Wolfgang Nolting

Theoretical Physics 3

Electrodynamics

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General Preface

The seven volumes of the series *Basic Course: Theoretical Physics* are thought to be textbook material for the study of university-level physics. They are aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. The conceptual design of the presentation is organized in such a way that

Classical Mechanics (volume 1)
Analytical Mechanics (volume 2)
Electrodynamics (volume 3)
Special Theory of Relativity (volume 4)
Thermodynamics (volume 5)

are considered as the theory part of an *integrated course* of experimental and theoretical physics as is being offered at many universities starting from the first semester. Therefore, the presentation is consciously chosen to be very elaborate and self-contained, sometimes surely at the cost of certain elegance, so that the course is suitable even for self-study, at first without any need of secondary literature. At any stage, no material is used which has not been dealt with earlier in the text. This holds in particular for the mathematical tools, which have been comprehensively developed starting from the school level, of course more or less in the form of recipes, such that right from the beginning of the study, one can solve problems in theoretical physics. The mathematical insertions are always then plugged in when they become indispensable to proceed further in the program of theoretical physics. It goes without saying that in such a context, not all the mathematical statements can be proved and derived with absolute rigour. Instead, sometimes a reference must be made to an appropriate course in mathematics or to an advanced textbook in mathematics. Nevertheless, I have tried for a reasonably balanced representation so that the mathematical tools are not only applicable but also appear at least 'plausible'.

The mathematical interludes are of course necessary only in the first volumes of this series, which incorporate more or less the material of a bachelor programme.

In the second part of the series which comprises the modern aspects of theoretical physics,

Quantum Mechanics: Basics (volume 6)
Quantum Mechanics: Methods and Applications (volume 7)
Statistical Physics (volume 8)
Many-Body Theory (volume 9),

mathematical insertions are no longer necessary. This is partly because, by the time one comes to this stage, the obligatory mathematics courses one has to take in order to study physics would have provided the required tools. The fact that training in theory has already started in the first semester itself permits inclusion of parts of quantum mechanics and statistical physics in the bachelor programme itself. It is clear that the content of the last three volumes cannot be part of an *integrated course* but rather the subject matter of pure theory lectures. This holds in particular for *Many-Body Theory* which is offered, sometimes under different names as, e.g. *advanced quantum mechanics*, in the eighth or so semester of study. In this part, new methods and concepts beyond basic studies are introduced and discussed which are developed in particular for correlated many particle systems which in the meantime have become indispensable for a student pursuing master's or a higher degree and for being able to read current research literature.

In all the volumes of the series *Basic Course: Theoretical Physics*, numerous exercises are included to deepen the understanding and to help correctly apply the abstractly acquired knowledge. It is obligatory for a student to attempt on his own to adapt and apply the abstract concepts of theoretical physics to solve realistic problems. Detailed solutions to the exercises are given at the end of each volume. The idea is to help a student to overcome any difficulty at a particular step of the solution or to check one's own effort. Importantly these solutions should not seduce the student to follow the *easy way out* as a substitute for his own effort. At the end of each bigger chapter, I have added self-examination questions which shall serve as a self-test and may be useful while preparing for examinations.

I should not forget to thank all the people who have contributed one way or an other to the success of the book series. The single volumes arose mainly from lectures which I gave at the universities of Muenster, Wuerzburg, Osnabrueck, and Berlin in Germany, Valladolid in Spain, and Warangal in India. The interest and constructive criticism of the students provided me the decisive motivation for preparing the rather extensive manuscripts. After the publication of the German version, I received a lot of suggestions from numerous colleagues for improvement, and this helped to further develop and enhance the concept and the performance of the series. In particular, I appreciate very much the support by Prof. Dr. A. Ramakanth, a long-standing scientific partner and friend, who helped me in many respects, e.g. what concerns the checking of the translation of the German text into the present English version.

Special thanks are due to the Springer company, in particular to Dr. Th. Schneider and his team. I remember many useful motivations and stimulations. I have the feeling that my books are well taken care of.

Berlin, Germany Wolfgang Nolting
May 2015

Preface to Volume 3

The main goal of this volume 3 (*Electrodynamics*) corresponds exactly to that of the total *Basic Course: Theoretical Physics*. It is thought to be an accompanying textbook material for the study of university-level physics. It is aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. It is presented in such a way that it enables self-study without the need for a demanding and laborious reference to secondary literature. For the understanding of the text, it is only presumed that the reader has a good grasp of what has been elaborated in the preceding volumes 1 and 2. Mathematical interludes are always presented in a compact and functional form and practiced when they appear indispensable for further development of the theory. For the whole text, it holds that I had to focus on the essentials, presenting them in a detailed and elaborate form, sometimes consciously sacrificing certain elegance. It goes without saying that after the basic course, secondary literature is needed to deepen the understanding of physics and mathematics.

Electrodynamics is presented here in its *inductive* version. That means that the fundamental *Maxwell equations* are motivated by some basic and consistent experimental facts. The conclusions derived from the Maxwell equations can then be compared to the corresponding experimental facts. The complete agreement found up to now provides a strong support of the validity of the concept. The mathematically demanding nature of *electrodynamics* makes practicing the application of concepts and methods especially mandatory. In this context, the exercises which are offered after all important subsections play an indispensable role for an effective learning. The elaborate solutions of exercises at the end of the book should not keep the learner from an independent treatment of the problems, but should only serve as a checkup of one's own efforts.

This volume on *electrodynamics* arose from lectures I gave at the German Universities in Muenster and Berlin. The animating interest of the students in my lecture notes has induced me to prepare the text with special care. This volume one as well as the other volumes is thought to be a textbook material for the study of basic physics, primarily intended for the students rather than for the teachers.

I am thankful to the Springer company, especially to Dr. Th. Schneider, for accepting and supporting the concept of my proposal. The collaboration was always delightful and very professional. A decisive contribution to the book was provided by Prof. Dr. A. Ramakanth from the Kakatiya University of Warangal (India). Many thanks for it!

Berlin, Germany
May 2015

Wolfgang Nolting

Contents

Chapter 1
Mathematical Preparations

In this introductory chapter we want to first discuss the **Dirac's δ-function** which is important for many practical applications. Then the following considerations will concern **Taylor expansions for fields** and **surface integrals**. Subsequently we shall deal with **vector analysis**, the decisive mathematical tool for working on electrodynamics.

1.1 Dirac's δ-Function

In order to motivate the introduction of the δ-function let us go back to classical mechanics. The *concept of the mass point* has proven under certain preconditions as rather useful. The center of mass theorem (Sect. 3.1.1, Vol. 1), for instance, states that the center of mass of a system of mass points moves as if the total mass of the system were concentrated in this point and all external forces would act exclusively on it. According to Eq. (4.4) in Vol. 1 the mass M of a body can be expressed in terms of the mass density $\rho(\mathbf{r})$:

$$M = \int\limits_V d^3r\,\rho(\mathbf{r}) \ .$$

But how does the mass density of a mass point look like? It can be unequal zero only in a single point:

$$\rho(\mathbf{r}) = 0 \quad \forall \mathbf{r} \neq \mathbf{r}_0 \ ,$$

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The volume integral, however,

$$\int_V d^3r\, \rho(\mathbf{r})$$

shall be nevertheless finite provided $\mathbf{r}_0$ lies within the volume V. We thus *'symbolize'* $\rho(\mathbf{r})$ as follows

$$\rho(\mathbf{r}) = M\,\delta(\mathbf{r}-\mathbf{r}_0) \tag{1.1}$$

and require:

$$\int_V d^3r\,\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} 1\ , & \text{if } r_0 \in V \\ 0 & \text{otherwise}\ , \end{cases} \tag{1.2}$$

$$\delta(\mathbf{r}-\mathbf{r}_0) = 0 \quad \forall\ \mathbf{r} \neq \mathbf{r}_0\ . \tag{1.3}$$

Equations (1.2) and (1.3) are the defining equations of the Dirac's δ-function (short: δ-function). Obviously, one must not understand the integral (1.2) as a usual Riemannian integral. Since, because of (1.3), the effective integration interval is of zero-width the integral should actually vanish. Sometimes one helps oneself with the idea that for $\mathbf{r} = \mathbf{r}_0$ the δ-function takes the value ∞ so that something finite results out of $0\cdot\infty$. This is of course only an auxiliary view. The δ-function is not a function in the ordinary mathematical sense which ascribes to each value of its definition range uniquely a certain value of the function. Instead it is **defined** by Eqs. (1.2) and (1.3). It is therefore denoted as an **improper** function or as a **distribution**. The related exact mathematical theory is called the **distribution theory**. It goes beyond the framework of our introductory presentation here and therefore must be settled by some plausibility considerations. For that we restrict ourselves at first to the one-dimensional case.

Let us consider a sequence of **Lorentz curves** (Fig. 1.1)

$$L_\eta(x-a) = \frac{1}{\pi}\,\frac{\eta}{\eta^2+(x-a)^2}\ , \qquad (\eta > 0)\ . \tag{1.4}$$

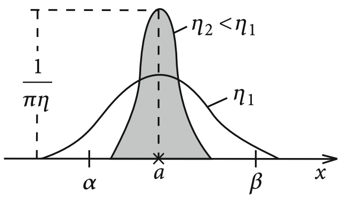


Fig. 1.1 Illustration of the δ-function as a limiting function of a sequence of Lorentzians

For the height of the maximum at $x = a$ we have

$$\frac{1}{\pi\eta} \underset{\eta\to 0^+}{\longrightarrow} \infty$$

and for the width of the peak (*'full width at half maximum'*)

$$2\eta \underset{\eta\to 0^+}{\longrightarrow} 0 \ .$$

The area under the Lorentz curve amounts to

$$\int\limits_\alpha^\beta dx \left[\frac{1}{\pi}\frac{\eta}{\eta^2 + (x-a)^2}\right] = \frac{1}{\pi}\left[\arctan\left(\frac{\beta - a}{\eta}\right) - \arctan\left(\frac{\alpha - a}{\eta}\right)\right]$$

$$\underset{\eta\to 0^+}{\longrightarrow} \begin{cases} 1 \ , \ \text{if } \alpha < a < \beta \ , \\ \frac{1}{2} \ , \ \text{if } a = \alpha \text{ or } a = \beta \ , \\ 0 \quad \text{otherwise } (a \neq \alpha, \beta) \ . \end{cases}$$

For $\eta \to 0^+$ L_η becomes arbitrarily narrow. Thus it is:

$$\lim_{\eta\to 0^+} L_\eta(x-a) = 0 \quad \forall\, x \neq a \ , \tag{1.5}$$

$$\lim_{\eta\to 0^+} \int\limits_\alpha^\beta L_\eta(x-a)\, dx = \begin{cases} 1 \ , \ \text{if } \alpha < a < \beta \ , \\ \frac{1}{2} \ , \ \text{if } a = \alpha \text{ or } a = \beta \ , \\ 0 \quad \text{otherwise } (a \neq \alpha, \beta) \ . \end{cases} \tag{1.6}$$

The sequence of integration and limiting process in Eq. (1.6) must not be interchanged. If we strictly obey that, then we can write *'abbreviatorily'*:

$$\delta(x-a) = \lim_{\eta\to 0^+} \frac{1}{\pi}\frac{\eta}{\eta^2 + (x-a)^2} \tag{1.7}$$

with

$$\delta(x-a) = 0 \quad \forall x \neq a \ ,$$

$$\int\limits_\alpha^\beta dx\, \delta(x-a) = \begin{cases} 1 \ , \ \text{if } \alpha < a < \beta \ , \\ \frac{1}{2} \ , \ \text{if } a = \alpha \text{ or } a = \beta \ , \\ 0 \quad \text{otherwise } (a \neq \alpha, \beta) \ . \end{cases} \tag{1.8}$$

The δ-function can be represented also by other limiting processes (see exercises!) where these processes have to fulfill only (1.5) and (1.6).

On the basis of Eqs. (1.5) to (1.7) one verifies the following properties of the δ-function:

1. Let $f(x)$ be a function which is continuous in the neighborhood of $x = a$. Then it holds:

$$\int_{\alpha}^{\beta} f(x)\,\delta(x-a)\,dx = \begin{cases} f(a)\,, & \text{if } \alpha < a < \beta\,, \\ \frac{1}{2} f(a)\,, & \text{if } a = \alpha \text{ or } a = \beta\,, \\ 0 & \text{otherwise } (a \neq \alpha, \beta)\,. \end{cases} \tag{1.9}$$

Proof From the *mean value theorem of integral calculus* (Sect. 1.2.3, Vol. 1) we first have:

$$F_\eta(a) = \int_{\alpha}^{\beta} L_\eta(x-a) f(x) dx = f(\xi) \int_{\alpha}^{\beta} L_\eta(x-a) dx\,, \quad \xi \in [\alpha, \beta]\,.$$

For $\eta \to 0^+$ $L_\eta(x-a)$ becomes an arbitrarily sharp peak around a and $F_\eta(a)$ does not change if the interval of integration is restricted to the region for which L_η is unequal to zero. ξ must lie within this effective region of integration which for $\eta \to 0^+$ contracts to the point a:

$$\lim_{\eta \to 0^+} F_\eta(a) = f(a) \lim_{\eta \to 0^+} \int_{\alpha}^{\beta} L_\eta(x-a)\,dx\,.$$

With (1.6) we then get the assertion (1.9).

2.

$$\delta\left[f(x)\right] = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x - x_i)\,, \tag{1.10}$$

x_i: **simple** zero of $f(x)$; $f(x_i) = 0$; $f'(x_i) \neq 0$.

We perform the proof as Exercise 1.7.3. One easily recognizes the following special cases:

(a)

$$\delta(ax) = \frac{1}{|a|}\,\delta(x)\,, \tag{1.11}$$

(b)

$$\delta\left(x^2 - a^2\right) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right]. \tag{1.12}$$

3.

$$g(x)\delta(x-a) = g(a)\delta(x-a) \; , \tag{1.13}$$

$$x\delta(x) = 0 \; . \tag{1.14}$$

4.

$$\int_{-\infty}^{x} d\bar{x}\,\delta(\bar{x}) = \Theta(x) = \begin{cases} 1 \text{ for } x > 0 \; , \\ 0 \text{ for } x < 0 \end{cases} \tag{1.15}$$

'step function'

5. **Derivative** of the δ-function [$a \in (\alpha, \beta)$]:

$$\int_{\alpha}^{\beta} \delta'(x-a) f(x)\,dx = f(x)\,\delta(x-a)\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \delta(x-a) f'(x)\,dx = -f'(a) \; .$$

Since $f(x)$ must only be differentiable, being otherwise arbitrary, this *formal* integration by parts leads to the following identity:

$$f(x)\,\delta'(x-a) = -f'(a)\delta(x-a) \; . \tag{1.16}$$

6. One can consider the δ-function also as the derivative of the step function:

$$\delta(x-a) = \frac{d}{dx}\Theta(x-a) \; . \tag{1.17}$$

because:

$$\int_{\alpha}^{\beta} \frac{d}{dx}\Theta(x-a)dx = \Theta(x-a)\Big|_{\alpha}^{\beta} = \begin{cases} 1 \;, \text{ if } \alpha < a < \beta \; , \\ 0 \quad \text{otherwise} \; , \end{cases}$$

$$\frac{d}{dx}\Theta(x-a) = 0 \quad \forall x \neq a \; .$$

But these are just the two defining equations of the δ-function.

7. **Multi-dimensional δ-function**

The three-dimensional δ-function is defined by (1.2) and (1.3).

(a) **Cartesian coordinates:**

$$\mathbf{r} = (x, y, z) \; ; \quad \mathbf{r}_0 = (x_0, y_0, z_0)$$

$$\int_V d^3r \ldots \longrightarrow \iiint_V dx\,dy\,dz \ldots$$

ansatz:

$$\delta(\mathbf{r} - \mathbf{r}_0) = \gamma(x, y, z)\, \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) \; .$$

$\gamma(x, y, z)$ must be chosen such that (1.2) is fulfilled:

$$\begin{aligned}
\int_V d^3r\, \delta(\mathbf{r} - \mathbf{r}_0) &= \iiint_V dx\,dy\,dz\, \gamma(x, y, z)\, \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) \\
&= \gamma(x_0, y_0, z_0) \iiint_V \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0)\, dx\,dy\,dz \\
&= \begin{cases} \gamma(x_0, y_0, z_0) \; , & \text{if } \mathbf{r}_0 \in V \; , \\ 0 & \text{otherwise} \; . \end{cases}
\end{aligned}$$

That means:

$$\gamma = 1$$

and therewith:

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) \; . \tag{1.18}$$

(b) **curvilinear coordinates** (u, v, w): According to ((1.367), Vol. 1) the volume element reads:

$$d^3r = dx\,dy\,dz = \underbrace{\frac{\partial(x, y, z)}{\partial(u, v, w)}}_{\text{Jacobian determinant}} \; du\,dv\,dw \; .$$

We choose a similar **ansatz** as in (a):

$$\delta(\mathbf{r} - \mathbf{r}_0) = \gamma(u, v, w)\, \delta(u - u_0)\, \delta(v - v_0)\, \delta(w - w_0) \; . \tag{1.19}$$

Because of (1.2) we then have to fulfill:

$$\int_V d^3r\,\delta(\mathbf{r}-\mathbf{r}_0) = \iiint_V du\,dv\,dw\,\frac{\partial(x,y,z)}{\partial(u,v,w)}\,\gamma(u,v,w)\,\delta(u-u_0)$$

$$\cdot\;\delta(v-v_0)\,\delta(w-w_0) \overset{!}{=} 1\,, \quad \text{if } \mathbf{r}_0 \in V\,.$$

That leads to:

$$\gamma = \left(\left.\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|_{\mathbf{r}_0}\right)^{-1}\,. \tag{1.20}$$

Examples
spherical coordinates (r, ϑ, φ) ((1.390), Vol. 1):

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{r_0^2 \sin\vartheta_0}\,\delta(r-r_0)\,\delta(\vartheta-\vartheta_0)\,\delta(\varphi-\varphi_0)\,. \tag{1.21}$$

cylindrical coordinates (ρ, φ, z) ((1.382), Vol. 1):

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{\rho_0}\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\delta(z-z_0)\,. \tag{1.22}$$

Very soon the great importance of the δ-function for Theoretical Physics will become evident. It is therefore absolutely mandatory to become familiar with this special function.

1.2 Taylor Expansion

Oftentimes it is unavoidable for a physicist to simplify certain mathematical functions in specific interesting regions in order to attain to concrete results for a given physical problem. Such a simplification should of course be *'physically reasonable'*, i.e. it should not falsify the actual results too roughly. In particular a reliable estimate of the mistake caused by the simplification would be desirable.

Let us first consider functions of one variable $f = f(x)$. If these are arbitrarily often differentiable, which is presumed in the following, then, normally, they can be expanded as a **power series** ((1.92), Vol. 1)

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the coefficients a_n are fixed by the behavior of the function at the point $x = 0$:

$$a_0 = f(0) \; ; \quad a_1 = f'(0) \; ; \quad a_2 = \frac{1}{1 \cdot 2} f''(0) \; ; \quad \dots \; ; \quad a_n = \frac{1}{n!} f^{(n)}(0) \; ; \quad \dots .$$

We thus have with $0! = 1! = 1$:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \; . \tag{1.23}$$

One says that $f(x)$ is expanded as a **Taylor series** around the point $x = 0$. A decisive precondition is, besides being arbitrarily often differentiable, the series should converge. The values of the variable x for which this is the case define the **region of convergence** of the power series.

For a series to be **convergent** the contributions of the summands must necessarily approach zero with increasing power of the variable. This allows for an approximation to the function $f(x)$ by terminating the series after a finite number of terms:

$$f(x) = \underbrace{\sum_{n=0}^{m} a_n x^n}_{\text{approximate polynomial of } m\text{-th degree}} + \underbrace{R_m(x)}_{\text{remainder term}} \; . \tag{1.24}$$

At which position the series is to be terminated that depends on the demand of accuracy.

Example

$$\begin{aligned} f(x) &= \sin x \; , \\ (\sin x)^{(2n)}\big|_{x=0} &= (-1)^n \sin 0 = 0 \; , \\ (\sin x)^{(2n+1)}\big|_{x=0} &= (-1)^n \cos 0 = (-1)^n \; . \end{aligned}$$

That yields with (1.23):

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

The accuracy of the approximation obviously increases with increasing n (Fig. 1.2).

The representation of the function $f(x)$ by an approximate polynomial of finite degree appears reasonable of course only when the remainder term

$$R_m(x) \underset{m \to \infty}{\longrightarrow} 0 \; .$$

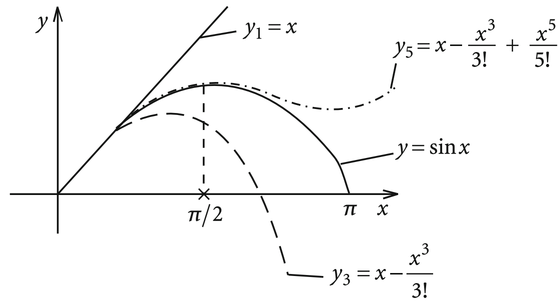


Fig. 1.2 Various approximations for the sine function by taking into account different numbers of terms of the exact series expansion

Unfortunately, in many practical applications it is not uniquely predictable. One knows various types of estimates for the remainder term, e.g. according to Lagrange:

$$R_m(x) = f^{(m+1)}(\xi) \frac{x^{m+1}}{(m+1)!} , \quad 0 < \xi < x . \tag{1.25}$$

We can take the value $\xi_0 \in (0, x)$ for which the right-hand side becomes maximal to get an upper bound for R_m.

If $f(x)$ is not to be expanded around $x = 0$ but around an arbitrary position $x = x_0$ then (1.23) must be changed accordingly:

$$u = x - x_0 \Longrightarrow f(x) = f(u + x_0) \equiv g(u) .$$

We expand $g(u)$ as above around $u = 0$:

$$g(u) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(0) u^n ,$$

$$g^{(n)}(0) = f^{(n)}(0 + x_0) .$$

The generalization to (1.23) thus reads:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n . \tag{1.26}$$

In electrodynamics the

Taylor expansion of fields”

turns out to be highly important, i.e. the expansion of functions of more than one variable.

Let $\varphi(\mathbf{r})$ be a scalar field and let us expand $\varphi(\mathbf{r} + \Delta\mathbf{r})$ around $\mathbf{r}$:

$$\varphi(\mathbf{r} + \Delta\mathbf{r}) = \varphi(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \equiv F(t = 1) .$$

Thereby we have defined:

$$F(t) = \varphi(x_1 + \Delta x_1 t, x_2 + \Delta x_2 t, x_3 + \Delta x_3 t) = \varphi(\mathbf{r} + \Delta \mathbf{r} t) \ .$$

According to (1.23) we have:

$$F(t) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) t^n \ .$$

The chain rule yields:

$$\begin{aligned}
F'(0) &= \sum_{j=1}^{3} \frac{\partial \varphi}{\partial x_j} \Delta x_j \ , \\
F''(0) &= \sum_{j,k} \Delta x_j \Delta x_k \frac{\partial^2}{\partial x_k \partial x_j} \varphi(\mathbf{r}) \\
&= \left(\sum_j \Delta x_j \frac{\partial}{\partial x_j} \right)^2 \varphi(\mathbf{r}) \ , \\
&\ \ \vdots \\
F^{(n)}(0) &= \left(\sum_j \Delta x_j \frac{\partial}{\partial x_j} \right)^n \varphi(\mathbf{r}) \ .
\end{aligned}$$

Therewith we have the **Taylor series** for scalar fields:

$$\begin{aligned}
\varphi(\mathbf{r} + \Delta \mathbf{r}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=1}^{3} \Delta x_j \cdot \frac{\partial}{\partial x_j} \right)^n \varphi(\mathbf{r}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\Delta \mathbf{r} \cdot \nabla \right)^n \varphi(\mathbf{r}) \\
&= \exp\left(\Delta \mathbf{r} \cdot \nabla \right) \varphi(\mathbf{r}) \ . \qquad (1.27)
\end{aligned}$$

We obtain an approximate polynomial of m-th degree for $\varphi(\mathbf{r} + \Delta \mathbf{r})$ if we quit the Taylor series after m summands. For the remainder term it follows according to (1.25):

$$R_m = R_m(t = 1) = F^{(m+1)}(\xi) \frac{1}{(m+1)!} \ , \quad 0 < \xi < 1 \ .$$

This means:

$$R_m = \frac{1}{(m+1)!} \left(\sum_{j=1}^{3} \Delta x_j \frac{\partial}{\partial x_j} \right)^{m+1} \varphi(\mathbf{r} + \xi \Delta \mathbf{r}) \ . \tag{1.28}$$

Example Let us expand

$$\varphi(\mathbf{r}) = \frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|} \quad \text{(Coulomb potential of a point charge)}$$

around $\mathbf{r} = 0$. $\mathbf{r}$ adopts here the role of $\Delta \mathbf{r}$ in (1.27):

$n = 0$:

$$\varphi_0 = \varphi(\mathbf{r} = 0) = \frac{\alpha}{r_0} \ , \tag{1.29}$$

$n = 1$:

$$\begin{aligned} \frac{\partial}{\partial x_j} \frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|} &= -\frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|^2} \frac{\partial}{\partial x_j} |\mathbf{r} - \mathbf{r}_0| = -\frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|^2} \frac{x_j - x_{j0}}{|\mathbf{r} - \mathbf{r}_0|} \\ \Longrightarrow \sum_j x_j \frac{\partial}{\partial x_j} \varphi(0) &= \frac{\alpha}{r_0^3} \sum_j x_j x_{j0} \ , \\ \varphi_1 &= \frac{\alpha}{r_0^3} (\mathbf{r} \cdot \mathbf{r}_0) \ , \end{aligned} \tag{1.30}$$

$n = 2$:

$$\begin{aligned} \sum_{j,k} x_j x_k \frac{\partial^2}{\partial x_k \partial x_j} \frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|} &= \sum_{j,k} x_j x_k \frac{\partial}{\partial x_k} \left[-\frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|^3} \left(x_j - x_{j0} \right) \right] \\ &= \sum_{j,k} x_j x_k \left[\frac{-\alpha\, \delta_{kj}}{|\mathbf{r} - \mathbf{r}_0|^3} + \frac{3\alpha}{|\mathbf{r} - \mathbf{r}_0|^5} (x_j - x_{j0})(x_k - x_{k0}) \right] \ , \end{aligned}$$

$$\begin{aligned} \left(\sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} \right)^2 \varphi(0) &= \sum_{jk} x_j x_k \left(-\frac{\alpha\, \delta_{kj}}{r_0^3} + 3\alpha \frac{x_{j0} x_{k0}}{r_0^5} \right) \\ &= \alpha \left[3 \frac{(\mathbf{r} \cdot \mathbf{r}_0)^2}{r_0^5} - \frac{r^2}{r_0^3} \right] \ , \\ \varphi_2 &= \frac{1}{2} \frac{\alpha}{r_0^5} \left[3(\mathbf{r} \cdot \mathbf{r}_0)^2 - r^2 r_0^2 \right] \ . \end{aligned} \tag{1.31}$$

Therewith we have found the following expansion

$$\varphi(\mathbf{r}) = \frac{\alpha}{|\mathbf{r} - \mathbf{r}_0|} = \alpha \left[\frac{1}{r_0} + \frac{\mathbf{r} \cdot \mathbf{r}_0}{r_0^3} + \frac{1}{2} \frac{3(\mathbf{r} \cdot \mathbf{r}_0)^2 - r^2 r_0^2}{r_0^5} + \cdots \right] , \tag{1.32}$$

which will be used in the course of this volume.

1.3 Surface Integrals

In connection with the definition of ‘work’ in Sect. 2.4.1 of Vol. 1 we came across the **line integral**. The **volume integral** was introduced in Sect. 1.2.5 of Vol. 1. Another multiple integral is the **surface integral**, which as a special type of multiple integrals is frequently used in electrodynamics. Therefore it should be considered here in proper detail.

1.3.1 Oriented Surface Elements

The position vector of any point of the space trajectory can be written as function of a **single** parameter (see Sect. 1.4.1, Vol. 1). Accordingly, surfaces are represented by **two** parameters:

$$F = \{\mathbf{r}(u, v) \; ; \; u, v \in D\} . \tag{1.33}$$

This can be made clear as follows: At first we keep u fixed and vary only v. That yields a special space curve. Then we change u to $u + du$. By a subsequent variation of v we get a second space curve and so on. A further family of curves we obtain when we fix v and vary u. That corresponds, by the way, to the *coordinate lines* of a suitably chosen system of coordinates introduced in Sect. 1.7.1 in Vol. 1. Following this way we can decompose the total area F into small elements (Fig. 1.3).

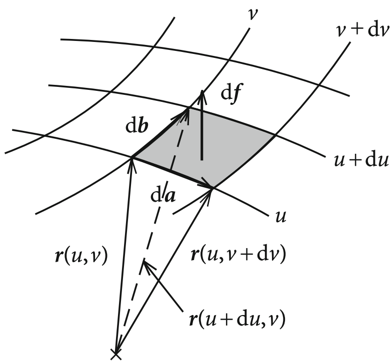
We now give each area element $d\mathbf{f}$ an **orientation**, i.e. we consider $d\mathbf{f}$ to be a vector such that the direction of $d\mathbf{f}$ is perpendicular to the area element. Of course we are then still left with two possibilities for the vector direction. To make it unique we agree that for surfaces $S(V)$ of a certain space region V the surface vector always points *outwards*. Now we have, maybe except for the sign:

$$d\mathbf{f} = d\mathbf{a} \times d\mathbf{b} ,$$

$$d\mathbf{a} = \mathbf{r}(u, v + dv) - \mathbf{r}(u, v) \approx \frac{\partial \mathbf{r}}{\partial v} dv ,$$

$$d\mathbf{b} = \mathbf{r}(u + du, v) - \mathbf{r}(u, v) \approx \frac{\partial \mathbf{r}}{\partial u} du .$$

Fig. 1.3 Representation of an oriented surface element



We can cut off the respective Taylor expansions in both cases after the linear term:

$$d\mathbf{f} = \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u}\right) du\, dv \ . \tag{1.34}$$

The two vectors

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} \quad \text{and} \quad \mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$$

span a plane which at the point $\mathbf{r}(u, v)$ is oriented tangentially to the area F. It is therefore denoted as

tangent plane

with the

surface normal

$$\mathbf{n}(\mathbf{r}) = \frac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_v \times \mathbf{r}_u|} \ . \tag{1.35}$$

Therewith it follows for the surface element:

$$d\mathbf{f} = df\, \mathbf{n}(\mathbf{r}) \ .$$

Example: Surface of a Sphere

Parametric representation:

$$F = \{\mathbf{r} = \mathbf{r}(r = R;\ \vartheta, \varphi)\ ;\ 0 \leq \vartheta \leq \pi\ ,\ 0 \leq \varphi \leq 2\pi\}\ .$$

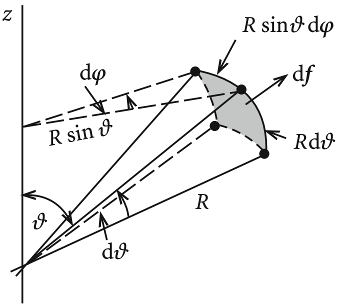


Fig. 1.4 Oriented surface element on the surface of a sphere

The transformation formulas ((1.389), Vol. 1),

$$
\begin{aligned}
x &= R\sin\vartheta\cos\varphi \ , \\
y &= R\sin\vartheta\sin\varphi \ , \\
z &= R\cos\vartheta \ ,
\end{aligned}
$$

lead to:

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \vartheta} &= R(\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \ , \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= R(-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0) \ .
\end{aligned}
$$

That means according to ((1.393), Bd. 1):

$$
\frac{\partial \mathbf{r}}{\partial \vartheta} = R\mathbf{e}_\vartheta \ ; \quad \frac{\partial \mathbf{r}}{\partial \varphi} = R\sin\vartheta\,\mathbf{e}_\varphi \ .
$$

With $\mathbf{e}_\vartheta \times \mathbf{e}_\varphi = \mathbf{e}_r$ the element of the surface of the sphere reads:

$$
d\mathbf{f} = \left(R^2\sin\vartheta\, d\vartheta\, d\varphi\right)\mathbf{e}_r \ . \tag{1.36}
$$

It is oriented radially outwards (Fig. 1.4).

Example: Cylindrical Barrel

Parametric representation (Fig. 1.5):

$$
F = \{\mathbf{r} = \mathbf{r}(\rho = R, \varphi, z) \ ; \ 0 \le \varphi \le 2\pi \ , \ -L/2 \le z \le +L/2\} \ .
$$

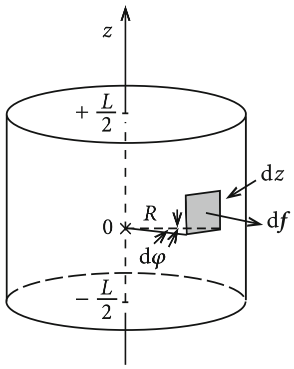


Fig. 1.5 Oriented surface element on a cylinder barrel

The transformation formulas ((1.381), Vol. 1),

$$x = R\cos\varphi \ ,$$
$$y = R\sin\varphi \ ,$$
$$z = z \ ,$$

lead to:

$$\frac{\partial \mathbf{r}}{\partial \varphi} = R(-\sin\varphi,\ \cos\varphi, 0) = R\mathbf{e}_\varphi \ ,$$
$$\frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) = \mathbf{e}_z \ .$$

Insertion into (1.34) yields with $\mathbf{e}_\varphi \times \mathbf{e}_z = \mathbf{e}_\rho$:

$$d\mathbf{f} = (R\,d\varphi\,dz)\mathbf{e}_\rho \ . \tag{1.37}$$

A befitting representation of the surface element requires the choice of a proper system of coordinates. It is therefore highly recommendable to look back on the method of transformation of variables which was introduced in Sect. 1.7.1 of Vol. 1.

1.3.2 Surface Integrals

Let

$$\mathbf{a}(\mathbf{r}) = (a_1(\mathbf{r}), a_2(\mathbf{r}), a_3(\mathbf{r}))$$

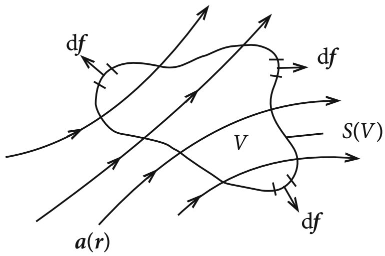


Fig. 1.6 To the definition of the flux of a vector field $\mathbf{a}(\mathbf{r})$ through the surface S of a volume V

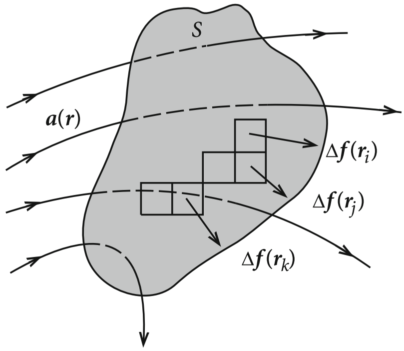


Fig. 1.7 Flux of a vector field through infinitesimally surface elements

be a vector field and V a volume with the closed surface $S(V)$. In electrodynamics very often the question arises how *strongly* the vector field $\mathbf{a}(\mathbf{r})$ penetrates the surface $S(V)$ from the inside to the outside and vice versa (Fig. 1.6).

Definition 1.3.1 Flux of $\mathbf{a}(\mathbf{r})$ through the area S:

$$\varphi_S(\mathbf{a}) = \int_S \mathbf{a}(\mathbf{r}) \cdot d\mathbf{f} \,. \tag{1.38}$$

At each point of the area S the scalar product of the vector field $\mathbf{a}(\mathbf{r})$ and the surface element $d\mathbf{f}$ is to be generated where the surface element has the direction of the outwardly directed normal. The flux is thus a scalar quantity and the surface integral a special case of a multiple integral introduced in Sect. 1.2.5 in Vol. 1.

Let us investigate the surface integral in (1.38) a bit more in detail. For an approximate calculation we decompose the area S in many small elements $\Delta\mathbf{f}(\mathbf{r}_i)$, where the argument $\mathbf{r}_i$ indicates at which position on S the surface element is to be found (Fig. 1.7). Then

$$\mathbf{a}(\mathbf{r}_i) \cdot \Delta\mathbf{f}(\mathbf{r}_i)$$

is the flux through the surface element $\Delta\mathbf{f}(\mathbf{r}_i)$. If the surface elements are sufficiently small we can assume the field $\mathbf{a}$ to be homogeneous on $\Delta\mathbf{f}$, i.e. we can replace it by a representative value $\mathbf{a}(\mathbf{r}_i)$. We obtain an approximate expression for the total flux

$\varphi_S(\mathbf{a})$ of the field $\mathbf{a}$ through the area S by adding up all partial fluxes through the small areas $\Delta\mathbf{f}(\mathbf{r}_i)$:

$$\varphi_S(\mathbf{a}) \simeq \sum_i \mathbf{a}(\mathbf{r}_i) \cdot \Delta\mathbf{f}(\mathbf{r}_i) \; ,$$

This expression can be improved step by step by a steady refinement of the elemental areas $\Delta\mathbf{f}$. The limiting value of these Riemannian sums of the elemental fluxes as a consequence of the described decomposition of the area into small pieces, which becomes finer and finer, is denoted as **surface integral**:

$$\varphi_S(\mathbf{a}) = \int\limits_S \mathbf{a}(\mathbf{r}) \cdot d\mathbf{f} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{a}(\mathbf{r}_i) \cdot \Delta\mathbf{f}(\mathbf{r}_i) \; . \tag{1.39}$$

Precondition here is that this limiting value does exist independently of the type of the decomposition into elemental areas. So the actual shape of the partial area $\Delta\mathbf{f}$ during the limiting process must be arbitrary.

The surface integral over a **closed** area is symbolized by a special integral sign:

$$\varphi_{S(V)}(\mathbf{a}) = \oint\limits_{S(V)} \mathbf{a}(\mathbf{r}) \cdot d\mathbf{f} \; . \tag{1.40}$$

Example: Flux of a Homogeneous Field Through a Cuboid

$$\mathbf{a} = (a_x, a_y, a_z) \; ; \quad a_x, a_y, a_z \text{ const}$$

$$\oint\limits_{S(V)} \mathbf{a} \cdot d\mathbf{f} = ab\, a_z - ab\, a_z + ca\, a_y - ca\, a_y + cb\, a_x - cb\, a_x = 0 \; .$$

We see that the flux of a homogeneous field through a cuboid is zero (Fig. 1.8). This result for homogeneous fields can obviously be generalized to arbitrary closed areas: *'The flux that is flowing into the volume V is also the flux flowing out'*.

Example: Flux of a Radially Symmetric Field Through the Surface of a Sphere

$$\mathbf{a}(\mathbf{r}) = a(r)\mathbf{e}_r \quad \text{(central field)} \; .$$

For $d\mathbf{f}$ we use (1.36):

$$\varphi(\mathbf{a}) = R^2 a(R) \int\limits_0^{\pi} \int\limits_0^{2\pi} \sin\vartheta \, d\vartheta \, d\varphi = 2\pi a(R) R^2 (-\cos\vartheta)\Big|_0^{\pi}$$

$$\Longrightarrow \varphi(\mathbf{a}) = 4\pi R^2 a(R) \; . \tag{1.41}$$

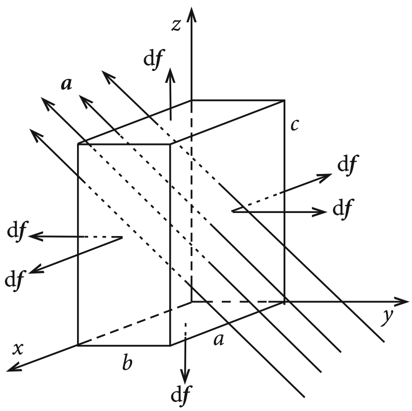


Fig. 1.8 Flux of a homogeneous vector field $\mathbf{a}(\mathbf{r})$ through a cuboid

Example: Flux Through Arbitrary Surfaces

Let S be parametrized by u, v. Then it follows with (1.34):

$$\varphi_S(\mathbf{a}) = \int_S \mathbf{a}\,[\mathbf{r}(u,v)] \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u} \right) du\, dv\ . \tag{1.42}$$

The resulting twofold integral is to be solved according to the rules of Sect. 1.2.5 in Vol. 1.

1.4 Differentiation Processes for Fields

After having discussed in the last section the methods of integration, we now discuss the relevant differentiation techniques for fields. The divergence (div $\equiv \nabla\cdot$) and the curl (rotation) (curl $\equiv \nabla\times$) have been already introduced in Sect. 1.5 of Vol. 1. They will be represented here once more but in a different manner.

1.4.1 Integral Representation of the Divergence

We perform the following considerations in connection with a physical example. Let $\Delta V(\mathbf{r})$ be a volume with the point $\mathbf{r}$ inside. Within this volume one finds the electrical charge $\Delta Q(\mathbf{r})$. The field lines of the electrical field strength $\mathbf{E}$ have their *sources* at positive charges and end at negative charges (*sinks*). If the surface $S(\Delta V)$

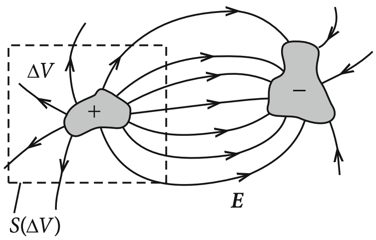


Fig. 1.9 Positive and negative charges as sources and sinks of an electrical field

encloses a positive charge density (Fig. 1.9) then the flux of $\mathbf{E}$ through $S(\Delta V)$ will be proportional to the enclosed charge ΔQ. One therefore calls

$$\frac{1}{\Delta V} \oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} \quad \textit{the average source density of the field } \mathbf{E} \textit{ in } \Delta V\,.$$

We are interested in the source density at a certain space point $\mathbf{r}$ which we determine by volumes $\Delta V(\mathbf{r})$ becoming progressively smaller around $\mathbf{r}$. We claim that this source density is identical to the **divergence** of the $\mathbf{E}$-field defined in ((1.278), Vol. 1):

$$\mathrm{div}\mathbf{E} \equiv \nabla \cdot \mathbf{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f}\,. \tag{1.43}$$

Let us consider a sequence of volumes ΔV_n which are centered around the point $\mathbf{r}_0$ contracting themselves onto this point for $n \to \infty$. For simplicity we think of cuboids with edge lengths $\Delta x_n,\ \Delta y_n,\ \Delta z_n$ which tend to zero for $n \to \infty$:

$$\begin{aligned}
\Delta\mathbf{f}_1 &= \Delta y\, \Delta z\, \mathbf{e}_x = -\Delta\mathbf{f}_2\,,\\
\Delta\mathbf{f}_3 &= \Delta x\, \Delta z\, \mathbf{e}_y = -\Delta\mathbf{f}_4\,,\\
\Delta\mathbf{f}_5 &= \Delta x\, \Delta y\, \mathbf{e}_z = -\Delta\mathbf{f}_6\,.
\end{aligned}$$

For the flux of $\mathbf{E}$ through the surface of the cuboid we find (Fig. 1.10):

$$\begin{aligned}
\oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} = &\iint\limits_{F_1} dy\, dz \left[E_x\left(x_0 + \frac{1}{2}\Delta x, y, z\right) - E_x\left(x_0 - \frac{1}{2}\Delta x, y, z\right)\right]\\
&+ \iint\limits_{F_3} dx\, dz \left[E_y\left(x, y_0 + \frac{1}{2}\Delta y, z\right) - E_y\left(x, y_0 - \frac{1}{2}\Delta y, z\right)\right]\\
&+ \iint\limits_{F_5} dx\, dy \left[E_z\left(x, y, z_0 + \frac{1}{2}\Delta z\right) - E_z\left(x, y, x_0 - \frac{1}{2}\Delta z\right)\right]\,.
\end{aligned}$$

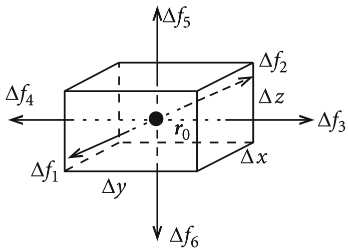


Fig. 1.10 Illustration of a special sequence of volumes for the justification of the integral representation of the divergence

To the integrand we now apply the Taylor expansion (1.26):

$$\oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} = \iint\limits_{F_1} dy\, dz \left[\frac{\partial E_x}{\partial x}(x_0, y, z)\Delta x + \mathcal{O}(\Delta x^3) \right]$$
$$+ \iint\limits_{F_3} dx\, dz \left[\frac{\partial E_y}{\partial y}(x, y_0, z)\Delta y + \mathcal{O}(\Delta y^3) \right]$$
$$+ \iint\limits_{F_5} dx\, dy \left[\frac{\partial E_z}{\partial z}(x, y, z_0)\Delta z + \mathcal{O}(\Delta z^3) \right] .$$

With the mean value theorem of integral calculus (see (1.117), Vol. 1) and

$$\Delta V = \Delta x\, \Delta y\, \Delta z$$

we can write:

$$\frac{1}{\Delta V} \oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} = \frac{\partial E_x}{\partial x}(x_0, y_1, z_1) + \mathcal{O}(\Delta x^2) + \frac{\partial E_y}{\partial y}(x_2, y_0, z_2) + \mathcal{O}(\Delta y^2)$$
$$+ \frac{\partial E_z}{\partial z}(x_3, y_3, z_0) + \mathcal{O}(\Delta z^2) .$$

Thereby we have:

$$x_2, x_3 \in \left[x_0 - \frac{1}{2}\Delta x, x_0 + \frac{1}{2}\Delta x \right] ,$$
$$y_1, y_3 \in \left[y_0 - \frac{1}{2}\Delta y, y_0 + \frac{1}{2}\Delta y \right] ,$$
$$z_1, z_2 \in \left[z_0 - \frac{1}{2}\Delta z, z_0 + \frac{1}{2}\Delta z \right] .$$

During the limiting process $\Delta V \to 0$ these intermediate values must approach x_0, y_0 and z_0, respectively, and the correction terms disappear. That means:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} = \frac{\partial E_x}{\partial x}(\mathbf{r}_0) + \frac{\partial E_y}{\partial y}(\mathbf{r}_0) + \frac{\partial E_z}{\partial z}(\mathbf{r}_0) = \mathrm{div}\mathbf{E}(\mathbf{r}_0) = \nabla \cdot \mathbf{E}(\mathbf{r}_0)\,. \tag{1.44}$$

Our derivation by use of a sequence of cuboids represents of course a certain restriction. In the *theory of differential forms* the general case is traced back by use of special mapping theorems to the above situation. Therewith it can be shown that the integral representation (1.43) of the divergence is valid for **all** types of sequences of volumes which are contracted onto the point $\mathbf{r}_0$.

Calculation rules for the **divergence** are presented in ((1.279) to (1.281), Vol. 1). The general representation by **curvilinear coordinates** is given in ((1.378), Vol. 1).

Let us further generalize the results of this section. For this purpose we choose in (1.43) $\mathbf{E} = \mathbf{a}\varphi$ ($\mathbf{a}$: constant vector; $\varphi(\mathbf{r})$: scalar field). Applying ((1.281), Vol. 1) we can use:

$$\mathrm{div}(\mathbf{a}\varphi) = \mathbf{a}\,\mathrm{grad}\varphi + \varphi\, \underbrace{\mathrm{div}\mathbf{a}}_{=0}\,.$$

This yields with (1.43) since $\mathbf{a}$ is arbitrary :

$$\mathrm{grad}\,\varphi = \nabla\varphi = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} d\mathbf{f}\,\varphi\,. \tag{1.45}$$

If we choose instead $\mathbf{E} = \mathbf{a} \times \mathbf{b}(\mathbf{r})$, where $\mathbf{a}$ is again a constant, while $\mathbf{b}(\mathbf{r})$ is a sufficiently often differentiable vector field, then it follows with (see Exercise 1.5.8, Vol. 1)

$$\mathrm{div}\,[\mathbf{a} \times \mathbf{b}(\mathbf{r})] = \mathbf{b}(\mathbf{r}) \cdot \underbrace{\mathrm{curl}\,\mathbf{a}}_{=0} - \mathbf{a} \cdot \mathrm{curl}\,\mathbf{b}(\mathbf{r}) = -\mathbf{a} \cdot \mathrm{curl}\,\mathbf{b}(\mathbf{r})$$

after insertion into (1.43):

$$\begin{aligned} -\mathbf{a} \cdot \mathrm{curl}\,\mathbf{b}(\mathbf{r}) &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} d\mathbf{f} \cdot [\mathbf{a} \times \mathbf{b}(\mathbf{r})] \\ &= \lim_{\Delta V \to 0} \frac{1}{\Delta V}\, \mathbf{a} \cdot \oint\limits_{S(\Delta V)} \mathbf{b}(\mathbf{r}) \times d\mathbf{f}\,. \end{aligned}$$

Since **a** is assumed to be arbitrary we can conclude:

$$\operatorname{curl}\mathbf{b}(\mathbf{r}) = \nabla \times \mathbf{b}(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} d\mathbf{f} \times \mathbf{b}(\mathbf{r}) \ . \tag{1.46}$$

We can gather (1.43), (1.45) and (1.46) to the following general

surface integral representation of the nabla-operator

$$\nabla \circ \ldots = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} d\mathbf{f} \circ \ldots \tag{1.47}$$

Thereby the small circle means

$$\begin{aligned} \circ &= \cdot \text{ for scalar fields } \varphi \iff (1.45) \\ &= \boldsymbol{\cdot} \text{ or } \times \text{ for vector fields } \mathbf{E}, \mathbf{b} \iff (1.43),\ (1.46)\ . \end{aligned}$$

1.4.2 Integral Representation of the Curl

By ((1.286), Vol. 1) we have introduced the curl, which ascribes to a vector field $\mathbf{a}(\mathbf{r})$ another vector field

$$\mathbf{b}(\mathbf{r}) = \operatorname{curl}\mathbf{a}(\mathbf{r}) = \nabla \times \mathbf{a}(\mathbf{r}) \ .$$

As in the last section for the divergence we are now looking for a corresponding integral representation which illustrates the geometrical meaning of the curl.

Definition 1.4.1

$$\begin{aligned} \mathbf{a}(\mathbf{r}) &: \ \text{vector field} \ , \\ C &: \text{closed curve without double-points } (\textit{`path'}) \ , \\ Z_C(\mathbf{a}) = \oint\limits_C \mathbf{a} \cdot d\mathbf{r} &: \ \textbf{circulation} \text{ of } \mathbf{a}(\mathbf{r}) \text{ along the way } C \ . \end{aligned} \tag{1.48}$$

The line integral needed for the calculation of Z_C has been introduced in Sect. 2.4.1 of Vol. 1.

The circulation can be considered in a descriptive manner as a measure of the **vorticity** of the vector field $\mathbf{a}(\mathbf{r})$ within the area F_C enclosed by the path C. One may interpret, for instance, **a** as the velocity field of a flowing liquid. In Fig. 1.11 the circulation along the circular paths C_1, C_2 is maximal for the field plotted in the left part of the figure, while it disappears for the field in the right part.

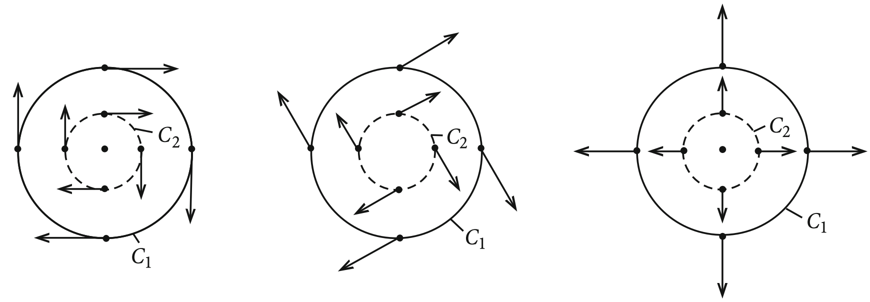


Fig. 1.11 Vector fields with different vorticities along the circular ways C_1 and C_2

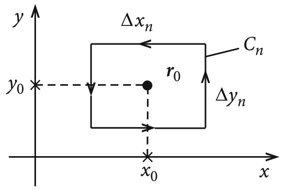


Fig. 1.12 Illustration of a special sequence of closed paths for calculating the circulation

We have learned in Sect. 2.4.2 of Vol. 1 that $Z_C(\mathbf{a})$ vanishes if $\nabla \times \mathbf{a} \equiv 0$. One therefore may expect that there does exist a close connection between circulation and curl which we will now derive.

Let C_n be a sequence of closed curves in a plane which for $n \to \infty$ are contracting to the point $\mathbf{r}_0$ and let F_{C_n} be the area enclosed by C_n. We calculate the circulation

$$Z_{C_n}(\mathbf{a}) = \oint\limits_{C_n} \mathbf{a} \cdot d\mathbf{r}$$

at first for a special sequence of paths C_n, namely for rectangles in the x, y-plane with edge lengths Δx_n, Δy_n, which are passed through in the mathematically positive sense (Fig. 1.12). The surface normals thus point into the z-direction:

$$Z_{C_n}(\mathbf{a}) = \int\limits_{x_0-\frac{1}{2}\Delta x_n}^{x_0+\frac{1}{2}\Delta x_n} dx \left\{ a_x\left(x, y_0 - \frac{1}{2}\Delta y_n, z_0\right) - a_x\left(x, y_0 + \frac{1}{2}\Delta y_n, z_0\right)\right\}$$

$$+ \int\limits_{y_0-\frac{1}{2}\Delta y_n}^{y_0+\frac{1}{2}\Delta y_n} dy \left\{ a_y\left(x_0 + \frac{1}{2}\Delta x_n, y, z_0\right) - a_y\left(x_0 - \frac{1}{2}\Delta x_n, y, z_0\right)\right\}$$

$$= \int\limits_{x_0-\frac{1}{2}\Delta x_n}^{x_0+\frac{1}{2}\Delta x_n} dx \left\{ -\frac{\partial a_x}{\partial y}(x, y_0, z_0)\Delta y_n + \mathcal{O}\left(\Delta y_n^3\right) \right\}$$
$$+ \int\limits_{y_0-\frac{1}{2}\Delta y_n}^{y_0+\frac{1}{2}\Delta y_n} dy \left\{ \frac{\partial a_y}{\partial x}(x_0, y, z_0)\Delta x_n + \mathcal{O}\left(\Delta x_n^3\right) \right\} .$$

In the last step we have applied the Taylor expansion. If we still exploit the mean value theorem of integral calculus, where

$$\bar{x} \in \left[x_0 - \frac{1}{2}\Delta x_n, x_0 + \frac{1}{2}\Delta x_n\right] ; \quad \bar{y} \in \left[y_0 - \frac{1}{2}\Delta y_n, y_0 + \frac{1}{2}\Delta y_n\right]$$

then it follows in the next step:

$$Z_{C_n}(\mathbf{a}) = -\frac{\partial a_x}{\partial y}(\bar{x}, y_0, z_0)\Delta x_n \Delta y_n + \mathcal{O}\left(\Delta x_n \Delta y_n^3\right) + \frac{\partial a_y}{\partial x}(x_0, \bar{y}, z_0)\Delta x_n \Delta y_n + \mathcal{O}\left(\Delta y_n \Delta x_n^3\right) .$$

By the limiting process $n \to \infty$,

$$\Delta x_n \to 0 , \quad \Delta y_n \to 0 ; \quad F_{C_n} = \Delta x_n \Delta y_n \to 0 ,$$

the way C_n contracts onto the point $\mathbf{r}_0$. The intermediate values $\bar{x}, \bar{y}$ reduce to x_0, y_0:

$$\lim_{n\to\infty} \frac{Z_{C_n}}{F_{C_n}} = \left[-\frac{\partial a_x}{\partial y}(\mathbf{r}_0) + \frac{\partial a_y}{\partial x}(\mathbf{r}_0)\right] + \lim_{n\to\infty} \left[\mathcal{O}\left(\Delta y_n^2\right) + \mathcal{O}\left(\Delta x_n^2\right)\right]$$
$$= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)(\mathbf{r}_0) = \left[\nabla \times \mathbf{a}(\mathbf{r}_0)\right]_z .$$

According to ((1.286), Vol. 1) the right-hand side represents just the z-component of the curl of $\mathbf{a}$.

We can now repeat the same consideration for sequences of areas F_{C_n}, which are oriented in, respectively, x- and y-direction obtaining then correspondingly the x and y components of the curl. These can be gathered by the following important **line integral representation** of the curl:

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) = \lim_{F_C \to 0} \frac{1}{F_C} \oint\limits_C \mathbf{a} \cdot d\mathbf{r} , \tag{1.49}$$

$\mathbf{n}$ is the surface normal of F_C. One can interpret the curl as *areal density of the circulation.*

Calculation rules for the **curl** are listed in ((1.287) to (1.293), Vol. 1). The representation in arbitrary **curvilinear coordinates** is given by ((1.380), Vol. 1).

In the last section we were able to derive from the integral representation of the divergence a general expression for the nabla-operator in the form of a surface integral. In a similar way we succeed in representing the nabla-operator by line integrals. Let

$$\mathbf{a}(\mathbf{r}) = \mathbf{b} \cdot \varphi(\mathbf{r}) \; ,$$

be a vector field where $\mathbf{b}$ is a constant vector and $\varphi(\mathbf{r})$ a scalar field. Then ((1.289), Vol. 1) can be used:

$$\operatorname{curl} \mathbf{a} = \varphi \underbrace{\operatorname{curl} \mathbf{b}}_{=0} + (\operatorname{grad}\varphi) \times \mathbf{b} = (\operatorname{grad}\varphi) \times \mathbf{b}$$

$$\Longrightarrow \mathbf{n} \cdot \operatorname{curl} \mathbf{a} = \mathbf{n} \cdot (\nabla\varphi \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{n} \times \nabla\varphi) \; .$$

Since the vector $\mathbf{b}$ is arbitrary, it follows from (1.49):

$$\mathbf{n} \times \nabla\varphi = \lim_{F_C \to 0} \frac{1}{F_C} \oint\limits_C \varphi(\mathbf{r}) d\mathbf{r} \; . \tag{1.50}$$

If we now choose

$$\mathbf{a}(\mathbf{r}) = \mathbf{b} \times \mathbf{E}(\mathbf{r}) \; .$$

then we get, by exploiting the cyclic invariance of the scalar triple product several times:

$$\begin{aligned} \mathbf{n} \cdot \operatorname{curl} \mathbf{a} &= \mathbf{n} \cdot [\nabla \times (\mathbf{b} \times \mathbf{E})] = [\nabla \times (\mathbf{b} \times \mathbf{E})] \cdot \mathbf{n} \\ &= (\mathbf{n} \times \nabla) \cdot (\mathbf{b} \times \mathbf{E}) = -(\mathbf{n} \times \nabla) \cdot (\mathbf{E} \times \mathbf{b}) \\ &= -\mathbf{b} \cdot [(\mathbf{n} \times \nabla) \times \mathbf{E}] \end{aligned}$$

Note that ∇ acts only on $\mathbf{E}$ since $\mathbf{b}$ is a constant vector! Inserting this result together with

$$[\mathbf{b} \times \mathbf{E}(\mathbf{r})] \cdot d\mathbf{r} = (\mathbf{E} \times d\mathbf{r}) \cdot \mathbf{b} = -\mathbf{b} \cdot (d\mathbf{r} \times \mathbf{E}) \; .$$

into (1.49) we come to:

$$(\mathbf{n} \times \nabla) \times \mathbf{E} = \lim_{F_C \to 0} \frac{1}{F_C} \oint\limits_C d\mathbf{r} \times \mathbf{E}(\mathbf{r}) \; . \tag{1.51}$$

From (1.49)–(1.51) we can read off a general

line integral representation of the nabla-operator

$$(\mathbf{n} \times \nabla) \triangle \ldots = \lim_{F_C \to 0} \frac{1}{F_C} \oint_C d\mathbf{r} \triangle \ldots \,, \tag{1.52}$$

where $\triangle$ stands for:

$$\begin{aligned} \cdot\varphi(\mathbf{r}) &\iff (1.50)\,, \\ \times\mathbf{E}(\mathbf{r}) &\iff (1.51)\,, \\ \cdot\mathbf{a}(\mathbf{r}) &\iff (1.49)\,. \end{aligned}$$

1.5 Integration Theorems

1.5.1 The Gauss Theorem

In connection with the introduction of the divergence in Sect. 1.4.1 we found:

$$\oint_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} = \Delta V \mathrm{div}\mathbf{E}(\mathbf{r}) + \Delta V \cdot \mathcal{O}\left(\Delta V^{2/3}\right) \,.$$

The remainder disappears in the limit $\Delta V \to 0$. We now place alongside the partial volume $\Delta V_i(\mathbf{r}_i)$ a further cuboid $\Delta V_{i+1}\left(\mathbf{r}_{i+1}\right)$ which has a side face in common with ΔV_i (Fig. 1.13):

$$\oint_{S(\Delta V_i)} \mathbf{E} \cdot d\mathbf{f} + \oint_{S(\Delta V_{i+1})} \mathbf{E} \cdot d\mathbf{f} = \Delta V_i \mathrm{div}\mathbf{E}(\mathbf{r}_i) + \Delta V_{i+1} \mathrm{div}\mathbf{E}(\mathbf{r}_{i+1}) + \text{remainder}\,.$$

The contributions of the common side face to the surface integrals on the left side of the equation just cancel each other because of the opposite directions of the respective surface normals (Fig. 1.13). What remains is only the surface integral

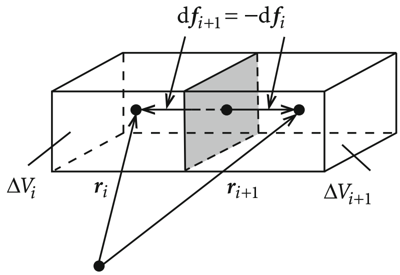


Fig. 1.13 Justification of the Gauss theorem by the flux of the vector field **E** through the surfaces of infinitesimally small volumes

over the *envelope surface* of the total volume. This procedure can be continued and a given volume V can be filled that way by a number of small cuboids ΔV_i. The contributions of the common side faces are not included anymore and we obtain an approximate expression for the flux of the vector field $\mathbf{E}$ through the surface $S(V)$:

$$\oint\limits_{S(V)} \mathbf{E} \cdot d\mathbf{f} \approx \sum_{i=1}^{n} \operatorname{div}\mathbf{E}(\mathbf{r}_i)\, \Delta V_i + \sum_{i=1}^{n} \Delta V_i\ \mathcal{O}\left(\Delta V_i^{2/3}\right) .$$

We can now let the decomposition of the volume become finer and finer. That does not change anything on the left-hand side, while on the right-hand side the first summand becomes a typical *Riemannian sum* and therewith finally a volume integral (Sect. 4.2, Vol. 1). The correction term on the right-hand side tends to zero:

$$\left| \sum_{i=1}^{n} \Delta V_i\ \mathcal{O}\left(\Delta V_i^{2/3}\right) \right| \le \sum_{i=1}^{n} \Delta V_i \left| \max_i \mathcal{O}\left(\Delta V_i^{2/3}\right) \right| \underset{n\to\infty}{\longrightarrow} 0 .$$

Therewith we eventually arrive at the important

Gauss theorem

Let $\mathbf{E}(\mathbf{r})$ *be a sufficiently often differentiable vector field and* V *be a volume with a closed surface* $S(V)$. *Then it holds:*

$$\int\limits_V \operatorname{div}\mathbf{E}(\mathbf{r})\, d^3r = \oint\limits_{S(V)} \mathbf{E} \cdot d\mathbf{f} . \tag{1.53}$$

This extremely useful theorem connects volume properties of a vector field with its surface properties. Let us add some special **remarks**:

(a) Vorticity flux through a closed area:

$$\oint\limits_{S(V)} \operatorname{curl}\mathbf{a} \cdot d\mathbf{f} = \int\limits_V \underbrace{\operatorname{div}\operatorname{curl}\mathbf{a}}_{=0}\, d^3r = 0 . \tag{1.54}$$

(b) Let $\mathbf{j}$ be the current density (*current per unit-area*) then $\oint_{S(V)} \mathbf{j} \cdot d\mathbf{f}$ will be the current through the surface of the volume V. Further let ρ be the charge density (*charge per unit-volume*) and therewith $\partial/\partial t \int_V \rho\, d^3r$ the temporal change of the total charge in V, then the latter must be opposite and equal to the charge current through the surface:

$$\int\limits_V d^3r \frac{\partial\rho}{\partial t} + \oint\limits_{S(V)} \mathbf{j} \cdot d\mathbf{f} \overset{!}{=} 0 .$$

We use the Gauss theorem to get:

$$\int_V d^3r \left(\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} \right) = 0 \, .$$

This relation is valid for **arbitrary** volumes and therefore can be correct only if:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0 \tag{1.55}$$

This is the fundamental **continuity equation**, the physical content of which will get an in-depth discussion later.

(c) Let us derive the Gauss theorem for scalar (!) fields. For this purpose we take in (1.53)

$$\mathbf{E}(\mathbf{r}) = \mathbf{A}\,\varphi(\mathbf{r})$$

where $\mathbf{A}$ is an arbitrary constant vector and $\varphi(\mathbf{r})$ a scalar field. With ((1.281), Vol. 1) we can write for the divergence:

$$\mathrm{div}\mathbf{E}(\mathbf{r}) = \varphi(\mathbf{r}) \underbrace{\mathrm{div}\mathbf{A}}_{=0} + \mathbf{A} \cdot \mathrm{grad}\varphi \, .$$

This inserted into (1.53) yields since $\mathbf{A}$ is an arbitrary vector:

$$\int_V \mathrm{grad}\varphi \, d^3r = \oint_{S(V)} \varphi \, d\mathbf{f} \, . \tag{1.56}$$

(d) We now choose

$$\mathbf{E}(\mathbf{r}) = \mathbf{A} \times \mathbf{b}(\mathbf{r}) \, ,$$

where $\mathbf{A}$ is again a constant vector and $\mathbf{b}(\mathbf{r})$ a vector field. We apply (see Exercise 1.5.8, Vol. 1)

$$\mathrm{div}(\mathbf{A} \times \mathbf{b}) = \mathbf{b} \cdot \underbrace{\mathrm{curl}\,\mathbf{A}}_{=0} - \mathbf{A} \cdot \mathrm{curl}\,\mathbf{b}$$

and find therewith:

$$\oint\limits_{S(V)} d\mathbf{f} \cdot (\mathbf{A} \times \mathbf{b}) = -\mathbf{A} \cdot \oint\limits_{S(V)} d\mathbf{f} \times \mathbf{b} \,,$$

$$\int\limits_{V} \operatorname{div}(\mathbf{A} \times \mathbf{b})\, d^3r = -\mathbf{A} \cdot \int\limits_{V} \operatorname{curl} \mathbf{b}\, d^3r \,.$$

Because of (1.53) we then have:

$$\int\limits_{V} \operatorname{curl} \mathbf{b}\, d^3r = \oint\limits_{S(V)} d\mathbf{f} \times \mathbf{b} \,. \tag{1.57}$$

Equations (1.53), (1.56) and (1.57) are different formulations of the Gauss theorem which can be combined symbolically as follows:

$$\oint\limits_{S(V)} d\mathbf{f} \circ \ldots = \int\limits_{V} d^3r\, \nabla \circ \ldots \tag{1.58}$$

Thereby the small circle means the same as in (1.47):

$$\begin{aligned} \circ = &\ \cdot \text{ for scalar fields } \varphi \,, \\ = &\ \cdot \text{ or } \times \text{ for vector fields } \mathbf{E}, \mathbf{b} \,. \end{aligned}$$

1.5.2 The Stokes Theorem

With a method of proof very similar to that of the last section we are now going to derive a theorem which combines, for an arbitrary vector field, the line integral over the boundary of an arbitrarily large and arbitrarily oriented area with the corresponding surface integral.

Let the area F be bordered by the boundary curve C. It need not thereby necessarily be about a **plane** area. However, the considered area can be approximately represented by a collection of n area elements $\Delta\mathbf{F}_i$, which may be so small that each of them may be considered as **plane**. The directions of the various area elements need **not** be parallel. The area elements are bordered by curves C_i which build with $\Delta\mathbf{F}_i$ a right-handed system (right-twisted screw) (Fig. 1.14).

On the way to the integral representation of the curl (1.49) we found the intermediate result:

$$\Delta\mathbf{F}_i \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}_i) + \Delta F_i\, \mathcal{O}(\Delta F_i) = \oint\limits_{C_i} \mathbf{a} \cdot d\mathbf{r} \,.$$

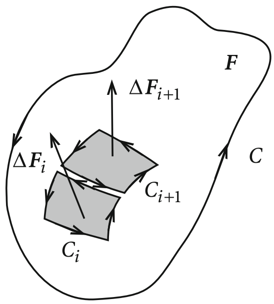


Fig. 1.14 To the justification of the Stokes theorem by inspecting the circulations of a vector field on the edges of infinitely small and arbitrarily oriented area elements

The area element $\Delta\mathbf{F}_{i+1}$ has together with $\Delta\mathbf{F}_i$ a common piece of boundary curve which, however, is run through on C_i and C_{i+1} in opposite directions (Fig. 1.14). If one adds to the last equation the corresponding equation for $i + 1$ then the contribution of this common piece to the total line integral vanishes. What is left is the integral over the boundary curve $C_{i+(i+1)}$ which runs around the total area $\Delta\mathbf{F}_i \vee \Delta\mathbf{F}_{i+1}$. Summing up the n area elements one gets:

$$\sum_{i=1}^{n} \Delta\mathbf{F}_i \cdot \operatorname{curl}\mathbf{a}(\mathbf{r}_i) + \sum_{i=1}^{n} \Delta F_i \; \mathcal{O}(\Delta F_i) = \oint\limits_{C_{1+2+\ldots+n}} \mathbf{a}\cdot d\mathbf{r} \,.$$

We now make the decomposition of the area finer and finer filling therewith F more and more exactly. The correction term on the left-hand side then disappears in the limit $n \to \infty$:

$$\sum_{i=1}^{n} \Delta F_i \; \mathcal{O}(\Delta F_i) \le F \cdot \max_i |\mathcal{O}(\Delta F_i)| \underset{n\to\infty}{\longrightarrow} 0 \,.$$

The first summand is again an ordinary *Riemannian sum* and the path $C_{1+2+\ldots+n}$ becomes identical to C. That eventually results in the

Stokes theorem

Let $\mathbf{a}(\mathbf{r})$ *be a sufficiently often differentiable vector field and* F *be an area with the boundary curve* $C(F) = \partial F$ *then it holds:*

$$\int\limits_{\partial F} \mathbf{a}\cdot d\mathbf{r} = \int\limits_{F} \operatorname{curl}\mathbf{a}\cdot d\mathbf{f} \,. \tag{1.59}$$

We want to follow up here also a first **discussion** of this fundamental theorem.

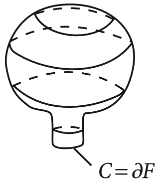


Fig. 1.15 To the calculation of the flux of a curl field through a closed area

(a) **Way-independence of line integrals**

With the discussion of conservative forces $\mathbf{F}$ in Sect. 2.4.2 of Vol. 1 we had found as possible criteria for the existence of a potential:

$$\operatorname{curl} \mathbf{F} \equiv 0 \quad \text{and} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

The equivalence can easily be proved by the use of Stokes theorem: Let $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for **arbitrary** closed paths C. That means with Stokes theorem $\oint_{F_C} \operatorname{curl} \mathbf{F} \cdot d\mathbf{f} = 0$ for **arbitrary** areas F_C. But this is possible only if $\operatorname{curl} \mathbf{F} \equiv 0$.

(b) **Vorticity flux through a closed area**

A **closed** area can be thought to be originated by a contraction of the boundary curve $C = \partial F$ onto a single point. But then it must be (Fig. 1.15)

$$\oint_{\partial F} \mathbf{a} \cdot d\mathbf{r} = 0 \;,$$

since the **length** of the boundary approaches zero. According to (1.59) it follows then

$$\oint_F \operatorname{curl} \mathbf{a} \cdot d\mathbf{f} = 0 \tag{1.60}$$

for **each** vector field $\mathbf{a}(\mathbf{r})$. The same result we could already derive with (1.54) from the Gauss theorem.

(c) **Special example**

Assume

$$\mathbf{a}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r} \;; \quad \mathbf{B} = \text{const} \;.$$

Then it holds (see Exercise 1.5.7, Vol. 1):

$$\operatorname{curl} \mathbf{a}(\mathbf{r}) = \mathbf{B} \;.$$

This leads with the Stokes theorem to

$$\int_{\partial F} \mathbf{a} \cdot d\mathbf{r} = \int_F \operatorname{curl} \mathbf{a} \cdot d\mathbf{f} = \mathbf{B} \cdot \int_F d\mathbf{f} = \mathbf{B} F_{\perp} \;. \tag{1.61}$$

The value of the integral turns out to be independent of the shape of the boundary curve ∂F, thus also independent of the shape of the area $\mathbf{F}$. Decisive is only the projection $F_\perp$, the area perpendicular to $\mathbf{B}$.

(d) **Stokes theorem for scalar fields**
We take

$$\mathbf{a}(\mathbf{r}) = \mathbf{A}\,\varphi(\mathbf{r}) \; ,$$

where $\mathbf{A}$ is an arbitrary constant vector and $\varphi(\mathbf{r})$ a scalar field. By use of ((1.289), Vol. 1):

$$\mathrm{curl}\,\mathbf{a} = \varphi \cdot \underbrace{\mathrm{curl}\mathbf{A}}_{=0} + \mathrm{grad}\varphi \times \mathbf{A} \; .$$

we get from (1.59):

$$\mathbf{A} \cdot \int\limits_{\partial F} \varphi \, d\mathbf{r} = \int\limits_{F} d\mathbf{f} \cdot (\mathrm{grad}\varphi \times \mathbf{A}) = \mathbf{A} \cdot \int\limits_{F} d\mathbf{f} \times \mathrm{grad}\varphi \; .$$

It follows since $\mathbf{A}$ is arbitrary:

$$\int\limits_{\partial F} \varphi \, d\mathbf{r} = \int\limits_{F} d\mathbf{f} \times \mathrm{grad}\varphi \; . \tag{1.62}$$

(e) **Stokes theorem for vector fields**
We now take

$$\mathbf{a}(\mathbf{r}) = \mathbf{A} \times \mathbf{b}(\mathbf{r}) \; ,$$

where $\mathbf{A}$ is again an arbitrary constant vector. After repeated application of the cyclic invariance of the scalar triple product one gets:

$$\begin{aligned}
\int\limits_{\partial F} (\mathbf{A} \times \mathbf{b}) \cdot d\mathbf{r} &= \mathbf{A} \cdot \int\limits_{\partial F} \mathbf{b} \times d\mathbf{r} \overset{(1.59)}{=} \int\limits_{F} \mathrm{curl}(\mathbf{A} \times \mathbf{b}) \cdot d\mathbf{f} \\
&= \int\limits_{F} [\nabla \times (\mathbf{A} \times \mathbf{b})] \cdot d\mathbf{f} = \int\limits_{F} (d\mathbf{f} \times \nabla) \cdot (\mathbf{A} \times \mathbf{b}) \\
&= -\int\limits_{F} (d\mathbf{f} \times \nabla) \cdot (\mathbf{b} \times \mathbf{A}) = -\mathbf{A} \cdot \int\limits_{F} [(d\mathbf{f} \times \nabla) \times \mathbf{b}(\mathbf{r})]
\end{aligned}$$

(∇ acts only on $\mathbf{b}(\mathbf{r})$!).

Therewith we have found:

$$\int_{\partial F} d\mathbf{r} \times \mathbf{b}(\mathbf{r}) = \int_F (d\mathbf{f} \times \nabla) \times \mathbf{b}(\mathbf{r}) \ . \tag{1.63}$$

Equations (1.59), (1.62) and (1.63) are different versions of the Stokes theorem which can be combined symbolically as follows:

$$\int_{\partial F} d\mathbf{r} \ \triangle \dots = \int_F (d\mathbf{f} \times \nabla) \ \triangle \dots \ . \tag{1.64}$$

The symbol $\triangle$ is to be understood as in (1.53):

$$\begin{aligned} \triangle: \quad \cdot\varphi(\mathbf{r}) &\iff (1.62)\ , \\ \cdot\mathbf{a}(\mathbf{r}) &\iff (1.59)\ , \\ \times\mathbf{b}(\mathbf{r}) &\iff (1.63)\ . \end{aligned}$$

In the second line $\operatorname{curl}\mathbf{a} \cdot d\mathbf{f} = (d\mathbf{f} \times \nabla) \cdot \mathbf{a}$ has been used.

