Classical Mathematics.
Baroque Mathematics.
Romantic Mathematics?
Mathematics *Jazz!*
Also Atonal, New Age, Minimalist, and Punk Mathematics.

Jim Henle[1]

Mathematicians love music. Nearly everyone likes music, of course, but mathematicians are typically extreme. Many are devotees. More than a few are amateur musicians of some skill and much energy. Some seem to love music more than their own discipline. Why?

The two are poles apart. Compare: science on the one hand, art on the other; reason versus passion, utility versus felicity. Then too, the nerd versus the rock star. Is it a case of opposites attracting?

Not at all. Certainly the feelings are not reciprocated. Musicians are not famous for their appreciation of mathematics. The few exceptions (academic composers who base works on esoteric formulae) only reinforce our instinct that the music we revere cannot be reduced to algorithms. A crude survey affirms these impressions. At Smith College (no distribution requirements) between 1987 and 1992, mathematics majors were 11% more likely to have taken a music course than the ordinary undergraduate. At the same time, music majors at Smith were 42% *less* likely to have taken a mathematics course than the ordinary undergraduate.

The affinity, however, is thousands of years old. From the Pythogoreans to Aristoxenus to Boethius, philosophers have imagined music as a branch of mathematics. In theory, there is much that is mathematical in music: the physics of sound, the arithmetic of rhythm, the algebra of scales. Is this the answer? Do mathematicians see music as an intellectual cousin?

I think not. Painting, sculpture and architecture all have their mathematical components, yet there is no mystical (or statistical) connection. Smith mathematics majors are 10% *less* likely to have taken a literature course and 15% *less* likely to have taken an art course than the ordinary undergraduate. I would argue, in fact, that cause and effect have been confused here. The existence of countless mathematical analyses of music is merely evidence that mathematicians have been around, picking at the corpus of music in an attempt to understand its appeal.

What then, is the explanation?

Thesis: In brief, it's not that music is *mathematical.* It's that mathematics is *musical.*

I contend that there is something profoundly similar about mathematics and music. I apprehend this by the way the disciplines respond to the intellectual

[1] I would like to acknowledge my colleagues at Smith and elsewhere who were kind enough to read early versions of this paper and call attention to my excesses: Gian-Luigi Bellini, Dan Isaacson, Marjorie Senechal, Ruth Solie, and Tom Tymoczko. Of course, none of these can be held responsible for any such excesses that remain.

currents in society. Over centuries, the patterns of their growth are remarkably alike. Specifically, I will argue:

1. Mathematics has many of the characteristics of an art.
2. Viewed as an art, it is possible to identify artistic periods in mathematics: Renaissance, Baroque, Classical, and Romantic.
3. These periods coincide nicely and share many characteristics with the corresponding musical epochs, but *are significantly different* from those of painting and literature.

The structure of this paper follows the outline above. In the first section we discuss briefly the aesthetic nature of mathematics. In the second, we locate the mathematics that corresponds to the major artistic periods. In the third section, we date the periods.

As the argument unfolds, the reader may note some leaps and stumbles. We will address them in the fourth section. The concluding sections contain some further speculations and the conclusion.

I emphasize **this is not a mathematical study of music**. Such studies are a staple of mathematicians, musicians and philosophers. While often fascinating, they are not relevant. They do not explain the real affinity between mathematics and music. That affinity is more emotional than procedural, more spiritual than intellectual. It is the cultural context that matters, not abstract principles.

Finally, let me add that I do not deny the validity of analogies between mathematics and poetry, nor do I wish to ignore those artists such as Helaman Ferguson now realizing mathematical ideas in stone and on canvas. I will, however, insist that in a very real but elusive sense mathematics, as a cultural enterprise, is closer to music than to literature, painting, or sculpture.

1. MATHEMATICS AS ART. Mathematicians today readily recognize in their discipline the attributes of art. The usual view of mathematics as a science, as rooted in the real world and as a purveyor of absolute truth has much to recommend it, but misses important features and is arguably false.

