other smaller decorations and a prominent "antenna."

Each of these large disks turns out to contain c-values for which Q_c admits an attracting cycle with period q and rotation number p/q. That is, the attracting cycle of Q_c tends to rotate about a central fixed point, turning on average p/q revolutions at each iteration. For this reason, this bulb is called the p/q bulb. Each of the c-values in this bulb has essentially the same dynamical behavior.

A perhaps surprising folk theorem says that we can recognize the p/q-bulb from the geometry of the bulb itself. That is, we can read off dynamical information from the geometric information contained in the Mandelbrot set.

For example, the $2/5$ bulb is displayed in Figure 2. For any c-value in this large disk, Q_c features an attracting cycle with rotation number $2/5$. The $2/5$ bulb possesses an antenna-like structure that features a junction point from which five spokes emanate. One of these spokes is attached directly to the $2/5$ bulb; we call this spoke the *principal spoke*. Now look at the "smallest" of the non-principal spokes. Note that this spoke is located, roughly speaking, $2/5$ of a turn in the counterclockwise direction from the principal spoke. This is how we identify this bulb as the $2/5$-bulb.

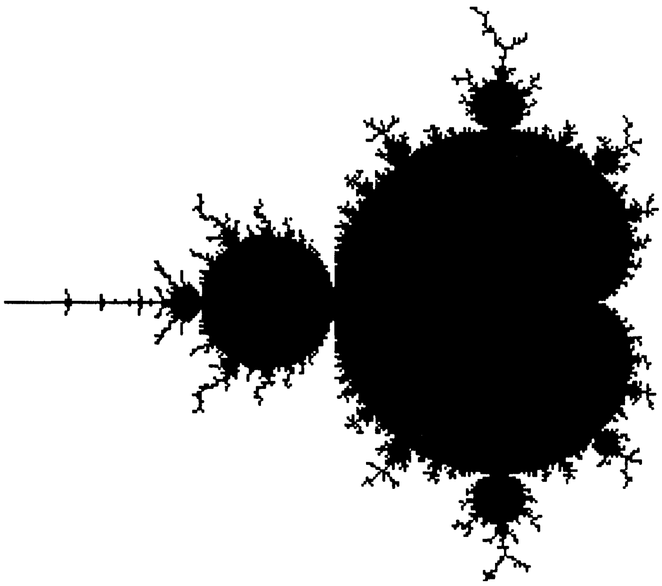

Figure 1. The Mandelbrot set.

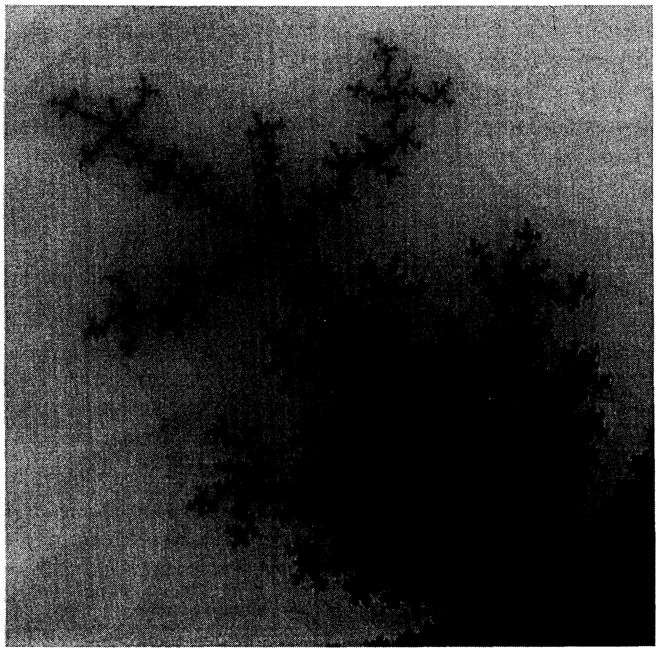

Figure 2. The $2/5$ bulb.

As another example, in Figure 3 we display the $3/7$ bulb. This bulb has 7 spokes emanating from the junction point, and the smallest is located $3/7$ of a turn in the counterclockwise direction from the principal spoke. This then is the folk theorem: You can recognize the p/q bulb by locating the "smallest" spoke in the antenna and determining its location relative to the principal spoke. Of course, the word "smallest" needs some clarification here; our goal is to make this notion precise.

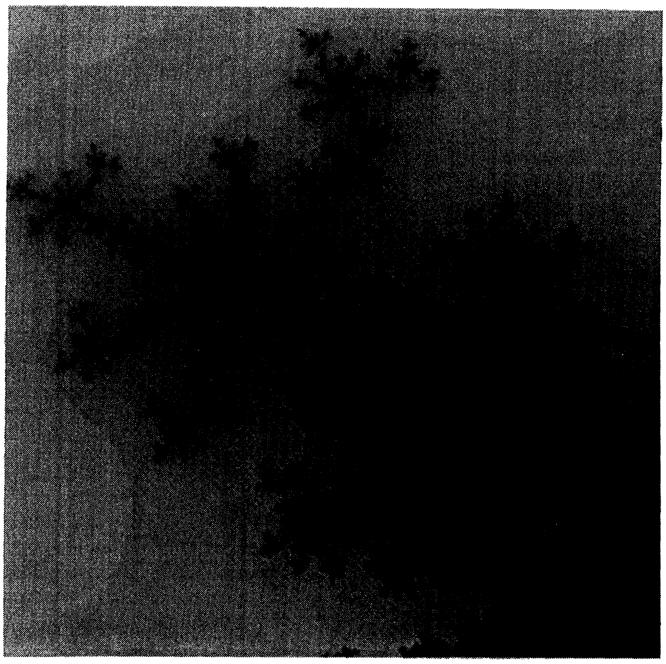

Figure 3. The $3/7$ bulb.