1.5.3 The Green Theorems

As simple applications of the Gauss theorem two valuable formulas can be derived which are called *'Green theorems'*, *'Green identities'* or *'Green laws'*.

Let φ, ψ be two at least twofold continuously differentiable scalar fields and V a volume with a closed surface $S(V)$. We define the vector field

$$\mathbf{E}(\mathbf{r}) = \varphi(\mathbf{r}) \operatorname{grad}\psi(\mathbf{r})$$

and apply to it the Gauss theorem (1.53). For this purpose we need div**E**, where we exploit ((1.281), Vol. 1) and ((1.282), Vol. 1):

$$\begin{aligned} \operatorname{div}\mathbf{E}(\mathbf{r}) = \operatorname{div}(\varphi \operatorname{grad}\psi) &= \varphi \operatorname{div}\operatorname{grad}\psi + \operatorname{grad}\psi \cdot \operatorname{grad}\varphi \\ &= \varphi\, \Delta\psi + \nabla\psi \cdot \nabla\varphi \ . \end{aligned}$$

We still introduce the (position-dependent!) surface normal $\mathbf{n}(\mathbf{r})$,

$$d\mathbf{f} = \mathbf{n}\, df \ ,$$

finding therewith:

$$\mathbf{E} \cdot d\mathbf{f} = \varphi(\nabla\psi \cdot \mathbf{n}) df \ .$$

Definition 1.5.1 **Normal derivative** of ψ on $S(V)$:

$$\nabla\psi \cdot \mathbf{n} \equiv \frac{\partial\psi}{\partial n} \,. \tag{1.65}$$

With these preparations the Gauss theorem (1.53) yields:

First Green Identity

$$\int_V \left(\varphi\,\Delta\psi + (\nabla\psi \cdot \nabla\varphi)\right) d^3r = \oint_{S(V)} \varphi \frac{\partial\psi}{\partial n} df \,. \tag{1.66}$$

In the above derivation, if we interchange the fields φ and ψ and subtract the resulting expression from the first Green identity (1.66) then we get the

Second Green Identity

$$\int_V (\varphi\Delta\psi - \psi\Delta\varphi) d^3r = \oint_{S(V)} \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) df \,. \tag{1.67}$$

If we finally still choose $\varphi \equiv 1$ in (1.67) then it follows a further useful identity:

$$\int_V \Delta\psi d^3r = \oint_{S(V)} \frac{\partial\psi}{\partial n} df \,. \tag{1.68}$$

1.6 Decomposition and Uniqueness Theorem for Vector Fields

In this section we want to prove two propositions which are of great importance for vector fields. When combined they tell us that under certain preconditions each vector field $\mathbf{a}(\mathbf{r})$ is **uniquely** determined by its source field $\operatorname{div}\mathbf{a}$ and its vortex field $\operatorname{curl}\mathbf{a}$. Or in other words: Each vector field can be **uniquely** represented as a sum of a curl-free and a source-free part. For the proof of these propositions some preparations are necessary.

Assertion

$$\delta(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi}\Delta\frac{1}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{1.69}$$

Proof We have to show that this representation of the δ-function fulfills the two relations (1.2) and (1.3):

(a) $\mathbf{r} \neq \mathbf{r}', \delta(\mathbf{r}-\mathbf{r}') = 0$:

$$\Delta \frac{-1}{|\mathbf{r}-\mathbf{r}'|} = \operatorname{div}\operatorname{grad}\frac{-1}{|\mathbf{r}-\mathbf{r}'|} = \operatorname{div}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$
$$\overset{((1.281),\ \text{Vol. 1})}{=} \frac{\operatorname{div}(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + (\mathbf{r}-\mathbf{r}')\cdot\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}$$
$$\overset{((1.284),\ \text{Vol. 1})}{=} \frac{3}{|\mathbf{r}-\mathbf{r}'|^3} - 3(\mathbf{r}-\mathbf{r}')\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{1}{|\mathbf{r}-\mathbf{r}'|^4}$$
$$= 0\ .$$

Therewith property (1.3) is verified.

(b) We still have to show:

$$\int_V d^3r\,\delta(\mathbf{r}-\mathbf{r}') = \begin{cases} 1\ , \text{ if } \mathbf{r}' \in V\ , \\ 0 \quad \text{otherwise} \end{cases}$$

We thus inspect:

$$\int_V d^3r\,\Delta_r \frac{1}{|\mathbf{r}-\mathbf{r}'|} \overset{\bar{\mathbf{r}}=\mathbf{r}-\mathbf{r}'}{=} \int_{\bar{V}} d^3\bar{r}\,\Delta_{\bar{r}}\frac{1}{\bar{r}}\ .$$

Because of part (a) the integrand is zero for $\bar{\mathbf{r}} \neq 0$. This leads to a first conclusion:

$$\int_{\bar{V}} d^3\bar{r}\,\Delta_{\bar{r}}\frac{1}{\bar{r}} = 0\ , \quad \text{if } \bar{\mathbf{r}} = 0 \notin \bar{V}\ .$$

If $\bar{V}$ contains the zero point then we are obviously allowed to replace, without changing the value of the integral, $\bar{V}$ by a sphere the center of which coincides with the origin:

$$\int_{\bar{V}} d^3\bar{r}\,\Delta_{\bar{r}}\frac{1}{\bar{r}} = \int_{V_K} d^3\bar{r}\operatorname{div}\left(\operatorname{grad}_{\bar{r}}\frac{1}{\bar{r}}\right) \overset{(1.53)}{=} \int_{S(V_K)} d\bar{\mathbf{f}}\cdot\left(-\frac{1}{\bar{r}^2}\mathbf{e}_{\bar{r}}\right)$$
$$\overset{(1.36)}{=} \int_0^{2\pi} d\bar{\varphi}\int_0^{\pi}\sin\bar{\vartheta}\,d\bar{\vartheta}\,\bar{r}_0^2\,\mathbf{e}_{\bar{r}}\cdot\left(-\frac{1}{\bar{r}_0^2}\mathbf{e}_{\bar{r}}\right) = -4\pi\ .$$

$\bar{r}_0$ is the radius of the sphere. We have therefore found all in all:

$$\int_V d^3r\,\Delta\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \begin{cases} -4\pi\ , \text{ if } \mathbf{r}' \in V\ , \\ 0\ , \quad\ \text{if } \mathbf{r}' \notin V\ . \end{cases} \tag{1.70}$$

This corresponds to (1.2). The assertion (1.69) is therewith proven.

1.6.1 Decomposition Theorem

Let $\mathbf{a}(\mathbf{r})$ be a vector field which is defined in the whole space and which, including its derivatives, tends to zero at infinity with *sufficiently high* order. Then $\mathbf{a}(\mathbf{r})$ can be written as a sum of a curl-free (*longitudinal*) and a divergence-free (*transversal*) part:

$$\mathbf{a}(\mathbf{r}) = \mathbf{a}_\mathrm{l}(\mathbf{r}) + \mathbf{a}_\mathrm{t}(\mathbf{r}) \, , \tag{1.71}$$

$$\mathrm{curl}\, \mathbf{a}_\mathrm{l} = \mathbf{0} \, ; \quad \mathrm{div}\, \mathbf{a}_\mathrm{t} = 0 \, . \tag{1.72}$$

The transversal part is thereby fixed by the curl of $\mathbf{a}(\mathbf{r})$ and the longitudinal part by the divergence of $\mathbf{a}(\mathbf{r})$:

$$\mathbf{a}_\mathrm{l}(\mathbf{r}) = \mathrm{grad}\, \alpha(\mathbf{r}) \, , \tag{1.73}$$

$$\mathbf{a}_\mathrm{t}(\mathbf{r}) = \mathrm{curl}\, \beta(\mathbf{r}) \, , \tag{1.74}$$

$$\alpha(\mathbf{r}) = -\frac{1}{4\pi} \int d^3r' \, \frac{\mathrm{div}\, \mathbf{a}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, , \tag{1.75}$$

$$\beta(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \, \frac{\mathrm{curl}\, \mathbf{a}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, . \tag{1.76}$$

Proof For the following reformulations we will apply several times the previously derived formulas:

$$\begin{aligned} \mathrm{curl}\, \mathrm{curl}\mathbf{A} &= \mathrm{grad}(\mathrm{div}\mathbf{A}) - \Delta\mathbf{A} \qquad ((1.293), \text{Vol. } 1) \, , \\ \mathrm{div}(\varphi\mathbf{A}) &= \varphi \mathrm{div}\mathbf{A} + \mathbf{A} \cdot \mathrm{grad}\varphi \qquad ((1.281), \text{Vol. } 1) \end{aligned}$$

If it is not unique onto which variables the differential operators act then we provide the respective symbols with additional indexes:

$$\begin{aligned} &\frac{1}{4\pi} \mathrm{curl}_r \mathrm{curl}_r \int d^3r' \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi} \mathrm{grad}_r \int d^3r' \mathrm{div}_r \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi} \int d^3r' \, \Delta_r \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi} \mathrm{grad}_r \int d^3r' \left\{ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \underbrace{\mathrm{div}_r \mathbf{a}(\mathbf{r}')}_{=0} + \mathbf{a}(\mathbf{r}') \cdot \mathrm{grad}_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right\} \\ &\quad + \int d^3r' \mathbf{a}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') \\ &= \mathbf{a}(\mathbf{r}) - \frac{1}{4\pi} \mathrm{grad}_r \int d^3r' \, \mathbf{a}(\mathbf{r}') \cdot \mathrm{grad}_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{a}(\mathbf{r}) - \frac{1}{4\pi}\text{grad}_r \int d^3r' \text{div}_{r'}\left(\mathbf{a}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \\
&\quad + \frac{1}{4\pi}\text{grad}_r \int d^3r' \frac{\text{div}_{r'}\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= \mathbf{a}(\mathbf{r}) - \mathbf{a}_l(\mathbf{r}) - \frac{1}{4\pi}\text{grad}_r \int_{S(V\to\infty)} d\mathbf{f}' \cdot \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ .
\end{aligned}
$$

In the last step we have used the Gauss theorem. Since according to our assumption the vector field $\mathbf{a}(\mathbf{r})$ vanishes at infinity *sufficiently rapidly*, the surface integral does not contribute:

$$
\mathbf{a}(\mathbf{r}) = \mathbf{a}_l(\mathbf{r}) + \frac{1}{4\pi}\text{curl}_r\,\text{curl}_r \int d^3r' \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ .
$$

We still manipulate the last summand a little bit:

$$
\begin{aligned}
\text{curl}_r \int d^3r' \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} &\overset{((1.289),\text{ Vol. 1})}{=} \int d^3r' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\underbrace{\text{curl}_r\mathbf{a}(\mathbf{r}')}_{=0} - \mathbf{a}(\mathbf{r}') \times \text{grad}_r \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \\
&= \int d^3r'\, \mathbf{a}(\mathbf{r}') \times \text{grad}_{r'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\int d^3r' \text{curl}_{r'}\left(\frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \int d^3r' \frac{\text{curl}_{r'}\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&\overset{(1.57)}{=} -\int_{S(V\to\infty)} d\mathbf{f}' \times \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + 4\pi\boldsymbol{\beta}(\mathbf{r}) \ .
\end{aligned}
$$

The surface integral vanishes in this case, too, so that it remains:

$$
\frac{1}{4\pi}\text{curl}_r\,\text{curl}_r \int d^3r' \frac{\mathbf{a}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \text{curl}\,\boldsymbol{\beta}(\mathbf{r}) = \mathbf{a}_t(\mathbf{r}) \ .
$$

Therewith the decomposition theorem (1.71) is proven.

1.6.2 Uniqueness Theorem

The vector field $\mathbf{a}(\mathbf{r})$ *is* ***uniquely*** *fixed if for all space points*

$$
\begin{aligned}
&\text{div}\,\mathbf{a}(\mathbf{r}) \qquad && \textit{sources} \ , \\
&\text{curl}\,\mathbf{a}(\mathbf{r}) \qquad && \textit{vortexes}
\end{aligned}
$$

are known.

Proof Let $\mathbf{a}_1(\mathbf{r})$, $\mathbf{a}_2(\mathbf{r})$ be two vector fields with

$$\operatorname{div}\mathbf{a}_1(\mathbf{r}) = \operatorname{div}\mathbf{a}_2(\mathbf{r}) \ ,$$
$$\operatorname{curl}\mathbf{a}_1(\mathbf{r}) = \operatorname{curl}\mathbf{a}_2(\mathbf{r}) \ .$$

For the difference vector

$$\mathbf{D}(\mathbf{r}) = \mathbf{a}_1(\mathbf{r}) - \mathbf{a}_2(\mathbf{r})$$

it then holds:

$$\operatorname{div}\mathbf{D} = 0 \ ; \quad \operatorname{curl}\mathbf{D} = \mathbf{0} \ .$$

The latter relation implies

$$\mathbf{D} = \nabla\psi \ ,$$

so that we have with the first relation:

$$\Delta\psi = 0 \ .$$

We now use the first Green identity (1.66) for $\varphi = \psi$:

$$\int \left[\psi\,\Delta\psi + (\nabla\psi)^2\right] d^3r = \oint\limits_{S(V\to\infty)} \psi\,\nabla\psi\cdot d\mathbf{f} = 0 \ .$$

The surface integral vanishes because of the presumption concerning the behavior of the fields at infinity. It remains:

$$\int (\nabla\psi)^2\, d^3r = 0 \iff \nabla\psi = \mathbf{0} = \mathbf{D} \ .$$

From this it follows the assertion $\mathbf{a}_1(\mathbf{r}) = \mathbf{a}_2(\mathbf{r})$.

Conclusions

1. A vortex-free field ($\operatorname{curl}\mathbf{a} = 0$) is a gradient-field! Because according to (1.71) and (1.76):

$$\mathbf{a}(\mathbf{r}) = \mathbf{a}_l(\mathbf{r}) = \operatorname{grad}\alpha(\mathbf{r}) \ .$$

2. A source-free field ($\operatorname{div}\mathbf{a} = 0$) is a curl-field! This follows from (1.71) and (1.75):

$$\mathbf{a}(\mathbf{r}) = \mathbf{a}_t = \operatorname{curl}\boldsymbol{\beta}(\mathbf{r}) \ .$$

3. In general $\mathbf{a}(\mathbf{r})$ is a superposition of a curl- and a gradient-field:

$$\mathbf{a}(\mathbf{r}) = \operatorname{grad}\alpha(\mathbf{r}) + \operatorname{curl}\beta(\mathbf{r}) \; .$$

4. The **scalar potential** $\alpha(\mathbf{r})$ is obtained from the sources of $\mathbf{a}(\mathbf{r})$:

$$\Delta\alpha(\mathbf{r}) = \operatorname{div}\mathbf{a}(\mathbf{r}) \; . \tag{1.77}$$

5. The **vector potential** $\beta(\mathbf{r})$ is obtained from the vortexes of $\mathbf{a}(\mathbf{r})$:

$$\Delta\beta(\mathbf{r}) = -\operatorname{curl}\mathbf{a}(\mathbf{r}) \; . \tag{1.78}$$

1.7 Exercises

Exercise 1.7.1 Show that Dirac's δ-function $\delta(x-a)$ can be written as limiting value of the function

$$f_\eta(x-a) = \frac{1}{\sqrt{\pi\eta}} \exp\left(-\frac{(x-a)^2}{\eta}\right)$$

for $\eta \to 0^+$.

Exercise 1.7.2 Verify the following representations of the δ-function:

$$\lim_{\eta\to 0^+} \operatorname{Im}\frac{1}{(x-a) \mp i\eta} = \pm\pi\delta(x-a)$$

(Im: imaginary part).

Exercise 1.7.3 Let $g(x)$ be a differentiable function with simple zeros $x_n \; (g(x_n) = 0, \; g'(x_n) \neq 0)$. Prove the following identity:

$$\delta\left(g(x)\right) = \sum_n \frac{1}{|g'(x_n)|}\,\delta(x-x_n) \; .$$

Exercise 1.7.4 Calculate the following integrals:

1. $\displaystyle\int_{-2}^{+5} (x^2 - 5x + 6)\,\delta(x-3)\,dx \; ,$

2. $\displaystyle\int_{\alpha}^{\beta} \left(f(x) - f(a)\right)\,\delta(x-a)\,dx \; ,$

$$3.\ \int_0^{\infty} x^2\,\delta(x^2-3x+2)\,dx\ ,$$

$$4.\ \int_0^{+\infty} \ln x\,\delta'(x-a)\,dx\ ,$$

$$5.\ \int_0^{\pi} \sin^3\vartheta\,\delta\left(\cos\vartheta-\cos\frac{\pi}{3}\right)\,d\vartheta\ .$$

Exercise 1.7.5 Write down the two-dimensional δ-function

1. in Cartesian coordinates,
2. in plane polar coordinates!

Exercise 1.7.6 Determine the Taylor series of the following scalar fields:

$$1.\ \varphi(\mathbf{r}) = \exp(i\,\mathbf{k}\cdot\mathbf{r}) \quad (\mathbf{k} = \text{const})\ ,$$

$$2.\ \varphi(\mathbf{r}) = |\mathbf{r}-\mathbf{r}_0| \quad \text{(up to second order)}\ .$$

Exercise 1.7.7 Integrate the function

$$f(x,y) = x^2y^3$$

1. over the triangle area $(0,0)\ -\ (1,0)\ -\ (1,1)$,
2. over the area of a circle around the origin with the radius R,
3. over an area for which the boundary consists of a circle around the origin with radius R plus the positive x- and the positive y-axis.

Exercise 1.7.8 Calculate for the rectangle with the edge points $\left(b,\frac{a}{\sqrt{2}},0\right)$, $\left(0,\frac{a}{\sqrt{2}},0\right)$, $\left(0,0,\frac{a}{\sqrt{2}}\right)$, $\left(b,0,\frac{a}{\sqrt{2}}\right)$

1. the vectorial surface element $d\mathbf{f}$,
2. the vector of the total area $\mathbf{F}$,
3. the flux of the field

$$\mathbf{a}(\mathbf{r}) = \left(y^2,\ 2xy,\ 3z^2-x^2\right)$$

through the area $\mathbf{F}$ of the rectangle.

Exercise 1.7.9 Calculate the flux of the vector field $\mathbf{a}(\mathbf{r})$ through the surface of a sphere with the radius R and the origin as its center:

1. $\mathbf{a}(\mathbf{r}) = 3\dfrac{\mathbf{r}}{r^2}$,
2. $\mathbf{a}(\mathbf{r}) = \dfrac{(x, y, z)}{\sqrt{\alpha + x^2 + y^2 + z^2}}$,
3. $\mathbf{a}(\mathbf{r}) = (3z,\ x,\ 2y)$.

Exercise 1.7.10 Given a cylinder with height L and radius R. The midpoint of the cylinder coincides with the origin of coordinates.

1. Find suitable parametrizations and **calculate** the vectorial element $d\mathbf{f}$ of the cylinder surface (barrel and front faces)!
2. Calculate **without** using the Gauss theorem the flux of the vector field

$$\mathbf{E}(\mathbf{r}) = \alpha\,\mathbf{r} \quad (\alpha = \text{const})$$

   through the cylinder surface.
3. Confirm the result of part 2. by applying the Gauss theorem!

Exercise 1.7.11 Calculate for the vector field

$$\mathbf{a}(\mathbf{r}) = \alpha\mathbf{r}$$

the vectorial double integral

$$\boldsymbol{\psi} = \int_F \mathbf{a}(\mathbf{r}) \times d\mathbf{f}$$

over a sphere (radius R, center at the origin of coordinates) and over a cylinder (radius R, length L).

Exercise 1.7.12 In case of known charge density $\rho(\mathbf{r})$ one can determine by

$$Q = \int d^3r\,\rho(\mathbf{r})$$

the total electric charge Q and by

$$\mathbf{p} = \int d^3r\,\mathbf{r}\,\rho(\mathbf{r})$$

the electric dipole moment $\mathbf{p}$ of the charge distribution. Calculate these quantities for a homogeneously charged sphere with radius R:

$$\rho(\mathbf{r}) = \begin{cases} \rho_0 \ , & \text{if } r \leq R \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Exercise 1.7.13 Prove the following useful relations:

1. Gradient of a scalar product:

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} + (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{b}) \ ,$$

2. Divergence of a vector product:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \ ,$$

3. Curl of a vector product:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) \ .$$

Exercise 1.7.14

1. Evaluate the divergence in arbitrary curvilinear-orthogonal coordinates y_1, y_2, y_3 (see (1.250), Vol. 1) using its integral representation

$$\operatorname{div}\mathbf{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\limits_{S(\Delta V)} \mathbf{E} \cdot d\mathbf{f} \ .$$

   In the process use for ΔV the volume of the differential scalar triple product built by the y_i-coordinate lines (Sect. 1.7., Vol. 1).
2. Express the divergence by cylindrical coordinates.
3. Formulate the divergence in spherical coordinates.

Exercise 1.7.15

1. Use for the evaluation of the curl in arbitrary curvilinear-orthogonal coordinates y_1, y_2, y_3 the integral representation

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{a}(\mathbf{r}) = \lim_{F_C \to 0} \frac{1}{F_C} \oint\limits_{C} \mathbf{a} \cdot d\mathbf{r} \ .$$

2. Express the curl by cylindrical coordinates.
3. Express the curl by spherical coordinates.

Exercise 1.7.16 Calculate

1. the components of grad $(\boldsymbol{\alpha} \cdot \mathbf{r})$ with spherical coordinates,
2. div $\mathbf{e_r}$, grad div$\mathbf{e_r}$, curl $\mathbf{e_r}$, div $\mathbf{e}_\varphi$, curl $\mathbf{e}_\vartheta$ with spherical coordinates,
3. the components of curl $(\boldsymbol{\alpha} \times \mathbf{r})$ by the use of cylindrical coordinates ($\boldsymbol{\alpha} = \text{const}$).

Exercise 1.7.17 Demonstrate that for a conservative force field $\mathbf{F}(\mathbf{r})$ the integral

$$\boldsymbol{\psi} = \oint\limits_{S(V)} \mathbf{F}(\mathbf{r}) \times d\mathbf{f}$$

over the closed surface of an arbitrary volume V always vanishes.

Exercise 1.7.18 Let the vectors $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ fulfill the relation

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \; .$$

At an arbitrary point of time t_0, $\mathbf{B} = 0$ for all $\mathbf{r}$. Show that for **all** times it must then hold div $\mathbf{B} = 0$.

Exercise 1.7.19 Given is an area $\mathbf{F} = F\mathbf{e}_z$ in the xy-plane:

1. $\mathbf{F}$: circle with radius R around the origin,
2. $\mathbf{F}$: rectangle with side lengths a and b.

Show first by a direct calculation and then alternatively by use of the Stokes theorem that

$$\int\limits_{\partial F} \mathbf{r} \times d\mathbf{r} = 2\mathbf{F} \; .$$

Interpret the result geometrically and find reasons why the relation is valid even for **arbitrary** areas $\mathbf{F}$!

Exercise 1.7.20 Given is a vector field

$$\mathbf{a}(\mathbf{r}) = (0, 0, y)$$

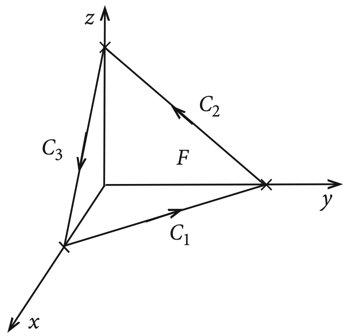
as well as the area F which is defined as the part of the plane

$$6x + 3y + 2z = 12 \; .$$

lying in the first octant (Fig. 1.16).

1. How does the parameter representation of the area F read? Find the vectorial area element $d\mathbf{f}$.
2. Calculate the flux of $\mathbf{a}$ through F.

Fig. 1.16 The area F as part of the plane $6x+3y+2z = 12$ framed by the way $C = C_1 + C_2 + C_3$



3. Find reasons why **a** can be represented as curl-field curl $\beta(\mathbf{r})$. Is the choice of $\beta(\mathbf{r})$ unique? Determine a possible $\beta(\mathbf{r})$!
4. Calculate once more the flux of **a** through F, but now via a line integral along the path $C = C_1 + C_2 + C_3$. Confirm therewith the result of part 2.! What influence does the non-uniqueness of $\beta(\mathbf{r})$ have?

Exercise 1.7.21 Prove the following relation, generally valid for vector fields $\mathbf{a}(\mathbf{r})$, $\mathbf{b}(\mathbf{r})$:

$$\int_V d^3r\, \mathbf{b} \cdot \operatorname{curl} \mathbf{a} = \int_V d^3r\, \mathbf{a} \cdot \operatorname{curl} \mathbf{b} + \oint_{S(V)} d\mathbf{f} \cdot (\mathbf{a} \times \mathbf{b}) \ .$$

Exercise 1.7.22 Calculate for the vector field

$$\mathbf{a}(\mathbf{r}) = \left(-y(x^2 + y^2),\, x\left(x^2 + y^2\right),\, xyz\right)$$

the line integral

$$\oint_C \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r}$$

along the circle lying in the xy-plane with its center at the origin of coordinates and the radius R.

Exercise 1.7.23 Are the following vector fields pure gradient-fields or pure curl-fields?

1. $\mathbf{a}(\mathbf{r}) = x\mathbf{e}_x + y\mathbf{e}_y$,
2. $\mathbf{a}(\mathbf{r}) = (6\alpha x,\, z\cos yz,\, y\cos yz)$,
3. $\mathbf{a}(\mathbf{r}) = (x(z-y),\, y(x-z),\, z(y-x))$,
4. $\mathbf{a}(\mathbf{r}) = (x^2y,\, \cos z^3,\, zy)$.

Exercise 1.7.24 Two scalar fields $\varphi_1(\mathbf{r})$, $\varphi_2(\mathbf{r})$ both fulfill within the volume V the differential equation:

$$\Delta\varphi(\mathbf{r}) = f(\mathbf{r}) \quad \textbf{Poisson-equation}$$

On the surface $S(V)$ it holds $\varphi_1(\mathbf{r}) = \varphi_2(\mathbf{r})$. Verify that then

$$\varphi_1(\mathbf{r}) \equiv \varphi_2(\mathbf{r}) \quad \text{in } V$$

Hint: Use the Green theorems for $\psi(\mathbf{r}) = \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r})$.

1.8 Self-Examination Questions

To Section 1.1

1. What do we call a distribution?
2. Which requirements do define Dirac's δ-function?
3. List the most important properties of the δ-function.
4. How does the three-dimensional δ-function read in curvilinear coordinates?

To Section 1.2

1. Under which conditions can a function $f(x)$ be expanded as a Taylor series?
2. What does one understand by the approximate polynomial and the remainder term, respectively, of a Taylor series?
3. How does the estimation of the remainder term according to Lagrange look like?
4. Give the Taylor expansion for a scalar field $\varphi(\mathbf{r})$.

To Section 1.3

1. What is to be understood by the orientation of an area element?
2. What is a tangent plane?
3. What is the parameter representation of the surface of the sphere (radius R)?
4. Calculate the oriented surface element of the cylindrical barrel!
5. Is the flux of the vector field $\mathbf{a}(\mathbf{r})$ through the area S a vector or a scalar? How is the flux defined?
6. Define the surface integral!
7. Formulate the flux of a homogeneous field through the surface $S(V)$ of an arbitrary volume V!
8. What is the flux of the field $\mathbf{a}(\mathbf{r}) = \alpha \cdot \frac{\mathbf{r}}{r}$ through the surface of sphere which has the radius R and its center at $r = 0$?

To Section 1.4

1. What does one understand by the average source density of the field $\mathbf{E}$ in the volume V?

2. How can one get from the average source density the divergence of the **E**-field?
3. How does the general surface-integral representation of the nabla-operator read?
4. What does the circulation of the field $\mathbf{a}(\mathbf{r})$ along the way C mean?
5. What is denoted as vorticity of a vector field?
6. Which connection does exist between circulation and curl?
7. Formulate the line-integral representation of the curl!
8. What is the general line-integral representation of the nabla-operator?

To Section 1.5

1. Formulate the Gauss theorem!
2. What can be said by use of the Gauss-theorem about the vorticity-flux $\operatorname{curl}\mathbf{E}$ of a vector field $\mathbf{E}(\mathbf{r})$ through a closed surface?
3. Which form does the Gauss theorem have for scalar fields?
4. Formulate the Stokes theorem!
5. Answer the question 2. by use of the Stokes theorem!
6. Which form has the Stokes theorem for scalar fields?
7. How can the Gauss and the Stokes theorem for all types of fields be written in a general (symbolic) form?
8. What does one understand by the normal-derivative of a scalar field on the area S?
9. What are the first and second Green identities? How can one derive the second identity from the first?

To Section 1.6

1. What does the decomposition theorem tell us?
2. What does one understand by the longitudinal, what by the transversal part of a vector field? By which properties of the field are they determined?
3. What is expressed by the uniqueness-theorem?
4. What can be said about a curl-free (source-free) field?
5. What does one understand by the scalar potential and what by the vector potential of a vector field?

Chapter 2
Electrostatics

2.1 Basic Concepts

2.1.1 Charges and Currents

The fundamental terms of classical mechanics,

mass, length, time

are more or less directly detectable by our sense organs and by our inherent sense of time, respectively. In a certain sense we can perceive them without any auxiliary experimental means. In electrodynamics there comes along as fourth basic quantity the

charge,

the observation of which, however, requires special auxiliary means. We do not have a sense organ for a direct perception of electrical phenomena. That makes them for the beginner *imperceptible* and conceptually quite difficult.

Already before Thales of Miletus (625 to 547 BC) it was known that certain bodies change their properties if they are rubbed against other bodies. If, for instance, a piece of the mineral amber (Greek: electron) is rubbed by a cloth then it is able to attract small, light bodies (grains, scraps of paper or the like). The thereby acting forces can not be explained mechanically. One therefore states, simply for a start, that the rubbed material finds itself in an

electric state

One observes further that this state can be transferred to another body by touching. This can be expressed most elegantly by introducing a *substance-like* quantity, namely the

electric charge Q

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This quantity is considered to cause the above-mentioned forces. By a proper contact it can *'flow'* as

electric current I

from one body to another.

Experimental experience tells us that two types of charges exist which, rather arbitrarily but appropriately, can be discriminated as **positive** and **negative**:

$$\begin{aligned} Q > 0&: \textit{positive} \text{ charge },\\ Q < 0&: \textit{negative} \text{ charge }. \end{aligned} \tag{2.1}$$

The sign of the charge is fixed in such a way that rubbing a rod of glass leaves behind the charge $Q > 0$ on the rod, while rubbing a rod of ebonite (hard rubber) leads to the charge $Q < 0$. This fixing has the consequence that the charge of the electron, which is chosen as the natural unit, is negative. As to additive and multiplicative numerical calculations the charges behave like usual negative and positive numbers.

Total Charge

$$Q = \sum_{i=1}^{n} q_i \,. \tag{2.2}$$

$Q = 0$ means firstly only that positive and negative charges are cancelling each other, and not necessarily that the whole body is built up by electrically *neutral* pieces. Removing positive charge makes the body negatively charged and vice versa.

For charges a **conservation law** is found:

In a closed system the sum of positive and negative charges is constant.

In the above-mentioned rubbing-experiments no charge is thus *created*; only positive and negative charges have been spatially separated from each other.

For a deeper insight into electromagnetic processes the decisive experimental finding is that the charge, as we know it from matter, possesses a *quantized, atomic* structure. There exists the smallest, no further fissionable

elementary charge e

Each other charge can be written as an integral multiple of **e**:

$$Q = n\,e\,; \quad n \in \mathbb{Z}\,. \tag{2.3}$$

Examples

$$\begin{aligned} &\text{electron:} && n = -1\,,\\ &\text{proton:} && n = +1\,,\\ &\text{neutron:} && n = 0\,,\\ &\text{atomic nucleus:} && n = Z \quad \text{(atomic number)}\,. \end{aligned}$$

Experimental proofs of the charge quantization are:

1. the electrolysis (Faraday's law),
2. the Millikan's experiment.

An important term for the electrodynamics is the

charge density $\rho(\mathbf{r})$**,**

which is to be understood as charge per unit volume. Using it the total charge Q in the volume V is calculated as

$$Q = \int_V d^3r\, \rho(\mathbf{r}) \ . \tag{2.4}$$

In strict analogy to the concept of the mass point in classical mechanics one introduces for the electrodynamics the

point charge q**,**

if the charge distribution is of negligible extension in all directions. We therefore have as charge density of a point charge:

$$\rho(\mathbf{r}) = q\, \delta(\mathbf{r} - \mathbf{r}_0) \ . \tag{2.5}$$

This abstraction frequently means a strong mathematical simplification, which, however, sometimes, has also to be handled with care.

The fact that charged bodies exert forces on each other can be used for measuring the charge (*electrometer*). One observes that charges with the same sign repel each other while charges with different signs attract. That can very easily be demonstrated by a *'charge balance'* (Fig. 2.1).

For a tentative (!) definition of the **unit of charge** we apply the concept of the point charge:

Two point charges of equal magnitudes, which exert on each other in the vacuum at a distance of 1 *m the force*

$$F = \frac{10^{12}}{4\pi \cdot 8{,}8543}\,\mathrm{N} \ , \tag{2.6}$$

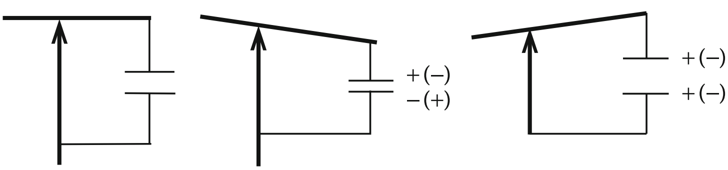


Fig. 2.1 Schematic representation of the charge balance

both possess the charge

$$1 \text{ coulomb } (1\,\mathrm{C}) = 1 \text{ ampere-second } (1\,\mathrm{A\,s}) \ .$$

The meaning of this definition will become clear later. For the elementary electric charge e it leads to the value:

$$e = 1{,}602 \times 10^{-19}\,\mathrm{C} \ . \tag{2.7}$$

As already mentioned **moving** charges build an electric current or a
current density $\mathbf{j}(\mathbf{r})$

$\frac{\mathbf{j}}{|\mathbf{j}|}$: *normal unit vector in the direction of the moving charge,*

$|\mathbf{j}|$: *charge which is transported per time unit through the unit of area perpendicular to the direction of the current.*

Example Homogeneous distribution of N particles each with the charge q in a volume V all of which have the same velocity $\mathbf{v}$:

$$\mathbf{j} = n\,q\,\mathbf{v} \ , \quad n = \frac{N}{V} \ . \tag{2.8}$$

As **(strength of) current** I through a given area one denotes then the following surface integral:

$$I = \int_F \mathbf{j} \cdot d\mathbf{f} \ . \tag{2.9}$$

The unit of current is *'ampere'* 1 A. A current of the strength of 1 A transports in 1 s just the charge of 1 C. Later it will be shown that the exact definition uses the mutual force acting between two conductors (wires) both carrying a definite current.

The conservation law of the charge can be formulated as **continuity equation**:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 \ . \tag{2.10}$$

We have already derived this relation earlier (1.55) by the use of the Gauss theorem. For the derivation we had presumed that the temporal change of the total charge in an arbitrary volume V must be oppositely equal to the charge current through the surface $S(V)$. However, this expresses nothing but the charge conservation in a closed system.

2.1.2 Coulomb's Law, Electric Field

We now investigate a bit more precisely the manner how charged particles interact with each other. Thereby we first rely exclusively on the experimental experience.

Let two charges q_1 and q_2 have the separation (Fig. 2.2)

$$r_{12} = |\mathbf{r}_{12}| = |\mathbf{r}_1 - \mathbf{r}_2| \; .$$

which is very much larger than the linear extensions of both the charge distributions so that it should be allowed to consider them as point charges. Then the force acting between the two charges is given by the **Coulomb's law**:

$$\mathbf{F}_{12} = k\, q_1\, q_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -\mathbf{F}_{21} \; . \tag{2.11}$$

$\mathbf{F}_{12}$ is the force exerted by the charge 2 on the charge 1. Equation (2.11) is to be considered as an experimentally uniquely verified matter of fact. The constant k depends on the medium in which the point charges are located, but of course also on the units which we use for measuring the basic electrical quantities. This will be explained below somewhat more precisely.

The **Coulomb-force** $\mathbf{F}_{12}$

1. is directly proportional to the charges q_1, q_2;
2. is inversely proportional to the square of the distance between the two charges;
3. acts along the connecting line, attractive for charges with opposite signs and repulsive for those with the same sign;
4. fulfills *action* = *reaction*.

Decisive precondition for the validity of (2.11) is that the considered charges are **at rest**. In case of moving charges additional terms appear which we will discuss in detail at a later stage of the course.

For the electro**statics** equation (2.11) is to be accepted as *experimentally established basic law*. The full formalism of the electrostatics is based on (2.11) together with the so-called **'principle of linear superposition'** which corresponds to Newton's fourth axiom for classical mechanics ((2.47),Vol. 1). It states that the Coulomb-forces on the charge q_1 due to several other charges q_j are adding together

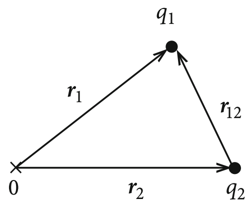


Fig. 2.2 Arrangement for the formulation of the Coulomb's law

vectorially:

$$\mathbf{F}_1 = k\, q_1 \sum_{j=2}^{n} q_j \frac{\mathbf{r}_1 - \mathbf{r}_j}{|\mathbf{r}_1 - \mathbf{r}_j|^3} \ . \tag{2.12}$$

The Coulomb's law connects charges with purely mechanical quantities what can be used for the definition of the unit of charge. Unfortunately, for electrodynamics there exists quite a series of different systems of units, with in principle are all equivalent being only suitable for different intended uses. Since the precise specifications, strictly speaking, can be understood only after one is familiar with the full electrodynamics, we restrict ourselves here to a few preliminary remarks:

(1) Gaussian System of Units (cgs-System)

This is defined by

$$k = 1 \ ,$$

where the charge unit (CU) derives itself uniquely with (2.11) from mechanical quantities, i.e does not represent a new basic quantity:

$$1\, CU = 1\,\mathrm{cm}\,\mathrm{dyn}^{1/2} \quad \left(1\,\mathrm{dyn} = 1\,\mathrm{g}\frac{\mathrm{cm}}{\mathrm{s}^2}\right) \ . \tag{2.13}$$

Two *unit charges* with a separation of 1 cm exert on each other a force of 1 dyn.

(2) SI: International System of Units (MKSA-System)

(SI because of *Systeme International d'Unités*). In addition to the mechanical basic units meter (**M**), kilogram (**K**), second (**S**) comes along as electrical unit the ampere (**A**) for the strength of current. That yields for the **charge unit**

$$1\,\mathrm{C}\,(\mathrm{coulomb}) = 1\,\mathrm{A\,s} \ .$$

The ampere is defined such that the constant k in (2.11) is given by:

$$k = 10^{-7}\, c^2\, \frac{\mathrm{N}}{\mathrm{A}^2} \ .$$

Thereby

$$c = 2{,}9979250 \cdot 10^8\, \frac{\mathrm{m}}{\mathrm{s}} \tag{2.14}$$

is the speed of light in vacuum. One takes

$$k = \frac{1}{4\pi\,\epsilon_0} \tag{2.15}$$

with the **permittivity of vacuum** ϵ_0 (also *dielectric constant of vacuum*)

$$\epsilon_0 = 8{,}8543 \cdot 10^{-12}\,\frac{\mathrm{A^2\,s^2}}{\mathrm{N\,m^2}} = 8{,}8543 \cdot 10^{-12}\,\frac{\mathrm{A\,s}}{\mathrm{V\,m}}\;. \tag{2.16}$$

Here we have further used

$$1\ \mathrm{V}\ (\text{volt}) = 1\,\frac{\mathrm{N\,m}}{\mathrm{A\,s}} \tag{2.17}$$

To keep the confusion to a minimum we apply from now on **exclusively** the SI system!

Although the actual measurand represents a force it appears to be highly useful to introduce the concept of the

electric field E(r)

It is created by a charge configuration and is defined by the force on a test charge q:

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q}\;. \tag{2.18}$$

Hence it is a vectorial quantity. The limiting process is necessary since the test charge itself modifies the field. But on the other hand, because of (2.3) it is of course also somewhat questionable (a mathematical abstraction which is physically not realizable). The unit of the electric field strength is then:

$$1\,\frac{\mathrm{N}}{\mathrm{C}} = 1\,\frac{\mathrm{V}}{\mathrm{m}}\;. \tag{2.19}$$

By the concept of the electric field the interaction process described by (2.11) consists of two steps. At first a given charge distribution creates **instantaneously** an electric field filling the whole space. This field exists independently of the point charge q which, in the second step, *reacts locally* on the already present field according to

$$\mathbf{F}(\mathbf{r}) = q\,\mathbf{E}(\mathbf{r}) \tag{2.20}$$

The idea goes back to M. Faraday (1791–1867) to illustrate the field concept by the use of a *pictorial language*, which, however, is more of qualitative than quantitative character. One introduces

field lines

and understands by these the paths along which small **positively** charged, initially resting bodies would propagate under the Coulomb force (2.11) and (2.20), respectively. Accordingly, the field lines of point charges are radial (Fig. 2.3).

At each point of space $\mathbf{r}$ the field

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\,\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \tag{2.21}$$

is oriented tangentially to the there existing field line.

When one brings two point charges closer together then the corresponding force lines will mutually alter each other, since the test body, whose path defines the lines, is now under the influence of **both** the point charges (Fig. 2.4).

The figures convey the impression that two charges of opposite sign execute a *'field-line tension'* on each other, i.e. they behave attractive, while charges of the same sign execute a *'pressure'*, i.e. they repel each other. From the definition of the field line as the path of a positively charged test-body follows:

Field lines do never intersect!

They start at positive charges and end at negative charges.

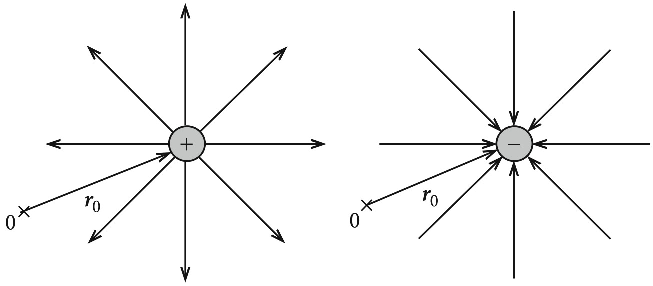


Fig. 2.3 Electric field lines of positive and negative point charges

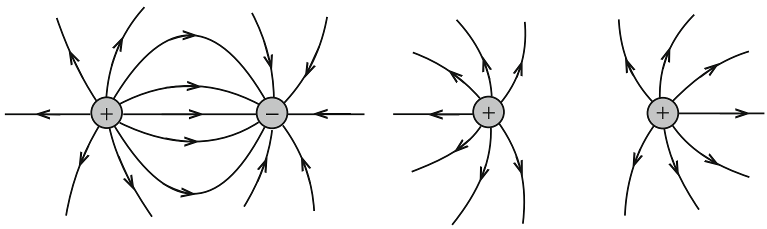


Fig. 2.4 Field lines produced by two, respectively, oppositely and similarly charged point charges

According to the principle of linear superposition (2.12) it holds for the **field of n point charges:**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0} \sum_{j=1}^{n} q_j \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3} \ . \tag{2.22}$$

The generalization to **continuous charge distributions** is then obvious (Fig. 2.5),

$$dq = \rho(\mathbf{r}')d^3r' \ ,$$

where dq means the charge in the volume element d^3r at $\mathbf{r}'$. dq creates at $\mathbf{r}$ the field:

$$d\mathbf{E}(\mathbf{r}) = \frac{dq}{4\pi\,\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ .$$

We *add together*:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0} \int d^3r'\,\rho\,(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ . \tag{2.23}$$

For the vector in the integrand we can also write:

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} \ .$$

The static electric field is therefore a pure gradient-field:

$$\mathbf{E}(\mathbf{r}) = -\nabla\,\varphi\,(\mathbf{r}) \ . \tag{2.24}$$

This relation defines the
scalar (electric) potential

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \ . \tag{2.25}$$

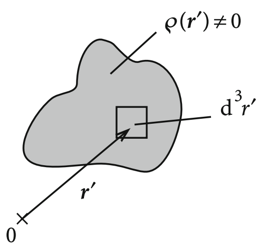


Fig. 2.5 Continuous charge density restricted to a finite space region

Because of (2.24) the field lines are perpendicular to the equipotential surfaces! From (2.24) follows also

$$\operatorname{curl}(q\,\mathbf{E}) \equiv 0\ ,$$

i.e. the Coulomb force (2.20) is conservative and thus possesses a potential V:

$$\mathbf{F} = -\nabla V\ ; \quad V = q\,\varphi(\mathbf{r})\ .$$

$\varphi(\mathbf{r})$ can be interpreted therewith as the potential energy of a unit charge $q = 1\,\mathrm{C}$ in the field $\mathbf{E}$ at the position $\mathbf{r}$.

The line integral of the electric field $\mathbf{E}$ must be independent of the path:

$$\varphi(\mathbf{r}) - \varphi(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'\ . \tag{2.26}$$

One denotes this **potential difference** as **voltage (electric tension)** $U(\mathbf{r}, \mathbf{r}_0)$. The unit of U and φ is the volt defined in (2.17).

Examples 1. N **point charges**

$$\rho(\mathbf{r}') = \sum_{j=1}^{N} q_j\,\delta(\mathbf{r}' - \mathbf{r}_j) \Longrightarrow \varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0} \sum_{j=1}^{N} \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|}\ . \tag{2.27}$$

With (2.24) it follows again (2.22).

2. **Homogeneously charged sphere, radius** R**, charge** Q

We put the origin at the center of the sphere (Fig. 2.6):

$$\rho(\mathbf{r}') = \begin{cases} \rho_0\ , & \text{if } r' \le R\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{2.28}$$

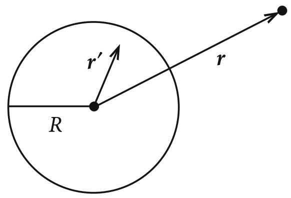


Fig. 2.6 To the calculation of the scalar potential of a homogeneously charged sphere

Let the direction of $\mathbf{r}$ define the z-axis. Then the scalar potential φ is calculated as follows:

$$
\begin{aligned}
\varphi(\mathbf{r}) &= \frac{\rho_0}{4\pi\,\epsilon_0} \int\limits_{\text{sphere}} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\rho_0}{4\pi\,\epsilon_0} \int\limits_0^R dr'\, r'^2 \int\limits_0^{2\pi} d\varphi' \int\limits_0^{\pi} d\vartheta' \sin\vartheta' \, \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\vartheta'}} \\
&= \frac{2\pi\,\rho_0}{4\pi\,\epsilon_0} \int\limits_0^R dr'\, r'^2 \int\limits_{-1}^{+1} d\cos\vartheta' \, \frac{d}{d\cos\vartheta'} \sqrt{r^2 + r'^2 - 2rr'\cos\vartheta'} \left(-\frac{1}{rr'}\right) \\
&= -\frac{2\pi\,\rho_0}{4\pi\,\epsilon_0} \frac{1}{r} \int\limits_0^R dr'\, r' \left(|r-r'| - |r+r'|\right) \\
&= \frac{2\pi\,\rho_0}{4\pi\,\epsilon_0} \frac{1}{r} \int\limits_0^R dr' \begin{cases} 2rr' \,, \text{ if } r < r' \,, \\ 2r'^2 \,, \text{ if } r \geq r' \end{cases} \\
&= \frac{\rho_0}{4\pi\,\epsilon_0} 4\pi \frac{1}{r} \begin{cases} \int\limits_0^R dr'\, r'^2 \,, & \text{if } r > R \,, \\ \int\limits_0^r dr'\, r'^2 + \int\limits_r^R dr'\, rr' \,, & \text{if } r \leq R \end{cases} \\
&= \frac{\rho_0}{4\pi\,\epsilon_0} \frac{4\pi}{r} \begin{cases} \dfrac{R^3}{3} \,, & \text{if } r > R \,, \\ \dfrac{r^3}{3} + \dfrac{r}{2}(R^2 - r^2) \,, & \text{if } r \leq R \,. \end{cases}
\end{aligned}
$$

This can be written as follows:

$$
\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0} \begin{cases} \dfrac{1}{r} & \text{for } r > R \,, \\ \dfrac{1}{2R^3}(3R^2 - r^2) & \text{for } r \leq R \,. \end{cases} \tag{2.29}
$$

Outside of the sphere the potential is identical to that of a point charge Q located at the origin of coordinates (Fig. 2.7).

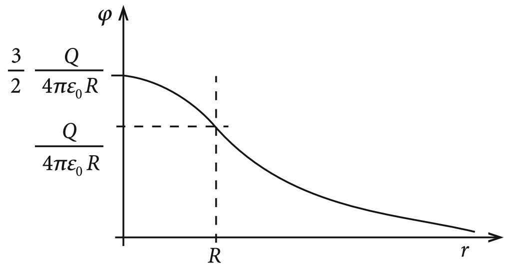


Fig. 2.7 Radial behavior of the scalar potential of a homogeneously charged sphere

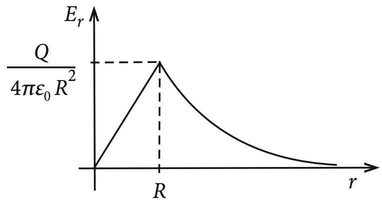


Fig. 2.8 Radial behavior of the electric field strength of a homogeneously charged sphere

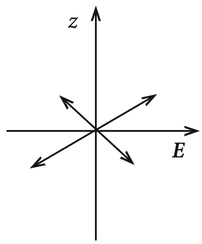


Fig. 2.9 To the calculation of the electric field strength of a homogeneously charged straight line

For the electric field we have (Fig. 2.8):

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_r \frac{1}{4\pi\epsilon_0} \begin{cases} \dfrac{Q}{r^2} & \text{for } r > R\,, \\ \dfrac{Q(r)}{r^2} & \text{for } r \leq R\,. \end{cases} \tag{2.30}$$

$Q(r)$ is here the charge which one finds inside the sphere with the radius $r < R$:

$$Q(r) = Q\frac{r^3}{R^3} = \rho_0 \frac{4\pi}{3} r^3 \quad (r \leq R)\,.$$

3. **Homogeneously charged straight line**
   Let the straight line define the z-axis (Fig. 2.9) and let κ be the charge per unit-length. For the calculation of the electric field according to (2.23) one uses conveniently the cylindrical coordinates (ρ, φ, z). We perform the explicit evaluation as Exercise 2.1.4:

$$\mathbf{E}(\mathbf{r}) = \frac{\kappa}{2\pi\,\epsilon_0\,\rho}\,\mathbf{e}_\rho\,. \tag{2.31}$$

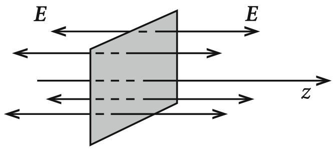


Fig. 2.10 Electric field of a homogeneously charged plane

This corresponds to the scalar potential:

$$\varphi(\mathbf{r}) = -\frac{\kappa}{2\pi\,\epsilon_0}\,\ln\rho + \text{const}\,. \tag{2.32}$$

4. **Homogeneously charged plane**
   Let it be the infinitely extended xy-plane with the homogeneous surface charge density σ (Fig. 2.10). The evaluation in Exercise 2.1.5 yields:

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\epsilon_0}\,\frac{z}{|z|}\,\mathbf{e}_z\,. \tag{2.33}$$

This corresponds to the scalar potential

$$\varphi(\mathbf{r}) = -\frac{\sigma}{2\epsilon_0}|z| + \text{const}\,. \tag{2.34}$$

2.1.3 Maxwell Equations of Electrostatics

Starting from the Coulomb's law (2.11) and (2.20), respectively, and the principle of superposition (2.12), which we consider as experimentally proven basic matter of facts, we now derive two fundamental **field equations** for **E**. We are dealing thereby in this chapter exclusively with time-**independent** fields in vacuum, which are generated by any charge distributions $\rho(\mathbf{r})$.

We use the general form (2.23) for the electric field $\mathbf{E}(\mathbf{r})$ and calculate its flux through the surface $S(V)$ of a preset volume V:

$$\begin{aligned}
\int\limits_{S(V)} \mathbf{E}(\mathbf{r})\cdot d\mathbf{f} &= \frac{1}{4\pi\,\epsilon_0}\int d^3r'\rho(\mathbf{r}')\int\limits_{S(V)} d\mathbf{f}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\\
&= \frac{-1}{4\pi\,\epsilon_0}\int d^3r'\rho(\mathbf{r}')\int\limits_{S(V)} d\mathbf{f}\cdot\nabla_r\frac{1}{|\mathbf{r}-\mathbf{r}'|}\\
&= \frac{-1}{4\pi\,\epsilon_0}\int d^3r'\rho(\mathbf{r}')\int\limits_{V} d^3r\Delta_r\frac{1}{|\mathbf{r}-\mathbf{r}'|}
\end{aligned}$$

$$= \frac{1}{\epsilon_0} \int d^3r' \rho(\mathbf{r}') \int_V d^3r \delta(\mathbf{r} - \mathbf{r}')$$

$$= \frac{1}{\epsilon_0} \int_V d^3r' \rho(\mathbf{r}') = \frac{1}{\epsilon_0} q(V) \ . \tag{2.35}$$

For the above rearranging we have applied ((1.282), Vol. 1), (1.58) and (1.69). The flux of the **E**-field through the surface of an arbitrary volume V is, except for an unessential factor, equal to the total charge $q(V)$ incorporated in V. This relation is called

'physical Gauss theorem'

If one applies to (2.35) once more the *mathematical* Gauss theorem (1.53) it follows:

$$\int_V d^3r \left(\operatorname{div} \mathbf{E} - \frac{\rho(\mathbf{r})}{\epsilon_0} \right) = 0 \ .$$

This is valid for arbitrary volumes V so that it must already hold:

$$\operatorname{div} \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} \rho(\mathbf{r}) \ . \tag{2.36}$$

We have derived therewith a first field equation. It expresses the fact that the sources of the electric field are indeed the electric charges.

We could have deduced the relation (2.36) also directly from the decomposition theorem (1.71) for general vector fields. According to (2.24) $\mathbf{E}(\mathbf{r})$ is a pure gradient-field, therefore it does not contain a transversal part. By comparison of (2.25) with (1.75) one finds immediately (2.36).

The second field equation follows automatically from (2.24):

$$\operatorname{curl} \mathbf{E} = 0 \ . \tag{2.37}$$

The electro**static** field is curl-free. For electric fields with an explicit time-dependence this relation has to be later modified.

With the Stokes theorem (1.60) one recognizes that the circulation of the **E**-field along an arbitrary closed path vanishes:

$$\int_{\partial F} \mathbf{E} \cdot d\mathbf{r} = \int_F \operatorname{curl} \mathbf{E} \cdot d\mathbf{f} = 0 \ . \tag{2.38}$$

Because of their decisive importance we summarize once more the field equations (2.35) to (2.38), which are called the

Maxwell equations of electrostatics

Differential representation:

$$\begin{aligned}\operatorname{div}\mathbf{E} &= \frac{1}{\epsilon_0}\,\rho\;,\\ \operatorname{curl}\mathbf{E} &= 0\;;\end{aligned} \tag{2.39}$$

Integral representation:

$$\begin{aligned}\int\limits_{S(V)} \mathbf{E}\cdot d\mathbf{f} &= \frac{1}{\epsilon_0}\,q(V)\;,\\ \int\limits_{\partial F} \mathbf{E}\cdot d\mathbf{r} &= 0\;.\end{aligned} \tag{2.40}$$

With the scalar potential $\varphi(\mathbf{r})$ introduced in (2.24) the two Maxwell equations (2.39) can be combined to give the so-called **Poisson equation**:

$$\Delta\varphi(\mathbf{r}) = -\frac{1}{\epsilon_0}\,\rho(\mathbf{r})\;. \tag{2.41}$$

The solution of this linear, inhomogeneous, partial differential equation of second order is considered as the **'basic problem of electrostatics'**. If $\rho(\mathbf{r}')$ is known for all $\mathbf{r}'$ and there are no boundary conditions for $\varphi(\mathbf{r})$ in the finiteness then the Poisson equation is solved by (2.25):

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;.$$

This can easily be proven by using (1.69):

$$\begin{aligned}\Delta\varphi(\mathbf{r}) &= \frac{1}{4\pi\,\epsilon_0}\int d^3r'\,\rho(\mathbf{r}')\,\Delta_r\frac{1}{|\mathbf{r}-\mathbf{r}'|}\\ &= -\frac{1}{\epsilon_0}\int d^3r'\,\rho(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}') = -\frac{1}{\epsilon_0}\,\rho(\mathbf{r})\;.\end{aligned}$$

Very often, however, the situation is different: $\rho(\mathbf{r}')$ is given in a **finite** volume V and the values for $\varphi(\mathbf{r})$ or for the derivatives of $\varphi(\mathbf{r})$ on the surface $S(V)$ are known. Therewith the potential $\varphi(\mathbf{r})$ for all $\mathbf{r}\in V$ is to be found. One speaks of a

boundary-value problem of the electrostatics

Typical boundary-value problems will be discussed in Sect. 2.3.

If the considered space region is free of charge then the **Laplace equation** must be solved:

$$\Delta\varphi(\mathbf{r}) \equiv 0 \tag{2.42}$$

The **general** solution of the Poisson equation can be written as the sum of any **special** solution of the Poisson equation and the **general** solution of the Laplace equation.

Final Remark

The physical Gauss theorem can serve to calculate elegantly and quite easily the **E**-fields of highly symmetric charge distributions. We demonstrate this by the example (2.28) of the homogeneously charged sphere. The choice of spherical coordinates appears to be obvious:

$$\mathbf{E}(\mathbf{r}) = E_r(r, \vartheta, \varphi)\mathbf{e}_r + E_\vartheta(r, \vartheta, \varphi)\,\mathbf{e}_\vartheta + E_\varphi(r, \vartheta, \varphi)\,\mathbf{e}_\varphi \ .$$

We simplify this expression first by elementary symmetry considerations:

1. Rotation around the z-axis does not change the charge distribution. The components E_r, E_ϑ, E_φ must therefore be independent of φ.
2. Because of the rotational symmetry around the x, y-axes there is no ϑ-dependence.

This yields as intermediate result:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r + E_\vartheta(r)\,\mathbf{e}_\vartheta + E_\varphi(r)\,\mathbf{e}_\varphi \ .$$

The charge distribution does not change, either, by a reflection at the xy-plane. Accordingly, **E** has therefore to be mirror-symmetrical, i.e.

$$(E_x, E_y, E_z) \stackrel{\vartheta\to\pi-\vartheta}{\longrightarrow} (E_x,\ E_y,\ -E_z) \ .$$

With ((1.393), Vol. 1) we find:

$$E_z = E_r(r)\cos\vartheta - E_\vartheta(r)\sin\vartheta \ .$$

With the transition $\vartheta \to \pi - \vartheta$ $\cos\vartheta$ changes its sign; $\sin\vartheta$ remains unchanged. In order to guarantee $E_z \to -E_z$ it must therefore be

$$E_\vartheta(r) \equiv 0$$

In addition, this is also needed to leave E_x and E_y unchanged by the reflection at the xy-plane ((1.393), Vol. 1).

Analogously, the mirror-symmetry at the yz-plane requires:

$$(E_x, E_y, E_z) \stackrel{\varphi\to\pi-\varphi}{\longrightarrow} (-E_x, E_y, E_z) \ .$$

Again with ((1.393), Vol. 1) one finds:

$$\begin{aligned} E_x &= E_r(r)\sin\vartheta\cos\varphi + E_\vartheta(r)\cos\vartheta\cos\varphi - E_\varphi(r)\sin\varphi \\ &= E_r(r)\sin\vartheta\cos\varphi - E_\varphi(r)\sin\varphi \ . \end{aligned}$$

With the transition $\varphi \to \pi - \varphi$ $\cos\varphi$ changes its sign, $\sin\varphi$ does not. That requires:

$$E_\varphi \equiv 0 \ .$$

After these symmetry considerations we are left with the strongly simplified **ansatz**:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r \ .$$

The field of a homogeneously charged sphere is *due to symmetry reasons* radially oriented. This holds of course for **all** spherically symmetric charge distributions.

We now calculate the flux of $\mathbf{E}(\mathbf{r})$ through the surface of a sphere with the radius r (Fig. 2.11):

$$\int\limits_{S(V_r)} \mathbf{E}\cdot d\mathbf{f} \stackrel{(1.37)}{=} E_r(r)\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\vartheta\,\sin\vartheta\, r^2 = 4\pi r^2 E_r(r) \ .$$

On the other hand, we have (2.35):

$$\int\limits_{S(V_r)} \mathbf{E}\cdot d\mathbf{f} = \frac{1}{\epsilon_0}\,q(V_r) = \frac{1}{\epsilon_0}\cdot\begin{cases} Q \ , & \text{if } r > R \ , \\ Q\dfrac{r^3}{R^3} \ , & \text{if } r \le R \ . \end{cases}$$

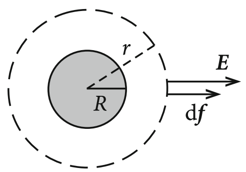


Fig. 2.11 To the calculation of the electric field of a homogeneously charged sphere by use of the Gauss theorem

The combination of the last two equations yields

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0}\,\mathbf{e}_r \cdot \begin{cases} \dfrac{1}{r^2}\ , & \text{if } r > R\ , \\[2mm] \dfrac{r}{R^3}\ , & \text{if } r \leq R\ , \end{cases}$$

where Q is the total charge of the sphere. This confirms our previous result (2.30).

2.1.4 Field-Behavior at Interfaces

What happens to the electrostatic field $\mathbf{E}(\mathbf{r})$ at interfaces which carry a finite surface charge density σ? The answer to this question can be easily found by use of the integral theorems. At first we place, as plotted in Fig. 2.12, at the interface a small box with the volume ΔV, half to the left of the interface and half to the right, which we call in the following

'Gauss-casket'

The edges perpendicular to the interface have the lengths Δx, which in a limiting process we let approach zero:

$$\int\limits_{\Delta V} d^3\,r\,\mathrm{div}\mathbf{E}(\mathbf{r}) = \int\limits_{S(\Delta V)} d\mathbf{f}\cdot\mathbf{E}(\mathbf{r}) \underset{\Delta x \to 0}{\longrightarrow} \Delta F\mathbf{n}\cdot(\mathbf{E}_\mathrm{a} - \mathbf{E}_\mathrm{i})\ .$$

$\mathbf{n}$ is the position-dependent normal of the interface. On the other hand it also holds:

$$\int\limits_{\Delta V} d^3r\ \mathrm{div}\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0}\int\limits_{\Delta V} d^3r\,\rho(\mathbf{r}) = \frac{1}{\epsilon_0}\sigma\Delta F\ .$$

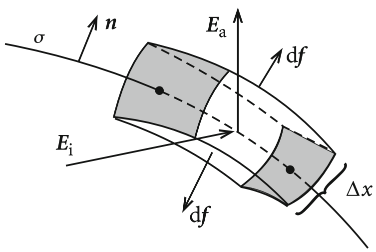


Fig. 2.12 'Gauss-casket' for the determination of the interface behavior of the normal component of the electric field

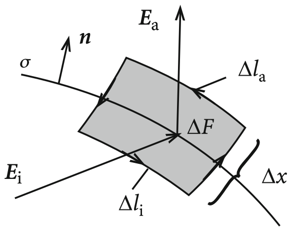


Fig. 2.13 'Stokes-area' for the determination of the interface behavior of the tangential component of the electric field

The comparison yields:

$$\mathbf{n} \cdot (\mathbf{E}_a - \mathbf{E}_i) = \frac{\sigma}{\epsilon_0} \ . \tag{2.43}$$

We see that the normal component of the electric field behaves discontinuously at the surface if $\sigma \neq 0$.

We investigate the behavior of the tangential component of the field by the use of the

'Stokes-area'

$\mathbf{t}$ = surface normal of ΔF, tangential at the interface: $\Delta \mathbf{F} = \Delta F \, \mathbf{t}$, $\Delta l_a = \Delta l (\mathbf{t} \times \mathbf{n}) = -\Delta l_i$ (Fig. 2.13).

With the aid of the Stokes theorem it first follows:

$$0 = \int\limits_{\Delta F} \operatorname{curl} \mathbf{E} \cdot d\mathbf{f} = \int\limits_{\partial \Delta F} d\mathbf{r} \cdot \mathbf{E} \underset{\Delta x \to 0}{\longrightarrow} \Delta l (\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{E}_a - \mathbf{E}_i) \ .$$

Here we read off

$$(\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{E}_a - \mathbf{E}_i) = 0 \tag{2.44}$$

i.e. the tangential component is in any case continuous at the interface!

2.1.5 *Electrostatic Field Energy*

According to (2.20) the force

$$\mathbf{F}(\mathbf{r}) = q \, \mathbf{E}(\mathbf{r})$$

acts on the point charge q at the position $\mathbf{r}$ in the electric field $\mathbf{E}(\mathbf{r})$. In order to shift the point charge q in the field $\mathbf{E}$ from the point B to the point A the work W_{AB} must be done:

$$W_{\mathrm{AB}} = -\int_{\mathrm{B}}^{\mathrm{A}} \mathbf{F} \cdot d\mathbf{r} = -q \int_{\mathrm{B}}^{\mathrm{A}} \mathbf{E} \cdot d\mathbf{r}$$

$$= q \int_{\mathrm{B}}^{\mathrm{A}} d\varphi = q\,[\varphi(\mathrm{A}) - \varphi(\mathrm{B})] = q\,U_{\mathrm{AB}}\ . \tag{2.45}$$

The work is counted as positive if it is *'absorbed by'* the system.

Definition 2.1.1 The energy of a charge configuration $\rho(\mathbf{r})$, which is restricted to a finite space region, corresponds to the work which is necessary to move the charges from infinity ($\varphi(\infty) = 0$) to the given configuration (2.25).

(1) *N* Point Charges

$(i-1)$ point charges q_j at the positions $\mathbf{r}_j$ create at $\mathbf{r}_i$ the potential

$$\varphi(\mathbf{r}_i) = \frac{1}{4\pi\,\epsilon_0} \sum_{j=1}^{i-1} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}\ .$$

The work to bring the additional i-th charge q_i from ∞ to $\mathbf{r}_i$ then amounts to

$$W_i = q_i\,\varphi(\mathbf{r}_i)\ , \quad [\varphi(\infty) = 0]\ .$$

We now add together these 'partial works' W_i from $i = 2$ to $i = N$. Note that the first charge ($i = 1$) is shifted with a zero-amount of work from ∞ to $\mathbf{r}_1$ since the space is then still field-free:

$$W = \frac{1}{4\pi\,\epsilon_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{8\pi\,\epsilon_0} {\sum_{i,j=1}^{N}}' \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}\ . \tag{2.46}$$

$\sum'$ means that the term $i = j$ is excluded.

(2) Continuous Charge Distributions

The corresponding generalization to (2.46) reads:

$$W = \frac{1}{8\pi\,\epsilon_0} \iint d^3r\,d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2} \int d^3r\,\rho(\mathbf{r})\varphi(\mathbf{r})\ . \tag{2.47}$$

$\varphi(\mathbf{r})$ is the electrostatic potential created by the charge density ρ itself. One can now express W, instead by ρ and φ, also by the electric field caused by the charge density ρ:

$$W = -\frac{\epsilon_0}{2}\int d^3r\,\Delta\varphi\,\varphi = -\frac{\epsilon_0}{2}\int d^3r\ \mathrm{div}(\varphi\,\nabla\varphi) + \frac{\epsilon_0}{2}\int d^3r\,(\nabla\varphi)^2$$
$$= -\frac{\epsilon_0}{2}\int d\mathbf{f}\cdot(\varphi\,\nabla\varphi) + \frac{\epsilon_0}{2}\int d^3r(\nabla\varphi)^2\ .$$

The surface integral is to be performed over an area which lies at infinity so that we can assume because of (2.25):

$$\varphi \sim \frac{1}{r}\ , \quad \varphi\,\nabla\varphi \sim \frac{1}{r^3}\ , \quad df \sim r^2\ .$$

Thus the surface integral does not contribute:

$$W = \frac{\epsilon_0}{2}\int d^3r|\mathbf{E}(\mathbf{r})|^2\ . \tag{2.48}$$

In the integrand we find the **energy density** of the electrostatic field:

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2\ . \tag{2.49}$$

The comparison of (2.46) and (2.48) poses a serious problem: In the *'field-formulation'* we always have $W \geq 0$, whereas for point charges according to (2.46) $W < 0$ is also possible. Is this a nasty contradiction? The cause is the **self-energy** of a point charge which is not accounted for in (2.46) ($\sum'_{ij}\ldots$), but it is implicit in (2.48). The two expressions are therefore not completely equivalent. Let us demonstrate this by an

Example Let q_1, q_2 be two point charges at $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{E} = \frac{1}{4\pi\,\epsilon_0}\left(q_1\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} + q_2\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3}\right)\ .$$

This yields for the energy density in the field-formulation:

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \frac{1}{32\pi^2\,\epsilon_0}\Bigg[\underbrace{\frac{q_1^2}{|\mathbf{r}-\mathbf{r}_1|^4} + \frac{q_2^2}{|\mathbf{r}-\mathbf{r}_2|^4}}_{\text{self-energy density}} + 2q_1q_2\underbrace{\frac{(\mathbf{r}-\mathbf{r}_1)\cdot(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_1|^3\ |\mathbf{r}-\mathbf{r}_2|^3}}_{\text{interaction-energy density}}\Bigg]$$
$$= w_\mathrm{S} + w_\mathrm{I}\ .$$

We discuss the interaction part:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_2 \,, \qquad \mathbf{R} - \mathbf{x} = \mathbf{r} - \mathbf{r}_1 \,, \quad d^3r = d^3R :$$

$$\int d^3r \, w_{\mathrm{I}} = \frac{q_1 q_2}{16\pi^2 \epsilon_0} \int d^3R \, \frac{\mathbf{R} \cdot (\mathbf{R} - \mathbf{x})}{R^3 |\mathbf{R} - \mathbf{x}|^3} \,.$$

The polar axis coincides with the vector $\mathbf{x}$ (Fig. 2.14):

$$\begin{aligned} \int d^3r \, w_{\mathrm{I}} &= \frac{q_1 q_2}{16\pi^2 \epsilon_0} \int_0^\infty R^2 \, dR \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\vartheta \, \frac{R^2 - Rx\cos\vartheta}{R^3 (R^2 + x^2 - 2Rx\cos\vartheta)^{3/2}} \\ &= \frac{q_1 q_2}{8\pi \, \epsilon_0} \int_{-1}^{+1} d\cos\vartheta \int_0^\infty dR \left(-\frac{d}{dR} \frac{1}{\sqrt{R^2 + x^2 - 2Rx\cos\vartheta}} \right) \\ &= \frac{q_1 q_2}{4\pi \, \epsilon_0} \, \frac{1}{x} = \frac{1}{4\pi \, \epsilon_0} \, \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \,. \end{aligned}$$

Without self-energy we therefore find according to (2.46) just the expected result for point charges.

The self-energy can be considered as the energy being necessary to contract the point charges out of an infinitely diluted 'charge cloud'. Thereto we calculate the energy of a homogeneously charged sphere and let then the radius of the sphere become arbitrarily small. With (2.30) we have:

$$W = \frac{\epsilon_0}{2} \, \frac{4\pi}{16\pi^2 \epsilon_0^2} \, Q^2 \left(\int_0^R dr \, r^2 \frac{r^2}{R^6} + \int_R^\infty dr \, r^2 \frac{1}{r^4} \right) = \frac{Q^2}{8\pi \, \epsilon_0} \left(\frac{1}{5} \, \frac{1}{R} + \frac{1}{R} \right) \,.$$

The electrostatic energy of the homogeneously charged sphere,

$$W = \frac{3}{5} \, \frac{Q^2}{4\pi \, \epsilon_0 R} \,, \tag{2.50}$$

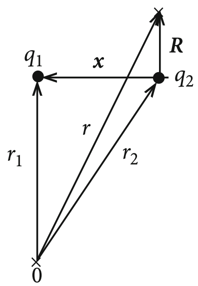


Fig. 2.14 Arrangement for the determination of the interaction part of the energy density of two point charges

thus diverges for $R \to 0$. That illustrates the divergence of the self-energy of a point charge, which represents even today an unsolved problem of the electrodynamics. One makes do with the 'auxiliary vision' that the self-energy of the point charges is *constant* and therewith physically uninteresting.

2.1.6 Exercises

Exercise 2.1.1 Explain the charge densities $\rho(\mathbf{r})$ for

1. a homogeneously charged sphere of the radius R,
2. a homogeneously charged very thin spherical shell of radius R,
3. a homogeneously charged, infinitely thin circular disc with the radius R and the total charge Q.

Exercise 2.1.2 An infinitely thin circular disc of the radius R lies in the xy-plane and possesses a constant surface charge density σ_0. The center of the disc coincides with the origin of coordinates. Calculate the electrostatic potential and the electric field on the z-axis. Discuss the behavior for $z = 0$ and $z \to \pm\infty$!

Exercise 2.1.3

1. The space between two concentric spheres with the radii R_{i} and R_{a} ($R_{\mathrm{i}} < R_{\mathrm{a}}$) is charged with the density

$$\rho(\mathbf{r}) = \begin{cases} \dfrac{\alpha}{r^2} & \text{for } R_{\mathrm{i}} < r < R_{\mathrm{a}}\ (\alpha > 0)\ , \\ 0 & \text{otherwise} \end{cases}$$

   Calculate the total charge.
2. Calculate for the charge distribution (*screened point charge*)

$$\rho(\mathbf{r}) = q\left[\delta(\mathbf{r}) - \frac{\alpha^2}{4\pi}\frac{e^{-\alpha r}}{r}\right]$$

   the total charge Q.
3. A hollow sphere with the radius R carries the charge density

$$\rho(\mathbf{r}) = \sigma_0 \cos\vartheta\, \delta(r - R)\ .$$

   Calculate the total charge Q and the *dipole moment* $\mathbf{p}$:

$$\mathbf{p} = \int \mathbf{r}\,\rho(\mathbf{r})\,d^3r\ .$$

Exercise 2.1.4 An infinitely thin, infinitely long straight wire carries the homogeneous line-charge κ (charge per unit length).

1. What is the spatial charge density $\rho(\mathbf{r})$?
2. Calculate directly (without using the Gauss theorem) the electric field strength and its potential.

Exercise 2.1.5 An infinitely extended plane carries the homogeneous surface charge density σ (charge per unit area). Calculate as in Exercise 2.1.4 the electric field and its potential.

Exercise 2.1.6 The stationary point charges $+q$ and $-q$ with the separation a represent an electrostatic dipole. By the dipole moment $\mathbf{p}$ one understands a vector with the magnitude qa in the direction from $-q$ to $+q$.

1. Calculate the potential of the dipole and express it for large distances $r \gg a$ approximately in terms of the dipole moment.
2. Formulate the electric field by use of spherical coordinates.

Exercise 2.1.7 Calculate for the charge density $\rho(\mathbf{r})$ from Exercise 2.1.3 (part 1.) the electric field strength $\mathbf{E}$ and the electrostatic potential in the three space regions

$$\text{(a)}\;\; 0 \le r < R_\mathrm{i}\;; \qquad \text{(b)}\;\; R_\mathrm{i} \le r \le R_\mathrm{a}\;; \qquad \text{(c)}\;\; R_\mathrm{a} < r\;.$$

Exercise 2.1.8 For the hydrogen atom in its ground state it holds approximately: The nuclear charge is centered point-like at the origin and the average electron charge density is given by

$$\rho_\mathrm{e}(\mathbf{r}) = -\frac{e}{\pi\, a^3}\exp\left(-\frac{2r}{a}\right)$$

(a = Bohr radius). Calculate the electric field strength $\mathbf{E}$ as well as its potential φ. Discuss the limiting cases $r \ll a$, $r \gg a$.