First of all, the idea of "absolute truth" has been severely damaged in the last two centuries by the discovery of alternative geometries, alternative number systems, and even alternative truth. While many mathematicians still regard their work as having universal validity, their "universe" has been narrowed and "validity" redefined.

Second, mathematicians are creators. While some mathematical worlds are forced into being by the need to understand some real phenomenon, others are created in the same spirit as poems, symphonies, and temples. They are meant to be admired and enjoyed.

Third, mathematicians exercise considerable taste in their research. Beyond truth, we seek beauty, harmony, and elegance. We have aesthetic standards for our proofs. We have standards for the fields we investigate and the theorems we publish.

Not surprisingly, taste differs among mathematicians. Mathematical aesthetics is quite complex, encompassing a greater emotional range than the public commonly imagines. While one mathematical structure is admired for its symmetry, another is prized for its singularity. Mathematicians are attracted not merely to the beautiful, but also to the grand, the picturesque, and the gothic.

This is not to say that mathematics *is* art. What mathematics *is* is difficult to say. It shares traits with the arts, but also with the sciences, with philosophy, with theology, with sport, with gastronomy, and so on. I argue here only that mathematics can be viewed as an art in a meaningful way, and that such a perspective will help us to understand mathematics—and understand art as well.

2. MATHEMATICAL PERIODS. A word on our procedure. We are going to apply the concepts of "Renaissance," "Baroque," etc. to mathematics. The meaning of these terms have evolved over centuries. We will not use the most authoritative versions, but popular ones, in fact, we choose as our references standard texts in music, art, literature, and mathematics (respectively, Grout: *History of Music*, Gardner: *Art Through the Ages*, Benet: *The Reader's Encyclopedia*, and Kline: *Mathematical Thought from Ancient to Modern*). This is appropriate. Any categorization of artistic periods is arbitrary, but our sources are contemporary and are products of the same cultural context, and context is what this study is all about.

Renaissance Mathematics. There was an identifiable Renaissance in mathematics. It was made possible by the recovery and revival of Greek mathematics, fueled by the rise of commerce, and marked by increasing secularization. It is characterized by work that goes beyond annotation and summarization of the extant classics.

Summa de Arithmetica, Geometria, Proportioni et Proportionalita, Pacioi, 1494, is what we would call typical pre-Renaissance. It organized all that was currently known and issued gloomy predictions for future progress. *Ars Magna*, Cardano, 1545, on the other hand, broke new ground in a subject which classical authors had generally left untouched: the solution of third and fourth-degree equations.

An important example of Renaissance mathematics is the geometry of perspective. The major figures at this time were Leone Battista Alberti (1404–1472), Piero della Francesca (c. 1410–1492), Leonardo (1452–1519), and Albrecht Durer (1471–1528). The mathematics was developed expressly for painters representing the three-dimensional world on a two-dimensional canvas. The ideas were new and the practitioners were generally ignorant of Greek authors.

Baroque Mathematics. Grout characterizes the "Baroque" in terms of a new means of expression. He writes: "Just as seventeenth-century philosophers were discarding outmoded ways of thinking about the world and establishing other more fruitful rationales, the contemporary musicians were seeking out other realms of the emotions and an expanded language in which to cope with the new needs of expression."

There is, at this time, a corresponding new mathematical language: algebra. From the earliest traces of mathematics in ancient civilizations, the common element had been geometry. Even arithmetic, highly developed by the Greeks, was for them firmly embedded in geometry. The great discovery by Descartes (1596–1650) and Fermat (1601–1665) was that geometric forms and ideas could be expressed algebraically. Their method, analytic geometry, was the engine that enabled mathematics to make great leaps forward.

Grout also sees in the Baroque the somewhat incompatible tendencies of emotional intensity and precision. "Baroque music shows conflict and tension between the centrifugal forces of freedom of expression and the centripetal forces of discipline and order in a musical composition. This tension, always latent in any work of art, was eventually made overt and consciously exploited by Baroque musicians; and this acknowledged dualism is the most important single principle which distinguishes between the music of this period and that of the Renaissance."

