UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

THE SHORTEST CURVE THAT MEETS ALL THE LINES THAT MEET A CONVEX BODY

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A telephone company, while repairing buried cable, has discovered that although the cable is buried 1 m deep, often the cable is not directly under the marker that is supposed to be erected above it. They do know that the cable is always within 2 m of the marker in the horizontal plane. To ensure finding the cable, even when its direction is unknown, the repairmen dig a 1-m-deep trench in a circle of radius 2 m about the marker (see Fig. 1). In 1974, M. Magidor showed us that Fig. 2 gives a more efficient way of finding the cable, and he asked for the length of the shortest trench that will find the cable (assuming that the cable is straight and *does* lie within 2 m of the marker). We demonstrated in [4] that for a continuous trench this method is optimal.

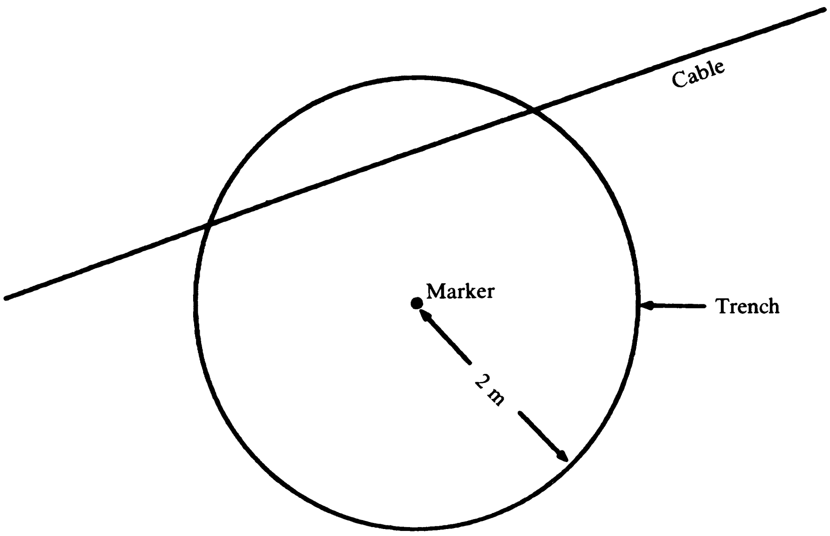


FIG. 1

What is the shortest curve that meets all the lines that meet a given circle? The answer may depend on our meaning of the words "curve" and "length," as we shall see below. Let $R \to S$ mean that R and S are sets in the plane (or more generally, n-dimensional Euclidean space) and every straight line meeting the set R meets the set S. Some short curves S such that $C \to S$ are shown in Figs. 3–9 for various convex sets C (lengths shown are percentages of the length of the boundary, where "length" will be defined below). Figs. 3–6 represent regular polygons. The