Exercise 2.1.9 An infinitely long circular cylinder is homogeneously charged. Calculate the electric field strength and the corresponding potential.

Exercise 2.1.10 Calculate the energy density and the total energy of the electrostatic fields which result from the following charge distributions:

1. Homogeneously charged thin spherical shell,
2.

$$\rho(\mathbf{r}) = \begin{cases} \dfrac{\alpha}{r^2} & \text{for } R_1 < r < R_2\; (\alpha > 0)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

2.2 Simple Electrostatic Problems

Let us discuss, as an interlude, a few simple applications of the theory of electrostatics developed so far.

2.2.1 Parallel-Plate Capacitor

By a *parallel-plate capacitor* we understand a system of two plates arranged parallel to each other with a distance d. They have both the area F (Fig. 2.15). In order to avoid later edge effects we assume

$$d \ll F^{1/2}$$

The two plates carry, homogeneously distributed, the equal and opposite charges $\pm Q$, i.e. the surface charge densities

$$\sigma(0) = \frac{Q}{F} = -\sigma(d) \; .$$

The electric field created by the lower plate will be oriented *from symmetry reasons*, except for edge effects, in positive or negative z-direction (Fig. 2.16) (see (2.33)):

$$\mathbf{E}_{+}(\mathbf{r}) = E_{+}(z)\frac{z}{|z|}\,\mathbf{e}_z \; .$$

We put a Gauss-casket with the volume $\Delta V = \Delta F\,\Delta z$ 'around the plate' so that the basal planes ΔF at the positions $\pm(1/2)\Delta z$ lie parallel to the plates of the capacitor. The side areas do not contribute to the flux of the $\mathbf{E}$-field through the surface $S(\Delta V)$

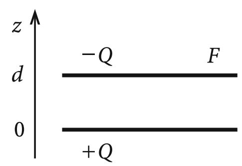


Fig. 2.15 Schematic arrangement of a parallel-plate capacitor

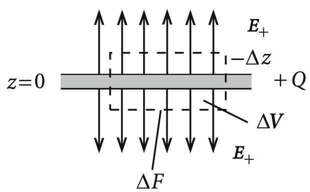


Fig. 2.16 Electric field (without edge effects) of the positively charged plate of a parallel-plate capacitor

since $\mathbf{E}$ and $d\mathbf{f}$ are orthogonal to each other there.

$$\int\limits_{S(\Delta V)} \mathbf{E}_+ \cdot d\mathbf{f} = 2\,E_+\left(z = \pm\frac{1}{2}\Delta z\right)\,\Delta F$$

$$\stackrel{!}{=} \frac{1}{\epsilon_0}\,q\,(\Delta V) = \frac{\sigma}{\epsilon_0}\,\Delta F \implies E_+(z) = \frac{\sigma}{2\,\epsilon_0}\,.$$

The result is a field homogeneous almost in the whole space which only reverses its direction at $z = 0$ (see (2.33)):

$$\mathbf{E}_+(\mathbf{r}) = \frac{\sigma}{2\,\epsilon_0}\,\frac{z}{|z|}\,\mathbf{e}_z\,.$$

The same consideration yields for the plate at $z = d$ (Fig. 2.17):

$$\mathbf{E}_-(\mathbf{r}) = -\frac{\sigma}{2\,\epsilon_0}\,\frac{z-d}{|z-d|}\,\mathbf{e}_z\,.$$

The resulting total field is then unequal zero only between the plates:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-(\mathbf{r}) = \begin{cases} \dfrac{\sigma}{\epsilon_0}\,\mathbf{e}_z & \text{for } 0 < z < d\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.51}$$

To this field an electrostatic potential belongs of the form:

$$\varphi(\mathbf{r}) = \begin{cases} \text{const}_1 & \text{for } z < 0\,, \\ \dfrac{-\sigma}{\epsilon_0}\,z + \text{const}_2 & \text{for } 0 \le z \le d\,, \\ \text{const}_3 & \text{for } z > d\,. \end{cases} \tag{2.52}$$

Between the plates we therefore find the voltage (electric tension):

$$U = \varphi(z = 0) - \varphi(z = d) = \frac{\sigma}{\epsilon_0}\,d = \frac{Q}{\epsilon_0 F}\,d\,. \tag{2.53}$$

Here we recognize a proportionality between the charge of the capacitor and the electric tension:

$$Q = C \cdot U\,. \tag{2.54}$$

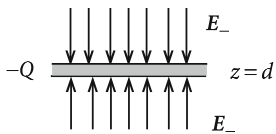


Fig. 2.17 Electric field (without edge effects) of the negatively charged plate of a parallel-plate capacitor

The constant of proportionality C is called **capacity** of the parallel-plate capacitor:

$$C = \epsilon_0 \frac{F}{d} \,. \tag{2.55}$$

We see that the capacity C is determined by purely geometrical material quantities. As **unit** one chooses:

$$[C] = 1\,\text{F (farad)} = 1\,\frac{\text{A s}}{\text{V}} \,. \tag{2.56}$$

It is about a rather huge unit since 1 farad corresponds to an area-to-distance ratio of about 10^{11} m ($1\,\text{F} = 10^6 \mu\,\text{F} = 10^9\,\text{nF} = 10^{12}\,\text{pF}$).

With (2.51) the **energy density** of the capacitor is easily calculated

$$w(\mathbf{r}) = \frac{\epsilon_0}{2} \left|\mathbf{E}(\mathbf{r})\right|^2 = \frac{\sigma^2}{2\,\epsilon_0} \tag{2.57}$$

for all points $\mathbf{r}$ between the plates. That leads to the **total energy**

$$W = w\,F d = \frac{1}{2}\frac{Q^2}{\epsilon_0\,F}\,d = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\,QU = \frac{1}{2}\,C\,U^2 \,. \tag{2.58}$$

2.2.2 Spherical Capacitor

In this case the capacitor consists of two concentric spherical shells with the radii R_1, R_2 and the homogeneously distributed charges $\pm Q$ (Fig. 2.18). The **charge density** (see Exercise 2.1.1)

$$\rho(\mathbf{r}) = \frac{Q}{4\pi\,R_1^2}\,\delta(r - R_1) - \frac{Q}{4\pi\,R_2^2}\,\delta(r - R_2) \tag{2.59}$$

is therewith restricted to a finite space region so that, according to (2.25), the potential vanishes at infinity. The charge distribution is spherically symmetric.

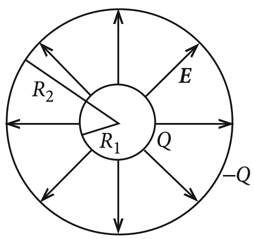


Fig. 2.18 Schematic plot of a spherical capacitor

Consequently it holds for the **E**-field:

$$\mathbf{E}(\mathbf{r}) = E(r)\,\mathbf{e}_r\ . \tag{2.60}$$

By use of the Gauss theorem we prove therewith in Exercise 2.2.1:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0}\,\mathbf{e}_r \begin{cases} 0\ , & \text{if } R_1 > r\ , \\ \dfrac{1}{r^2}\ , & \text{if } R_2 > r > R_1\ , \\ 0\ , & \text{if } r > R_2\ . \end{cases} \tag{2.61}$$

Together with the physical boundary conditions

$$\varphi(r \to \infty) = 0\ ; \qquad \varphi \text{ continuous at } r = R_1 \text{ and } r = R_2$$

we find for the scalar potential (Fig. 2.19):

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0} \begin{cases} \dfrac{1}{R_1} - \dfrac{1}{R_2}\ , & \text{if } r < R_1\ , \\ \dfrac{1}{r} - \dfrac{1}{R_2}\ , & \text{if } R_1 \le r \le R_2\ , \\ 0\ , & \text{if } R_2 \le r\ . \end{cases} \tag{2.62}$$

It thus appears as **electric tension (voltage)** between the spherical shells:

$$U = \varphi(R_1) - \varphi(R_2) = \frac{Q}{4\pi\,\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\ .$$

The spherical capacitor has therewith the capacity

$$C = 4\pi\,\epsilon_0 \frac{R_1\,R_2}{R_2 - R_1}\ . \tag{2.63}$$

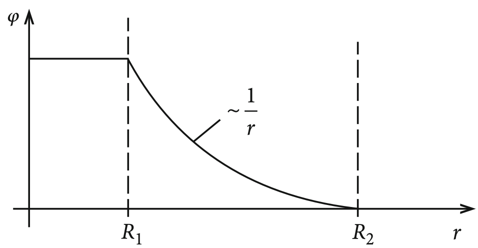


Fig. 2.19 Radial dependence of the scalar potential of a spherical capacitor

The **energy density** is restricted to the space between the two concentric spherical shells:

$$w(\mathbf{r}) = \frac{Q^2}{32\pi^2 \epsilon_0} \frac{1}{r^4} \quad \text{for } R_1 \le r \le R_2 \, .$$

That yields formally the same **total energy** as for the parallel-plate capacitor:

$$\begin{aligned} W &= \frac{Q^2}{32\pi^2 \epsilon_0} \cdot 4\pi \int_{R_1}^{R_2} r^2 dr \, \frac{1}{r^4} \\ &= \frac{Q^2}{8\pi \epsilon_0} \frac{R_2 - R_1}{R_2 \cdot R_1} = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} Q\, U = \frac{1}{2} C\, U^2 \, . \end{aligned} \tag{2.64}$$

2.2.3 Cylindrical Capacitor

The formation consists of two coaxial cylinders both with the height h and with radii $R_1 < R_2$ (Fig. 2.20). We neglect again the stray fields at the edges and can therefore act on the assumption that the **E**-field is axially symmetric. Using cylindrical coordinates (ρ, φ, z) leads then to the following plausible ansatz:

$$\mathbf{E}(\mathbf{r}) = E(\rho)\, \mathbf{e}_\rho \, .$$

Let us consider a further coaxial cylinder Z_ρ and calculate the flux of the **E**-field through its surface. The front sides do not contribute since **E** and $d\mathbf{f}$ are perpendicular to each other. On the cylinder jacket it holds (1.37):

$$d\mathbf{f} = (\rho \, d\varphi \, dz)\, \mathbf{e}_\rho \, .$$

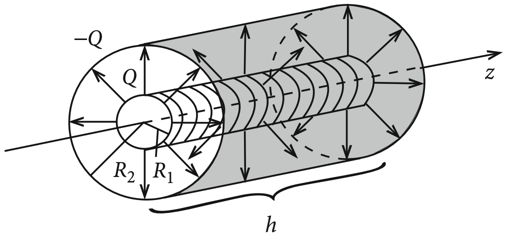


Fig. 2.20 Schematic plot of a cylindrical capacitor

That leads to:

$$\int\limits_{S(Z_\rho)} \mathbf{E} \cdot d\mathbf{f} = \rho \, E(\rho) \, 2\pi \, h \overset{!}{=} \frac{1}{\epsilon_0} \int\limits_{Z_\rho} d^3r' \, \rho \, (\mathbf{r}')$$

$$= \frac{1}{\epsilon_0} \begin{cases} 0 \,, & \text{if } \rho < R_1 \,, \\ Q \,, & \text{if } R_1 < \rho < R_2 \,, \\ 0 \,, & \text{if } R_2 < \rho \,. \end{cases}$$

The **electric field** is thus restricted to the inside space:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{2\pi \, \epsilon_0 h} \frac{1}{\rho} \mathbf{e}_\rho \begin{cases} 0 \,, & \text{if } \rho < R_1 \,, \\ 1 \,, & \text{if } R_1 < \rho < R_2 \,, \\ 0 \,, & \text{if } R_2 < \rho \,. \end{cases} \tag{2.65}$$

From that we get the **electrostatic potential** by fulfilling all physical boundary conditions:

$$\varphi(\mathbf{r}) = \varphi(\rho) = \frac{Q}{2\pi \, \epsilon_0 h} \begin{cases} \ln \dfrac{R_2}{R_1} \,, & \text{if } \rho < R_1 \,, \\ \ln \dfrac{R_2}{\rho} \,, & \text{if } R_1 < \rho < R_2 \,, \\ 0 \,, & \text{if } R_2 < \rho \,. \end{cases} \tag{2.66}$$

Between the two cylinders there appears the **voltage**:

$$U = \frac{Q}{2\pi \, \epsilon_0 h} \ln \frac{R_2}{R_1} \,. \tag{2.67}$$

The cylindrical capacitor therefore has the capacity:

$$C = \frac{2\pi \, \epsilon_0 h}{\ln(R_2/R_1)} \,. \tag{2.68}$$

The **energy density** follows directly from (2.65):

$$w(\mathbf{r}) = \frac{Q^2}{8\pi^2 \epsilon_0 h^2} \begin{cases} \dfrac{1}{\rho^2} \,, & \text{if } R_1 \leq \rho \leq R_2 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Therewith the **total energy** is easily calculated:

$$W = \int \rho\, d\rho\, d\varphi\, dz\, w(\mathbf{r}) = 2\pi h \frac{Q^2}{8\pi^2 \epsilon_0 h^2} \int\limits_{R_1}^{R_2} d\rho\, \frac{1}{\rho}$$

$$= \frac{Q^2}{4\pi\, \epsilon_0 h} \ln \frac{R_2}{R_1} \,. \tag{2.69}$$

We see that in this case, too, it holds:

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} Q\, U = \frac{1}{2} C\, U^2 \,.$$

2.2.4 *The Dipole*

An arrangement of two equal and opposite point charges $\pm q$ is called a **dipole** (Fig. 2.21). If $\mathbf{a}$ is the distance vector oriented from $-q$ to $+q$ one denotes as **dipole moment** the vector $\mathbf{p} = q\mathbf{a}$. This is the usual definition which we will, from reasons which become clear at a later stage, formulate here a bit more precisely.

Definition 2.2.1 **'Dipole'** This is an arrangement of two equal and opposite point charges $\pm q$, the distance a of which is approaching zero with simultaneously increasing charge q in such a way that the **dipole moment**

$$\mathbf{p} = \lim_{\substack{a \to 0 \\ q \to \infty}} q\, \mathbf{a} \tag{2.70}$$

thereby remains constant and finite.

The so defined dipole is then located in a fixed space point. Not only charges (*monopoles*) but also such **dipoles are sources of electrostatic fields** which we are now going to inspect a bit in more detail.

Let $\mathbf{a}$ at first still be finite and let the charge $-q$ be at the origin (Fig. 2.22). Then the two charges produce the following potential:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\, \epsilon_0} \left(-\frac{q}{r} + \frac{q}{|\mathbf{r} - \mathbf{a}|} \right) \,.$$

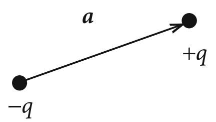


Fig. 2.21 Simplest arrangement of a dipole consisting of two point charges

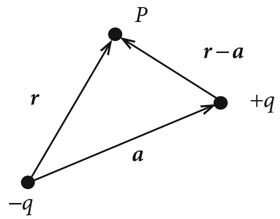


Fig. 2.22 Arrangement for the calculation of the scalar potential of a dipole

For the second summand we use its Taylor-expansion (1.32):

$$\frac{1}{|\mathbf{r}-\mathbf{a}|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{a}}{r^3} + \frac{1}{2}\frac{3(\mathbf{r}\cdot\mathbf{a})^2 - r^2a^2}{r^5} + \dots$$

$$\Longrightarrow \varphi(\mathbf{r}) = \frac{q}{4\pi\,\epsilon_0}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{a})^2 - r^2a^2}{2r^5} + \dots\right) .$$

If we now let the distance between the charges become arbitrarily small with simultaneously, in the sense of (2.70), increasing q then the second summand and all higher terms of the expansion vanish:

$$\varphi_\mathrm{D}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\,\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} . \tag{2.71}$$

An electrostatic charge configuration with such a scalar potential is called a *dipole*. The corresponding electric field $\mathbf{E}(\mathbf{r})$ is conveniently expressed by spherical coordinates where the polar axis is chosen to be the direction of the dipole $\mathbf{p}$:

$$\varphi_\mathrm{D}(r,\vartheta,\varphi) = \frac{1}{4\pi\,\epsilon_0}\,\frac{p\cos\vartheta}{r^2} .$$

The components of the electric field are then:

$$\begin{aligned} E_r^\mathrm{D} &= -\frac{\partial\varphi_\mathrm{D}}{\partial r} = \frac{p}{4\pi\,\epsilon_0}\,\frac{2\cos\vartheta}{r^3} ,\\ E_\vartheta^\mathrm{D} &= -\frac{1}{r}\,\frac{\partial\varphi_\mathrm{D}}{\partial\vartheta} = \frac{p}{4\pi\,\epsilon_0}\,\frac{\sin\vartheta}{r^3} ,\\ E_\varphi^\mathrm{D} &= -\frac{1}{r\sin\vartheta}\,\frac{\partial\varphi_\mathrm{D}}{\partial\varphi} = 0 . \end{aligned} \tag{2.72}$$

The field obviously possesses rotational symmetry around the dipole-axis (Fig. 2.23)!

One should notice that the **E**-field of two point charges with a finite distance a (see Exercise 2.1.6) represents a real dipole field only in the *far zone* ($r \gg a$). It looks very much different in the *near zone*. The electric flux, created by a dipole, through a closed area, which encases the dipole, is of course zero because the total charge of the dipole vanishes (2.35).

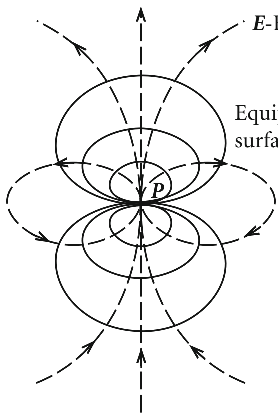


Fig. 2.23 Equipotential surfaces and electric field lines of a dipole

Let us write the dipole field still in a somewhat more compact form:

$$\mathbf{E}^{\mathrm{D}}(\mathbf{r}) = -\nabla\,\varphi_{\mathrm{D}}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\,\nabla\left(\mathbf{p}\cdot\nabla\frac{1}{r}\right)\,.$$

In Exercise 1.7.13 we have shown:

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} + (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{b}\times\mathrm{curl}\,\mathbf{a} + \mathbf{a}\times\mathrm{curl}\,\mathbf{b}\,.$$

It follows therewith for the field:

$$\begin{aligned}\mathbf{E}^{\mathrm{D}}(\mathbf{r}) &= \frac{1}{4\pi\,\epsilon_0}\left[(\mathbf{p}\cdot\nabla)\,\nabla\,\frac{1}{r} + \mathbf{p}\times\mathrm{curl}\left(\nabla\frac{1}{r}\right)\right]\\ &= \frac{-1}{4\pi\,\epsilon_0}(\mathbf{p}\cdot\nabla)\frac{\mathbf{r}}{r^3} = -\frac{1}{4\pi\,\epsilon_0}\sum_i p_i\frac{\partial}{\partial x_i}\,\frac{\mathbf{r}}{r^3}\\ &= -\frac{1}{4\pi\,\epsilon_0}\sum_i p_i\left(\frac{\mathbf{e}_i}{r^3} - \frac{3\mathbf{r}}{r^4}\,\frac{x_i}{r}\right)\,.\end{aligned}$$

Finally we have:

$$\mathbf{E}^{\mathrm{D}}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\left[\frac{3(\mathbf{r}\cdot\mathbf{p})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right]\,. \tag{2.73}$$

In analogy to the charge densities of the electric monopoles one can introduce also a **dipole density**:

$$\Pi(\mathbf{r}) = \sum_{j=1}^{N}\mathbf{p}_j\,\delta(\mathbf{r}-\mathbf{R}_j)\,. \tag{2.74}$$

The total potential of N discrete dipoles $\mathbf{p}_j$ results from a superposition of the individual contributions according to (2.71):

$$\begin{aligned} \varphi_{\mathrm{D}}(\mathbf{r}) &= -\frac{1}{4\pi\,\epsilon_0}\sum_{j=1}^{N}\mathbf{p}_j\cdot\nabla_r\,\frac{1}{|\mathbf{r}-\mathbf{R}_j|} \\ &= -\frac{1}{4\pi\,\epsilon_0}\int d^3r'\;\Pi(\mathbf{r}')\,\nabla_r\frac{1}{|\mathbf{r}-\mathbf{r}'|}\ . \end{aligned} \tag{2.75}$$

The last step can be understood as the generalization of the microscopic to the continuous dipole density analogously to that of the charge densities performed with (2.23). We will encounter this expression again in Sect. 2.4 when we discuss the electrostatic field in macroscopic media.

What a force does act on a dipole in the electrostatic field? We answer this question most easily by considering the two point charges $\pm q$ with the at first finite distance a. Let the charge $-q$ be located at $\mathbf{r}$ and the charge $+q$ at $\mathbf{r}+\mathbf{a}$ (Fig. 2.24). $\mathbf{E}(\mathbf{r})$ is an external field! Then the force that acts on the dipole is:

$$\mathbf{F}(\mathbf{r}) = -q\mathbf{E}(\mathbf{r}) + q\mathbf{E}(\mathbf{r}+\mathbf{a})\ .$$

A Taylor-expansion according to (1.27) yields:

$$\mathbf{E}(\mathbf{r}+\mathbf{a}) = \mathbf{E}(\mathbf{r}) + (\mathbf{a}\cdot\nabla)\,\mathbf{E}\,(\mathbf{r}) + \frac{1}{2}\,(\mathbf{a}\cdot\nabla)^2\,\mathbf{E}(\mathbf{r})\ + \dots$$

It follows for the total force:

$$\mathbf{F}(\mathbf{r}) = q(\mathbf{a}\cdot\nabla)\,\mathbf{E}(\mathbf{r}) + \frac{1}{2}q\,(\mathbf{a}\cdot\nabla)^2\,\mathbf{E}(\mathbf{r}) + \dots$$

Performing the limiting process (2.70) we are left with the first term:

$$\mathbf{F}_{\mathrm{D}}(\mathbf{r}) = (\mathbf{p}\cdot\nabla)\,\mathbf{E}\,(\mathbf{r})\ . \tag{2.76}$$

We see that in a **homogeneous** field the dipole does **not** experience a force but it does feel a **torque M**:

$$\begin{aligned} \mathbf{M}(\mathbf{r}) &= -q\,[\mathbf{0}\times\mathbf{E}(\mathbf{r})] + q\ [\mathbf{a}\times\mathbf{E}(\mathbf{r}+\mathbf{a})] \\ &= q\,\mathbf{a}\times\mathbf{E}(\mathbf{r}) + q\,\mathbf{a}\times(\mathbf{a}\cdot\nabla)\,\mathbf{E}\,(\mathbf{r})\ + \dots \end{aligned}$$

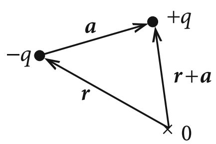


Fig. 2.24 To the calculation of the force which acts on a dipole in an electrostatic field

It then results with the limiting process (2.70):

$$\mathbf{M}_\mathrm{D}(\mathbf{r}) = \mathbf{p} \times \mathbf{E}(\mathbf{r}) \ . \tag{2.77}$$

The torque tries to rotate the dipole into an energetically favourable position, i.e. to a position of minimal potential energy V. The latter is determined simply as follows:

Starting with the general vector relation proven in Exercise 1.7.13, which was already used above, we can write because of $\mathbf{p} = \mathrm{const}$:

$$\nabla(\mathbf{p} \cdot \mathbf{E}) = (\mathbf{p} \cdot \nabla)\, \mathbf{E}\,(\mathbf{r}) + \mathbf{p} \times \underbrace{\mathrm{curl}\, \mathbf{E}(\mathbf{r})}_{=0} \ .$$

This yields with (2.76) the alternative representation of the force on the dipole located at $\mathbf{r}$:

$$\mathbf{F}_\mathrm{D}(\mathbf{r}) = \nabla(\mathbf{p} \cdot \mathbf{E}) \ . \tag{2.78}$$

Via the general connection between (conservative) force and potential energy V_D ((2.233), Vol. 1),

$$\mathbf{F}_\mathrm{D}(\mathbf{r}) = -\nabla\, V_\mathrm{D}(\mathbf{r}) \ ,$$

one finds by comparison:

$$V_\mathrm{D}(\mathbf{r}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}) \ . \tag{2.79}$$

The state of the minimal energy is stable. It corresponds to a parallel orientation of dipole and field.

We could have found the expression (2.79) also directly by evaluating

$$V_\mathrm{D}(\mathbf{r}) = -q\,[\varphi(\mathbf{r}) - \varphi(\mathbf{r} + \mathbf{a})]$$

with a Taylor-expansion and the subsequent limiting process (2.70). The reader should check that!

2.2.5 Dipole-Layer

By a *dipole-layer* one understands an area being filled with dipoles the axes of which have everywhere the direction of the respective area-normals. We want to find out how the electrostatic potential behaves when crossing such a dipole-layer.

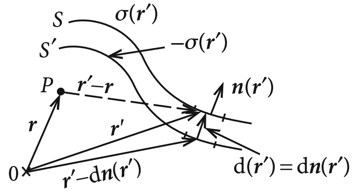


Fig. 2.25 Representation of a dipole-layer

We realize the dipole-layer by two parallel surfaces S and S' with equal and opposite surface charge densities $\sigma(\mathbf{r}')$ and $-\sigma(\mathbf{r}')$. Let $\mathbf{n}(\mathbf{r}')$ be the position-dependent surface normal (Fig. 2.25).

Definition 2.2.2 Dipole surface density

$$\mathbf{D}(\mathbf{r}') = \lim_{d\to 0} \left[\sigma(\mathbf{r}')\,\mathbf{d}(\mathbf{r}')\right] \tag{2.80}$$

$\mathbf{d}(\mathbf{r}') = d\,\mathbf{n}\,(\mathbf{r}')$. In order to keep $\mathbf{D}$ constant during this limiting process the surface charge density has obviously to grow indefinitely (cf. (2.70)).

According to (2.25) the dipole-layer creates the following potential at the point P at $\mathbf{r}$:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\left[\int\limits_S df'\,\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \int\limits_{S'} df'\,\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'+\mathbf{d}(\mathbf{r}')|}\right].$$

Since d becomes infinitely small we can cut the respective Taylor-expansion for the second summand after the linear term. We use (1.27):

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'+\mathbf{d}(\mathbf{r}')|} &= \frac{1}{|\mathbf{r}-\mathbf{r}'|} + (\mathbf{d}\cdot\nabla)\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \dots \\ &= \frac{1}{|\mathbf{r}-\mathbf{r}'|} - d\frac{\mathbf{n}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \dots\end{aligned}$$

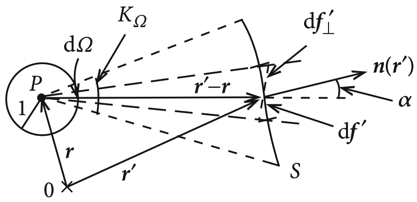
That yields for the potential:

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\,\epsilon_0}\int df'\left[\sigma(\mathbf{r}')\,d\right]\frac{\mathbf{n}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \dots \\ &\underset{d\to 0}{\longrightarrow} \frac{-1}{4\pi\,\epsilon_0}\int df'\,\mathbf{D}(\mathbf{r}')\cdot\frac{(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|^3}\end{aligned} \tag{2.81}$$

(cf. (2.75)).

We want to further evaluate the integral by simple geometric considerations. For this purpose we consider an element $d\mathbf{f}'$ on the area S (Fig. 2.26). This appears for the point under consideration $\mathbf{r}$ under the solid angle $d\Omega$. For the projection $df'_\perp$

Fig. 2.26 Geometrical illustration to the calculation of the surface integral in (2.81)



perpendicular to $\mathbf{r}' - \mathbf{r}$ it then holds obviously:

$$df'_\perp = df' \left(\mathbf{n} \cdot \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|} \right) = df' \cos\alpha$$

$$\simeq d\Omega \, |\mathbf{r}' - \mathbf{r}|^2 \quad \text{(for } \textit{sufficiently small } d\Omega) \, .$$

If the point under consideration $\mathbf{r}$ lies, differently from Fig. 2.26, on the *positive* side of the double-layer directly in the face of P on the prolongation of the vector $\mathbf{r}' - \mathbf{r}$, then we have

$$\mathbf{n} \cdot \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|} = \cos(\pi - \alpha) = -\cos\alpha \, .$$

We embrace both cases by $\pm d\Omega$ where

$$d\Omega = df' \left(\mathbf{n} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

is to be calculated with the geometry from Fig. 2.26. That means in (2.81):

$$\varphi_\pm(\mathbf{r}) = \pm \frac{1}{4\pi\,\epsilon_0} \int\limits_{K_\Omega} d\Omega \, D(\mathbf{r}') \, . \tag{2.82}$$

The integration is done over the part K_Ω of the unit-sphere which appears to be covered by the area S. The minus-sign is valid if, as in Fig. 2.26, the point P lies on the negatively charged side of the dipole (double)-layer and the plus-sign if it is on the positively charged side.

For simplicity let us assume that

$$D(\mathbf{r}') = D = \text{const} \quad \text{on } S \, ,$$

then the potential φ is given by the product of dipole density D and the solid angle $\Omega_S(\mathbf{r})$, under which the area S is seen from the point $\mathbf{r}$. The actual shape of S is thereby immaterial:

$$\varphi_\pm(\mathbf{r}) = \pm \frac{D}{4\pi\epsilon_0} \, \Omega_S(\mathbf{r}) \, . \tag{2.83}$$

Let us additionally assume that the area S is **plane** and let the point $\mathbf{r}$ approach $\mathbf{r}'$ on the dipole-layer. Then $\Omega_S(\mathbf{r})$ tends to 2π. Passing through the dipole-layer the potential thus performs a jump of

$$\Delta\varphi = \varphi_- - \varphi_+ = -\frac{1}{\epsilon_0} D \,. \tag{2.84}$$

This result can now easily be generalized for the case that a) S is not everywhere plane and b) $D(\mathbf{r}')$ is not everywhere constant on S. To show this one decomposes at first the total area S into a small piece of area ΔF around the point under consideration, at which the behavior of the potential will be investigated, and the rest. ΔF shall be chosen so small that ΔF can be considered as plane and $D(\mathbf{r}')$ as constant on ΔF. The potential $\varphi(\mathbf{r})$ is then a superposition of the contributions of this piece of area ΔF and all the rest of S. If the point under consideration $\mathbf{r}$ now approaches the area ΔF, then the potential produced by ΔF performs a jump according to (2.84). On the other hand, for points $\mathbf{r}'$ outside ΔF the vector $\mathbf{r}' - \mathbf{r}$ behaves continuously at the transition of $\mathbf{r}$ through the piece ΔF of the dipole-layer. That holds then also for $\mathbf{n} \cdot (\mathbf{r}' - \mathbf{r})/|\mathbf{r}' - \mathbf{r}|^3$. The potential created by the dipole-layer S *without* ΔF is continuous for $\mathbf{r} \in \Delta F$ and does not contribute to the potential-jump. Altogether it thus holds for the discontinuity of the potential:

$$\Delta\varphi = -\frac{1}{\varepsilon_0} D(\mathbf{r}) \quad \mathbf{r} \in S \,. \tag{2.85}$$

One can interpret this potential-jump as a potential-decline **within** the dipole-layer. If one considers this as a small parallel-plate capacitor at ΔF with the distance d of the plates then (2.85) corresponds exactly to (2.53).

2.2.6 *The Quadrupole*

In Sect. 2.2.4 we have built the dipole on the basis of the limiting process (2.70) by the use of two equal and opposite point charges $\pm q$. By a similar limiting process two equal and anti-parallel dipoles can be composed to build a **quadrupole** (Fig. 2.27). We define as

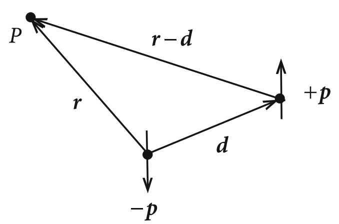


Fig. 2.27 To the definition of the quadrupole moment

quadrupole moments

$$q_{ij} = \lim_{\substack{d_i \to 0 \\ p_j \to \infty}} d_i p_j \,, \tag{2.86}$$

and require that these remain finite during the indicated limiting process. i, j are indexes for Cartesian components. We determine the potential of such a quadrupole as superposition of the potentials of both the dipoles which are at first at a finite distance d. At the end we perform the limiting process (2.86). According to (2.71) we find:

$$\begin{aligned} 4\pi\,\epsilon_0\,\varphi(\mathbf{r}) &= \mathbf{p}\cdot\nabla_r\left(\frac{1}{r} - \frac{1}{|\mathbf{r}-\mathbf{d}|}\right) \\ &= \mathbf{p}\cdot\nabla_r\left[\frac{1}{r} - \frac{1}{r} + (\mathbf{d}\cdot\nabla_r)\,\frac{1}{r} \pm \ldots\right] \\ &= \mathbf{p}\cdot\nabla_r\left[(\mathbf{d}\cdot\nabla_r)\,\frac{1}{r}\right] + \ldots \end{aligned}$$

Higher terms do not play a role because of (2.86). An expression of this form we have already dealt with in preparation of (2.73):

$$\begin{aligned} \nabla_r\left[\mathbf{d}\cdot\nabla_r\frac{1}{r}\right] &= (\mathbf{d}\cdot\nabla)\,\nabla\,\frac{1}{r} + \mathbf{d}\times\underbrace{\operatorname{curl}\left(\nabla\frac{1}{r}\right)}_{=0} \\ &= \frac{1}{r^5}\left[3(\mathbf{r}\cdot\mathbf{d})\mathbf{r} - r^2\mathbf{d}\right] \,. \end{aligned}$$

Therewith we have the potential:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\,\frac{1}{r^5}\left[3(\mathbf{r}\cdot\mathbf{d})(\mathbf{r}\cdot\mathbf{p}) - r^2(\mathbf{d}\cdot\mathbf{p})\right] + \ldots$$

If we express the scalar products by Cartesian components and perform the limiting process (2.86) then we get the quadrupole potential:

$$\varphi_{\mathrm{Q}}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\,\frac{1}{r^5}\sum_{i,j} q_{ij}\,\left(3x_i x_j - r^2\delta_{ij}\right) \,. \tag{2.87}$$

We agree to call an electrostatic charge configuration, which leads to such a scalar potential, a **quadrupole**.

Let us discuss, as an illustration, a concrete **realization of the quadrupole by point charges** as we did it similarly for the dipole. For this purpose we now need a system of **four** charges which are of the same magnitude being arranged as in

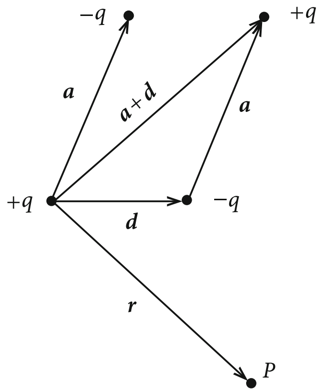


Fig. 2.28 Simple realization of a quadrupole by four point charges

Fig. 2.28. Each two of them carry the charge $+q$ and $-q$, respectively. We will see that the potential of this arrangement represents in the far zone a quadrupole potential of the form (2.87) if we take

$$q_{ij} \simeq q\, a_i\, d_j$$

That can be seen as follows:

$$\begin{aligned}
4\pi\epsilon_0\varphi(\mathbf{r}) &= q\left(\frac{1}{r} + \frac{1}{|\mathbf{r}-\mathbf{a}-\mathbf{d}|} - \frac{1}{|\mathbf{r}-\mathbf{a}|} - \frac{1}{|\mathbf{r}-\mathbf{d}|}\right) \\
&= q\left\{\frac{1}{r} + \frac{1}{r} - [(\mathbf{a}+\mathbf{d})\cdot\nabla]\,\frac{1}{r} + \frac{1}{2}\,[(\mathbf{a}+\mathbf{d})\cdot\nabla]^2\,\frac{1}{r} + \ldots\right. \\
&\qquad -\frac{1}{r} + (\mathbf{a}\cdot\nabla)\frac{1}{r} - \frac{1}{2}(\mathbf{a}\cdot\nabla)^2\frac{1}{r} + \ldots \\
&\qquad \left. -\frac{1}{r} + (\mathbf{d}\cdot\nabla)\frac{1}{r} - \frac{1}{2}(\mathbf{d}\cdot\nabla)^2\frac{1}{r} + \ldots\right\}
\end{aligned}$$

Contributions of monopoles and dipoles, respectively, are compensating each other:

$$\begin{aligned}
4\pi\epsilon_0\varphi(\mathbf{r}) &= \frac{1}{2}q\,[(\mathbf{a}\cdot\nabla)(\mathbf{d}\cdot\nabla) + (\mathbf{d}\cdot\nabla)(\mathbf{a}\cdot\nabla)]\,\frac{1}{r} + \ldots \\
&= +q\sum_{i,j} a_i d_j \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r} + \ldots = \sum_{i,j} q_{ij}\frac{\partial}{\partial x_i}\left(-\frac{x_j}{r^3}\right) + \ldots \\
&= \sum_{i,j} q_{ij}\left(\frac{3x_j x_i}{r^5} - \delta_{ij}\frac{1}{r^3}\right) + \ldots
\end{aligned} \tag{2.88}$$

For the far zone ($r \gg a, d$) we can neglect the higher terms. It then remains the pure quadrupole potential (2.87) which in the near zone, however, will exhibit strong modifications.

Special case: *stretched (linear) quadrupole (Fig. 2.29)*

$$\mathbf{a} = (0, 0, a) \; ; \quad \mathbf{d} = (0, 0, a) \; .$$

This yields in the far zone according to (2.88) the potential:

$$\begin{aligned} 4\pi\epsilon_0 \varphi_{\mathrm{Q}}(\mathbf{r}) &= q\, a^2 \frac{3z^2 - r^2}{r^5} \\ &= q\, a^2 \frac{3\cos^2\vartheta - 1}{r^3} \; . \end{aligned} \tag{2.89}$$

As expected the potential is axially symmetric, i.e. it is φ-**in**dependent. Taking the gradient we find the components of the electrical field (Fig. 2.30):

$$\begin{aligned} E_r^Q &= -\frac{\partial \varphi_{\mathrm{Q}}}{\partial r} = \frac{3q\, a^2}{4\pi\epsilon_0} \frac{3\cos^2\vartheta - 1}{r^4} \; , \\ E_\vartheta^Q &= -\frac{1}{r}\frac{\partial \varphi_{\mathrm{Q}}}{\partial \vartheta} = \frac{6q\, a^2}{4\pi\epsilon_0} \frac{\cos\vartheta \sin\vartheta}{r^4} \; , \\ E_\varphi^Q &= 0 \; . \end{aligned} \tag{2.90}$$

It should be reminded once more that within the near zone the field of the above system of point charges does really look completely different. The **pure** quadrupole-

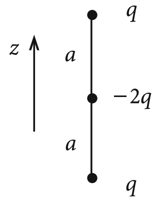


Fig. 2.29 The stretched quadrupole being built by three point charges

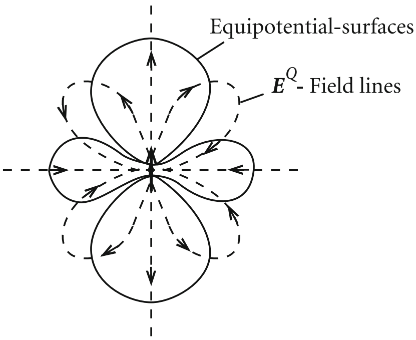


Fig. 2.30 Equipotential-surfaces and electric field lines of the stretched quadrupole in the far zone

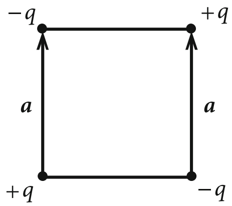


Fig. 2.31 Square arrangement of a quadrupole consisting of four point charges

field (2.90) does exist only in the limit

$$\begin{matrix} a \to 0 \\ q \to \infty \end{matrix} \quad \text{with } q\,a^2 = \text{const}\,,$$

because then the terms neglected in the above expansion vanish **exactly**.

Another realization of a quadrupole could be the following system of point charges (Fig. 2.31):

$$\begin{aligned} \mathbf{a} &= (0, 0, a)\,, \\ \mathbf{d} &= (0, a, 0)\,, \\ q_{32} &= q\,a^2\,; \qquad \text{all the others} \quad q_{ij} = 0 \end{aligned}$$

It holds for the potential of this arrangement:

$$4\pi\epsilon_0\varphi_{\mathrm{Q}}(\mathbf{r}) = q\,a^2\frac{3zy}{r^5} = q\,a^2\frac{3\cos\vartheta\,\sin\vartheta\,\sin\varphi}{r^3}\,.$$

In this case the potential is of course **not** axially symmetric.

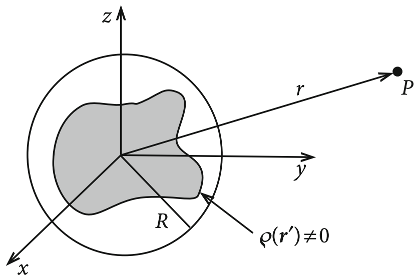
2.2.7 Multipole Expansion

We now discuss the potential and the electric field of a spatially restricted charge distribution $\rho(\mathbf{r}')$, i.e. we presume that the total $\rho \neq 0$-region can be embedded into a sphere of finite radius R (Fig. 2.32). If there is no need to fulfill the boundary conditions of finiteness then (2.25) is valid:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,.$$

The evaluation of such a volume-integral is not always simple. On the other hand, often one is interested only in the asymptotic behavior of φ and $\mathbf{E}$ in the *far zone* ($r \gg R$), i.e. far outside the $\rho \neq 0$-region. It therefore suggests itself a Taylor-

Fig. 2.32 Charge density within a sphere of finite radius R



expansion of the integrand with respect to powers of $\frac{r'}{r}$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \exp(-\mathbf{r}'\cdot\nabla)\frac{1}{r} = \frac{1}{r} - (\mathbf{r}'\cdot\nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r} \pm \dots$$
$$\overset{(1.32)}{=} \frac{1}{r} + \frac{\mathbf{r}'\cdot\mathbf{r}}{r^3} + \frac{3(\mathbf{r}'\cdot\mathbf{r})^2 - r'^2r^2}{2r^5} + \dots$$

This we insert into the above expression for $\varphi(\mathbf{r})$:

$$4\pi\epsilon_0\varphi(\mathbf{r}) = \frac{1}{r}\int d^3r'\rho(\mathbf{r}') + \frac{1}{r^3}\mathbf{r}\cdot\int d^3r'\mathbf{r}'\rho(\mathbf{r}')$$
$$+\frac{1}{2r^5}\int d^3r'\rho(\mathbf{r}')\left(3(\mathbf{r}\cdot\mathbf{r}')^2 - r'^2r^2\right) + \dots$$

We still rewrite the third summand a little bit:

$$\int d^3r'\rho(\mathbf{r}')\left(3(\mathbf{r}\cdot\mathbf{r}')^2 - r'^2r^2\right) = \int d^3r'\rho(\mathbf{r}')\left(\sum_{i,j}3x_ix_i'x_jx_j' - r'^2\sum_{i,j}\delta_{ij}x_ix_j\right)$$
$$= \sum_{i,j}x_ix_j\int d^3r'\rho(\mathbf{r}')\left(3x_i'x_j' - r'^2\delta_{ij}\right) \; .$$

One now defines the following

moments of the charge distribution

$$\textbf{total charge (monopole):}\;\; q = \int d^3r'\rho(\mathbf{r}') \; , \tag{2.91}$$

$$\textbf{dipole moment:}\;\; \mathbf{p} = \int d^3r'\mathbf{r}'\rho(\mathbf{r}') \; , \tag{2.92}$$

$$\textbf{quadrupole moment:}\;\; Q_{ij} = \int d^3r'\rho(\mathbf{r}')(3x_i'x_j' - r'^2\delta_{ij}) \tag{2.93}$$

$$\dots$$

The resulting expansion of the potential

$$4\pi\epsilon_0\varphi(\mathbf{r}) = \frac{q}{r} + \frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots \tag{2.94}$$

shows that the potential of an arbitrary charge distribution is composed of the potentials of a point charge, a dipole, a quadrupole, octupole and so on. One speaks of a **multipole expansion**. For points which are very far from the $\rho \neq 0$-region the charge distribution acts as a point charge located in the origin since the first term of the expansion dominates. The closer one approaches the $\rho \neq 0$-region the more terms of the expansion are to be taken into account.

Discussion

1. If $q \neq 0$ then the **monopole term** dominates in the far zone:

$$\varphi_{\mathrm{M}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r}\ . \tag{2.95}$$

   The **E**-field corresponds to that of a point charge q in the origin ((2.21) with $\mathbf{r}_0 = \mathbf{0}$).
2. If $q = 0$ then the **dipole term** dominates:

$$\varphi_{\mathrm{D}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}\ , \tag{2.96}$$

   which we have extensively discussed subsequent to (2.71). A simple realization of a charge distribution with $q = 0$ is a pair of equal and opposite point charges,

$$\rho(\mathbf{r}) = -q\,\delta(\mathbf{r}) + q\,\delta(\mathbf{r}-\mathbf{a})\ ,$$

   with the dipole moment:

$$\mathbf{p} = -q\cdot\mathbf{0} + q\,\mathbf{a} = q\,\mathbf{a}\ .$$

   In the far zone, as soon as the higher multipoles become unimportant, the corresponding field is a pure dipole field (2.73).

   The dipole moment $\mathbf{p}$ (2.92) is invariant with respect to rotations of the system of coordinates, but normally not invariant with respect to translations, i.e. linear shifts of the origin as in (Fig. 2.33):

$$\begin{aligned}
\bar{\mathbf{p}} &= \int d^3r''\mathbf{r}''\bar{\rho}(\mathbf{r}'')\ ,\\
d^3r'' &= d^3r'\ ,\\
\bar{\mathbf{p}} &= \int d^3r'\mathbf{r}'\rho(\mathbf{r}') - \mathbf{d}\int d^3r''\bar{\rho}(\mathbf{r}'') = \mathbf{p} - \mathbf{d}\,q\ .
\end{aligned} \tag{2.97}$$

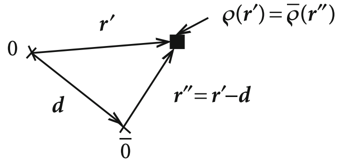


Fig. 2.33 To the dependence of the dipole moment on the choice of the origin of coordinates

In the case that the total charge q is zero then the dipole moment is invariant with respect to translations, but only then.

Mirror-symmetrical charge distributions

$$\rho(\mathbf{r}') = \rho(-\mathbf{r}')$$

do not have a dipole moment:

$$\mathbf{p} = \int d^3r'\mathbf{r}'\rho(\mathbf{r}') \overset{\mathbf{r}'\to-\mathbf{r}'}{=} \int d^3r'(-\mathbf{r}')\rho(-\mathbf{r}') = -\int d^3r'\,\mathbf{r}'\rho(\mathbf{r}') = -\mathbf{p}$$

$$\Longrightarrow \quad \mathbf{p} = \mathbf{0}\,.$$

3. If $q = 0$ and $\mathbf{p} = \mathbf{0}$ then the **quadrupole term** dominates:

$$\varphi_{\mathrm{Q}}(\mathbf{r}) = \frac{1}{8\pi\epsilon_0}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5}\,. \tag{2.98}$$

The Q_{ij}, defined in (2.93), are components of the **quadrupole tensor**:

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}\,.$$

The tensor concept we have already introduced in Sect. 4.3.3, Vol. 1 (see Exercise 2.2.12). From (2.93) one reads off some elementary properties of the quadrupole tensor $\mathbf{Q}$:

(a) **Traceless**

By the *'trace'* of a matrix one understands the sum of its diagonal elements:

$$\sum_i Q_{ii} = \int d^3r'\rho(\mathbf{r}')\left(3\sum_i x_i'^2 - 3r'^2\right) = 0\,. \tag{2.99}$$

(b) **Symmetric**, i.e. $Q_{ij} = Q_{ji}$

That means that $\mathbf{Q}$ has only five independent elements.

(c) The quadrupole moments q_{ij} derived in Sect. 2.2.6 from an illustrative model are not completely compatible with the Q_{ij} presented here. If one compares (2.87) with the expression directly before (2.91) one comes to:

$$q_{ij} = \frac{1}{2}\int d^3r'\rho(\mathbf{r}')x_i'x_j' \; . \tag{2.100}$$

By this the q_{ij} get their meaning for arbitrary charge distributions. The comparison with (2.93) leads to:

$$Q_{ij} = 6\,q_{ij} - 2\delta_{ij}\sum_k q_{kk} \; . \tag{2.101}$$

(d) **Spherically symmetric charge distributions** $\rho(\mathbf{r}') = \rho(r')$ have **no finite** quadrupole moment. For it follows at first from symmetry reasons

$$Q_{11} = Q_{22} = Q_{33}$$

and therewith because of (2.99) $Q_{ii} = 0$, $i = 1, 2, 3$. That $Q_{ij} = 0$ for $i \neq j$, one sees by a direct angle integration. So it holds for $i \neq j$:

$$Q_{ij} = \int d^3r'\rho(r')\left(3x_i'x_j'\right) = 3\int_0^\infty dr'r'^2\rho(r')\int_{-1}^{+1} d\cos\vartheta \int_0^{2\pi} d\varphi x_i'x_j' \; .$$

That means, for instance, for the xy-element:

$$x'y' = r'^2\sin^2\vartheta'\cos\varphi'\sin\varphi' = \frac{1}{2}r'^2\sin^2\vartheta'\frac{d}{d\varphi'}\sin^2\varphi' \; .$$

The φ'-integration thus let this term vanish. That can be shown in this way for all off-diagonal terms.

(e) **Example:** *stretched point quadrupole* (Fig. 2.29).

charge density:

$$\rho(\mathbf{r}) = q\,\delta(x)\delta(y)\left(\delta(z) - 2\delta(z-a) + \delta(z-2a)\right) \; .$$

total charge:

$$q = 0 \; .$$

dipole moment:

$$\mathbf{p} = q\int_{-\infty}^{+\infty} dz'(0,0,z')\left[\delta(z') - 2\delta(z'-a) + \delta(z'-2a)\right] = \mathbf{0} \; .$$

quadrupole moments:

$$Q_{ij} = 0 \quad \text{for } i \neq j \,,$$

$$Q_{11} = \int d^3r'\, \rho(\mathbf{r}')\left[3x'^2 - r'^2\right]$$

$$= q \int\limits_{-\infty}^{+\infty} dz'(-z'^2)\left[\delta(z') - 2\delta(z'-a) + \delta(z'-2a)\right]$$

$$= -2q\,a^2 = Q_{22}\,,$$

$$Q_{33} = q \int\limits_{-\infty}^{+\infty} dz' 2z'^2\left[\delta(z') - 2\delta(z'-a) + \delta(z'-2a)\right] = 4q\,a^2\,.$$

The quadrupole tensor therewith reads:

$$\mathbf{Q} = 2q\,a^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\,. \tag{2.102}$$

2.2.8 *Interaction of a Charge Distribution with an External Electric Field*

The external charge distribution ρ_{ex} creates an electric field with which the charge distribution $\rho(\mathbf{r})$ interacts (Fig. 2.34). According to (2.47) it holds for the electrostatic field energy of the total charge density:

$$W = \frac{1}{8\pi\epsilon_0}\iint d^3r\, d^3r'\, \frac{[\rho(\mathbf{r}) + \rho_{\mathrm{ex}}(\mathbf{r})]\,[\rho(\mathbf{r}') + \rho_{\mathrm{ex}}(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|}\,.$$

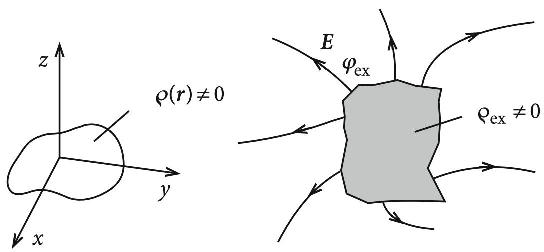


Fig. 2.34 Schematic plot of a charge density being restricted to a finite space-region in the field of an external charge density

The interaction part therewith reads:

$$W_1 = \frac{1}{4\pi\epsilon_0}\iint d^3r\, d^3r' \frac{\rho(\mathbf{r})\rho_{\text{ex}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \int d^3r\, \rho(\mathbf{r})\varphi_{\text{ex}}(\mathbf{r})\ . \tag{2.103}$$

φ_{ex} is the scalar potential produced by ρ_{ex}. We assume that the $\rho \neq 0$-region is small enough so that φ_{ex} can be considered there as approximately constant:

$$\begin{aligned}\varphi_{\text{ex}}(\mathbf{r}) &= \varphi_{\text{ex}}(0) + (\mathbf{r}\cdot\nabla)\varphi_{\text{ex}}(0) + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\varphi_{\text{ex}}(0) + \dots \\ &= \varphi_{\text{ex}}(0) - \mathbf{r}\cdot\mathbf{E}(0) + \frac{1}{2}\sum_{i,j} x_i x_j \left.\frac{\partial^2\varphi_{\text{ex}}}{\partial x_j \partial x_i}\right|_{\mathbf{r}=0} + \dots\end{aligned}$$

Within the $\rho \neq 0$-region there do not exist charges which create the field $\mathbf{E}$. Hence there we have div$\mathbf{E} = 0$. That means:

$$0 = \sum_i \frac{\partial}{\partial x_i}E_i = -\sum_i \frac{\partial^2\varphi_{\text{ex}}}{\partial x_i^2} = -\sum_{i,j}\delta_{ij}\frac{\partial^2\varphi_{\text{ex}}}{\partial x_j \partial x_i}\ .$$

Such a term can therefore be confidently added to the above expression:

$$\varphi_{\text{ex}}(\mathbf{r}) = \varphi_{\text{ex}}(0) - \mathbf{r}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_i(0)}{\partial x_j} + \dots$$

This is now inserted into (2.103):

$$W_1 = q\,\varphi_{\text{ex}}^{(0)} - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_i(0)}{\partial x_j} + \dots \tag{2.104}$$

The charge (monopole) interacts with the external potential, the dipole moment with the external field $\mathbf{E}$, and the quadrupole moment with the spatial field-derivatives.

We can use this relation to determine the interaction between two dipoles. For this purpose we insert into the second summand the field (2.73) of another dipole:

$$W_{12} = \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r_{12}^3} - 3\frac{(\mathbf{r}_{12}\cdot\mathbf{p}_1)\,(\mathbf{r}_{12}\cdot\mathbf{p}_2)}{r_{12}^5}\right] \tag{2.105}$$

($\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$). This important equation shows that the dipole-dipole interaction can be attractive as well as repulsive depending on the relative orientation of the two dipoles.

2.2.9 Exercises

Exercise 2.2.1

1. Calculate the energy density and the total energy of the electric field in a spherical capacitor. The two spherical shells carry the charges Q and $-Q$.
2. How does the energy in the capacitor change when the inner shell carries the charge Q, the outer $-Q/2$, and vice versa?
3. Which pressure is exerted in both cases on the shells of the spherical capacitor?

Exercise 2.2.2 The two plates of a capacitor (distance d, area F, the left plate at $x = 0$, the right plate at $x = d$) are charged with Q_1 and Q_2, respectively. Between these two plates there is at the position x_0 a third, isolated plate with the known charge Q.

1. Calculate the electric field inside the capacitor! Which voltage U is applied on the capacitor?
2. Which electrostatic force acts on the middle plate? What happens in the case of a short circuit $U = 0$?
3. Let the middle plate with the constant charge Q and the mass M be freely movable (no gravitational force!). How does its mechanical equation of motion look like?

Exercise 2.2.3 A dipole with the moment $\mathbf{p}$ is located at $\mathbf{r}$ and a point charge q at the origin of coordinates (Fig. 2.35).

1. Calculate the potential energy of the dipole.
2. Calculate the force which acts on the dipole.
3. Consider whether or not the third Newtonian axiom is fulfilled.

Exercise 2.2.4 Given is a cylindrical capacitor with the inner radius a and the outer radius b. A voltage $U = \varphi(a) - \varphi(b)$ is applied.

1. Calculate the electric field $\mathbf{E}(\mathbf{r})$, the potential $\varphi(\mathbf{r})$, and the capacity per unit length.
2. For which value of a does the field strength at the inner cylinder become minimal for a given U?

Exercise 2.2.5 Two concentric metallic spherical shells with radii R_1 and R_2 possess the potential values Φ_1 and Φ_2, respectively (Fig. 2.36).

1. Determine the potential $\Phi(r)$ in the whole space for given Φ_1 and Φ_2.
2. Which charges Q_1 and Q_2 are on the spherical shells?

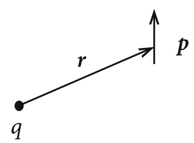


Fig. 2.35 Dipole $\mathbf{p}$ and point charge q shifted relative to each other by the vector $\mathbf{r}$

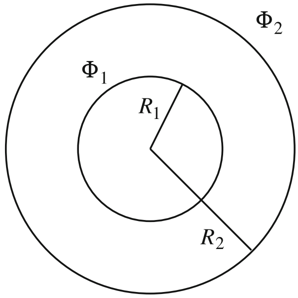


Fig. 2.36 Concentric metallic spherical shells with different potentials Φ_1 and Φ_2

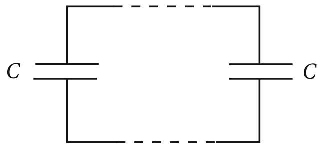


Fig. 2.37 Two parallel connected capacitors each of them with the capacity C

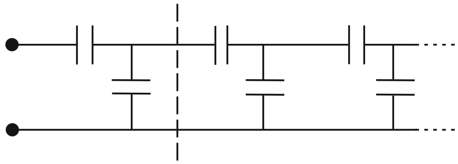


Fig. 2.38 Infinite chain of ordered capacitors each with the capacity C

Exercise 2.2.6 A capacitor C is charged to the voltage U_0 and then separated from the voltage source. How much are the charge and the stored energy? In the next step a second but uncharged capacitor of the same capacity is connected in parallel (Fig. 2.37). How much are now the voltage and the total energy of the two capacitors ($C = 100\,\mu\text{F}$; $U_0 = 1000\,\text{V}$)?

Exercise 2.2.7 Calculate the capacity of an infinitely long chain of capacitors as given in Fig. 2.38, each of them has the same capacity C.

Hint: By cutting one unit (broken line in Fig. 2.38) the capacity C_∞ of the **infinitely** long array does not change.

Exercise 2.2.8 A hollow sphere with the radius R carries the charge density

$$\rho(\mathbf{r}) = \sigma_0 \cos^2\theta\, \delta(r - R) \ .$$

Calculate

1. the total charge q,
2. the dipole moment $\mathbf{p}$,
3. the components Q_{ij} of the quadrupole tensor,
4. the electrostatic potential $\varphi(\mathbf{r})$ and the electric field $\mathbf{E}(\mathbf{r})$ up to quadrupole terms.

Exercise 2.2.9 Let an electric dipole $\mathbf{p}_1$ be at the origin of the coordinates and point into the z-direction. A second electric dipole $\mathbf{p}_2$ is at the position $(x_0, 0, z_0)$. Which direction does $\mathbf{p}_2$ take in the field of $\mathbf{p}_1$?

Exercise 2.2.10 Four charges q are arranged in a Cartesian system of coordinates at the points

$$(0, d, 0),\ (0, -d, 0),\ (0, 0, d),\ (0, 0, -d)$$

and four further charges $-q$ at the points

$$(-d, 0, 0),\ \left(-\frac{d}{2}, 0, 0\right),\ (d, 0, 0),\ (2d, 0, 0)\ .$$

Calculate the dipole moment $\mathbf{p}$ and the quadrupole tensor $\mathbf{Q}$ of this charge-arrangement.

Exercise 2.2.11 A given charge distribution $\rho(\mathbf{r})$ possesses axial symmetry about the z-axis.

1. Show that the quadrupole tensor is diagonal!
2. Verify: $Q_{xx} = Q_{yy} = -(1/2)Q_{zz}$.
3. Calculate the potential and the electric field strength of the quadrupole as a function of Q_{zz}.

Exercise 2.2.12 Why does the 3×3-matrix of the quadrupole moments Q_{ij} (2.93) have to be a **tensor of the second rank**?

2.3 Boundary-Value Problems in Electrostatics

2.3.1 Formulation of the Boundary-Value Problem

In Sect. 2.1.3 we identified the solution of the Poisson equation (2.41) as the basic problem of electrostatics. All considerations therefore aim at developing solution methods for this linear, inhomogeneous, partial differential equation of second order.

If the charge density $\rho(\mathbf{r}')$, which creates the potential $\varphi(\mathbf{r})$, is known and there are no boundary conditions to be fulfilled at interfaces (bounding surfaces) for finiteness, then the general solution (2.25) is fully sufficient:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad \textbf{(Poisson integral)}\ .$$

If ρ is spatially restricted then it holds in particular

$$\varphi \underset{r\to\infty}{\longrightarrow} 0\,; \quad \nabla\varphi \underset{r\to\infty}{\longrightarrow} 0\,.$$

For many practical problems, however, this is not the actual starting point.

Definition 2.3.1 'boundary-value problem'

given: $\rho(\mathbf{r}')$ in a certain space region V,
φ or $\frac{\partial\varphi}{\partial n} = -\mathbf{E}\cdot\mathbf{n}$
on certain interfaces and bounding surfaces in V.

to be found: the scalar potential $\varphi(\mathbf{r})$ at all points $\mathbf{r}$
within the interesting space-region V.

Let us first investigate under which conditions an electrostatic boundary-value problem has a unique mathematical solution. For this purpose we use as essential auxiliary means the two Green theorems (1.66) and (1.67) by which we transform the Poisson equation (2.41) into an integral equation. If we insert

$$\varphi \to \varphi(\mathbf{r}')\,; \quad \psi \to \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,,$$

into (1.67) then it follows:

$$\begin{aligned}
&\int\limits_V \left[\varphi(\mathbf{r}')\Delta_{r'}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\Delta_{r'}\varphi(\mathbf{r}')\right]d^3r' \\
&\quad = -4\pi\int\limits_V d^3r'\varphi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') + \frac{1}{\epsilon_0}\int\limits_V d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&\quad = \int\limits_{S(V)} df'\left[\varphi(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\varphi}{\partial n'}\right].
\end{aligned}$$

In the second step we have exploited (1.69) and inserted the Poisson equation (2.41). The normal-derivatives are thought as in (1.65).

If now $\mathbf{r}\in V$ then it remains as solution for the potential:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\int\limits_{S(V)} df'\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\varphi}{\partial n'} - \varphi(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]. \tag{2.106}$$

We want to discuss this relation:

1. ρ in V and, respectively, φ and $\partial\varphi/\partial n = \mathbf{n} \cdot \nabla\varphi$ on $S(V)$ ($\mathbf{n}$: surface normal) determine the potential in the full region V. Existing charges **outside** V enter the surface integrals only implicitly.
2. If V is free of charges then it holds for $\mathbf{r} \in V$:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \int\limits_{S(V)} df' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial\varphi}{\partial n'} - \varphi(\mathbf{r}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) . \tag{2.107}$$

   Thus φ is completely determined by its own values and those of its normal-derivative on $S(V)$.
3. If V is the entire space and

$$\varphi(\mathbf{r}') \underset{r' \to \infty}{\longrightarrow} \frac{1}{r'} ,$$

   i.e.

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial\varphi}{\partial n'} \underset{r' \to \infty}{\longrightarrow} \frac{1}{r'^3} ,$$

$$\varphi(\mathbf{r}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \underset{r' \to \infty}{\longrightarrow} \frac{1}{r'^3} ,$$

   then the surface integral vanishes, only the volume integral is left which one calls the **Poisson integral**, i.e. just the known result (2.25).
4. By **both** the data φ **and** $\partial\varphi/\partial n$ on $S(V)$ (**Cauchy-boundary conditions**) the **problem is overdetermined**. We will see that in general they cannot be fulfilled simultaneously. Thus Eq. (2.106) can not yet be considered as a solution of the boundary-value problem. It is actually an integral equation which is equivalent to the Poisson equation.

2.3.2 Classification of the Boundary Conditions

One distinguishes two types of boundary conditions:

Dirichlet-Boundary Conditions

$$\varphi \qquad \text{given on } S(V)!$$

Neumann-Boundary Conditions

$$\frac{\partial \varphi}{\partial n} = -\mathbf{n} \cdot \mathbf{E} \qquad \text{given on } S(V)!$$

One speaks of **mixed** boundary conditions if these have on $S(V)$ piecewise Dirichlet and piecewise Neumann character.

Before we think about the physical origin of such boundary conditions we demonstrate the **uniqueness** of the solutions resulting from them (see Exercise 1.7.24):

Let $\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r})$ be solutions of the Poisson equation

$$\Delta \varphi_{1,2}(\mathbf{r}) = -\frac{1}{\epsilon_0}\rho(\mathbf{r})$$

with

$$\varphi_1 \equiv \varphi_2 \quad \text{on } S(V) \quad \text{(Dirichlet)}$$

or

$$\frac{\partial \varphi_1}{\partial n} \equiv \frac{\partial \varphi_2}{\partial n} \quad \text{on } S(V) \quad \text{(Neumann)}\ .$$

For

$$\psi(\mathbf{r}) = \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r})$$

it then holds

$$\Delta\, \psi \equiv 0$$

with

$$\psi \equiv 0 \quad \text{on } S(V) \quad \text{(Dirichlet)}$$

or

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on } S(V) \quad \text{(Neumann)}\ .$$

The first Green theorem (1.66) then reads for $\varphi = \psi$:

$$\int_V d^3r \left[\psi\, \Delta\, \psi + (\nabla\, \psi)^2\right] = \oint_{S(V)} \psi \frac{\partial \psi}{\partial n} df\ .$$

For both types of boundary conditions the right-hand side vanishes. We therefore have:

$$\int_V d^3r(\nabla\psi)^2 = 0 \implies \nabla\psi \equiv \mathbf{0} \implies \psi = \text{const}\,.$$

Dirichlet:

$$\psi = 0 \text{ on } S(V) \implies \psi \equiv 0 \text{ in } V \implies \varphi_1(\mathbf{r}) \equiv \varphi_2(\mathbf{r}) \text{ in } V$$

Neumann:

$$\psi = \text{const in } V \text{ and } \frac{\partial\psi}{\partial n} = 0 \text{ on } S(V) \implies \varphi_1(\mathbf{r}) = \varphi_2(\mathbf{r}) + C\,.$$

The constant C is of no importance. It vanishes, for instance, when one goes by performing the gradient to the actually interesting field strength $\mathbf{E}$. Both types of boundary conditions thus fix *physically uniquely* the solution of the Poisson equation. This is valid also for *mixed* boundary conditions.

Why are Dirichlet- or Neumann-boundary conditions of practical interest? Where and when do they become relevant? To answer these questions some preliminary considerations are necessary: One can roughly divide the materials which can carry charges into two classes:

1. **Non-conductors (insulators)**: These are substances whose charged constituents are fixed at certain space points. Even the application of an electric field cannot release them of their bonds. One may think of the $Na^+$and Cl^- ions of a NaCl-crystal. Excess charges brought onto a non-conductor remain localized even if electric Coulomb-forces are acting.
2. **Conductors (metals)**: These are materials in which electric charges (e.g. electrons of a partially filled energy band in a solid) can almost freely be shifted. They immediately react to an electric field. That holds in particular for excess charges that are additionally brought in.

If the conductor finds itself in an electrostatic, i.e. time-**in**dependent field, the charges will rearrange themselves in such a way as to give an equilibrium state in which all the charges on the surface as well as inside the conductor are at rest. But that means necessarily:

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r}) &\equiv 0 \\ \varphi(\mathbf{r}) &= \text{const} \end{aligned}\right\} \text{ in the conductor}\,. \tag{2.108}$$

What happens at the interface between the conductor and the vacuum? On the inside of the conductor surface it must hold according to our above considerations:

$$E_i^{(n)} = E_i^{(t)} = 0\,. \tag{2.109}$$

Tangential as well as normal component of the **E**-field are zero. We have found in (2.44) that the tangential component behaves continuously at the interface:

$$E_{\mathrm{a}}^{(t)} = 0\,; \quad E_{\mathrm{a}}^{(n)} = \frac{\sigma}{\epsilon_0}\,. \tag{2.110}$$

Important conclusion: The electric field is always oriented perpendicular to the surface of the conductor (Fig. 2.39), i.e.

Surface of the conductor ≡ equipotential surface

From (2.109) it follows with the physical Gauss theorem that the interior of an electric conductor is always charge-neutral. This fact does not change even if the conductor is hollowed out. The resulting hole remains field-free (**Faraday cage**).

When we bring an electric conductor into an external electrostatic field the quasi-free charge carriers will be shifted until the resulting total field enters the surface of the conductor perpendicularly, i.e. the tangential component of **E** vanishes. That means that the external field will be deformed. If, however, $E_{\mathrm{a}}^{(n)} \neq 0$ then it follows from (2.110) that a proper surface charge density σ must have been built. One says:

The external field *induces* charges on the surface of an conductor!

We now come back to our boundary-value problem. We are looking for the electrostatic potential $\varphi(\mathbf{r})$ as the solution of the Poisson equation in a certain space-region V (Fig. 2.40). The Poisson equation is defined by a

charge density $\rho(\mathbf{r})$.

Its solution is influenced by boundary conditions on the

1. surfaces of conductors $\Longleftrightarrow \varphi = \mathrm{const}$,
2. charged areas $\Longleftrightarrow \frac{\partial\varphi_{\mathrm{a}}}{\partial n} - \frac{\partial\varphi_{\mathrm{i}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$,
3. dipole layers $\Longleftrightarrow \varphi_{\mathrm{a}} - \varphi_{\mathrm{i}} = \pm\frac{1}{\epsilon_0}D$.

The following considerations aim at such cases for which the boundary conditions, which are to be fulfilled, are of the Dirichlet- or the Neumann-type.

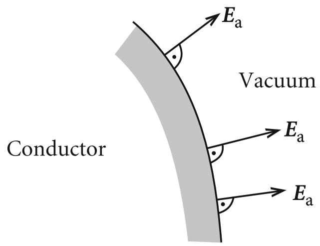


Fig. 2.39 To the behavior of the electric field at interfaces between a conductor and the vacuum

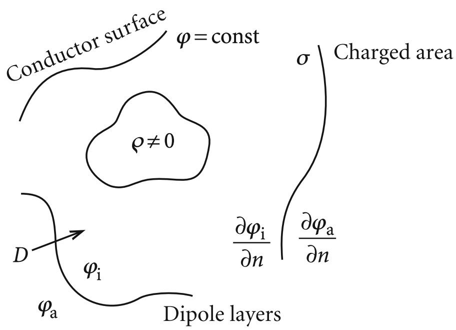


Fig. 2.40 Typical boundary conditions for the solution of the Poisson equation

2.3.3 *Green's Function*

We want to solve the boundary-value problem at first formally, and that by the use of the so-called *Green's function* $G(\mathbf{r}, \mathbf{r}')$.

Green's function: Solution of the Poisson equation for a point charge $q = 1$:

$$\Delta_r G(\mathbf{r}, \mathbf{r}') = -\frac{1}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}') \,. \tag{2.111}$$

Obviously it is about a function which is symmetric in $\mathbf{r}$ and $\mathbf{r}'$; i.e. we can let the Laplace operator act also on the variable $\mathbf{r}'$. With (1.69) one easily shows that (2.111) has in the interesting space-region V the solution

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - \mathbf{r}'|} + f(\mathbf{r}, \mathbf{r}') \tag{2.112}$$

where $f(\mathbf{r}, \mathbf{r}')$ is an almost arbitrary, symmetric function in $\mathbf{r}$ and $\mathbf{r}'$ which has only to fulfill **within** V

$$\Delta_r f(\mathbf{r}, \mathbf{r}') = 0 \,. \tag{2.113}$$

We will later exploit the freedom to choose f almost arbitrarily for realizing special boundary conditions.

We use once more the second Green identity (1.67):

$$\begin{aligned}
&\int\limits_V d^3r' \left[\varphi(\mathbf{r}')\Delta_{r'} G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\Delta_{r'}\varphi(\mathbf{r}')\right] \\
&\qquad = -\frac{1}{\epsilon_0}\int\limits_V d^3r' \varphi(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') + \frac{1}{\epsilon_0}\int\limits_V d^3r'\, G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') \\
&\qquad = \int\limits_{S(V)} df' \left[\varphi(\mathbf{r}')\frac{\partial G}{\partial n'} - G(\mathbf{r}, \mathbf{r}')\frac{\partial \varphi}{\partial n'}\right] .
\end{aligned}$$

For $\mathbf{r} \in V$ that means:

$$\varphi(\mathbf{r}) = \int\limits_V d^3r' \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') - \epsilon_0 \int\limits_{S(V)} df' \left[\varphi(\mathbf{r}') \frac{\partial G}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi}{\partial n'} \right] . \tag{2.114}$$

This relation is of course completely equivalent to (2.106); but we now have the possibility to remove via the still freely choosable function $f(\mathbf{r}, \mathbf{r}')$ the over-determination of the problem.

1. **Dirichlet**-boundary conditions
   In the case that $\varphi(\mathbf{r}')$ on $S(V)$ is given one chooses $f(\mathbf{r}, \mathbf{r}')$ such that
$$\int\limits_{S(V)} df' \, G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi}{\partial n'} = 0 . \tag{2.115}$$
   Often, but not necessarily always, we realize this by
$$G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}') \equiv 0 \quad \forall \, \mathbf{r}' \in S(V) . \tag{2.116}$$
   We then have for the scalar potential:
$$\varphi(\mathbf{r}) = \int\limits_V d^3r' \rho(\mathbf{r}') G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}') - \epsilon_0 \int\limits_{S(V)} df' \varphi(\mathbf{r}') \frac{\partial G_{\mathrm{D}}}{\partial n'} . \tag{2.117}$$
   Since φ on $S(V)$ and ρ in V are known the solution of the problem is herewith traced back to the determination of the Green's function. The latter must fulfill (2.115) and (2.116), respectively.
2. **Neumann**-boundary conditions
   In the case that $\frac{\partial \varphi}{\partial n} = -\mathbf{E} \cdot \mathbf{n}$ on $S(V)$ is given one chooses $f(\mathbf{r}, \mathbf{r}')$ such that
$$\epsilon_0 \int\limits_{S(V)} df' \varphi(\mathbf{r}') \frac{\partial G_N(\mathbf{r}, \mathbf{r}')}{\partial n'} = -\varphi_0 , \tag{2.118}$$
   where φ_0 may be an arbitrary constant. The at first glance obvious requirement to choose, in analogy to (2.116), the function $f(\mathbf{r}, \mathbf{r}')$ such that
$$\frac{\partial}{\partial n'} G_N(\mathbf{r}, \mathbf{r}') \equiv 0 \quad \forall \, \mathbf{r}' \in S(V)$$
   leads to a contradiction. That can be seen as follows:
$$\int\limits_V d^3r' \Delta_{r'} G_N(\mathbf{r}, \mathbf{r}') = -\frac{1}{\epsilon_0} \int\limits_V d^3r' \delta(\mathbf{r} - \mathbf{r}') = -\frac{1}{\epsilon_0} , \quad \text{if } \mathbf{r} \in V .$$

When we evaluate the integral on the left by the use of the Gauss theorem,

$$\int_V d^3r' \Delta_{r'} G_N(\mathbf{r}, \mathbf{r}') = \int_{S(V)} d\mathbf{f}' \cdot \nabla_{r'} G_N(\mathbf{r}, \mathbf{r}') = \int_{S(V)} df' \frac{\partial G_N}{\partial n'} ,$$

we get by comparison

$$\int_{S(V)} df' \frac{\partial G_N}{\partial n'} = -\frac{1}{\epsilon_0} , \quad \text{if } \mathbf{r} \in V . \tag{2.119}$$

That would be in obvious contradiction to the assumption that the normal-derivative of G_N on $S(V)$ identically vanishes. In the case of Neumann-boundary conditions one therefore often chooses $f(\mathbf{r}, \mathbf{r}')$ such that

$$\frac{\partial}{\partial n'} G_N(\mathbf{r}, \mathbf{r}') = -\frac{1}{\epsilon_0 S} \quad \forall\, \mathbf{r}' \in S(V) . \tag{2.120}$$

Then the in principle irrelevant constant φ_0 in (2.118) can be interpreted as the average value of φ on the closed surface $S(V)$:

$$\varphi_0 = \frac{1}{S} \int_{S(V)} \varphi(\mathbf{r}') df' . \tag{2.121}$$

We are then left with the formal solution for the scalar potential:

$$\varphi(\mathbf{r}) - \varphi_0 = \int_V d^3r' \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') + \epsilon_0 \int_{S(V)} df' G_N(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi}{\partial n'} . \tag{2.122}$$

Since $\partial\varphi/\partial n'$ on $S(V)$ and ρ in V are known, the problem to be solved is traced back in this case, too, to the determination of a Green's function, i.e. to the determination of the potential of a point charge $q = 1$. The Green's function $G_N(\mathbf{r}, \mathbf{r}')$ must now fulfill the boundary condition (2.120) and (2.118).