We can see this dualism in mathematics. The duality is between algebra and geometry. Geometry was the disciplined side of mathematics. It was connected directly to the masterpieces of Greek science, works of rigor and power which were unmatched in the sixteenth century. In algebra was the "freedom of expression," especially the enthusiastic and almost careless use of infinites and infinitesimals, concepts explicitly forbidden by the Greeks.

The tension was very real. In the hands of intuitive geniuses such as Leibniz, the Bernoulis, and later, Euler (1707–1783), algebraic techniques produced a wealth of fundamental results. With others, however, contradictions and falsehoods beckoned alarmingly on every side. The gap between the certainty of geometry and the apparent lack of substance in algebra worried and divided mathematicians and philosophers. Thomas Hobbes (1588–1679) felt the excesses of algebra were totally unjustified. He described Wallis' *Arithmetica Infinitorum* as "a scab of symbols." The most famous attack was by Bishop Berkeley (1685–1753) who savagely mocked imprecision in the work of Newton and others.

Classical Mathematics. Here is Grout on Classical music: "The ideal music of the middle and later eighteenth century, then, might be described as follows: its language should be universal, not limited by national boundaries; it should be noble as well as entertaining; it should be expressive within the bounds of decorum; it should be 'natural', in the sense of being free of needless technical complications and capable of immediately pleasing any normally sensitive listener."

The music of this period spoke immediately to its listeners. So too must the mathematics we call "Classical." Rather than concerning itself with theory, with philosophy, it must be well-motivated by the real world. What Grout and others mean in discussing Classical art, is that the artists communicate easily and directly, that their work is understood and appreciated at once. Compare this to Kline: "Far more than in any other century, the mathematical work of the eighteenth was directly inspired by physical problems. In fact, one can say that the goal of the work was not mathematics, but rather the solution of physical problems; mathematics was just a means to physical ends."

The tension mentioned earlier remained throughout this era, but at a low level. Mathematicians "dared merely to apply the rules and yet assert the reliability of their conclusions." [Kline]. The justification was that it worked; worked in the sense that when applied to physical problems, the new analysis provided answers that were physically verified. Thus, mathematics was "natural" in the sense that it was grounded in the natural world. "The physical meaning of the mathematics guided the mathematical steps and often supplied partial arguments to fill in nonmathematical steps."

Mathematicians of this era sensed there were formal inadequacies in their methods, but on the whole, did not consider them a problem. The employment of mathematics, like the enjoyment of music, tended to be "free of needless technical complications."

Romantic Mathematics. "Romantic" is seemingly the most incongruous adjective we could apply to mathematics. Public impressions of mathematics are far removed from romance, and yet the term can be most apt. Here is Grout describing some of the chief characteristics of the romantic: "... romantic art differs from classic art by its greater emphasis on the qualities of remoteness and strangeness...," "Another fundamental trait of romanticism is boundlessness...," "romantic art aspires... to seize eternity...," "... Romanticism cherishes freedom, movement, passion, and endless pursuit of the unattainable. Just because its goal can never be attained, romantic art is haunted by a spirit of longing, of yearning after an impossible fulfillment."

Two very mathematical, very modern ideas are expressed here, the concept of the *infinite*, and the concept of the *impossible*. These embody the essence of what I call "romantic mathematics." Both flowered in the nineteenth century.

Let's take infinity first. For thousands of years, the human race had alternately shunned and flirted with the absolute infinite. It was rejected by Aristotle, relished by Lucretius, embraced by Bruno, discarded by Galileo, etc. etc., never rising above philosophy or religion. The first tentative steps to mathematical status were taken by Augustin Cauchy early in the nineteenth century. By 1900, Georg Cantor had lain the foundation for a theory of infinite numbers that would have scandalized the ancient world.