As an additional disclaimer, this folk theorem is only about 80% true using the Euclidean notion of "smallness" or Lebesgue measure. We provide a somewhat different framework in which this result is always true.

There is more to the story of interplay between the geometry of the Mandelbrot set and the corresponding dynamics. In Figure 4, we display the $1/2$ and $1/3$ bulbs. The $1/2$ bulb is the large bulb to the left; the $1/3$ bulb is the topmost bulb. In between these two bulbs are infinitely many smaller bulbs, but the largest we recognize as the $2/5$ bulb. Now note that $2/5$ can be obtained from $1/2$ and $1/3$ by "Farey addition":

$$\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}.$$

As a second example, note that

$$\frac{2}{5} \oplus \frac{1}{3} = \frac{3}{8}$$

and that the $3/8$ bulb is the largest between the $2/5$ and $1/3$ bulbs; see Figure 5.

That is, to obtain the largest bulb between two given bulbs (in a particular ordering), we simply add the corresponding fractions just the way we always wanted to add them, namely by adding the numerators and adding the denominators. This is the second of the folk theorems we want to discuss. In particular, it follows that the size of bulbs is determined by the Farey tree, as we show in Section 6.

Figures 4 and 5 represent the beginning of a very special sequence of p/q bulbs in the Mandelbrot set

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \ldots$$

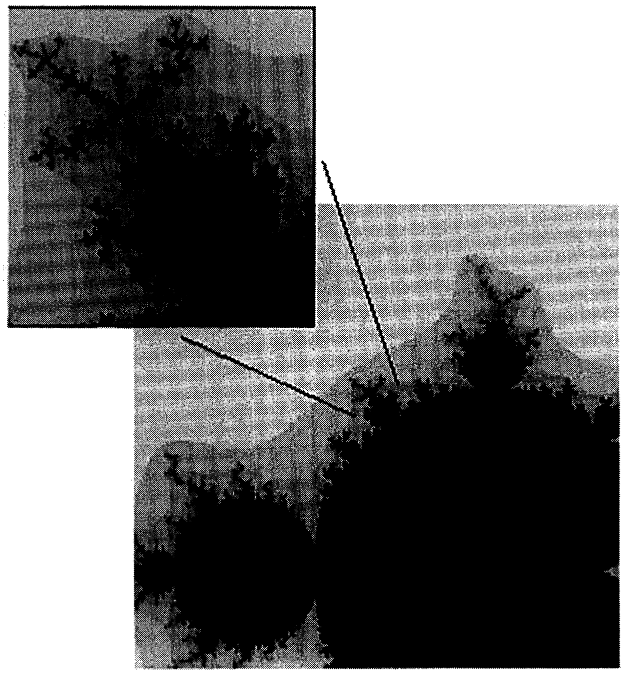

Figure 4. $\frac{1}{2} \oplus \frac{1}{3} = \frac{2}{5}$.

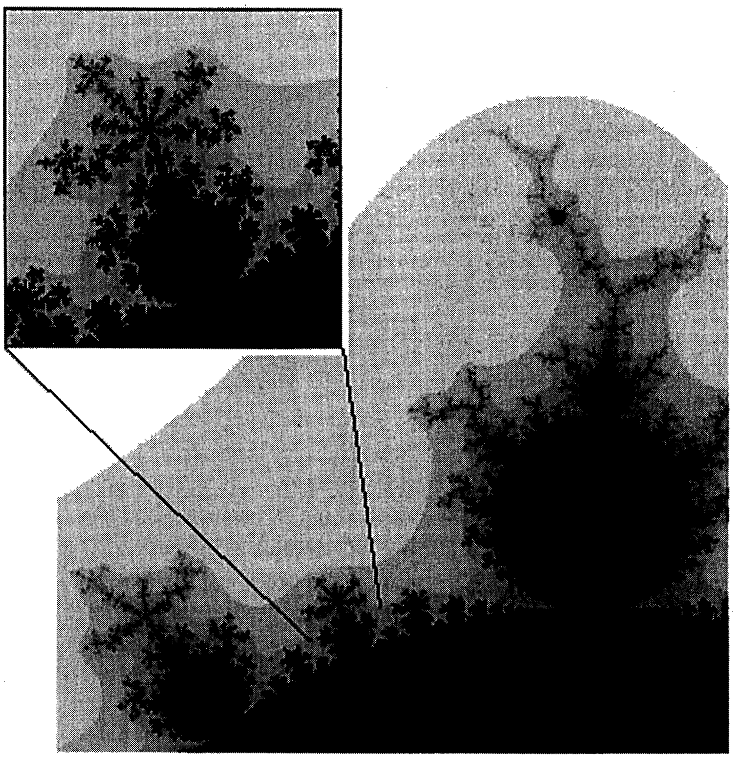

Figure 5. $\frac{2}{5} \oplus \frac{1}{3} = \frac{3}{8}$.

whose numerators and denominators correspond to the Fibonacci sequence. We discuss this connection in more detail in Section 8.