Application Example

We consider a certain charge density ρ located in front of an in the xy-plane infinitely extended, conducting and grounded plate (Fig. 2.41). We search for the potential in $V =$ half space ($z \geq 0$). The boundary conditions to be fulfilled are of the Dirichlet-type:

$$\varphi(x, y, z = 0) = 0 \quad \text{(grounded metallic plate, see (2.108))} ,$$
$$\varphi(x = \pm\infty, y, z > 0) = \varphi(x, y = \pm\infty, z > 0) = \varphi(x, y, z = +\infty) = 0 .$$

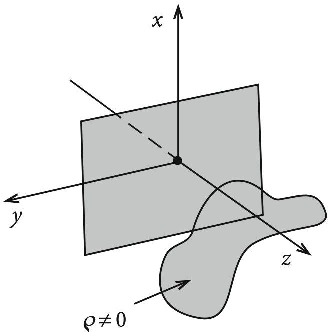


Fig. 2.41 Charge density in front of an infinitely extended, grounded metallic plate

φ is therefore identical to zero on the surface $S(V)$. According to (2.117) the task is the determination of the Green's function $G_{\mathrm{D}}(\mathbf{r},\mathbf{r}')$ for which, as a starting point, (2.112) must hold:

$$G_{\mathrm{D}}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\,\frac{1}{|\mathbf{r}-\mathbf{r}'|} + f_{\mathrm{D}}(\mathbf{r},\mathbf{r}') \, .$$

By definition we can perceive the G_{D} as the potential of a point charge at $\mathbf{r}' \in V$. Thereby the following conditions are to be fulfilled:

$$\Delta_r f_{\mathrm{D}}(\mathbf{r},\mathbf{r}') = 0 \qquad \forall\, \mathbf{r} \in V \, ,$$

$$\int\limits_{S(V)} df'\, G_{\mathrm{D}}(\mathbf{r},\mathbf{r}')\frac{\partial\varphi}{\partial n'} = 0 \, .$$

We try to realize the second condition by (2.116), i.e.

$$G_{\mathrm{D}}(\mathbf{r},\mathbf{r}') \stackrel{!}{=} 0 \quad \text{for } \mathbf{r}' \in S(V) \, .$$

We inspect first the xy-plane, which represents a part of $S(V)$. There it must hold:

$$f_{\mathrm{D}}(\mathbf{r},\mathbf{r}')\big|_{z'=0} \stackrel{!}{=} \frac{-1}{4\pi\epsilon_0\sqrt{(x-x')^2+(y-y')^2+z^2}} \, .$$

This suggests for $f_{\mathrm{D}}(\mathbf{r},\mathbf{r}')$ the following **ansatz**:

$$f_{\mathrm{D}}(\mathbf{r},\mathbf{r}') = \frac{-1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'_{\mathrm{B}}|} \, .$$

$\mathbf{r}'_B$ shall thereby originate from $\mathbf{r}'$ by reflection at the xy-plane:

$$\mathbf{r}' = (x', y', z') \implies \mathbf{r}'_B = (x', y', -z') \, . \tag{2.123}$$

That means:

$$f_D(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi\epsilon_0 \sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \, .$$

By applying the Laplace operator to the so-defined $f_D(\mathbf{r}, \mathbf{r}')$ we get:

$$\Delta_r f_D(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}'_B) = \frac{1}{\epsilon_0}\delta(x-x')\delta(y-y')\delta(z+z') = 0 \quad \forall\, \mathbf{r}, \mathbf{r}' \in V \, .$$

Therewith the first requirement on $f_D(\mathbf{r}, \mathbf{r}')$ is fulfilled. With our ansatz for f_D the full Green's function reads:

$$\begin{aligned} G_D(\mathbf{r}, \mathbf{r}') &= \frac{1}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'_B|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right. \\ &\qquad \left. - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}\right] . \end{aligned} \tag{2.124}$$

On the xy-plane ($z = 0$) the two summands within the bracket compensate each other while on the boundary surfaces of V which lie at the infinity each summand itself is already zero:

$$G_D(\mathbf{r}, \mathbf{r}') = 0 \qquad \forall\, \mathbf{r}' \in S(V), \; \mathbf{r} \in V \, .$$

Therewith all the requirements are fulfilled. We are now able with (2.124) and (2.117) to write down the full result for the scalar potential φ of the charge density ρ:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r' \rho(\mathbf{r}') \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'_B|}\right] \tag{2.125}$$

($\mathbf{r} = (x, y, z)$; $\mathbf{r}' = (x', y', z')$; $\mathbf{r}'_B = (x', y' - z')$). Note that, as expected, the Green's function $G_D(\mathbf{r}, \mathbf{r}')$ is symmetric with respect to an interchange of $\mathbf{r}$ and $\mathbf{r}'$.

One should remember that the result (2.125) has been derived for the special boundary condition $\varphi \equiv 0$ on $S(V)$. That was the reason why the second term in (2.117) vanishes. If we change the boundary conditions such that the $z = 0$-plane is no longer to be interpreted as grounded metal plate with therefore vanishing potential, but instead carrying an arbitrary $\varphi(x, y, z = 0) \neq 0$, then nothing changes for the Green's function (2.124). However, we still need for the second summand in (2.117) its normal-derivative:

$$\left.\frac{\partial G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}')}{\partial n'}\right|_S = -\left.\frac{\partial G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}')}{\partial z'}\right|_{z'=0} = -\frac{1}{4\pi\varepsilon_0} \frac{2z}{\left((x-x')^2 + (y-y')^2 + z^2\right)^{3/2}} \,.$$

The potential then reads:

$$\begin{aligned}\varphi(\mathbf{r}) &= \varphi(x, y, z) \\ &= \int_V d^3r' G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') + \frac{z}{2\pi} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \frac{\varphi(x', y', 0)}{\left((x-x')^2 + (y-y')^2 + z^2\right)^{3/2}} \,.\end{aligned}$$

2.3.4 *Method of Image Charges*

In the last section we could trace back the formal solution of the boundary-value problem completely to the determination of the Green's function,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} + f(\mathbf{r}, \mathbf{r}') \,,$$

that means to the determination of the potential of a point charge $q = 1$. The actual problem therefore concerns the specification of the function $f(\mathbf{r}, \mathbf{r}')$, which has to fulfill on $S(V)$ the conditions (2.116) and (2.120), respectively. Inside the interesting space-region V the function f must be the solution of the Laplace equation:

$$\Delta_r f(\mathbf{r}, \mathbf{r}') = 0 \quad \forall\, \mathbf{r}, \mathbf{r}' \in V \,.$$

That suggests the following **physical interpretation**:

$f(\mathbf{r}, \mathbf{r}')$: Scalar potential of a charge distribution **outside** V, that together with the potential $(4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|)^{-1}$ of the point charge $q = 1$ at $\mathbf{r}'$ realizes the given boundary conditions on $S(V)$.

The position of this *fictitious* charge distribution depends of course on the position $\mathbf{r}'$ of the *real* charge $q = 1$.

This interpretation is the starting point for the **method of image charges**: One places outside of V fictitious charges, so-called *'image charges'*, at certain spots which are determined by the geometry of the underlying problem. These image charges serve to fulfill the required boundary conditions. On the other hand, they lie outside of V and therefore do not *disturb* the Poisson equation within V (Fig. 2.42).

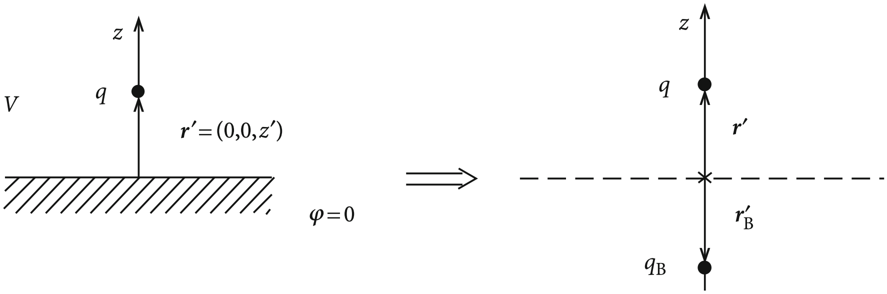


Fig. 2.42 Replacement of the boundary conditions of an electrostatic problem by introducing suitable image charges, demonstrated for the simple example of a point charge over a grounded metallic plate

$\rho(\mathbf{r}')$ **plus** boundary conditions	$\Longrightarrow$	$\rho(\mathbf{r}')$ plus image charges **without** boundary conditions

We try to become familiar with the procedure by considering some examples!

Example: *Point Charge over a Grounded, Infinitely Extended Metallic Plate*
We already discussed this problem in the last section in a slightly more general form:

$$V: \quad \text{half space } z \geq 0\ .$$

The boundary conditions are of the Dirichlet-type:

$$\varphi = 0 \quad \text{on } S(V)\ .$$

We can always choose the system of coordinates such that the point charge q lies on the z-axis. We can realize the condition $\varphi = 0$ on the xy-plane by an image charge q_{B} **outside** of V. It is plausible to assume that this image charge must also be a point charge on the z-axis (Fig. 2.42). We therefore start with the following ansatz for the potential:

$$4\pi\epsilon_0\varphi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{q_{\mathrm{B}}}{|\mathbf{r}-\mathbf{r}'_{\mathrm{B}}|}$$

($\mathbf{r}' = (0,0,z')$; $\mathbf{r}'_{\mathrm{B}} = (0,0,z'_{\mathrm{B}})$). We have to determine $q_{\mathrm{B}}, \mathbf{r}'_{\mathrm{B}}$ in such a way that $\mathbf{r}'_{\mathrm{B}} \notin V$ and

$$\varphi(\mathbf{r}) = 0 \quad \forall\, \mathbf{r} = (x,y,0)$$

This however means:

$$0 \stackrel{!}{=} \frac{q}{\sqrt{x^2 + y^2 + (-z')^2}} + \frac{q_B}{\sqrt{x^2 + y^2 + (-z'_B)^2}} .$$

From this one gets immediately:

$$\begin{aligned} q_B &= -q\,; \quad z'_B = -z' \iff \mathbf{r}'_B = -\mathbf{r}' \,, \\ \varphi(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}+\mathbf{r}'|}\right) . \end{aligned} \tag{2.126}$$

Because of

$$\Delta_r \frac{1}{|\mathbf{r}+\mathbf{r}'|} = -4\pi\,\delta(\mathbf{r}+\mathbf{r}') = 0 \quad \forall\,\mathbf{r},\mathbf{r}' \in V$$

it is a solution of the Poisson equation which fulfills the Dirichlet-boundary condition $\varphi = 0$ on $S(V)$. It is therewith unique.

Let us discuss the result with respect to physics:

1. **Electric field**

   We have to get the negative gradient of (2.126):

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{(x, y,\ z - z')}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{(x, y,\ z + z')}{|\mathbf{r}+\mathbf{r}'|^3}\right] .$$

   The surface of the metal is an equipotential area ($\varphi = 0$). The field $\mathbf{E}$ is therefore perpendicular to it (Fig. 2.43), corresponding to our general considerations (2.110):

$$\mathbf{E}(\mathbf{r};\ z = 0) = -\frac{q}{2\pi\epsilon_0}\,\frac{z'}{(x^2 + y^2 + z'^2)^{3/2}}\mathbf{e}_z . \tag{2.127}$$

2. **Induced surface charge density**

   For this it holds according to (2.110):

$$\sigma = \epsilon_0 E(\mathbf{r};\ z = 0) = -\frac{q}{2\pi}\,\frac{z'}{(x^2 + y^2 + z'^2)^{3/2}} . \tag{2.128}$$

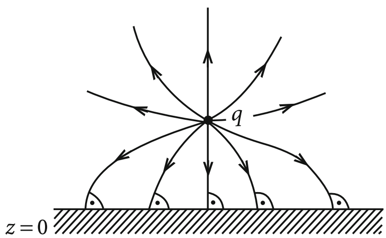


Fig. 2.43 Behavior of the electric field of a point charge in front of a grounded metallic plate

We obtain the total induced surface charge by integration over the metal surface

$$\bar{q} = \int\limits_{z=0} df\,\sigma\ ,$$

where conveniently plane polar coordinates are used:

$$\begin{aligned} df &= \rho\, d\rho\, d\varphi \\ \Longrightarrow \bar{q} &= -q \int\limits_0^\infty d\rho\, \rho \frac{z'}{(\rho^2 + z'^2)^{3/2}} \\ &= -q\, z' \int\limits_0^\infty d\rho \left(-\frac{d}{d\rho} \frac{1}{(\rho^2 + z'^2)^{1/2}} \right) = -q\ . \end{aligned} \tag{2.129}$$

The total induced surface charge is just equal to the image charge $q_\mathrm{B} = -q$.

3. **Image force**

The surface charge σ which is induced on the metal plate by the point charge q itself executes a force on the point charge.

The element $d\mathbf{f}$ of the metal surface has the direction $\mathbf{e}_z$ and carries the charge $\sigma\, df$. By q it experiences the force

$$d\bar{\mathbf{F}} = \mathbf{e}_z(\sigma\, df)\widetilde{E}(z = 0)\ .$$

$\widetilde{E}(z = 0)$ is the contribution of the point charge q exclusively to the field at $z = 0$. Since the field *below* the plate ($z < 0$) vanishes, because there the contributions due to q and due to σ compensate each other, it holds:

$$\widetilde{E}(z = 0) = \frac{1}{2} \frac{\sigma}{\epsilon_0}$$

(see the considerations about the fields $\mathbf{E}_\pm(\mathbf{r})$ in the parallel-plate capacitor in Sect. 2.2.1; the strict justification of the factor $\frac{1}{2}$ comes later: *Maxwell's stress tensor*). According to *action* $=$ *reaction* it follows then for the force $\mathbf{F}$ on the point charge:

$$\mathbf{F} = -\int\limits_{z=0} d\bar{\mathbf{F}} = -\mathbf{e}_z \frac{1}{2\epsilon_0} \int\limits_{z=0} df\,\sigma^2\ .$$

Using (2.128) we have:

$$\mathbf{F} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{(2z')^2} \mathbf{e}_z\ . \tag{2.130}$$

We see that $\mathbf{F}$ is always attractive and corresponds exactly to the Coulomb-force which the fictitious image charge q_B at $\mathbf{r}'_\mathrm{B}$ would exert on the charge q at $\mathbf{r}'$. One calls $\mathbf{F}$ the **image force**.

Example: ***Point Charge over a Grounded Metallic Sphere***

V is the space between two concentric spheres with radii R and $R' \to \infty$. We simulate the boundary condition

$$\varphi = 0 \quad \text{on } S(V)$$

by introducing an image charge q_B, which should not lie in V. It has therefore to be located within the metallic sphere (Fig. 2.44). Due to symmetry reasons we expect:

$$\mathbf{r}'_\mathrm{B} \uparrow\uparrow \mathbf{r}' \quad (r'_\mathrm{B} < R)\ .$$

This leads to the **ansatz**:

$$4\pi\epsilon_0\varphi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{q_\mathrm{B}}{|\mathbf{r}-\mathbf{r}'_\mathrm{B}|} = \frac{\frac{q}{r}}{\left|\mathbf{e}_r - \frac{r'}{r}\mathbf{e}_{r'}\right|} + \frac{q_\mathrm{B}/r'_\mathrm{B}}{\left|(r/r'_\mathrm{B})\mathbf{e}_r - \mathbf{e}_{r'}\right|}$$

$(\mathbf{e}_r \cdot \mathbf{e}_{r'} = \cos\alpha)$. We fulfill the boundary condition $\varphi(r = R) = 0$,

$$0 = \frac{q}{R}\left(1 + \frac{r'^2}{R^2} - 2\frac{r'}{R}\cos\alpha\right)^{-1/2} + \frac{q_\mathrm{B}}{r'_\mathrm{B}}\left(\frac{R^2}{r'^2_\mathrm{B}} + 1 - 2\frac{R}{r'_\mathrm{B}}\cos\alpha\right)^{-1/2}\ ,$$

by

$$\frac{q}{R} = -\frac{q_\mathrm{B}}{r'_\mathrm{B}}; \quad \frac{r'}{R} = \frac{R}{r'_\mathrm{B}}\ .$$

Therefore the solution is evident:

$$r'_\mathrm{B} = \frac{R^2}{r'} \le R; \quad q_\mathrm{B} = -q\frac{R}{r'}\ . \tag{2.131}$$

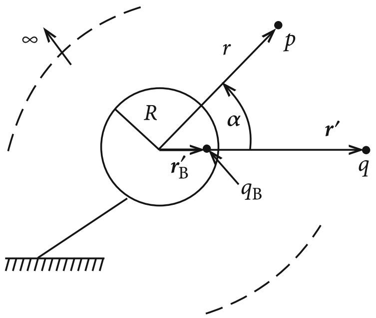


Fig. 2.44 Point charge q over a grounded metallic sphere with suitably chosen image charge q_B

The closer q approaches the surface of the sphere the larger is the magnitude of the image charge and the farther the image charge shifts away from the center of the sphere towards its surface.

The potential

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}\right) \tag{2.132}$$

solves in V the Poisson equation and fulfills on $S(V)$ the Dirichlet-boundary conditions representing therewith a unique solution of the potential problem.

We now can read off from (2.132) the

Green's function of the sphere

$$\begin{aligned} G_{\mathrm{D}}(\mathbf{r},\mathbf{r}') &= \frac{1}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\left|\frac{r'}{R}\mathbf{r} - \frac{R}{r'}\mathbf{r}'\right|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\Bigg[\left(r^2 + r'^2 - 2r\,r'\,\mathbf{e}_r\cdot\mathbf{e}_{r'}\right)^{-1/2} \\ &\qquad - \left(\frac{r^2 r'^2}{R^2} + R^2 - 2r\,r'\,\mathbf{e}_r\cdot\mathbf{e}_{r'}\right)^{-1/2}\Bigg] . \end{aligned} \tag{2.133}$$

for which it obviously holds:

$$\begin{aligned} G_{\mathrm{D}}(\mathbf{r},\mathbf{r}') &= G_{\mathrm{D}}(\mathbf{r}',\mathbf{r}) , \\ G_{\mathrm{D}}(\mathbf{r},\mathbf{r}') &= 0 \quad \forall\,\mathbf{r}' \in S(V) \quad \text{and} \quad \mathbf{r}\in V . \end{aligned} \tag{2.134}$$

Therewith we have solved automatically via our special example a big class of essentially more general potential problems. The Green's function $G_{\mathrm{D}}(\mathbf{r},\mathbf{r}')$ according to our general theory (2.117) is all that we need in order to calculate the potential $\varphi(\mathbf{r})$ of an arbitrary charge distribution $\rho(\mathbf{r}')$ above a sphere with the radius R, on the surface of which φ is arbitrary but known. So the sphere must not necessarily be grounded ($\varphi = 0$). For the complete solution we still need the normal derivative of G_{D}. Thereby it is to be kept in mind that the normal unit vector points perpendicular on $S(V)$ outwards, that means according to our choice of V into the inside of the sphere:

$$\left.\frac{\partial G_{\mathrm{D}}}{\partial n'}\right|_{S(V)} = -\left.\frac{\partial G_{\mathrm{D}}}{\partial r'}\right|_{r'=R} = -\frac{1}{4\pi\epsilon_0 R}\,\frac{r^2 - R^2}{\left(r^2 + R^2 - 2r\,R\,\mathbf{e}_r\cdot\mathbf{e}_{r'}\right)^{3/2}} .$$

If the charge density ρ is known in V and also the surface potential $\varphi(r') = \varphi(R, \vartheta', \varphi')$ on $S(V)$ then the problem is completely solved ($df' = R^2 \sin\vartheta' d\vartheta' d\varphi'$):

$$\varphi(\mathbf{r}) = \varphi(r, \vartheta, \varphi) = \int\limits_V d^3r' \rho(\mathbf{r}') G_{\mathrm{D}}(\mathbf{r}, \mathbf{r}')$$

$$+ \frac{R(r^2 - R^2)}{4\pi} \int\limits_{-1}^{+1} d\cos\vartheta' \int\limits_0^{2\pi} d\varphi' \frac{\varphi(R, \vartheta', \varphi')}{(r^2 + R^2 - 2rR\,\mathbf{e}_r \cdot \mathbf{e}_{r'})^{3/2}} \,. \tag{2.135}$$

In both integrands a possibly complicated angle-dependence appears because of

$$\mathbf{e}_r \cdot \mathbf{e}_{r'} = \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi') + \cos\vartheta \cos\vartheta'$$

Let us come back once more to our special example of the point charge q in front of the grounded metallic sphere:

1. **Surface charge density**
   It holds:

$$\sigma = +\epsilon_0 \frac{\partial\varphi}{\partial n}\bigg|_{r=R} = +\epsilon_0 \mathbf{n} \cdot \nabla\varphi|_{r=R} = -\epsilon_0 \frac{\partial\varphi}{\partial r}\bigg|_{r=R} \,.$$

   The same calculation as that above for $\partial G_{\mathrm{D}}/\partial n'$ leads to:

$$\sigma = -\frac{q}{4\pi R^2} \left(\frac{R}{r'}\right) \frac{1 - R^2/r'^2}{(1 + R^2/r'^2 - 2(R/r')\cos\alpha)^{3/2}} \,. \tag{2.136}$$

   σ is rotational-symmetric around the direction $\mathbf{e}_{r'}$ and is maximal for $\alpha = 0$ (α as defined in Fig. 2.44). The smaller the distance between the point charge and the surface of the sphere the larger is the concentration of the induced surface charge around the $\mathbf{e}_{r'}$-direction (Fig. 2.45).

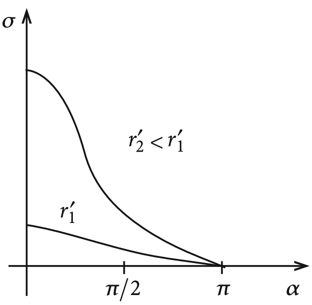


Fig. 2.45 The surface charge density σ induced by a point charge q on a grounded metallic sphere as the function of the angle α defined in Fig. 2.44

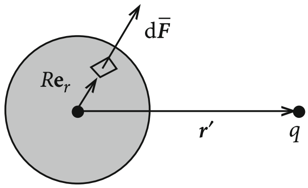


Fig. 2.46 To the calculation of the image force between a grounded metallic sphere and a point charge over the sphere

One should figure out that the total induced surface charge $\bar{q}$ amounts to:

$$\bar{q} = \int\limits_{\text{sphere}} df\,\sigma = -q\frac{R}{r'} = q_{\text{B}}\ . \tag{2.137}$$

2. **Image force**
   The metal surface is an equipotential-surface, the electric field $\mathbf{E}$ is therefore perpendicular on it. The force, which is exerted by q on the area-element, is thus oriented in the radial direction. As justified in the preceding example, it then holds for the force which acts on the area-element $d\mathbf{f}$

$$d\bar{\mathbf{F}} = \frac{\sigma^2}{2\epsilon_0}d\mathbf{f}\ .$$

σ is rotational-symmetric around the $\mathbf{e}_{r'}$-direction. When we integrate $d\bar{\mathbf{F}}$ over the entire surface of the sphere (Fig. 2.46) then the components perpendicular to $\mathbf{e}_{r'}$ are averaged out:

$$\mathbf{F} = -\int\limits_{r=R} d\bar{\mathbf{F}} = -\mathbf{e}_{r'}\frac{1}{2\epsilon_0}\int\limits_{\text{sphere}} df\,\sigma^2\cos\alpha\ .$$

After a simple calculation (see Exercise 2.3.1) it is found, as in (2.130), the familiar Coulomb-force between charge and image charge:

$$\mathbf{F} = \mathbf{e}_{r'}\frac{1}{4\pi\epsilon_0}\frac{q\,(-qR/r')}{(r' - R^2/r')^2} = \mathbf{e}_{r'}\frac{q\cdot q_{\text{B}}}{4\pi\epsilon_0\,|\mathbf{r}' - \mathbf{r}'_{\text{B}}|^2}\ . \tag{2.138}$$

It is always attractive ($q \cdot q_{\text{B}} < 0$).

2.3.5 Expansion in Orthogonal Functions

The explicit solution of a potential problem can often be found or at least simplified by an expansion of the solution function in terms of suitable systems of orthogonal functions. What we have to understand in this connection by *'suitable'* is, they are determined by the symmetry of the boundary conditions. At first let us compile a list of terms and concepts which are also of importance for other disciplines of Theoretical Physics.

$$U_n(x),\ n = 1, 2, 3\ldots: \quad \text{real or complex, square-integrable functions in the interval } [a, b]\ .$$

Two terms are decisive for the following: *orthonormality* and *completeness*.

(1) Orthonormality

is given if it holds

$$\int_a^b dx\, U_n^*(x) U_m(x) = \delta_{nm}\ . \tag{2.139}$$

(2) Completeness

This requires a little more discussion. Let

$$f(x) \text{ be a square-integrable function}\ .$$

We then define

$$f_N(x) = \sum_{n=1}^N c_n U_n(x)$$

and ask ourselves how the c_n must be chosen in order that $f_N(x)$ approximates the given function $f(x)$ *as closely as possible*. That means that we require

$$\int_a^b dx\, |f(x) - f_N(x)|^2 \stackrel{!}{=} \text{minimal}\ .$$

We reformulate

$$\int_a^b dx\, |f(x) - f_N(x)|^2 = \int_a^b dx f^*(x) f(x) - \sum_{n=1}^N c_n^* \int_a^b dx\, U_n^*(x) f(x)$$
$$- \sum_{n=1}^N c_n \int_a^b dx\, U_n(x) f^*(x) + \sum_{n=1}^N c_n^* c_n\ .$$

and build

$$0 \stackrel{!}{=} \frac{\partial}{\partial c_n} \ldots = -\int_a^b dx\, U_n(x) f^*(x) + c_n^* \,,$$

$$0 \stackrel{!}{=} \frac{\partial}{\partial c_n^*} \ldots = -\int_a^b dx\, U_n^*(x) f(x) + c_n \,.$$

The *'best'* choice for the coefficients c_n thus is:

$$c_n = \int_a^b dx\, U_n^*(x) f(x) \,. \tag{2.140}$$

Intuitively one would expect that the approximation of $f(x)$ by $f_N(x)$ becomes better and better the more terms of the system of functions $\{U_n(x)\}$ are taken into account. One speaks of

convergence in the mean

if

$$\lim_{N\to\infty} \int_a^b dx\, |f(x) - f_N(x)|^2 = 0 \,. \tag{2.141}$$

That is just the case for the so-called *'complete'* systems of functions.

Definition 2.3.2 An orthonormal system of functions $U_n(x)$, $n = 1, 2, \ldots,$ is called **complete**, if for **each** square-integrable function $f(x)$ the series $f_N(x)$ *converges in the mean* towards $f(x)$ so that it holds with the c_n from (2.140):

$$f(x) = \sum_{n=1}^{\infty} c_n U_n(x) \tag{2.142}$$

The exact proof that a certain system of functions is complete is not always a trivial task! If we insert (2.140) into (2.142),

$$f(x) = \sum_{n=1}^{\infty} \int_a^b dy\, U_n^*(y) f(y) U_n(x) \,,$$

we recognize the so-called **completeness relation**

$$\sum_{n=1}^{\infty} U_n^*(y) U_n(x) = \delta(x-y) \ . \tag{2.143}$$

Examples 1. Interval $[-x_0, x_0]$

$$U_n(x): \qquad \frac{1}{\sqrt{2x_0}} \ ; \ \frac{1}{\sqrt{x_0}} \sin\left(\frac{n\pi}{x_0}x\right), \frac{1}{\sqrt{x_0}} \cos\left(\frac{n\pi}{x_0}x\right) \tag{2.144}$$
$$(n=0) \qquad (n = 1, 2, \ldots) \ .$$

This is a complete orthonormal system, i.e. each in $[-x_0, x_0]$ square-integrable function can be expanded in this:

$$f(x) = C + \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi}{x_0}x\right) + b_n \cos\left(\frac{n\pi}{x_0}x\right) \right]$$

(Fourier series).

2. **Functions of the spherical surface**

With spherical coordinates (r, ϑ, φ) the Laplace operator can be written as follows:

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{\vartheta,\varphi} \ ,$$
$$\Delta_{\vartheta,\varphi} = \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \sin\vartheta \frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \ . \tag{2.145}$$

The eigen-functions of the operators $\Delta_{\vartheta,\varphi}$ and $i\frac{\partial}{\partial\varphi}$,

$$\Delta_{\vartheta,\varphi} Y_{lm}(\vartheta,\varphi) = -l(l+1)\, Y_{lm}(\vartheta,\varphi)$$
$$i\frac{\partial}{\partial\varphi} Y_{lm}(\vartheta,\varphi) = -m\, Y_{lm}(\vartheta,\varphi)$$

are called **spherical harmonics**:

$$Y_{lm}(\vartheta,\varphi) \ ; \quad l = 0, 1, 2, \ldots \ , \quad m = -l, -l+1, \ldots, l-1, l \ . \tag{2.146}$$

They constitute a **complete system on the unit sphere**. We list here their most important properties without intending to prove them in all detail:

(a)

$$Y_{lm}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\vartheta)\, e^{im\varphi} \; ,$$

$$Y_{l-m}(\vartheta,\varphi) = (-1)^m Y_{lm}^*(\vartheta,\varphi) \; . \tag{2.147}$$

(b) $P_l^m(z)$: **associated Legendre functions**

$$P_l^m(z) = (-1)^m (1-z^2)^{m/2} \frac{d^m}{dz^m} P_l(z) \; ,$$

$$P_l^{-m}(z) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(z) \; . \tag{2.148}$$

These are the solutions of the so-called **generalized Legendre equation:**

$$\frac{d}{dz}\left[(1-z^2)\frac{dP}{dz}\right] + \left[l(l+1) - \frac{m^2}{1-z^2}\right] P(z) = 0 \; . \tag{2.149}$$

(c) $P_l(z)$: **Legendre polynomials**

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l}(z^2-1)^l \; . \tag{2.150}$$

They are solutions of the so-called **ordinary Legendre equation:**

$$\frac{d}{dz}\left[(1-z^2)\frac{dP}{dz}\right] + l(l+1)P(z) = 0 \; . \tag{2.151}$$

They build a complete orthogonal system in the interval $[-1,+1]$. However, they are **not** normalized to 1; moreover it holds:

$$P_l(\pm 1) = (\pm 1)^l \; . \tag{2.152}$$

(d) **Orthogonality relations:**

$$\int_{-1}^{+1} dz\, P_l(z) P_k(z) = \frac{2}{2l+1}\delta_{lk} \; , \tag{2.153}$$

$$\int_{-1}^{+1} dz\, P_l^m(z) P_k^m(z) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{lk} \; , \tag{2.154}$$

$$\int_0^{2\pi} d\varphi\, e^{i(m-m')\varphi} = 2\pi\, \delta_{m\,m'} \ , \tag{2.155}$$

$$\int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\vartheta\ Y^*_{l'm'}(\vartheta,\varphi) Y_{l\,m}(\vartheta,\varphi) = \delta_{l\,l'}\delta_{m\,m'} \ . \tag{2.156}$$

(e) **Completeness relations:**

$$\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)P_l(z')P_l(z) = \delta(z-z') \ , \tag{2.157}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y^*_{l\,m}(\vartheta',\varphi')Y_{l\,m}(\vartheta,\varphi) = \\ = \delta(\varphi-\varphi')\delta(\cos\vartheta - \cos\vartheta') \ . \tag{2.158}$$

(f) **Expansion theorem:**

$$f(\mathbf{r}) = f(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} R_{l\,m}(r)Y_{l\,m}(\vartheta,\varphi) \ , \tag{2.159}$$

$$R_{l\,m}(r) = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\vartheta\, f(r,\vartheta,\varphi)Y^*_{l\,m}(\vartheta,\varphi) \ . \tag{2.160}$$

(g) **Addition theorem:**

$$\sum_{m=-l}^{+l} Y^*_{l\,m}(\vartheta',\varphi')Y_{l\,m}(\vartheta,\varphi) = \frac{2l+1}{4\pi}P_l(\cos\gamma) \tag{2.161}$$

$(\gamma = \sphericalangle(\vartheta'\varphi',\ \vartheta\varphi))$.

(h) **Special functions:**

$$P_0(z) = 1 \ ,$$

$$P_1(z) = z \ ,$$

$$P_2(z) = \frac{1}{2}(3z^2-1) \ ,$$

$$P_3(z) = \frac{1}{2}(5z^3-3z) \ , \dots \ ;$$

$$
\begin{aligned}
Y_{00} &= \frac{1}{\sqrt{4\pi}} \;, \\
Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin\vartheta \, e^{i\varphi} \;, \\
Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos\vartheta \;, \\
Y_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta \, e^{i\,2\varphi} \\
Y_{21} &= -\sqrt{\frac{15}{8\pi}} \sin\vartheta \, \cos\vartheta \, e^{i\varphi} \;, \\
Y_{20} &= \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\vartheta - \frac{1}{2} \right) \;, \ldots
\end{aligned}
$$

Use (2.147) for $Y_{1,-1}$; $Y_{2,-2}$; $Y_{2,-1}$; ...

2.3.6 *Separation of Variables*

We are looking for further solution methods for the Poisson equation,

$$
\Delta\,\varphi(\mathbf{r}) = -\frac{1}{\epsilon_0}\rho(\mathbf{r}) \;,
$$

which represents a linear, partial, inhomogeneous differential equation of second order in a region, on the boundary of which certain conditions are prescribed. The **method of separation** appears to be conceptually rather simple. It consists essentially only of a special **solution ansatz:**

$\varphi(\mathbf{r})$ is written as a combination (e.g. a product) of functions, each of which depends only on ***one*** *independent coordinate (variable), for instance: $\varphi(\mathbf{r}) = f(x)g(y)h(z)$.*

One tries to achieve therewith that the partial differential equation reduces to several ordinary ones which can be usually solved more easily. We demonstrate this procedure by two examples:

(1) Laplace Equation with Boundary Conditions

We discuss the **two-dimensional** problem sketched in Fig. 2.47:

$$
\Delta\,\varphi = 0 \quad \text{in } V \;.
$$

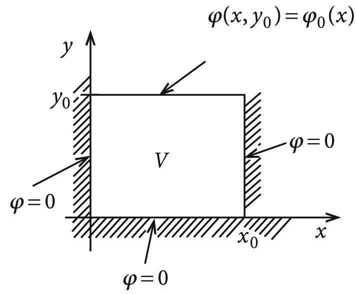


Fig. 2.47 Two-dimensional electrostatic problem with boundary conditions for points within a charge-free space V

We try to find the potential $\varphi(\mathbf{r})$ for all $\mathbf{r} \in V$ under the boundary conditions on $S(V)$, which can be read off from Fig. 2.47 being all of Dirichlet-type. It is obviously convenient to choose Cartesian coordinates,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} ,$$

as well as the **separation ansatz**:

$$\varphi(x, y) = f(x)g(y) .$$

When we insert this ansatz into the Laplace equation and then divide the resulting expression by φ then we get:

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = 0 .$$

Since the first summand depends only on x and the second exclusively on y, both terms on their own must already be constant:

$$\frac{1}{g}\frac{d^2 g}{dy^2} = \alpha^2 = -\frac{1}{f}\frac{d^2 f}{dx^2} .$$

Therewith we already know the *structure* of the solution:

$$\begin{aligned} g(y) &: \; a\cosh(\alpha y) + b\sinh(\alpha y) , \\ f(x) &: \; \bar{a}\cos(\alpha x) + \bar{b}\sin(\alpha x) . \end{aligned}$$

We have to fulfill the boundary conditions:

$$\begin{aligned} \varphi(0, y) \equiv 0 \quad &\Longrightarrow \quad \bar{a} = 0 , \\ \varphi(x, 0) \equiv 0 \quad &\Longrightarrow \quad a = 0 , \\ \varphi(x_0, y) \equiv 0 \quad &\Longrightarrow \quad \alpha \to \alpha_n = \frac{n\pi}{x_0} ; \quad n \in \mathbb{N} . \end{aligned}$$

A special solution which fulfills all these three conditions would then be:

$$\varphi_n(x, y) = \sinh(\alpha_n y)\sin(\alpha_n x) \ .$$

The **general solution** thus looks like:

$$\varphi(x, y) = \sum_n c_n \sinh\left(\frac{n\pi}{x_0} y\right)\sin\left(\frac{n\pi}{x_0} x\right) \ .$$

We fix the coefficients c_n by the not yet used fourth boundary condition:

$$\varphi_0(x) = \sum_n c_n \sinh\left(\frac{n\pi}{x_0} y_0\right)\sin\left(\frac{n\pi}{x_0} x\right) \ .$$

We multiply this equation with sin $(m\pi x/x_0)$, integrate from 0 to x_0, and exploit the orthonormality relation (2.139) of the complete system of equations (2.144):

$$\begin{aligned}
&\sum_n c_n \sinh\left(\frac{n\pi}{x_0} y_0\right)\int\limits_0^{x_0} dx \sin\left(\frac{n\pi}{x_0} x\right)\sin\left(\frac{m\pi}{x_0} x\right) \\
&\quad = \sum_n c_n \sinh\left(\frac{n\pi}{x_0} y_0\right)\frac{x_0}{2}\delta_{nm} = \frac{x_0}{2} c_m \sinh\left(\frac{m\pi}{x_0} y_0\right) \ .
\end{aligned}$$

This leads to

$$c_m = \frac{2}{x_0 \sinh\left(\frac{m\pi}{x_0} y_0\right)}\int\limits_0^{x_0} dx\, \varphi_0(x)\sin\left(\frac{m\pi}{x_0} x\right) \ ,$$

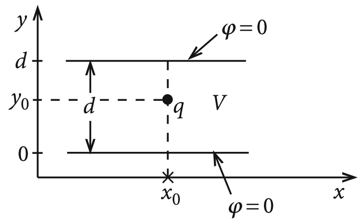
so that the problem is completely solved.

(2) Poisson Equation with Boundary Conditions

The formal solutions (2.117) and (2.122) of the boundary-value problem are completely determined by the corresponding Green's function. We therefore can limit the discussion to the case of point charges. The following example will show how the solution of the Poisson equation can be traced back to that of a corresponding Laplace equation.

Let there be two infinitely extended, parallel, grounded metallic plates between which a homogeneously charged wire is located, parallel to the plates and with the distance y_0 from the lower plate (Fig. 2.48). We are interested in the potential everywhere between the plates.

Fig. 2.48 Point charge between two parallel grounded metal plates as an example of a Dirichlet-boundary value problem



The problem is independent of the z-coordinate. We therefore can consider it to be an actual two-dimensional problem so that the homogeneously charged wire becomes a point charge:

$$V = \{\mathbf{r} = (x, y);\ x \text{ arbitrary};\ 0 \leq y \leq d\} ,$$
$$\rho(\mathbf{r}) = q\,\delta(\mathbf{r} - \mathbf{r}_0); \quad \mathbf{r}_0 = (x_0, y_0) .$$

Boundary conditions:
$\varphi = 0$ on the plates and for $x \to \pm\infty$. Thus it is about a **Dirichlet-boundary problem**.

But let us slightly reformulate the problem. We decompose the interesting space-region V into two partial volumes V_+ and V_-,

$$V_+ = V(x > x_0) ; \quad V_- = V(x < x_0) ,$$

and solve the Laplace equation in each of the partial volumes $V_\pm$, where the point charge at $\mathbf{r}_0$ can formally be interpreted as surface charge:

$$\Delta\varphi = 0 \quad \text{in } V_-, V_+ .$$

Boundary conditions:

$$\text{(a)}\ \ \varphi \underset{x\to\pm\infty}{\longrightarrow} 0 ,$$
$$\text{(b)}\ \ \varphi(x, y = 0) = 0 ,$$
$$\text{(c)}\ \ \varphi(x, y = d) = 0 ,$$
$$\text{(d)}\ \ \sigma(x_0, y) = q\,\delta(y - y_0) = -\epsilon_0 \left(\frac{\partial\varphi_+}{\partial x} - \frac{\partial\varphi_-}{\partial x}\right)\bigg|_{x=x_0} .$$

The last condition is due to the interface-behavior (2.43) of the normal component of the electric field:

$$\sigma = \varepsilon_0 \mathbf{n} \cdot (\mathbf{E}_a - \mathbf{E}_i) .$$

One notes that it must be chosen for V_+

$$\mathbf{n} = -\mathbf{e}_x \; ; \quad \mathbf{E}_\mathrm{a} = -\nabla\varphi_-|_{x_0} \; ; \quad \mathbf{E}_\mathrm{i} = -\nabla\varphi_+|_{x_0}$$

and for V_-

$$\mathbf{n} = \mathbf{e}_x \; ; \quad \mathbf{E}_\mathrm{a} = -\nabla\varphi_+|_{x_0} \; ; \quad \mathbf{E}_\mathrm{i} = -\nabla\varphi_-|_{x_0} \; .$$

In both the cases one finds the same boundary condition (d).

Furthermore, φ must be continuous at $x = x_0$ ($y \neq y_0$). Hence we now have to realize mixed boundary conditions. (a) to (c) are of Dirichlet-type, (d) of Neumann-type.

We start with a **separation-ansatz:**

$$\varphi(x, y) = f(x)g(y) \; .$$

The Laplace equation

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\frac{1}{g}\frac{d^2 g}{dy^2} = \beta^2$$

has the special solution:

$$\begin{aligned} f(x) &= a\, e^{\beta x} + b\, e^{-\beta x} \; , \\ g(y) &= \bar{a}\cos(\beta y) + \bar{b}\sin(\beta y) \; ; \quad \beta > 0 \; . \end{aligned}$$

We now fit the boundary conditions.

Boundary condition (b) yields:

$$\bar{a} = 0 \; .$$

From condition (c) it follows:

$$\beta \to \beta_n = \frac{n\pi}{d} \; ; \quad n \in \mathbb{N} \; .$$

Boundary condition (a) leads to:

$$\varphi_\pm = \sum_{n=1}^{\infty} A_n^{(\pm)} e^{\mp\frac{n\pi}{d}x} \sin\left(\frac{n\pi}{d}y\right) \; .$$

The continuity at $x = x_0$ requires:

$$a_n \equiv A_n^{(+)} e^{-\frac{n\pi}{d}x_0} = A_n^{(-)} e^{+\frac{n\pi}{d}x_0} \quad \forall\, n \; .$$

Therewith we have the following intermediate result:

$$\varphi(x, y) = \sum_{n=1}^{\infty} a_n e^{-\frac{n\pi}{d}|x-x_0|} \sin\left(\frac{n\pi}{d} y\right) .$$

The coefficients a_n we derive from the not yet used boundary condition (d):

$$\sigma(x_0, y) = -\epsilon_0 \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{d} y\right)\left(-\frac{n\pi}{d} - \frac{n\pi}{d}\right) .$$

We exploit once more the orthonormality relation:

$$\frac{2\pi\,\epsilon_0}{d} a_m\, m = \frac{2}{d}\int_0^d dy\, \sigma(x_0, y) \sin\left(\frac{m\pi}{d} y\right) = \frac{2q}{d}\int_0^d dy\, \delta(y - y_0) \sin\left(\frac{m\pi}{d} y\right) .$$

This leads to

$$a_m = \frac{q}{\pi\,\epsilon_0}\,\frac{\sin\left(\frac{m\pi}{d} y_0\right)}{m}$$

and therewith to the complete solution for the potential:

$$\varphi(x, y) = \frac{q}{\pi\,\epsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{d} y_0\right)\sin\left(\frac{n\pi}{d} y\right) e^{-\frac{n\pi}{d}|x-x_0|} .$$

For not too small $|x - x_0|$ one can restrict oneself, because of the exponential function, to the first few summands (Fig. 2.49).

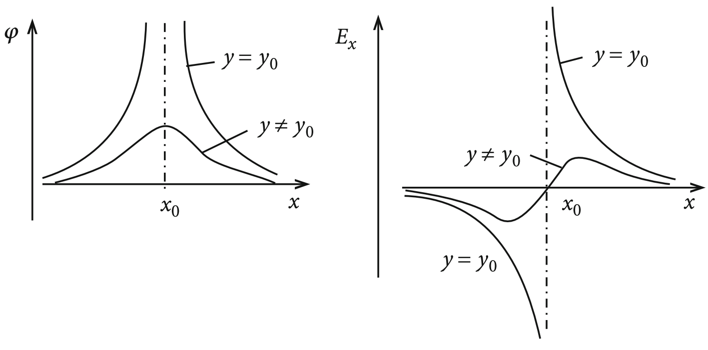


Fig. 2.49 Qualitative solution of the boundary-value problem of Fig. 2.48 for the scalar potential φ and the x-component of the electric field strength E_x

2.3.7 Solution of the Laplace Equation in Spherical Coordinates

Boundary conditions are often to be fulfilled on surfaces which exhibit a special symmetry. Then for the description one should use the corresponding coordinates and should expand the potential in functions which fit to these coordinates. Let us try to find in this section, as an important example, the general solution of the Laplace equation

$$\Delta\Phi(r, \vartheta, \varphi) = 0$$

in spherical coordinates. The suitable complete system of coordinates are here the spherical harmonics (2.146). We use the **expansion theorem** (2.159) in order to express the potential Φ by these functions:

$$\Phi(r, \vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} R_{lm}(r) Y_{lm}(\vartheta, \varphi) \ . \tag{2.162}$$

We apply the Laplace operator (2.145):

$$\begin{aligned} 0 = \Delta\Phi &= \sum_{l,m} \left\{ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R}{r^2} \Delta_{\vartheta,\varphi} \right\} Y_{lm}(\vartheta, \varphi) \\ &= \sum_{l,m} \left\{ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \frac{l(l+1)}{r^2} R \right\} Y_{lm}(\vartheta, \varphi) \ . \end{aligned}$$

Because of the orthonormality of the spherical harmonics each summand itself must already be zero. This leads to the so-called **radial equation**:

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dR}{dr} \right] - \frac{l(l+1)}{r^2} R = 0 \ . \tag{2.163}$$

We solve it with the ansatz ($r \neq 0$):

$$R(r) = \frac{1}{r} u(r) \ .$$

Inserted into (2.163) we have therewith

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) u(r) = 0 \ .$$

This equation has the solution

$$u(r) = A\, r^{l+1} + B\, r^{-l}\ .$$

According to (2.162) the potential Φ has then the general form:

$$\Phi(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(A_{lm}r^{l} + B_{lm}r^{-(l+1)}\right) Y_{lm}(\vartheta,\varphi)\ . \tag{2.164}$$

The coefficients must be fixed with respect to the current physical boundary conditions. An oftentimes given special case is that of **azimuthal symmetry** of the boundary conditions. Then the solution of the Laplace equation must exhibit the same symmetry, i.e. must be φ-independent. According to (2.147) that is fulfilled only by the $m = 0$-spherical harmonics. Then (2.164) changes with (2.147) to :

$$\Phi(r,\vartheta) = \sum_{l=0}^{\infty}(2l+1)\left[A_l r^{l} + B_l r^{-(l+1)}\right] P_l(\cos\vartheta)\ . \tag{2.165}$$

For many boundary-value problems of the electrostatics the expressions (2.164) and (2.165) represent extremely useful starting points.

Example: ***Potential of a Sphere with Azimuthal-Symmetric Surface Charge Density***

In case of azimuthal symmetry the Legendre polynomials $P_l(\cos\vartheta)$ (2.150) represent a suitable complete orthogonal system on the sphere. It is therefore recommendable to expand the given surface density $\sigma(\vartheta)$ also in these functions:

$$\sigma(\vartheta) = \sum_{l=0}^{\infty}(2l+1)\sigma_l P_l(\cos\vartheta)\ . \tag{2.166}$$

The factor $(2l + 1)$ is, as in (2.165), without any special meaning, it is there only for utility reasons. $\sigma(\vartheta)$ is given and therefore is known. With the aid of the orthogonality relation (2.153) for the Legendre polynomials, we can derive all the coefficients σ_l from $\sigma(\vartheta)$:

$$\sigma_l = \frac{1}{2}\int_{-1}^{+1} d\cos\vartheta\, \sigma(\vartheta) P_l(\cos\vartheta)\ . \tag{2.167}$$

For the scalar potential $\Phi(r,\vartheta,\varphi)$ we start with (2.165). Let us decompose Φ:

$\Phi_{\mathrm{i}}(\mathbf{r})$: potential inside the sphere,

$\Phi_{\mathrm{a}}(\mathbf{r})$: potential outside the sphere.

The following conditions are to be fulfilled:

1. Φ_i is regular at $r = 0$:

$$\Longrightarrow \quad \Phi_\mathrm{i}(r, \vartheta) = \sum_{l=0}^{\infty} (2l+1) A_l^{(\mathrm{i})}\, r^l P_l(\cos\vartheta) \ ,$$

2. $\Phi_\mathrm{a} \to 0$ for $r \to \infty$:

$$\Longrightarrow \quad \Phi_\mathrm{a}(r, \vartheta) = \sum_{l=0}^{\infty} (2l+1) B_l^{(\mathrm{a})}\, r^{-(l+1)} P_l(\cos\vartheta) \ ,$$

3. Φ is continuous at the surface of the sphere:

$$\Longrightarrow \quad \Phi_\mathrm{i}(r = R, \vartheta) = \Phi_\mathrm{a}(r = R, \vartheta) \quad \Longrightarrow \quad B_l^{(\mathrm{a})} = A_l^{(\mathrm{i})} R^{2l+1} \ ,$$

4. The surface of the sphere has the charge density $\sigma(\vartheta)$. That means according to (2.43):

$$\begin{aligned} \sigma(\vartheta) &= -\epsilon_0 \left(\frac{\partial \Phi_\mathrm{a}}{\partial r} - \frac{\partial \Phi_\mathrm{i}}{\partial r} \right)\Bigg|_{r=R} \\ &= -\epsilon_0 \sum_{l=0}^{\infty} (2l+1) P_l(\cos\vartheta) \left[-(l+1) B_l^{(\mathrm{a})} R^{-l-2} - l A_l^{(\mathrm{i})} R^{l-1} \right] \ . \end{aligned}$$

   It follows then:

$$\sigma(\vartheta) = \epsilon_0 \sum_{l=0}^{\infty} (2l+1)^2 A_l^{(\mathrm{i})} R^{l-1} P_l(\cos\vartheta) \ .$$

   The comparison with (2.166) yields, since the P_l represent an orthogonal system:

$$\sigma_l = \epsilon_0 (2l+1) A_l^{(\mathrm{i})} R^{l-1} \ .$$

   Therewith the A_l are all fixed so that we can formulate the complete solution:

$$\begin{aligned} \Phi_\mathrm{i}(r, \vartheta) &= \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \sigma_l \left(\frac{r}{R} \right)^l P_l(\cos\vartheta) \ , \\ \Phi_\mathrm{a}(r, \vartheta) &= \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \sigma_l \left(\frac{R}{r} \right)^{l+1} P_l(\cos\vartheta) \ . \end{aligned} \tag{2.168}$$

2.3.8 Potential of a Point Charge, Spherical Multipole Moments

In Sect. 2.2.7 we got the multipole expansion of the electrostatic potential $\Phi(\mathbf{r})$, for the case that there are no boundary conditions regarding the finiteness, by a Taylor expansion of the term $1/|\mathbf{r}-\mathbf{r}'|$ in the integrand of the Poisson integral. There does exist an alternative multipole expansion when one expands this term in spherical harmonics.

Let us discuss this expansion at first under a somewhat more general aspect, namely in connection with the potential of a point charge q at the site $\mathbf{r}_0$. We imagine a sphere whose center is at the origin of coordinates and is of the radius r_0 (Fig. 2.50):

$$\Phi_>(\mathbf{r}): \text{ potential for } r > r_0 \ ,$$

$$\Phi_<(\mathbf{r}): \text{ potential for } r < r_0 \ .$$

We take the general form (2.164) for the solution of the Laplace equation inside and outside the sphere and determine the coefficients A_{lm} and B_{lm} by regarding the point charge q as surface charge on the virtual sphere:

$$\sigma(r_0, \vartheta, \varphi) = \frac{q}{r_0^2}\delta(\varphi - \varphi_0)\,\delta(\cos\vartheta - \cos\vartheta_0) \ .$$

Boundary conditions:

(1) Φ regular at $r = 0$,
(2) $\Phi \to 0$ for $r \to \infty$,
(3) Φ continuous at $r = r_0$ for $(\vartheta, \varphi) \neq (\vartheta_0, \varphi_0)$,
(4) $\sigma = -\epsilon_0 \left(\frac{\partial\Phi_>}{\partial r} - \frac{\partial\Phi_<}{\partial r}\right)_{r=r_0}$.

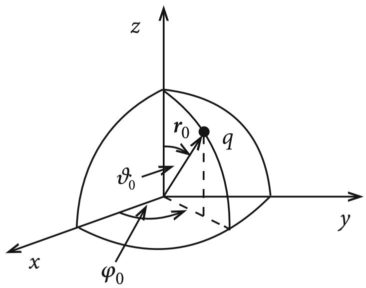


Fig. 2.50 To the representation of the potential of a point charge by spherical coordinates

These boundary conditions must be consistent with (2.164):

$$\text{from (1)} \implies \quad \Phi_< = \sum_{l,m} A_{lm} r^l Y_{lm}(\vartheta, \varphi) \ ,$$

$$\text{from (2)} \implies \quad \Phi_> = \sum_{l,m} B_{lm} r^{-(l+1)} Y_{lm}(\vartheta, \varphi) \ ,$$

$$\text{from (3)} \implies \quad A_{lm} r_0^l = B_{lm} r_0^{-(l+1)} = \frac{1}{r_0} a_{lm} \ .$$

We still introduce the following notation:

$$\text{inside } (r < r_0)\text{: } r = r_< \ , r_0 = r_> \ ,$$

$$\text{outside } (r > r_0)\text{: } r = r_> \ , r_0 = r_< \ .$$

Therewith we get the intermediate result:

$$\Phi(\mathbf{r}) = \frac{1}{r_>} \sum_{l,m} a_{lm} \left(\frac{r_<}{r_>} \right)^l Y_{lm}(\vartheta, \varphi) \ .$$

The coefficients are fixed by the fourth boundary condition. To see this we first exploit the completeness relation (2.158):

$$\begin{aligned}
\sigma(r_0, \vartheta, \varphi) &= \frac{q}{r_0^2} \sum_{l,m} Y_{lm}^*(\vartheta_0, \varphi_0) Y_{lm}(\vartheta, \varphi) = -\epsilon_0 \left(\frac{\partial \Phi_>}{\partial r_>} - \frac{\partial \Phi_<}{\partial r_<} \right)_{r_> = r_< = r_0} \\
&= -\epsilon_0 \sum_{l,m} a_{lm} Y_{lm}(\vartheta, \varphi) \left[-(l+1) \frac{r_<^l}{r_>^{l+2}} - l \frac{r_<^{l-1}}{r_>^{l+1}} \right]_{r_> = r_< = r_0} \\
&= \frac{\epsilon_0}{r_0^2} \sum_{l,m} a_{lm} (2l+1) Y_{lm}(\vartheta, \varphi) \ .
\end{aligned}$$

The comparison of the first with the last line yields:

$$\epsilon_0 a_{lm} (2l+1) = q \, Y_{lm}^*(\vartheta_0, \varphi_0) \ .$$

Therewith we have found the potential of the point charge:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_0|} = \frac{q}{\epsilon_0 r_>} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{1}{2l+1} \left(\frac{r_<}{r_>} \right)^l Y_{lm}^*(\vartheta_0, \varphi_0) Y_{lm}(\vartheta, \varphi) \ . \tag{2.169}$$

The useful feature of this representation lies in the complete factorization of the two sets of coordinates (r, ϑ, φ) and $(r_0, \vartheta_0, \varphi_0)$. This can be of great value when, for instance, the one set contains the integration variables while the other set represents the coordinates of a fixed point under consideration.

We can now repeat the same considerations once more for the special situation that the point charge lies on the z-axis. Then we have azimuthal symmetry and therefore can start from the representation (2.165) for Φ. The above boundary condition (4) then reads

$$\sigma(r_0, \vartheta) = \frac{q}{2\pi r_0^2}\delta(\cos\vartheta - 1) \overset{(2.157)}{=} \frac{q}{4\pi r_0^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta) \ .$$

By a completely analogous calculation one finds:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\vartheta) \ . \tag{2.170}$$

Since the axes can always be chosen such that q lies on the z-axis, the two relations (2.169) and (2.170) must of course be completely equivalent. If one replaces in (2.170) ϑ by

$$\gamma = \sphericalangle(\mathbf{r}, \mathbf{r}_0) \ ,$$

then the comparison leads to the important **addition theorem for spherical harmonics** (2.161):

$$\frac{1}{4\pi}P_l(\cos\gamma) = \frac{1}{2l+1}\sum_{m=-l}^{+l} Y_{lm}^*(\vartheta_0, \varphi_0)Y_{lm}(\vartheta, \varphi) \ . \tag{2.171}$$

We now come to the multipole expansion mentioned at the start. If we take $q = 1$ in (2.169) and multiply the expression by $4\pi\epsilon_0$, then we have the expansion of $|\mathbf{r} - \mathbf{r}_0|^{-1}$ in spherical harmonics that we need for the Poisson integral

$$4\pi\epsilon_0\Phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

of a spatially limited charge distribution. We observe the electric field and the scalar potential Φ, respectively, far outside the charge-region $\rho \neq 0$. It is therefore to be inserted into (2.169):

$$r \gg r' \iff r' = r_<, \ r = r_>$$

We obtain:

$$4\pi\epsilon_0\Phi(\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\vartheta, \varphi) \tag{2.172}$$

with the **spherical multipole moments**

$$q_{lm} = \int d^3r' \, \rho(\mathbf{r}') r'^l Y^*_{lm}(\vartheta', \varphi') \,, \tag{2.173}$$

for which it obviously holds because of (2.147):

$$q_{l,-m} = (-1)^m q^*_{lm} \,. \tag{2.174}$$

(2.172) is equivalent to (2.94). The multipole moments are, however, defined somewhat differently:

(1) Monopole ($l = 0$)

It follows with $Y_{00} = (4\pi)^{-1/2}$:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int d^3r' \, \rho(\mathbf{r}') = \frac{q}{\sqrt{4\pi}} \,. \tag{2.175}$$

Except for the unessential factor $1/\sqrt{4\pi}$ this agrees with (2.91).

(2) Dipole ($l = 1$)

Via the spherical harmonics

$$\begin{aligned}
Y_{10}(\vartheta, \varphi) &= \sqrt{\frac{3}{4\pi}} \cos\vartheta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \,, \\
Y_{11}(\vartheta, \varphi) &= -\sqrt{\frac{3}{8\pi}} \sin\vartheta \, e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x + i y}{r} \,, \\
Y_{1-1}(\vartheta, \varphi) &= -Y^*_{11}(\vartheta, \varphi)
\end{aligned}$$

the following connection appears between the spherical and the Cartesian dipole moments (2.92):

$$
\begin{aligned}
q_{10} &= \sqrt{\frac{3}{4\pi}} p_z \,, \\
q_{11} &= -\sqrt{\frac{3}{8\pi}} (p_x + i\, p_y) = -q_{1-1}^* \,.
\end{aligned}
\tag{2.176}
$$

(3) Quadrupole ($l = 2$)

With the spherical harmonics

$$
\begin{aligned}
Y_{20}(\vartheta, \varphi) &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\vartheta - 1) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,\frac{3z^2 - r^2}{r^2} \,, \\
Y_{21}(\vartheta, \varphi) &= -\sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta\, e^{i\varphi} = -\sqrt{\frac{15}{8\pi}}\,\frac{z}{r^2}(x + i\, y) \,, \\
Y_{22}(\vartheta, \varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\vartheta\; e^{i\,2\varphi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\frac{(x + i\, y)^2}{r^2}
\end{aligned}
$$

one gets the following five independent components of the quadrupole tensor (Q_{ij} from (2.93)):

$$
\begin{aligned}
q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33} \,, \\
q_{21} &= -\frac{1}{3}\sqrt{\frac{15}{8\pi}} (Q_{31} - i\, Q_{32}) = -q_{2-1}^* \,, \\
q_{22} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - Q_{22} - 2i\, Q_{12}) = q_{2-2}^* \,.
\end{aligned}
\tag{2.177}
$$

2.3.9 Exercises

Exercise 2.3.1 Calculate the image force $\mathbf{F}$ between a grounded metallic sphere and a point charge q above the sphere. Verify Eq. (2.138)!

Exercise 2.3.2 Let a point charge q be inside a grounded metallic hollow sphere. Calculate the potential $\varphi(\mathbf{r})$ within the sphere and the surface charge density induced on the inner side of the hollow sphere. How large is the total induced charge?

Exercise 2.3.3 A point charge q is at the site $\mathbf{r}'$ above an insulated metallic sphere which carries the total charge Q. R is the radius of the sphere. Calculate the potential $\varphi(\mathbf{r})$ outside the sphere and discuss the force $\mathbf{F}$ on the point charge.

Exercise 2.3.4

1. Calculate the Green's function for a two-dimensional potential problem **without** boundary conditions regarding finiteness.
   Hint: Use plane polar coordinates (ρ, φ) with the Laplace operator:

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} \; .$$

   Solve then for $\rho \neq 0$ the Laplace equation

$$\Delta\, G(\rho, \varphi) = \Delta\, G(\rho) = 0$$

   and show by use of the (two-dimensional) Gauss theorem that it holds:

$$G(\rho) = -\frac{1}{2\pi\epsilon_0} \ln c\, \rho$$

2. Calculate the potential of a point charge q at $\mathbf{r}_0 = (x_0, y_0)$ for the two-dimensional boundary-value problem sketched in Fig. 2.51. Use thereto the method of the image charges.

Exercise 2.3.5 Solve by separation of variables the two-dimensional boundary-value problem plotted in Fig. 2.52. Let the region G be charge-free. On the two legs of the angle α $\Phi = 0$ and on the circular arc $\Phi = \Phi_0(\varphi)$. Calculate the potential $\Phi(\mathbf{r}) = \Phi(\rho, \varphi)$ inside G.

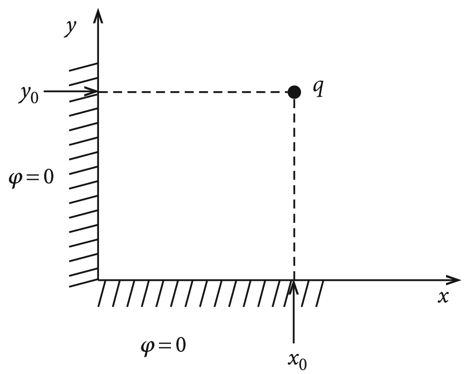


Fig. 2.51 Two-dimensional electrostatic boundary-value problem for the scalar potential due to a point charge

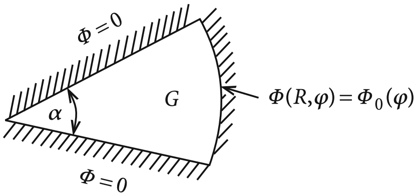


Fig. 2.52 Two-dimensional electrostatic boundary-value problem for a charge-free region G

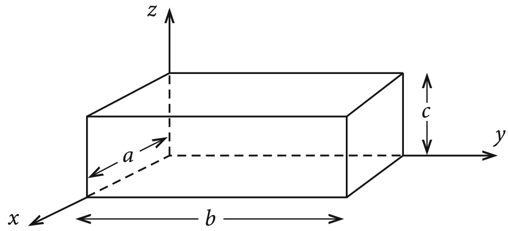


Fig. 2.53 For the calculation of the potential in a charge-free rectangular box

Exercise 2.3.6 On the surface of a sphere with the radius R one finds the surface charge density

$$\sigma(\vartheta) = \sigma_0(3\cos^2\vartheta - 1) \ .$$

Calculate the potential inside and outside the sphere.

Exercise 2.3.7 Given a rectangular box (cuboid) with edge lengths a, b, c in x, y, z-direction (Fig. 2.53). Calculate the potential $\varphi(x, y, z)$ in the inside of the charge-free box obeying the following boundary conditions:

$$\begin{aligned}
\varphi(0, y, z) &= \varphi(a, y, z) = 0 \\
\varphi(x, 0, z) &= \varphi(x, b, z) = 0 \\
\varphi(x, y, 0) &= \varphi(x, y, c) = \varphi_0 \ .
\end{aligned}$$

Exercise 2.3.8 Try to solve the (ordinary) Legendre equation (2.151) by the power-series ansatz

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

1. Derive a recursion formula for the coefficients a_n. Show by it that the two linearly independent solutions consist of a polynomial of l-th degree (Legendre polynomial) and a not terminated power series (Legendre function of the second kind).

2. Calculate with the condition

$$P_l(1) = 1$$

the Legendre polynomials $P_4(z)$ and $P_5(z)$!

Exercise 2.3.9

1. Assume that the for the solution of an electrostatic problem interesting space-region V is charge-free. The electrostatic potential $\Phi(r, \vartheta, \varphi)$ therefore fulfills the Laplace equation

$$\Delta\,\Phi = 0\ .$$

The given boundary conditions possess azimuthal symmetry which is transferred to the potential Φ:

$$\Phi = \Phi(r, \vartheta)\ .$$

Show that in such a case the solution can be expanded in Legendre polynomials $P_l(\cos\vartheta)$ as follows:

$$\Phi(r, \vartheta) = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \beta_l r^{-(l+1)}\right)\ P_l(\cos\vartheta)\ .$$

2. With the result of part 1., show that in the inside of a metallic grounded hollow sphere $(\Phi(r = R, \vartheta) = 0)$ one always has:

$$\Phi_{\mathrm{i}} = 0$$

3. Let the grounded metal hollow sphere from part 2. be in an electric field that induces on the sphere the surface charge density:

$$\sigma = \varepsilon_0\sigma_0\cos\vartheta$$

Determine the potential Φ_{a} in the exterior space!

Exercise 2.3.10

1. Consider a hollow sphere with azimuthal-symmetric surface charge density $\sigma(\theta)$. This can surely be expanded in Legendre polynomials:

$$\sigma(\vartheta) = \sum_{l=0}^{\infty} \sigma_l P_l(\cos\vartheta)\ .$$

Calculate the potential inside and outside the sphere!

2. Find the result for the potential in case of the special surface charge density:

$$\sigma(\vartheta) = \sigma_0(2\cos^2\vartheta + \cos\vartheta - \sin^2\vartheta)$$

Exercise 2.3.11 A grounded metallic hollow sphere is situated in a homogeneous electric field

$$\mathbf{E} = E_0\mathbf{e}_z \ .$$

1. Calculate the potential $\varphi(\mathbf{r})$!
2. Determine the surface charge density on the sphere!

Exercise 2.3.12 Demonstrate by a direct calculation that the scalar product $\mathbf{r}\cdot\mathbf{r}'$ of the two space vectors

$$\mathbf{r} = \mathbf{r}(r, \vartheta, \varphi)$$
$$\mathbf{r}' = \mathbf{r}'(r', \vartheta', \varphi')$$

can be expressed by spherical harmonics as follows:

$$\mathbf{r}\cdot\mathbf{r}' = \frac{4\pi}{3} r r' \sum_{m=-1,0,1} Y^*_{1m}(\vartheta', \varphi') Y_{1m}(\vartheta, \varphi) \ .$$

Test the result by the use of the addition theorem for spherical harmonics.

Exercise 2.3.13 A hollow sphere of radius R carries on its surface the charge density

$$\rho(\mathbf{r}) = \sigma_0 \cos\vartheta \, \delta(r - R) \ .$$

Calculate the electrostatic potential Φ and the electric $\mathbf{E}$ inside and outside the sphere. For this purpose, start at the Poisson integral,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \ ,$$

and use for $|\mathbf{r} - \mathbf{r}'|^{-1}$ the expansion in spherical harmonics (2.169)!

Exercise 2.3.14 Consider an electric dipole $\mathbf{p}$ at distance a in front of a plane grounded metal surface which is assumed to be infinitely extended (Fig. 2.54).

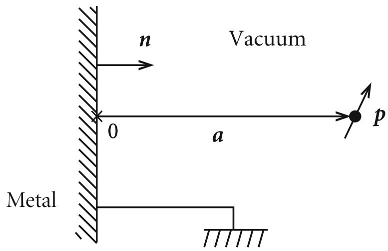


Fig. 2.54 Electrostatic boundary-value problem concerning the electric field which arises by an electric dipole in front of a grounded metal plane

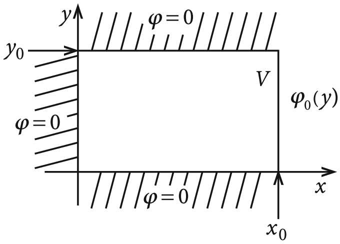


Fig. 2.55 Two-dimensional boundary-value problem for the scalar potential in an otherwise charge-free space

1. How do the potential and the electric field of a (point-like) dipole in the free space look like if the dipole is located
   (a) at the origin of the coordinates,
   (b) at the position **a**?
2. Calculate with the method of image charges the potential in the space to the right of the metal plate (vacuum) (Fig. 2.54) thereby fulfilling the boundary conditions.
3. Calculate the electric field $\mathbf{E}(\mathbf{r})$ and the density $\sigma(\mathbf{r})$ of the induced charge on the metal surface.
4. Discuss the sign of the induced charge density for the cases that the dipole moment is oriented
   (a) perpendicular to the surface,
   (b) parallel to the surface.

   Sketch qualitatively the shape of the electric field strength for both the cases.
5. Calculate for both the above cases the total induced charges, and that separately for each sign.

Exercise 2.3.15 Consider the two-dimensional boundary-value problem sketched in Fig. 2.55. The region V is free of charges. On three of the edges of V it is $\varphi = 0$, while on the fourth side of the rectangle it holds

$$\varphi_0(y) = \sin\left(\frac{\pi}{y_0} y\right) .$$

Determine the scalar potential everywhere in V!

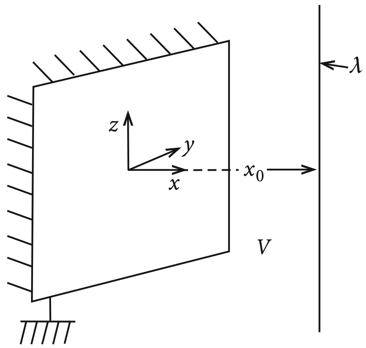


Fig. 2.56 Homogeneously charged wire in front of a grounded metal plate

Exercise 2.3.16 A straight, long, thin wire, which is homogeneously charged (λ = charge per unit length) is located with the distance x_0 parallel to a very large grounded metal plate (Fig. 2.56).

1. Calculate the scalar potential φ of the wire at first **without** the conducting plate (Hint: Gauss theorem with **proper** symmetry considerations).
2. Determine in the next step for the given arrangement the potential φ in the half-space V to the right of the plate by means of the method of image charges.
3. How large is the surface charge density induced on the plate?

2.4 Electrostatics of Dielectrics (Macroscopic Media)

Our considerations up to now referred exclusively to electric fields **in the vacuum**, described by the two Maxwell equations (2.39),

$$\mathrm{div}\mathbf{E} = \frac{\rho}{\epsilon_0} \; ; \quad \mathrm{curl}\,\mathbf{E} = 0 \; .$$

The goal now is to derive the corresponding field equations of matter. Matter consists for the most part of charged particles (protons, electrons, ions, ...), which naturally react to external electric fields, i.e. being more or less strongly shifted out of their equilibrium positions. That leads to induced multipoles and therewith to excess fields within the matter which superimpose on the external one. It is clear that the manner how the charged particles *will adjust themselves* with the external field will distinctly modify the up to now discussed electrostatics of the vacuum. We now therefore want to try to formulate the Maxwell equations in such a way that the extremely complicated microscopic correlations in matter are taken into account in a proper manner. Strictly speaking, for this we have to manage two subtasks:

1. *Setting of a theoretical model for the analysis of the atomic interactions,*
2. *Definition of macroscopic field quantities on the basis of atomic data.*

The atomic model can be developed correctly only within the framework of quantum mechanics. We have to restrict ourselves here therefore to certain broad indications, only.

The considerations of this section deal exclusively with insulators (**dielectrics**), i.e. substances which do not contain freely mobile charges and which consist of stable sub-units, as e.g. atoms, molecules or unit cells of a crystal, with vanishing total charge.

2.4.1 Macroscopic Field Quantities

We begin with the second point and discuss therefore at first the relevant macroscopic observables. Starting point is the fundamental **postulate:**

The Maxwell equations in the vacuum are microscopically universal laws!

$$\operatorname{div}\mathbf{e} = \frac{\rho_m}{\epsilon_0}\ ; \quad \operatorname{curl}\mathbf{e} = 0\ , \tag{2.178}$$

e: microscopic electric field; ρ_m: microscopic charge density.

If we knew the microscopic fields and charge distributions then there would not be any reason to change the theory developed so far. This knowledge, however, we do not have since there are typically about 10^{23} molecular (atomic, subatomic) particles per cubic centimeter performing quick and in general disordered motions (lattice oscillations of ions, electron motions on atomic orbitals, ...). They give rise to spatially as well as temporally strongly oscillating microscopic fields whose exact determination appears to be absolutely hopeless. On the other hand, however, a macroscopic measurement means in general a *'rough sampling'* of a microscopically huge region and therewith automatically an averaging over a certain finite space-time sector by which rapid microscopic fluctuations are *smoothed out* to a certain degree. A theory is therefore actually reasonable only for averaged quantities. A **microscopically exact** theory is on the one hand not realizable, but on the other hand not necessary, either. It would contain very much *superfluous, since experimentally not accessible* information. How do we describe theoretically the just mentioned experimental averaging process?

Definition 2.4.1 ***Phenomenological average***

$$\overline{f(\mathbf{r},t)} = \frac{1}{v(\mathbf{r})}\int\limits_{v(\mathbf{r})} d^3r' f(\mathbf{r}',t) = \frac{1}{v}\int\limits_{v(0)} d^3r' f(\mathbf{r}'+\mathbf{r},t)\ . \tag{2.179}$$

$f(\mathbf{r},t)$: microscopic field quantity,
$v(\mathbf{r})$: microscopically large, macroscopically tiny volume of a sphere with its center at **r** (e.g. $v \approx 10^{-6}\,\text{cm}^3$ with on an average still 10^{17} particles).

One can show that because of the large number of particles in the macroscopic volume $v(\mathbf{r})$ the rapid temporal fluctuations are smoothed out by the spatial averaging. Equation (2.179) is not the only possibility for averaging. It is, however, for our purpose here especially convenient. The physical results must be and will be independent of the type of averaging.

It is an important assumption for the following steps that

$$\nabla\bar{f} = \overline{\nabla f} \qquad \left(\text{later also: } \frac{\partial}{\partial t}\bar{f} = \overline{\frac{\partial f}{\partial t}}\right) , \tag{2.180}$$

which obviously applies to the proposed averaging process (2.179). We now define:

$$\mathbf{E}(\mathbf{r}) = \overline{\mathbf{e}(\mathbf{r})}: \quad \textbf{macroscopic electrostatic field.} \tag{2.181}$$

Because of (2.180) we then have:

$$\operatorname{curl}\bar{\mathbf{e}} = \overline{\operatorname{curl}\mathbf{e}} \; ; \quad \operatorname{div}\bar{\mathbf{e}} = \overline{\operatorname{div}\mathbf{e}} \; .$$

By averaging in (2.178) we then obtain the

macroscopic Maxwell equations

$$\operatorname{div}\mathbf{E} = \frac{\bar{\rho}_m}{\epsilon_0} \; ; \quad \operatorname{curl}\mathbf{E} = 0 \; . \tag{2.182}$$

We can write the macroscopic field $\mathbf{E}$, too, as a gradient of a scalar potential:

$$\mathbf{e} = -\nabla\varphi \implies \bar{\mathbf{e}} = -\overline{\nabla\varphi} = -\nabla\bar{\varphi} \implies \mathbf{E}(\mathbf{r}) \implies = -\nabla\overline{\varphi(\mathbf{r})} \; . \tag{2.183}$$

We still have to fix $\overline{\varphi(\mathbf{r})}$. For that we calculate at first the potential φ_j of a single *'particle'* (ion, molecule, ...), which is composed of atomic electrons and nuclei which on their part can be assumed to be point charges $q_n^{(j)}$. That shall also hold for the *excess charges* (*free* charges) which are found momentarily in the space region of the j-th particle.