More dramatic is the story of impossibility. In the early nineteenth century, a series of problems which had plagued and motivated mathematicians from ancient times were found to have no solution. Three famous examples come to mind: the trisection of angles by straight-edge and compass, the representation of pi with radicals ("squaring the circle"), and the proof of Euclid's fifth postulate. All were found to be impossible.

Let's look more closely at the last example. Euclid's *Elements* was a compilation and distillation of the known mathematical world. It was a self-contained system that began with five postulates, statements that were accepted as obvious and not requiring proof. Almost from the start, however, there were questions about one of them: the fifth. This statement lacked the simplicity and compelling nature of the first four. Later authors felt that it could and should be proved from the others. For two thousand years, the problem festered. At times, it was put aside, at times work was intense. At times, a solution seemed near. Several writers, indeed, felt they had proved the fifth postulate. Toward the end of the 18th century, the possibility began to be considered that the task was *impossible*.

This was a giant step to take, and one which seemed in open rebellion with the Enlightenment. From our perspective, it is understandable that this would be problematic before the Romantic era. Indeed, it is possible that Gauss, the most powerful and respected mathematician of his age, had come to the conclusion that the problem was impossible over forty years before publishing.

Consider now the logical implications. If one believes that the fifth postulate can't be proved, then one must imagine the existence of a geometry in which the first four axioms are true and fifth false. Otherwise, the acceptance of the first four *compels the acceptance of the fifth*. If we can't prove the fifth then there must exist a geometry unlike any previously known. Here is the important point: mathematicians could accept that the axiom was not provable, yet they could not make the logical step and imagine a different geometry. This step was taken early in the nineteenth century.

The existence of new and strange worlds: *romantic mathematics*. The importance of this particular example is seen in the fact that while all the pieces of the puzzle were in the hands of mathematicians certainly for hundreds of years and arguably for thousands, it remained unsolved. When it *was* solved, it was solved independently by at least three, possibly five researchers, Gauss (perhaps around 1813), Bolyai (c. 1823), Lobachevsky (1827), Schweikart (c. 1812), and Young (Canada, 1860). *Why?*

The immediate suspicion is that something besides mathematics was at work. Our explanation is that the very environment, the intellectual climate of nineteenth century romanticism is the reason. Einstein said "Imagination is more important than knowledge." Indeed, the world had all the requisite knowledge, but until the nineteenth century, it did not have the requisite imagination.

After Romanticism. We should end with the romantic era. We are still too close to the twentieth century to understand it, especially when we are trying to grasp a phenomenon as delicate as the artistic milieu of mathematics. There are many identifiable movements: Post-Romanticism, Neo-Classicism, Impressionism, atonal music, and jazz. It is not clear that strong parallels should be drawn before we have the perspective to see what was really going on. Nonetheless, I will make a few irresponsible guesses at the end of this paper.

3. LOOKING AT CORRESPONDING PERIODS. I have set our discussion of mathematical history in a musical context, and so the reader may be unimpressed if the chronologies fit. I claim, nonetheless, that the correspondence is non-trivial. My definition of each mathematical period is inspired by the attributes of the musical one, not by the time frame. For example, I examined Grout's characteristics of the Baroque to decide what mathematics belongs to the Baroque.

In any case, the fit is very good. It is especially good when compared to the chronologies of painting and literature. Running through intellectual history is a significant time lag. Musical periods are generally late. They seem to appear noticeably later than the corresponding periods in the other arts. As we shall see, mathematics appears to match this gap.

Comparing Renaissances. Grout places the Renaissance in music in the range 1450–1600. I place the Renaissance in mathematics at 1500–1600. Gardner puts the Renaissance in art in the fifteenth and sixteenth centuries, with a "proto-Renaissance beginning as early as the late thirteenth century." Benet marks the Renaissance "from the mid-14th century to the end of the 16th century," although it may be referring to a general Renaissance not particularized to literature.