While we do not give complete proofs of each of these folk theorems, we do indicate some of the combinatorial arguments involved in making the statements precise. For more folk theorems and complete proofs, see **[5]**.

2. THE FAREY TREE. Before discussing the Mandelbrot set, we recall a few facts about the *Farey tree*, which is a tree containing all of the rationals between 0 and 1. At each stage of its construction, the Farey tree consists of a finite list of rationals. Adjacent rationals in this list are called *Farey neighbors*. The inductive step in the construction of the tree is: Each pair of Farey neighbors produces a *Farey child*, which is the rational between the two whose denominator is the smallest. Naturally, the rationals that produce a Farey child are called its *Farey parents*.

One of the most intriguing features of the Farey tree is that we obtain Farey children by Farey addition. That is, the fraction between the Farey neighbors α/β and γ/δ is given by

$$\frac{\alpha}{\beta} \oplus \frac{\gamma}{\delta} = \frac{\alpha+\gamma}{\beta+\delta}.$$

So, to obtain the fraction between two Farey neighbors whose denominator is the smallest, we simply add the numerators and add the denominators of the parents to obtain the child. For a proof that this yields the fraction between the parents with smallest denominator, we refer to **[8]**.

We begin the construction of the tree with the pair of rationals 0 and 1, which we write as $0/1$ and $1/1$. Their child is $1/2$, so the second stage of the construction gives the list

$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}.$$

At the next stage we obtain two new Farey children

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}.$$

At generation four we find

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}.$$

The Farey tree contains all rationals; see **[8]** or **[9]** for more details.

One other fact that we use is that α/β and γ/δ are Farey neighbors if and only if $\alpha\delta - \gamma\beta = \pm 1$. Consequently, we have

$$\left|\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right| = \frac{1}{\beta\delta}. \tag{1}$$

This equality is easily proved by induction.

3. THE MANDELBROT SET. The Mandelbrot set is

$$\mathscr{M} = \{c | Q_c^n(0) \text{ is bounded}\}$$

for $Q_c(z) = z^2 + c$. Thus $\mathscr{M}$ gives a picture of those c-values for which the orbit of 0 under Q_c does not tend to ∞.

The visible bulbs in $\mathscr{M}$ correspond to c-values for which Q_c has an attracting cycle of some given period. For example, the main central cardioid in $\mathscr{M}$ consists of c-values for which Q_c has an attracting fixed point. This can be seen by solving for the fixed points ($z^2 + c = z$) that are *attracting*: $|Q_c'(z)| = |2z| < 1$. Solving these equations simultaneously, we see that the boundary of this region is given by $c = z - z^2$, where $z = \frac{1}{2}e^{2\pi i\theta}$. That is,

$$c(\theta) = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$$

parametrizes the boundary of the cardioid with $0 \le \theta \le 1$. At $c(\theta)$, $Q_{c(\theta)}$ has a fixed point that is neutral; the derivative of $Q_{c(\theta)}$ at this fixed point is $e^{2\pi i\theta}$.

For each rational value of θ, there is a bulb tangent to the main cardioid at $c(\theta)$. For c-values in the bulb attached to the cardioid at $c(p/q)$, Q_c has an attracting cycle of period q. We call this bulb the p/q bulb attached to the main cardioid and denote it by $B_{p/q}$.

It is known that, as c passes from the main cardioid, through $c(p/q)$, and into $B_{p/q}$, Q_c undergoes a p/q-bifurcation. By this we mean: when c lies in the main cardioid near $c(p/q)$, Q_c has an attracting fixed point with a nearby repelling cycle of period q. At $c(p/q)$ the attracting fixed point and repelling cycle merge to produce a neutral fixed point with derivative $e^{2\pi ip/q}$. When c lies in $B_{p/q}$, Q_c now has an attracting cycle of period q and a repelling fixed point.

When $c = c(p/q)$, the local (linearized) dynamics are given by rotation through angle $2\pi(p/q)$. As a consequence, for nearby $c \in B_{p/q}$, the attracting cycle rotates about the repelling fixed point by jumping approximately $2\pi(p/q)$ radians at each iteration. For more details see **[2]**.

4. ANGLE DOUBLING MOD 1. To prepare to use the fundamental results of Douady and Hubbard **[6]** regarding the Mandelbrot set we digress to recall some facts about the *doubling function*, which is defined on the circle considered as the reals modulo one and is given by $D(\theta) = 2\theta \bmod 1$.

Fact 1: The angle θ is periodic under D if and only if θ is a rational of the form p/q (in lowest terms) with q odd.