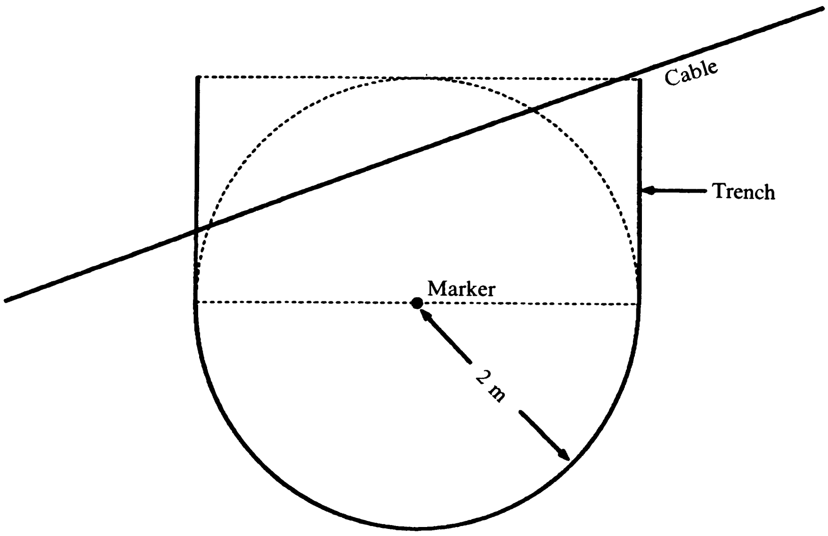
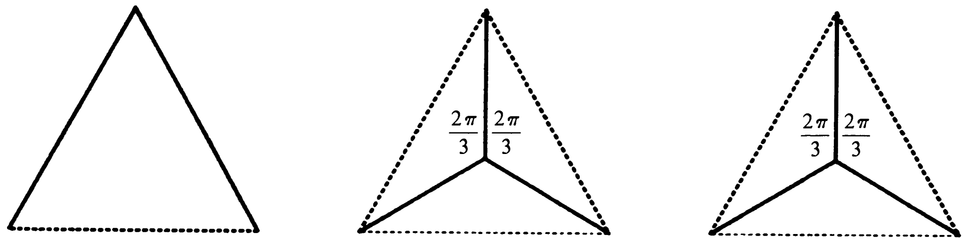
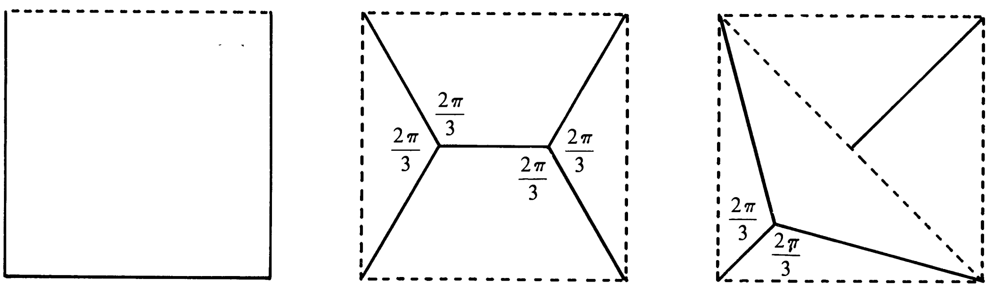


FIG. 2



(a) $\frac{2}{3} = .666666666\ldots$ (b) $\frac{\sqrt{3}}{3} = .577350269\ldots$ (c) $\frac{\sqrt{3}}{3} = .577350269\ldots$

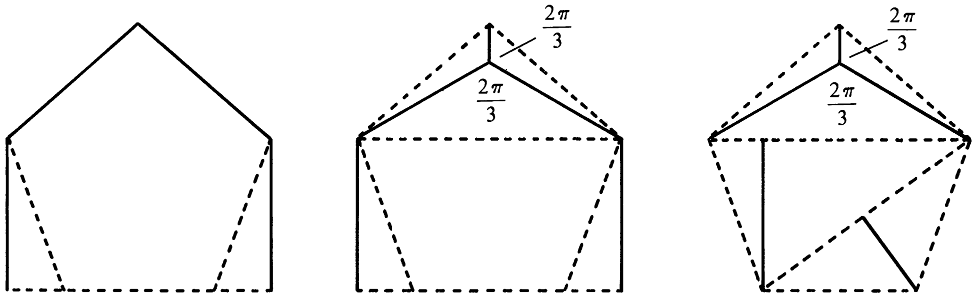
FIG. 3



(a) $\frac{3}{4} = .75$ (b) $\frac{1+\sqrt{3}}{4} = .683012702\ldots$ (c) $.659739609\ldots$

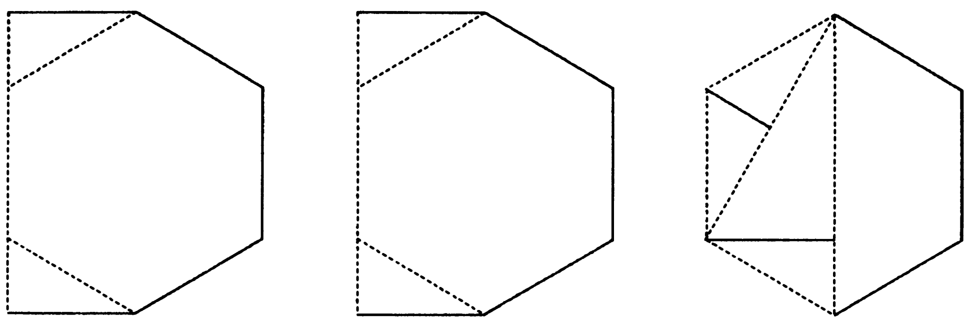
FIG. 4

figures (a) illustrate the shortest known arcs, the figures (b) illustrate the shortest known connected sets, and the figures (c) illustrate the shortest known closed sets. Note that in Figs. 3(b) and (c), 4(b), and 7(b), the shortest known sets join all the vertices of the given polygon—the so called *Steiner span* of the vertices (see **[1]** and **[8]**). (The Steiner span seems to be different from the shortest connected set meeting all the lines that meet a given polygon when that polygon has many sides.)