The fit is not merely chronological. Consider the following from Kline: "The Renaissance did not produce any brilliant new results in mathematics. The minor progress in this area contrasts with the achievements in literature, painting, and architecture, where masterpieces that still form part of our culture were created . . . " *Kline does not mention music.* In fact, the Renaissance did produce artists and writers whose works remain important and significant today. Consider Dante, Petrarch, Boccaçio, and Shakespeare, Giotto, Botticelli, Michelangelo, Durer, and Leonardo. At the same time, there are almost no comparable figures in music. Only Monteverdi has much popularity today. The names of Palestrina, Praetorius, di Lasso, Byrd, and Gabrielli are known, perhaps, but not to the general public, still less, their work.

Comparing Baroque Periods. Grout places the Baroque loosely between 1600 and 1750. "Baroque" does not seem to be recognized as a literary period. Close in feeling, however, are the metaphysical poets, Donne, Crawshaw, Marvell, also such writers as John Bunyan, all 17th century. Painting and architecture are roughly similar.

Analytic geometry, was developed around 1630. The crowning achievement of the mathematical Baroque, however, was the calculus. Many of the pieces had lain at hand for thousands of years. The critical ideas, however, could hardly be expressed, let alone discovered, without the language of analytic geometry. Not surprisingly, these ideas were discovered independently by the two greatest minds of the age, Newton (1642–1727) and Leibniz (1646–1716) soon after that language had been established.

One could argue that the Baroque in music and mathematics was not "mature" until the second half of the seventeenth century. In art, maturity was reached much earlier with such figures as Caravaggio (1573–1610), Velasquez (1599–1660), Rubens (1577–1640), Hals (c. 1580–1666), Poussin (1594–1665), and Rembrandt (1606–69).

Comparing Classical Periods. Grout sets the Classical period in music from 1770 to somewhere between 1800 and 1830. Gardner makes no reference to the Classical in art, but the Rococo might be considered equivalent. Indeed, the painter most often compared with Mozart (1756–1791) in spirit and feeling is Watteau (1684–1721). Gardner begins the Rococo early in the eighteenth century.

Literature is more difficult to place, but probably its Classical period is significantly earlier than music's. It might begin with figures such as Dryden (late 17th c). Benet describes Dryden's first works as essentially Baroque: "extravagant late metaphysical," and his later works as Classical: "restrained and natural," "orderly, lucid."

Mathematicians characteristic of the mood and content of the Classical period are Euler (1707–1783), Lagrange (1736–1813), Legendre (1752–1833), and Laplace (1749–1827). This seems to fit better with music than with art.

Consider also the subsidence of the Baroque tension. The great debate over the foundations of calculus, over the infinite, over the place of algebra simply faded away. The issues had in no way been resolved. It has been something of a mystery why the argument died down. In the light of our analysis, however, it does not seem strange. If mathematics is indeed like music, it is natural that mathematicians might simply move on. Grout quotes J. J. Quantz, a critic, writing in 1752, "... the old composers were too much absorbed with musical 'tricks' and carried them too far, so that they neglected the essential thing in music, which is to move and please." On the other hand, the debate raged in the first half of the eighteenth century—further evidence that the Classical period in mathematics begins late in the century, as with music.

Comparing Romantic Periods. The Romantic period is an excellent case for the consanguinity of mathematics and music. Gardner dates the first stirrings of Romantic art early in the eighteenth century in English Neo-Gothic. In Europe, it takes hold as Neo-classicism by, say, 1750. Romanticism in literature is also early, late eighteenth century. Rousseau (1712–1778) is sometimes called the father of romanticism. The first gothic novels appeared in the late eighteenth century.

The Romantic era in both mathematics and music did not begin until early in the nineteenth century. I consider the case for affinity here is especially strong because the identification of romantic mathematics is especially apt.

Summarizing (abbreviations indicate roughly the middle of the period):

	1400		*1500*		*1600*		*1700*		*1800*		*1900*
Music				Ren			Bar		Cla	Rom	
Mathematics				Ren			Bar		Cla	Rom	
Art		Ren				Bar		Cla	Rom		
Literature		Ren				Bar		Cla	Rom		

4. LEAPS, STUMBLES, AND ANTECEDENTS

Am I being fair? My definitions of "Renaissance mathematics," "Baroque mathematics," etc. were not derived in complete ignorance of the relative histories. This was unavoidable. It is certainly conceivable that I was corrupted in the process.