$q_j = \sum_n^{(j)} q_n^{(j)}$: **total charge** of the j-th particle ,

$\rho_j = \sum_n^{(j)} q_n^{(j)} \delta(\mathbf{r} - \mathbf{r}_n)$: **charge density** in the j-th particle ,

$\mathbf{p}_j = \int d^3r\, \rho_j(\mathbf{r})\,(\mathbf{r} - \mathbf{R}_j)$: **dipole moment** of the j-th particle .

The distances within the given particle are of atomic dimensions and therefore small compared to the distance between the center of gravity $\mathbf{R}_j$ and the space point

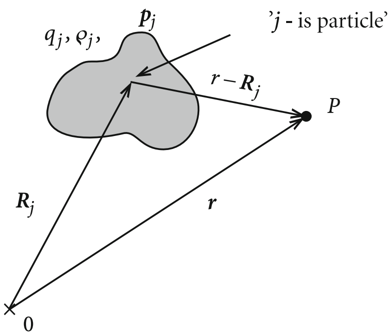


Fig. 2.57 Schematic illustration of the total charge and total dipole moment of a 'particle' which is composed of point-like 'sub-particles'

P (Fig. 2.57). It therefore recommends itself a **multipole expansion** of the scalar potential $\varphi_j(\mathbf{r})$ around $\mathbf{R}_j$, which we will terminate after the dipole term (2.94):

$$4\pi\epsilon_0\varphi_j(\mathbf{r}) \approx \frac{q_j}{|\mathbf{r}-\mathbf{R}_j|} + \frac{\mathbf{p}_j \cdot (\mathbf{r}-\mathbf{R}_j)}{|\mathbf{r}-\mathbf{R}_j|^3} \ .$$

This expansion actually holds only for a fixed point in time t because it is of course $\mathbf{R}_j = \mathbf{R}_j(t)$. This time-dependence, however, is segregated, as already mentioned, by the subsequent averaging process and therefore we do no longer consider it explicitly in the following.

We further introduce an *effective* charge density

$$\rho_e(\mathbf{r}) = \sum_{j=1}^{N} q_j \delta(\mathbf{r}-\mathbf{R}_j) \ ,$$

where N is the total number of particles, as well as an *effective* dipole density (see (2.74)):

$$\Pi_e(\mathbf{r}) = \sum_{j=1}^{N} \mathbf{p}_j \delta(\mathbf{r}-\mathbf{R}_j) \ .$$

Then we can write for the **total** scalar potential $\varphi(\mathbf{r})$ produced by **all** particles:

$$4\pi\epsilon_0\varphi(\mathbf{r}) = \int d^3r' \left[\frac{\rho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \Pi_e(\mathbf{r}') \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \right] \ .$$

We now perform the averaging on this expression:

$$
\begin{aligned}
4\pi\epsilon_0\overline{\varphi(\mathbf{r})} &= \frac{1}{v}\int\limits_{v(0)} d^3x \int d^3r' \left[\frac{\rho_e(\mathbf{r}')}{|\mathbf{r}+\mathbf{x}-\mathbf{r}'|} + \Pi_e(\mathbf{r}')\frac{\mathbf{r}+\mathbf{x}-\mathbf{r}'}{|\mathbf{r}+\mathbf{x}-\mathbf{r}'|^3}\right] \\
&= \frac{1}{v}\int\limits_{v(0)} d^3x \int d^3r'' \left[\frac{\rho_e(\mathbf{r}''+\mathbf{x})}{|\mathbf{r}-\mathbf{r}''|} + \Pi_e(\mathbf{r}''+\mathbf{x})\frac{\mathbf{r}-\mathbf{r}''}{|\mathbf{r}-\mathbf{r}''|^3}\right] \\
&= \int d^3r'' \left[\frac{\overline{\rho_e(\mathbf{r}'')}}{|\mathbf{r}-\mathbf{r}''|} + \overline{\Pi_e(\mathbf{r}'')}\cdot\frac{\mathbf{r}-\mathbf{r}''}{|\mathbf{r}-\mathbf{r}''|^3}\right] .
\end{aligned}
$$

This is the potential relevant for the electrostatics of dielectrics.

Definition 2.4.2 ***Macroscopic charge density***

$$
\rho(\mathbf{r}) = \overline{\rho_e(\mathbf{r})} = \frac{1}{v(\mathbf{r})}\sum_{j\in v} q_j . \tag{2.184}
$$

Note that $\rho(\mathbf{r})$ results from an averaging which includes **all** charges in $v(\mathbf{r})$. Normally the bounded charges of the solid will compensate each other so that $\rho(\mathbf{r})$ will finally be due to the free excess charges, only.

Definition 2.4.3 ***Macroscopic polarization***

$$
\mathbf{P}(\mathbf{r}) = \overline{\Pi_e(\mathbf{r})} = \frac{1}{v(\mathbf{r})}\sum_{j\in v} \mathbf{p}_j . \tag{2.185}
$$

This at this stage is only a definition of the term $\mathbf{P}(\mathbf{r})$. It arises from the effect of internal and external fields and will therefore be later calculated as functional of these fields by the use of proper theoretical models.

With these definitions, the averaged scalar potential reads:

$$
4\pi\epsilon_0\overline{\varphi(\mathbf{r})} = \int d^3r' \left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}')\cdot\nabla_{r'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] .
$$

For the Maxwell equations we need div$\mathbf{E}$:

$$
\begin{aligned}
4\pi\epsilon_0\mathrm{div}\mathbf{E} &= -4\pi\epsilon_0\Delta\,\overline{\varphi} = -\int d^3r'\left[\rho(\mathbf{r}')\Delta_r\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}')\cdot\nabla_{r'}\Delta_r\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] \\
&\overset{(1.69)}{=} 4\pi\,\rho(\mathbf{r}) + 4\pi\int d^3r'\mathbf{P}(\mathbf{r}')\cdot\underbrace{\nabla_{r'}\delta(\mathbf{r}-\mathbf{r}')}_{-\nabla_r\delta(\mathbf{r}-\mathbf{r}')} \\
&= 4\pi\left[\rho(\mathbf{r}) - \nabla\cdot\mathbf{P}(\mathbf{r})\right] .
\end{aligned}
$$

We have found therewith the important result:

$$\operatorname{div}(\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho(\mathbf{r}) \; . \tag{2.186}$$

Definition 2.4.4 ***(Di)electric displacement***

$$\mathbf{D}(\mathbf{r}) = \epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) \; . \tag{2.187}$$

Therewith we have the general

Maxwell equations of electrostatics

$$\operatorname{div}\mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}) \; ; \quad \operatorname{curl}\mathbf{E}(\mathbf{r}) = 0 \; . \tag{2.188}$$

Note that **D** is created by the *'true'* excess charges being thus **independent** of the material under consideration. The field **E**, in contrast, does depend on the material because of **P**. Two electrostatic fields of the same geometry with the same excess charges have identical dielectric displacements **D**.

The relations (2.187) and (2.188) suggest the definition of a

$$\textbf{polarization charge density} \quad \rho_p = -\operatorname{div}\mathbf{P} \; . \tag{2.189}$$

We can write therewith the Maxwell equation also as follows:

$$\operatorname{div}\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0}\left[\rho(\mathbf{r}) + \rho_p(\mathbf{r})\right] \; . \tag{2.190}$$

We see that the electric field reacts on the **actual** local charge density in the matter, in contrast to the **D**-field which is exclusively due to the *excess-charge density* $\rho(\mathbf{r})$. Hence it is clear that the actual experimental measurand is the **E**-field, while **D** is only an auxiliary quantity. The polarization **P** acts like an additional internal field $\mathbf{E}_p$ which supplements the field $\mathbf{E}_0$ generated by the excess charges so that the total field is given by:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_p \; , \\ \mathbf{E}_p &= -\frac{1}{\epsilon_0}\mathbf{P} \; . \end{aligned} \tag{2.191}$$

From the above derivation we have to conclude that the polarization field results from induced dipoles (Fig. 2.58). During this induction process charge is neither added nor led away. The total polarization charge must therefore vanish:

$$Q_p = \int\limits_V d^3r\, \rho_p(\mathbf{r}) = -\int\limits_V d^3r\, \operatorname{div}\mathbf{P}(\mathbf{r}) = -\int\limits_{S(V)} d\mathbf{f}\cdot\mathbf{P} = 0 \; . \tag{2.192}$$

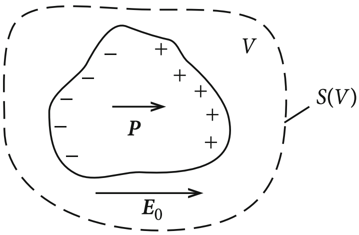


Fig. 2.58 Illustration of the polarization generated in condensed matter by an external electric field. The 'fictitious' volume V serves for the calculation of the total polarization charge

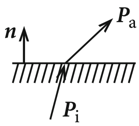


Fig. 2.59 To the calculation of the surface charge density generated by the electric polarization

P is of course unequal to zero only inside the (condensed) matter. Although the total charge Q_p vanishes, there appears, however, **locally** a finite polarization charge density $\rho_p(\mathbf{r})$ as soon as $\operatorname{div}\mathbf{P}(\mathbf{r}) \neq 0$. This is for instance the case at the surface (Fig. 2.58). There $\mathbf{P}(\mathbf{r})$ induces a surface charge density σ_p which can be calculated by applying the Gauss theorem as done for (2.43) (Fig. 2.59):

$$\mathbf{n} \cdot (\mathbf{P}_a - \mathbf{P}_i) = -\sigma_p \ .$$

Since only $\mathbf{P}_i = \mathbf{P} \neq 0$ we have:

$$\sigma_p = \mathbf{n} \cdot \mathbf{P} \ . \tag{2.193}$$

One should bear in mind, however, that local polarization charges always occur when $\operatorname{div}\mathbf{P} \neq 0$, i.e. not necessarily only at the surface.

Up to now the term **P** is only *defined*. We still have to think about the physical reasons for $\mathbf{P} \neq 0$. One distinguishes different types of polarizations according to which one can classify the dielectrics:

(1) (Ordinary) Dielectric

The external field shifts the positive and negative charges, bound in a *'particle'*, relatively to each other so that electric dipoles are created. One speaks of **deformation polarization**.

Example In the neutral atom without an external electric field the charge centers of gravity of the negative electron cloud and the positive nucleus coincide. However, in the field **E** these centers are shifted against each other thereby creating a resultant finite dipole moment **p** (Fig. 2.60).

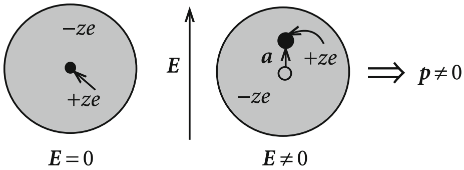


Fig. 2.60 Illustration of a deformation polarization

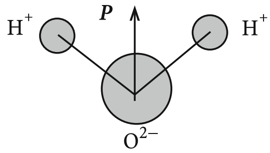


Fig. 2.61 Molecular structure of water (H_2O) (schematic)

(2) Paraelectric

If the material contains **permanent** dipoles, e.g. on grounds of the structure of the molecule as in water (H_2O) (Fig. 2.61), ammonia (NH_3), ..., then **in the absence of** an external field the directions of the vectorial moments will be statistically distributed, therefore compensate themselves. An external field $\mathbf{E}_0 \neq 0$, however, causes a certain alignment of the moments since the potential energy of the system will therewith decrease according to (2.79). One speaks of **orientation polarization**. This ordering tendency is opposed by the disordering tendency due to the thermal motion. Both tendencies lead to a temperature-dependent compromise.

(3) Ferroelectric

This phenomenon is observed in materials which contain permanent dipoles which orient themselves spontaneously, i.e. without the presence of an external field, below a critical temperature T_C (T_C: *Curie temperature*). Examples are:

$$\text{seignette salt:}\quad NaKC_4H_4O_6 \cdot 4H_2O\ ,$$

$$\text{barium titanate:}\quad BaTiO_3\ .$$

These substances exhibit a rather complicated field-dependent behavior. They are therefore **not** included in the following considerations.

For dielectrics of the type (1) or (2) it holds in any case

$$\mathbf{P} = \mathbf{P}(\mathbf{E}) \quad \text{with} \quad \mathbf{P}(0) = 0\ . \tag{2.194}$$

We expand $\mathbf{P}$ in powers of E:

$$P_i = \sum_{j=1}^{3} \gamma_{ij} E_j + \sum_{j,k=1}^{3} \beta_{ijk} E_j E_k + \dots\ , \qquad i, j, k \in \{x, y, z\}\ . \tag{2.195}$$

γ_{ij} (tensor of the second rank), β_{ijk} (tensor of the third rank), ... are material quantities. The experimental experience tells us that for not too high fields the first term of the expansion is already sufficient:

$$P_i \approx \sum_{j=1}^{3} \gamma_{ij} E_j: \quad \textbf{anisotropic dielectric} \, ,$$

$$P_i \approx \gamma \, E_i: \quad \textbf{isotropic dielectric} \, .$$

In what follows we consider exclusively isotropic dielectrics (strong restriction!) for which **E** and **P** are parallel:

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \, . \tag{2.196}$$

χ_e is denoted as **electric (dielectric) susceptibility**, which as a so-called *response-function* describes the reaction of the system to the electric field **E**:

$$\mathbf{D} = (1 + \chi_e)\epsilon_0 \mathbf{E} \equiv \epsilon_r \epsilon_0 \mathbf{E} \, , \tag{2.197}$$

$\epsilon_r = 1 + \chi_e$: **(relative) dielectric constant (permittivity).** For not polarizable media ($\chi_e = 0$) $\epsilon_r = 1$.

A simple demonstration of the theory developed so far represents the **capacitor with dielectric medium** (Fig. 2.62). For its capacity it holds according to (2.54):

$$C = \frac{Q}{U} \, .$$

Q is here the *'true'* excess charge on the right plate, $-Q$ that on the left plate. The area of the capacitor-plate is F and therewith the surface charge density $\sigma = Q/F$. For the latter we derive from (2.188) by applying the Gauss theorem as in (2.43):

$$\sigma = \mathbf{D} \cdot \mathbf{n} = D \, . \tag{2.198}$$

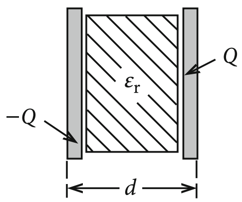


Fig. 2.62 Plane-parallel capacitor with dielectric medium

Let the dielectric be homogeneous so that between the plates homogeneous **D** and **E**-fields are formed:

$$U = E\,d = \frac{d}{\epsilon_r \epsilon_0} D \,,$$
$$Q = \sigma F = D F \,.$$

It follows:

$$C = \epsilon_r \epsilon_0 \frac{F}{d} \,. \tag{2.199}$$

The comparison with (2.55) shows that the dielectric medium enhances the capacity of the capacitor by the factor $\epsilon_r > 1$! This must be understood as follows:

(1) *U* Fixed

Without dielectric $Q = Q_0 = C_0 U$. With dielectric between the plates polarization charges arise at its surface (Fig. 2.63) which, to keep U constant, must be compensated by the source:

$$C = \frac{Q_0 + Q_p}{U} = C_0 + C_0 \frac{Q_p}{Q_0} = C_0 \left(1 + \frac{\sigma_p}{\sigma_0}\right)$$
$$= C_0 \left(1 + \frac{P}{\epsilon_0 E}\right) = C_0 \frac{D}{\epsilon_0 E} = \epsilon_r C_0 \,.$$

(2) *Q* Constant

Without dielectric we have now $U_0 = Q/C_0$. With dielectric the electric tension between the plates decreases because of the opposing field created by the

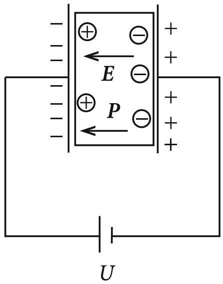


Fig. 2.63 Electric field and polarization in the dielectric between the plates of a capacitor

polarization charges:

$$C = \frac{Q}{U_0 - U_p} = \frac{\sigma}{\sigma - \sigma_p} C_0 = \frac{D}{D - P} C_0 = \epsilon_r C_0 \ .$$

The now still remaining task consists in the development of model-pictures for the macroscopic parameters χ_e and ϵ_r.

2.4.2 *Molecular Polarizability*

For the polarization $\mathbf{P}(\mathbf{r})$ we found in (2.185):

$$\mathbf{P}(\mathbf{r}) = \overline{\Pi_e(\mathbf{r})} = n\,\overline{\mathbf{p}(\mathbf{r})} \ . \tag{2.200}$$

Here

$$n(\mathbf{r}) = \frac{N(v)}{v(\mathbf{r})} \tag{2.201}$$

shall be the particle density in the averaging-volume $v(\mathbf{r})$, while $\overline{\mathbf{p}(\mathbf{r})}$ is the average dipole moment per particle in $v(\mathbf{r})$. For the **exact** field acting at the particle-position $\mathbf{r}$ we can write:

$$\mathbf{E}_{ex}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) + \mathbf{E}_i(\mathbf{r}) \ . \tag{2.202}$$

$\mathbf{E}(\mathbf{r})$ is here the averaged macroscopic field discussed in the last section and $\mathbf{E}_i(\mathbf{r})$ is an additional internal field, in a certain sense the microscopic correction field.

Definition 2.4.5 ***Molecular polarizability***

$$\overline{\mathbf{p}(\mathbf{r})} = \alpha\, \mathbf{E}_{ex}(\mathbf{r}) \ . \tag{2.203}$$

Our first goal is to express the atomic characteristic α by macroscopic quantities like ϵ_r and n. In the next step we then have to develop microscopic models for α itself.

We first try to determine the exact field $\mathbf{E}_{ex}$ at the particle-position. Let the considered particle be at the origin of coordinates. The origin may simultaneously be the center of a spherical volume V (Fig. 2.64), which is chosen as *microscopically large* and *macroscopically tiny*. The exact as well as the averaged field at the particle-position are both generated by the *external* field $\mathbf{E}_0$ of the excess charges and by the polarization of the dielectric medium. The difference between the exact and the averaged field at the particle-position mainly results from how the polarizability is treated. The resulting field is in any case a superposition of field-contributions which stem from each single particle of the material. As to the

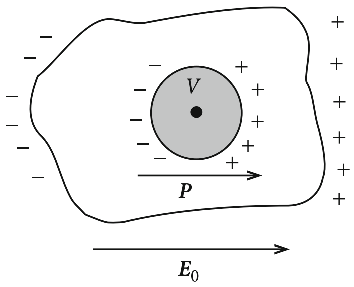


Fig. 2.64 Auxiliary construction for the determination of the molecular polarizability of a dielectric medium

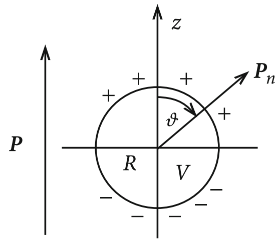


Fig. 2.65 Arrangement for the calculation of the field contribution due to polarization by the 'auxiliary sphere' in Fig. 2.64

contributions of the particles far away to the field at $\mathbf{r} = 0$ it will be rather unimportant whether we average or not. The observable difference between the exact and the averaged field will predominantly stem from the nearest neighbors, e.g. from the particles in V (Fig. 2.64). The following ansatz therefore appears to be plausible:

$$\mathbf{E}_{\mathrm{i}}(\mathbf{0}) \approx \mathbf{E}_{\mathrm{p,ex}}^{(V)}(\mathbf{0}) - \mathbf{E}_{\mathrm{p}}^{(V)}(\mathbf{0}) \; . \tag{2.204}$$

$\mathbf{E}_{\mathrm{p}}^{(V)}$ is the macroscopic averaged contribution of the charges in V to the polarization field, while $\mathbf{E}_{\mathrm{p,ex}}^{(V)}$ is their actual contribution.

We start with the discussion of $\mathbf{E}_{\mathrm{p}}^{(V)}(\mathbf{0})$. $\mathbf{P}$ is a macroscopic field term and V was chosen as macroscopically tiny. We therefore can assume that $\mathbf{P}$ is practically constant within the sphere. According to (2.193) $\mathbf{P}$ induces a surface charge on the (fictitious) sphere (Fig. 2.65):

$$\sigma_{\mathrm{p}} = P_n = P \cos \vartheta \; .$$

We can interpret this as space-charge density:

$$\rho_{\mathrm{p}}(\mathbf{r}) = P \cos \vartheta \cdot \delta(r - R) \; .$$

$\rho_p(\mathbf{r})$ creates at $(0, 0, 0)$ the following field:

$$\mathbf{E}_p^{(V)}(\mathbf{0}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \rho_p(\mathbf{r}')\frac{(-\mathbf{r}')}{|-\mathbf{r}'|^3}$$
$$= \frac{-P}{4\pi\epsilon_0}\int_0^\infty dr'\delta(r'-R)\int_0^{2\pi} d\varphi' \int_{-1}^{+1} d\cos\vartheta' \cos\vartheta' \cdot \begin{pmatrix}\sin\vartheta'\cos\varphi'\\ \sin\vartheta'\sin\varphi'\\ \cos\vartheta'\end{pmatrix} .$$

It therefore holds for the averaged contribution of the sphere:

$$\mathbf{E}_p^{(V)}(\mathbf{0}) = -\frac{P}{3\epsilon_0}\mathbf{e}_z = -\frac{1}{3\epsilon_0}\mathbf{P}(\mathbf{0}) . \tag{2.205}$$

The calculation of the second field term in (2.204) requires a little more effort, in particular the actual arrangement of the lattice sites, the so-called lattice structure will play a role. Let us agree upon the following assumptions (simplifications!):

1. All atomic dipoles $\mathbf{p}_\mathrm{i}$ **within** the volume V have the same magnitude and the same direction.
2. The dipoles are arranged on a simple cubic lattice (Fig. 2.66) with the lattice constant a.

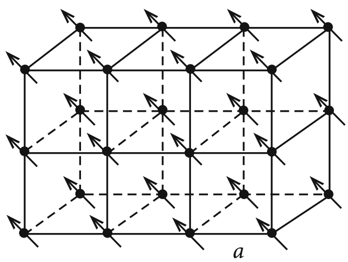
For the positions of the dipoles we can write:

$$\mathbf{r}_{ijk} = a(i, j, k) \; ; \quad i, j, k \in \mathbb{Z} .$$

The dipole at $\mathbf{r}_{ijk}$ then creates according to (2.73) the following field at $\mathbf{0}$:

$$\mathbf{E}_{ijk} = \frac{3\mathbf{r}_{ijk}(\mathbf{p}\cdot\mathbf{r}_{ijk}) - \mathbf{p}\, r_{ijk}^2}{4\pi\epsilon_0 r_{ijk}^5} .$$

Fig. 2.66 Atomic dipoles on a simple cubic lattice



We get the total field by summing over all i, j, k permitted in V. This means, for instance, for the x-component:

$$ {}_xE^{(V)}_{\mathrm{p,ex}}(\mathbf{0}) = \sum_{ijk}^{V} E^x_{ijk} = \frac{1}{a^3}\sum_{ijk}^{V} \frac{3i(i\,p_x + j\,p_y + k\,p_z) - p_x(i^2 + j^2 + k^2)}{4\pi\epsilon_0(i^2 + j^2 + k^2)^{5/2}} \; . $$

It holds obviously

$$ \sum_{ijk}^{V} \frac{i\,j}{4\pi\epsilon_0(i^2 + j^2 + k^2)^{5/2}} = \sum_{ijk}^{V} \frac{i\,k}{4\pi\epsilon_0(i^2 + j^2 + k^2)^{5/2}} = 0 \; , $$

since i, j, k run in V through the same positive as well as negative integers. Furthermore, the cubic symmetry leads to:

$$ \sum_{ijk}^{V} \frac{i^2}{(i^2 + j^2 + k^2)^{5/2}} = \sum_{ijk}^{V} \frac{j^2}{(i^2 + j^2 + k^2)^{5/2}} = \sum_{ijk}^{V} \frac{k^2}{(i^2 + j^2 + k^2)^{5/2}} \; . $$

It remains therewith for the x-component of the resulting field

$$ {}_xE^{(V)}_{\mathrm{p,ex}}(\mathbf{0}) = \frac{1}{a^3}\sum_{ijk}^{V} \frac{3i^2 p_x - 3i^2 p_x}{4\pi\epsilon_0(i^2 + j^2 + k^2)^{5/2}} = 0 \; . $$

The same can be shown for the two other components so that we finally have:

$$ \mathbf{E}^{(V)}_{\mathrm{p,ex}}(\mathbf{0}) = 0 \; . \tag{2.206} $$

We can now insert Eq. (2.204) to (2.206) into (2.202):

$$ \mathbf{E}_{\mathrm{ex}}(\mathbf{0}) = \mathbf{E}(\mathbf{0}) + \mathbf{E}_i(\mathbf{0}) = \mathbf{E}(\mathbf{0}) + \frac{1}{3\epsilon_0}\mathbf{P}(\mathbf{0}) \; . \tag{2.207} $$

With the definition Eqs. (2.200) and (2.203) the polarizability now comes into play,

$$ \mathbf{P}(\mathbf{0}) = n\,\bar{\mathbf{p}}(\mathbf{0}) = n\,\alpha\,\mathbf{E}_{\mathrm{ex}}(\mathbf{0}) = n\,\alpha\left[\mathbf{E}(\mathbf{0}) + \frac{1}{3\epsilon_0}\mathbf{P}(\mathbf{0})\right] \; , $$

and via (2.196) the susceptibility χ_{e}:

$$ \chi_{\mathrm{e}}\epsilon_0\mathbf{E}(\mathbf{0})\left(1 - \frac{n\,\alpha}{3\epsilon_0}\right) = n\,\alpha\mathbf{E}(\mathbf{0}) \; . $$

It follows:

$$\chi_e = \frac{n\,\alpha}{\epsilon_0 - \dfrac{n\,\alpha}{3}} \; . \tag{2.208}$$

When we further introduce by $\chi_e = \epsilon_r - 1$ the dielectric constant, then we have the useful

Clausius-Mossotti relation

$$\alpha = \frac{3\epsilon_0}{n}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \; , \tag{2.209}$$

which relates the atomic characteristic α to the macroscopic parameters ϵ_r and n.

There does exist a series of more or less precise model interpretations of the polarizability α each of them aiming at a special type of dielectric. An extensive discussion here, however, is beyond the scope of our presentation. The mentioned models connect α to atomic physical measurands. The value of the relation (2.209) can be seen, among other things, in the fact that atomic properties can be understood by measuring macroscopic quantities such as ϵ_r and n.

2.4.3 Boundary-Value Problems, Electrostatic Energy

The macroscopic Maxwell equations (2.188) are structurally unchanged compared to those in the vacuum (2.39) and (2.40). The basic task always is to determine the **E**-field. In principle the same considerations and procedures are valid which we have developed in detail in Sect. 2.3 for the case of the vacuum.

If the relative dielectric constant ϵ_r is space-independent then the Poisson equation is to be solved in the form

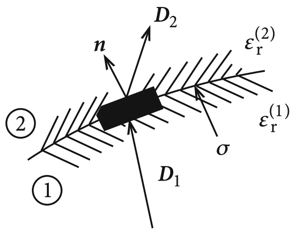
$$\Delta\,\varphi = -\frac{\rho}{\epsilon_r \epsilon_0} \; . \tag{2.210}$$

That means nothing new compared to Sect. 2.3. The charge density simply gets the add by $1/\epsilon_r$. If, however, the interesting space region is filled by different dielectrics with different $\epsilon_r^{(i)}$, then it is vital for the accomplishment of the basic task to know the behavior of the **D**- and the **E**-fields at the interfaces (Fig. 2.67). Exactly the same considerations as in Sect. 2.1.4 then lead to the following statements:

- With $\operatorname{div}\mathbf{D} = \rho$ and by use of the Gauss theorem it follows as in (2.43):

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \; . \tag{2.211}$$

Fig. 2.67 Behavior of the dielectric displacement at the interface between two different dielectrics



σ here is the surface density of the excess charges, polarization charges are thus excluded.

- From curl $\mathbf{E} = 0$ follows the unchanged Eq. (2.44):

$$(\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{E}_2 - \mathbf{E}_1) = 0 \, . \tag{2.212}$$

The notation is the same as in Sect. 2.1.4.

It therefore holds on **uncharged** interfaces ($\sigma = 0$):

$$\begin{aligned} D_{1n} = D_{2n} \quad &\Longleftrightarrow \quad E_{1n} = \frac{\epsilon_r^{(2)}}{\epsilon_r^{(1)}} E_{2n} \, , \\ E_{1t} = E_{2t} \quad &\Longleftrightarrow \quad D_{1t} = \frac{\epsilon_r^{(1)}}{\epsilon_r^{(2)}} D_{2t} \, . \end{aligned} \tag{2.213}$$

If $\epsilon_r^{(1)} \neq \epsilon_r^{(2)}$ then it is obviously such that both the fields cannot be simultaneously continuous at the interface.

Let us close this section with a few considerations on the **electrostatic energy**. For the vacuum we found in (2.47):

$$W_{\text{vacuum}} = \frac{1}{2} \int d^3r \, \rho(\mathbf{r}) \varphi(\mathbf{r}) \, .$$

This expression cannot be directly adopted since in the dielectric the buildup of the polarization charges also requires energy.

The charge $\delta\rho(\mathbf{r}) d^3r$ possesses in the potential $\varphi(\mathbf{r})$, created by other charges, the energy

$$\varphi(\mathbf{r}) \delta\rho(\mathbf{r}) d^3r \, .$$

The work which is needed to change the charge density from ρ to $\rho + \delta\rho$ amounts therefore to:

$$\delta W = \int d^3r \, \varphi(\mathbf{r}) \delta\rho(\mathbf{r}) \, .$$

$\varphi(\mathbf{r})$ is thereby thought to be created by $\rho(\mathbf{r})$. With

$$\varphi\,\delta\rho = \varphi\,\mathrm{div}(\delta\,\mathbf{D}) = \mathrm{div}(\varphi\,\delta\,\mathbf{D}) - \nabla\varphi\cdot\delta\,\mathbf{D}$$

it further follows:

$$\delta\,W = \int d^3r\,\mathrm{div}(\varphi\,\delta\mathbf{D}) + \int d^3r\,\mathbf{E}\cdot\delta\,\mathbf{D}\,.$$

We rewrite the first summand by the use of the Gauss theorem into a surface integral which disappears for the case $\varphi(r\to\infty) = 0$ (at least as $1/r$). We then have for the total energy:

$$W = \int d^3r \int_0^{\mathrm{D}} \mathbf{E}\cdot\delta\,\mathbf{D}\,. \tag{2.214}$$

If we assume in addition an isotropic, linear medium, i.e. $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$, then we can further reformulate:

$$\mathbf{E}\cdot\delta\,\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}\cdot\,\delta\mathbf{E} = \frac{1}{2}\epsilon_r\epsilon_0\delta(\mathbf{E}^2) = \frac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})\,.$$

The relation (2.47) is therefore replaced in the case of a dielectric medium by:

$$W = \frac{1}{2}\int d^3r\;\mathbf{E}\cdot\mathbf{D}\,. \tag{2.215}$$

2.4.4 Exercises

Exercise 2.4.1 In a neutral hydrogen atom, which is in its ground state, the charge density of the orbital electron is described by

$$\rho_e(\mathbf{r}) = -\frac{e}{\pi\,a^3}\exp\left(-\frac{2r}{a}\right)\,.$$

e is the magnitude of the electron charge, r the distance between the electron and the proton. If an electric field $\mathbf{E}_0$ is applied it holds in first approximation that the charge cloud of the electron is shifted without any deformation relative to the proton by the vector $\mathbf{r}_0$.

1. Express the dipole moment $\mathbf{p}$ of the hydrogen atom in the field $\mathbf{E}_0$ in terms of $\mathbf{r}_0$.
2. Calculate the restoring force on the proton due to the shifted charge cloud of the electron. Express it for $r_0/a \ll 1$ by the dipole moment $\mathbf{p}$. Find then a

representation of $\mathbf{p}$ as function of the field via the equilibrium condition for the force executed on the proton by the electric field $\mathbf{E}_0$.

3. Calculate the relative dielectric constant ϵ_r for a dielectric medium consisting of N hydrogen atoms homogeneously distributed in the volume V.

Exercise 2.4.2 A dielectric sphere ($\epsilon_r^{(2)}$, radius R) is surrounded by a homogeneous, isotropic dielectric medium ($\epsilon_r^{(1)}$) and is located in an (originally) homogeneous field (Fig. 2.68)

$$\mathbf{E}_0 = E_0\,\mathbf{e}_z\ .$$

Find the resulting field inside and outside the sphere. What is the dipole moment of the sphere?

Exercise 2.4.3 A parallel-plate capacitor (plate-area F, plate-distance d) is completely filled with a dielectric of the permittivity $\epsilon_r(z)$. Calculate its capacity. How does the capacity look like if the dielectric medium consists of two layers with thicknesses d_1 and d_2 and permittivities $\epsilon_r^{(1)}$ and $\epsilon_r^{(2)}$?

Exercise 2.4.4 A dielectric with the permittivity $\epsilon_r > 1$ is pushed into a parallel-plate capacitor (area $F = a \cdot b$, distance of the plates d) by a distance x ((I); see Fig. 2.69). The remaining space (II) between the plates is empty. The charge on the lower plate is Q, that on the upper plate $-Q$. Edge-effects such as stray fields shall be disregarded.

1. Which relations do exist between the electric field $\mathbf{E}$ and the dielectric displacement $\mathbf{D}$ in (I) and (II)?
2. What can be said about D_I/D_{II} and E_I/E_{II}?
3. Which connection does exist between D_I, D_{II} and the surface charge densities σ_I, σ_{II}?

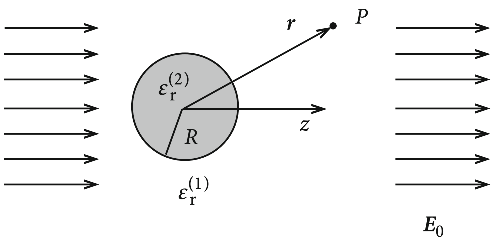


Fig. 2.68 Dielectric sphere in a homogeneous electric field

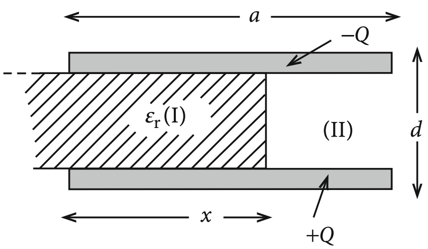


Fig. 2.69 Parallel-plate capacitor partially filled with a dielectric medium

4. Calculate the **E**- and the **D**-field for the whole space between the plates!
5. Calculate the electrostatic field energy W!
6. Determine the force which acts on the dielectric by inspecting the change in energy in consequence of shifting the dielectric by the way dx.

Exercise 2.4.5 Given is a dielectric sphere of the radius R. Let it be homogeneously polarized:

$$\mathbf{P}(\mathbf{r}) = \begin{cases} \mathbf{P}_0 = P_0 \mathbf{e}_z & \text{for } r < R\,, \\ 0 & \text{for } r > R\,. \end{cases}$$

The density of the *'excess charges'* $\rho(\mathbf{r})$ is zero.

1. Calculate the scalar potential $\varphi(\mathbf{r})$ inside and outside the sphere!
2. Calculate and plot the electric field strength **E**!
3. Determine the polarization charge density $\rho_\mathrm{P}(\mathbf{r})$!
4. How can the starting situation be realized?

Exercise 2.4.6 The yz-plane separates vacuum (left half-space) and a dielectric medium ($\varepsilon_\mathrm{r} > 1$) in the right half-space. At the position $\mathbf{r}_0 = -a\mathbf{e}_x$ ($a > 0$) in the vacuum-region a positive point charge q is located. Calculate the electrostatic field **E** in the whole space and discuss the polarization **P** and the polarization charge density $\rho_\mathrm{P}(\mathbf{r})$ of the dielectric (Fig. 2.70).

Proposal Use the method of image charges to fulfill the boundary conditions (continuity-conditions at the interface).

2.5 Self-Examination Questions

To Section 2.1

1. What does the law of conservation of the charge mean?
2. What is denoted as elementary charge?

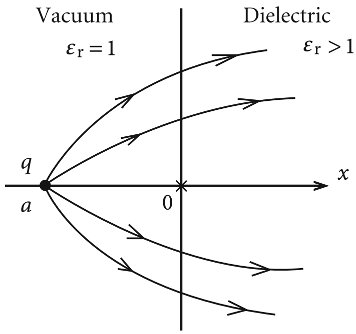


Fig. 2.70 Electric field of a point charge in front of a dielectric medium

3. What does one understand by charge density? How is it connected to the total charge?
4. Give the charge density of a point charge q.
5. Formulate the charge conservation as a continuity equation.
6. Which are the experimental facts of experience the electrostatics is based on?
7. What is the Coulomb law for point charges?
8. How is the electrostatic field defined?
9. What is the electric field of a point charge, of n point charges, of a continuous charge distribution?
10. How do we understand in the framework of the *field concept* the interaction process between point charges?
11. Is the Coulomb force conservative?
12. How is the scalar electric potential defined?
13. How are the electric field lines oriented relative to the equipotential areas?
14. What are the scalar potential and the electric field of a homogeneously charged sphere (radius R, total charge Q)?
15. What does one understand by the *physical* Gauss theorem?
16. Formulate the Maxwell equations of the electrostatics in differential and in integral form!
17. What do we consider as the basic problem of the electrostatics?
18. Give the connection between the Maxwell equations and the Poisson equation!
19. How do normal and tangential components of the electrostatic field behave during the transition through an interface which carries a finite surface charge density σ?
20. How is the energy of a static charge configuration defined?
21. What is the definition of the energy density of an electrostatic field?

To Section 2.2

1. What is a parallel-plate capacitor?
2. How is the capacity of a capacitor defined?
3. Which energy density is found in a spherical capacitor? What is its total energy?
4. How is the electric field oriented in a cylindrical capacitor?
5. What do we understand by a dipole? How does the scalar potential of a dipole $\mathbf{p}$ look like?
6. Which r-dependence does the electric dipole-field have?
7. Which force and which torque act on a dipole in an homogeneous electrostatic field? For which orientation has the dipole the least potential energy?
8. What is a dipole layer?
9. Which jump of the scalar potential does appear when crossing a dipole layer with the dipole surface density $\mathbf{D}(\mathbf{r})$?
10. What is a quadrupole? How does the potential of a quadrupole look like?
11. Sketch the equipotential areas and the electric field lines of the stretched (linear) quadrupole!
12. What is understood by a multipole expansion?

13. Define the dipole moment and the quadrupole moment of a general charge density $\rho(\mathbf{r})$!
14. How does the dipole moment behave in case of a rotation or a translation of the system of coordinates?
15. List some special properties of the quadrupole tensor!
16. Do spherical-symmetric charge distributions have a dipole moment and a quadrupole moment? Give reasons!

To Section 2.3

1. What do we understand by a boundary-value problem?
2. Define and characterize Dirichlet- and Neumann-boundary conditions.
3. Describe physical situations for which, respectively, Dirichlet- and Neumann-boundary conditions are relevant.
4. How is the Green's function defined in electrostatics?
5. How does the Green's function determine the electrostatic potential in case of given Dirichlet- (Neumann-) boundary conditions?
6. Describe the method of image charges.
7. What do we understand by an induced charge density?
8. What is an image force? How strong is it for a point charge q in front of an infinitely extended, grounded metallic plate?
9. A point charge q is located in front of a grounded metallic sphere with the radius R at a distance of r from the center of the sphere ($r > R$). How large is the total surface charge induced on the sphere? Is the image force attractive or repulsive?
10. When do we call a system of functions $u_n(x)$ to be complete?
11. Formulate the completeness relation!
12. Cite some examples of complete systems of functions!
13. What does one understand by a separation ansatz?
14. How does the general solution of the Laplace equation read in spherical coordinates?
15. Which complete system of functions appears to be convenient for the solution of the Laplace equation with boundary conditions of azimuthal symmetry?
16. What do we understand by spherical multipole moments?

To Section 2.4

1. How is the macroscopic electric field connected with the microscopic field?
2. What do we understand by the macroscopic polarization?
3. How do the dielectric displacement, the electric field, and the polarization stick together?
4. Comment on the difference between **D** and **E**!
5. What are the Maxwell equations of electrostatics in matter?
6. What is the actual measurand: **D** or **E**?
7. What is to be understood by deformation-polarization and what by orientation-polarization?
8. Define the terms dielectrics, paraelectrics, ferroelectrics!

9. How is the electric susceptibility defined?
10. What is the relation between susceptibility and dielectric constant (permittivity)?
11. Comment on the meaning of the molecular polarizability!
12. Give the Clausius-Mossotti formula! Sketch the derivation of this formula!
13. How does the electrostatic field energy look like in a space-region filled with matter?

Chapter 3
Magnetostatics

Electrostatic fields arise from electric charges at rest and can be observed through the actions of forces.

Magnetostatic fields arise from stationary electric currents, i.e. from moving electric charges. The observation is that an on the whole uncharged but current-carrying conductor exerts a force. Since an uncharged system cannot produce an electric field one ascribes to this force another type of field which is called the *magnetic field*.

In the following we will recognize again and again that there do exist distinct analogies between electrostatic and magnetostatic phenomena. However, there are also characteristic differences. Most of them rely on the experimental fact that there are free electric charges (monopoles) but there do **not** exist free magnetic monopoles. The basic unit of the magnetism is not any elementary charge but the magnetic dipole **m**. Hence the magnetic field cannot be gauged by any magnetic 'test-particle' but only through the torque **M** which is exerted on a given magnetic system by a certain known moment **m**. For this it holds, fully analogously to Eq. (2.77) which is valid for the torque on an electric dipole **p**:

$$\mathbf{M} = \mathbf{m} \times \mathbf{B} \,. \tag{3.1}$$

This relation will later be derived explicitly. **B** is the so-called **magnetic induction**, the relevant field of magnetism. The definition of **B** provokes conceptually many more difficulties than that of the analogous electric field **E**. It is obtained from the fact that **B** is generated by currents.

The fundamental task of magnetostatics consists of the calculation of the magnetic induction **B** from a given current density **j**.

W. Nolting, *Theoretical Physics 3*, DOI 10.1007/978-3-319-40168-3_3

3.1 The Electric Current

In metallic conductors, according to our considerations in Sect. 2.3.2, an electro**static** field cannot exist. However, we can definitely generate in such conductors a temporally constant potential difference ($\cong$ temporally constant electric field) by a steady supply of energy (external voltage source!). This field differs from the electrostatic field on the outside by the following features:

1. heat development,
2. transport of electric charge (*current*),
3. set-up of a magnetic field.

The terms current density $\mathbf{j}(\mathbf{r})$ and the strength of current I have already been introduced in Sect. 2.1. We are well-accustomed to them from experimental physics. We therefore restrict ourselves here to a compilation in note form.

3.1.1 Electric Current: Ordered Motion of Electric Charges

Possible realizations:

(a) Shift of a charged body (conductor or dielectric) in the space $\Longrightarrow$ **convection current**.
(b) Creation of a potential difference between the endings of a metallic wire $\Longrightarrow$ action of force on quasi-free charge carriers.
Assumptions:

$$
\begin{aligned}
&v: && \text{average particle velocity along } z\text{-direction}\,,\\
&n = \tfrac{N}{V}: && \text{temporally constant, homogeneous particle density}\,,\\
&q: && \text{charge of the particle}\,,\\
&F: && \text{cross section of the conductor}\,.
\end{aligned}
$$

$dQ = (F\,v\,dt)\,n\,q$ is therewith the charge flowing in the time dt through the cross section of the conductor (Fig. 3.1).

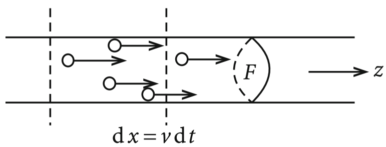


Fig. 3.1 Simple arrangement for the determination of the current strength due to charged particles

3.1.2 Current Intensity *I*

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \; . \tag{3.2}$$

I is thus the amount of charge which penetrates the cross section of the conductor in unit time. For the simple example given above the strength of current is therewith

$$I = n\, F\, v\, q \; .$$

The unit of the strength of current was already introduced after Eq. (2.9):

$$[I] = \text{ampere} = 1\,\text{A} = 1\,\text{C/s}$$

3.1.3 Current Density **j**

This is a vector whose direction is given by the direction of the motion of the electric charge and whose magnitude corresponds to the charge transported per unit-time through the unit-area perpendicular to the direction of the current. For the above example that means:

$$j = \frac{I}{F} = n\, q\, v \; .$$

$n\, q$ is the in this example homogeneous charge density. In the general case the **current density** is a **time-dependent vector field**,

$$\mathbf{j}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) \; , \tag{3.3}$$

which is connected via the charge density $\rho(\mathbf{r}, t)$ with the velocity field $\mathbf{v}(\mathbf{r}, t)$ of the system. The current intensity I through a given area F is the surface integral of $\mathbf{j}$ over F (Fig. 3.2):

$$I = \int_F \mathbf{j} \cdot d\mathbf{f} \; . \tag{3.4}$$

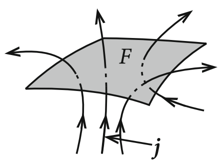


Fig. 3.2 Strength of current as surface integral over the current density

3.1.4 Continuity Equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 \,. \tag{3.5}$$

This important relation which we already derived in (2.10) is directly related to the **law of conservation of charge**.

For the magneto**statics** only the **stationary** case $\frac{\partial \rho}{\partial t} = 0$ is interesting, which because of (3.5) entails

$$\operatorname{div} \mathbf{j} \equiv 0 \tag{3.6}$$

We will exploit this relation several times in the course of this section. Two consequences are mentioned right here:

(a) In the stationary case (3.6) the same current is running through **each kind** of cross section. This we prove by calculating the surface integral over the surface $S(V)$ of a volume V, which may contain the two cross sections F_1 and F_2 (see Fig. 3.3):

$$0 = \int_V d^3 r \operatorname{div} \mathbf{j} = \int_{S(V)} d\mathbf{f} \cdot \mathbf{j} = \int_{\mathrm{F}_1} \mathbf{j} \cdot d\mathbf{f} + \int_{\mathrm{F}_2} \mathbf{j} \cdot d\mathbf{f}$$
$$= I_1 - I_2 \implies I_1 = I_2 \,.$$

(b) Kirchhoff's current law (node rule)

$$0 = \int_V d^3 r \operatorname{div} \mathbf{j} = \int_{S(V)} d\mathbf{f} \cdot \mathbf{j}$$
$$= -I_1 - I_2 + I_3 + I_4 \implies I_1 + I_2 = I_3 + I_4 \,.$$

At the node of some conductors (Fig. 3.4) the sum of the currents into the node is equal to the sum of the currents flowing out of the node.

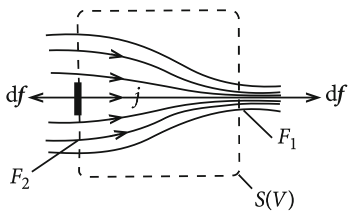


Fig. 3.3 Behavior of the current in a conductor with variable cross section

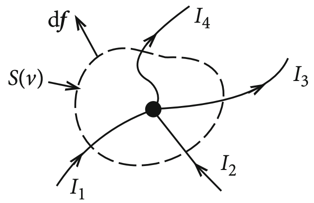


Fig. 3.4 Illustration of Kirchhoff's current law

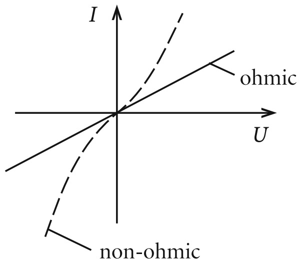


Fig. 3.5 Schematic current-voltage characteristic for ohmic and non-ohmic conductors

3.1.5 Ohm's Law

The experimental observation teaches us that under 'normal conditions' the current I running through an electric conductor is proportional to the applied voltage U:

$$U = I \cdot R \,. \tag{3.7}$$

The proportionality factor R is called **electric** or **ohmic resistance** with the unit:

$$[R] = \left[\frac{U}{I}\right] = 1\ \frac{\mathrm{V}}{\mathrm{A}} = 1\,\Omega\ \ (\mathrm{ohm})\,. \tag{3.8}$$

In general R is temperature-dependent so that, strictly speaking, (3.7) implies that the temperature is kept constant in spite of the unavoidable heat development.

The Ohm's law is not a physical law in the strict sense. It is not at all always fulfilled by all electric conductors. One therefore sometimes divides the conductors into two classes, the **ohmic conductors** and the **non-ohmic conductors** (Fig. 3.5).

The resistance R is not a material constant. In fact it depends on the physical dimensions of the electric conductor. We come to a corresponding material constant, however, when we formulate the Ohm's law in *local* quantities. In this sense the voltage corresponds to the electric field strength $\mathbf{E}(\mathbf{r})$ and the current to the current density $\mathbf{j}(\mathbf{r})$:

$$\mathbf{j}(\mathbf{r}) = \sigma(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\,. \tag{3.9}$$

The actual statement of the Ohm's law is that in an *ohmic* conductor the

electric conductivity $\sigma(\mathbf{r})$

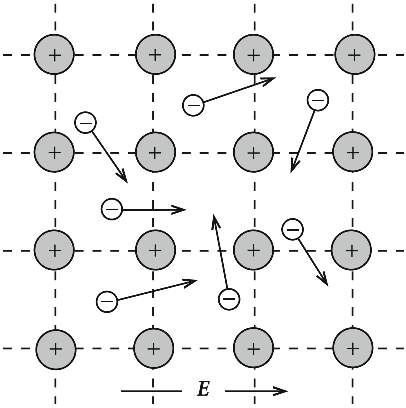


Fig. 3.6 Simple model of a metallic solid

does **not** depend on the field **E**. The reciprocal electric conductivity is called the

$$\textbf{specific electric resistance } \rho(\mathbf{r}) = \sigma^{-1}(\mathbf{r})$$

where ρ should not be confused with the charge density, not any more than σ with the surface charge density.

We finally calculate σ in the framework of a simple model for a metal: The particles which build the metallic solid are located as positively charged ions at lattice sites of a highly symmetric structure (Fig. 3.6). They are positively charged since the rather weakly bound valence electrons of the outermost electron shell have detached themselves from the parent atom to move quasi-free within the lattice. One says that they constitute an **'electron gas'**. Their velocity vectors $\mathbf{v}_j$ do **not** have a preferred direction if there is **no** external field. However, if there is a field $\mathbf{E} \neq 0$ then the velocities are superimposed by a field-parallel component which increases with time, until the electron is decelerated again to zero by a particle collision. If t_j is the time which has been passed for the j-th particle since the last collision then it holds for its average velocity:

$$\bar{\mathbf{v}} \simeq \frac{1}{N} \sum_j \left(\mathbf{v}_j - \frac{e}{m} \mathbf{E}\, t_j \right) .$$

One defines

$$\tau = \frac{1}{N} \sum_j t_j \quad \text{as } \textbf{average collision time} .$$

Since the first term in the sum for $\bar{\mathbf{v}}$ vanishes we get:

$$\bar{\mathbf{v}} = -\frac{e\tau}{m} \mathbf{E} .$$

Therewith we have as current density

$$\mathbf{j} = -n\,e\,\bar{\mathbf{v}} = \frac{e^2\,n\,\tau}{m}\mathbf{E}\;,$$

an expression which is in accordance with the Ohm's law (3.9) where

$$\sigma = \frac{e^2\,n\,\tau}{m}\;. \tag{3.10}$$

3.1.6 Thread of Current

In the electrostatics the concept of the point charge has proven to be very useful. The analog for the current is the *thread of current*, by which one understands a linearly current I along a path C (Fig. 3.7). We parametrize C by the arc length s and place at each point of the path the *moving trihedron* (Sect. 1.4.4, Vol. 1). $\hat{\mathbf{t}}$: tangent-unit vector; $\mathbf{r} = \mathbf{r}(s)$. In this local Cartesian system of coordinates it holds:

$$d\mathbf{r} = ds\,\hat{\mathbf{t}}\;; \quad d\mathbf{f} = df\,\hat{\mathbf{t}}\;; \quad d^3r = d\mathbf{f}\cdot d\mathbf{r} = df\,ds\;; \quad \mathbf{j} = j\,\hat{\mathbf{t}}\;; \quad I = \mathbf{j}\cdot d\mathbf{f} = j\,df\;.$$

It follows therewith:

$$\mathbf{j}\,d^3r = j\,\hat{\mathbf{t}}\,df\,ds = j\,df\,d\mathbf{r} = I\,d\mathbf{r}\;.$$

The transition to the thread of current is thus performed by the replacement:

$$\mathbf{j}\,d^3r \;\Longrightarrow\; I\,d\mathbf{r}\;. \tag{3.11}$$

3.1.7 Electric Power

When one shifts the charge q in an electric field $\mathbf{E}$ by the line segment $d\mathbf{r}$ then **on** the charge one has to carry out the work

$$dW = \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = q\,\mathbf{E}(\mathbf{r})\cdot d\mathbf{r}\;.$$

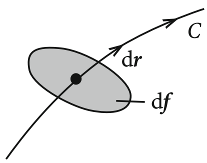


Fig. 3.7 To the introduction of the concept of the 'thread of current'

If this is done within the time interval dt the charge possesses the velocity $\mathbf{v} = d\mathbf{r}/dt$, and the field provides the power

$$\frac{dW}{dt} = q\,\mathbf{E}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) \; .$$

Let us now consider a general charge density $\rho(\mathbf{r})$. Then the power provided by the field on the charged volume element is:

$$dP = \left[\rho(\mathbf{r}) d^3 r\right] \mathbf{E}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r})\, d^3 r \; .$$

This leads immediately to the term of the

power density $\quad \mathbf{j}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$

The total **power** P carried out by the field $\mathbf{E}$ on the system in the volume V then amounts to:

$$P = \int\limits_V \mathbf{j}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, d^3 r \; . \tag{3.12}$$

3.1.8 Special Case: Very Thin Wire $\Longrightarrow$ Thread of Current

For this it follows with (3.11) and (3.12):

$$P = I \int\limits_C \mathbf{E} \cdot d\mathbf{r} = I\,U = R\,I^2 = \frac{1}{R} U^2 \; , \quad \text{in the case of an ohmic conductor} \; . \tag{3.13}$$

In the stationary case the average velocity of the charge carriers in an ohmic conductor **does not** increase. That means that the effect of the field does no longer cause an enhancement of the kinetic energy of the carriers but is rather transferred via collision processes to the lattice components (ions). That manifests itself as heat energy = **Joule(an) heat**. One therefore also speaks of

$$P = R\,I^2 \quad \textbf{power loss} \; ,$$

which appears during the passage of the current I through the **ohmic load** R.

Unit

$$[P] = 1\,\mathrm{VA} = 1\,\mathrm{W} = 1\,\mathrm{J/s} \; . \tag{3.14}$$

3.2 Basics of Magnetostatics

3.2.1 Biot-Savart Law

The Coulomb law (2.11) represents, as an experimentally uniquely verified fact, the basis of the whole of electrostatics. In magnetostatics this role is played by the

Ampère's law

which describes the interaction between two current-carrying conductors (*threads of current*) (Fig. 3.8):

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{r}_{12})}{r_{12}^3} \,, \tag{3.15}$$

μ_0: **magnetic field constant (permeability of vacuum)**

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{Vs}}{\mathrm{Am}} \approx 1{,}2566 \cdot 10^{-6} \frac{\mathrm{N}}{\mathrm{A}^2} \,. \tag{3.16}$$

Comparing this quantity with the definition (2.15) of the dielectric constant ϵ_0 (permittivity of vacuum) we recognize:

$$\epsilon_0 \, \mu_0 \, c^2 = 1 \,. \tag{3.17}$$

c is thereby the speed of light in vacuum (2.14). The two constants ϵ_0 and μ_0 are therefore not independent of each other. In the special theory of relativity the distinction between resting and moving charges is only a question of the reference system which implies an equivalence of the Coulomb and the Ampère law. The relation (3.17) between μ_0 and ϵ_0 is therefore not accidental.

For certain purposes it proves to be convenient to reformulate the force law (3.15) a little bit:

$$d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{r}_{12}) = d\mathbf{r}_2(d\mathbf{r}_1 \cdot \mathbf{r}_{12}) - \mathbf{r}_{12}(d\mathbf{r}_1 \cdot d\mathbf{r}_2) \,.$$

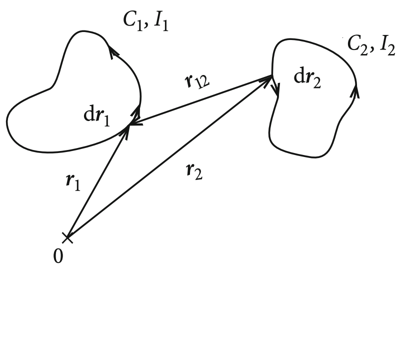


Fig. 3.8 Interaction between two current-carrying closed conductors

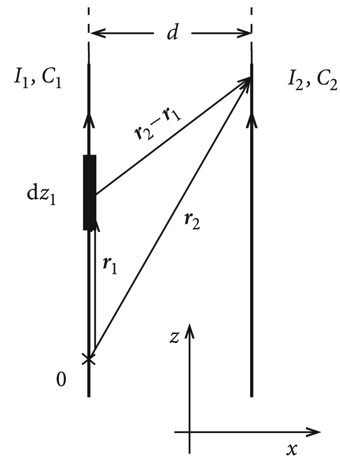


Fig. 3.9 Illustration of the mutual influence of two parallel current-carrying wires

The first summand in (3.15) does not contribute:

$$\oint_{C_1} d\mathbf{r}_1 \cdot \frac{\mathbf{r}_{12}}{r_{12}^3} = -\oint_{C_1} d\mathbf{r}_1 \cdot \nabla \frac{1}{r_{12}} = -\int_{A_{C_1}} d\mathbf{f} \cdot \operatorname{curl}\left(\operatorname{grad} \frac{1}{r_{12}}\right) = 0 \; .$$

It remains as force between the two threads of current:

$$\mathbf{F}_{12} = -\mu_0 \frac{I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} d\mathbf{r}_1 \cdot d\mathbf{r}_2 \; \frac{\mathbf{r}_{12}}{r_{12}^3} \; . \tag{3.18}$$

This representation reveals the symmetry between the two interacting partners. Obviously Newton's third axiom (law of reaction) is also fulfilled:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \; .$$

Example We consider two long, parallel, straight wires with the separation d through which the currents I_1 and I_2, respectively, are running (Fig. 3.9). Which force does the current-carrying conductor C_2 exert on the element dz_1 of the conductor C_1? In order to be allowed to apply (3.18) we bring to our mind that C_1, C_2 are both closed at infinity by large semicircles so that the contributions of the semicircles to the force on dz_1 do not play a role:

$$d\mathbf{F}_{12} = -\mu_0 \frac{I_1 I_2}{4\pi} dz_1 \int_{-\infty}^{+\infty} dz_2 \; \frac{\mathbf{r}_{12}}{r_{12}^3}$$

$$
\begin{aligned}
&= -\mu_0 \frac{I_1 I_2}{4\pi} dz_1 \int_{-\infty}^{+\infty} dz_2 \frac{-d\mathbf{e}_x - (z_2 - z_1)\mathbf{e}_z}{\left[d^2 + (z_2 - z_1)^2\right]^{3/2}} \\
&= \mu_0 d \frac{I_1 I_2}{4\pi} dz_1 \mathbf{e}_x \int_{-\infty}^{+\infty} \frac{dz_2}{\left[d^2 + (z_2 - z_1)^2\right]^{3/2}} \\
&= \mu_0 d \frac{I_1 I_2}{4\pi} dz_1 \mathbf{e}_x \left\{ \frac{(z_2 - z_1)}{d^2 \left[d^2 + (z_2 - z_1)^2\right]^{1/2}} \right\}_{z_2=-\infty}^{z_2=+\infty} \\
&= \mu_0 \frac{I_1 I_2}{2\pi d} dz_1 \mathbf{e}_x \; .
\end{aligned}
$$

The **force per length** exerted by C_2 on C_1,

$$
\mathbf{f}_{12} = \mu_0 \frac{I_1 I_2}{2\pi\, d} \mathbf{e}_x \; , \tag{3.19}
$$

thus acts perpendicular to both the current directions, and is attractive if the currents are in the same direction, and is repulsive if they are oppositely directed. This relation serves in the system SI for fixing the unit of measurement of the electric current. One considers two infinitely long, parallel, straight threads of current with a distance of 1 m, through which the same currents $I_1 = I_2 = I$ are running. The current I amounts to just 1 A if then according to (3.19) on a 1 m long conductor-piece a force of 2×10^{-7} N is exerted.

We now use (3.15) to define the **magnetic induction** generated by the current I_2 in the loop C_2,

$$
\mathbf{B}_2(\mathbf{r}_1) = \mu_0 \frac{I_2}{4\pi} \oint_{C_2} \frac{d\mathbf{r}_2 \times \mathbf{r}_{12}}{r_{12}^3} \; , \tag{3.20}
$$

analogously to the procedure which we used in (2.20) to introduce the electric field $\mathbf{E}(\mathbf{r})$ via the Coulomb force between point charges. The current I_1 in the conductor loop C_1 interacts with the $\mathbf{B}$-field generated by the current I_2:

$$
\mathbf{F}_{12} = I_1 \oint_{C_1} d\mathbf{r}_1 \times \mathbf{B}_2(\mathbf{r}_1) \; . \tag{3.21}
$$

Example Magnetic induction of a straight conductor:

$$
\mathbf{B}(\mathbf{r}) = \mu_0 \frac{I}{4\pi} \int_C \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \; .
$$

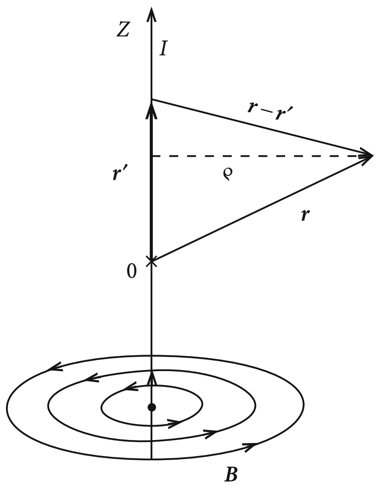


Fig. 3.10 For the calculation of the magnetic induction of a linear electric conductor

Cylindrical coordinates (Fig. 3.10):

$$\mathbf{r}' = z'\mathbf{e}_z \implies d\mathbf{r}' = dz'\mathbf{e}_z \;,$$
$$\mathbf{r} - \mathbf{r}' = \rho\,\mathbf{e}_\rho + (z - z')\mathbf{e}_z$$
$$\implies \mathbf{B}(\mathbf{r}) = \mu_0 \frac{I}{4\pi}\rho\,\mathbf{e}_\varphi \int\limits_{-\infty}^{+\infty} \frac{dz'}{[\rho^2 + (z - z')^2]^{3/2}} \;.$$

It follows:

$$\mathbf{B}(\mathbf{r}) = \mu_0 \frac{I}{2\pi\,\rho}\,\mathbf{e}_\varphi \;. \tag{3.22}$$

The **B**-lines are thus concentric circles around the linear conductor. They revolve round the current in the sense of a right-hand helix (Fig. 3.10). The magnitude of the magnetic induction is proportional to the strength of current I and is inversely proportional to the perpendicular distance ρ from the conductor.

Formula (3.20), the so-called **Biot-Savart law**, shall now be extended to arbitrary current densities in a similar way as was done in the electrostatics with the transition from point charges to spatial charge distributions $\rho(\mathbf{r})$ (2.23). For that we simply use (3.11) in (3.20):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r'\,\mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \;. \tag{3.23}$$

The analogy to the electrostatics is obvious. The comparison of the above result with the expression (2.23) for the electric field $\mathbf{E}(\mathbf{r})$ shows that the product of charge density ρ and vector $(\mathbf{r} - \mathbf{r}')$ is now replaced by the vector product of $\mathbf{j}$ and $(\mathbf{r} - \mathbf{r}')$. The transition from (3.20) to (3.23) has implicitly needed the postulate of the **superposition property of magnetic fields** which represents together with the Ampère's law (3.15) the experimental basis of the magnetostatics.

Eventually we still find with (3.11) in (3.21) the force which a current density $\mathbf{j}(\mathbf{r})$ experiences due to the $\mathbf{B}$-field of **another** current density:

$$\mathbf{F} = \int [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\, d^3r \,. \tag{3.24}$$

Example

$$\begin{aligned} &\text{point charge:} \quad \rho(\mathbf{r}) = q\,\delta(\mathbf{r} - \mathbf{r}_0)\,; \quad \mathbf{j}(\mathbf{r}) = q\,\mathbf{v}(\mathbf{r})\,\delta(\mathbf{r} - \mathbf{r}_0) \\ &\Longrightarrow \quad \mathbf{F} = q\,\mathbf{v}(\mathbf{r}_0) \times \mathbf{B}(\mathbf{r}_0)\,. \end{aligned} \tag{3.25}$$

That is the so-called **Lorentz force**, strictly speaking it is its *magnetic part*.

The magnetic induction $\mathbf{B}(\mathbf{r})$ exerts further a torque $\mathbf{M}$ on the current density $\mathbf{j}$:

$$\mathbf{M} = \int \{\mathbf{r} \times [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\}\, d^3r \,. \tag{3.26}$$

Finally we still have to introduce the unit of the magnetic induction:

$$[\mathbf{B}] : 1\,\frac{\mathrm{N}}{\mathrm{A\,m}} = 1\,\frac{\mathrm{V\,s}}{\mathrm{m}^2} = 1\ \text{tesla} \ \ (1\,\mathrm{T})\,. \tag{3.27}$$

3.2.2 Maxwell Equations

The fundamental Biot-Savart law (3.23) can be further rearranged (see (1.289), Vol. 1). If one uses

$$\begin{aligned} \nabla_r \times \left[\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right] &= \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla_r \times \mathbf{j}(\mathbf{r}') - \mathbf{j}(\mathbf{r}') \times \nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= \mathbf{j}(\mathbf{r}') \times \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \end{aligned}$$

in (3.23) then one realizes that $\mathbf{B}$ can be written as the curl of a vector field:

$$\mathbf{B}(\mathbf{r}) = \nabla_r \times \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{3.28}$$

The magnetic induction is therefore a pure curl-field and thus source-free:

$$\operatorname{div} \mathbf{B} = \nabla \cdot \mathbf{B} = 0 \,. \tag{3.29}$$

This is the **homogeneous** Maxwell equation of the magnetostatics. The corresponding integral form is found with the aid of the Gauss theorem:

$$\oint\limits_{S(V)} \mathbf{B}(\mathbf{r}) \cdot d\mathbf{f} = 0 \,. \tag{3.30}$$

The flux through the surface $S(V)$ of an **arbitrary** volume V equals zero. That expresses the fact that magnetic charges (monopoles) do not exist.

According to the general decomposition theorem (1.71) for vector fields it must hold for $\mathbf{B}(\mathbf{r})$ because of (3.29):

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl}_r \left[\frac{1}{4\pi} \int d^3r' \frac{\operatorname{curl}_{r'} \mathbf{B}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] .$$

The comparison with (3.28) yields the **inhomogeneous** Maxwell equation of the magnetostatics:

$$\operatorname{curl} \mathbf{B} = \nabla \times \mathbf{B} = \mu_0 \, \mathbf{j} \,. \tag{3.31}$$

Using the Stokes theorem leads to the equivalent integral form:

$$\int\limits_{\partial F} \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int\limits_{F} \mathbf{j} \cdot d\mathbf{f} = \mu_0 I \,. \tag{3.32}$$

I is the current through the area F. This so-called **Ampère's magnetic flux law** can be rather useful for the calculation of the **B**-field in the case of highly symmetric current distributions, similarly to the Gauss theorem (2.35) in electrostatics. (One consider once more as a simple application example the magnetic induction of a straight conductor (3.22).)

3.2.3 Vector Potential

Equation (3.28) demonstrates that the magnetic induction $\mathbf{B}(\mathbf{r})$ can be written as the curl of the vector field **A**:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{3.33}$$

The so-called **'vector potential'** $\mathbf{A}(\mathbf{r})$ adopts for the magnetostatics the role which the scalar potential $\varphi(\mathbf{r})$ plays for the electrostatics. Note the formal similarity of (3.33) with (2.25). We have:

$$\mathbf{B} = \text{curl}\mathbf{A} \ . \tag{3.34}$$

The vector potential is by the above ansatz not uniquely determined, though. The physically relevant field quantity $\mathbf{B}$ is obviously invariant under a

gauge transformation

of the vector potential:

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \text{grad}\chi \ . \tag{3.35}$$

χ may be an arbitrary scalar function which can be fixed merely depending on the utility-points of view since in any case it holds:

$$\text{curl}\,\text{grad}\,\chi = 0$$

Example Homogeneous $\mathbf{B}$-field: $\mathbf{B} = B_0\,\mathbf{e}_z$.
We have shown as Exercise 1.5.7, Vol. 1 that then

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{1}{2}B_0(-y, x, 0)$$

is an allowed choice which leads to curl $\mathbf{A} = \mathbf{B}$.

An often applied convention is the

$$\textbf{Coulomb gauge:} \quad \text{div}\,\mathbf{A} = 0 \ . \tag{3.36}$$

(The notation will become clear later!) With this gauge the inhomogeneous Maxwell equation (3.31) reads:

$$\text{curl}\,\mathbf{B} = \text{grad}\,\text{div}\,\mathbf{A} - \Delta\,\mathbf{A} = -\Delta\,\mathbf{A} = \mu_0\,\mathbf{j} \ .$$

From this we come to the

'basic problem of magnetostatics'

given: 1. $\mathbf{j}$ in a relevant space-region V
2. boundary conditions on $S(V)$

to be found: solution of the partial, inhomogeneous, linear differential equation of second order for each component of $\mathbf{A}$:

$$\Delta\,\mathbf{A} = -\mu_0\,\mathbf{j} \ . \tag{3.37}$$

From a formal mathematical point of view this is the same way of looking at a problem as we encountered it in connection with the Poisson equation of electrostatics. That means the solution methods developed in Sect. 3.2 can directly be adopted.

It does not make sense to discuss already at this stage concrete boundary-value problems of the magnetostatics since we have formulated the latter **so far only for the vacuum**. Typical boundary conditions, however, are found more or less exclusively in matter!

3.2.4 Exercises

Exercise 3.2.1 A current is evenly distributed over the cross section of a straight conducting wire with the radius R. Determine by use of the Maxwell equations and by using simple symmetry considerations the $\mathbf{B}$-field inside and outside the conductor!

Exercise 3.2.2 A current-carrying, plane loop of conductor generates a magnetic induction $\mathbf{B}(\mathbf{r})$. The current-element at P interacts with the $\mathbf{B}$-field which is created by other current-elements. Calculate the total force which the conductor loop exerts on itself. Consider the conductor as a 'thread of current'.

Exercise 3.2.3 Consider a cylindrically-symmetric current-distribution in spherical coordinates:

$$\mathbf{j} = j(r, \vartheta)\mathbf{e}_\varphi \ .$$

1. Show that then the vector potential, too, has this structure:

   $$\mathbf{A}(\mathbf{r}) = A(r, \vartheta)\mathbf{e}_\varphi$$

   Hint: Expand $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ in the integral formula for $\mathbf{A}(\mathbf{r})$ in spherical harmonics.
2. Which scalar differential equation must be obeyed by $A(r, \vartheta)$?

Exercise 3.2.4 Through an infinitely long hollow cylinder with the inner radius R_1 and the outer radius $R_2 > R_1$ (Fig. 3.11) flows a homogeneous current I. Calculate the magnetic induction $\mathbf{B}$ everywhere in space. Sketch $|\mathbf{B}|$ as function of the distance from the z-axis.

Exercise 3.2.5 A coaxial cable consists of an infinitely long straight wire with a circular cross section of the radius ρ_1, which is coaxially surrounded by a metal cylinder with inner radius ρ_2 and outer radius ρ_3 (Fig. 3.12). Inner and outer conductor carry the currents I_1 and I_2, respectively, which are homogeneously distributed over the respective conductor-cross section.

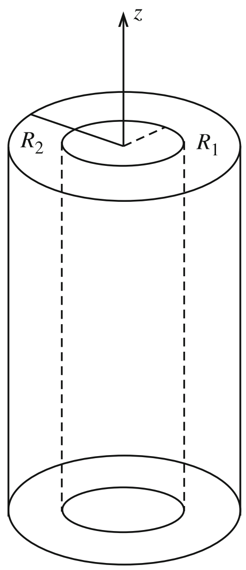


Fig. 3.11 Current-carrying, metallic hollow cylinder

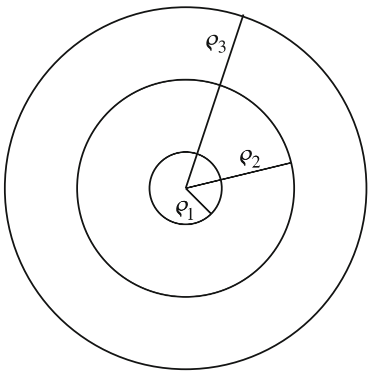


Fig. 3.12 Current-carrying coaxial cable

1. Calculate the **B**-field in the whole space! Sketch the field behavior for $I_1 = 0$ and $I_2 < 0$ as well as for $I_1 > 0$ and $I_2 = 0$!
2. What does happen for the special case $I_1 = -I_2 = I$? Sketch for this special case, too, the field behavior!

Exercise 3.2.6

1. For an infinitely extended conducting circular cylinder (radius R), which carries a current of constant density (Fig. 3.13)

$$\mathbf{j} = j\,\mathbf{e}_z \ ,$$

determine the magnetic induction for the whole space!

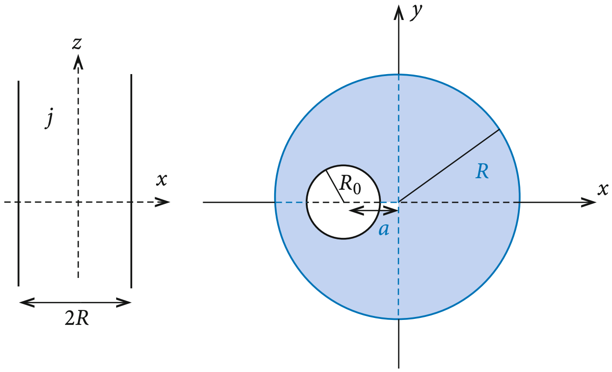


Fig. 3.13 Current-carrying, circular-cylindrical conductor with a paraxial drill hole

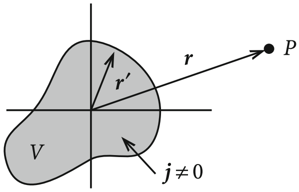


Fig. 3.14 System of coordinates for the calculation of the magnetic induction of a current-density distribution which is restricted to a finite spatial region

2. The conductor may have additionally a paraxially drilled hole (radius R_0) at the distance a ($a + R_0 < R$) from the cylinder-axis (Fig. 3.13). Determine the magnetic induction $\mathbf{B}$ within the drilled hole!

3.3 Magnetic Moment

3.3.1 Magnetic Induction of a Local Current Distribution

We consider a current-density distribution $\mathbf{j}(\mathbf{r})$ which is confined to a finite spatial region and creates at the observation point P a magnetic induction $\mathbf{B}(\mathbf{r})$ (Fig. 3.14). Let the distance of the point P from the $\mathbf{j} \neq 0$-region be very large compared to the linear dimensions of this region.

