AN INTERESTING CANTOR SET

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1. Introduction. Cantor sets are all the same topologically, since any two are homeomorphic. But the classical Cantor set also enjoys a number of properties which are not preserved under homeomorphism. These properties extend in part to the Cantor set described in this paper. I was led to its definition by a study of recent work of Feigenbaum [**2**], [**3**]. However, it is of interest in its own right and will be presented from this point of view here. The discussion illustrates a variety of significant topics, such as the notion of dimension, Fourier–Stieltjes transforms, 2-adic integers, and ergodic theory.

2. The Two-Ratio Cantor Set. We define a *Cantor set* C to be a compact metric space which has no isolated point and which has the property that for any two distinct points a, b, there exist disjoint closed sets A, B with union C containing a, b respectively. A compact subset of the real line is a Cantor set if and only if it has no isolated point and contains no interval.

Let r_1, r_2 be positive numbers with sum less than 1. We construct a Cantor set $C = C(r_1, r_2)$ in the following way. Put

$$\begin{aligned} E_0 &= [0,1], \\ E_1 &= [0, r_1] \cup [1 - r_2, 1], \\ E_2 &= [0, r_1^2] \cup [r_1(1 - r_2), r_1] \\ &\quad \cup [1 - r_2, 1 - r_2(1 - r_2)] \cup [1 - r_1 r_2, 1], \\ &\quad \ldots . \end{aligned}$$

In general, E_k is a union of 2^k disjoint compact intervals. If $[a, b]$ is a typical interval in E_k, the intervals in E_{k+1} are given by $[r_1 a, r_1 b]$ and $[1 - r_2 b, 1 - r_2 a]$. It should be observed that if we choose c so that $r_1 < c < 1 - r_2$, then all intervals $[r_1 a, r_1 b]$ lie to the left of c and all intervals $[1 - r_2 b, 1 - r_2 a]$ lie to the right of c. By induction we easily see that $E_{k+1} \subset E_k$ and that E_k has Lebesgue measure $(r_1 + r_2)^k$. More precisely, if the lengths of the $m = 2^k$ intervals in E_k are, from left to right, $d_1, \ldots, d_m$, then the lengths of the $2m$ intervals in E_{k+1}, also from left to right, are

$$r_1 d_1, r_2 d_1, r_2 d_2, r_1 d_2, \ldots, r_1 d_{m-1}, r_2 d_{m-1}, r_2 d_m, r_1 d_m.$$

Thus if we set

$$C = \bigcap_{k=0}^{\infty} E_k,$$

then C is a nonempty compact set of Lebesgue measure zero. A point belongs to C if and only if it is either an endpoint of an interval of some set E_k or a limit of such endpoints. It follows readily that C is a Cantor set.

For $r_1 = r_2 = \frac{1}{3}$ we obtain the original set of Cantor himself. The more general case $r_1 = r_2 = r$ is of some interest in harmonic analysis and is discussed, for example, in Kahane and Salem [**6**].

We note first that *the set* $C = C(r_1, r_2)$ *has Hausdorff dimension* α, *where* α *is the unique root between* 0 *and* 1 *of the equation*

$$r_1^\alpha + r_2^\alpha = 1. \tag{1}$$

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The definition of Hausdorff measure is given by Kahane and Salem, and a proof may be modelled on one given there for the case $r_1 = r_2 = r$. In that case $\alpha = \log 2/\log r^{-1}$ can be expressed in closed form. The result is also contained in Theorem II of Moran [7].

There are other reasonable definitions of dimension besides that of Hausdorff. We show next that one proposed by Besicovitch and Taylor [**1**] gives the same value in the present case, although in general the two definitions do not agree.

In forming E_1 we exclude from E_0 an open interval of length τ. In forming E_2 we exclude from E_1 two open intervals of lengths $r_1\tau$ and $r_2\tau$. In general, in forming E_{k+1} we exclude from E_k 2^k open intervals of which one has length $r_1^k\tau$, a further k have length $r_1^{k-1}r_2\tau$, another $k(k-1)/2$ have length $r_1^{k-2}r_2^2\tau, \dots,$ and one has length $r_2^k\tau$. For any β such that $0 < \beta \leqslant 1$ the sum of the βth powers of the lengths of these intervals is

$$\tau^\beta \sum_{j=0}^{k} \binom{k}{j} r_1^{(k-j)\beta} r_2^{j\beta} = \tau^\beta \eta^k, \tag{2}$$

where we have put $\eta = r_1^\beta + r_2^\beta$. Thus the sum of the βth powers of the lengths of all excluded intervals is

$$\sigma_\beta = \tau^\beta(1 + \eta + \eta^2 + \cdots).$$

Hence

$$\begin{aligned} \sigma_\beta &= \tau^\beta(1-\eta)^{-1} \text{ if } \eta < 1, \\ &= \infty \text{ if } \eta \geqslant 1. \end{aligned} \tag{3}$$

