Reflections on Rippling Water

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1. INTRODUCTION. On a summer evening standing beside a large lake that extends to the horizon, we observe the moon's reflection on the rippled surface of the water. When the moon is low, but nonetheless completely above the horizon, the reflection may still appear as a long uninterrupted yellow column which stretches from some point on the lake to the horizon. Its length can be considered as infinite. Later, when the moon rises higher up in the sky the reflection changes aspect and becomes a shorter beam, in the shape of a narrow oval. It is closer to us and no longer extends to the horizon. Its length is now finite.

Stars may appear in this evening sky. If a gentle breeze is blowing, each one of these stars will appear to be reflected an odd number of times in a given direction on the surface of the lake.

These evocative images raise interesting mathematical questions. At what angle does the moon's reflection change from an infinite image to a finite one? Is it possible to see exactly two reflections of the same star? The object of this paper is to answer these questions. Our analysis only requires simple trigonometry.

2. THE THEORY. Suppose an observer at height H_1 sees a reflected object on the wavelet M at a distance x across the water.

Let $\alpha = \alpha(x)$ be the angle measured in radians between the normal MN to the wave with the vertical V. Let α_0 be the maximal value of $|\alpha(x)|$ and define $\varphi(x)$ by $\alpha(x) = \alpha_0\varphi(x)$ so that $|\varphi(x)| \leq 1$. Let i be the angle of reflection (Figure 1 and 2).

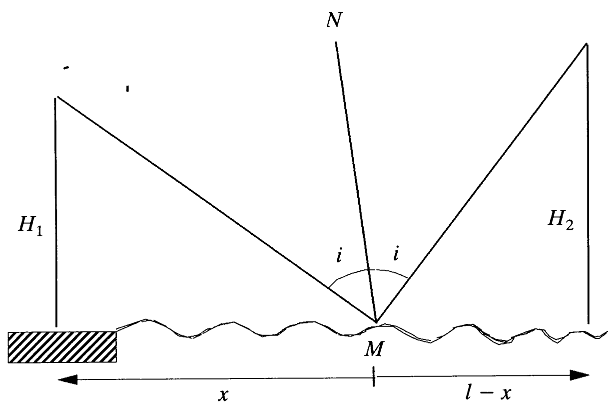

Figure 1

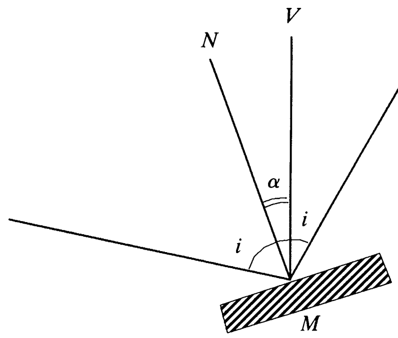

Figure 2

Trivial trigonometry shows that

$$\begin{cases} \tan(i+\alpha) = \dfrac{x}{H_1} \\ \tan(i-\alpha) = \dfrac{l-x}{H_2}. \end{cases}$$

Hence

$$\begin{aligned} \tan 2\alpha &= \tan[(\alpha+i)-(i-\alpha)] \\ &= \frac{\tan(\alpha+i)-\tan(i-\alpha)}{1+\tan(\alpha+i)\tan(i-\alpha)} \\ &= \frac{\dfrac{x}{H_1}+\dfrac{x-l}{H_2}}{1+\dfrac{x(l-x)}{H_1H_2}} = \frac{(H_1+H_2)x-lH_1}{H_1H_2+lx-x^2}. \end{aligned}$$

We now assume that α_0 is small. Then

$$2\alpha \approx \frac{(H_1+H_2)x-lH_1}{H_1H_2+lx-x^2}.$$

Finally,

$$\varphi(x) \approx \frac{(H_1+H_2)x-lH_1}{2\alpha_0\left(H_1H_2+lx-x^2\right)}. \tag{1}$$

Before exploiting the relationship given by (1), let us analyze the corresponding equation

$$\varphi(x) = \frac{x(H_1+H_2)-lH_1}{2\alpha_0\left(H_1H_2+lx-x^2\right)}. \tag{2}$$

Note that each side of (2) has a physical interpretation. The function φ describes the shape of the waves while the right-hand side represents the distance at which a reflection occurs. So, for a given shape φ, the solutions of (2) are the approximate distances at which a reflection occurs. In particular, the number of solutions is the number of reflected images we see.