For example, the D-orbit of $1/3$ is

$$\frac{1}{3} \to \frac{2}{3} \to \frac{1}{3} \to \cdots,$$

which has period 2. The rational $1/7$ has period 3 under doubling:

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7} \to \cdots,$$

while $1/5$ has period 4:

$$\frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5} \to \cdots.$$

The rationals with even denominator are eventually periodic but not periodic. For example, $1/6$ lies on an eventual 2-cycle

$$\frac{1}{6} \to \frac{1}{3} \to \frac{2}{3} \to \frac{1}{3} \to \cdots,$$

and $1/8$ is eventually fixed:

$$\frac{1}{8} \to \frac{1}{4} \to \frac{1}{2} \to 1 \to 1 \to \cdots.$$

A second important fact about doubling is that we can read off the binary expansion of θ by noting the *itinerary* of θ in the circle relative to D. To define the itinerary, we denote the upper semicircle $0 \leq \theta < 1/2$ by I_0 and the lower semicircle $1/2 \leq \theta < 1$ by I_1. Given θ, we attach an infinite string of 0's and 1's to θ as follows: The itinerary of θ is $B(\theta) = (s_0 s_1 s_2 \dots)$, where s_j is either 0 or 1: $s_j = 0$ if $D^j(\theta) \in I_0$, and $s_j = 1$ if $D^j(0) \in I_1$. That is, we simply watch the orbit of θ in the circle under doubling and assign 0 or 1 to the itinerary whenever $D^j(\theta)$ lands in the arc I_0 or I_1.

Fact 2: The itinerary $B(\theta)$ is the binary expansion of θ.

For example, if $\theta = 1/3$, then $\theta \in I_0$, while $D(\theta) \in I_1$ and $D^2(\theta) = \theta$. Hence $B(1/3)$ is the repeating sequence $\overline{01}$, which of course is the binary expansion of $1/3$. Similarly, $B(1/7) = \overline{001}$ while $B(1/5) = \overline{0011}$.

5. EXTERNAL RAYS. In order to make precise the folk theorems mentioned in the introduction, we recall some beautiful results of Douady and Hubbard [7] concerning the external rays of the Mandelbrot set.

Let $E = \{z | \, |z| > 1\}$ denote the exterior of the unit circle in the plane. According to Douady and Hubbard, there is a unique analytic isomorphism Φ that maps E to the exterior of the Mandelbrot set. The mapping Φ takes positive reals to positive reals. This mapping is the uniformization of the exterior of the Mandelbrot set, or the exterior Riemann map.

The importance of Φ stems from the fact that the image under Φ of the straight rays $\theta =$ constant in E have dynamical significance. In the Mandelbrot set, we define the *external ray with external angle* θ_0 to be the Φ-image of $\theta = \theta_0$. It is known that an external ray whose angle θ_0 is rational actually "lands" on $\mathcal{M}$. That is $\lim_{r \to 1} \Phi(re^{2\pi i \theta_0})$ exists and is a unique point on the boundary of $\mathcal{M}$. This c-value is called the *landing point* of the ray with angle θ_0.

For example, the ray with angle 0 lies on the real axis and lands on $\mathcal{M}$ at the cusp of the main cardioid, namely $c = 1/4$. Also, the ray with angle $1/2$ lies on the negative real axis and lands on $\mathcal{M}$ at the tip of the "tail" of $\mathcal{M}$, which can be shown to be $c = -2$.

Consider now the interior of $\mathcal{M}$. The interior consists of infinitely many simply connected regions. A *bulb* of $\mathcal{M}$ is a component of the interior of $\mathcal{M}$ in which each c-value corresponds to a quadratic function that admits an attracting cycle. The period of this cycle is constant over each bulb. In many cases, a bulb is attached to a component of lower period at a unique point called the *root point* of the component.

An important result of Douady and Hubbard is:

Theorem 1. *Suppose a bulb B consists of c-values for which the quadratic map has an attracting q-cycle. Then the root point of this bulb is the landing point of exactly* 2 *rays, and the angles of each of these rays have period q under doubling.*

Thus, the angles of the external rays of $\mathcal{M}$ determine the ordering of the bulbs in $\mathcal{M}$. For example, the large bulb directly to the left of the main cardioid is the

$1/2$ bulb, so two rays with period 2 under doubling must land there. Now the only angles with period 2 under doubling are $1/3$ and $2/3$, so these are the angles of the rays that land at the root point of $B_{1/2}$.

Now consider the $1/3$ bulb atop the main cardioid. This bulb lies "between" the rays 0 and $1/3$. There are only two angles between 0 and $1/3$ that have period 3 under doubling, namely $1/7$ and $2/7$, so these are the rays that land at the root point of $B_{1/3}$.

The $2/5$ bulb lies between the $1/3$ and $1/2$ bulbs. Hence the rays that land at $c(2/5)$ must have period 5 under doubling and lie between $2/7$ and $1/3$. The only angles that have this property are $9/31$ and $10/31$, so these rays must land at $c(2/5)$; see Figure 6.