On the other hand, I am not making extravagant statements. I am only calling attention to parallels in the development of mathematics and music. These are certainly there. I don't claim there aren't others as well. Intellectual history is not a science. Perhaps others should attempt to chart similar analogies, between, say, painting and mathematics, or between music and physics. Such a project might well be successful without invalidating this study, however. As I remarked earlier, mathematics is "like" many fields, and in different ways. A new approach could simply reveal a correspondence of an entirely new sort.

Am I being simple-minded? Most of our definitions are based on descriptions from the third edition of Grout's *A History of Western Music*, a popular text first published in 1960. It's not a profound work, and it's probably dated.

Simple-minded yes, but appropriately so. Our focus is on the cultural context of mathematics and music. Mainstream and contemporaneous works such as Grout, Gardner and Kline provide excellent vantage points. In fact, the study could have been performed using texts a hundred years old, as long as they were largely reflective of the prevailing views.

Haven't I missed something? I have mapped the analogy in detail, but I haven't said a word about the mechanism. If music and mathematics develop in parallel, how and why does this happen?

I don't know. That may be difficult to discover. The effect I claim to have noticed here is delicate in the extreme. It is only by looking at the sweep of hundreds of years that I feel confident there is something there at all.

Am I being parochial? I've looked exclusively at western culture. Isn't it possible the conclusions do not carry over to other societies?

I deliberately restricted attention to areas in which I felt I had some knowledge. It may be that the phenomena I observed are not present in other cultural traditions. I would be most interested in studies which explored this.

And anyhow, is this new? Surely others have considered the cultural context of mathematics.

There are antecedents. First of all, there are works which locate mathematics in the stream of intellectual history. Kline's *Mathematics in Western Culture* is a good example. Such studies tend to treat mathematics as a science, however, and concentrate on its relationship with physics, chemistry, etc.

Many writers discuss mathematics and music. The approach is usually philosophical; the analogies between the disciplines as they have actually developed are not noticed. An exception is Tymoczko's paper, "Value Judgements in Mathematics: Can We Treat Mathematics as an Art?" which explores the two as they are practiced, asking such questions as: what corresponds in mathematics to musical performance? In recent work Tymoczko has pioneered and promoted the conception of mathematics as a cultural activity.

Especially significant is the work of Yves Hellegouarch who has written several papers on mathematics and music. Many treat music mathematically, but a few move in the direction of this study. Most intriguing is "Le Romantisme des les mathematiques" which discusses the romantic features of mathematics, especially in the nineteenth century, and draws parallels to other artistic movements. Prof. Hellegouarch has also linked mathematics and music in education.

There are works which discuss the Romantic movement and the sciences. I found Knight's "Romanticism and the Sciences" intriguing. It focuses on the life of Humphry Davy, poet and chemist using his life span, 1778–1829 to date the period. This seems in serious disagreement with the analysis in this paper, which starts romanticism in mathematics much later. There is, however, *no mention* of mathematics in the paper, or for that matter, in the volume of papers in which it appears. The impulse to separate mathematics from the physical sciences (quite common in the history of science) is significant. It seems likely to me that science, or physics for example, correlates better with literature than with music (or with mathematics).

There have been studies which attempt to discern cultural differences among the mathematics of various countries. This sort of work is called 'ethnomathematics'. Though it is not quite what we are doing here, it is predicated on similar principles.

Jamie James' recent book, *The Music of the Spheres*, deserves special mention. He traces the history in our culture of the conception of the "musical universe," the belief that everything in our world has an explanation, a purpose, a meaning, and a relationship with everything else. It is a theme at least as old as Pythagoras in which "music" and "harmony" have a larger meaning and unite the physical universe with the spiritual. The effect of this theme on music, however, is the chief focus, and music is related more to science than mathematics.