Starting point is the expression (3.33) for the vector potential $\mathbf{A}(\mathbf{r})$. For the denominator in the integrand a Taylor expansion appears to make sense as in Sect. 2.2.7:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}' \cdot \mathbf{r}}{r^3} + \dots \quad (\cong \text{ multipole expansion}) \,.$$

This means at first:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi\, r}\int d^3r'\,\mathbf{j}(\mathbf{r}') + \frac{\mu_0}{4\pi}\frac{1}{r^3}\int d^3r'(\mathbf{r}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}') + \ldots \tag{3.38}$$

For the further evaluation the following **lemma** will be useful:

Let $f(\mathbf{r}), g(\mathbf{r})$ be continuously differentiable, but otherwise arbitrary scalar fields. Then it holds in the magnetostatics:

$$\widehat{I} = \int d^3r\,[f(\mathbf{r})\,\mathbf{j}\cdot\nabla g + g(\mathbf{r})\,\mathbf{j}\cdot\nabla f] = 0\,. \tag{3.39}$$

Proof

$$\mathrm{div}\,(gf\,\mathbf{j}) = (gf)\underbrace{\mathrm{div}\,\mathbf{j}}_{=0\ (3.6)} + \mathbf{j}\cdot\mathrm{grad}(gf) = f\,(\mathbf{j}\cdot\nabla g) + g(\mathbf{j}\cdot\nabla f)$$

$$\Longrightarrow\ \widehat{I} = \int d^3r\,\mathrm{div}\,(gf\,\mathbf{j}) = \int\limits_{S(V)\to\infty} d\mathbf{f}\cdot(gf\,\mathbf{j})$$

$$= 0\,,\ \text{since the current density vanishes at infinity!}$$

At first we use (3.39) for

$$f \equiv 1 \quad \text{and} \quad g = x,\ y \ \text{ or }\ z$$

and obtain:

$$\int d^3r\;\mathbf{j}\cdot\mathbf{e}_{x,y,z} = 0\,.$$

This means:

$$\int d^3r\;\mathbf{j}(\mathbf{r}) = 0\,. \tag{3.40}$$

The first summand in (3.38), the **monopole-term**, thus vanishes. This is once more a confirmation of the fact that magnetic charges do not exist. It remains to be calculated:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{r}}{r^3}\int d^3r'\mathbf{r}'\,\mathbf{j}(\mathbf{r}')\ \ldots \tag{3.41}$$

For this purpose we use once more the lemma (3.39), namely for $f = x_i$, $g = x_j$, where $x_i, x_j \in \{x, y, z\}$:

$$0 = \int d^3r \left(x_i j_j + x_j j_i\right) \implies \int d^3r\, x_j j_i = -\int d^3r\, x_i j_j \,.$$

Therewith we calculate (**a** arbitrary vector):

$$\begin{aligned}\mathbf{a} \cdot \int d^3r' \mathbf{r}' j_i(\mathbf{r}') &= \sum_j a_j \int d^3r' x'_j j_i(\mathbf{r}') = -\frac{1}{2} \sum_j a_j \int d^3r' \left(x'_i j_j - x'_j j_i\right) \\ &= -\frac{1}{2} \sum_{j,k} \epsilon_{ijk} a_j \int d^3r' (\mathbf{r}' \times \mathbf{j})_k \,.\end{aligned}$$

In the last step we have exploited the definition of the vector product (see (1.95), Vol. 1). Applying the same formula once more we realize that the following vector-identity is valid:

$$\int d^3r' (\mathbf{a} \cdot \mathbf{r}')\, \mathbf{j}(\mathbf{r}') = -\frac{1}{2} \left\{ \mathbf{a} \times \int \left[\mathbf{r}' \times \mathbf{j}(\mathbf{r}')\right] d^3r' \right\} \,. \tag{3.42}$$

With the

Definition 3.3.1 *'magnetic moment'*

$$\mathbf{m} = \frac{1}{2} \int d^3r \,[\mathbf{r} \times \mathbf{j}(\mathbf{r})] \tag{3.43}$$

the vector potential has then for large distances from the $\mathbf{j} \neq 0$-region the following form ($\mathbf{a} = \mathbf{r}$):

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} + \dots \tag{3.44}$$

We restrict ourselves here to the lowest non-vanishing term of the expansion. Since the monopole-term is zero it is the dipole-term.

For the calculation of the magnetic induction we apply the formulas

$$\begin{aligned}\operatorname{curl}(\mathbf{a}\varphi) &= \varphi \operatorname{curl} \mathbf{a} - \mathbf{a} \times \nabla \varphi \,, \qquad ((1.289), \text{Vol. } 1) \\ \operatorname{curl}(\mathbf{a} \times \mathbf{b}) &= (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a} \operatorname{div} \mathbf{b} - \mathbf{b} \operatorname{div} \mathbf{a} \qquad (\text{Exercise } 1.7.13)\end{aligned}$$

and get with $\mathbf{m} = \text{const}$:

$$\operatorname{curl} \mathbf{A} = \frac{\mu_0}{4\pi} \left[\frac{1}{r^3} \operatorname{curl}(\mathbf{m} \times \mathbf{r}) - (\mathbf{m} \times \mathbf{r}) \times \nabla \frac{1}{r^3} \right]$$

$$= \frac{\mu_0}{4\pi}\left\{\frac{1}{r^3}\left[(\mathbf{r}\cdot\nabla)\,\mathbf{m} - (\mathbf{m}\cdot\nabla)\,\mathbf{r} + \mathbf{m}\mathrm{div}\,\mathbf{r} - \mathbf{r}\mathrm{div}\,\mathbf{m}\right] + \frac{3}{r^5}(\mathbf{m}\times\mathbf{r})\times\mathbf{r}\right\}$$

$$= \frac{\mu_0}{4\pi}\left\{-\frac{1}{r^3}(\mathbf{m}\cdot\nabla)\,\mathbf{r} + \frac{1}{r^3}\mathbf{m}\mathrm{div}\,\mathbf{r} - \frac{3}{r^5}\left[\mathbf{m}r^2 - \mathbf{r}(\mathbf{m}\cdot\mathbf{r})\right]\right\}$$

$$= \frac{\mu_0}{4\pi}\left[-\frac{\mathbf{m}}{r^3} + \frac{3}{r^5}\mathbf{r}(\mathbf{m}\cdot\mathbf{r})\right] .$$

B has therefore the same mathematical form as the analogous electrostatic dipole field $\mathbf{E}^D(\mathbf{r})$ (2.73):

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{r}\cdot\mathbf{m})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right] . \tag{3.45}$$

The magnetic induction **B** generated by **j** behaves, sufficiently far away from the current density distribution, always like a dipole field provided that the dipole **m** is defined as in (3.43).

Let us calculate explicitly the dipole moment **m** for two examples:

(1) Closed Plane Circuit of Current

We interpret the current loop as a thread of current and therefore use (3.11) in (3.43) (Fig. 3.15):

$$\mathbf{m} = \frac{1}{2} I \int_C (\mathbf{r}\times d\mathbf{r}) = I\,\mathbf{F} . \tag{3.46}$$

This simple result holds independently of the shape of the enclosed area (see Exercise 1.7.19). **m** stands perpendicularly on the conductor plane (right-handed screw rule!).

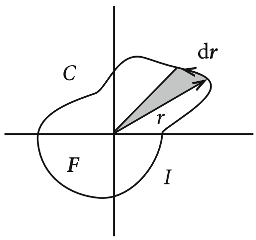


Fig. 3.15 For the determination of the magnetic moment of a closed plane current circuit

(2) System of Point Charges

The current density $\mathbf{j}$ is now produced by a great many of charged particles which are all considered as point charges. Let us assume that each of the particles has the same charge q and the same mass M. Let the i-th particle at the time t be at the position $\mathbf{R}_i(t)$ and have the velocity $\mathbf{v}_i(t)$:

$$\mathbf{j}(\mathbf{r}) = q \sum_{i=1}^{N} \mathbf{v}_i \, \delta(\mathbf{r} - \mathbf{R}_i) \; . \tag{3.47}$$

Inserted into (3.43) this yields:

$$\mathbf{m} = \frac{1}{2} q \sum_{i=1}^{N} (\mathbf{R}_i \times \mathbf{v}_i) = \frac{q}{2M} \sum_{i=1}^{N} \mathbf{l}_i \; , \tag{3.48}$$

$\mathbf{l}_i$ is the orbital angular momentum of the i-th particle.

The ratio of magnetic moment and total angular momentum $\mathbf{L} = \sum_i \mathbf{l}_i$ is denoted as **'gyromagnetic ratio'**. The factor $q/2M$, which is derived here purely classically, remains valid even down to the atomic region, i.e. even for the electrons in an atom, so long as it is about their orbital motion. For the intrinsic angular momentum (*spin*) $\mathbf{S}$ of the electron, however, the corresponding magnetic moment $\mathbf{m}_S$ is rather exactly twice as large compared to what would be expected according to (3.48):

$$\mathbf{m}_S = \frac{-e}{M} \mathbf{S} \; . \tag{3.49}$$

An explanation for this deviation from the classical expectation, which is called **magnetomechanical anomaly**, needs the relativistic Dirac theory of the electron which is however, beyond the scope of the presentation in this volume. We have to shift it to Sect. 5.2 in Vol. 5/2.

3.3.2 Force and Torque on a Local Current Distribution

On the current density $\mathbf{j}(\mathbf{r})$, according to (3.24) and (3.26), an external magnetic induction $\mathbf{B}(\mathbf{r})$ exerts the force (Fig. 3.16)

$$\mathbf{F} = \int [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})] \, d^3r$$

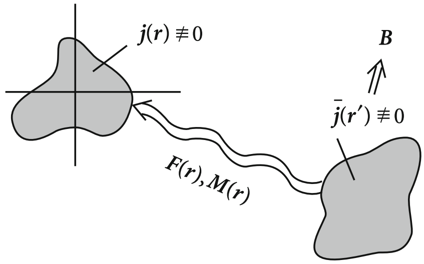


Fig. 3.16 Interaction between two local current-density distributions

and the torque

$$\mathbf{M} = \int \{\mathbf{r} \times [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\}\, d^3r\ .$$

We will investigate these relations now for the case that **B** changes only very little within the $\mathbf{j} \neq 0$-region, which is assumed to be locally restricted. Then a Taylor expansion of **B** with respect to the origin $\mathbf{r} = 0$ located in the $j \neq 0$-region will be useful:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + (\mathbf{r} \cdot \nabla)\mathbf{B}(\mathbf{r})|_{\mathbf{r}=0} + \dots$$

This yields for **F**:

$$\mathbf{F} = -\mathbf{B}(0) \times \int \mathbf{j}(\mathbf{r})\, d^3r + \int [\mathbf{j}(\mathbf{r}) \times (\mathbf{r} \cdot \nabla)\mathbf{B}(0)]\, d^3r + \dots$$

Because of (3.40) the first summand vanishes, i.e. **a homogeneous B-field does not execute a force on a stationary current distribution.**

We calculate the i-th component of the force:

$$F_i \approx -\int d^3r\, [(\mathbf{r} \cdot \nabla)\mathbf{B}(0) \times \mathbf{j}(\mathbf{r})]_i = -\sum_{j,k} \epsilon_{ijk} \left(\int \mathbf{r}\, j_k(\mathbf{r})\, d^3r \right) \cdot \left[\nabla B_j(0)\right]\ .$$

At this stage we can exploit the vector-identity (3.42) where we take $\mathbf{a} = \nabla B_j$:

$$\begin{aligned} F_i &\approx +\frac{1}{2} \sum_{j,k} \epsilon_{ijk} \left\{ \left[\nabla B_j(0)\right] \times \int [\mathbf{r} \times \mathbf{j}(\mathbf{r})]\, d^3r \right\}_k \\ &= -\sum_{jk} \epsilon_{ijk} \left[\mathbf{m} \times \nabla B_j(0)\right]_k = -\sum_{j,k} \epsilon_{ijk} [\mathbf{m} \times \nabla]_k B_j(0) \\ &= \sum_{j,k} \epsilon_{ijk} [\mathbf{m} \times \nabla]_j B_k(0) = [(\mathbf{m} \times \nabla) \times \mathbf{B}(0)]_i\ . \end{aligned}$$

We have found therewith the following expression for the force $\mathbf{F}$ on the current distribution $\mathbf{j}$:

$$\mathbf{F} \approx (\mathbf{m} \times \nabla) \times \mathbf{B}(0) \ . \tag{3.50}$$

However, one should not forget that it is the result with only the first non-vanishing term of an infinite expansion:

$$\mathbf{F} \approx -\mathbf{m}\,[\nabla \cdot \mathbf{B}(0)] + \nabla\,[\mathbf{m} \cdot \mathbf{B}(0)] \ .$$

It then remains with $\operatorname{div} \mathbf{B} = 0$:

$$\mathbf{F} \simeq \nabla(\mathbf{m} \cdot \mathbf{B}) \ . \tag{3.51}$$

Note that this expression, too, formally agrees exactly with the analogous relation (2.78) of the electrostatics. Generally the (conservative) force is defined as the negative gradient of a potential energy V. This means:

$$V = -\mathbf{m} \cdot \mathbf{B} \ . \tag{3.52}$$

The dipole will try to orient itself parallel to the $\mathbf{B}$-field in order to reach the state of minimal energy.

The magnetic induction $\mathbf{B}$ therefore exerts on the current distribution $\mathbf{j}$ a torque $\mathbf{M}$. In contrast to the force $\mathbf{F}$, already the first term of the above field expansion contributes to the torque. Let us restrict ourselves here to this leading term:

$$\begin{aligned}\mathbf{M} &\approx \int \{\mathbf{r} \times [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(0)]\}\, d^3r \\ &= \int d^3r\, \{\mathbf{j}(\mathbf{r})(\mathbf{r} \cdot \mathbf{B}) - \mathbf{B}\,[\mathbf{r} \cdot \mathbf{j}(\mathbf{r})]\}\end{aligned}$$

We apply once more the lemma (3.39), now with $f = g = r$:

$$0 = 2\int d^3r(r\,\mathbf{j} \cdot \nabla r) = 2\int d^3r(r\,\mathbf{j} \cdot \mathbf{e}_r) = 2\int d^3r(\mathbf{j} \cdot \mathbf{r}) \ .$$

Therewith the second summand in the $\mathbf{M}$-relation vanishes:

$$\mathbf{M} \approx \int d^3r\,[\mathbf{r} \cdot \mathbf{B}(0)]\;\mathbf{j}(\mathbf{r}) \ .$$

At this stage we use again the vector-identity (3.42), now with $\mathbf{a} = \mathbf{B}(0)$:

$$\mathbf{M} \approx -\frac{1}{2}\left\{\mathbf{B}(0) \times \int d^3r\,[\mathbf{r} \times \mathbf{j}(\mathbf{r})]\right\} \ .$$

With the definition (3.43) of the magnetic moment $\mathbf{m}$ the leading term in the expansion of the torque takes the form

$$\mathbf{M} \approx \mathbf{m} \times \mathbf{B}(0) \ , \tag{3.53}$$

which we have already discussed in the introduction of this chapter with (3.1) as a possibility to measure direction and magnitude of the field $\mathbf{B}$.

3.3.3 Exercises

Exercise 3.3.1 Calculate the vector potential $\mathbf{A}(\mathbf{r})$ and the magnetic induction $\mathbf{B}(\mathbf{r})$ of a circular conducting loop (thread of current!). The current density reads in cylindrical coordinates (ρ, φ, z):

$$\mathbf{j}(\mathbf{r}) = I\delta(\rho - R)\delta(z)\mathbf{e}_\varphi \ .$$

The calculation of $\mathbf{A}(\mathbf{r})$ leads to an elliptic integral which cannot be solved elementarily. Estimate this integral for the limits $\rho \ll R$ and $\rho \gg R$ by the use of suitable Taylor expansions. Show that for $\rho \gg R$ a dipole field emerges! Find the corresponding magnetic dipole moment!

Exercise 3.3.2 On the surface of a hollow sphere with radius R a charge q is homogeneously distributed. The sphere rotates with constant angular velocity $\boldsymbol{\omega}$ around one of its diameter.

1. Determine the current density $\mathbf{j}(\mathbf{r})$ generated by the rotation!
2. Calculate the magnetic moment of the sphere caused by $\mathbf{j}$!
3. Determine the vector potential $\mathbf{A}(\mathbf{r})$ inside and outside the hollow sphere! Express $\mathbf{A}(\mathbf{r})$ and the magnetic induction $\mathbf{B}(\mathbf{r})$ for $r > R$ by the magnetic moment of the sphere!

Exercise 3.3.3 Let the surface charge density of a hollow sphere with the radius R be distributed as

$$\sigma = \sigma_0 \cos\vartheta \ .$$

Calculate the magnetic moment of the sphere for the case that it

1. is moved translationally with the velocity $\mathbf{v}$ in x-direction,
2. rotates with the angular velocity $\boldsymbol{\omega}$ around an arbitrary axis through its center.

Exercise 3.3.4 Calculate the magnetic moments of the following systems:

1. Solid sphere (radius R, charge Q), which rotates with constant angular velocity ω around a space-fixed axis through the center of the sphere.

2. Hollow sphere (radius R) with the charge density

$$\rho(\mathbf{r}) = \sigma_0 \delta(r - R) \cos^2 \vartheta \ ,$$

which rotates with constant angular velocity ω around a space-fixed axis through the center of the sphere ($\vartheta = \sphericalangle(\boldsymbol{\omega}, \mathbf{r})$).

Exercise 3.3.5 Given a densely wrapped coil of the length L (coil radius R, number of turns n), through which a direct current I flows.

1. Calculate the magnetic induction on the axis (z-direction).
2. Discuss the limiting cases $L \gg R$ and $L \ll R$.
3. Calculate the magnetic moment $\mathbf{m}$ of the coil.
4. How does the magnetic induction $\mathbf{B}(\mathbf{r})$ look like at a great distance from the coil center?

3.4 Magnetostatics in Matter

So far we have always presumed that the current density $\mathbf{j}$ is a given and therewith a known quantity. This presumption can, strictly speaking, no longer be taken as the starting point when we investigate the magnetostatics in matter. The electrons of the atomic elements of the solid build complicated, rapidly fluctuating, microscopic currents which, according to (3.46), give rise to magnetic moments which, in turn, provide, according to (3.45), contributions to the magnetic induction $\mathbf{B}$. The quantitative registration of these contributions appears to be impossible. But as already explained in detail in the corresponding Sect. 2.4 of the electrostatics, averaged field quantities are entirely sufficient (see (2.179)).

3.4.1 Macroscopic Field Quantities

We start again with the assumption that the Maxwell equations in the vacuum (3.29) and (3.31) are, from a microscopic point of view, universally valid:

$$\operatorname{div} \mathbf{b} = 0; \qquad \operatorname{curl} \mathbf{b} = \mu_0 \, \mathbf{j}_\mathrm{m} \ . \tag{3.54}$$

$\mathbf{j}_\mathrm{m}$ is the microscopic current density and $\mathbf{b}$ the microscopic magnetic induction. The averaging process for field quantities introduced and explained in (2.179) defines the

macroscopic magnetic induction

$$\mathbf{B}(\mathbf{r}) = \overline{\mathbf{b}(\mathbf{r})} \ . \tag{3.55}$$

Because of the permutability of averaging and differentiation the homogeneous Maxwell equation remains formally unchanged after the averaging:

$$\operatorname{div} \mathbf{B} = 0 \,. \tag{3.56}$$

The macroscopic **B**-field is therewith, just as the microscopic one, a pure curl-field, i.e. as in (3.34) we can define a vector potential $\mathbf{A}(\mathbf{r})$:

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \,. \tag{3.57}$$

However, what is the averaged current density $\overline{\mathbf{j}}_m$? It consists of two components. There will be contributions from free, i.e. not bound (*susceptible to manipulation*) charges. Think, for instance, of the quasi-free conduction electrons. But the bound charges will also react to the fields. They will be shifted and therewith will create certain currents:

$$\overline{\mathbf{j}}_m = \mathbf{j}_f + \mathbf{j}_{\text{bound}} \,. \tag{3.58}$$

If ρ_f is the charge density of the not bound particles then it holds for its contribution to the current density:

$$\mathbf{j}_f = \overline{\rho_f \, \mathbf{v}} \,. \tag{3.59}$$

It is expedient to decompose the *'bound'* current density still into two further constituents:

$$\mathbf{j}_{\text{bound}} = \overline{\mathbf{j}}_p + \overline{\mathbf{j}}_{\text{mag}} \,. \tag{3.60}$$

$\overline{\mathbf{j}}_p$ is the current density of the polarization charges. The polarization $\mathbf{P}(\mathbf{r})$ brings about, according to (2.189), a polarization charge density

$$\rho_p(\mathbf{r}) = -\operatorname{div} \mathbf{P} \,,$$

which of course obeys a continuity equation:

$$\frac{\partial}{\partial t} \rho_p(\mathbf{r}) + \operatorname{div} \overline{\mathbf{j}}_p = 0$$

That explains the current density $\overline{\mathbf{j}}_p$:

$$\overline{\mathbf{j}}_p(\mathbf{r}) = \frac{\partial}{\partial t} \mathbf{P} \,. \tag{3.61}$$

As partial time-derivative it does not play, however, a role for the magneto**statics**. We will have to come back to this point later when treating **dynamic** phenomena.

Here the **magnetization-current density** $\overline{\mathbf{j}}_{\mathrm{mag}}$ is decisive. It results from the motions of the atomic electrons in their stationary orbits around the respective positively charged nuclei. Each of these motions represents a tiny magnetic dipole. Without an external field, the directions of these dipoles are randomly oriented, on an average therefore mutually compensating their actions. According to (3.53) an external field will execute a torque on the elementary dipole, therewith taking care for a certain ordering that eventually leads to an additional internal field $\mathbf{B}_{\mathrm{mag}}$. We imagine that this additional field is due to a current density $\mathbf{j}_{\mathrm{mag}}$. Let $\mathbf{j}_{\mathrm{mag}}^{(i)}(\mathbf{r})$ be the magnetization current density of the i-th *particle* which we assume to be *stationary* (3.6):

$$\operatorname{div} \mathbf{j}_{\mathrm{mag}}^{(i)} = 0 \; . \tag{3.62}$$

Furthermore, it generates the moment $\mathbf{m}_i$:

$$\mathbf{m}_i = \frac{1}{2} \int d^3r \left[(\mathbf{r} - \mathbf{R}_i) \times \mathbf{j}_{\mathrm{mag}}^{(i)}(\mathbf{r}) \right] \; . \tag{3.63}$$

$\mathbf{R}_i$ shall be the position of the dipole. The fulfillment of these two equations succeeds with the following, relatively general ansatz:

$$\mathbf{j}_{\mathrm{mag}}^{(i)}(\mathbf{r}) = -\mathbf{m}_i \times \nabla f_i(\mathbf{r}) = \operatorname{curl}\, (\mathbf{m}_i f_i(\mathbf{r})) \; . \tag{3.64}$$

The function f_i is thought here only as an in-between quantity. Its precise meaning is not so important. It has to fulfill only the following conditions:

1. f_i is *smooth* within the volume occupied by the i-th particle and outside identical to zero.
2.
$$\int\limits_{\text{particle } (v_i)} d^3r f_i(\mathbf{r}) = 1 \; . \tag{3.65}$$

That the ansatz (3.64) fulfills the condition (3.62) is immediately clear and Eq. (3.63) can be verified by insertion:

$$\begin{aligned}
\mathbf{m}_i &= -\frac{1}{2} \int d^3r \, (\mathbf{r} - \mathbf{R}_i) \times (\mathbf{m}_i \times \nabla f_i) \\
&= -\frac{1}{2} \int d^3r \, \{ \mathbf{m}_i \, [(\mathbf{r} - \mathbf{R}_i) \cdot \nabla f_i] - \nabla f_i \, [(\mathbf{r} - \mathbf{R}_i) \cdot \mathbf{m}_i] \} \\
&= -\frac{1}{2} \mathbf{m}_i \int d^3r \, (\mathbf{r} - \mathbf{R}_i) \nabla f_i + \frac{1}{2} \int d^3r \, \nabla f_i \, [(\mathbf{r} - \mathbf{R}_i) \cdot \mathbf{m}_i]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\mathbf{m}_i \int d^3r \mathrm{div}\, [(\mathbf{r}-\mathbf{R}_i) f_i(\mathbf{r})] + \frac{1}{2}\mathbf{m}_i \int d^3r f_i(\mathbf{r}) \mathrm{div}\, (\mathbf{r}-\mathbf{R}_i) \\
&\quad + \frac{1}{2}\int d^3r\, \nabla \left\{ f_i \left[(\mathbf{r}-\mathbf{R}_i)\cdot \mathbf{m}_i \right] \right\} - \frac{1}{2}\int d^3r f_i \nabla \left[\mathbf{m}_i \cdot (\mathbf{r}-\mathbf{R}_i) \right] \\
&= -\frac{1}{2}\mathbf{m}_i \int\limits_{S(V\to\infty)} d\mathbf{f}\cdot(\mathbf{r}-\mathbf{R}_i) f_i(r) + \frac{3}{2}\mathbf{m}_i \\
&\quad + \frac{1}{2}\int\limits_{S(V\to\infty)} d\mathbf{f} f_i(\mathbf{r}) \left[(\mathbf{r}-\mathbf{R}_i)\cdot \mathbf{m}_i \right] - \frac{1}{2}\mathbf{m}_i \int d^3r f_i(\mathbf{r}) \\
&= \mathbf{m}_i
\end{aligned}
$$

The surface integrals do not contribute since, because of 1., f disappears at infinity. In the penultimate step we have used the Gauss theorem in its ordinary form (1.53) and in its form (1.56). Moreover, condition 2. has been fulfilled.

We are now convinced that Eq. (3.64) is for our purposes here a reasonable ansatz. We perform the averaging,

$$
\overline{\mathbf{j}_{\mathrm{mag}}(\mathbf{r})} = \mathrm{curl}\left(\overline{\sum_i \mathbf{m}_i f_i} \right) = \mathrm{curl}\, \mathbf{M}(\mathbf{r}) \ ,
$$

and define

$$
\mathbf{M}(\mathbf{r}) = \sum_i \overline{\mathbf{m}_i f_i(\mathbf{r})} \tag{3.66}
$$

as **magnetization.**

The function f_i has, because of (3.65), the dimension *1/volume*, the magnetization therewith the dimension *magnetic moment per volume*. Performing the averaging explicitly we get:

$$
\begin{aligned}
\mathbf{M}(\mathbf{r}) &= \frac{1}{v}\int\limits_{v(\mathbf{r})} d^3r' \left(\sum_i \mathbf{m}_i f_i(\mathbf{r}') \right) \\
&= \frac{1}{v(\mathbf{r})} \sum_{i=1}^{N(v(\mathbf{r}))} \mathbf{m}_i \int\limits_{v_i} d^3r' f_i(\mathbf{r}') \ .
\end{aligned}
$$

v_i is the volume of the i-th particle. It follows:

$$
\mathbf{M}(\mathbf{r}) = \frac{1}{v(\mathbf{r})} \sum_{i=1}^{N(v(\mathbf{r}))} \mathbf{m}_i \ . \tag{3.67}
$$

This expression yields the illustrative meaning of the **magnetization** as **average magnetic moment per volume**. One should not forget, however, that just as Eq. (2.185) for the macroscopic polarization $\mathbf{P}(\mathbf{r})$, Eq. (3.67) *actually only defines* the magnetization $\mathbf{M}(\mathbf{r})$. The magnetic moments $\mathbf{m}_i$ are influenced by internal as well as external fields, i.e., $\mathbf{M}(\mathbf{r})$ will be a functional of these fields and as such a functional has to be calculated on the basis of appropriate theoretical models.

After these foregoing considerations we are now able to formulate the macroscopic inhomogeneous Maxwell equation. After averaging in (3.54) we have:

$$\operatorname{curl}\mathbf{B} = \mu_0\,\bar{\mathbf{j}}_{\mathrm{m}} = \mu_0\left(\mathbf{j}_{\mathrm{f}} + \bar{\mathbf{j}}_{\mathrm{p}} + \bar{\mathbf{j}}_{\mathrm{mag}}\right) = \mu_0\,\mathbf{j}_{\mathrm{f}} + \mu_0\dot{\mathbf{P}} + \mu_0\operatorname{curl}\mathbf{M}\,. \tag{3.68}$$

$\dot{\mathbf{P}}$ drops out in the case of magneto**statics**. We introduce a new field quantity:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \quad (\textbf{magnetic field})\,. \tag{3.69}$$

This definition of the macroscopic magnetic field $\mathbf{H}$ is chosen completely analogously to that of the dielectric displacement $\mathbf{D}$ in the electrostatics (2.187). In both cases they are, strictly speaking, only auxiliary quantities. The real measurands are $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})\,. \tag{3.70}$$

The **inhomogeneous Maxwell equation** now reads:

$$\operatorname{curl}\mathbf{H} = \mathbf{j}_{\mathrm{f}}\,. \tag{3.71}$$

$\mathbf{H}$ is connected to the *free* current, $\mathbf{B}$ to the real (total) current(comparison is valid here also with the electrostatics). $\mathbf{H}$ and $\mathbf{M}$ have the same dimension:

$$[\mathbf{H}] = [\mathbf{M}] = \frac{\mathrm{A}}{\mathrm{m}}\,. \tag{3.72}$$

Under specific preconditions (isotropic, linear medium) we can, in analogy to the relation (2.196), choose the ansatz:

$$\mathbf{M} = \chi_{\mathrm{m}}\mathbf{H}\,, \tag{3.73}$$

which defines the

magnetic susceptibility $\chi_{\mathbf{m}}$.

Because of (3.70) one finally introduces the

relative permeability $\mu_{\mathbf{r}} = 1 + \chi_{\mathbf{m}}$:

$$\mathbf{B} = (1 + \chi_{\mathrm{m}})\,\mu_0\mathbf{H} = \mu_{\mathrm{r}}\mu_0\mathbf{H}\,. \tag{3.74}$$

Non-magnetizable materials have $\chi_m = 0$. That holds in particular for vacuum:

$$\mathbf{B} = \mathbf{B}_0 = \mu_0 \mathbf{H} \quad \text{(vacuum!)} . \tag{3.75}$$

3.4.2 Classification of Magnetic Materials

The magnetic susceptibility χ_m is excellently suited to classify the magnetic materials. Contrary to its electrical counterpart χ_e it can also be negative.

(1) Diamagnetism

This manifestation of magnetism is characterized by:

$$\chi_m < 0 \, ; \quad \chi_m = \text{const} . \tag{3.76}$$

The diamagnetism turns out to be a pure induction effect. Diamagnets do not contain any permanent magnetic dipoles. Only when a magnetic field is switched on such dipoles appear as *induced* moments. According to Lenz's law, which we are going to discuss in the next section, the induced dipoles (vectors!) are opposed to the applied field. χ_m is therefore negative. Further on, typical is that χ_m is almost temperature- and field-**independent** as well as it is very small in magnitude:

$$|\chi_m| \approx 10^{-5} .$$

Diamagnetism is a property of **all** substances. One speaks of diamagnetism, however, only if there are not additional components of paramagnetism or collective magnetism (see below) present, which bury the relatively weak diamagnetism.

Examples

- almost all organic substances,
- noble metals such as Zn, Hg,
- nonmetals as S, I, Si,
- superconductors (Meissner-Ochsenfeld effect: $\chi_m = -1 \implies$ ideal diamagnets).

(2) Paramagnetism

Decisive precondition for paramagnetism is the existence of **permanent** magnetic dipoles which in a field are aligned more or less (cf with the orientation polarization of the paraelectrica). To this orientating tendency, the disordering tendency due to

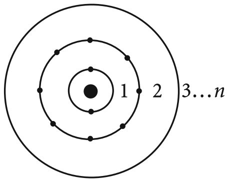


Fig. 3.17 Simplest idea of the electron shells in an atom

the thermal motion is opposed. It is therefore typical for paramagnets:

$$\chi_\mathrm{m} > 0 \; ; \quad \chi_\mathrm{m} = \chi_\mathrm{m}(T) \; . \tag{3.77}$$

These permanent dipoles can be strictly localized at certain lattice sites. That is in particular the case when an **inner** electron shell of an atomic constituent of the system is not completely filled (Fig. 3.17). An electron shell can accept at most $2n^2$ electrons, where the so-called principal quantum number n runs *from the center outwards* through the values $n = 1, 2, 3, \ldots$. Each electron has an orbital angular momentum $\mathbf{l}_i$. For a closed, i.e. completely occupied electron shell, the angular momenta compensate each other to yield the total angular momentum $\mathbf{L} = \sum_i \mathbf{l}_i = 0$. If the shell is **not** completely occupied then $\mathbf{L} \neq 0$ and therewith according to (3.48) a magnetic moment $\mathbf{m}$ results. – This situation is typical for **magnetic insulators**, the susceptibility of which obeys for high temperatures the

Curie Law

$$\chi_\mathrm{m}(T) = \frac{C}{T} \tag{3.78}$$

Even the quasi-free conduction electrons of a metallic solid carry, due to their spins, a permanent magnetic moment (see (3.49)). That leads to the so-called *Pauli paramagnetism* with a susceptibility which in contrast to (3.78) is practically temperature-**in**dependent.

(3) Collective Magnetism

The susceptibility here is in general a very complicated function of the external magnetic field as well as the temperature:

$$\chi_\mathrm{m} = \chi_\mathrm{m}(T, H) \; . \tag{3.79}$$

Precondition is as in (2) the existence of permanent magnetic dipoles which, however, in consequence of an only quantummechanically understandable *exchange interaction* align themselves in an ordered manner below a critical temperature T^*, and that spontaneously, i.e. without being forced by an external field. The permanent magnetic moments can be

localized (Gd, EuO, Rb_2MnCl_4 . . .)

but also

mobile (*itinerant*) (Fe, Co, Ni, . . .).

The collective magnetism can further be grouped into three big subclasses:

(3.1) Ferromagnetism

In this case the critical temperature is called

$$T^* = T_C: \quad \textbf{Curie temperature} .$$

At the absolute zero ($T = 0$) all moments are parallelly aligned ($\uparrow\uparrow\uparrow\uparrow\uparrow$), for $0 < T < T_C$ a certain disorder sets in which becomes stronger with increasing temperature ($\nearrow\nwarrow\nearrow\nwarrow$), where, however, a non-zero total magnetization still exists. For $T > T_C$ the ferromagnet behaves like a normal paramagnet. The Curie temperatures of some prominent ferromagnets are shown in the following table:

substance:	Fe	Co	Ni	Gd	EuO	$CrBr_3$
T_C [K]:	1043	1393	631	290	69	37

A typical feature of a ferromagnet is on the one hand the rather large absolute value of the susceptibility χ_m and, on the other, its strong dependence on the *'pre-treatment', 'history'* of the material that leads to the so-called

hysteresis curve

(Fig. 3.18). When switching on the magnetic field the 'virgin' material will at first be magnetized along the

initial (magnetization) curve (a),

in order to reach finally a

saturation M_S (b)

After switching off the field, there remains a finite rest-magnetization which is called

remanence (c)

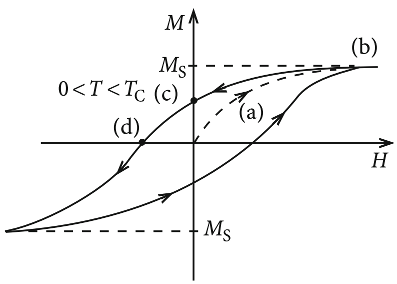


Fig. 3.18 Hysteresis curve of a ferromagnet

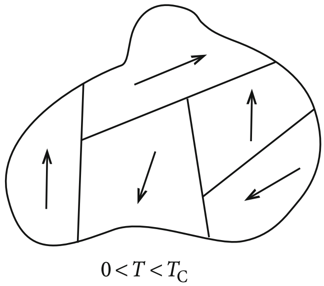


Fig. 3.19 Weiss domains of a ferromagnet. The *arrows* indicate the directions of the magnetization

To remove it one needs an opposing field, the

coercive force (d),

The property (c) defines the

permanent magnet.

The hysteresis-loop is caused, in the final analysis, by the fact that the macroscopic material decomposes into small, microscopic regions, the so-called

Weiss domains.

The respective domains are spontaneously magnetized but in different directions because of thermodynamic, energetic reasons (Fig. 3.19). The external field H brings these more and more in line until they attain the final parallel orientation ('*saturation*'). It goes without saying that for ferromagnets the linear relation (3.73) **does not** hold.

(3.2) Ferrimagnetism

In this case the lattice of the solid consists of two ferromagnetic sublattices A and B with different magnetizations

$$\mathbf{M}_\mathrm{A} \neq \mathbf{M}_\mathrm{B}$$

where

$$\mathbf{M} = \mathbf{M}_\mathrm{A} + \mathbf{M}_\mathrm{B} \neq 0 \quad \text{for } 0 \leq T < T_\mathrm{C} \, .$$

(3.3) Antiferromagnetism

It is a special case of the ferrimagnetism. The two sub-lattices have equal but opposite magnetizations. The critical temperature is denoted here as

$$T^* = T_\mathrm{N} : \quad \textbf{Néel temperature.}$$

The total magnetization

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_{\mathrm{A}} + \mathbf{M}_{\mathrm{B}} \\ \mathbf{M}_{\mathrm{A}} &= -\mathbf{M}_{\mathrm{B}} \end{aligned} \qquad (\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow)$$

is thus always zero. For temperatures above T_{N} the antiferromagnet, too, becomes a normal paramagnet. The linear relation (3.73) is **not** applicable.

3.4.3 Field-Behavior at Interfaces

With (3.37) we have formulated the basic problem of the magnetostatics. Concrete boundary conditions come often into play by the special behavior of the fields **B** and **H** at interfaces. That shall now be investigated more precisely where we apply the integral theorems of Sect. 2.1.4. We locate around the interface a

Gauss-casket (Fig. 3.20)

with the volume $\Delta V \approx \Delta F \cdot \Delta x$. Then we get:

$$0 = \int\limits_{\delta V} d^3 r \operatorname{div} \mathbf{B} = \int\limits_{S(\delta V)} d\mathbf{f} \cdot \mathbf{B} \underset{\delta x \to 0}{\longrightarrow} \Delta F\, \mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1)\ .$$

The normal component of the magnetic induction is therefore continuous at the interface:

$$\mathbf{B}_{2n} = \mathbf{B}_{1n}\ . \tag{3.80}$$

In case of different permeabilities $\mu_{\mathrm{r}}^{(1)}, \mu_{\mathrm{r}}^{(2)}$ of the two substances, however, this does not at all hold for the magnetic fields:

$$H_{2n} = \frac{\mu_{\mathrm{r}}^{(1)} H_{1n}}{\mu_{\mathrm{r}}^{(2)}}\ . \tag{3.81}$$

We now put around the interface a small

Stokes-area.

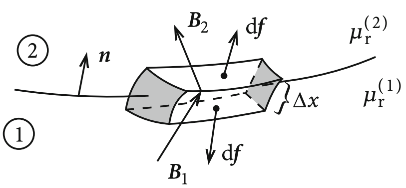


Fig. 3.20 Gauss-casket for the determination of the behavior of the magnetic induction at interfaces between two materials with different permeabilities

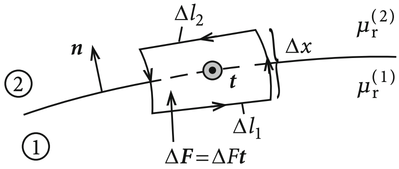


Fig. 3.21 'Stokes-area' for the determination of the behavior of the magnetic field and the magnetic induction at interfaces between two substances of different permeabilities

Let $\mathbf{t}$ be the surface normal of $\Delta\mathbf{F}$, directed tangentially on the interface. Then it holds (Fig. 3.21):

$$\Delta\mathbf{l}_2 = \Delta l(\mathbf{t} \times \mathbf{n}) = -\Delta\mathbf{l}_1 \ .$$

$\mathbf{j}_F$ is the surface-current density, i.e. the current per unit length on the interface.

$$\int_{\delta F} d\mathbf{f} \cdot \operatorname{curl} \mathbf{H} = \int_{\delta F} d\mathbf{f} \cdot \mathbf{j}_F \underset{\delta x \to 0}{\longrightarrow} (\mathbf{j}_F \cdot \mathbf{t})\delta l \ ,$$

$$\int_{\delta F} d\mathbf{f} \cdot \operatorname{curl} \mathbf{H} = \int_{\partial\delta F} d\mathbf{s} \cdot \mathbf{H} \underset{\delta x \to 0}{\longrightarrow} \Delta l(\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{H}_2 - \mathbf{H}_1) \ .$$

The comparison yields in this case:

$$(\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_F \cdot \mathbf{t} \ . \tag{3.82}$$

$(\mathbf{t} \times \mathbf{n})$ is a unit vector parallel to the Stokes-area. Thus, if there is no surface-current density then the tangential component of the $\mathbf{H}$-field is continuous:

$$\mathbf{j}_F = 0 : \ H_{2t} = H_{1t} \iff B_{2t} = \frac{\mu_r^{(2)}}{\mu_r^{(1)}} B_{1t} \ . \tag{3.83}$$

The magnetic induction, however, has even for $\mathbf{j}_F = 0$, a discontinuous tangential component.

3.4.4 Boundary-Value Problems

We had formulated in (3.37) the basic problem of the magnetostatics for the vacuum. That must now be still debated for matter. Starting point are the two Maxwell equations

$$\operatorname{div}\mathbf{B} = 0\,; \qquad \operatorname{curl}\mathbf{H} = \mathbf{j}\,,$$

where from now on we suppress, as usual, the index f on the current density $\mathbf{j}$. Of course it is always meant to be the current density of the unbound charges. Let us present and discuss, in form of a list, several typical statements of the problem.

(1) μ_r = const in the Whole Interesting Space-Region V

Then it is an isotropic, linear medium

$$\operatorname{curl}\mathbf{B} = \mu_r\mu_0\,\mathbf{j}\,. \tag{3.84}$$

The problem has, thereby, not changed compared to (3.37). By the use of the Coulomb gauge, the differential equation to be solved reads:

$$\Delta\,\mathbf{A} = -\mu_r\mu_0\,\mathbf{j}\,. \tag{3.85}$$

On the right-hand side only the constant factor μ_r is added.

(2) V Consists of Partial Regions V_i with Pairwise Different but Within V_i Constant $\mu_r^{(i)}$

The problem has to be solved in each partial volume V_i as described in (1), where finally the partial solutions are fitted to each other by fulfilling the boundary conditions (3.80) and (3.82).

(3) $\mathbf{j} \equiv 0$ in V with Boundary Conditions on $S(V)$

In this case we can define, because of $\operatorname{curl}\mathbf{H} = 0$, a scalar magnetic potential φ_m:

$$\mathbf{H} = -\nabla\varphi_m\,. \tag{3.86}$$

When we assume again a linear medium with at least piecewise constant μ_r then we get from $\operatorname{div}\mathbf{B} = 0$:

$$\operatorname{div}\left(\mu_r\mu_0\,\nabla\varphi_m\right) = 0 \iff \Delta\varphi_m = 0\,. \tag{3.87}$$

That is the Laplace equation known from electrostatics which is to be solved along with the given boundary conditions.

(4) $\mathbf{M(r)} \neq 0$ with $\mathbf{j} \equiv 0$ in V

This situation can be realized, for instance, in a ferromagnet for $T < T_C$. Because of $\operatorname{curl} \mathbf{H} = 0$, it can be defined as in (3.86), a scalar potential φ_m so that the second Maxwell equation can be rewritten as follows:

$$0 = \operatorname{div} \mathbf{B} = \mu_0 \operatorname{div} (\mathbf{H} + \mathbf{M}) \implies \Delta\varphi_m = \operatorname{div} \mathbf{M} \,. \tag{3.88}$$

This corresponds to the Poisson equation of the electrostatics where $\operatorname{div} \mathbf{M(r)}$ adopts the role of $(-1/\epsilon_0)\rho(\mathbf{r})$. If there are no boundary conditions in a finite region we then find as in (2.25) (*Poisson integral*):

$$\varphi_m(\mathbf{r}) = -\frac{1}{4\pi} \int d^3r' \, \frac{\operatorname{div} \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{3.89}$$

We assume that $\mathbf{M}$ is restricted to a finite space-region and use then in (3.89):

$$\begin{aligned}
\frac{\operatorname{div} \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} &= \operatorname{div} \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla_{r'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\
&= \operatorname{div} \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) + \mathbf{M}(\mathbf{r}') \cdot \nabla_{r} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \,.
\end{aligned}$$

The first summand, inserted into (3.89), can be written by the use of the Gauss theorem as a surface integral which vanishes because of the localization of $\mathbf{M}$. It remains:

$$\varphi_m(\mathbf{r}) = -\frac{1}{4\pi} \nabla_r \int d^3r' \, \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{3.90}$$

If the point of observation $\mathbf{r}$ lies far away from the $\mathbf{M} \neq 0$-region then we can terminate the already several times applied expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \ldots$$

after the first summand. The leading term yields:

$$\varphi_m(\mathbf{r}) \approx -\frac{1}{4\pi} \left(\nabla_r \frac{1}{r} \right) \int d^3r' \, \mathbf{M}(\mathbf{r}') \,.$$

The integral is just the total magnetic moment $\mathbf{m}_{tot}$ of the system

$$\mathbf{m}_{tot} = \int d^3r' \, \mathbf{M}(\mathbf{r}') \,. \tag{3.91}$$

Therewith the scalar magnetic potential gets a form already known to us:

$$\varphi_{\mathrm{m}}(\mathbf{r}) \approx \frac{1}{4\pi} \frac{\mathbf{r} \cdot \mathbf{m}_{\mathrm{tot}}}{r^3} \,. \tag{3.92}$$

This corresponds to the electrostatic dipole potential $\varphi_{\mathrm{D}}(\mathbf{r})$ (2.71). Since $\mathbf{H}$ follows from φ_{m} as $\mathbf{E}^{\mathrm{D}}(\mathbf{r})$ follows from $\varphi_{\mathrm{D}}(\mathbf{r})$, we can adopt directly the calculation that led to (2.73). $\mathbf{H}$ has the typical form of a dipole field:

$$\mathbf{H} \approx \frac{1}{4\pi} \left[\frac{3\,(\mathbf{r} \cdot \mathbf{m}_{\mathrm{tot}})\,\mathbf{r}}{r^5} - \frac{\mathbf{m}_{\mathrm{tot}}}{r^3} \right] . \tag{3.93}$$

One should not forget that the results (3.89) and (3.90) are valid only if there are no boundary conditions in a finite region. However, if there are boundary conditions to be fulfilled on $S(V)$, e.g. by

$$\frac{\partial \varphi_{\mathrm{m}}}{\partial n} = \mathbf{n} \cdot \nabla \varphi_{\mathrm{m}} = +\mathbf{n} \cdot \mathbf{M} \,,$$

then we have to use the same considerations as done in Sect. 2.3 for the boundary-value problems of the electrostatics. We find in analogy to (2.122):

$$\varphi_{\mathrm{m}}(\mathbf{r}) = -\frac{1}{4\pi} \int\limits_V d^3r' \frac{\operatorname{div} \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi} \int\limits_{S(V)} \frac{d\mathbf{f}' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{3.94}$$

The reader should check that!

Example: ***'homogeneously magnetized sphere' (Fig. 3.22)***

We use (3.90):

$$\varphi_{\mathrm{m}}(\mathbf{r}) = -\frac{M_0}{4\pi} \frac{d}{dz} \int\limits_{V_K} d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \,.$$

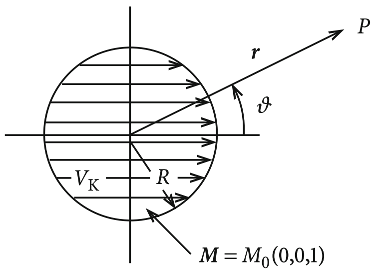


Fig. 3.22 For the determination of the scalar magnetic potential of a homogeneously magnetized sphere

For the evaluation of the integral it is recommendable to choose **r** as polar axis:

$$\int\limits_{V_K} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = 2\pi \int\limits_0^R r'^2 dr' \int\limits_{-1}^{+1} dx \frac{1}{(r^2 + r'^2 - 2rr'x)^{1/2}}$$

$$= -\frac{2\pi}{r} \int\limits_0^R r' dr' \, (r^2 + r'^2 - 2rr'x)^{1/2}\Big|_{x=-1}^{x=+1}$$

$$= \frac{4\pi}{r} \int\limits_0^R r'^2 dr' = \frac{4\pi}{3r} R^3 \, .$$

With

$$\frac{d}{dz}\frac{1}{r} = -\frac{z}{r^3} = -\frac{\cos\vartheta}{r^2} \; ; \quad \vartheta = \sphericalangle(\mathbf{r}, \mathbf{M})$$

it follows then:

$$\varphi_m(\mathbf{r}) = \frac{1}{3} M_0 R^3 \frac{\cos\vartheta}{r^2} \, .$$

The total magnetic moment of the sphere is easily calculated since **M** was assumed to be homogeneous:

$$\mathbf{m}_{tot} = \frac{4\pi}{3} R^3 \mathbf{M} = \frac{4\pi}{3} R^3 M_0 \mathbf{e}_z \, . \tag{3.95}$$

Therewith we have the result:

$$\varphi_m(\mathbf{r}) = \frac{1}{4\pi} \frac{\mathbf{m}_{tot} \cdot \mathbf{r}}{r^3} \, , \tag{3.96}$$

which agrees exactly with (3.92). The scalar magnetic potential as well as the corresponding **H**- or **B**-field thus do not change if one replaces the homogeneously magnetized sphere by a dipole at the origin of coordinates with the moment (3.95). For this highly symmetric special case we therefore do find the dipole field (3.93) not only asymptotically for great distances, but even in the immediate neighborhood of the sphere.

3.4.5 Exercises

Exercise 3.4.1 A small ferromagnet with the dipole moment $\mathbf{m}$ is fixed at $\mathbf{x}_0 = x_0$ $\mathbf{e}_x$ in such a way that it is freely rotatable in the xy-plane. On it acts a homogeneous field $\mathbf{B}_0 = B_0\mathbf{e}_x$. In addition there is the field $\mathbf{B}_1$ of a linear thread of current of the current intensity I. Determine the angle α between the dipole moment and the x-axis.

Exercise 3.4.2 Consider a sphere of the radius R with the permeability μ_r. Inside it is homogeneously magnetized:

$$\mathbf{M} = M_0\mathbf{e}_z \; .$$

The current density $\mathbf{j} \equiv 0$ inside and outside the sphere.

1. State a reason why the magnetic field can be written as

$$\mathbf{H} = -\nabla\varphi_m \; .$$

   Calculate the magnetic potential φ_m in the exterior space of the sphere!
2. Calculate the magnetic field $\mathbf{H}$ inside and outside the sphere!
3. Assume that the magnetization $\mathbf{M}$ of the sphere is generated by a surface current density $\mathbf{j}$. Provide evidence that this must be of the form

$$\mathbf{j} = \alpha(\vartheta)\delta(r-R)\mathbf{e}_\varphi; .$$

   Express $\alpha(\vartheta)$ by M_0!

Exercise 3.4.3 A straight, long, and thin wire lies at a distance a parallel to a very large plate of the permeability $\mu_r^{(1)}$ (Fig. 3.23). The region to the right of the plate has the permeability $\mu_r^{(2)}$. The dc-current I runs through the wire.

1. Under which conditions is the introduction of a scalar magnetic potential φ_m with $\mathbf{H} = -\nabla\varphi_m$ possible and reasonable?
2. How large are $\mathbf{H}$ and φ_m in case of an at first absent plate ($\text{curl}\mathbf{A} = \mu_r\mu_0\mathbf{H}$)?

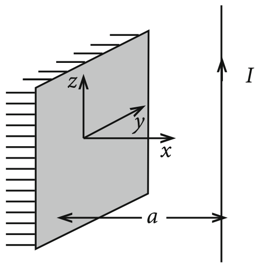


Fig. 3.23 Thin wire in front of a large plate, where the region in front of the plate and the plate itself possess different relative permeabilities

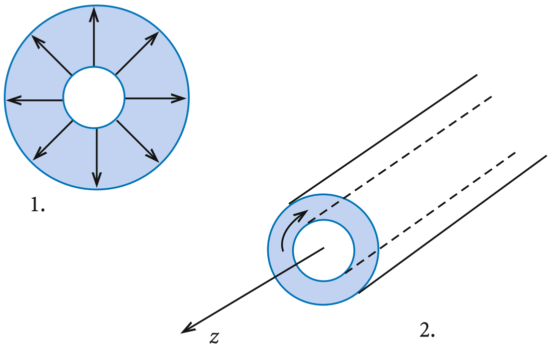


Fig. 3.24 Radially magnetized spherical shell, azimuthally magnetized hollow cylinder (Exercise 3.4.5)

3. Formulate the problem of the determination of φ_m for the given arrangement as a boundary-value problem!
4. For the solution of the problem introduce *image currents* I_1 at $x = -a$ and $y = 0$ as well as I_2 at $x = a$ and $y = 0$ so that I_1 together with I realizes the potential in the region 2 and I_2 alone in the region 1. Express φ_m, $\mathbf{H}$ and $\mathbf{A}$ by I_1, I_2.
5. Determine I_1, I_2 from the boundary conditions for the fields.
6. How strong is the force per unit length that acts on the wire?

Exercise 3.4.4 An infinitely long solid cylinder ($\mu_r = 1$) of the radius R carries a constant current density $\mathbf{j}_0$. Calculate the vector potential and the magnetic field intensity inside and outside the conductor by solving the Poisson equation for the vector potential. Check the result for the magnetic field by applying the Stokes theorem.

Exercise 3.4.5 Determine the **H**- and **B**-fields in the whole space

1. for a spherical shell which is magnetized in radial direction (Fig. 3.24)

$$\mathbf{M}(\mathbf{r}) = M(r)\,\mathbf{e}_r \ ,$$

2. for a hollow cylinder being magnetized in azimuthal direction (Fig. 3.24):

$$\mathbf{M}(\mathbf{r}) = \widehat{M}(\rho)\,\mathbf{e}_\varphi \ .$$

3.5 Self-Examination Questions

To Section 3.1

1. How are current density and current intensity defined?
2. What is expressed by the continuity equation?
3. Derive Kirchhoff's node rule!
4. What is the Ohm's law?

5. Is the electric resistance R a material constant?
6. What is to be understood by a thread of current?
7. How large is the power generated by an electric field **E** on a current density?
8. What is meant by power loss?

To Section 3.2

1. Which experimental observation does provide the basis of magnetostatics?
2. Formulate the force between two conductor loops C_1, C_2 carrying the stationary currents I_1 and I_2. Is the rule *action=reaction* fulfilled?
3. How can the force acting between currents be used to fix the measurement unit of the electric current?
4. How is the magnetic induction defined?
5. How do the **B**-lines of a straight conductor look like?
6. Which force and which torque are exerted by a magnetic induction $\mathbf{B}(\mathbf{r})$ on a current density $\mathbf{j}(\mathbf{r})$?
7. State the Maxwell equations of magnetostatics?
8. What does the Ampère's magnetic flux law say?
9. What is the connection between the vector potential and the current density?
10. What is to be understood by a gauge transformation?
11. How is the Coulomb gauge defined?
12. Formulate the basic problem of the magnetostatics!

To Section 3.3

1. Define the magnetic moment of a current density $\mathbf{j}(\mathbf{r})$!
2. Which form does the vector potential **A** have sufficiently far away from the $\mathbf{j} \neq 0$-region?
3. What is the magnetic moment of an arbitrary, closed, plane current loop?
4. Which force is exerted by a homogeneous magnetic field on a stationary current distribution?
5. Which potential energy does a magnetic moment **m** possess in the field of the magnetic induction **B**?

To Section 3.4

1. Interpret the term magnetization-current density!
2. What does one understand by magnetization? Which connection does exist between magnetization, magnetic field, and magnetic induction?
3. What are the macroscopic Maxwell equations of the magnetostatics?
4. Which analogies do exist between the field quantities **E**, **D** and **P** of the electrostatics and **A**, **H** and **M** of the magnetostatics? Which are the actual measurands?
5. Which physical quantity is in particular appropriate for a classification of magnetic materials? By what are diamagnetism and paramagnetism, respectively, characterized and by what do they differ from each other?

6. List some typical features of ferromagnetism!
7. What do we understand by ferri- and anti-ferromagnetism?
8. How do **B** and **H** behave at interfaces?
9. When is it reasonable to define a scalar magnetic potential? Under which conditions does it obey a Laplace and a Poisson equation, respectively?
10. How does the magnetic field of a homogeneously magnetized sphere look like?

Chapter 4
Electrodynamics

Chapters 2 and 3 have shown that electrostatic and magnetostatic problems can be treated completely independent of each other. Certain formal analogies, though, allow one to apply to a large extent identical calculation techniques to solve the basic problems, but that does not lead to any direct dependency. This will now change when we consider time-dependent phenomena, i.e. the decoupling of electric and magnetic fields ha to be set aside. Therefore one should speak from now on of **electromagnetic** fields rather than of electric and magnetic fields, separately. The deep understanding of the close correlation between electric and magnetic fields will be provided in the framework of the theory of relativity.

4.1 Maxwell Equations

At first we want to generalize the fundamental field equations of the **electrostatics**

$$\operatorname{div}\mathbf{D} = \rho\ ; \quad \operatorname{curl}\mathbf{E} = 0$$

and the **magnetostatics**,

$$\operatorname{div}\mathbf{B} = 0\ ; \quad \operatorname{curl}\mathbf{H} = \mathbf{j}$$

to time-dependent phenomena. Thereby the starting point of our considerations shall again be an experimental fact which is believed to be uniquely verified.

4.1.1 Faraday's Law of Induction

The Biot-Savart law (3.23) includes the statement that a current density $\mathbf{j}$ generates a magnetic induction $\mathbf{B}$. In the year 1831 Faraday was concerned with the problem

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whether it might be also conversely possible to create a current by **B**. His famous experiments on the behavior of currents in temporally changeable magnetic fields led to the following observations: In a conductor loop C_1 a current is generated if

1. a permanent magnet is moved relative to the loop,
2. a second, a constant current carrying conductor loop C_2 is moved relative to C_1,
3. the current in a conductor loop C_2 (*'at rest'* relative to C_1) is changed.

This direct experimental observation concerns electric currents. In the range of validity of the *Ohm's law* (3.9),

$$\mathbf{j} = \sigma\,\mathbf{E}\ ,$$

it is directly transferred to electric fields. Let us try to combine Faraday's observations into a compact mathematical formula.

Definition 4.1.1

Electromotive force (emf):

$$(\text{emf}) = \oint\limits_{\mathrm{C}} \mathbf{E}\cdot d\mathbf{r}\ , \tag{4.1}$$

Magnetic flux through the area F_C (Fig. 4.1):

$$\Phi = \int\limits_{F_C} \mathbf{B}\cdot d\mathbf{f}\ . \tag{4.2}$$

Faraday's experiments *'prove'* the proportionality between $\dot{\Phi}$ and (emf).

Faraday's law of induction:

$$\oint\limits_{C} \mathbf{E}\cdot d\mathbf{r} = -k\frac{d}{dt}\int\limits_{F_C} \mathbf{B}'\cdot d\mathbf{f}\ . \tag{4.3}$$

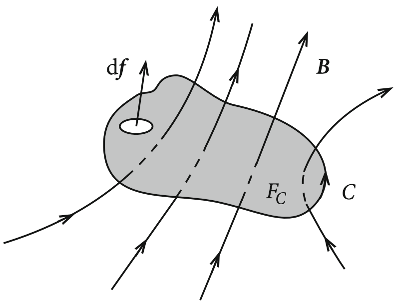


Fig. 4.1 Thumbsketch for the definition of the magnetic flux

Here $\mathbf{E}$ is to be understood as the electric field in the ('co-moving') system of coordinates, in which the conductor loop C is 'at rest'. $\mathbf{B}'$ is the magnetic induction in the 'fixed' laboratory system and $\frac{d}{dt}$ mediates the temporal change of the magnetic flux through the conductor loop as seen from the point of view of the 'co-moving' observer.

The law (4.3) holds not only for the case where C is a real conductor loop, but also even when C represents a fictitious closed geometrical loop.

However, we still have to fix the proportionality constant k. For this purpose we use the following consideration: Let the current circuit C, in which the induced current is observed, move with the velocity $\mathbf{v}$, which is constant both in direction and magnitude, relative to the laboratory (Fig. 4.2). In the lab we find the (possibly time-dependent) magnetic induction $\mathbf{B}'$. In contrast, as mentioned, in the Faraday's law of induction (4.3) the field $\mathbf{E}$ at $\mathbf{r}$ is meant as that in the **'co-moving'** reference system, where the conductor-element $d\mathbf{r}$ 'is at rest'. The total time derivative on the right-hand side of Eq. (4.3) refers to the view of an observer who 'co-moves' with C. It can therefore contribute in two ways:

$$\frac{d}{dt}: \quad \begin{array}{l} \text{(1) explicit rate of } \mathbf{B}'\text{-change} \,, \\ \text{(2) position-change of the conductor loop} \,. \end{array}$$

Formally this can be seen as follows. It holds for the temporal change of $\mathbf{B}'$ seen from the co-moving observer:

$$\frac{d}{dt}\mathbf{B}' = \frac{\partial}{\partial t}\mathbf{B}' + (\mathbf{v}\cdot\nabla)\mathbf{B}'$$

and also, since $\mathbf{v}$ is constant in direction and magnitude:

$$\operatorname{curl}(\mathbf{B}'\times\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{B}' - (\mathbf{B}'\cdot\nabla)\mathbf{v} + \mathbf{B}'\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{B}' = (\mathbf{v}\cdot\nabla)\mathbf{B}' \,.$$

That means:

$$\frac{d}{dt}\mathbf{B}' = \frac{\partial}{\partial t}\mathbf{B}' + \operatorname{curl}(\mathbf{B}'\times\mathbf{v}) \,.$$

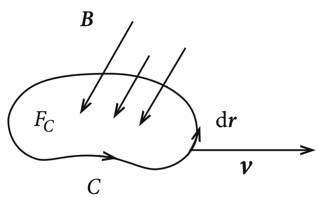


Fig. 4.2 Simple arrangement (magnetic induction, closed conductor loop) for fixing the constant k in the Faraday's law of induction (4.3)

The Stokes theorem yields:

$$\int_{F_C} d\mathbf{f} \cdot \operatorname{curl}\left(\mathbf{B}' \times \mathbf{v}\right) = \oint_C d\mathbf{r} \cdot \left(\mathbf{B}' \times \mathbf{v}\right) = \oint_C \mathbf{B}' \cdot \left(\mathbf{v} \times d\mathbf{r}\right) .$$

For the temporal change of the magnetic flux we thus have:

$$\frac{d}{dt} \int_{F_C} \mathbf{B}' \cdot d\mathbf{f} = \underbrace{\int_{F_C} \frac{\partial \mathbf{B}'}{\partial t} \cdot d\mathbf{f}}_{\cong 1)} + \underbrace{\oint_C \mathbf{B}' \cdot \left(\mathbf{v} \times d\mathbf{r}\right)}_{\cong 2)} . \tag{4.4}$$

Here we have used the fact that F_C does not change in the co-moving reference system. The time differentiation can therefore be shifted into the integrand. Therewith (4.3) can be written as:

$$\oint_C \left[\mathbf{E} - k(\mathbf{v} \times \mathbf{B}')\right] \cdot d\mathbf{r} = -k \int_{F_C} \frac{\partial \mathbf{B}'}{\partial t} \cdot d\mathbf{f} . \tag{4.5}$$

In a second thought experiment ('gedankenexperiment') we fix the conductor loop somewhere in space ($\mathbf{v} = 0$), such that it has the same coordinates which the moving conductor possesses in the co-moving reference system. Then the field in the rest-system of the conductor is identical to the field $\mathbf{E}'$ observed in the laboratory:

$$\oint_C \mathbf{E}' \cdot d\mathbf{r} = -k \int_{F_C} \frac{\partial \mathbf{B}'}{\partial t} \cdot d\mathbf{f} . \tag{4.6}$$

The comparison of (4.5) and (4.6) leads to:

$$\mathbf{E} = \mathbf{E}' + k(\mathbf{v} \times \mathbf{B}') . \tag{4.7}$$

Decisive precondition for the derivation of this relation has been the assumption that the Faraday's law of induction (4.3) is equally valid in all reference systems which move relatively to each other with constant velocities $\mathbf{v}$, i.e that it is

Galilean-invariant

(Sect. 2.2.3, Vol. 1). That is an acceptable assumption in the non-relativistic region $v^2/c^2 \ll 1$. In order to finally fix the quantity k, we consider the force on a single point charge q, which is at rest in the moving conductor. It experiences the force

$$\mathbf{F} = q\,\mathbf{E} .$$

Seen from the laboratory the point charge represents a current,

$$\mathbf{j} = q\,\mathbf{v}\,\delta(\mathbf{r} - \mathbf{R}_0) \;,$$

onto which the magnetic induction $\mathbf{B}'$ exerts according to (3.24) the force:

$$\int d^3r\,(\mathbf{j} \times \mathbf{B}') = q(\mathbf{v} \times \mathbf{B}')$$

The total force acting on the particle as seen from the laboratory is then:

$$\mathbf{F}' = q\left[\mathbf{E}' + (\mathbf{v} \times \mathbf{B}')\right] \;.$$

The Galilean invariance requires $\mathbf{F} = \mathbf{F}'$ and therewith

$$\mathbf{E} = \mathbf{E}' + (\mathbf{v} \times \mathbf{B}') \;. \tag{4.8}$$

This important relation for the electric field $\mathbf{E}$ in a reference system, which moves relative to the laboratory with the velocity $\mathbf{v}$, brings out the close connection between magnetic and electric fields. One should keep in mind, however, that because of the assumed Galilean invariance the relation is free of doubts only non-relativistically.

By (4.7) and (4.8) eventually the constant k in the law of induction (4.3) is now fixed:

$$k = 1 \;, \quad (\mathrm{SI}) \;. \tag{4.9}$$

The final version of the law of induction therewith reads:

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt}\int\limits_{F_C} \mathbf{B}' \cdot d\mathbf{f} \;. \tag{4.10}$$

Let us assume that the laboratory system is also the rest-system of the conductor ($\mathbf{B} = \mathbf{B}'$) so that $\mathbf{E}$ and $\mathbf{B}$ are defined in the same reference system, then we can reformulate (4.10) by use of the Stokes theorem:

$$\int\limits_{F_C} d\mathbf{f} \cdot (\mathrm{curl}\,\mathbf{E} + \dot{\mathbf{B}}) = 0 \;.$$

This holds for arbitrary areas F_C. Therefore we can further conclude:

$$\mathrm{curl}\,\mathbf{E} = -\dot{\mathbf{B}} \;. \tag{4.11}$$

This is the generalization of the homogeneous Maxwell equation of the electrostatics (2.188) for time-dependent phenomena.

4.1.2 Maxwell's Supplement

Let us summarize here the basic equations which are so far at our disposal for the description of electromagnetic phenomena:

$$\begin{aligned} \operatorname{div}\mathbf{D} &= \rho \quad \text{(Coulomb)}\,, \\ \operatorname{curl}\mathbf{E} &= -\dot{\mathbf{B}} \ \text{(Faraday)}\,, \\ \operatorname{curl}\mathbf{H} &= \mathbf{j} \quad \text{(Ampère)}\,, \\ \operatorname{div}\mathbf{B} &= 0\,. \end{aligned}$$

Except for the Faraday's law all these rules have been deduced from experiments which concern *static* charge distributions and *stationary* currents, respectively. It is therefore not at all astonishing that for non-stationary fields certain contradictions might appear. This is indeed the case in connection with the Ampère's law. We had intentionally introduced, when discussing the magnetostatics in matter in Sect. 3.4, the magnetic field $\mathbf{H}$ without the term $\dot{\mathbf{P}}$ since in connection with (3.68) we were interested only in magneto**static** phenomena. The relation $\operatorname{curl}\mathbf{H} = \mathbf{j}$ can therefore not be valid for the general case. That can be illustrated immediately by applying the divergence to this equation:

$$0 = \operatorname{div}\operatorname{curl}\mathbf{H} = \operatorname{div}\mathbf{j}\,.$$

For non-stationary currents this is a clear contradiction to the continuity equation (3.5):

$$\operatorname{div}\mathbf{j} = -\frac{\partial\rho}{\partial t}\,.$$

Maxwell solved this contradiction by the following ansatz which is called the **Maxwell's supplement**:

$$\operatorname{curl}\mathbf{H} = \mathbf{j} + \mathbf{j}_0\,. \tag{4.12}$$

$\mathbf{j}_0$ is at first only a hypothetical additional current for which it must hold:

$$\operatorname{div}\mathbf{j}_0 = \operatorname{div}\operatorname{curl}\mathbf{H} - \operatorname{div}\mathbf{j} = \frac{\partial\rho}{\partial t} = \operatorname{div}\dot{\mathbf{D}}\,.$$

Hence the mentioned contradiction is removed when we replace the Ampère relation by

$$\operatorname{curl}\mathbf{H} = \mathbf{j} + \dot{\mathbf{D}}\,. \tag{4.13}$$

The static limit is obviously not violated. According to Maxwell one calls $\dot{\mathbf{D}}$ the

electric displacement current

We have performed here the extension of the Maxwell equations to time-dependent phenomena directly for the macroscopic field equations. When deriving the macroscopic field equations in the electro- and magneto**statics** we chose a different way. At first we started at the corresponding Maxwell equations in the vacuum and then generalized these properly for matter. The same way we could have chosen in principle here also. The identical consideration to that used above (*Maxwell's supplement*) would have given for the vacuum instead of (4.13):

$$\operatorname{curl}\mathbf{B} = \mu_0\,\mathbf{j} + \epsilon_0\mu_0\dot{\mathbf{E}}\ .$$

From this relation one assumes again that, from a microscopic point of view, it is universally valid. That means that the equation would work even in the matter if one knew the corresponding microscopic currents:

$$\operatorname{curl}\mathbf{b} = \mu_0\,\mathbf{j}_m + \epsilon_0\mu_0\dot{\mathbf{e}}\ .$$

By applying the averaging process (2.179) one gets the corresponding macroscopic equation:

$$\operatorname{curl}\mathbf{B} = \mu_0\,\bar{\mathbf{j}}_{\mathrm{m}} + \epsilon_0\mu_0\dot{\mathbf{E}}\ .$$

We calculated the averaged current density $\bar{\mathbf{j}}_{\mathrm{m}}$ with (3.68):

$$\bar{\mathbf{j}}_{\mathrm{m}} = \mathbf{j}_{\mathrm{f}} + \dot{\mathbf{P}} + \operatorname{curl}\mathbf{M}\ .$$

It follows:

$$\operatorname{curl}(\mathbf{B} - \mu_0\mathbf{M}) = \mu_0\mathbf{j}_{\mathrm{f}} + \mu_0(\epsilon_0\dot{\mathbf{E}} + \dot{\mathbf{P}})\ .$$

If we further use the definitions (2.187) and (3.69) for the auxiliary fields **D** and **H**, then we indeed arrive at (4.13). But bear in mind that in (4.13) the term **j** always means the *free* current density. From now on we suppress, however, the index f.

Therewith we have now at hand the complete set of basic electromagnetic equations:

Maxwell equations:

homogeneous:

$$\operatorname{div}\mathbf{B} = 0 \tag{4.14}$$

$$\operatorname{curl}\mathbf{E} + \dot{\mathbf{B}} = 0 \tag{4.15}$$

inhomogeneous:

$$\operatorname{div}\mathbf{D} = \rho \tag{4.16}$$

$$\operatorname{curl}\mathbf{H} - \dot{\mathbf{D}} = \mathbf{j} \tag{4.17}$$

material equations: $$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \longrightarrow \mu_\mathrm{r}\mu_0\mathbf{H} \tag{4.18}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \longrightarrow \epsilon_\mathrm{r}\epsilon_0\mathbf{E} \tag{4.19}$$

The arrows in (4.18) and (4.19) point to the special expressions for **linear** media.

4.1.3 Electromagnetic Potentials

The typical problem of electrodynamics consists of the calculation of electromagnetic fields generated by given charge- and/or current density distributions on the basis of the fundamental Maxwell equations. That can be done by directly starting with the Maxwell equations, i.e. by trying to solve a coupled system of four partial differential equations of **first** order. Sometimes, however, it appears to be more comfortable to introduce electromagnetic potentials $(\varphi, \mathbf{A})$ which *automatically* fulfill the homogeneous Maxwell equations, however at the cost that the inhomogeneous equations are now transferred into a set of two partial differential equations of **second** order. The general concept is already known to us from electrostatics.

The homogeneous Maxwell equation

$$\operatorname{div}\mathbf{B} = 0$$

is trivially solved when we write the magnetic induction, as we did in the magnetostatics (3.34), now also as the curl of a vector field, the

vector potential $\mathbf{A}(\mathbf{r}, t)$

$$\mathbf{B}(\mathbf{r}, t) = \operatorname{curl}\mathbf{A}(\mathbf{r}, t) \ . \tag{4.20}$$

We insert this into the second homogeneous Maxwell equation (4.15),

$$\operatorname{curl}(\mathbf{E} + \dot{\mathbf{A}}) = 0 \ ,$$

which suggests the following ansatz for the electric field:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \dot{\mathbf{A}}(\mathbf{r}, t) \ . \tag{4.21}$$

The **scalar potential** $\varphi(\mathbf{r}, t)$ and the **vector potential** $\mathbf{A}(\mathbf{r}, t)$ must be determined by the inhomogeneous Maxwell equations. Both are actually auxiliary quantities which fix by (4.20) and (4.21) the real physical observables $\mathbf{E}$ and $\mathbf{B}$.

The magnetic induction $\mathbf{B}(\mathbf{r}, t)$ obviously does not change when we go over from $\mathbf{A}$ to

$$\overline{\mathbf{A}}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t) \ ,$$

where χ can be an arbitrary scalar field. Such a 'non-uniqueness' of the vector potential can bring about non-trivial computational advantages. However, we have to take into consideration that such a transformation would affect in general also the electric field $\mathbf{E}$, if we kept $\varphi(\mathbf{r}, t)$ fixed. φ must therefore be properly co-transformed:

$$\nabla\varphi + \dot{\mathbf{A}} \stackrel{!}{=} \nabla\overline{\varphi} + \dot{\overline{\mathbf{A}}} = \nabla\overline{\varphi} + \dot{\mathbf{A}} + \nabla\dot{\chi} \ .$$

Except for an unimportant constant, which we are free to put equal to zero, we end up with the following, always allowed

gauge transformation:

$$\mathbf{A}(\mathbf{r}, t) \mapsto \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t) \ , \tag{4.22}$$

$$\varphi(\mathbf{r}, t) \mapsto \varphi(\mathbf{r}, t) - \dot{\chi}(\mathbf{r}, t) \ . \tag{4.23}$$

Thereby the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, which are the actual measurands, remain unchanged.

In order to recognize which gauging could be useful, let us inspect now the inhomogeneous Maxwell equations where we restrict ourselves to the case of the vacuum:

$$\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0} \ ; \quad \operatorname{curl}\mathbf{B} = \mu_0\,\mathbf{j} + \mu_0\epsilon_0\dot{\mathbf{E}} \ . \tag{4.24}$$

We insert (4.20) and (4.21):

$$-\Delta\varphi - \operatorname{div}\dot{\mathbf{A}} = \frac{\rho}{\epsilon_0} \ ,$$

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{grad}(\operatorname{div}\mathbf{A}) - \Delta\mathbf{A} = \mu_0\,\mathbf{j} - \mu_0\epsilon_0\nabla\dot{\varphi} - \mu_0\epsilon_0 d\dot{\mathbf{A}} \ .$$

We use (3.17): $c^2 = (\mu_0\epsilon_0)^{-1}$:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} - \nabla\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\dot{\varphi}\right) = -\mu_0\mathbf{j} \ ,$$

$$\left[\Delta\varphi + \frac{\partial}{\partial t}(\operatorname{div}\mathbf{A})\right] = \frac{-\rho}{\epsilon_0} \ . \tag{4.25}$$

This system of equations can be simplified by proper gauge transformations:

(1) Coulomb-Gauge

One chooses the gauge-function χ such that

$$\operatorname{div} \mathbf{A} = 0 \; . \tag{4.26}$$

According to (4.25) the scalar potential then fulfills a differential equation,

$$\Delta\varphi = \frac{-\rho}{\epsilon_0} \; , \tag{4.27}$$

which, provided that there are no boundary conditions in the finiteness, is formally identical to the Poisson equation of the electrostatics. We therefore can immediately quote its solution:

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \; . \tag{4.28}$$

$\varphi(\mathbf{r}, t)$ is the *instantaneous* Coulomb potential of the charge density $\rho(\mathbf{r}, t)$. One therefore speaks of *'Coulomb-gauge'*.