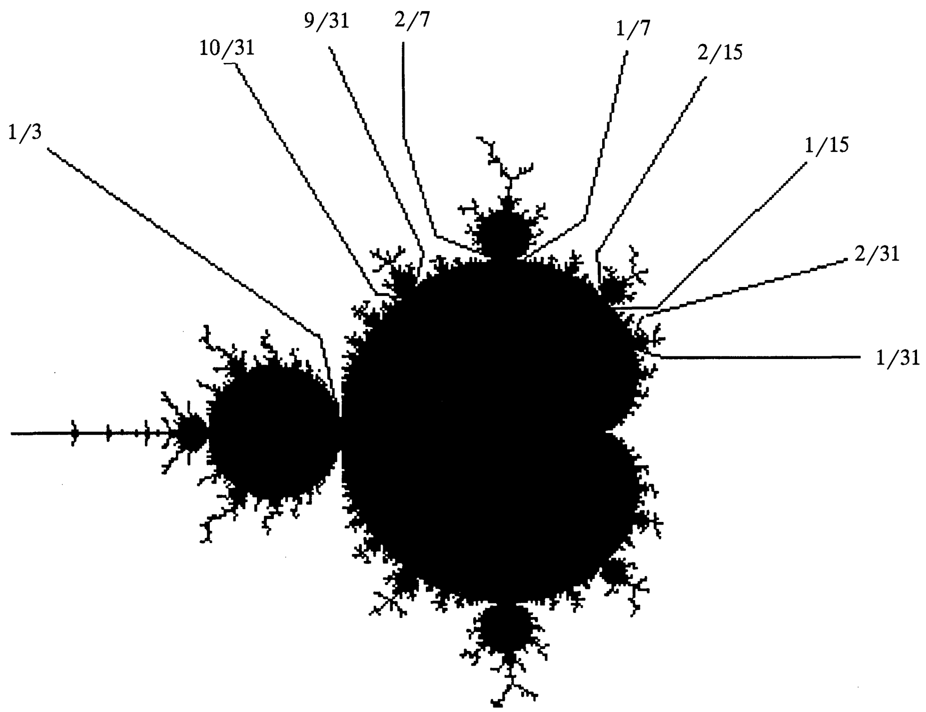

Figure 6. Rays landing on the Mandelbrot set.

These ideas allow us to measure the "largeness" or "smallness" of portions of the Mandelbrot set. Suppose we have two rays with angles θ_- and θ_+ that both land at a point c_* in the boundary of $\mathcal{M}$.

Then, by the isomorphism Φ, all rays with angles between θ_- and θ_+ must approach the component of $M - \{c_*\}$ cut off by θ_- and θ_+. (It is not known that all such rays actually land on $\mathcal{M}$—indeed, this is the major open conjecture about $\mathcal{M}$.) Thus it is natural to measure the size of this portion of $\mathcal{M}$ by the length of the interval $[\theta_-, \theta_+]$.

The root point of the p/q bulb of $\mathcal{M}$ divides $\mathcal{M}$ into two sets. The component containing the p/q bulb is called the *p/q limb*. We can then measure the size of the p/q limb if we know the external rays that land on the root point of the p/q bulb. We compute these rays in the next section.

6. RAYS LANDING ON THE p/q BULB. In order to make the notion of "large" or "small" precise in the statement of the folk theorems, we need a way to determine the angles of the rays landing at the root point of $B_{p/q}$. We denote the angles of these two rays in binary by $\overline{l_{\pm}(p/q)}$, where $\overline{l_{-}(p/q)} < \overline{l_{+}(p/q)}$. We call $\overline{l_{-}(p/q)}$ the *lower angle* of $B_{p/q}$ and $\overline{l_{+}(p/q)}$ the *upper angle.*

As we will see, $l_{\pm}(p/q)$ is a string of q digits (0 or 1) and so $\overline{l_{\pm}(p/q)}$ denotes the infinite repeating sequence whose basic block is $l_{\pm}(p/q)$. Douady and Hubbard **[6]** have a geometric method involving Julia sets to determine these angles. Our method is more combinatorial and resembles algorithms due to Atela **[1]**, LaVaurs **[10]**, and Lau and Schleicher **[11]**.

To describe this algorithm, let $R_{p/q}$ denote rotation of the unit circle through p/q turns, i.e.,

$$R_{p/q}(\theta) = e^{2\pi i(\theta + p/q)}.$$

We consider the itineraries of points in the unit circle under R using two different partitions of the circle.

The *lower partition* of the circle is defined as follows. Let $I_0^- = \{\theta | 0 < \theta \leq 1 - p/q\}$ and $I_1^- = \{\theta | 1 - p/q < \theta \leq 1\}$. The boundary point 0 belongs to I_1^- and $-p/q = 1 - p/q$ belongs to I_0^-. We then define $\overline{s_{-}(p/q)}$ to be the itinerary of p/q under $R_{p/q}$ relative to this partition. We call the basic repeating block of this itinerary, $s_{-}(p/q)$, the *lower itinerary of* p/q. That is, $s_{-}(p/q) = s_1 \dots s_q$ where s_j is either 0 or 1 and the digit s_j is 0 if and only if $R_{p/q}^{j-1}(p/q) \in I_0^-$. Otherwise, $s_j = 1$.