Finally, I note Scott Buchanan's *Poetry and Mathematics*. Buchanan develops a rather ideosyncratic view of mathematics, but makes a strong case for its similarity to poetry. The emphasis on economy of expression is a particularly telling point.

5. GUESSES

Atonal mathematics: This century swirls, from the present perspective, in turbulent musical and mathematical currents. Nonetheless, a reasonably strong argument can be made linking the atonal movement in music with the drive towards formalism in mathematics. Russell and Whitehead's *Principia Mathematica*, for example, the work of Frege, Peano, and Zermelo, mark a radical departure in mathematics. Foundations are now studied for their own sake. In both mathematics and music, it was formerly the case that the work preceded the theory. First Mozart, then Kochel. First Leibniz, then Cauchy. Now it is reversed: first, twelve-tone theory, then Shoenberg. First Zermelo-Fraenkel Set theory, then large cardinals.

Mathematics Jazz: Musically, jazz is the product of two entirely different musical traditions. It is characterized by standard ideas, chords, rhythms, instruments, combined and used in radical and creative ways. A mathematical candidate for jazz is topology. Topology brings together geometry, analysis, algebra, and combinatorics in a field that is totally new. Coincidentally, the jazz player's careless (to the classical musician) treatment of *meter* corresponds neatly to the topologist's disregard of *metric*.

6. IRRESPONSIBLE GUESSES. What follows now is *highly* speculative. As noted earlier, the phenomenon we study is most subtle. Any attempt to discern it in recent movements, isolated cultures, or individuals is extremely risky!

Minimalist Mathematics: In the last 15 years or so, a field of mathematics has arisen that concerns itself with finding the minimum assumptions needed to prove various theorems. It is called "reverse mathematics." Within mathematics, it has not achieved wide recognition, though it is known to most atonal (formal) mathematicians.

New Age Mathematics: Fractal geometry certainly seems in the right mood, with the gauzy pictures it produces of imaginary landscapes. On the other hand, the related subjects of chaos theory and catastrophe theory may be more appropriate. These fields model life qualitatively, not quantitatively, recognizing that most aspects of the world cannot be understood in terms of precise numerical structures.

Punk Mathematics: So-called "brute force" techniques might be described as "punk." So also might the recent practice of using computers to assist in proofs, and to gather inductive evidence when a deductive proof is out of reach.

Here is another: the gradual expansion of what we call "numbers" is an important thread in the history of mathematics. In recent years, however, the operative number system consists merely of those numbers expressible in so many bits by a computer. This a finite system containing no square roots, no irrational numbers, not even *pi*. In reality, most such systems do not even contain the fraction 1/3, just an approximation, .333333. Punk?

Numbers: Number lies at the very heart of mathematics. The history of number is a thread that runs through the history of mathematics from the very earliest times. Is there a possible correspondence between number and musical instruments? A case in point might be the development of keyboard instruments.

Personalities: For the reasons discussed earlier, it is probably *not* a good idea to look for the musical Gauss or the mathematical Beethoven, though it is certainly tempting (especially in the case of Gauss and Beethoven). Chance must definitely play a role in the development of discipline, and that role is magnified in personalities.

On the other hand, there are a few persons who have operated in several fields. I was tempted to look for lessons from this, but on reflection it does not seem justified. Rousseau, for example, composed music. The "father of romanticism" wrote pieces that are thoroughly Baroque. Is this further evidence that musical periods lag behind literary ones? Not necessarily. One could argue that he merely wrote and composed in the style of the times. I have labelled one style "Romantic" and the other "Baroque," but it is the labelling we are debating. If I had labelled that sort of music "Romantic," would the example of Rousseau support a claim that musical periods coincide with literary ones?

For what it is worth, the great English architect Christopher Wren (1632–1723) was also a mathematician. He is described as a transitional figure, his buildings having characteristics of the Baroque and the Classical. They seem more Classical to me. In any case, his mathematics was almost pre-Baroque.