Now let us analyze the equation. We start by looking at the simplest case when there are no waves at all: $\varphi \equiv 0$. Then

$$x = \frac{lH_1}{H_1 + H_2}.$$

Thus there is only one reflection so that the observer sees a perfect image of the object. If, in particular $H_1 = H_2$, then $x = l/2$ and the light ray is reflected at the midpoint between the observer and the object. This is of course well known.

Let us now discuss the general case where φ stays small. We solve the equation (2) graphically.

Let β be the curve

$$\beta \colon x \mapsto \frac{x(H_1 + H_2) - lH_1}{2\alpha_0(H_1H_2 + lx - x^2)}.$$

In the interval $(0, l)$, β is continuous and increasing. Furthermore

$$\beta(0) = -\frac{l}{2\alpha_0 H_2} \quad \text{and} \quad \beta(l) = \frac{l}{2\alpha_0 H_1}.$$

We assume that both H_1 and H_2 are strictly less than $l/2\alpha_0$ (l is large and α_0 is small).

The curve $x \mapsto \varphi(x)$ oscillates in the horizontal strip $y = -1$, $y = +1$. Supposing φ is continuous in the interval $[0, l]$, both curves intersect either at an odd number of points or infinitely often. Thus, whatever the shape of the waves may be, one should see either an odd number of reflections, or infinitely many. (This last case may indeed occur if, for example, φ has a singularity of the type $(x - a)\sin(x - a)^{-1}$ in the neighbourhood of some $a \in (0, l)$).

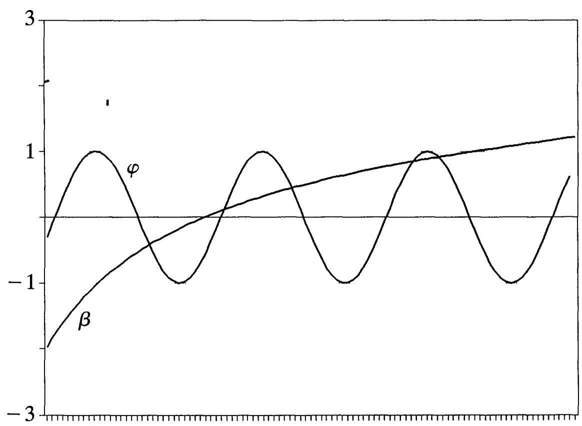

Figure 3

3. LIMITING CASES. Let us study the solutions of equation (2) when $l = +\infty$ (reflection of the moon or the sun...).

We denote by w the angle at which the infinitely far away object is seen. Then H_1/l is negligible and $H_2/l = \tan w$. Thus rewriting the right hand side of equation (1) in the form

$$\frac{x\left(H_1 l^{-1} + H_2 l^{-1}\right) - H_1}{2\alpha_0\left(-\dfrac{x^2}{l} + x + \dfrac{H_1 H_2}{l}\right)}$$

we have

$$\varphi(x) \approx \frac{x \tan w - H_1}{2\alpha_0(x + H_1 \tan \omega)} = \gamma(x).$$

As before we solve the equation $\varphi(x) = \gamma(x)$ graphically and we suppose that φ oscillates a great many times, say

$$\varphi(x) = \sin \lambda x$$

where λ is large. We solve equation (3) for $x \in (0, l)$

$$\sin \lambda x = \gamma(x). \tag{3}$$

Since $\gamma(x)$ is monotonically increasing we know that the smallest solution x_S of (3) occurs when this function is -1 and the largest solution x_L occurs when the function is $+1$.

When $\tan w < 2\alpha_0$, we see from Figure 4 that the two curves intersect infinitely many times and the smallest solution is approximately

$$x_S \approx \max\left\{0, H_1 \frac{1 - 2\alpha_0 \tan w}{\tan w + 2\alpha_0}\right\}.$$

In this case, the reflection on the water extends from x_S to the horizon. When $\tan w > 2\alpha_0$, the solution is shown on Figure 5.