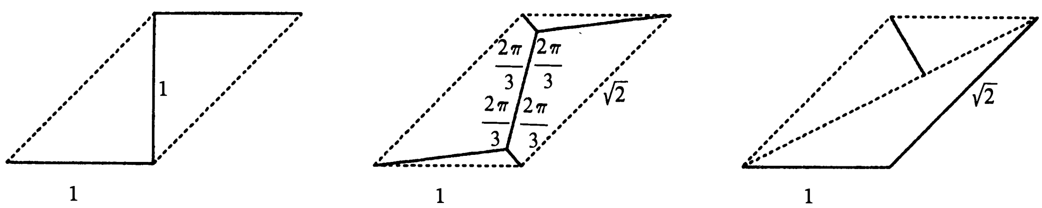
(a) .780422607... (b) .778231365... (c) .705577112

FIG. 5



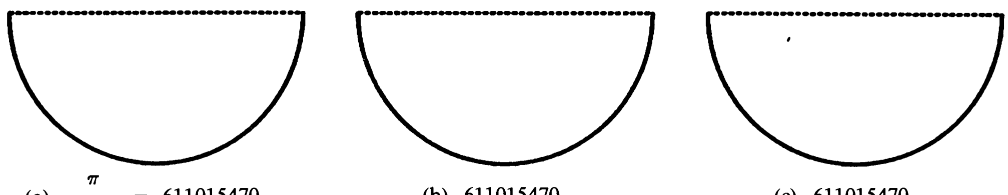
(a) $\frac{3+\sqrt{3}}{6} = .788675135\ldots$ (b) .788675135... (c) $\frac{7+\sqrt{3}}{12} = .727670901\ldots$

FIG. 6



(a) $\frac{3}{2+2\sqrt{2}} = .621320344\ldots$ (b) .602538433... (c) .592620968...

FIG. 7



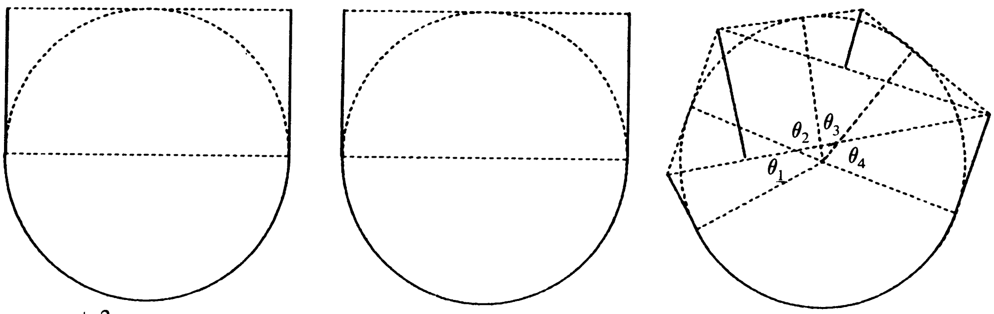
(a) $\frac{\pi}{\pi+2} = .611015470\ldots$ (b) .611015470... (c) .611015470...

FIG. 8

If we restrict our attention to *paths*, that is, continuous functions $f(t) = (x(t), y(t))$ from the unit interval I into the plane, the natural length to consider is *path length*:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left\{ \left(x((k+1)/n) - x(k/n)\right)^2 + \left(y((k+1)/n) - y(k/n)\right)^2 \right\}^{1/2}.$$

A path f is called an *arc* if f is one-to-one. It was shown in [4] that the shortest path that meets all the lines that meet the unit circle is an arc and has length $\pi + 2$ [Fig. 9(a)].



(a) $\dfrac{\pi+2}{2\pi} = .818309886\ldots$ (b) $.818309886\ldots$ (c) $.763926466\ldots$

$\theta_1 = .96 \quad \theta_2 = 1.04$

$\theta_3 = .7 \quad \theta_4 = 1.2$

FIG. 9

A natural generalization is the *n-arc*; that is, a sequence $P_n = (f_1, f_2, \ldots, f_n)$ where each f_i is an arc. If P_n is an n-arc, its *arc length* is the sum of the arc lengths of the f_i's.