Using the Coulomb-gauge in (4.25), we find for the vector potential $\mathbf{A}(\mathbf{r}, t)$ the following differential equation:

$$\Box\, \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \nabla\dot{\varphi} - \mu_0 \mathbf{j} \; . \tag{4.29}$$

Here we have introduced the

d'Alembert operator:

$$\Box \equiv \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tag{4.30}$$

We insert (4.28) for φ into the right-hand side of (4.29) and exploit the continuity equation (3.5):

$$\Box\, \mathbf{A}(\mathbf{r}, t) = -\mu_0\, \mathbf{j} - \frac{\mu_0}{4\pi} \nabla_r \int d^3r' \, \frac{\operatorname{div} \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \; . \tag{4.31}$$

According to the general decomposition theorem (1.72) for vector fields, the current density $\mathbf{j}(\mathbf{r}, t)$ consists of a longitudinal ($\mathbf{j}_l$) and a transverse part ($\mathbf{j}_t$):

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_l(\mathbf{r}, t) + \mathbf{j}_t(\mathbf{r}, t) , \tag{4.32}$$

$$\mathbf{j}_l(\mathbf{r}, t) = -\frac{1}{4\pi} \nabla_r \int d^3 r' \, \frac{\operatorname{div} \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} , \tag{4.33}$$

$$\mathbf{j}_t(\mathbf{r}, t) = \frac{1}{4\pi} \nabla_r \times \int d^3 r' \, \frac{\operatorname{curl} \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} . \tag{4.34}$$

With (4.31) one realizes that the vector potential is completely determined by the transverse current density $\mathbf{j}_t$:

$$\Box \, \mathbf{A}(\mathbf{r}, t) = -\mu_0 \, \mathbf{j}_t(\mathbf{r}, t) . \tag{4.35}$$

The Coulomb-gauge is therefore also referred to as **transverse gauge**. It is not Lorentz-invariant, i.e. observers in different reference systems moving relative to each other are gauging differently. In principle that is irrelevant since the gauging is completely optional. On the other hand, it will turn out to be inconvenient for the treatment of relativistic problems.

A rather simple consideration makes clear that the Coulomb-gauge can always be fulfilled, namely if

$$\operatorname{div} \mathbf{A}(\mathbf{r}, t) = a(\mathbf{r}, t) \neq 0$$

then one may choose instead of $\mathbf{A}$

$$\overline{\mathbf{A}}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla \chi(\mathbf{r}, t) ,$$

where $\chi(\mathbf{r}, t)$ shall guarantee:

$$\operatorname{div} \overline{\mathbf{A}} = \operatorname{div} \mathbf{A} + \Delta\chi \stackrel{!}{=} 0 .$$

That means

$$\Delta\chi = -a(\mathbf{r}, t) .$$

That is again a Poisson equation with the solution:

$$\chi(\mathbf{r}, t) = \frac{1}{4\pi} \int d^3 r' \, \frac{a(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} . \tag{4.36}$$

(2) Lorenz-Gauge

This gauge leads to a complete decoupling of both the differential equations (4.25) for φ and $\mathbf{A}$, which then in addition assume very symmetric forms.

Lorenz-condition

$$\operatorname{div} \mathbf{A} + \frac{1}{c^2}\dot{\varphi} = 0 \,. \tag{4.37}$$

Inserted into (4.25) it results:

$$\Box\, \mathbf{A}(\mathbf{r}, t) = -\mu_0\, \mathbf{j} \,, \tag{4.38}$$

$$\Box\, \varphi(\mathbf{r}, t) = \tfrac{-\rho}{\epsilon_0} \,. \tag{4.39}$$

One can show that this gauging is independent of the reference system (inertial system) therefore being Lorentz-invariant and thus convenient for the theory of relativity (see Vol. 4).

We can convince ourselves that the condition (4.37), too, is always realizable. Let us assume that

$$\operatorname{div} \mathbf{A} + \frac{1}{c^2}\dot{\varphi} = a(\mathbf{r}, t) \neq 0$$

then it follows with (4.22) and (4.23):

$$\operatorname{div} \overline{\mathbf{A}} + \frac{1}{c^2}\dot{\overline{\varphi}} = a(\mathbf{r}, t) + \Delta\chi - \frac{1}{c^2} d\dot{\chi} \,.$$

Equation (4.37) is thus realizable if the gauge function $\chi(\mathbf{r}, t)$ is chosen such that it fulfills the *inhomogeneous wave equation*

$$\Box\, \chi(\mathbf{r}, t) = -a(\mathbf{r}, t) \,.$$

We notice that even the choice of χ is not yet unique, since obviously one can add to χ any solution Λ of the *homogeneous wave equation*

$$\Box\, \Lambda(\mathbf{r}, t) = 0 \,.$$

The Lorenz-condition defines therewith a whole **gauge-class**.

4.1.4 Field Energy

Let us discuss as a first important consequence of the Maxwell equations the

energy law of electrodynamics.

For that we first consider a particle with the charge q (point charge), which according to (2.20) and (3.25) experiences in the electromagnetic field the **Lorentz force:**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,. \tag{4.40}$$

For a displacement by $d\mathbf{r}$ the field exerts work on the particle. This work is therefore counted as positive:

$$dW = \mathbf{F} \cdot d\mathbf{r} = q\, \mathbf{E} \cdot d\mathbf{r}$$

By this process field energy is converted into kinetic energy. This corresponds to *power*

$$\frac{dW}{dt} = q\, \mathbf{v} \cdot \mathbf{E} \,. \tag{4.41}$$

Only the electric part of the force $\mathbf{F}$ is involved in the energy exchange between the field and the particle. The magnetic field component does not exert any work because it is always perpendicular to the particle velocity $\mathbf{v}$.

The same statements are valid also for continuous charge distributions $\rho(\mathbf{r}, t)$ with the velocity field $\mathbf{v}(\mathbf{r}, t)$, which experience in the field the **force density**

$$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \left[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \right] \,. \tag{4.42}$$

The corresponding
power density

$$\mathbf{f}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \tag{4.43}$$

is determined only by the electric field $\mathbf{E}$ and the current density $\mathbf{j}$. The total work performed by the field in the volume V then amounts to

$$\frac{dW_V}{dt} = \int_V d^3r\, \mathbf{j} \cdot \mathbf{E} \,. \tag{4.44}$$

This relation becomes physically more transparent when we rearrange it further by use of the Maxwell equation (4.17):

$$\mathbf{j} \cdot \mathbf{E} = \mathbf{E} \cdot \operatorname{curl} \mathbf{H} - \mathbf{E} \cdot \dot{\mathbf{D}} \,.$$

Because of

$$\operatorname{div}(\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \operatorname{curl} \mathbf{E} - \mathbf{E} \cdot \operatorname{curl} \mathbf{H} = -\mathbf{H} \cdot \dot{\mathbf{B}} - \mathbf{E} \cdot \operatorname{curl} \mathbf{H}$$

we then have:

$$\frac{dW_V}{dt} = \int\limits_V d^3r \left[-\mathbf{H} \cdot \dot{\mathbf{B}} - \mathbf{E} \cdot \dot{\mathbf{D}} - \operatorname{div}(\mathbf{E} \times \mathbf{H}) \right] .$$

We now introduce two important quantities:

Definition 4.1.2 ***Poynting vector***

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) . \tag{4.45}$$

We will see that **S** has the meaning of an **energy current density**

Definition 4.1.3 ***Energy density of the electromagnetic field***

$$w(\mathbf{r}, t) = \frac{1}{2} \left[\mathbf{H}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t) + \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{D}(\mathbf{r}, t) \right] . \tag{4.46}$$

This definition includes the special case (2.215) of electrostatics. Whether or not this definition is really reasonable must be confirmed by the following considerations. At least the dimensions are correct because not only $\mathbf{E} \cdot \mathbf{D}$ but also the product $\mathbf{H} \cdot \mathbf{B}$ has the dimension of an energy density:

$$[\mathbf{H} \cdot \mathbf{B}] = 1 \ \frac{\mathrm{A}}{\mathrm{m}} \frac{\mathrm{Vs}}{\mathrm{m}^2} = 1 \ \frac{\mathrm{J}}{\mathrm{m}^3} .$$

In any case Eq. (4.46) applies only to the so-called *linear, homogeneous* media,

$$\mathbf{D} = \epsilon_\mathrm{r} \epsilon_0 \mathbf{E} ; \quad \mathbf{B} = \mu_\mathrm{r} \mu_0 \mathbf{H} \quad (\epsilon_\mathrm{r} = \mathrm{const} , \ \mu_\mathrm{r} = \mathrm{const}) ,$$

for which further holds:

$$\mathbf{H} \cdot \dot{\mathbf{B}} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B}) ,$$

$$\mathbf{E} \cdot \dot{\mathbf{D}} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}) .$$

After these definitions and rearrangements the power of the field in the volume V reads:

$$\frac{dW_V}{dt} = \int\limits_V d^3r \, \mathbf{j} \cdot \mathbf{E} = -\int\limits_V d^3r \left(\frac{\partial w}{\partial t} + \operatorname{div} \mathbf{S} \right) .$$

Since V is arbitrary it must even be satisfied the following **continuity equation**:

$$\frac{\partial w}{\partial t} + \operatorname{div} \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \,. \tag{4.47}$$

This relation is known as **Poynting's theorem**. Provided that we accept the definitions and interpretations of w as energy density and $\mathbf{S}$ as energy current density then the Poynting's theorem leads to the statement that the field energy in the volume V

$$\frac{dW_V^{(\text{field})}}{dt} = \int_V d^3r \, \frac{\partial w}{\partial t}$$

changes, on the one hand, by conversion into mechanical particle energy and therewith via particle collisions finally into Joule-heat energy,

$$\frac{dW_V^{(\text{mech})}}{dt} = \int_V d^3r \; \mathbf{j} \cdot \mathbf{E} \,,$$

and, on the other hand by an energy current (*radiation*) through the surface of V:

$$\int_V d^3r \operatorname{div} \mathbf{S} = \int_{S(V)} d\mathbf{f} \cdot \mathbf{S} \,.$$

The total energy balance, written in integral form, is then:

$$\frac{d}{dt}\left(W_V^{(\text{mech})} + W_V^{(\text{Field})}\right) = -\int_{S(V)} d\mathbf{f} \cdot \mathbf{S} \,. \tag{4.48}$$

We close this section with a remark on the Poynting vector $\mathbf{S}$, which we could interpret, obviously reasonably, as energy current density. Note, however, that this vector enters our considerations via (4.47) exclusively in the form of $\operatorname{div} \mathbf{S}$. The given physical meaning refers only to this expression, i.e. $\mathbf{S}$ itself is actually not unique since a transformation of the form

$$\mathbf{S} \to \mathbf{S} + \operatorname{curl} \boldsymbol{\alpha}$$

does not change $\operatorname{div} \mathbf{S}$. It is therefore quite possible that $\mathbf{S} \neq \mathbf{0}$ without involving any energy radiation.

Example

$$\mathbf{E} = (E, 0, 0) \,; \quad \mathbf{H} = (0, 0, H) \quad \text{homogeneous!}$$

$$\Longrightarrow \quad \mathbf{S} = \mathbf{E} \times \mathbf{H} = (0, -E\,H, 0) \neq \mathbf{0} \,.$$

However, since $\operatorname{div}\mathbf{S} = 0$ there is no energy radiation through the surface of V:

$$0 = \int_V d^3r \operatorname{div}\mathbf{S} = \int_{S(V)} d\mathbf{f}\cdot\mathbf{S}\ .$$

4.1.5 Field Momentum

After the energy theorem we now want to discuss the

momentum theorem of electrodynamics

as a further important consequence of the Maxwell equations. We consider a system of charged particles onto which only the Lorentz force of the electromagnetic field acts. Then it holds according to Newton's second axiom, if $\mathbf{P}_V^{(\mathrm{mech})}$ is the total momentum of all particles in V:

$$\frac{d}{dt}\mathbf{P}_V^{(\mathrm{mech})} = \int_V d^3r\,\rho(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \int_V d^3r\,(\rho\,\mathbf{E}+\mathbf{j}\times\mathbf{B})\ . \tag{4.49}$$

We eliminate ρ and $\mathbf{j}$ by the inhomogeneous Maxwell equations (4.16) and (4.17):

$$\begin{aligned}\rho\,\mathbf{E}+\mathbf{j}\times\mathbf{B} &= \mathbf{E}\operatorname{div}\mathbf{D} + \operatorname{curl}\mathbf{H}\times\mathbf{B} - \dot{\mathbf{D}}\times\mathbf{B}\\ &= \mathbf{E}\operatorname{div}\mathbf{D} + \mathbf{H}\operatorname{div}\mathbf{B} + \operatorname{curl}\mathbf{H}\times\mathbf{B} - \frac{d}{dt}(\mathbf{D}\times\mathbf{B}) - \mathbf{D}\times\operatorname{curl}\mathbf{E}\ .\end{aligned}$$

In the last step we have added a *'proper zero'* ($\mathbf{H}\operatorname{div}\mathbf{B}$) and applied the homogeneous Maxwell equation (4.15).

We define *tentatively:*

Definition 4.1.4 ***Momentum of the electromagnetic field***

$$\mathbf{p}_V^{(\mathrm{field})} = \int_V d^3r\,(\mathbf{D}\times\mathbf{B})\ . \tag{4.50}$$

By (4.49) we find therewith the following intermediate result:

$$\frac{d}{dt}\left(\mathbf{p}_V^{(\mathrm{mech})} + \mathbf{p}_V^{(\mathrm{Field})}\right) = \int_V d^3r\,(\mathbf{E}\operatorname{div}\mathbf{D} - \mathbf{D}\times\operatorname{curl}\mathbf{E} + \mathbf{H}\operatorname{div}\mathbf{B} - \mathbf{B}\times\operatorname{curl}\mathbf{H})\ . \tag{4.51}$$

The right-hand side is symmetric with respect to electric and magnetic quantities. We should try to represent them as **momentum flux** through the surface $S(V)$ in order to be able to interpret (4.51) as a balance of momentum. For this purpose we again presume a linear homogeneous medium ($\epsilon_\mathrm{r} = \mathrm{const}$, $\mu_\mathrm{r} = \mathrm{const}$) and denote by x_1, x_2, x_3 the Cartesian coordinates:

$$\begin{aligned}(\mathbf{E}\operatorname{div}\mathbf{D} - \mathbf{D}\times\operatorname{curl}\mathbf{E})_1 &= \epsilon_\mathrm{r}\epsilon_0\Bigg[E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) \\ &\quad - E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) + E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)\Bigg] \\ &= \epsilon_\mathrm{r}\epsilon_0\Bigg[\frac{\partial}{\partial x_1}\left(\frac{1}{2}E_1^2 - \frac{1}{2}E_2^2 - \frac{1}{2}E_3^2\right) \\ &\quad + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3)\Bigg] .\end{aligned}$$

We get the corresponding expressions for the other two components:

$$(\mathbf{E}\operatorname{div}\mathbf{D} - \mathbf{D}\times\operatorname{curl}\mathbf{E})_i = \epsilon_\mathrm{r}\epsilon_0\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) .$$

Strictly analogously one finds for the magnetic part in (4.51):

$$(\mathbf{H}\operatorname{div}\mathbf{B} - \mathbf{B}\times\operatorname{curl}\mathbf{H})_i = \frac{1}{\mu_\mathrm{r}\mu_0}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right) .$$

We define:

Maxwell stress tensor $\overline{T} = (T_{ij})$:

$$T_{ij} = \epsilon_\mathrm{r}\epsilon_0 E_iE_j + \frac{1}{\mu_\mathrm{r}\mu_0}B_iB_j - \frac{1}{2}\delta_{ij}\left(\epsilon_\mathrm{r}\epsilon_0E^2 + \frac{1}{\mu_\mathrm{r}\mu_0}B^2\right) . \tag{4.52}$$

With the elements of this symmetric tensor of second rank ($T_{ij} = T_{ji}$) it follows from (4.51):

$$\frac{d}{dt}\left(\mathbf{p}_V^{(\mathrm{mech})} + \mathbf{p}_V^{(\mathrm{field})}\right)_i = \int_V d^3r \sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij} . \tag{4.53}$$

If we understand the i-th row of the tensor $\overline{T}$ as a vector $\mathbf{T}_i$,

$$\mathbf{T}_i = (T_{i1},\, T_{i2},\, T_{i3}) ,$$

then the sum on the right-hand side of (4.53) represents the divergence of $\mathbf{T}_i$ so that we can apply the Gauss theorem for a further reformulation:

$$\frac{d}{dt}\left(\mathbf{p}_V^{(\mathrm{mech})} + \mathbf{p}_V^{(\mathrm{field})}\right)_i = \int\limits_V d^3r \operatorname{div} \mathbf{T}_i = \int\limits_{S(V)} d\mathbf{f} \cdot \mathbf{T}_i \ . \tag{4.54}$$

Let $\mathbf{n} = (n_1, n_2, n_3)$ be the normal-unit vector on $S(V)$ directed outwards and be in general position-dependent, i.e.

$$d\mathbf{f} = df\, \mathbf{n} \ ,$$

then we can also write:

$$\frac{d}{dt}\left(\mathbf{p}_V^{(\mathrm{mech})} + \mathbf{p}_V^{(\mathrm{field})}\right)_i = \int\limits_{S(V)} df \sum_{j=1}^{3} T_{ij} n_j \ : \quad \textbf{momentum theorem} \ . \tag{4.55}$$

The expression

$$\sum_{j=1}^{3} T_{ij} n_j$$

is obviously to be interpreted in this balance of momentum as the i-th component of the **momentum flux** through the unit-area on $S(V)$. Since the left-hand side represents the total force acting on the system in V the above expression also means:

$$\sum_{j=1}^{3} T_{ij} n_j = i\text{-th component of the force acting on } S(V) \text{ per unit-area.}$$

One can exploit this fact to calculate the force on an arbitrary physical body in an electromagnetic field. To do this one chooses as $S(V)$ a proper area enwrapping the considered body.

Example (Plane-Parallel Capacitor)

Field contributions appear only between the plates:

$$\mathbf{B} = \mathbf{0} \ ; \quad \mathbf{E} = (0, 0, -E) \ .$$

Therewith the Maxwell stress tensor reads

$$\bar{T} = (T_{ij}) = \frac{1}{2}\varepsilon_{\mathrm{r}}\varepsilon_0 E^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \ .$$

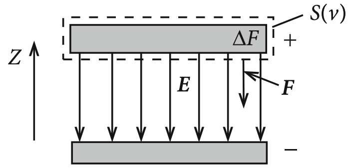


Fig. 4.3 For the calculation of the force on one of the plates of a plane-parallel capacitor by use of the Maxwell stress tensor

For the area-normal on the inside of the upper plate of the capacitor it holds according to Fig. 4.3:

$$\mathbf{n} = (0, 0, -1) \ .$$

Therewith it follows for the force components:

$$\left(\frac{\mathbf{F}}{\Delta F}\right)_x = \sum_{j=1}^{3} T_{1j} n_j = -T_{13} = 0 \ ,$$

$$\left(\frac{\mathbf{F}}{\Delta F}\right)_y = \sum_{j=1}^{3} T_{2j} n_j = -T_{23} = 0 \ ,$$

$$\left(\frac{\mathbf{F}}{\Delta F}\right)_z = \sum_{j=1}^{3} T_{3j} n_j = -T_{33} = -\frac{1}{2}\varepsilon_r \varepsilon_0 E^2 \ .$$

We thus have found for the force acting on the unit-area of the inside of the upper plate of the capacitor:

$$\frac{\mathbf{F}}{\Delta F} = \left(0, 0, -\frac{1}{2}\varepsilon_r \varepsilon_0 E^2\right) \ .$$

4.1.6 Exercises

Exercise 4.1.1 Let Σ, Σ' be two inertial systems. Furthermore, let the electromagnetic field be $\mathbf{E}$, $\mathbf{B}$ in Σ and $\mathbf{E}'$, $\mathbf{B}'$ in Σ'. The field $\mathbf{E}$ has a constant direction in the whole space. Σ' is moving relative to Σ with the constant velocity $\mathbf{v}_0$ parallel to $\mathbf{E}$ ($\mathbf{v}_0 = \alpha \mathbf{E}$). Show that the component of $\mathbf{E}'$ in the direction of $\mathbf{E}$ is equal to E.

Exercise 4.1.2 Let the electromagnetic potentials in the vacuum be defined by

$$\mathbf{A}(\mathbf{r}, t) = \alpha (x - ct)^2 \, \mathbf{e}_z \ ; \quad \varphi(\mathbf{r}, t) \equiv 0 \quad (\alpha > 0) \ .$$

Calculate the field-energy density $w(\mathbf{r}, t)$ and the Poynting vector $S(\mathbf{r}, t)$!

Exercise 4.1.3 If there are no currents and charges then the scalar potential $\varphi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ both do fulfill in the Lorenz-gauge the homogeneous wave equation

$$\Box\, \varphi(\mathbf{r}, t) = 0\ ,$$
$$\Box \mathbf{A}(\mathbf{r}, t) = 0\ ,$$

where $\Box\ = \Delta - (1/c^2)(\partial^2/\partial t^2)$.

1. Demonstrate that the electric field intensity $\mathbf{E}(\mathbf{r}, t)$ and the magnetic induction $\mathbf{B}(\mathbf{r}, t)$ obey the same differential equation.
2. The expressions

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)\ ,$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

   solve the wave equation. Which relation does exist then between ω and $\mathbf{k}$? Investigate the relative orientation of the vectors $\mathbf{k}$, $\mathbf{E}_0$ and $\mathbf{B}_0$!
3. How large is the energy current density (energy flux) parallel and perpendicular, respectively, to $\mathbf{k}$?
4. How large is the field-energy density?

Exercise 4.1.4 Show, starting from the Maxwell equations, that the fields $\mathbf{E}$ and $\mathbf{B}$ fulfill in the vacuum the inhomogeneous wave equations of the form

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \Box\,\mathbf{E} = \lambda_1(\mathbf{r}, t)$$
$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \Box\,\mathbf{B} = \lambda_2(\mathbf{r}, t)\ ,$$

Determine λ_1 and λ_2.

Exercise 4.1.5 Given an arrangement of two parallel circular metallic plates of negligible thicknesses with radii R with a separation of d. The space between the plates is filled by a dielectric with a dielectric constant which has a space dependence according to

$$\epsilon_r(z) = \epsilon_1 + \frac{1}{2}\Delta\epsilon\left(1 + 2\,\frac{z}{d}\right)\ .$$

We assume in addition that $R \gg d$.

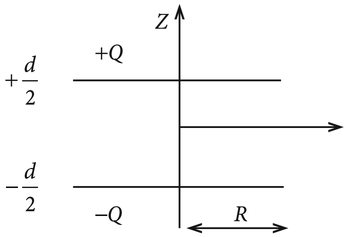


Fig. 4.4 Capacitor (charge $\pm Q$) with circular plates of radius R and a separation d of the plates

1. Calculate the capacity of the capacitor, the area-charge density at $z = \pm d/2$ as well as the volume density of the charges bound in the dielectric (Fig. 4.4).
2. The plates are equal and oppositely charged ($\pm Q$). How strong are the electrostatic forces which act on the plates? Assume thereto that the formula (4.52) derived under the precondition $\epsilon_r = \text{const}$ for the stress tensor is still approximately valid. (Weak z-dependence of ϵ_r!)

Exercise 4.1.6 A homogeneously charged sphere with the total charge q and the radius R is rotating with constant angular velocity ω around an axis through the center of the sphere. Calculate

1. the electric field $\mathbf{E}$ in the whole space,
2. the current density $\mathbf{j}$ caused by the sphere rotation,
3. the vector potential $\mathbf{A}$ in the whole space,
4. the magnetic induction $\mathbf{B}$ in the whole space,
5. the field momentum density $\bar{\mathbf{p}}_{\text{field}} = \mathbf{D} \times \mathbf{B}$,
6. the angular momentum of the electromagnetic field

$$\mathbf{L}_{\text{Field}} = \int d^3r(\mathbf{r} \times \bar{\mathbf{p}}_{\text{Field}}) \; !$$

4.2 Quasi-Stationary Fields

In Chaps. 2 and 3 we have discussed how typical problems of magnetostatics and electrostatics can be solved. Starting points were always the Maxwell equations which for static phenomena exhibit somewhat simplified structures. For time-dependent problems we have to integrate the full set of Maxwell equations (4.14) to (4.17). Because of their great technological importance, however, we restrict ourselves here at first to relatively slowly varying fields, i.e. on so-called **quasi-stationary** fields which can be worked on by an approximated set of Maxwell equations. The approximation consists in the neglect of the displacement current $\dot{\mathbf{D}}$

in (4.17). The law of induction (4.15), in contrast, is fully taken into consideration:

Maxwell equations in the quasi-stationary approximation

$$\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}\;;\quad \operatorname{curl}\mathbf{H} \approx \mathbf{j}\;,$$
$$\operatorname{div}\mathbf{D} = \rho\;;\quad \operatorname{div}\mathbf{B} = 0\;. \tag{4.56}$$

The approximation $\dot{\mathbf{D}} \approx 0$ is equivalent to $\dot{\rho} \approx 0$ and therewith, because of the continuity equation, to $\operatorname{div}\mathbf{j} \approx 0$, which in turn according to (3.6) is formally identical to the stationarity condition of the magnetostatics. That is the reason for the nomenclature **quasi-stationary**. With this simplification the equations for the magnetic fields will have the same structure as in magnetostatics!

What do we understand by *'slowly varying fields'*? Since $\dot{\mathbf{D}} \approx 0$ follows from $\dot{\rho} \approx 0$ we should better ask for *'slowly fluctuating local charge distributions'*.

The question can be answered of course only if we know *'slowly compared to what?'*. Later we will see that electromagnetic fields propagate with the velocity of light c. One therefore considers $\rho(\mathbf{r}, t)$ as *'slowly fluctuating'* if ρ is changing only very little during the time $\Delta t = d/c$, which the light needs to run through the linear dimension d of the charge arrangement. One can then assume that at each point of the field such a state essentially exists which corresponds to an infinitely fast field propagation. *Retardation effects* of the fields, which will be discussed later, can then be neglected at present.

4.2.1 Mutual Induction and Self-Induction

According to the law of induction (4.10) the temporal change of the magnetic flux Φ through the area F_C,

$$\Phi = \int_{F_C} \mathbf{B} \cdot d\mathbf{f}\;,$$

corresponds to an *electromotive force (emf)* along the boundary C which is also called **induction voltage**:

$$U_{\mathrm{ind}} = \oint_C \mathbf{E} \cdot d\mathbf{r}\;. \tag{4.57}$$

For an illustrative interpretation we imagine the path C to be realized by a conductor loop which we think for a moment to be disconnected between two close-by points P_1 and P_2 (Fig. 4.5). According to Faraday's observation an induction current is flowing along C. But that can be possible only if an electric field $\mathbf{E}$ is present in

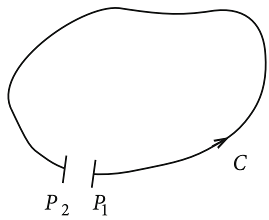


Fig. 4.5 Thumbsketch for the interpretation of the induction voltage

the conductor. Let the conductor be linear then **E** is oriented along C. We have previously interpreted the *voltage* as the work to be performed for shifting the unit-charge $q = 1$ between two space-points (see e.g. (2.45)). The work which is needed to shift $q = 1$ **against** the field from P_1 to P_2,

$$W_{21}(q = 1) = -\int\limits_{\substack{1 \\ (-C)}}^{2} \mathbf{E} \cdot d\mathbf{r} \;, \tag{4.58}$$

is therefore to be understood as the voltage between these points (Fig. 4.5). In reality it appears, e.g., as a spark-over:

$$U_{\text{ind}} = W_{21}(q = 1) = +\int\limits_{\substack{1 \\ (C)}}^{2} \mathbf{E} \cdot d\mathbf{r} = \oint\limits_{C} \mathbf{E} \cdot d\mathbf{r} \;.$$

In the last step we exploited the fact that the points P_1, P_2 are closely adjacent. Note that in electro**statics** the integral on the right-hand side vanishes. The induced field, however, is no longer curl-free.

The voltage in C will be unequal zero as long as the flux Φ through the conductor loop **temporally changes**,

$$U_{\text{ind}} = \int_{F_C} \operatorname{curl} \mathbf{E} \cdot d\mathbf{f} = -\int_{F_C} \dot{\mathbf{B}} \cdot d\mathbf{f} = -\dot{\Phi} \;, \tag{4.59}$$

where the minus sign is a manifestation of the **Lenz's law**:

The induced electric field is always oriented in such a way that it weakens the cause of its origin.

Example Let the change of the magnetic induction **B** be such that

$$d\mathbf{B} \downarrow\uparrow d\mathbf{f} \iff d\Phi < 0 \;.$$

This means:

$$U_{\text{ind}} = \oint\limits_{C} \mathbf{E} \cdot d\mathbf{r} > 0 \;.$$

The induction current I_{ind} therefore flows parallel to C. I_{ind} itself generates a magnetic induction $\mathbf{B}_{\text{ind}}$, which according to the right-handed screw rule (see (3.22), Fig. 4.6) is opposed to $d\mathbf{B}$.

Of great importance is the mutual induction of **different** conductor loops. A time-dependent current $I_i(t)$ in a closed conductor C_i will create a magnetic induction $\mathbf{B}_i(\mathbf{r}, t)$. If its field lines penetrate another conductor loop C_j then a voltage is induced in this loop. That shall be investigate now a bit more precisely:

Let $C_1, \dots, C_i, \dots, C_j, \dots, C_n$ be closed conductor loops whose directions of winding are defined by the directions of the currents. We denote as $\mathbf{F}_i$ the area which is enclosed by C_i (right-handed screw rule!). According to (4.2) the various currents cause the magnetic flux through $\mathbf{F}_j$ (Fig. 4.7):

$$\Phi_j = \int_{F_j} d\mathbf{f} \cdot \mathbf{B}^{(j)} \ . \tag{4.60}$$

$\mathbf{B}^{(j)}$ is the total magnetic induction, penetrating the area $\mathbf{F}_j$:

$$\mathbf{B}^{(j)} = \sum_{m=1}^{n} \mathbf{B}_m = \sum_{m=1}^{n} \operatorname{curl} \mathbf{A}_m \ . \tag{4.61}$$

The vector potentials $\mathbf{A}_m$ assigned to the various currents are determinable as in the magnetostatics since the differential equation to be solved in the quasi-stationary approximation is formally identical to the basic problem (3.37) of the magnetostatics. If we use the Coulomb-gauge then the vector potential obeys the

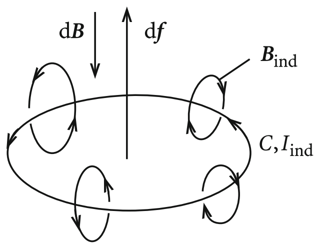


Fig. 4.6 Field profile of the magnetic induction of a current induced in a closed conductor (illustration of the Lenz's law)

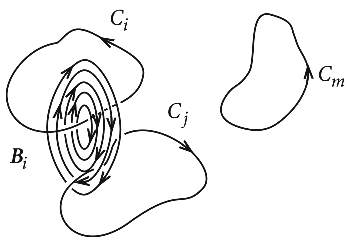


Fig. 4.7 Schematic field behavior of the magnetic inductions of different closed conductor loops for the discussion of the mutual induction

Poisson equation,

$$\Delta \mathbf{A}_m(\mathbf{r},t) = -\mu_r\mu_0\,\mathbf{j}_m(\mathbf{r},t)\ ,$$

whose solution is already known to us:

$$\mathbf{A}_m(\mathbf{r},t) = \frac{\mu_r\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}_m(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{4.62}$$

We assume that the current is homogeneously distributed over the cross section of the conductor so that the concept of the 'thread of current' (3.11) is applicable:

$$\int d^3r'\,\frac{\mathbf{j}_m(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \implies I_m(t)\int\limits_{C_m} d\mathbf{r}'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\ .$$

We therewith get:

$$\begin{aligned}\Phi_j &= \sum_{m=1}^{n}\int\limits_{F_j} d\mathbf{f}\cdot\operatorname{curl}\mathbf{A}_m = \sum_{m=1}^{n}\oint\limits_{C_j} d\mathbf{r}\cdot\mathbf{A}_m\\ &= \frac{\mu_r\mu_0}{4\pi}\sum_{m=1}^{n} I_m(t)\oint\limits_{C_j}\oint\limits_{C_m} d\mathbf{r}\cdot d\mathbf{r}'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\\ &= \sum_{m=1}^{n} L_{jm}I_m(t)\ .\end{aligned} \tag{4.63}$$

The coefficient, which depends only on the geometry of the conductor loops and the permeability of the medium,

$$L_{jm} = \frac{\mu_r\mu_0}{4\pi}\oint\limits_{C_j}\oint\limits_{C_m}\frac{d\mathbf{r}\cdot d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = L_{mj} \tag{4.64}$$

is called **'coefficient of induction'** or more precisely:

L_{jj}: **self-inductance** ,

L_{jm}; $j \neq m$: **mutual inductance.** .

According to (4.59) it then holds for the voltage induced in the conductor loop C_j:

$$U_{\mathrm{ind}}^{(j)}(t) = -\sum_{m=1}^{n} L_{jm}\dot{I}_m(t)\ . \tag{4.65}$$

The induced voltage for a certain loop thus consists of two contributions: One is caused by current changes in the other conductor loops and the other by that in the considered loop itself. Even if there is only a single conductor loop, an induced voltage appears as a consequence of a current change since the magnetic flux which penetrates the area of the current circuit will alter. That is described by the self-inductance:

$$U_{\text{ind}}^{(j)}(t) = -L_{jj}\dot{I}_j(t)\ . \tag{4.66}$$

The calculation of the self-inductance according to (4.64) encounters serious difficulties because the double-integral is divergent. The reason lies of course in the concept of the thread of current applied. This is unproblematic when calculating the mutual inductance since normally one can assume that the distances between the conductors are large compared to the diameters of their cross sections. Such an assumption, however, does not work for the self-inductance, for which the expression (4.64) is therefore only of formal nature and can not be used for a direct calculation. One has to apply other methods. For instance, one could decompose the current, which runs through the considered conductor loop, into several threads of currents and regard the mutual interference of these threads. In any case, one has to take into consideration the **finite** cross section of the conductor loop. The determination of the self-inductance therefore turns out to be much more troublesome than that of the mutual inductance.

Sometimes, though, the derivation of the self-inductance can be carried out by directly utilizing the relation (4.66) and (4.65), respectively:

Example: ***Self-Inductance of a Long Coil***

Let us consider a coil of the length l and the radius R of the cross-section with $l \gg R$ so that stray fields are negligible. For the description we choose cylindrical coordinates (ρ, φ, z), where the coil axis defines the z-direction. Because of *symmetry reasons* and with the result (3.22) for the single wire the following ansatz for the magnetic induction $\mathbf{B}$ should be valid:

$$\mathbf{B} = B(\rho)\,\mathbf{e}_z\ .$$

Let I be the current in the coil and $I(F_C)$ the total current through the area F_C (Fig. 4.8). Then we can make use of

$$\operatorname{curl}\mathbf{B} \approx \mu_r\mu_0\,\mathbf{j} \iff \oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_r\mu_0\,I(F_C)$$

in order to integrate the magnetic induction along the way plotted in Fig. 4.8. The contributions from C_a and C_c vanish. It thus holds:

$$\int_{C_b} \mathbf{B}\cdot d\mathbf{r} + \int_{C_d} \mathbf{B}\cdot d\mathbf{r} = n\,\hat{l}\,\mu_r\mu_0 I\ .$$

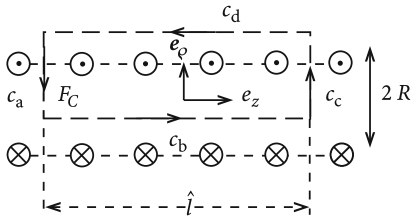


Fig. 4.8 Schematic representation of a current-carrying coil

n here is the number of coil windings per unit-length. $\hat{l}$ is defined in Fig. 4.8. The distances of the two partial paths C_b and C_d from the coil axis obviously do not play a role. **B** must therefore be homogeneous inside as well as outside the coil. Since **B** is in any case zero at infinity we have to conclude:

$$\mathbf{B} \equiv 0 \quad \text{for } \rho > R . \tag{4.67}$$

Inside the coil we then have:

$$\int_{C_b} \mathbf{B} \cdot d\mathbf{r} = B\,\hat{l} = \mu_r \mu_0\, n\, \hat{l}\, I .$$

The magnetic induction inside the coil thus is:

$$\mathbf{B} = \mu_r \mu_0\, n\, I\, \mathbf{e}_z \quad (n = N/l) . \tag{4.68}$$

So the field inside the long coil consisting of N windings is homogeneous. This means for the magnetic flux through the cross section F:

$$\Phi = B\,F = \mu_r \mu_0 n\, F\, I .$$

The voltage induced in the whole coil then reads in the case of N windings according to (4.59):

$$U_{\text{ind}} = -N\,\dot{\Phi} = -\mu_r \mu_0 \frac{N^2}{l} F\,\dot{I} .$$

The comparison with (4.66) yields the **self-inductance of the coil**,

$$L = \mu_r \mu_0 \frac{N^2}{l} F , \tag{4.69}$$

which, as expected, depends only on the geometry of the coil and on the permeability of the filling-material within the coil.

4.2.2 Magnetic Field Energy

When we inspect a system of current-carrying conductors then we find that its energy is first of all given by the magnetic energy of the various conductors. The electric energy is, in contrast to that, practically negligible because, in such cases, only very weak electric field intensities are present. The expression for the magnetic part of the field energy can be put with self-inductance and mutual inductance into a special form which is very useful for many purposes.

According to (4.46) it holds for the magnetic field energy:

$$W_{\mathrm{m}} = \frac{1}{2}\int d^3r\, \mathbf{H}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) = \frac{1}{2}\int d^3r\, \mathbf{H}(\mathbf{r},t)\cdot \operatorname{curl}\mathbf{A}(\mathbf{r},t)\ .$$

Because of

$$\operatorname{div}(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\operatorname{curl}\mathbf{A} - \mathbf{A}\cdot\operatorname{curl}\mathbf{H}$$

we can also write:

$$W_{\mathrm{m}} = \frac{1}{2}\int d^3r\, \mathbf{A}(\mathbf{r},t)\cdot\operatorname{curl}\mathbf{H}(\mathbf{r},t) + \frac{1}{2}\int d^3r \operatorname{div}(\mathbf{A}\times\mathbf{H})\ .$$

We reformulate the second integral using the Gauss theorem:

$$\int d^3r \operatorname{div}(\mathbf{A}\times\mathbf{H}) = \int\limits_{S(V\to\infty)} d\mathbf{f}\cdot(\mathbf{A}\times\mathbf{H}) = 0\ .$$

The surface integral vanishes because of the asymptotic behavior of the various terms: $d\mathbf{f} \sim r^2$, $\mathbf{A} \sim 1/r$ (3.33), $\mathbf{H} \sim 1/r^2$ (3.23). Hence in the quasi-stationary approximation we are left with:

$$W_{\mathrm{m}} = \frac{1}{2}\int d^3r\, \mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)\ . \tag{4.70}$$

Note that in this expression the vector potential $\mathbf{A}$ is generated by the current density $\mathbf{j}$, i.e. we can insert (4.62):

$$W_{\mathrm{m}} = \frac{\mu_{\mathrm{r}}\mu_0}{8\pi}\int d^3r \int d^3r'\, \frac{\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{4.71}$$

If the whole system consists exclusively of ‘thread-like’ conductors then we can apply (3.11):

$$W_{\mathrm{m}} = \frac{\mu_{\mathrm{r}}\mu_0}{8\pi}\sum_{i,j} I_i(t) I_j(t) \oint\limits_{C_i}\oint\limits_{C_j} \frac{d\mathbf{r}\cdot d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\ .$$

When we still insert the coefficients of induction L_{ij} according to (4.64) then we eventually arrive at:

$$W_\mathrm{m} = \frac{1}{2}\sum_{i,j} L_{ij} I_i(t) I_j(t) \ . \tag{4.72}$$

For the special case of a single conductor loop this reads:

$$W_\mathrm{m} = \frac{1}{2} L I^2 \ . \tag{4.73}$$

4.2.3 Alternating Currents (AC)

We consider a current circuit with a periodically *impressed* alternating voltage $U_\mathrm{e}(t)$ (generator), an inductance L (coil), a capacity C (capacitor) and an ohmic resistance R (Fig. 4.9). The circuit carries a *'thread-like'* current $I(t)$.

The partial voltages at the various components of the circuit are known to us or are at least easily calculable. So it holds for the voltage drop at the ohmic resistance:

$$\int\limits_{(R)} \mathbf{E} \cdot d\mathbf{r} = \frac{1}{\sigma} \int\limits_{(R)} \mathbf{j} \cdot d\mathbf{r} = \frac{l}{\sigma\, F} I \ .$$

l is the length and F the cross-section area of the resistance.

With (3.7)

$$U_R = I\, R$$

and the *specific resistance* $\rho = 1/\sigma$ the *ohmic resistance* R is written as

$$R = \rho \frac{l}{F} \ . \tag{4.74}$$

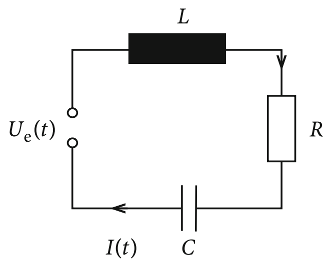


Fig. 4.9 Current circuit with ohmic resistance R, capacity C, and inductance L

At the capacitor we have according to (2.54) the voltage

$$U_C = \frac{Q}{C} \,,$$

being opposed to the impressed voltage as one can easily understand. At the coil it appears the induced voltage

$$U_L = -L\dot{I} \,.$$

Altogether it therefore holds:

$$U_\mathrm{e} - L\dot{I} - \frac{Q}{C} = IR$$

or

$$L\dot{I} + RI + \frac{Q}{C} = U_\mathrm{e} \,. \tag{4.75}$$

Furthermore, we still have the connection between current and charge:

$$I = \dot{Q} \,. \tag{4.76}$$

We have found a coupled system of linear inhomogeneous differential equations of first order for the determination of the time-dependent current $I(t)$ as consequence of a given impressed voltage $U_\mathrm{e}(t)$. Differentiating once more in (4.75) with respect to the time and inserting (4.76) we can combine the two equations to **one** differential equation of **second** order for $I(t)$:

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = \dot{U}_\mathrm{e} \,. \tag{4.77}$$

In the frequent case of a purely periodic impressed voltage

$$U_\mathrm{e} = U_0 \cos\omega t \tag{4.78}$$

we have to solve a differential equation,

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = -U_0\,\omega \sin\omega t \,,$$

which is already known to us from the mechanics ('harmonic oscillator', (2.189), Vol. 1). There we have already seen that a complete solution in the body of real numbers is of course possible but rather cumbersome. It is recommendable to perform the calculation in complex numbers since the exponential function is much easier to handle than the trigonometric functions (addition theorems!). One therefore chooses, instead of (4.78), the complex ansatz,

$$U_\mathrm{e} = U_0 e^{i\omega t} \ ,$$

to find therewith from (4.77):

$$I(t) = I_0 \, e^{i(\omega t - \varphi)} \ .$$

Of course physical measurands are always real. Therefore one has to interpret only the real part of the complex solution of (4.77) as the actual *physical* result. Since the differential equation (4.77) is linear, real and imaginary parts do not mix. If, for instance, $I = I_0 \, e^{i(\omega t - \varphi)}$ solves (4.77) for $U_\mathrm{e} = U_0 \, e^{i\omega t}$ then, because R, L, and C are real quantities, $I^*(t)$ must obviously be a solution for $U_\mathrm{e}^*(t)$. Because of the superposition principle it is then certainly

$$\mathrm{Re} I(t) = \frac{1}{2} \left(I(t) + I^*(t) \right) = I_0 \cos(\omega t - \varphi)$$

a solution of (4.77) for

$$\mathrm{Re} U_\mathrm{e}(t) = \frac{1}{2} \left(U_\mathrm{e}(t) + U_\mathrm{e}^*(t) \right) = U_0 \cos \omega t \ .$$

As a logical consequence of the complex notation for I and U one defines also a **complex resistance** Z:

$$Z = \frac{U}{I} = \frac{U_0}{I_0} \, e^{i\varphi} = |Z| \, e^{i\varphi} \ . \tag{4.79}$$

One uses the following notations:

impedance: $|Z| = U_0/I_0 = \sqrt{(\mathrm{Re}Z)^2 + (\mathrm{Im}Z)^2} \ ,$

effective resistance: $\mathrm{Re}Z \ ,$

reactance: $\mathrm{Im}Z \ .$

One visualizes these quantities in the so-called **vector diagram** (Fig. 4.10). The complex resistance Z has a **phase shift** φ:

$$\tan \varphi = \frac{\mathrm{Im}Z}{\mathrm{Re}Z} \ .$$

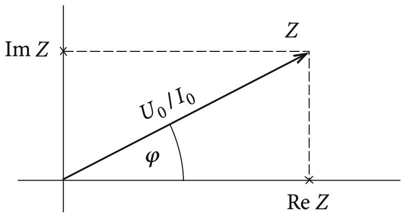


Fig. 4.10 Effective resistance ReZ and reactance ImZ in the vector diagram

In the context of alternating-currents one frequently uses the **root-mean-square (rms) values** of the current and the voltage, respectively, which are defined as the roots of the time-averaged squares of U and I, i.e. for instance (τ: periodic time):

$$U_{\text{eff}}^2 = \frac{1}{\tau} U_0^2 \int_0^\tau \cos^2 \omega t \, dt \qquad (\omega\tau = 2\pi)$$

$$= \frac{U_0^2}{2\pi} \int_0^{2\pi} \cos^2 x \, dx = \frac{U_0^2}{2} \, .$$

Thus it is:

$$U_{\text{eff}} = \frac{U_0}{\sqrt{2}} \; ; \quad I_{\text{eff}} = \frac{I_0}{\sqrt{2}} \, . \tag{4.80}$$

For the calculation of the

power P in the alternating-current circuit

we can not use the above complex ansatze but have to apply the real terms because P is not linear in U and I. The *momentary* power arises out of the solution of (4.77):

$$P(t) = U(t)\,I(t) = U_0 I_0 \, \cos \omega t \, \cos(\omega t - \varphi) \, . \tag{4.81}$$

The time-averaged power $\overline{P(t)}$ is, however, more important, for which we get with

$$\frac{1}{\tau} \int_0^\tau dt \, \cos \omega t \, \cos(\omega t - \varphi) = \cos \varphi \frac{1}{\tau} \int_0^\tau dt \, \cos^2 \omega t + \sin \varphi \frac{1}{\tau} \int_0^\tau dt \, \cos \omega t \, \sin \omega t$$

$$= \frac{1}{2} \cos \varphi$$

the following expression:

$$\overline{P(t)} = \frac{1}{2} U_0 I_0 \, \cos \varphi = U_{\text{eff}} I_{\text{eff}} \cos \varphi \, . \tag{4.82}$$

Before we start to find the general solution of the differential equation (4.77) let us still inspect some special cases:

(1) Alternating-Current Circuit with Ohmic Resistance (Fig. 4.11)

Z is because of

$$U_\mathrm{e}(t) = I\,R$$

a real quantity:

$$Z = R = \mathrm{Re}Z = |Z|\ .$$

The phase shift between current and voltage is zero:

$$\varphi = 0\ . \tag{4.83}$$

The time-averaged power consumption in such a case is maximal:

$$\overline{P(t)} = \frac{1}{2}U_0 I_0 = U_\mathrm{eff} I_\mathrm{eff}\ . \tag{4.84}$$

In terms of the root-mean-square values the power has the same structure as in the direct-current (DC) case.

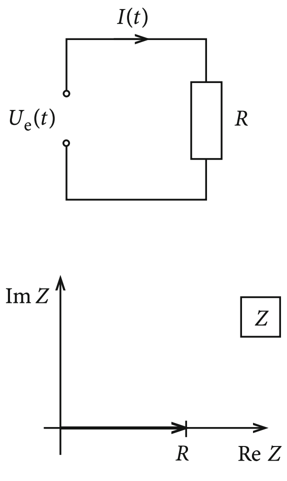


Fig. 4.11 Alternating-current circuit with an ohmic resistance and the corresponding vector diagram

(2) Alternating-Current Circuit with Inductance (Fig. 4.12)

Equation (4.77) simplifies to

$$U_e(t) = U_0\, e^{i\omega t} = L\dot{I} = i\omega L\, I_0\, e^{i(\omega t - \varphi)} = i\,\omega L\, I(t)\ .$$

The complex resistance is now purely imaginary and vanishes in the case of a direct current ($\omega = 0$):

$$Z = i\,\omega L\ ; \quad |Z| = \omega L\ . \tag{4.85}$$

The current is lagging behind the voltage by $\pi/2$:

$$\varphi = \frac{\pi}{2}\ . \tag{4.86}$$

Because of $\cos \pi/2 = 0$ the time-averaged power is zero:

$$\overline{P} = 0 \tag{4.87}$$

(*watt-less current*).

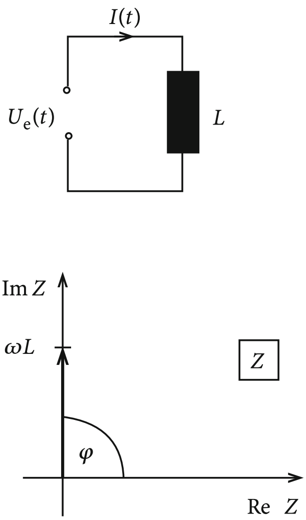


Fig. 4.12
Alternating-current circuit with inductance and the corresponding vector diagram

(3) Alternating-Current Circuit with Capacity (Fig. 4.13)

(4.77) simplifies in this case to:

$$
\begin{aligned}
U_{\mathrm{e}} = \frac{Q}{C} \iff \quad \dot{U}_{\mathrm{e}} &= \frac{1}{C} I \\
\iff i\omega U_{\mathrm{e}} &= \frac{1}{C} I \,, \\
U_{\mathrm{e}}(t) &= -\frac{i}{\omega C} I(t) \,.
\end{aligned}
$$

Z is again purely imaginary:

$$Z = -\frac{i}{\omega C} \,; \quad |Z| = \frac{1}{\omega C} \,. \tag{4.88}$$

In this case the current leads the voltage by $\pi/2$:

$$\varphi = -\frac{\pi}{2} \,. \tag{4.89}$$

For the direct current ($\omega = 0$) the impedance is infinitely large, since the direct current cannot flow through the capacitor. The time-averaged power is again zero!

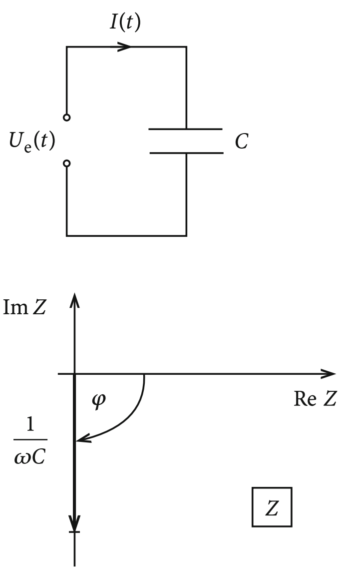


Fig. 4.13 Alternating-current circuit with capacity and the corresponding vector diagram

(4) Series Connection of Complex Resistances (Fig. 4.14)

The same current $I(t)$ flows through all resistances and the partial voltages add up:

$$U(t) \stackrel{!}{=} Z\,I(t) = U_1 + U_2 + \ldots + U_n = (Z_1 + Z_2 + \ldots + Z_n)I(t)\ .$$

The resistances therefore add together:

$$Z = Z_1 + Z_2 + \ldots + Z_n\ . \tag{4.90}$$

Example (Fig. 4.15)

$$Z = Z_R + Z_L + Z_C = R + i\left(\omega L - \frac{1}{\omega C}\right)\ . \tag{4.91}$$

(5) Parallel Connection of Complex Resistances (Fig. 4.16)

The voltage drop $U(t)$ is the same across each of the resistors and the partial currents add up ($\operatorname{div}\mathbf{j} \approx 0!$):

$$I(t) = \frac{1}{Z}U(t) = I_1 + I_2 + \ldots + I_n = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n}\right)U(t)\ .$$

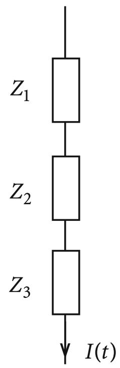


Fig. 4.14 Connection in series of three complex resistances

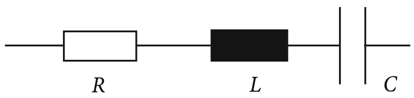


Fig. 4.15 Series connection of an ohmic, an inductive, and a capacitive resistance

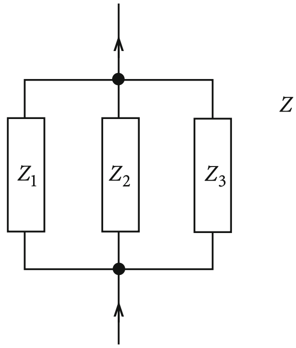


Fig. 4.16 Parallel connection of three complex resistances

We see that the complex conductances are adding together:

$$\frac{1}{Z} = \sum_{i=1}^{n} \frac{1}{Z_i} \ . \tag{4.92}$$

4.2.4 The Oscillator Circuit

Figure 4.9 represents a so-called **series-resonant circuit**. It consists of an *external* voltage source $U_e(t)$ and the in series connected ohmic resistance R, coil of inductance L, and capacitor of capacity C. The voltage $U_e(t)$ is assumed to be known. One looks for the current $I(t)$ as the solution of the inhomogeneous differential equation of second order (4.77) and (4.75), respectively. The general solution can be composed by the general solution of the respective homogeneous differential equation and a special solution of the inhomogeneous equation. We thus discuss in this section at first a situation which refers to the homogeneous differential equation, i.e.

series-resonant circuit ***without*** *external voltage source*

for which (4.75) is to be solved in the form:

$$L\dot{I} + RI + Q/C = 0 \ ,$$
$$I = \dot{Q} \tag{4.93}$$

We think this to be realized, for instance, by the arrangement sketched in Fig. 4.17. In case of the switch setting (1) the capacitor is brought to the voltage U_0 by the direct-current-voltage source. When we turn the switch over into the position (2) the current circuit is short-circuited and the voltage source is decoupled. We then observe the time variation of the current $I(t)$, e.g., by using an oscilloscope via the voltage $U_R(t) = RI(t)$ dropping at R. This issue corresponds to the following initial

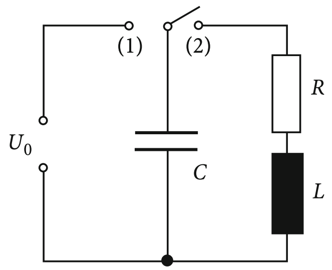


Fig. 4.17 Realization of a series-resonant circuit without voltage source (switch setting (2))

conditions which can be formulated most easily for the voltage $U_C(t)$ that drops at the capacitor:

$$\begin{aligned} U_C(0) &= U_0 \,, \\ \dot{U}_C(0) &= \frac{1}{C}\dot{Q} = \frac{1}{C}I(0) = 0 \,. \end{aligned} \tag{4.94}$$

We therefore rewrite the differential equation (4.93) using U_C as the variable ($I(t) = \dot{Q}(t) = C\dot{U}_C(t)$):

$$L\,C\ddot{U}_C + R\,C\dot{U}_C + U_C = 0 \,.$$

With the definitions

$$\begin{aligned} 2\beta &= \tfrac{R}{L}\text{:} \quad \textbf{damping} \,, \\ \omega_0^2 &= \tfrac{1}{LC}\text{:} \; \textbf{eigenfrequency} \end{aligned} \tag{4.95}$$

it becomes a differential equation,

$$\ddot{U}_C + 2\beta\,\dot{U}_C + \omega_0^2\,U_C = 0 \,, \tag{4.96}$$

which is formally identical to the equation of motion ((2.170), Vol. 1) of the free, damped, linear harmonic oscillator. Hence we already know the method of solution. Starting point is the complex ansatz:

$$U_C \sim e^{i\overline{\omega}t} \,, \tag{4.97}$$

by which (4.96) turns into:

$$-\overline{\omega}^2 + 2i\beta\overline{\omega} + \omega_0^2 = 0 \,.$$

This equation is solved by:

$$\begin{aligned} i\,\overline{\omega}_{1,2} &= -\beta \pm i\,\omega\;, \\ \omega &= \sqrt{\omega_0^2 - \beta^2} = \sqrt{\frac{1}{L\,C} - \frac{R^2}{4L^2}}\;. \end{aligned} \tag{4.98}$$

Therewith the **general solution** of the differential equation (4.96) reads:

$$U_C(t) = e^{-\beta t}\left(U_0^{(1)}\,e^{i\omega t} + U_0^{(2)}\,e^{-i\omega t}\right)\;. \tag{4.99}$$

Let us evaluate it further by explicitly using the initial conditions (4.94):

$$\begin{aligned} U_0^{(1)} &= \frac{1}{2}U_0\left(1 - i\,\frac{\beta}{\omega}\right)\;, \\ U_0^{(2)} &= \frac{1}{2}U_0\left(1 + i\,\frac{\beta}{\omega}\right)\;. \end{aligned} \tag{4.100}$$

Via the frequency ω (real, imaginary or zero) one recognizes, just like for the harmonic oscillator, three types of solution:

(1) Weak Damping (Oscillatory Case)

This is the case if

$$\beta^2 < \omega_0^2 \iff R^2 < 4\,\frac{L}{C} \tag{4.101}$$

The frequency ω is then real and (4.99) and (4.100) can be combined to

$$\begin{aligned} U_C(t) &= e^{-\beta t}U_0\left(\cos\omega t + \frac{\beta}{\omega}\sin\omega t\right) \\ &= e^{-\beta t}U_0\frac{\omega_0}{\omega}\left(\frac{\omega}{\omega_0}\cos\omega t + \frac{\beta}{\omega_0}\sin\omega t\right) \\ &= e^{-\beta t}U_0\frac{\omega_0}{\omega}\sin(\omega t + \varphi)\;, \end{aligned} \tag{4.102}$$

where it must hold for the phase (cf. (2.178), Vol. 1):

$$\sin\varphi = \frac{\omega}{\omega_0}\;;\quad \cos\varphi = \frac{\beta}{\omega_0} \tag{4.103}$$

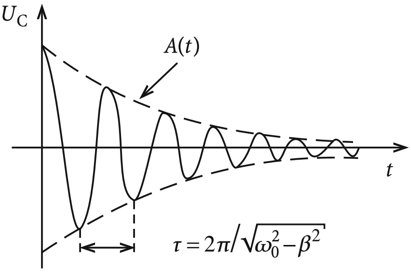


Fig. 4.18 Time-dependence of the voltage drop across the capacitor of a series-resonant circuit in the case of weak damping

The voltage at the capacitor performs a damped oscillation with an exponentially decaying amplitude (Fig. 4.18):

$$A = U_0 \frac{\omega_0}{\omega} e^{-\beta t} = A(t) \ .$$

By differentiation in (4.102) with respect to time we obtain the actually interesting current in the oscillator circuit:

$$\begin{aligned} I(t) &= C\dot{U}_C(t) = CU_0 \ e^{-\beta t} \left(-\beta \left(\cos \omega t + \frac{\beta}{\omega} \sin \omega t \right) - \omega \sin \omega t + \beta \cos \omega t \right) \\ &= CU_0 \ e^{-\beta t} \left(-\frac{1}{\omega} \right) \underbrace{(\beta^2 + \omega^2)}_{\omega_0^2 = \frac{1}{LC}} \sin \omega t \\ &= -\frac{U_0}{\omega L} \ e^{-\beta t} \sin \omega t \\ &= \frac{U_0}{\omega L} \ e^{-\beta t} \sin(\omega t + \pi) \ . \end{aligned} \tag{4.104}$$

The current is of course also exponentially damped, where the damping increases with R and decreases with L.

For very weak damping $\beta \ll \omega_0$ ($R \approx 0$) the above solution simplifies to:

$$\omega \approx \omega_0 \ ; \quad \varphi \approx \frac{\pi}{2} \ ,$$

$$U_C(t) \approx U_0 \, e^{-\beta t} \sin \left(\omega_0 t + \frac{\pi}{2} \right) \ ,$$

$$I(t) \approx U_0 \sqrt{\frac{C}{L}} \, e^{-\beta t} \sin(\omega_0 t + \pi) \ .$$

We see that the current runs ahead the voltage by about $\pi/2$. The oscillations, performed by $U_C(t)$ and $I(t)$, bring about a steady exchange between electric field energy W_e (capacitor!) and magnetic field energy W_m (coil!):

$$W_\mathrm{e} = \frac{1}{2} C\, U_C^2 \sim e^{-2\beta t} \cos^2 \omega_0 t \; ,$$

$$W_\mathrm{m} = \frac{1}{2} L I^2 \sim e^{-2\beta t} \sin^2 \omega_0 t \; ,$$

$$\begin{array}{lll}
t = 0: & I = 0, U_C \text{ maximal} & \Longrightarrow\; W_\mathrm{m} = 0,\; \text{only } W_\mathrm{e} \neq 0 \; , \\
t = \tau_0/4: & U_C = 0, I \text{ maximal} & \Longrightarrow\; W_\mathrm{e} = 0,\; \text{only } W_\mathrm{m} \neq 0 \; , \\
t = \tau_0/2: & I = 0, U_C \text{ maximal} & \text{(capacitor but oppositely charged} \\
 & & \text{compared to the } t = 0 \text{ case)} \\
 & & \Longrightarrow\; W_\mathrm{m} = 0,\; \text{only } W_\mathrm{e} \neq 0 \; , \\
t = (3/4)\tau_0: & U_C = 0, I \text{ maximal} & \text{(but opposed to the current at } \tau_0/4) \\
 & & \Longrightarrow\; W_\mathrm{e} = 0,\; \text{only } W_\mathrm{m} \neq 0 \; .
\end{array}$$

The ohmic resistance R (*resistive load*) causes energy dissipation. Via R the field energy is converted into Joule heat:

$$\begin{aligned}
\frac{d}{dt} W_\mathrm{field}(t) &= \frac{d}{dt}\left(\frac{1}{2} C\, U_C^2 + \frac{1}{2} L I^2\right) = C\, U_C \dot{U}_C + L I \dot{I} \\
&\overset{(4.93)}{=} U_C I + I(-R I - U_C) = -R I^2 \; . \qquad (4.105)
\end{aligned}$$

This according to (3.12) and (3.13), respectively, is the power loss which manifests itself as Joule heat.

(2) Critical Damping (Aperiodic Limiting Case)

There exists an interesting limiting case:

$$\beta^2 = \omega_0^2 \iff \omega = 0 \iff R^2 = 4\,\frac{L}{C} \; . \qquad (4.106)$$

In this case the two roots $\overline{\omega}_{1,2}$ in (4.98) are identical. However, we cannot simply adopt (4.99) with $\omega = 0$ since the solution then would contain only one independent parameter:

$$U_C(t) = \alpha\; e^{-\beta t} \; .$$

That would then be only a special solution, not the general one. However, we can come to the general solution when we use this special solution as ansatz with a time-dependent pre-factor (cf. (2.182), Vol. 1):

$$\alpha = \alpha(t)$$

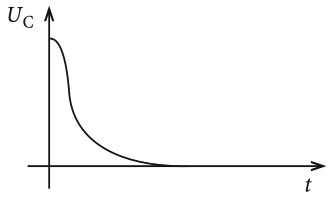


Fig. 4.19 Time-dependence of the voltage across the capacitor of a series-resonant circuit in the case of critical damping

With (4.106) and (4.96) this ansatz leads to

$$\ddot{\alpha}(t) = 0 \iff \alpha(t) = a_1 + a_2 t \ ,$$

where we fit the two independent parameters a_1 and a_2 to the boundary conditions (4.94):

$$U_C(t) = U_0(1 + \beta t)\, e^{-\beta t} \ . \tag{4.107}$$

The voltage at the capacitor does in this special case no longer perform oscillations instead being very rapidly exponentially damped without any zero-crossing (Fig. 4.19). The current intensity $I(t)$ of course behaves accordingly:

$$I(t) = C\, \dot{U}_C(t) = -\beta^2 C\, U_0 \cdot t\ e^{-\beta t} \ . \tag{4.108}$$

(3) Strong Damping (Creeping Case)

This case concerns

$$\beta^2 > \omega_0^2 \iff R^2 > 4\,\frac{L}{C} \ . \tag{4.109}$$

The frequency ω (4.98) is now purely imaginary:

$$\omega = i\gamma \ ; \quad \gamma = \sqrt{\beta^2 - \omega_0^2} \ . \tag{4.110}$$

That means at first with (4.99):

$$U_C(t) = e^{-\beta t} \left(U_0^{(1)}\, e^{-\gamma t} + U_0^{(2)}\, e^{\gamma t} \right) \ , \tag{4.111}$$

where according to (4.100) the coefficients are found to be:

$$U_0^{(1)} = \frac{U_0}{2}\left(1 - \frac{\beta}{\gamma}\right) \ ; \quad U_0^{(2)} = \frac{U_0}{2}\left(1 + \frac{\beta}{\gamma}\right) \ . \tag{4.112}$$

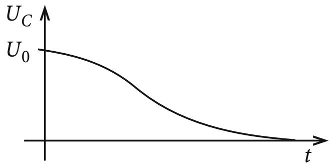


Fig. 4.20 Time-dependence of the voltage across the capacitor of a series-resonant circuit in the case of strong damping

The voltage $U_C(t)$ is in this case, too, not capable of performing oscillations (Fig. 4.20). It rather decays exponentially with a characteristic time constant

$$\tau = \frac{1}{\beta - \gamma} \; ,$$

which turns out to be larger than that of the aperiodic limiting case (there $\tau = 1/\beta$). The voltage U_C across the capacitor of the oscillator circuit drops down after the short-circuit definitely most quickly in the aperiodic limiting case.

4.2.5 Resonance

The oscillation process, which is performed by the current $I(t)$ as discussed in the last section, is exponentially damped due to the ohmic resistance R ($\Longrightarrow$ *friction*). If the oscillation is to be maintained then an additional external periodic voltage source must be applied. Because of the reasons explained in Sect. 4.2.3 we start with a complex ansatz for the external source:

$$U_\mathrm{e}(t) = U_0 \, e^{i\overline{\omega}t}$$

and calculate the electric current with (4.77). After a certain *settling time*, to which we will not refer here in detail, the current $I(t)$ in the series-resonant circuit will follow the *'exciting'* voltage, i.e. it will oscillate with the same frequency $\overline{\omega}$. We therefore choose the ansatz:

$$I(t) = I_0 \, e^{i(\overline{\omega}t - \varphi)} \; .$$

That means we do not try to solve the full inhomogeneous differential equation of second order (4.77) but rather simplify the problem by the assumption that the 'build-up process' is completed. Insertion of the ansatz into (4.77) then yields an equation of determination for the amplitude I_0:

$$I_0 \left(-L\overline{\omega}^2 + iR\overline{\omega} + \frac{1}{C} \right) = i\overline{\omega} U_0 \, e^{i\varphi}$$

$$\Longrightarrow \quad I_0 \left[R + i \left(\overline{\omega} L - \frac{1}{\overline{\omega} C} \right) \right] = U_0 \, e^{i\varphi} \; .$$

For the complex resistance (4.79) we read off, not surprisingly, the expression (4.91):

$$Z = \frac{U_{\rm e}(t)}{I(t)} = \frac{U_0}{I_0}\, e^{i\varphi} = R + i\left(\overline{\omega} L - \frac{1}{\overline{\omega} C}\right) . \tag{4.113}$$

The amplitude of the current I_0 therewith becomes a function of the frequency $\overline{\omega}$ of the applied voltage (Fig. 4.21)

$$I_0 = \frac{U_0}{|Z|} = \frac{U_0}{\sqrt{R^2 + (\overline{\omega} L - 1/\overline{\omega} C)^2}} . \tag{4.114}$$

There is a particular frequency, namely the so-called **resonant frequency**

$$\overline{\omega}_{\rm R} = \omega_0 = \frac{1}{\sqrt{LC}} , \tag{4.115}$$

at which the amplitude of the current becomes maximal:

$$I_0(\overline{\omega} = \omega_0) = \frac{U_0}{R} .$$

The resonant frequency corresponds according to (4.95) to the eigenfrequency of the oscillator circuit.

Current and voltage oscillate, though, with the same frequency $\overline{\omega}$, but are phase-shifted with respect to each other by the angle φ:

$$\tan\varphi = \frac{{\rm Im} Z}{{\rm Re} Z} = \frac{\overline{\omega} L - 1/\overline{\omega} C}{R} .$$

At the resonance the phase shift is zero (Fig. 4.22).

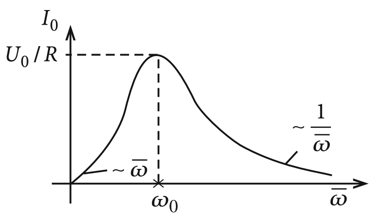


Fig. 4.21
Frequency-dependence of the current amplitude in the oscillator circuit with external voltage source

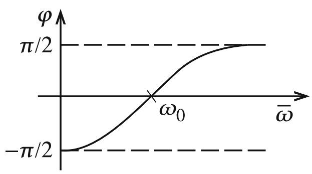


Fig. 4.22
Frequency-dependence of the phase shift between current and voltage in the oscillator circuit with external voltage source

For the averaged *power consumption* $\overline{P}$ of the resonant circuit it holds according to (4.82):

$$\begin{aligned}\overline{P} &= \frac{1}{2}U_0 I_0 \cos\varphi \\ &= \frac{1}{2}U_0 I_0 \frac{\mathrm{Re}Z}{|Z|} = \frac{1}{2}U_0 I_0 \frac{R}{|Z|} = \frac{(1/2)U_0^2 R}{R^2 + (\overline{\omega}L - 1/\overline{\omega}C)^2} \,. \end{aligned} \tag{4.116}$$

$\overline{P}$ is therefore also frequency-dependent with a maximum at the resonance $\overline{\omega}_R = \omega_0$:

$$\overline{P}_{\max} = \overline{P}(\overline{\omega} = \omega_0) = \frac{1}{2}\frac{U_0^2}{R} \,. \tag{4.117}$$

By *'resonance'* one understands, strictly speaking, just this fact that there is a frequency with maximal power consumption.

The frequencies $\overline{\omega}_{1,2}$, at which $\overline{P}$ still amounts to only half the maximum value,

$$\overline{P}\,(\overline{\omega} = \overline{\omega}_{1,2}) \overset{!}{=} \frac{1}{2}\overline{P}_{\max}\,; \quad \overline{\omega}_{1,2} = \mp\frac{R}{2L} + \sqrt{\omega_0^2 + \frac{R^2}{4L^2}}\,,$$

define the **resonance width** or the **half width (full width at half maximum)**

$$\Delta\omega_{1,2} = \overline{\omega}_2 - \overline{\omega}_1 = \frac{R}{L}(= 2\beta) \,. \tag{4.118}$$

The resonance curve therewith is the sharper the smaller the *damping* of the circuit. For very weak damping, power is consumed more or less exclusively in the interval

$$\Delta\overline{\omega} = \omega_0 \pm \frac{R}{2L} \,.$$

By a change of $\omega_0 = 1/\sqrt{LC}$, e.g., with the aid of a variable capacity of the capacitor, one can extract a definite frequency interval from a mixture of alternating voltages of different frequencies. Just in this way a radio receiving set tunes itself to a certain emitter.

4.2.6 Switching Processes

We want to discuss at the end as a simple application example the build-up and break-down of a direct current in an *RL*-circuit (Fig. 4.23).

At the time $t = t_0$ the direct-current voltage is turned on by flipping a switch:

$$U = \text{const}$$

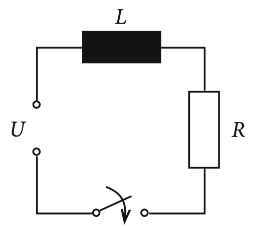


Fig. 4.23 Schematic set-up for the investigation of the build-up process in an *RL*-circuit

The then starting current obeys according to (4.75) the following differential equation:

$$L\dot{I} + RI = U = \text{const} , \quad \text{if } t \geq t_0 . \tag{4.119}$$

The general solution of the corresponding homogeneous equation reads:

$$I_{\text{hom}}(t) = A \; e^{-(R/L)t} .$$

A special solution of the inhomogeneous differential equation (4.119) one directly realizes:

$$I_{\text{S}} = \frac{U}{R} .$$

This result can of course also be *'physically guessed'*. The direct current, set in after the build-up process, must of course also be a solution of (4.119) and should fulfill the Ohm's law.

The general solution of the homogeneous and a special solution of the inhomogeneous differential equation build the general solution of the inhomogeneous equation:

$$I(t) = \frac{U}{R} + A \; e^{-(R/L)t} .$$

The initial condition $I(t = t_0) = 0$ fixes A:

$$I(t) = \frac{U}{R} \left(1 - e^{-(R/L)(t-t_0)}\right) \quad (t \geq t_0) . \tag{4.120}$$

The current achieves its saturation value, strictly speaking, only at $t \rightarrow \infty$ (Fig. 4.24):

$$I_\infty = \frac{U}{R} .$$

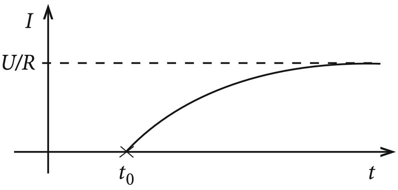


Fig. 4.24 Time-dependence of the current intensity in an *RL*-circuit when the voltage U is switched on at $t = t_0$ by flipping on a switch

The build-up process is characterized by the

$$\textbf{characteristic time constant } \tau = L/R$$

The process is the more time consuming the smaller R and the bigger L. It is lastly the self-inductance L which prevents the current to reach its full strength.

The energy which is taken from the source of the direct current is not only converted into Joule heat but also partly used to generate the magnetic field within the coil. That can be seen when (4.119) is multiplied by I and then integrated from t_0 to $t > t_0$:

$$U \int_{t_0}^{t} I(t')dt' = \frac{1}{2} L I^2(t) + R \int_{t_0}^{t} I^2(t')\, dt' \,. \tag{4.121}$$

The left-hand side represents the energy given by the source. The first term on the right-hand side is just the energy needed to build up the magnetic field, and the second terms represents the Joule heat produced in the resistive load.

We finally investigate the analogous **switch-off process**. For this we have to solve the homogeneous differential equation

$$\dot{I}(t) + \frac{R}{L} I(t) = 0 \,, \tag{4.122}$$

with the boundary condition

$$I(t) = \frac{U}{R} \quad \text{for} \quad t \leq t_1$$

what obviously leads to

$$I(t) = \frac{U}{R}\, e^{-(R/L)(t-t_1)} \quad (t \geq t_1) \,. \tag{4.123}$$

The current therefore does not disappear immediately with the switching-off of the voltage source but rather decays exponentially with the same time constant as for the switching-on process (Fig. 4.25).

Fig. 4.25 Time-dependence of the current intensity in an *RL*-circuit when the source of the voltage is switched off at the time $t = t_1$

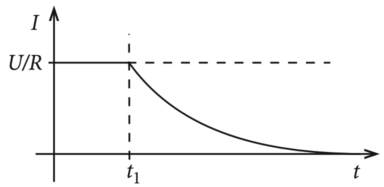


Fig. 4.26 Schematic representation of a hollow conductor

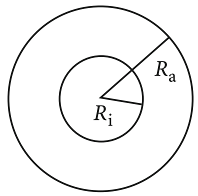
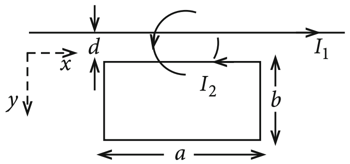


Fig. 4.27 Rectangular conductor loop in the area of influence of the magnetic field of a current-carrying thin conductor



4.2.7 Exercises

Exercise 4.2.1 Given a hollow conductor with the inner radius R_{i} and the outer radius R_{a} (Fig. 4.26). In the inner hollow conductor a current I flows and in the outer an equal and opposite current $-I$.