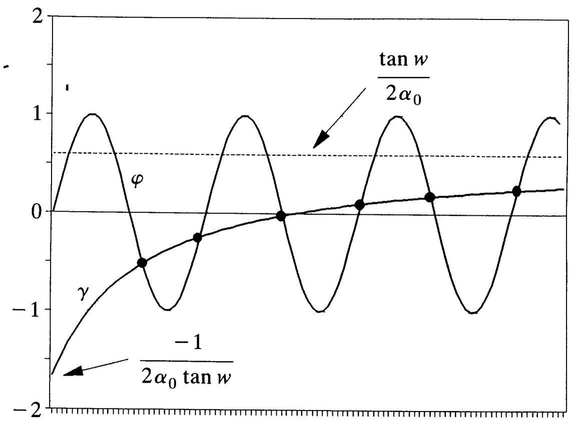

Figure 4

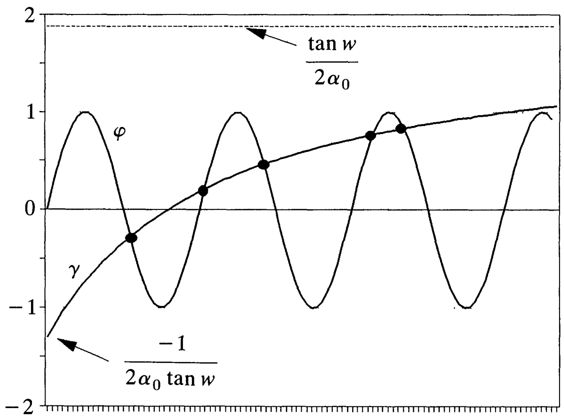

Figure 5

In this case there is only a finite odd number of reflections and the reflections lie between x_S and x_L

$$x_L \approx \frac{H_1}{\tan w - 2\alpha_0}(1 + 2\alpha_0 \tan w).$$

It follows that the critical w_c at which the reflection ceases to be infinite is therefore

$$w_c = \tan^{-1}(2\alpha_0).$$

As α_0 is assumed to be small, we have

$$w_c \approx 2\alpha_0.$$

Finally, if one is given the shape of the waves as a Cartesian equation

$$y = \psi(x),$$

then

$$\alpha(x) = \tan^{-1}\psi'(x),$$

provided ψ is differentiable. As $|\alpha(x)|$ is small, this entails $\alpha(x) \simeq \psi'(x)$ so that the critical angle is

$$w_c \approx 2\max_x |\psi'(x)|.$$

4. AN APPLICATION. The amplitudes of real waves on the ocean, far away from the coast (say 100 yards or more) are difficult to measure: they move rapidly and their size may be small, especially if we are discussing wavelets or even ripples. On the other hand, w_c can be quite easily measured at sunset: observe at what angle w_c the reflection starts to touch the horizon. If we assume that during that time of waiting, the waves keep approximately the same shape, say

$$\varphi(x) = \alpha_0 \sin \lambda x \cos \lambda ct$$

where c is the velocity of the wave and t is time, then the knowledge of w_c and of the frequency λ gives us the amplitude A of the waves

$$A = \frac{w_c}{2\lambda}.$$

Our analysis is also valid for studying the microscopic structure of a glossy surface. The macroscopic observation of a reflecting luminous point provides information on the fine structure of the surface. Determining w_c measures the product $A\lambda$.

It was only after completing this work that I discovered M. Minnaert's delightful book [1] on the "Nature of Light & Colour in the open air." It discusses related topics and I highly recommend it (see in particular pp. 23–26). I wish to thank the referee and Jacques Harthong for helping me to improve the exposition and the graphs.

Addendum. Many authors have studied the reflection on rippling water. I would like to single out M. V. Berry's beautiful article "Disruption of images: the caustic-touching theorem," *J. Opt. Soc. Am. A*. 4, 1987, pp. 561–569.

REFERENCE

1. M. Minnaert, *The Nature of Light & Colour in the Open Air*, Dover Publ., Inc., 1954.

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PICTURE PUZZLE
(from the collection of Paul Halmos)

Are they related?
(see page 809.)