For example, $s_{-}(1/3) = 001$ since

$$I_0^- = (0, 2/3], \qquad I_1^- = (2/3, 1],$$

and the orbit $\frac{1}{3} \to \frac{2}{3} \to 1 \to \frac{1}{3} \to \cdots$ lies in I_0^-, I_0^-, I_1^-, respectively.

Similarly, $s_{-}(2/5) = 01001$ since

$$I_0^- = (0, 3/5], \qquad I_1^- = (3/5, 1],$$

and the orbit is $\frac{2}{5} \to \frac{4}{5} \to \frac{1}{5} \to \frac{3}{5} \to 0 \to \frac{2}{5} \to \cdots$.

We also define the *upper partition* I_0^+ and I_1^+ as follows:

$$I_0^+ = [0, 1 - p/q), \qquad I_1^+ = [1 - p/q, 1).$$

The *upper itinerary of* p/q, $s_{+}(p/q)$, is then the repeating block of the itinerary of p/q relative to this partition. Note that I_0^+ and I_1^+ differ from I_0^- and I_1^- only at the endpoints.

For example, $s_{+}(1/3) = 010$ since the orbit is $\frac{1}{3} \to \frac{2}{3} \to 0 \cdots$ and

$$I_0^+ = [0, 2/3), \qquad I_1^+ = [2/3, 1).$$

This orbit starts in I_0^+, hops to I_1^+, and then returns to I_0^+ before cycling. For $2/5$, we have

$$I_0^+ = [0, 3/5), \qquad I_1^+ = [3/5, 1)$$

and $s_{+}(2/5) = 01010$.

The following theorem provides an algorithm for computing the angles of rays landing at $c(p/q)$. For a proof, we refer to **[5]** and **[6]**.

Theorem 2. *The rays $\overline{l_{\pm}(p/q)}$ landing at the root point $c(p/q)$ of the p/q bulb are given by $\overline{s_{-}(p/q)}$ and $\overline{s_{+}(p/q)}$.*

Note that $s_{\pm}(p/q)$ differ only in their last two digits (provided $q \geq 2$). Indeed we may write

$$s_{-}(p/q) = s_1 \dots s_{q-2} 0\, 1$$
$$s_{+}(p/q) = s_1 \dots s_{q-2} 1\, 0$$

The reason for this is that the upper and lower itineraries are the same except at $R_{p/q}^{q-2}(p/q) = -p/q$ and $R_{p/q}^{q-1}(p/q) = 0$, which form the endpoints of the two partitions of the circle.

We now define the *size of the p/q limb* to be the length of the interval $[s_{-}(p/q), s_{+}(p/q)]$. That is, the size of the p/q limb is given by the number of external rays that approach this limb. We may compute size of these bulbs explicitly by using the fact that $s_{\pm}(p/q)$ differ only in the last two digits.

Theorem 3. *The size of the p/q limb is $1/(2^q - 1)$. That is*

$$\overline{s_{+}(p/q)} - \overline{s_{-}(p/q)} = \frac{1}{2^q - 1}. \tag{2}$$

Proof: We write the binary expansion of the difference in the form

$$\begin{aligned}\overline{s_{+}(p/q)} - \overline{s_{-}(p/q)} &= \frac{1}{2^{q-1}} + \frac{1}{2^{2q-1}} + \frac{1}{2^{3q-1}} \\ &\quad + \cdots - \left(\frac{1}{2^q} + \frac{1}{2^{2q}} + \frac{1}{2^{3q}} + \cdots\right) \\ &= \frac{1}{2^{q-1}} \cdot \frac{2^q}{2^q - 1} - \frac{1}{2^q} \cdot \frac{2^q}{2^q - 1} = \frac{1}{2^q - 1}. \qquad \blacksquare\end{aligned}$$

As we see in Figure 7, the visual size of the bulbs does indeed correspond to the size as defined above.

7. THE SIZE OF LIMBS AND THE FAREY TREE.

In this section we relate the size of a p/q limb to the size of the limbs corresponding to the Farey parents of p/q. The following proposition relates the upper and lower itineraries of p/q and its Farey parents.

Proposition 1. *Suppose α/β and γ/δ are the Farey parents of p/q and that $0 < \alpha/\beta < \gamma/\delta \leq 1$. Then the lower itinerary $s_{-}(p/q)$ consists of the first q digits of the upper angle $\overline{s_{+}(\alpha/\beta)}$ of the smaller parent, and the upper itinerary $s_{+}(p/q)$ consists of the first q digits of the lower angle $\overline{s_{-}(\gamma/\delta)}$ of the larger parent.*

Proof: We consider only $s_{+}(p/q)$; the proof for $s_{-}(p/q)$ is similar.