We can measure the length of any Borel set in n-dimensional Euclidean space by means of the *one-dimensional Hausdorff measure* λ_1. The *α-dimensional Hausdorff outer measure* (where α is a positive real number) of any subset S of a metric space is

$$\lambda_\alpha(S) = \lim_{\delta\to 0}\left(\inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} E_i)^\alpha \,\Big|\, \bigcup_{i=1}^{\infty} E_i = S \text{ and } \operatorname{diam} E_i \leqslant \delta \text{ for all } i\right\}\right).$$

For example, if C is the Cantor set, then $\lambda_1(C) = 0$ but $\lambda_1(C^2) = \infty$.

Little is known about the relationship $R \to S$. In [3] and [4], it is shown that: (1) if R is closed and bounded, then there is a shortest (in terms of λ_1) closed connected set S such that $R \to S$ and its length is $\pi + 2$ (see Fig. 9); (2) if R is convex and ∂R is its boundary, then any closed set S such that $R \to S$ satisfies $\lambda_1(S) \geqslant \frac{1}{2}\lambda_1(\partial R)$; (3) if R is a compact convex set in the plane, then the shortest path f such that $R \to fI$ is one-to-one. Now we shall prove a theorem that answers a question stated in Section 2 of [4].

THEOREM. *If R is a compact set in the plane, then there exists a closed set S satisfying $R \to S$ and of minimal length (in terms of λ_1) among those with at most n connected components.*

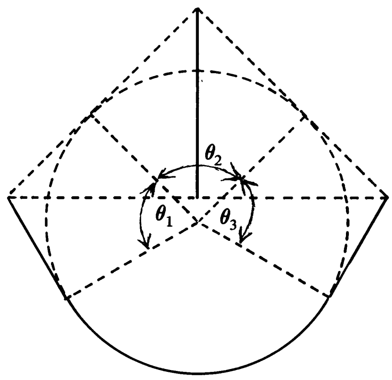
Proof. Let λ_0 be the infimum of the lengths $\lambda_1(S)$ for all closed S of at most n components with $R \to S$. Let S_i have at most n components and satisfy $R \to S_i$ and $\lim \lambda_1(S_i) = \lambda_0$. By choosing an appropriate subsequence of the S_i, we can assume that $S_{i,1}, \ldots, S_{i,k}$ are components of S_i whose distance from R is bounded as $i \to \infty$, while $S_{i,k+1}, \ldots, S_{i,n}$ are the components whose distance from R is unbounded. By again choosing an appropriate subsequence of the S_i, we can assume that, for $j \leqslant k$, S_{ij} converges in Hausdorff distance to certain closed sets S_j^* (as $i \to \infty$) and that, for $j > k$, the slopes of the lines meeting S_{ij} and R converge to some angles α_j. Let $S^* = S_1^* \cup \cdots \cup S_k^*$. Thus the only lines meeting R but missing S^* would have one of the slopes $\alpha_{k+1}, \ldots, \alpha_n$. But, since S^* is compact, if there exists any line L meeting R and missing S^*, then for any $p \in L$, any L' containing p and such that the angle between L and L' is small enough also misses S^*. Thus a finite number of exceptional slopes is impossible, and $R \to S^*$ follows.

But since the S_{ij} are connected and the sequence S_{ij} converges in Hausdorff distance to S_j^*, it follows that the S_j^* are connected and

$$\lambda_1(S_j^*) \leqslant \liminf_{i \to \infty} \lambda_1(S_{ij})$$

(for a simple proof and references to related inequalities, see for example [**4**, Theorem 3]). Hence $\lambda_1(S^*) \leqslant \lambda_0$, and, since $R \to S^*$, $\lambda_1(S^*) = \lambda_0$. This concludes the proof of the theorem.

Now we list some open questions. Let B_2 be the unit disk.



$\theta_1 = \theta_3 = 1.28652\ldots$

$\theta_2 = 1.19106\ldots$

FIG. 10

Q1. Does the shortest n-arc P_n such that $B_2 \to P_n I$, have exactly n components? (A short 2-arc is shown in Fig. 10 and the shortest we know is a 3-arc shown in Fig. 9(c).)

Q2. Does there exist a shortest closed set S in the plane such that $B_2 \to S$?

Q3. For the regular polygon C_n with n sides, what is the shortest connected closed set S such that $C_n \to S$?