The Besicovitch–Taylor dimension is the infimum α of all β for which σ_β is finite and thus satisfies $r_1^\alpha + r_2^\alpha = 1$.

Since the Hausdorff and Besicovitch–Taylor dimensions are equal, it follows by a result of Hawkes [**4**] that the entropy dimension of C also exists and has the same value α. The entropy dimension is defined by dividing $[0, 1]$ into n equal subintervals, counting the number c_n of subintervals which contain points of C, and taking the limit of $\log c_n/\log n$ as $n \to \infty$.

3. The Corresponding Cantor Function. We are now going to construct a distribution function ψ with support on the Cantor set C. Let $I_{k,j}(j = 1, \dots, 2^k - 1)$ denote the open intervals complementary to E_k, numbered from left to right. Let ψ_k be the unique continuous function on $E_0 = [0, 1]$ such that $\psi_k(0) = 0$, $\psi_k(1) = 1$, ψ_k is linear on each interval of E_k, and

$$\psi_k(x) = j2^{-k} \text{ if } x \in I_{k,j}(j = 1, \dots, 2^k - 1).$$

Then ψ_k is nondecreasing, $\psi_{k+1} = \psi_k$ on $I_{k,j}(j = 1, \dots, 2^k - 1)$, and

$$|\psi_{k+1}(x) - \psi_k(x)| < 2^{-k}$$

for all $x \in E_0$. Hence the sequence $\{\psi_k\}$ converges uniformly on E_0. Its limit ψ is a continuous, nondecreasing function such that $\psi(0) = 0$, $\psi(1) = 1$, and ψ is constant on any open interval complementary to any set E_k.

This construction can be applied to an arbitrary Cantor set on the real line. However, the special structure of the set C makes it possible to say more. In fact ψ satisfies the functional equations

$$\begin{aligned} \psi(r_1 x) &= \frac{1}{2}\psi(x), \\ \psi(1 - r_2 x) &= 1 - \frac{1}{2}\psi(x), \quad (0 \leqslant x \leqslant 1) \\ \psi[(1-x)r_1 + x(1-r_2)] &= \frac{1}{2}. \end{aligned} \tag{4}$$

The first two relations are easily verified if x is an endpoint of an interval $I_{k,j}$ and extend at once to arbitrary $x \in E_0$.

The functional equations (4) completely characterize ψ. In fact if ψ^* is any bounded function which satisfies (4), then the difference $\phi = \psi - \psi^*$ satisfies

$$\begin{aligned} \phi(x) &= \frac{1}{2}\phi(x/r_1) \quad (0 \leqslant x \leqslant r_1), \\ &= -\frac{1}{2}\phi[(1-x)/r_2] \quad (1 - r_2 \leqslant x \leqslant 1), \\ &= 0 \quad (r_1 \leqslant x \leqslant 1 - r_2). \end{aligned}$$

If we denote by μ, μ_1, μ_2 the supremum of $|\phi(x)|$ over the intervals $[0,1], [0, r_1], [1 - r_2, 1]$ respectively, then $\mu = \max(\mu_1, \mu_2)$. On the other hand $\mu_1 = \frac{1}{2}\mu$, $\mu_2 = \frac{1}{2}\mu$. It follows that $\mu = 0$.