From (1), we have

$$\frac{\gamma}{\delta} - \frac{p}{q} = \frac{1}{q\delta}.$$

Consider the orbits of p/q and γ/δ relative to the respective rotations $R_{p/q}$ and $R_{\gamma/\delta}$. Since γ/δ rotates faster than p/q, the distance between these orbits advances by $1/q\delta$ at each iteration. We thus have

$$R_{\gamma/\delta}^{j}(\gamma/\delta) - R_{p/q}^{j}(p/q) = \frac{j+1}{q\delta}.$$

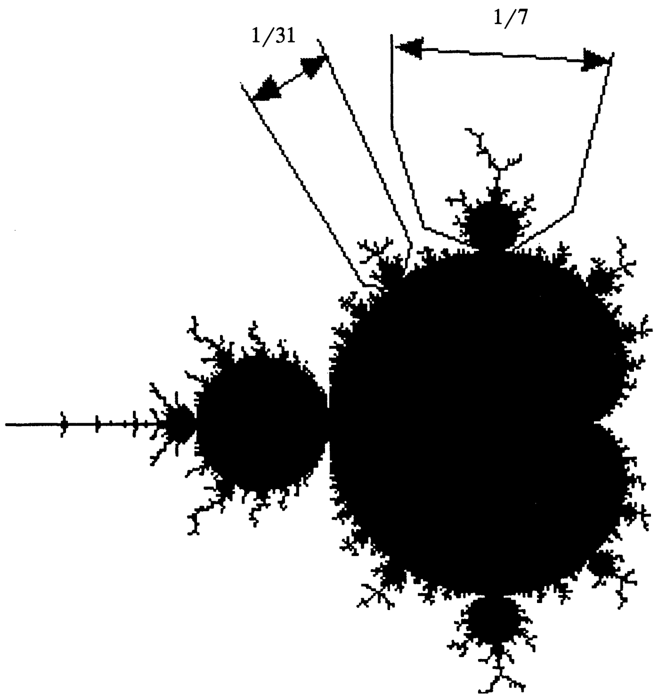

Figure 7. Size of the 2/5 and 1/3 limbs of $\mathscr{M}$.

It follows that $R^j_{p/q}(p/q)$ lies within $1/\delta$ units of $R^j_{\gamma/\delta}(\gamma/\delta)$ provided $j < q - 1$. Since points on the orbit of γ/δ under $R_{\gamma/\delta}$ lie exactly $1/\delta$ units apart on the circle, it follows that the first $q - 1$ entries in the itineraries of p/q and γ/δ are the same, provided we choose the lower itinerary for γ/δ and the upper itinerary for p/q. The reason for this is that the orbit of γ/δ lies ahead of that of p/q in the counterclockwise direction, but by no more than $1/\delta$ units. Choosing the upper itinerary for p/q and the lower for γ/δ forces the corresponding digits to be the same.

When $j = q - 1$, we have $R_{p/q}(p/q) = 0$ and

$$R^{q-1}_{\gamma/\delta}(\gamma/\delta) - R^{q-1}_{p/q}(p/q) = \frac{q}{q\delta} = \frac{1}{\delta}.$$

Hence

$$R^{q-1}_{\gamma/\delta}(\gamma/\delta) = \frac{1}{\delta}.$$

Therefore the qth digit in $s_+(p/q)$ is 0 and so is the qth digit of γ/δ, as long as $\gamma/\delta \neq 1$. ■

If one of the Farey parents is 0 or 1, we must modify Proposition 1.

Proposition 2. *Suppose that* 0 *is a Farey parent of* p/q. *Then the* q *digits in the lower itinerary of* p/q *are* $s_-(p/q) = 0\ldots01$. *If* 1 *is a Farey parent of* p/q *then* $s_+(p/q) = 1\ldots10$.

Proof: For $s_-(p/q)$, we first note that, since $0/1$ is a Farey parent, we must have $p = 1$. Thus, $s_-(p/q)$ is given by the itinerary of $1/q$ under counterclockwise rotation by $1/q$ units. We therefore have

$$I_0^- = (0, (q-1)/q], \qquad I_1^- = ((q-1)/q, 1].$$

It follows that the first $q - 1$ digits of $s_-(1/q)$ are 0, and the last digit is 1. If a Farey parent is $1/1$, the proof is similar, since in this case $p = q - 1$. ∎

We now complete the proof of one of the folk theorems mentioned in the introduction.

Theorem 4. *Suppose α/β and γ/δ are the Farey parents of p/q and that $0 \le \alpha/\beta < \gamma/\delta \le 1$. Then the size of the p/q limb is larger than the size of any other limb between the α/β and γ/δ limbs.*

Proof: Assume first that neither parent is 0 or 1. Propositions 1 and 2 ensure that $s_-(p/q)$ and $s_+(\alpha/\beta)$ agree in their first q digits. Using these binary representations, we have

$$\overline{s_-(p/q)} - \overline{s_+(\alpha/\beta)} \le \frac{1}{2^q}.$$

Similarly

$$\overline{s_-(\gamma/\delta)} - \overline{s_+(p/q)} \le \frac{1}{2^q}.$$

This implies that the arc of rays between the p/q limb and either of its parents' limbs has length no larger than $1/2^q$. Thus any limb between them has size smaller than $1/2^q$.

From (2), we know that

$$\overline{s_+(p/q)} - \overline{s_-(p/q)} = \frac{1}{2^q - 1}.$$

As this quantity is larger than $1/2^q$, it follows that the p/q limb attracts the largest number of rays between its two parents.

