MATH 4200 HW: PROBLEM SET ZERO

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General Comment: Our approach to set theory in this course will be highly naive, and except for peripheral matters you will not need to know more than the basic trichotomy between finite, countably infinite and uncountably infinite sets. In case anyone is considerably set-theoretically sophisticated: you should assume the Axiom of Choice in all matters.

Exercise 0.1. *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*
a) Show: if f and g are injective, so is $g \circ f$. If f and g are surjective, so is $g \circ f$. If f and g are bijective, so is $g \circ f$.
b) Show: if $g \circ f : X \to Z$ is injective, then f is injective. Show by example that $g \circ f$ may be surjective without f being surjective.
c) Show that if $g \circ f$ is surjective, then g is surjective. Show by example that $g \circ f$ may be injective without g being injective.
d) Suppose g is bijective. Show that f is surjective iff $g \circ f$ is surjective and f is bijective iff $g \circ f$ is bijective.
e) Suppose f is bijective. Show that g is injective iff $g \circ f$ is injective and g is bijective iff $g \circ f$ is bijective.

Exercise 0.2. [Ka, Exc. 1.2.7, Exc. 1.2.8] *Let A, B, C be sets.*
a) Show: $C \setminus (A \cup B) = (C \setminus A) \cup (C \setminus B)$.
b) Show: $C \setminus (A \cap B) = (C \setminus A) \cap (C \setminus B)$.
c) Show: $B \setminus (B \setminus A) = A \cap B$.
d) Show: $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.
*(Note that this set is called the **symmetric difference** of A and B.)*
e) Show that each of union and intersection distributes over the other:

$$(A \cap B) \cup C = (A \cap B) \cup (A \cap C).$$

$$(A \cup B) \cap C = (A \cup B) \cup (A \cup C).$$

f) In this part, all capital letters denote subsets of some fixed set X. Show:

$$A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i).$$

$$A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i).$$

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} X \setminus A_i.$$

$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} X \setminus A_i.$$

Let $f : X \to Y$, let $A \subset X$ and let $B \subset Y$. We define the **direct image**

$$f(A) = \{f(x) \mid x \in A\} \subset Y$$

and the **inverse image**

$$f^{-1}(B) = \{x \in A \mid f(x) \in Y\}.$$

Much of higher mathematics involves some calisthenics with direct and inverse images. The general rule of thumb is that inverse images are better!

Exercise 0.3. [Ka, Exc. 1.4.9] *We take notation as above.*
a) Show: if $A_1 \subset A_2 \subset X$, then $f(A_1) \subset f(A_2)$.
b) Show: if $B_1 \subset B_2 \subset Y$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$.
c) Show: if $B \subset Y$, then $f(f^{-1}(B)) \subset B$.
d) Show that in part c), equality holds for all $B \subset Y$ iff f is surjective.
e) Show: if $A \subset Y$, then $f^{-1}(f(A)) \supset A$.
f) Show that in part e), equality holds for all $A \subset X$ iff f is injective.

Exercise 0.4. [Ka, Exc. 1.4.10] *We take notation as above. Let $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$.*
a) Show: $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
b) Show: $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
c) Show: $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Show that equality holds for all A_1 and A_2 iff f is injective.
d) Show: $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
(Inverse images are better!)
e) Generalize all parts of this exercise to arbitrary families of subsets $\{A_i\}_{i \in I}$ of X and $\{B_j\}_{j \in J}$ of Y.

Exercise 0.5. [Ka, Exc. 1.4.13] *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*
a) Show: $g \circ f$ is injective iff f is injective and $g|_{f(X)}$ is injective.
b) Show: $g \circ f$ is surjective iff $g|_{f(X)}$ is surjective.

Exercise 0.6.
a) Explain how an equivalence relation on a set X induces a partition of X.
b) Explain how a partition of a set X induces an equivalence relation on X.

Exercise 0.7. *Let $f : X \to Y$ be a function. For $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$ and call it the **fiber over y**.*
a) Show that $\{f^{-1}(y)\}_{y \in Y}$ is a partition of X. What is the associated equivalence relation?
b) Show that f is surjective iff there are no empty fibers and injective iff no fiber has more than one element.

Exercise 0.8. *Let $f : X \to Y$ and $g : X \to Z$ be maps. We say that **f factors through g** if there is a map $q : Y \to Z$ such that $g = q \circ f$.*
a) Suppose f is surjective. Show that if f factors through g, the map $q : Y \to Z$ is unique.
b) Show that the following are equivalent:
(i) f factors through g.
(ii) For all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $g(x_1) = g(x_2)$.
(iii) Let E_f be the equivalence relation determined by the fibers of f and E_g be the equivalence relation determined by the fibers of g. Then $E_f \subset E_g$.

Exercise 0.9. *We denote by $\leq$ a binary relation on a set X. Consider the following properties:*

(Reflexivity) For all $x \in X$, $x \leq x$.

(Anti-Symmetry) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

(Transitivity) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

(Totality) For all $x, y \in X$, either $x \leq y$ or $y \leq x$.

*A relation which satisfies reflexivity and transitivity is called a **quasi-ordering**. A relation which satisfies reflexivity, anti-symmetry and transitivity is called a **partial ordering**. A relation which satisfies all four properties is called an **ordering** (sometimes a **total** or **linear ordering**).*

a) Show that the divisibility relation on the integers is a quasi-ordering but not a partial ordering.

b) Show that for any set X, $\subset$ is a partial ordering on the power set 2^X.

c) Let $\leq$ be a quasi-ordering on X, and define $x \sim y$ if $x \leq y$ and $y \leq x$. Show that this is an equivalence relation. Let $Y = X/\sim$ be the set of equivalence classes. Show that the relation $\sim$ descends to Y in the following sense: if $x_1 \sim x_2$ and $y_1 \sim y_2$ are elements of X, then $x_1 \leq y_1$ iff $x_2 \leq y_2$. Thus $\leq$ defines a binary relation on Y. Show that it is a partial ordering.

Exercise 0.10. *An ordered set $(X, \leq)$ is **well-ordered** if every nonempty subset has a least element.*

a) Show that for an ordered set X, exactly one of the following holds:

(i) X is well-ordered.

(ii) There is an infinite descending chain in X: i.e., there is a function $f : \mathbb{Z}^+ \to X$ such that for all $m < n$, $f(m) > f(n)$.

*b) Prove the **Monotonicity Lemma**: for any infinite ordered set X, at least one of the following holds:*

(i) There is an infinite ascending chain in X: i.e., a function $f : \mathbb{Z}^+ \to X$ such that for all $m < n$, $f(m) < f(n)$.

(ii) There is an infinite descending chain in X.

REFERENCES

[Ka]   I. Kaplansky, *Set theory and metric spaces*. Allyn and Bacon Series in Advanced Mathematics. Allyn and Bacon, Inc., Boston, Mass., 1972.