There is no difficulty in principle in determining the moments of ψ. It is readily shown that

$$\int_0^1 x\, d\psi_k(x) = \frac{1}{2}(1 + h + \cdots + h^k),$$

where $h = (r_1 - r_2)/2$. Letting $k \to \infty$ we obtain

$$\int_0^1 x\, d\psi(x) = \frac{1}{2}(1-h)^{-1}. \tag{5}$$

It may further be shown that

$$(2 - r_1^2 - r_2^2)\int_0^1 x^2\, d\psi(x) = 1 - r_2(1-h)^{-1}. \tag{6}$$

It is also of interest to consider the Fourier–Stieltjes transform

$$\hat{\psi}(\lambda) = \int_0^1 e^{-i\lambda x}\, d\psi(x) \quad (-\infty < \lambda < \infty).$$

We have

$$\begin{aligned} \hat{\psi}(r_1\lambda) &= \int_0^1 e^{-i\lambda r_1 x}\, d\psi(x) \\ &= 2\int_0^1 e^{-i\lambda r_1 x}\, d\psi(r_1 x) \\ &= 2\int_0^{r_1} e^{-i\lambda x}\, d\psi(x) \end{aligned}$$

and similarly

$$e^{-i\lambda}\hat{\psi}(-r_2\lambda) = 2\int_{1-r_2}^1 e^{-i\lambda x}\, d\psi(x).$$

It follows that $\hat{\psi}$ satisfies the functional equation

$$\hat{\psi}(r_1\lambda) + e^{-i\lambda}\hat{\psi}(-r_2\lambda) = 2\hat{\psi}(\lambda). \tag{7}$$

Harmonic analysts may be interested to determine for what values of r_1 and r_2 the set C is a set of uniqueness.

4. Connection with the 2-adic Integers. We define a 2-adic integer to be an infinite sequence $\alpha = (a_0, a_1, a_2, \dots)$, where $a_i = 0$ or 1 for all i. If $\beta = (b_0, b_1, b_2, \dots)$ is another such sequence the sum

$$\alpha + \beta = (c_0, c_1, c_2, \dots)$$

is defined in the following way. If $a_0 + b_0 < 2$, then $c_0 = a_0 + b_0$, but if $a_0 + b_0 \geqslant 2$, then $c_0 = a_0 + b_0 - 2$ and we carry 1 to the next position. The terms $c_1, c_2, \dots$ are successively determined in the same fashion. With this definition of addition the set J of all 2-adic integers is an abelian group.

We can also define a metric on J by setting $d(\alpha, \alpha) = 0$ and $d(\alpha, \beta) = 2^{-k}$ if $\alpha \neq \beta$ and k is the least integer such that $a_k \neq b_k$. This metric is invariant and nonarchimedean, i.e., for all $\alpha, \beta, \gamma \in J$

$$d(\alpha + \gamma, \beta + \gamma) = d(\alpha, \beta),$$
$$d(\alpha + \beta, 0) \leqslant \max[d(\alpha, 0), d(\beta, 0)].$$

Moreover J is now a compact topological group.

If $\delta = (1, 0, 0, \dots)$, then the multiples $n\delta (n = 0, 1, 2, \dots)$ consist precisely of all $\alpha = (a_0, a_1, a_2, \dots)$ with $a_i = 0$ for all large i. Hence the semigroup J_0 formed by these multiples is dense in J.

Now let u and v denote the maps of the unit interval into itself defined by

$$u(x) = r_1 x, v(x) = 1 - r_2 x \quad (0 \leqslant x \leqslant 1).$$

Then every endpoint, other than 0 and 1, of an interval of the set E_k can be uniquely represented in the form

$$w_m \circ \cdots \circ w_1(1),$$

where $w_i = u$ or v for each i and $1 \leqslant m \leqslant k$. For example, $v \circ v \circ u(1)$ represents the endpoint $1 - r_2(1 - r_1 r_2)$ of E_3. To such an endpoint we make correspond the 2-adic integer

$$\alpha = (a_0, a_1, a_2, \dots),$$

where $a_i = 0$ for $i > m$, $a_m = 1$, and $a_i = 0$ or 1 according as $w_{m-i} = u$ or v for $0 \leqslant i < m$. To the endpoints 1 and 0 we make correspond the 2-adic integers $\delta = (1, 0, 0, \dots)$ and $0 = (0, 0, 0, \dots)$. In this way we define a 1-1 map ω of the set C_0 of all endpoints of intervals of the sets E_k onto the set J_0 of all 2-adic integers $\alpha = (a_0, a_1, a_2, \dots)$ with $a_i = 0$ for all large i.