7. CONCLUSIONS. I have presented, I believe, a *prima facie* case for the parallel development of mathematics and music. It embodies a very different understanding of what is popularly imagined a "scientific" discipline. It suggests that the dynamics of mathematical invention resemble the dynamics of artistic invention.

Many of us in mathematics have long felt our field is misunderstood. Children are taught from an early age that mathematics is exact, unforgiving, and arbitrary. Their relationship to it is complex and troubled. They see it as a trap, a contest in which they must guess what the teacher is thinking. For most, it is alternately boring and painful, a game they cannot win, a puzzle where the solution is rarely complete. The situation is perpetuated by the large number of teachers who share, sometimes consciously, sometimes unconsciously, this grim view of mathematics.

It's a terrible way to educate children. Few can ever become fond of a subject treated this way. The import of this study is that it is worse than bad strategy, it is a fundamentally flawed picture of mathematics. I would argue that mathematics should be taught as music is taught. Students should make mathematics *together* (as in fact professional mathematicians do), not alone. Creative ideas should be stressed over the "right way to do it." Mathematics should be a treat, not a chore. And finally, students should perform mathematics; they should *sing* mathematics and *dance* mathematics.

A lovely example of what can be accomplished in this way is Zoltan Dienes' *Mathematics Through the Senses*, unfortunately out of print. Dienes presents an astonishing variety of settings for exploring significant mathematics at the grade-school level. The very first, the "three cornered waltz" is a perfect example. The basic dance form is presented which grade-school children, in groups of three, can perform and vary. The steps are simple, but they lead the young dancers into an exploration of the symmetric group S_3.

As an art, mathematics has not had a wide audience. Its passions and its pleasures are denied to most. To the public, it is cold and barren. To poets, it is cold and austere. Both are quite wrong. Mathematics is not divine; it is mortal. Mathematics is not law; it is taste. Mathematics is not calculation, but communication. The best mathematics is not *true*; it is *beautiful.*

REFERENCES

Benet, William Rose, *The Reader's Encyclopedia*, New York: Thomas Y. Crowell Co., 1965.

Buchanan, Scott, *Poetry and Mathematics*, John Day Co. 1929.

Dienes, Zoltan, *Mathematics Through the Senses, Games, Dance, and Art*, NFER Pub. Co. Ltd., 1973.

Gardner, Helen, *Art Through the Ages*, 7th edition, revised by Hosrt De La Croix and Richard G. Tansey, New York: Harcourt, Brace, 1980.

Grout, Donald Jay, *History of Music*, 3rd edition, New York: W. W. Norton & Co., 1980.

Hellegouarch, Yves, "Le Romantisme dans les Mathematiques, ou un Regard Oblique sur les Mathematiques du XIX^{e} Siecle," Yves Hellegouarch, in *Destin del l'Art Desseins de la Science*, Bessot, Hellegouarch, le Goff, eds.

James, Jamie, *The Music of the Spheres*, Grove Press, 1993.

Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford: Oxford University Press, 1972.

Kline, Morris, *Mathematics in Western Culture*, Oxford: Oxford University Press, 1953.
Knight, David, "Romanticism and the Sciences," in *Romanticism and the Sciences*, Andrew Cunningham, ed., Cambridge: Cambridge University Press, 1990.
Tymoczko, Thomas, "Humanistic and Utilitarian Aspects of Mathematics," to appear in the *Proceedings of the 7th International Congress of Mathematical Education.*
Tymoczko, Thomas, "Value Judgements in Mathematics: Can We Treat Mathematics As An Art?", to appear in *Essays in Humanistic Mathematics*, Alvin White, ed., MAA Monographs.

Clark Science Center
Smith College
Northampton, MA 01063
jhenle@smith.edu.

PICTURE PUZZLE
(from the collection of Paul Halmos)

This picture was taken almost a quarter of a century ago, and he still hasn't learned to spell his name "right".
(see page 50)