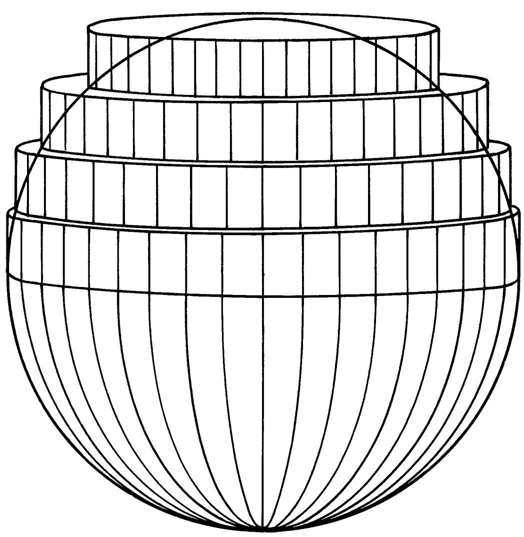


FIG. 11

A similar problem is

Q4. Does there exist a closed set S in 3-dimensional space such that $\lambda_2(S)$ is minimal and $B_3 \to S$, where B_3 is the unit ball?

In 1974, R. Laver found that for every $\varepsilon > 0$ there exists a set S with $B_3 \to S$ and $\lambda_2(S) < 2\pi + \pi^2/2 + \varepsilon$. This set S (see Fig. 11) consists of the lower hemisphere (of area 2π) plus vertical rings standing on the surface of the sphere. The first ring stands on the equator, the second ring stands on the circle formed by the intersection of the plane of the top of the first ring and the sphere, etc. The height of the last ring extends to the north pole. Since

$$\int_0^{\pi/2} 2\pi \cos\alpha \, d(\sin\alpha) = \frac{\pi^2}{2},$$

it is clear that if the consecutive rings are narrow enough, then their joint area is less than $\pi^2/2 + \varepsilon$. It is also clear that every line that intersects the sphere intersects S. We do not know if there exists an S with $B_3 \to S$ and $\lambda_2(S) = 2\pi + \pi^2/2$.

Editorial note. The authors have since shown that the Steiner tree of a triangle T is the shortest connected set which meets all the lines which meet T.

The problem has already appeared [**2, 3, 5, 7**] in various forms: a hunter lost in a dense forest who knows he is within a mile of a straight boundary; a swimmer at sea in a thick fog who knows she is within a mile of a straight shoreline. Each has zero visibility, but can do dead reckoning navigation. The title of Croft's paper [**2**] doesn't immediately suggest a connexion, but see his section 3; he gives Ogilvy [**7**] as his source. Eggleston [**3**] solves the question originally asked by Croft, and also by the authors and others:

Q0. What is the shortest (in terms of λ_1) connected closed set such that $B_2 \to S$?

by maximizing the radius of the disc, rather than by minimizing the length of the connected set. The answer is as shown in Fig. 2; a special case was earlier considered by Joris [**5**]. Moran considers a related problem and mentions others at the end of his paper [**6**].

References

1. F. R. K. Chung and R. L. Graham, Steiner trees for ladders, Ann. Discrete Math., 2 (1978) 173–200; MR 58#315.

2. H. T. Croft, Curves intersecting certain sets of great circles on the sphere, J. London Math. Soc. (2) 1(1969) 461–469; MR 40#865.

3. H. G. Eggleston, The maximal inradius of the convex cover of a plane connected set of given length, Proc. London Math. Soc. (3) 45(1982) 456–478; MR 84e:52004.

4. Vance Faber, Jan Mycielski, and Paul Pedersen, On the shortest curve which meets all the lines which meet a circle, Ann. Polon. Math., 44(1984) 249–266.

5. H. Joris, Le chasseur perdu dans la forêt: un problème de géométrie plane, Elem. Math., 35(1980) 1–14; MR 81d:52001.

6. P. A. P. Moran, On a problem of S. Ulam, J. London Math. Soc., 21(1946) 175–179; MR 8, 597n.

7. C. Stanley Ogilvy, Tomorrow's Math, Unsolved Problems for the Amateur, 2nd ed., Oxford University Press, 1972 (1st ed. 1962, pp. 23–24).

8. R. Spira, The isogonic and Erdős-Mordell points of a simplex, this MONTHLY, 78(1971) 856–864; MR 44#5861.

ANSWER TO PHOTO ON PAGE 792

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