We will show that this map ω is uniformly continuous. If D is an interval of the set E_k, then it has the form

$$D = w_k \circ \cdots \circ w_1 E_0,$$

where $w_1, \dots, w_k$ are uniquely determined u's or v's. One endpoint of D is $\xi = w_k \circ \cdots \circ w_1(1)$. If $w_i = u$ for $1 \leqslant i \leqslant k$, then the other endpoint of D is $\bar{\xi} = 0$. Otherwise there exists an h $(1 \leqslant h \leqslant k)$ such that $w_h = v$ and $w_i = u$ for all $i < h$, and the other endpoint of D is then

$$\bar{\xi} = w_k \circ \cdots \circ w_{h+1}(1).$$

If $\alpha = \omega(\xi)$ and $\bar{\alpha} = \omega(\bar{\xi})$, then in any event we have $d(\alpha, \bar{\alpha}) \leqslant 2^{-k}$. Any point ξ' of C_0 in the interior of D has the form

$$\xi' = w_l' \circ \cdots \circ w_1'(1),$$

where $w_1', \dots, w_l'$ are uniquely determined u's or v's and $l > k$. Moreover

$$D' = w_l' \circ \cdots \circ w_1' E_0 \subset D = w_k \circ \cdots \circ w_1 E_0.$$

Since also

$$D' \subset w_l' \circ \cdots \circ w_{l-k+1}' E_0$$

we must have $w_{l-i}' = w_{k-i}$ for $0 \leqslant i < k$. If $\alpha' = \omega(\xi')$, it follows that $d(\alpha, \alpha') \leqslant 2^{-k}$.

Put

$$r_0 = \min(r_1, r_2, 1 - r_1 - r_2).$$

Then the distance between any two distinct endpoints of intervals of E_k is at least r_0^k. Thus if two distinct points ξ_1, ξ_2 of C_0 are distant less than r_0^k, then they lie in the same interval D of E_k. Hence, by what we have just shown, the corresponding 2-adic integers α_1, α_2 satisfy $d(\alpha_1, \alpha_2) \leqslant 2^{-k}$.

This proves that ω is uniformly continuous. The inverse map ω^{-1} is also uniformly continuous. For if $d(\alpha_1, \alpha_2) \leqslant 2^{-k}$, then $\xi_1 = \omega^{-1}(\alpha_1)$ and $\xi_2 = \omega^{-1}(\alpha_2)$ lie in a common interval $D = w_k \circ \cdots \circ w_1 E_0$ and hence are distant at most r^k, where $r = \max(r_1, r_2)$.

It follows that the map ω admits a unique continuous extension, which we will still denote by ω, mapping the whole of C into J. Moreover ω must map C onto J, since its range is compact and contains a dense subset of J. Since the inverse map also admits a unique continuous extension, ω is actually a homeomorphism of C onto J. If we now define the sum $\xi = \xi_1 \oplus \xi_2$ of two elements of C by $\omega(\xi) = \omega(\xi_1) + \omega(\xi_2)$, then C acquires the structure of a compact topological abelian group. Moreover this structure is naturally connected to the definition of the set C.

Conversely, the measure ν on C determined by the distribution function ψ can be transferred to a measure μ on J. In fact this is precisely the Haar measure on J. This may be shown without difficulty from the explicit form for Haar measure on J, given in Hewitt and Ross [**5**, p. 202]. Returning to C, we see that the measure ν is invariant under the group action just defined. If we set $T\xi = \xi \oplus 1$, for any $\xi \in C$, then it follows from the ergodic theory of group rotations, described in Walters [**9**, pp. 160–162], that for any continuous function $f: C \to \mathbb{R}$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\xi) \to \int_C f(x)\, d\psi(x) \text{ uniformly as } n \to \infty.$$

It may be noted that the measure ν is also invariant under the piecewise linear transformation S of the unit interval defined by

$$\begin{aligned} Sx &= r_1^{-1}x \quad \text{for } 0 < x < r_1, \\ &= (1 - r_1 - r_2)^{-1}(x - r_1) \quad \text{for } r_1 < x < 1 - r_2, \\ &= r_2^{-1}(1 - x) \quad \text{for } 1 - r_2 < x < 1. \end{aligned}$$

In fact the inverse image of $(0, x)$ is the union of the intervals

$$(0, r_1 x), \quad (r_1, r_1 + (1 - r_1 - r_2)x), \quad (1 - r_2 x, 1),$$

whose total ν-measure is

$$\psi(r_1 x) + 1 - \psi(1 - r_2 x) = \psi(x).$$

Transformations of this type have been extensively studied in ergodic theory; see, e.g., Parry [**8**] and Wilkinson [**10**].

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