If one of the parents of $1/q$ is 0, then the size of the $1/q$ bulb is again $1/(2^q - 1)$, while the gap between 0 and $\overline{s_-(p/q)} = \overline{0\ldots01}$ is also $1/(2^q - 1)$. But then any limb between the $1/q$ limb and the cusp of the cardioid must have size strictly smaller than $1/(2^q - 1)$, again showing that the $1/q$ limb is the largest. The case of Farey parent 1 is handled similarly. ∎

8. THE FIBONACCI SEQUENCE. Theorem 4 shows that the Fibonacci sequence appears in the Mandelbrot set. As we have seen in Figures 4 and 5, the largest bulb between the $1/2$ and $1/3$ bulb is the $2/5$ bulb, and the largest between the $1/3$ and $2/5$ bulbs is the $3/8$ bulb. This progression continues, with the numerators (and denominators) forming the Fibonacci sequence. For example, we next have

$$\frac{2}{5} \oplus \frac{3}{8} = \frac{5}{13}.$$

The corresponding bulbs are shown in Figure 8.

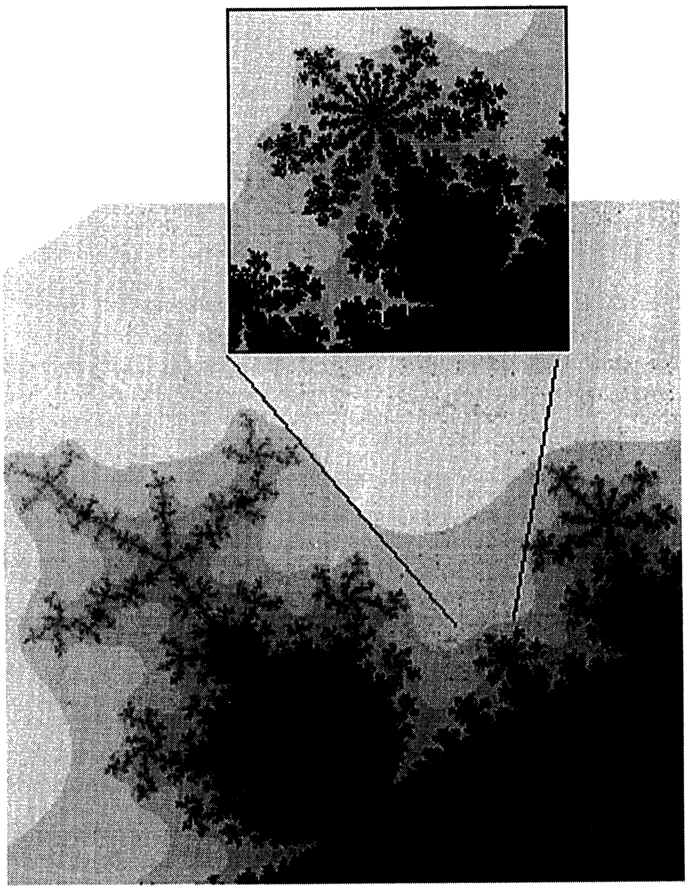

Figure 8. $\frac{2}{5} \oplus \frac{3}{8} = \frac{5}{13}$.

This sequence of bulbs actually converges to a single point on the boundary of the main cardioid. At this particular c-value, Q_c is known to have a fixed point z_0 with $Q_c'(z_0) = e^{2\pi i\theta}$, where θ is related to the golden ratio and hence is *highly irrational.* The dynamics of complex functions near such fixed points is the subject of the Fields' Medal work of J.-C. Yoccoz **[12]**.

9. CONCLUSION. The technique of measuring the size of certain portions of the Mandelbrot set by the length of the interval of rays that land on that portion provides justification for other folk theorems involving the size of $\mathcal{M}$. For example, this technique is used to identify the p/q bulb using the "lengths" of the spokes in its antenna. Once we know these rays, we can easily compute the lengths of the various spokes.

For example, it can be shown that the two rays that land at the junction point of the antenna adjacent to the principal spoke are given by $s_-\overline{s_+}$ and $s_+\overline{s_-}$, where we have dropped the p/q for clarity. These two rays are therefore given by preperiodic binary sequences that begin to repeat only after the qth entry. Thus, the vast majority of rays that land on the p/q limb actually approach the spokes of the antenna. For we have the following ordering of the rays landing on the p/q bulb:

$$\overline{s_-} < s_-\overline{s_+} < s_+\overline{s_-} < \overline{s_+}.$$

It is easy to check that the length of the arc of rays approaching the antenna between $s_-\overline{s_+}$ and $s_+\overline{s_-}$ is

$$\frac{1}{2^{q-1}} - \frac{2}{2^q(2^{q-1})}.$$

This number is much larger (for large q) than the length of the arc between $\overline{s_-}$ and $s_-\overline{s_+}$ or between $\overline{s_+}$ and $s_+\overline{s_-}$, each of which has length

$$\frac{1}{2^q(2^{q-1})}.$$

We can also use the two rays separating the principal spoke from the rest of the antenna to determine a list of the q rays that land on the junction point. Then we can determine that the shortest is located p/q turns in the counterclockwise direction from the principal spoke. See **[5]** for details.

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