1. Calculate the magnetic induction in the whole space!
2. Determine the self-inductance per unit-length!

Exercise 4.2.2 A rectangular conductor loop (length a, width b), which carries a current I_2, is located within the magnetic field of a thin wire which in turn carries a current I_1 (Fig 4.27).

1. Calculate the coefficient of the mutual inductance L_{12}.
2. Which force is exerted by the current I_1 on the conductor loop?
3. The current I_1 in the wire is switched on at the time $t = 0$ according to

$$I_1(t) = I_0\left(1 - e^{-\alpha t}\right) .$$

Calculate the voltage induced in the conductor loop. (The self-inductance of the circuit is assumed to be negligible.)

Exercise 4.2.3 Consider the *RL*-circuit plotted in Fig. 4.28 where the ohmic resistance $R = R(t)$ is time-dependent.

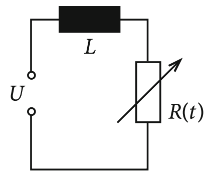


Fig. 4.28 *RL*-circuit with temporally variable ohmic resistance

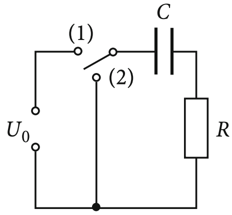


Fig. 4.29 *RC*-circuit with a switch for connecting and disconnecting a voltage source

1. **Switching-on process:** Let τ be the duration of the switching-on process which starts at $t = 0$. For the resistivity R it holds:

$$R(t) = \begin{cases} \infty & \text{for } t < 0 \,, \\ R_0\tau/t & \text{for } 0 \leq t \leq \tau \,, \\ R_0 & \text{for } \tau \leq t \,. \end{cases}$$

   Calculate the current $I(t)$ for $t \geq 0$. What is the condition for, respectively, quick and slow switching on?
2. **Switching-off process:** This also starts at $t = 0$ being terminated at $t = \tau$.

$$R(t) = \begin{cases} R_0 & \text{for } t \leq 0 \,, \\ R_0\frac{\tau}{\tau - t} & \text{for } 0 \leq t < \tau \,, \\ \infty & \text{for } \tau < t \,. \end{cases}$$

   Calculate $I(t)$ for $0 \leq t < \tau$, where before the switching-off $I(t) = U/R_0 =$ const.

Exercise 4.2.4 Given a switching circuit consisting of a direct-current-voltage source U_0, an ohmic resistance R and a capacity C (Fig. 4.29).

1. At the time $t = t_0$ a direct voltage U_0 is switched on (switch-position (1)). Calculate the voltages U_C across the capacitor and U_R across the ohmic resistance as well as the current I as functions of the time t.
2. At the time $t = t_1$ $(U_C(t_1) = U_0)$ the voltage source is disconnected, the loop short-circuited (switch-position (2)). Calculate again $U_C(t)$, $U_R(t)$, $I(t)$.

Exercise 4.2.5 Calculate for the arrangement in Fig. 4.30 (capacitively coupled oscillator circuits) the eigenfrequencies!

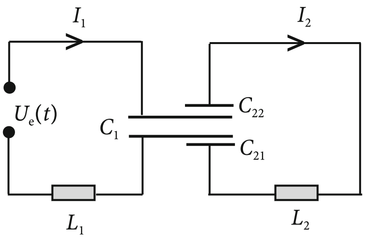


Fig. 4.30 Two capacitively coupled oscillator circuits both with inductances

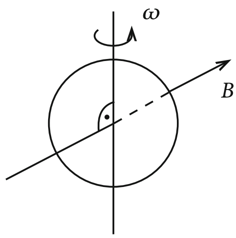


Fig. 4.31 A wire ring rotating with the frequency ω in the field of a homogeneous magnetic induction B

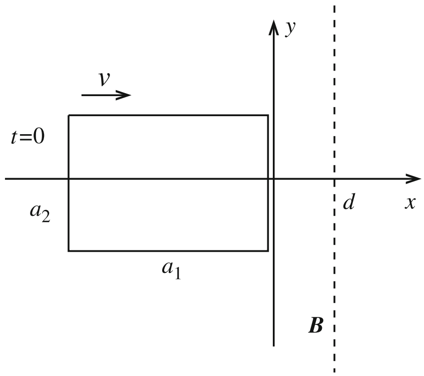


Fig. 4.32 Moving conductor loop in a homogeneous magnetic induction

Exercise 4.2.6 A circular ring with the radius R is rotating with constant angular velocity around a diameter of the ring. Perpendicular to the rotation axis a homogeneous magnetic induction $\mathbf{B}$ is present (Fig. 4.31).

1. Calculate the induction voltage generated in the ring as a function of time.
2. The ring consists of a metallic wire of conductivity σ. Which current $I(t)$ flows through the ring if one assumes that it is homogeneously distributed over a cross section A?

Exercise 4.2.7

1. A rectangular conductor loop with the side-lengths a_1 and a_2 lies in the xy-plane and moves with constant velocity v in x-direction (Fig. 4.32). A homogeneous magnetic induction $\mathbf{B} = B_0\mathbf{e}_z$ acts in the region $0 \le x \le d < a_1$. At the time $t = 0$ the right rectangle-side of the conductor is located at $x = 0$. Calculate the voltage U_{ind} induced in the conductor loop!

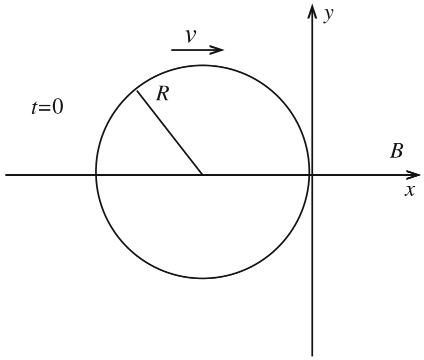


Fig. 4.33 Circular conductor loop in the field of a homogeneous magnetic induction

2. Solve the same problem for a circular conductor loop (radius R) where a homogeneous magnetic induction $\mathbf{B} = B_0 \mathbf{e}_z$ acts in the region $x > 0$ (Fig. 4.33). The conductor loop moves in this case also in x-direction with $\mathbf{v} =$ const. Sketch $U_{\text{ind}}(t)$!

Exercise 4.2.8 A charge q is homogeneously distributed on the surface of a hollow sphere with the radius R. As in Exercise 3.3.2 it rotates at first with constant angular velocity ω_0 around one of its diameter. Starting at $t = 0$ it is decelerated according to

$$\omega(t) = \omega_0 \, e^{-\gamma t} \quad (\gamma > 0) \; .$$

1. Which electric field is thereby induced in the quasi-stationary approximation ($\dot{D} \approx 0$) in the outer space ($r > R$)?
2. Under which conditions can the induced field be neglected compared to the electrostatic field ($t < 0$)?
3. What is the energy 'emitted' by the sphere per unit-time?
4. What is the total energy given away during the deceleration process?

4.3 Electromagnetic Waves

The finding that electromagnetic fields by themselves can propagate independently of any charges and currents even in the vacuum with the velocity of light, belongs to the most impressive and fundamental successes of the Maxwell's theory. That means that the fields are not only just auxiliary quantities for the interaction processes between charges and between currents, in the manner we had at first introduced them, but possess a self-contained physical reality. This assertion will be proven by the fact that the complete set of Maxwell equations possess solutions for the fields $\mathbf{E}$ and $\mathbf{B}$, which depending on their type correspond to waves propagating in the entire space. There is no need to specially emphasize the great technological importance which the discovery of the electromagnetic waves has achieved.

4.3.1 Homogeneous Wave Equation

In order to study non-stationary processes at first in an as simple framework as possible we exclude for the present electrical conductors and investigate the **electromagnetic fields in an uncharged insulator (e.g. vacuum)**:

$$\rho_{\mathrm{f}} \equiv 0\,, \quad j_{\mathrm{f}} \equiv 0\,, \quad \sigma = 0\,. \tag{4.124}$$

We presume furthermore, as usual, a linear and homogeneous medium:

$$\mathbf{B} = \mu_{\mathrm{r}}\mu_0\mathbf{H}\,; \quad \mathbf{D} = \epsilon_{\mathrm{r}}\epsilon_0\mathbf{E}\,.$$

For this situation the **Maxwell equations** read:

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0\,; \quad \operatorname{div}\mathbf{B} = 0\,, \\ \operatorname{curl}\mathbf{E} &= -\dot{\mathbf{B}}\,; \operatorname{curl}\mathbf{B} = \epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0\dot{\mathbf{E}}\,. \end{aligned} \tag{4.125}$$

The displacement current can now no longer be neglected. That means we are now going beyond the quasi-stationary approximation.

Equation (4.125) represents a coupled system of linear, partial, **homogeneous** differential equations of first order for the fields **E** and **B**. We will see that the system can be exactly decoupled so that in this case it is not necessary to revert to the auxiliary quantities φ and **A**:

From (4.125) we obtain by a repeated application of the curl to the equations of the second line:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\underbrace{(\operatorname{div}\mathbf{E})}_{=0} - \Delta\mathbf{E} = -\operatorname{curl}\dot{\mathbf{B}} = -\epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0\ddot{\mathbf{E}}\,,$$

$$\operatorname{curl}\operatorname{curl}\mathbf{B} = \operatorname{grad}\underbrace{(\operatorname{div}\mathbf{B})}_{=0} - \Delta\mathbf{B} = \epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0\operatorname{curl}\dot{\mathbf{E}} = -\epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0\ddot{\mathbf{B}}\,.$$

The constant

$$u = \frac{1}{\sqrt{\epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0}} = \frac{c}{\sqrt{\epsilon_{\mathrm{r}}\mu_{\mathrm{r}}}} = \frac{c}{n} \tag{4.126}$$

has the dimension of velocity. One calls

$$n = \sqrt{\epsilon_r \mu_r} \tag{4.127}$$

the **index of refraction** of the medium marked by ϵ_r, μ_r. u will turn out to be the velocity of light in this medium.

Under the precondition (4.124) each component of $\mathbf{E}$ and $\mathbf{B}$ as well as also each component of the vector potential $\mathbf{A}(\mathbf{r}, t)$ (in both gauges!) and the scalar potential $\varphi(\mathbf{r}, t)$ (in the Lorenz-gauge) fulfills the

homogeneous wave equation

$$\Box\, \psi(\mathbf{r}, t) = 0\ . \tag{4.128}$$

The *d'Alembert operator* $\Box$ is here defined as in (4.30) if one only replaces the velocity of light in the vacuum $c = (\epsilon_0 \mu_0)^{-1/2}$ by that in the medium:

$$\Box \equiv \Delta - \frac{1}{u^2} \frac{\partial^2}{\partial t^2}\ . \tag{4.129}$$

The differential equation (4.128) is of similar fundamental importance as the Laplace equation of electrostatics. We will have to investigate extensively, in the following, this linear, partial, homogeneous differential equation of second order.

One should bear in mind that the wave equation (4.128) arose out of the Maxwell equations by a curl-*application*. Its solution set must therefore not necessarily be identical to that of the Maxwell equations. However, only those solutions of the for $\mathbf{E}$ and $\mathbf{B}$ decoupled wave equations are interesting for us which simultaneously reproduce the couplings between $\mathbf{E}$ and $\mathbf{B}$ as required by the Maxwell equations.

4.3.2 Plane Waves

The homogeneous wave equation (4.128) is obviously fulfilled by each function of the form

$$\psi(\mathbf{r}, t) = f_-(\mathbf{k} \cdot \mathbf{r} - \omega t) + f_+(\mathbf{k} \cdot \mathbf{r} + \omega t) \tag{4.130}$$

where f_- and f_+ are sufficiently often differentiable, but otherwise arbitrary functions of the **phase**

$$\varphi_\mp(\mathbf{r}, t) = \mathbf{k} \cdot \mathbf{r} \mp \omega t\ . \tag{4.131}$$

Without any loss of generality we can therefore assume $\omega \geq 0$, since by the ansatz (4.130) both signs are already implied. Equation (4.130) is, however, only then a solution when a certain relation between ω and k is realized which we easily find by insertion into the wave equation:

$$\Delta\psi = k^2\psi'' \ ; \quad \frac{\partial^2}{\partial t^2}\psi = \omega^2\psi'' \ .$$

Here it is meant with ψ'' the second derivative with respect to the phase $\varphi_{\mp}$, i.e. with respect to the full argument. Therewith the wave equation takes the form

$$\left(k^2 - \frac{\omega^2}{u^2}\right)\psi''(\mathbf{r}, t) = \left(k^2 - \frac{\omega^2}{u^2}\right)\left(\frac{d^2 f_-}{d\varphi_-^2} + \frac{d^2 f_+}{d\varphi_+^2}\right) \equiv 0$$

and is solved by

$$\omega = u\,k \ . \tag{4.132}$$

Let us investigate the solution (4.130) in more detail, but restricting ourselves here to the partial solution f_-.

For a constant phase $\varphi_-(\mathbf{r}, t)$ f_- is obviously also constant, i.e. areas of equal phases are also areas of constant f_--values. Let us consider a *snap-shot* at $t = t_0$:

$$\varphi_-(\mathbf{r}, t_0) = \mathbf{k}\cdot\mathbf{r} - \omega t_0 \ ,$$

The area of constant phase φ_- is then defined by the condition

$$\mathbf{k}\cdot\mathbf{r} = \text{const} \ .$$

This, however, is the equation of a plane (**wavefront**) perpendicular to $\mathbf{k}$ (Fig. 4.34). f_- has the same value for all points $\mathbf{r}$ with the same projection $\mathbf{k}\cdot\mathbf{r}$ onto the direction of $\mathbf{k}$.

When we consider the total space-time process then the condition for the motion of a plane of constant phase $\varphi_-^{(0)}$ reads:

$$\mathbf{k}\cdot\mathbf{r} - \omega t = k r_{\|} - \omega t = \varphi_-^{(0)} \overset{!}{=} \text{const}$$

$$\Longrightarrow \quad r_{\|} = \frac{\mathbf{r}\cdot\mathbf{k}}{k} = \frac{1}{k}\varphi_-^{(0)} + \frac{\omega}{k}t \ .$$

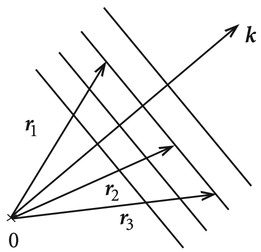


Fig. 4.34 Wavefronts of a plane wave

This obviously moves with the **phase velocity**

$$\frac{dr_{\|}}{dt} = \frac{\omega}{k} = u \tag{4.133}$$

in the direction of $\mathbf{k}$. $\mathbf{k}$ is therefore called the **propagation vector**.

Hence, the partial solution $f_{-}(\mathbf{k} \cdot \mathbf{r} - \omega t)$ in (4.130) describes the propagation of a *'disturbance'* with plane fronts, in the direction of $\mathbf{k}$ with the phase velocity u. $f_{+}(\mathbf{k} \cdot \mathbf{r} + \omega t)$ then expresses the corresponding motion in the opposite, i.e. the $(-\mathbf{k})$-direction.

Since **both** f_{-} and f_{+}, which are of the form given in (4.130), solve the wave equation, this must in particular hold for the periodic functions:

$$\begin{aligned} f_{-}(\mathbf{r}, t) &= A\ e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \ , \\ f_{+}(\mathbf{r}, t) &= B\ e^{i(\mathbf{k}\cdot\mathbf{r} + \omega t)} \ . \end{aligned} \tag{4.134}$$

Spatiotemporally periodic functions like these, for which at fixed values t the points of equal phase build a plane, are referred to as

plane waves

Let us at first use again here the expedient complex notation where we agree upon, as usual, to consider only the real parts as the physically relevant terms.

In the case of plane waves, the areas of equal $f_{\pm}$-values recur for a fixed time periodically in the space with distance-vectors $\Delta\mathbf{r}_n$:

$$\Delta\mathbf{r}_n \cdot \mathbf{k} = 2\pi n \ ; \quad n \in \mathbb{Z} \ .$$

One denotes the perpendicular distance between next adjacent wavefronts with the same $f_{\pm}$-value (Fig. 4.35),

$$\lambda = \frac{2\pi}{k} \ , \tag{4.135}$$

as the **wavelength** and the propagation vector $\mathbf{k}$ also as **wave vector**.

If we now keep the space-position fixed instead of the time, i.e. if we observe from a fixed space-point $\mathbf{r}_0$ ψ and $f_{\pm}$, respectively, they will change at this point as function of time, and will reach after the time interval

$$\tau = \frac{2\pi}{\omega} \tag{4.136}$$

again its starting value. τ is therefore called the **(oscillation) period**,

$$\nu = \frac{1}{\tau} \tag{4.137}$$

is the **frequency** and $\omega = 2\pi\nu$ the **angular frequency**.

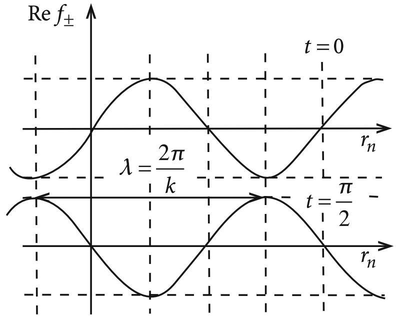


Fig. 4.35 To the definition of the wavelength of a plane wave

By combination of Eqs. (4.133), (4.135)–(4.137) one finds the important connection:

$$u = \lambda \nu = \frac{\lambda}{\tau} \ . \tag{4.138}$$

We now transfer these general results onto the **electromagnetic field** which we are actually interested in.

Thereby we learn all the essentials already by inspecting the partial solutions:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \ , \\ \mathbf{B} &= \mathbf{B}_0 \ e^{i(\overline{\mathbf{k}}\cdot\mathbf{r}-\overline{\omega} t)} \ . \end{aligned} \tag{4.139}$$

It is a decisive requirement now that the plane waves do fulfill not only the homogeneous wave equation but simultaneously have to obey the couplings in the Maxwell equations.

In a first step it follows from curl $\mathbf{E} = -\dot{\mathbf{B}}$:

$$i(\mathbf{k} \times \mathbf{E}_0) \, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\overline{\omega}\mathbf{B}_0 \ e^{i(\overline{\mathbf{k}}\cdot\mathbf{r}-\overline{\omega} t)} \ .$$

Since this has to be valid for all space-time points we obviously have to start with

$$\omega = \overline{\omega} \ ; \quad \mathbf{k} = \overline{\mathbf{k}} \ .$$

Then we have:

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0 \ . \tag{4.140}$$

div $\mathbf{E} = 0$ leads to:

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \ . \tag{4.141}$$

From div $\mathbf{B} = 0$ we get:

$$\mathbf{k} \cdot \mathbf{B}_0 = 0 \,. \tag{4.142}$$

Finally we still have curl $\mathbf{B} = (1/u^2)\dot{\mathbf{E}}$:

$$\mathbf{k} \times \mathbf{B}_0 = -\frac{\omega}{u^2}\mathbf{E}_0 \,. \tag{4.143}$$

Let us square this equation in order to get statements about the magnitudes of the field components:

$$(\mathbf{k} \times \mathbf{B}_0)^2 = \frac{\omega^2}{u^4} E_0^2 = k^2 B_0^2 = \frac{\omega^2}{u^2} B_0^2 \implies E_0^2 = u^2 B_0^2 \,.$$

The vectors $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{k}$ build in this order an orthogonal right-system, i.e. $\mathbf{E}$ and $\mathbf{B}$ are always and everywhere perpendicular to $\mathbf{k}$ and to each other (Fig. 4.36). One therefore speaks of

transverse waves

Without loss of generality we can assume that the wave vector $\mathbf{k}$ defines the z-direction:

$$\mathbf{k} = k\,\mathbf{e}_z \,.$$

Then the solutions of the wave equation which also satisfy the Maxwell equations (4.125) are:

$$\begin{aligned} \mathbf{E} &= \left(E_{0x}\mathbf{e}_x + E_{0y}\mathbf{e}_y\right)\, e^{i(kz-\omega t)} \,, \\ \mathbf{B} &= \frac{1}{u}\left(-E_{0y}\mathbf{e}_x + E_{0x}\mathbf{e}_y\right)\, e^{i(kz-\omega t)} \,. \end{aligned} \tag{4.144}$$

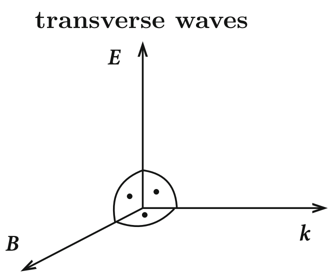


Fig. 4.36 Relative directions of the electric field, the magnetic induction, and the propagation vector in an electromagnetic transverse wave

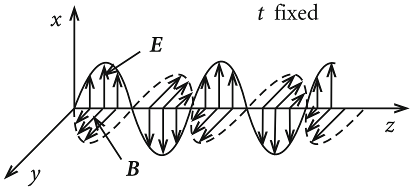


Fig. 4.37 Periodic space dependence of the **E**- and the **B**-field where the propagation vector is parallel to the z-axis

The final form of the wave is determined by E_{0x}, E_{0y} which are, however, in general complex quantities. We consider as an example the *physical* solutions for real E_{0x} and $E_{0y} = 0$ (Fig. 4.37):

$$\begin{aligned}\mathbf{E} &= E_{0x}\cos(kz - \omega t)\mathbf{e}_x \ , \\ \mathbf{B} &= \frac{1}{u}E_{0x}\cos(kz - \omega t)\mathbf{e}_y \ . \end{aligned} \tag{4.145}$$

As a further characteristic, transverse waves have a so-called **polarization** which will be the topic of the next section.

4.3.3 Polarization of the Plane Waves

The solution (4.144) of the Maxwell equations (4.125) represents a monochromatic (i.e. single-frequency) plane wave which propagates in the positive z-direction. It is the spatial extension of a harmonic oscillation. We notice that the electromagnetic wave is obviously completely determined by the **E**-vector (or equivalently by the **B**-vector) alone. It is therefore sufficient to refer the following discussion exclusively to the electric field vector **E**.

We firstly recognize that, in general, both the coefficients E_{0x}, E_{0y} are about complex quantities:

$$E_{0x} = |E_{0x}|\, e^{i\varphi} \ ; \quad E_{0y} = \left|E_{0y}\right|\, e^{i(\varphi+\delta)} \ .$$

Thus we have as actual *physical* **E**-field:

$$\mathbf{E} = E_x\mathbf{e}_x + E_y\mathbf{e}_y \tag{4.146}$$

with

$$\begin{aligned}E_x &= |E_{0x}|\cos(kz - \omega t + \varphi) \ , \\ E_y &= \left|E_{0y}\right|\cos(kz - \omega t + \varphi + \delta) \ . \end{aligned} \tag{4.147}$$

As to the *relative* phase δ, several cases can be distinguished:

(1) $\delta = 0$ or $\delta = \pm\pi$

Then it is obviously

$$\mathbf{E} = \left(|E_{0x}|\,\mathbf{e}_x \pm \left|E_{0y}\right|\mathbf{e}_y\right)\cos(kz - \omega t + \varphi)\ ,$$
$$|\mathbf{E}| = \sqrt{|E_{0x}|^2 + \left|E_{0y}\right|^2}\ . \tag{4.148}$$

The coefficient is a space- and time-**in**dependent vector, i.e. the electric field intensity **E** oscillates in a **fixed** direction relative to the direction of propagation. In such a case one calls the wave

linearly polarized

and the direction of **E** is the **direction of polarization**. It is inclined towards the x-axis by the angle α (Fig. 4.38):

$$\tan\alpha = \frac{\pm\left|E_{0y}\right|}{|E_{0x}|}\ . \tag{4.149}$$

Obviously one can understand each of the two terms in (4.146) as a linearly polarized plane wave. This means that every arbitrarily polarized plane wave can be represented as a superposition of two linearly independent, linearly polarized plane waves.

(2) $\delta = \pm\pi/2$; $|E_{0x}| = |E_{0y}| = E$

In this case it follows from (4.147):

$$\mathbf{E} = E\left[\cos(kz - \omega t + \varphi)\,\mathbf{e}_x \mp \sin(kz - \omega t + \varphi)\,\mathbf{e}_y\right]\ . \tag{4.150}$$

The upper sign holds for $\delta = +\pi/2$, the lower for $\delta = -\pi/2$. The bracket is for a fixed space point $z = z^*$ just the parameter representation of the unit-circle. The **E**-vector runs through a circle of radius E with the angular velocity ω in the plane

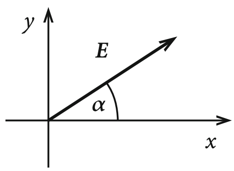


Fig. 4.38 The polarization direction of the electric field in the case of a linearly polarized plane wave

perpendicular to the direction of the propagation. One therefore calls this type of waves:

circularly polarized

Depending on the sign of δ the circle is run through in one of the two possible directions, namely clockwise or counter-clockwise.

In Fig. 4.39 the **k**-vector points perpendicularly out of the plane of the paper (z-direction). By convention the observer looks in the $-z$-direction, i.e. against the propagation direction of the arriving wave. He then sees that the **E**-vector rotates for $\delta = +\pi/2$ to the left (counter-clockwise) and for $\delta = -\pi/2$ to the right (clockwise). In this sense one speaks of a right or left circularly polarized wave. If one considers the full space-time motion then the **E**-vector describes a circular helix (Fig. 4.40):

(3) $\delta = \pm\pi/2$; $|E_{0x}| \neq |E_{0y}|$

In this case it follows from (4.147):

$$
\begin{aligned}
E_x &= |E_{0x}| \cos(kz - \omega t + \varphi) \ , \\
E_y &= \mp \left|E_{0y}\right| \sin(kz - \omega t + \varphi) \ .
\end{aligned}
\tag{4.151}
$$

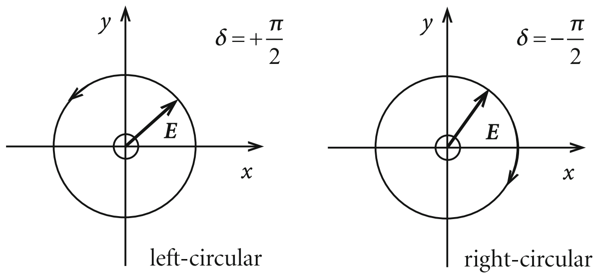


Fig. 4.39 Projection of the electric field vector of a, respectively, left and right circularly polarized wave which propagates in the z-direction

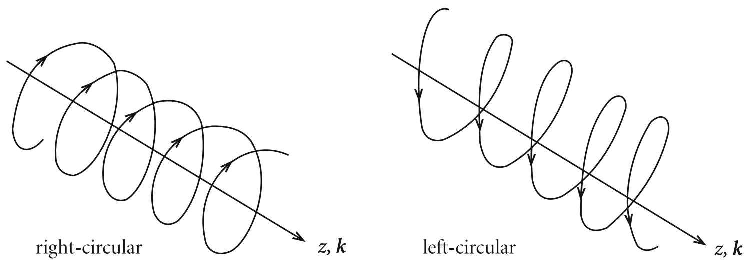


Fig. 4.40 Space-time behavior of the electric field vector in the case of a, respectively, right and left circularly polarized plane wave which propagates in the z-direction

That can be combined to:

$$\left(\frac{E_x}{|E_{0x}|}\right)^2 + \left(\frac{E_y}{|E_{0y}|}\right)^2 = 1 \; . \tag{4.152}$$

This is the equation of an ellipse with the semiaxes $|E_{0x}|$ and $|E_{0y}|$, which lie in x- and y-direction, respectively. One therefore speaks of

elliptically polarized waves.

The **E**-vector runs through an *elliptical helix* and its amplitude is obviously no longer constant (Fig. 4.41). The inclination angle α of the **E**-vector with the x-axis is now, in contrast to the case of linear polarization, space- and time-dependent:

$$\tan\alpha = \frac{\mp|E_{0y}|}{|E_{0x}|}\tan(kz - \omega t + \varphi) \; .$$

(4) δ Arbitrary; $|E_{0x}| \neq |E_{0y}|$

That is the most general and naturally the most complicated case since now the ellipse is even twisted with respect to the xy-coordinate axes (Fig. 4.42). But the wave is still called

elliptically polarized.

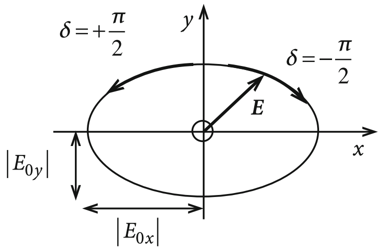


Fig. 4.41 Behavior of the electric field vector in an elliptically polarized wave with a phase shift by $\pi/2$ between x- and y-component

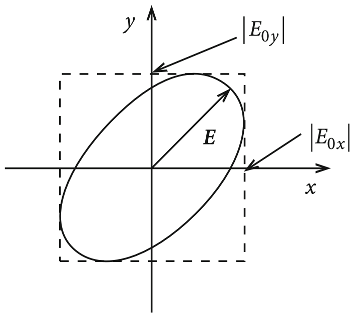


Fig. 4.42 Behavior of the electric field vector in an elliptically polarized wave with an arbitrary (!) phase shift between the x- and the y-component

At the start of this discussion we have already realized that an arbitrary elliptically polarized wave can be thought as being built up by two linearly polarized waves which are perpendicular to each other. At the end of this section we want to show that the wave can also be built up by two oppositely circularly polarized partial waves.

We start with the complex vectors

$$\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}} \left(\mathbf{e}_x \pm i\, \mathbf{e}_y\right) \ ,$$

by which we express the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$:

$$\mathbf{e}_x = \frac{1}{\sqrt{2}} \left(\mathbf{e}_+ + \mathbf{e}_-\right) \ ; \quad \mathbf{e}_y = \frac{-i}{\sqrt{2}} \left(\mathbf{e}_+ - \mathbf{e}_-\right) \ .$$

That can be used for:

$$E_{0x}\mathbf{e}_x + E_{0y}\mathbf{e}_y = \frac{1}{\sqrt{2}} \left[\left(E_{0x} - i\, E_{0y}\right) \mathbf{e}_+ + \left(E_{0x} + i\, E_{0y}\right) \mathbf{e}_-\right] \ .$$

The terms in the parentheses are complex quantities:

$$E_{0x} \pm i\, E_{0y} = E_{\pm}\, e^{i\gamma_{\pm}} \ .$$

Note that E_+ and E_- are now real. The general plane wave (4.144) can therewith be brought into the following form:

$$\mathbf{E} = \frac{1}{\sqrt{2}} \left[E_-\, e^{i(kz-\omega t+\gamma_-)}\mathbf{e}_+ + E_+\, e^{i(kz-\omega t+\gamma_+)}\mathbf{e}_-\right] \ .$$

Only the real part is *physically relevant*:

$$\begin{aligned} \mathrm{Re}\mathbf{E} = &\frac{1}{2}E_- \left[\cos(kz-\omega t+\gamma_-)\, \mathbf{e}_x - \sin(kz-\omega t+\gamma_-)\, \mathbf{e}_y\right] \\ &+\frac{1}{2}E_+ \left[\cos(kz-\omega t+\gamma_+)\, \mathbf{e}_x + \sin(kz-\omega t+\gamma_+)\, \mathbf{e}_y\right] \ . \end{aligned} \tag{4.153}$$

This is just the sum of two oppositely circularly polarized waves with different amplitudes (cf. (4.150)).

4.3.4 Wave Packets

As the general solution of the wave equation (4.128) we had found expressions of the form

$$f_{\pm}(kz \pm \omega t) \ ,$$

where the propagation direction was identified as the z-direction. There was no need to be precise with respect to the wave vector k and the angular frequency ω. Only the relation (4.132) for the **phase velocity** u has to be valid:

$$u = \frac{\omega}{k} \ .$$

One can, for instance, consider k as an independent variable. But then, however, because of this relation, ω can no longer be chosen arbitrarily. But this means also that besides $f_{\pm}$ each linear superposition of such functions belonging to different wave vectors k solve the wave equation if only the above relation is respected. A still more general solution would therefore be

$$F_{\pm}(z, t) = \int\limits_{-\infty}^{+\infty} a(k) f_{\pm}(kz \pm \omega t)\, dk \qquad (4.154)$$

with a completely arbitrary **weight function** $a(k)$.

For practical purposes this is an important point. In the last section we have discussed monochromatic plane waves, i.e. waves with sharply defined (k, ω). For the practitioner that is not realistic since even the thinkably best real source cannot emit monochromatically but rather in the form of more or less sharp *'bunches of frequencies'*. However, because of (4.154) this does not mean any fundamental difficulty for our theory. In contrast, additional considerations are unavoidable for the so-called **dispersive media**:

$$\textbf{dispersion} \iff \epsilon_{\mathbf{r}} = \epsilon_{\mathbf{r}}(\omega).$$

Because of (4.126) the phase velocity u is then frequency-dependent. In systems with dispersion one therefore has to consider ω as some function of k:

$$\omega = \omega(k) \ .$$

The partial waves $f_{\pm}$, which build up $F_{\pm}$ in (4.154), then propagate with different velocities. One cannot have a uniform phase velocity: That will lead us to a new kind of velocity, the so-called **group velocity**.

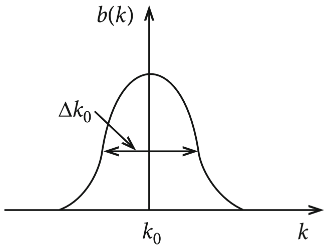


Fig. 4.43 Weight function for a superposition of plane waves being concentrated on a narrow region of wave vectors

Weighted superpositions of **plane waves** are of practical importance in this connection (Fig. 4.43),

$$H_{\pm}(z,t) = \int_{-\infty}^{+\infty} b(k)\, e^{i(kz \pm \omega t)} dk \,, \tag{4.155}$$

for which the weight function $b(k)$ represents a function being concentrated on a relatively narrow region Δk_0 around a certain k_0. The main contribution to the above integral is then due to this wave-vector region. We therefore perform a Taylor expansion of $\omega(k)$ around k_0, presuming thereby that $\omega(k)$ is about a *'well-behaved'* function of k:

$$\omega(k) = \omega(k_0) + (k - k_0) \left.\frac{d\omega}{dk}\right|_{k_0} + \ldots$$

We write $\omega(k_0) = \omega_0$ and define:

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k=k_0} \quad : \quad \textbf{group velocity} \,. \tag{4.156}$$

In dispersion-less media the group velocity is identical to the phase velocity u. When we insert the expansion into the exponent of the exponential-function,

$$e^{i(kz \pm \omega t)} = e^{i(k_0 z \pm \omega_0 t)}\, e^{i\,q(z \pm v_{\mathrm{g}} t)} + \ldots \quad (q = k - k_0)\,,$$

then we can, in case of a sharply *peaked* weight function in (4.155), truncate the Taylor expansion of $\omega(k)$ after the linear term since wave vectors k which deviate strongly from k_0 hardly contribute to the integral because $b(k) \approx 0$:

$$\begin{aligned} H_{\pm}(z,t) &\approx e^{i(k_0 z \pm \omega_0 t)} \int_{-\infty}^{+\infty} dq\, b(k_0 + q)\, e^{i\,q(z \pm v_{\mathrm{g}} t)} \\ &= e^{i(k_0 z \pm \omega_0 t)} \widehat{H}_{\pm}(z \pm v_{\mathrm{g}}\, t) \,. \end{aligned} \tag{4.157}$$

That is a plane wave whose wavelength and frequency refer to the maximum of the weight function $b(k)$, modulated, however, by a space- and time-dependent function $\widehat{H}_{\pm}$. The **modulation function** $\widehat{H}_{\pm}$ moves with the velocity v_g in, respectively, positive and negative z-direction because a constant modulation phase

$$z \pm v_g t = \text{const}$$

means:

$$\frac{dz}{dt} = \mp v_g \ .$$

A plane wave modulated in such a way is called a

wave packet

In such a wave packet the plane wave propagates with the phase velocity u_0, but the total *packet* propagates with the group velocity v_g (Fig. 4.44).

Example: *Gaussian Wave Packet*

We assume a Gaussian distribution (Fig. 4.45) as the weight function :

$$b(k) = \frac{2}{\Delta k_0 \sqrt{\pi}}\, exp\left(-\frac{4(k-k_0)^2}{\Delta k_0^2}\right) \ . \tag{4.158}$$

The **maximum** lies at $k = k_0$:

$$b_{\max} = \frac{2}{\Delta k_0 \sqrt{\pi}} \ .$$

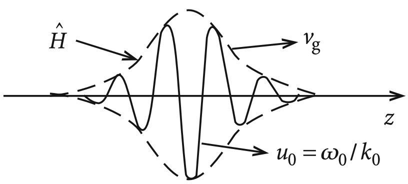


Fig. 4.44 Phase and group velocity of a modulated plane wave

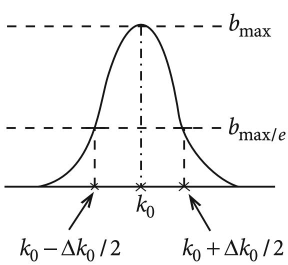


Fig. 4.45 Weight function of the Gaussian wave packet

The separation of the two points, located symmetrically to k_0, at which $b(k)$ decreases to only the e-th fraction of its maximum, amounts to just Δk_0. The **area under the Gaussian bell curve** is always 1, since:

$$\int_{-\infty}^{+\infty} dk\, b(k) = \frac{2}{\Delta k_0 \sqrt{\pi}} \int_{-\infty}^{+\infty} dk\, exp\left(-\frac{4(k-k_0)^2}{\Delta k_0^2}\right) = \frac{1}{\sqrt{\pi}} I \ ,$$

where

$$I = \int_{-\infty}^{+\infty} dy\ e^{-y^2} \ .$$

To calculate I we use the following trick:

$$I^2 = i \int_{-\infty}^{+\infty} dx\, dy\ e^{-(x^2+y^2)} = 2\pi \int_0^{\infty} d\rho\, \rho\ e^{-\rho^2} = -\frac{1}{2} 2\pi \int_0^{\infty} d\rho \frac{d}{d\rho}\, e^{-\rho^2} = \pi \ .$$

Thus it holds:

$$I = \sqrt{\pi} \iff \int_{-\infty}^{+\infty} dk\, b(k) = 1 \ . \tag{4.159}$$

(4.158) allows for a possible limiting-value representation of the δ-function:

$$\delta(k - k_0) = \lim_{\Delta k_0 \to 0} b(k) \ . \tag{4.160}$$

We now insert the Gaussian $b(k)$ into the modulation function $\widehat{H}_\pm$:

$$\widehat{H}_\pm(z \pm v_g t) = \frac{2}{\Delta k_0 \sqrt{\pi}} \int_{-\infty}^{+\infty} dq\ e^{-(4q^2/\Delta k_0^2)}\, e^{iq(z \pm v_g t)} \ ,$$

$$\frac{4q^2}{\Delta k_0^2} - iq(z \pm v_g t) = \left(\frac{2q}{\Delta k_0} - \frac{i}{4} \Delta k_0 (z \pm v_g t)\right)^2 + \frac{\Delta k_0^2}{16} (z \pm v_g t)^2 \ .$$

We substitute:

$$y = \frac{2q}{\Delta k_0} - \frac{i}{4} \Delta k_0 (z \pm v_g t) \ .$$

It proves to be correct, although we can not strictly verify at this stage, that the integration, in spite of the imaginary part of y, has to be performed from $-\infty$ to $+\infty$:

$$\widehat{H}_{\pm}(z \pm v_{\mathrm{g}}t) = \frac{1}{\sqrt{\pi}} I \; e^{(-\Delta k_0^2/16)(z \pm v_{\mathrm{g}}t)^2} \, .$$

With (4.159) and (4.157) it follows then:

$$H_{\pm}(z,t) = e^{i(k_0 z \pm \omega_0 t)} \, e^{(-\Delta k_0^2/16)(z \pm v_{\mathrm{g}}t)^2} \, . \tag{4.161}$$

This is a plane wave whose amplitude depends **gaussian-like** on $(z \pm v_{\mathrm{g}}t)$. The *Gaussian bell curve* moves rigidly with the velocity v_{g} in $\mp z$-direction. One speaks of an **aperiodic wave train**. The width of the wave packet, defined analogously to that of $b(k)$, is obviously:

$$\Delta z = \frac{8}{\Delta k_0} \, .$$

That means:

$$\Delta z \cdot \Delta k_0 = \mathrm{const} \, . \tag{4.162}$$

For the wave packet we realize that the broader the k-distribution the narrower the z-region and vice versa. A distribution, sharply localized in the *k-space* $b(k) = \delta(k - k_0)$, i.e. $\Delta k_0 \to 0$, means in the position space an unmodulated plane wave,

$$H_{\pm}(z,t) \underset{\Delta k_0 \to 0}{\longrightarrow} e^{i(k_0 z \pm \omega_0 t)} \, ,$$

being therefore not localizable. On the other hand, spatially sharply locatable means $1/\Delta k_0 \to 0$ or $\Delta k_0 \to \infty$. The distribution in the k-space is therewith completely smeared over. All the wave vectors then appear with the same weight.

This section has shown that a wave is characterized by two types of propagation velocities:

$$\begin{aligned} &\textbf{phase velocity: } u = \frac{\omega(k)}{k} \, , \\ &\textbf{group velocity: } v_{\mathrm{g}} = \frac{d\omega(k)}{dk} \, . \end{aligned} \tag{4.163}$$

The former describes the propagation of a plane wave, the latter that of a wave packet. v_{g} corresponds to the velocity by which energy and information (signals) can be transported in a wave. The special relativity teaches us that the velocity of light c in the vacuum represents an upper bound for v_{g}:

$$v_{\mathrm{g}} \leq c \, . \tag{4.164}$$

This does not necessarily hold for the phase velocity u.

One speaks of **dispersion** exactly then when $u \neq v_{\mathrm{g}}$. One should, however, bear in mind that the *concept of the group velocity* is reasonable only as long as the approximations carried out from (4.155) to (4.157) are really allowed.

4.3.5 Spherical Waves

The plane waves discussed in the last section represent only a special, though very important type of solution of the homogeneous wave equation (4.128). Another class of solution is given by the spherical waves. We come to these functions when we formulate the wave equation with spherical coordinates (2.145):

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{\vartheta\varphi}\ ,$$
$$\Delta_{\vartheta\varphi} = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\ .$$

We proceed by assuming spherically-symmetric solutions

$$\psi(\mathbf{r},t) = \psi(r,t) \implies \Delta_{\vartheta\varphi}\psi \equiv 0$$

and verify this assumption by insertion into the wave equation:

$$\Box\,\psi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right]\psi = 0\ .$$

With

$$\frac{\partial^2}{\partial r^2}(r\,\psi) = \frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r} + \psi\right) = r\frac{\partial^2\psi}{\partial r^2} + 2\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)$$

and

$$\frac{\partial^2}{\partial t^2}\psi = \frac{1}{r}\frac{\partial^2}{\partial t^2}(r\,\psi)$$

as well as the substitution

$$v(r,t) = r\,\psi(r,t)$$

we have the following differential equation,

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)v(r,t) = 0\ ,$$

which is solved by all functions of the type

$$v(r,t) = v_+(kr + \omega t) + v_-(kr - \omega t) ,$$

if only

$$k^2 = \frac{\omega^2}{u^2} \iff \omega = k\,u \quad (\omega \geq 0) \tag{4.165}$$

as in the case of plane waves. Hence, this means that

$$\psi(\mathbf{r},t) = \frac{1}{r}\left[v_+(kr + \omega t) + v_-(kr - \omega t)\right] \tag{4.166}$$

represents a further class of solutions of the homogeneous wave equation. Let us briefly discuss these functions:

1. The **phase**

$$\varphi_\pm = kr \pm \omega t \tag{4.167}$$

   depends only on the magnitude of the position vector $\mathbf{r}$. At a fixed time $t = t_0$ the points of equal phase and therewith of equal ψ-values all have the same distance from the origin, i.e. lying on a spherical surface of the radius r.
2. The amplitude decreases according to $1/r$ with increasing distance from the origin.
3. If $v_\pm(kr \pm \omega t)$ is in addition periodic, for instance

$$v_\pm \sim e^{i(kr \pm \omega t)} , \tag{4.168}$$

   then one speaks of **spherical waves**:

$$\psi_\pm(\mathbf{r},t) = \frac{A_\pm}{r}\, e^{i(kr \pm \omega t)} . \tag{4.169}$$

4. How do the areas of constant phase $\varphi_\pm^{(0)}$ move?

$$kr \pm \omega t = \varphi_\pm^{(0)} \overset{!}{=} \text{const} .$$

   This leads to the **phase velocity:**

$$\frac{dr}{dt} = \mp\frac{\omega}{k} = \mp u = \mp\frac{c}{n} . \tag{4.170}$$

The solution (4.169) represents the propagation of a *'disturbance'* with spherical wavefronts and the phase velocity u:

$$r(t) = r_0 - u\,t\text{: }\textbf{incoming}\text{ spherical wave ,}$$
$$r(t) = r_0 + u\,t\text{: }\textbf{outgoing}\text{ spherical wave .}$$

5. Spherical waves of equal phases have at a fixed time $t = t_0$ the radial separation Δr:

$$k\,\Delta r = 2\pi\,n\,; \quad n \in \mathbb{N}\,. \tag{4.171}$$

The shortest distance ($n = 1$) (Fig. 4.46) defines the same

$$\textbf{wavelength} \quad \lambda = \frac{2\pi}{k} \tag{4.172}$$

as for the plane waves (4.135).

If we keep the space-point fixed then the phase changes periodically with time with the period

$$\tau = \frac{2\pi}{\omega}\,,$$

again as for the plane waves (Fig. 4.47).

6. Finally, the solutions of the homogeneous wave equation have still to fulfill the special couplings required by the Maxwell equations. For instance, the fields

$$\mathbf{E} = \mathbf{E}_0\frac{1}{r}\,e^{i(kr-\omega t)}\,,$$
$$\mathbf{B} = \mathbf{B}_0\frac{1}{r}\,e^{i(kr-\omega t)} \tag{4.173}$$

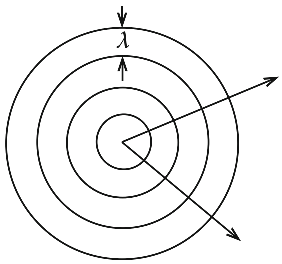


Fig. 4.46 Wavelength of spherical waves

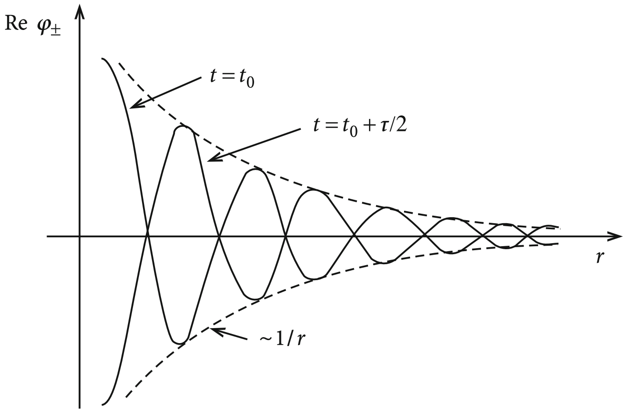


Fig. 4.47 Space-dependence of two spherical waves temporally shifted by one half of the oscillation period

have to satisfy:

$$\operatorname{div} \mathbf{E} = 0 \quad \text{and} \quad \operatorname{div} \mathbf{B} = 0$$

Let us check:

$$\begin{aligned}
\operatorname{div} \mathbf{E} &= \left(E_{0x} \frac{\partial r}{\partial x} + E_{0y} \frac{\partial r}{\partial y} + E_{0z} \frac{\partial r}{\partial z} \right) \frac{d}{dr} \left[\frac{1}{r} \, e^{i(kr - \omega t)} \right] \\
&= \left[E_{0x} \frac{x}{r} + E_{0y} \frac{y}{r} + E_{0z} \frac{z}{r} \right] \frac{d}{dr} \left[\frac{1}{r} \, e^{i(kr - \omega t)} \right] \\
&= (\mathbf{E}_0 \cdot \mathbf{r}) \frac{1}{r} \frac{d}{dr} (\ldots) = 0 \implies \mathbf{E}_0 \cdot \mathbf{r} = 0 \, .
\end{aligned}$$

This condition, however, is not at all satisfiable for a constant vector $\mathbf{E}_0$, which is not the zero vector, and an arbitrary position vector $\mathbf{r}$. The same is true for the relation

$$\mathbf{B}_0 \cdot \mathbf{r} = 0 \, ,$$

which analogously follows from $\operatorname{div} \mathbf{B} = 0$. The spherical waves (4.173) **do not fulfill** the Maxwell equations in this form.

4.3.6 Fourier Series, Fourier Integrals

We have recognized electromagnetic plane waves as special solutions of the *source-free* Maxwell equations (4.125). It turns out, however, that **any arbitrary** solution of the homogeneous wave equation can be expanded in these plane waves. To see this, we now introduce a new mathematical tool which encompasses in Theoretical Physics a wide region of application and shall therefore be investigated here in detail. We are talking about the

Fourier transformation

by which we will be able to find the most general solution of the wave equation.

We already got a certain *'foretaste'* with the considerations of the wave packet in Sect. 4.3.4. There we had provided plane waves of different wave vectors k with a known weight function $b(k)$ and then *'bunched'* them by summing together (integrating) to a packet moving in space. Very often, however, it is also interesting to pose the problem *'the other way round'*, namely, when we ask how such a weight function $b(k)$ has to be built up in order to realize a given wave packet. That can be answered by the use of the Fourier transformation.

In Sect. 2.3.5, which dealt with orthogonal and complete systems of functions, we had seen that the orthonormal system of the trigonometric functions (2.144),

$$\frac{1}{\sqrt{2a}}\ ; \quad \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{a}x\right)\ ; \quad \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{a}x\right)\ ; \quad n = 1, 2, \ldots\ ,$$

represents, in the interval $[-a, +a]$, a complete system in which any square-integrable function $f(x)$ can be expanded:

$$f(x) = f_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{a}x\right) + b_n\sin\left(\frac{n\pi}{a}x\right)\right]\ . \tag{4.174}$$

This representation of the function $f(x)$ is called its **Fourier series**. According to (2.140) we find for the coefficients (explicit proof as Exercise 4.3.7):

$$\begin{aligned} f_0 &= \frac{1}{2a}\int\limits_{-a}^{+a} f(x)dx\ , \\ a_n &= \frac{1}{a}\int\limits_{-a}^{+a} f(x)\cos\left(\frac{n\pi}{a}x\right)dx\ , \\ b_n &= \frac{1}{a}\int\limits_{-a}^{+a} f(x)\sin\left(\frac{n\pi}{a}x\right)dx\ . \end{aligned} \tag{4.175}$$

If $f(x)$ is periodic with the period $2a$, which we will presume at first,

$$f(x + 2a) = f(x)\ ,$$

then (4.174) is valid even for all x. Special cases are:

(1) Even Functions

$$f(x) = f(-x) \implies b_n = 0 \quad \forall n \; .$$

(2) Odd Functions

$$f(x) = -f(-x) \implies f_0 = 0 \; ; \quad a_n = 0 \quad \forall n \; .$$

Example: ***Fourier Series of the Relaxation Oscillation***

$$f(x) = \begin{cases} \dfrac{1}{\pi}x + 1 \text{ for } -\pi \le x < 0 \; , \\ \\ \dfrac{1}{\pi}x - 1 \text{ for } 0 \le x \le \pi \; . \end{cases}$$

$f(x)$ is defined at first in the interval $[-\pi, +\pi]$, however, with the periodicity and symmetry (Fig. 4.48):

$$f(x + 2\pi) = f(x) \; ; \quad f(-x) = -f(x) \; .$$

Thus, only the values of the coefficients b_n are to be calculated:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{\pi}x - 1\right)\sin(nx)dx + \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{1}{\pi}x + 1\right)\sin(nx)dx \\ &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} x \, \sin(nx)\, dx - \frac{2}{\pi}\int_0^{\pi}\sin(nx)\, dx \end{aligned}$$

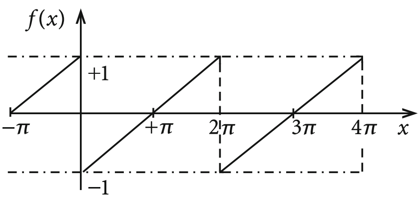


Fig. 4.48 Representation of a relaxation oscillation

$$= \frac{1}{\pi^2}\left(-\frac{1}{n}x\,\cos(nx)\right)\Bigg|_{-\pi}^{+\pi} + \frac{1}{n\pi^2}\underbrace{\int\limits_{-\pi}^{+\pi}\cos(nx)dx}_{=0} + \frac{2}{n\pi}\cos(nx)\Bigg|_0^{\pi}$$

$$= \frac{1}{n\pi^2}\left[-\pi(-1)^n - \pi(-1)^n\right] + \frac{2}{n\pi}\left[(-1)^n - 1\right] = -\frac{2}{n\pi}\,.$$

Therewith we have found the

Fourier series of the relaxation oscillation

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}\,. \tag{4.176}$$

Let us now continue to investigate the general Fourier series (4.174). With the Euler's formula ((2.146), Vol. 1) and the definition

$$v_n(x) = \frac{1}{\sqrt{2a}}\exp\left(i\frac{n\pi}{a}x\right)\,,\quad n = 0, \pm 1, \pm 2, \ldots \tag{4.177}$$

we can write:

$$\frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{a}x\right) = \frac{1}{\sqrt{2}}\left(v_n(x) + v_{-n}(x)\right)\,,$$

$$\frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{a}x\right) = \frac{-i}{\sqrt{2}}\left(v_n(x) - v_{-n}(x)\right)\,.$$

Inserting that into (4.174) we can state firstly that the $v_n(x)$, too, represent a **complete** system of functions. Moreover, we can show, as follows, that it is even an **orthonormal system**:

$$\int\limits_{-a}^{+a} v_n^*(x)v_{\mathrm{m}}(x)dx \overset{n\neq m}{=} \frac{1}{2a}\int\limits_{-a}^{+a} e^{-i(\pi/a)(n-m)x}\,dx$$

$$= \frac{1}{2a}\int\limits_{-a}^{+a}\cos\left[\frac{\pi}{a}(n-m)x\right]dx - \frac{i}{2a}\int\limits_{-a}^{+a}\sin\left[\frac{\pi}{a}(n-m)x\right]dx$$

$$= \frac{1}{2\pi(n-m)}\sin\left(\frac{\pi}{a}(n-m)x\right)\Bigg|_{-a}^{+a} + \frac{i}{2\pi(n-m)}\cos\left(\frac{\pi}{a}(n-m)x\right)\Bigg|_{-a}^{+a}$$

$$= 0\,.$$

The case $n = m$ is trivial because of $|v_n(x)|^2 = 1/(2a)$ so that we have indeed:

$$\int_{-a}^{+a} v_n^*(x) v_\mathrm{m}(x) dx = \delta_{nm} \ . \tag{4.178}$$

The completeness of the function $v_n(x)$ can be demonstrated by insertion into the Fourier series (4.174):

$$\begin{aligned} f(x) &= f_0 + \sum_{n=1}^{\infty} \left(a_n \sqrt{\frac{a}{2}} (v_n(x) + v_{-n}(x)) + b_n \left(-i \sqrt{\frac{a}{2}} \right) (v_n(x) - v_{-n}(x)) \right) \\ &= f_0 + \sum_{n=1}^{\infty} \sqrt{\frac{a}{2}} \left((a_n - i b_n)\, v_n(x) + (a_n + i b_n)\, v_{-n}(x) \right) \ . \end{aligned}$$

One reads off the relations (4.175):

$$a_n = a_{-n} \ ; \quad b_n = -b_{-n} \ ; \quad a_0 = 2 f_0 \ ; \quad b_0 = 0 \ .$$

Therewith, the Fourier series can be further compressed,

$$f(x) = \sqrt{\frac{a}{2}} \sum_{n=-\infty}^{+\infty} (a_n - i b_n) v_n(x) \ ,$$

which makes the completeness of the $v_n(x)$ become clear since $f(x)$ is an arbitrary square-integrable function. Note that the $n = 0$-term is just the constant f_0:

$$\sqrt{\frac{a}{2}} (a_0 - i b_0)\, v_0(x) = \sqrt{\frac{a}{2}}\, 2 f_0 \, \frac{1}{\sqrt{2a}} = f_0 \ .$$

For each periodic function $f(x)$ with the period $2a$, which is square-integrable in the interval $[-a, +a]$, one therefore can also write:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{+\infty} \alpha_n \, e^{i(n\pi/a)x} \ , \\ \alpha_n &= \frac{1}{2}(a_n - i b_n) = \frac{1}{2a} \int_{-a}^{+a} f(x)\, e^{-i(n\pi/a)x} dx \ . \end{aligned} \tag{4.179}$$

In particular we get for the δ-distribution,

$$\delta(x - x_0) \quad \text{with} - a < x_0 < +a \ ,$$

the expansion:

$$\delta(x - x_0) = \frac{1}{2a} \sum_{n=-\infty}^{+\infty} e^{i(n\pi/a)(x-x_0)} . \tag{4.180}$$

We now introduce a couple of new abbreviations:

$$k_n = \frac{n\pi}{a} \; ; \quad \tilde{f}_n = \alpha_n a \sqrt{\frac{2}{\pi}} \; ; \quad \Delta k = \frac{\pi}{a} \; .$$

Therewith (4.179) changes accordingly:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \tilde{f}_n \, e^{ik_n x} \Delta k \; , \tag{4.181}$$

$$\tilde{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} f(x) \, e^{-ik_n x} dx \; . \tag{4.182}$$

Δk is the separation of adjacent k_n. If one now goes over to non-periodic functions, i.e. formally, to functions with a *'periodicity interval'* $[-a, a]_{a\to\infty}$, then one has to replace the sum in (4.181) *'in the Riemannian sense'* ($\Delta k \to 0$) by the corresponding integral:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \tilde{f}(k) \, e^{ikx} \; , \tag{4.183}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx f(x) \, e^{-ikx} \; . \tag{4.184}$$

One denotes $\tilde{f}(k)$ as the **Fourier transform** or also as the **spectral function** of the function $f(x)$. Let us list some of its most important properties:

1. $f(x)$ **even**: $f(x) = f(-x)$
   Then obviously:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx f(x) \cos(kx) \; .$$

This means that $\tilde{f}(k)$, too, is an even function:

$$\tilde{f}(k) = \tilde{f}(-k) \ . \tag{4.185}$$

If $f(x)$ is in addition real then that holds for $\tilde{f}(k)$, too!

2. $f(x)$ **odd**: $f(x) = -f(-x)$

   Then:

$$\tilde{f}(k) = \frac{-i}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx f(x) \sin kx \ .$$

   Hence, $\tilde{f}(k)$, too, is odd:

$$\tilde{f}(k) = -\tilde{f}(-k) \ . \tag{4.186}$$

   If $f(x)$ is in addition real then $\tilde{f}(k)$ is purely imaginary!

3. $f(x)$ **real**

   In such a case $\tilde{f}(k)$ can obviously be decomposed as follows,

$$\tilde{f}(k) = \tilde{f}_1(k) - i\tilde{f}_2(k) \ ,$$

   with real $\tilde{f}_{1,2}(k)$, where $\tilde{f}_1$ is an even function of k and $\tilde{f}_2(k)$ an odd function:

$$\tilde{f}(-k) = \tilde{f}_1(k) + i\tilde{f}_2(k) = \tilde{f}^*(k) \ . \tag{4.187}$$

4. **Fourier transformation is linear**

   One reads off directly from the definition that the Fourier transform $\tilde{g}(k)$ of

$$g(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$$

   is given by:

$$\tilde{g}(k) = \alpha_1 \tilde{f}_1(k) + \alpha_2 \tilde{f}_2(k) \ ,$$

   provided that $\tilde{f}_{1,2}(k)$ are the Fourier transforms of $f_{1,2}(x)$.

5. **Convolution theorem**

   Let $\tilde{f}_1(k)$, $\tilde{f}_2(k)$ again be the Fourier transforms of the functions $f_1(x)$, $f_2(x)$. Then we have

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dk' \tilde{f}_2(k') \tilde{f}_1(k - k') \tag{4.188}$$

   as the Fourier transform of the product $f_1(x) f_2(x)$. We perform the proof as Exercise 4.3.10.

6. **δ-function**

   Because of

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx\, \delta(x - x_0)\; e^{-ikx} = \frac{1}{\sqrt{2\pi}}\; e^{-ikx_0}$$

we find after Fourier inversion the important representation of the **δ-function as Fourier integral**:

$$\delta(x - x_0) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dk\; e^{ik(x-x_0)}\ . \tag{4.189}$$

Analogous considerations lead to:

$$\delta(k - k_0) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dx\; e^{-i(k-k_0)x}\ .$$

7. **The factors in front of the Fourier integrals**

   are chosen in (4.183) and (4.184) symmetrically. Their choice is, however, widely arbitrary, only the product of the pre-factors for back and forth transformation must always give $1/2\pi$. That can be seen as follows: The abbreviations, introduced before (4.181), could have also been

$$\tilde{f}_n = \gamma \alpha_n a \;\curvearrowright\; \Delta k \tilde{f}_n = \gamma \pi \alpha_n$$

with an at first arbitrary real γ. All the following considerations would have been the same and would have led in (4.183) and (4.184) to

$$f(x) = \frac{1}{\gamma\pi} \int\limits_{-\infty}^{+\infty} \tilde{f}(k)\; e^{ikx} dk$$

$$\tilde{f}(k) = \frac{1}{2}\gamma \int\limits_{-\infty}^{+\infty} f(x)\; e^{-ikx} dx\ .$$

The product of the pre-factors

$$\frac{1}{\gamma\pi} \cdot \frac{1}{2}\gamma = \frac{1}{2\pi}$$

is thereby unique and fixed, while γ can be chosen arbitrarily. The choice $\gamma = \sqrt{2/\pi}$ results in the symmetric pre-factors used so far. But often one takes also $\gamma = 1/\pi$ or $\gamma = 2$:

$$\tilde{f}(k) = \frac{1}{2\pi}\int dx \ldots \iff f(x) = \int dk \ldots$$

$$\tilde{f}(k) = \int dx \ldots \iff f(x) = \frac{1}{2\pi}\int dk \ldots$$

Notice, however, that the pre-factors in front of both (!) Fourier integrals of the δ-function (point 6.) are not arbitrary being instead fixed at $1/2\pi$!

8. **Signs in the exponents**
   The signs in the exponents of the exponential functions in (4.183) and (4.184) are also rather arbitrary. They have only to be opposite for $f(x)$ and $\tilde{f}(k)$.
9. **Transformation of a time-function** $f(t)$
   The rules, which we have derived above for the pair of variables (x, k), are valid in completely analogous manner for *'times and frequencies'* ($k = 2\pi/\lambda \iff \omega = 2\pi/\tau$). Normally, but in principle completely unimportant, one interchanges the signs in the exponents compared to (4.183), (4.184):

$$f(t) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} d\omega\, \tilde{f}(\omega)\, e^{-i\omega t} ,$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} dt\, f(t)\, e^{i\omega t} . \tag{4.190}$$

10. **Multi-dimensional functions**
    So far we have defined the Fourier transformation only for functions of one variable. The generalization, however, is obvious, e.g.:

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^2}\int d^3k \int\limits_{-\infty}^{+\infty} d\omega\, \tilde{f}(\mathbf{k}, \omega)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} ,$$

$$\tilde{f}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2}\int d^3r \int\limits_{-\infty}^{+\infty} dt\, f(\mathbf{r}, t)\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} . \tag{4.191}$$

Final Remark

The definition of the Fourier transform $\tilde{f}(k)$ in (4.184) is certainly reasonable only if the integral does exist for all k. Thereto we have to surely require a sufficiently rapid vanishing of the function $f(x)$ for $|x| \to \infty$. That restricts enormously, though, the class of functions, which allow a Fourier transformation. As an example, the function

$$f(x) \equiv c = \text{const}$$

would not be transformable. One therefore extends the definition (4.184) by a **convergence generating factor**:

$$\tilde{f}(k) = \lim_{\eta \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; e^{-ikx - \eta x^2} f(x) \; . \tag{4.192}$$

At first this extension does not at all influence those functions which can already be transformed according to (4.184). The class of transformable functions, however, is now substantially larger. As an example, let us inspect the above-mentioned function $f(x) \equiv c$:

$$\tilde{f}(k) = \lim_{\eta \to 0^+} \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; e^{-ikx - \eta x^2} \; .$$

We have already calculated an integral of this type in connection with (4.161):

$$\tilde{f}(k) = \lim_{\eta \to 0^+} \frac{c}{\sqrt{2\eta}} \, e^{-k^2/4\eta} = \sqrt{2\pi}\, c \, \delta(k) \; . \tag{4.193}$$

In the last step we exploited (4.160) (see Exercise 1.7.1). The back-transformation is then automatically fulfilled with (4.183). But this is valid only for this example.

In general one has to of course perform the same limiting process for the inverse function, too:

$$f(x) = \lim_{\tilde{\eta} \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \; e^{ikx - \tilde{\eta} k^2} \tilde{f}(k) \; . \tag{4.194}$$

The same calculation as above delivers then to $f(x) = \delta(x)$ the Fourier transform $\tilde{f}(k) = 1/\sqrt{2\pi}$, which coincides with (4.189).

The limiting processes (4.192) and (4.194) are in general not explicitly stated, but are always implied. In this sense the integrals in (4.183) and (4.184) are to be understood symbolically.

4.3.7 General Solution of the Wave Equation

Let us come back once more to the initial problem, namely to the solution of the homogeneous wave equation (4.128),

$$\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{r},t) = 0\;,$$

for which we want to presume **initial conditions** of the form

$$\psi(\mathbf{r},t=0) = \psi_0(\mathbf{r})\;;\quad \dot{\psi}(\mathbf{r},t=0) = v_0(\mathbf{r})\;. \tag{4.195}$$

Let $\tilde{\psi}(\mathbf{k},\omega)$ be the Fourier transform of the required solution:

$$\psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int d^3k \int\limits_{-\infty}^{+\infty} d\omega\, \tilde{\psi}(\mathbf{k},\omega)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;. \tag{4.196}$$

Thus, $\psi(\mathbf{r},t)$ represents a superposition of plane waves without being, however, itself necessarily a plane wave, since **all** propagation directions $\mathbf{k}/k$ are in principle allowed.

We insert the ansatz (4.196) into the wave equation and use:

$$\begin{aligned}
\frac{\partial}{\partial x}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= ik_x\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;,\\
\frac{\partial}{\partial y}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= ik_y\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;,\\
\frac{\partial}{\partial z}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= ik_z\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;,\\
\Longrightarrow \nabla\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= i\mathbf{k}\; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\
\Longrightarrow \Delta\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= -k^2\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\
\frac{\partial^2}{\partial t^2}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= -\omega^2\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;.
\end{aligned}$$

Therewith it follows:

$$\frac{1}{(2\pi)^2}\int d^3k \int\limits_{-infty}^{+infty} d\omega\, \left(-k^2 + \frac{\omega^2}{u^2}\right)\tilde{\psi}(\mathbf{k},\;\omega)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0\;.$$

The Fourier inversion then leads to:

$$\left(\frac{\omega^2}{u^2} - k^2\right) \tilde{\psi}(\mathbf{k}, \omega) = 0 \; . \tag{4.197}$$

This is a remarkable result since we succeeded to replace the original partial differential equation for $\psi(\mathbf{r}, t)$ by a purely algebraic equation for $\tilde{\psi}(\mathbf{k}, \omega)$. Obviously $\tilde{\psi}$ can be unequal to zero only for

$$\omega = \pm u\, k \; . \tag{4.198}$$

In that case it must indeed be $\tilde{\psi} \neq 0$ because otherwise it would be $\psi(\mathbf{r}, t) \equiv 0$. This leads to the **ansatz**

$$\tilde{\psi}(\mathbf{k}, \omega) = a_+(\mathbf{k})\, \delta(\omega + u\, k) + a_-(\mathbf{k}) \delta(\omega - u\, k) \; , \tag{4.199}$$

yielding as **preliminary solution:**

$$\psi(\mathbf{r}, t) = \frac{1}{(2\pi)^2} \int d^3k \left[a_+(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} + kut)} + a_-(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - kut)}\right] \; .$$

This now we fit to the **initial conditions** (4.195):

$$\psi_0(\mathbf{r}) = \frac{1}{(2\pi)^2} \int d^3k \; e^{i\,\mathbf{k}\cdot\mathbf{r}} \left(a_+(\mathbf{k}) + a_-(\mathbf{k})\right) \; ,$$

$$v_0(\mathbf{r}) = \frac{i}{(2\pi)^2} \int d^3k \; e^{i\,\mathbf{k}\cdot\mathbf{r}} ku \left(a_+(\mathbf{k}) - a_-(\mathbf{k})\right) \; .$$

The Fourier inversion then leads to:

$$\frac{1}{\sqrt{2\pi}} \left(a_+(\mathbf{k}) + a_-(\mathbf{k})\right) = \frac{1}{(2\pi)^{3/2}} \int d^3r \; e^{-i\,\mathbf{k}\cdot\mathbf{r}} \psi_0(\mathbf{r}) \; ,$$

$$\frac{1}{\sqrt{2\pi}} \left(a_+(\mathbf{k}) - a_-(\mathbf{k})\right) = \frac{-i}{ku(2\pi)^{3/2}} \int d^3r \; e^{-i\,\mathbf{k}\cdot\mathbf{r}} v_0(\mathbf{r}) \; .$$

The weight functions $a_\pm(\mathbf{k})$ are therewith determined:

$$a_\pm(\mathbf{k}) = \frac{1}{4\pi} \int d^3r \; e^{-i\,\mathbf{k}\cdot\mathbf{r}} \left(\psi_0(\mathbf{r}) \mp \frac{i}{ku} v_0(\mathbf{r})\right) \; . \tag{4.200}$$

Insertion into the above preliminary solution yields:

$$\psi(\mathbf{r},t) = \frac{1}{2(2\pi)^3}\int d^3k \int d^3r'\; e^{i\,\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\cdot$$
$$\cdot\left[e^{i\,kut}\left(\psi_0(\mathbf{r}') - \frac{i}{ku}v_0(\mathbf{r}')\right) + e^{-i\,kut}\left(\psi_0(\mathbf{r}') + \frac{i}{ku}v_0(\mathbf{r}')\right)\right] .$$

With the **abbreviation**:

$$D(\mathbf{r},t) = \frac{-i}{2(2\pi)^3}\int \frac{d^3k}{ku}\; e^{i\,\mathbf{k}\cdot\mathbf{r}}\left(e^{i\,kut} - e^{-i\,kut}\right) \tag{4.201}$$

it eventually follows the **solution:**

$$\psi(\mathbf{r},t) = \int d^3r'\;\left(\dot{D}(\mathbf{r}-\mathbf{r}',t)\psi_0(\mathbf{r}') + D(\mathbf{r}-\mathbf{r}',t)v_0(\mathbf{r}')\right) . \tag{4.202}$$

We want to investigate the function $D(\mathbf{r},t)$ in somewhat more detail. With $\mathbf{r}$ as polar axis the integrations in (4.201) can be performed as follows:

$$\begin{aligned}
D(\mathbf{r},t) &= \frac{-i}{2(2\pi)^2}\int_0^\infty dk\,\frac{k}{u}\int_{-1}^{+1} dx\; e^{i\,krx}\left(e^{i\,kut} - e^{-i\,kut}\right)\\
&= \frac{-1}{2(2\pi)^2}\,\frac{1}{ur}\left\{\int_0^\infty dk\left[e^{i\,kr}\left(e^{i\,kut} - e^{-i\,kut}\right) - e^{-i\,kr}\left(e^{i\,kut} - e^{-i\,kut}\right)\right]\right\}\\
&= \frac{-1}{2(2\pi)^2 ur}\int_{-\infty}^{+\infty} dk\;\left(e^{i\,k(r+ut)} - e^{i\,k(r-ut)}\right)\\
&= \frac{-1}{4\pi ur}\left[\delta(r+ut) - \delta(r-ut)\right] .
\end{aligned}$$

In the last step we have used the representation (4.189) of the δ-function. Because of $r > 0$ and $u > 0$ finally we get:

$$D(\mathbf{r},t) = \frac{1}{4\pi ur}\begin{cases}\delta(r-ut)\,, & \text{if } t > 0\,,\\ -\delta(r+ut)\,, & \text{if } t < 0\,.\end{cases} \tag{4.203}$$

For $t = 0$ we have to go back to the definition (4.201):

$$D(\mathbf{r},t=0) = 0 . \tag{4.204}$$

The homogeneous wave equation is therewith completely solved.

4.3.8 Energy Transport in Wave Fields

Because of mathematical expedience in the preceding sections we have used the complex notation for the electromagnetic fields. That was allowed since the linear operations in the relevant differential equations do not mix real and imaginary parts. One can therefore perform the transition to the actual *physical result* ($\Rightarrow$ real part) only at the very end.

However, we are now interested in the **energy-current density** (4.45), the **energy density** (4.46) and the **momentum density** (4.50) of the electromagnetic wave field. These terms are all scalar or vector products of field vectors and thus are non-linear expressions. We have therefore to treat the fields in these cases with special care from the very beginning.

Let us consider as an example the scalar product of two complex vectors **a** and **b**:

$$(\mathrm{Re}\,\mathbf{a}) \cdot (\mathrm{Re}\,\mathbf{b}) = \frac{1}{4}\left(\mathbf{a} + \mathbf{a}^*\right) \cdot \left(\mathbf{b} + \mathbf{b}^*\right) = \frac{1}{4}\left(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}^* + \mathbf{a}^* \cdot \mathbf{b} + \mathbf{a}^* \cdot \mathbf{b}^*\right) \ .$$

Very often we are interested in situations in which the fields exhibit **harmonic time-dependences**,

$$\mathbf{a}(\mathbf{r}, t) = \hat{\mathbf{a}}_0(\mathbf{r})\ e^{-i\omega t} \ ,$$
$$\mathbf{b}(\mathbf{r}, t) = \hat{\mathbf{b}}_0(\mathbf{r})\ e^{-i\omega t} \ ,$$

and which are needed only as **time average**:

$$\overline{A}(t) = \frac{1}{\tau} \int_t^{t+\tau} A(t')\, dt' \ . \tag{4.205}$$

The averaging is done over a characteristic period τ $(\omega\tau = 2\pi)$ where the first and the last summand in the above equation vanish:

$$\overline{\mathbf{a} \cdot \mathbf{b}(t)} = \frac{1}{\tau}\, \hat{\mathbf{a}}_0 \cdot \hat{\mathbf{b}}_0 \int_t^{t+\tau} dt'\ e^{-2i\omega t'} = i \frac{\hat{\mathbf{a}}_0 \cdot \hat{\mathbf{b}}_0}{2\omega\tau}\ \ e^{-2i\omega t'}\Big|_t^{t+\tau} = 0 \ . \tag{4.206}$$

On the other hand, for the two other summands we get:

$$\overline{\mathbf{a}^* \cdot \mathbf{b}(t)} = \hat{\mathbf{a}}_0^* \cdot \hat{\mathbf{b}}_0 \ ; \quad \overline{\mathbf{a} \cdot \mathbf{b}^*(t)} = \hat{\mathbf{a}}_0 \cdot \hat{\mathbf{b}}_0^* \ ,$$

which finally leads to:

$$\overline{(\text{Re}\mathbf{a})\cdot(\text{Re}\mathbf{b})(t)} = \frac{1}{4}\left(\hat{\mathbf{a}}_0^*\cdot\hat{\mathbf{b}}_0+\hat{\mathbf{a}}_0\cdot\hat{\mathbf{b}}_0^*\right)$$
$$= \frac{1}{2}\text{Re}\left(\hat{\mathbf{a}}_0^*\cdot\hat{\mathbf{b}}_0\right) = \frac{1}{2}\text{Re}\left(\hat{\mathbf{a}}_0\cdot\hat{\mathbf{b}}_0^*\right)\ . \tag{4.207}$$

Analogously one finds for the corresponding vector product:

$$\overline{(\text{Re}\mathbf{a})\times(\text{Re}\mathbf{b})(t)} = \frac{1}{2}\text{Re}\left(\hat{\mathbf{a}}_0\times\hat{\mathbf{b}}_0^*\right) = \frac{1}{2}\text{Re}\left(\hat{\mathbf{a}}_0^*\times\hat{\mathbf{b}}_0\right)\ . \tag{4.208}$$

Hence, if the electromagnetic fields exhibit a harmonic time-dependence then we have for the **energy density** (4.46):

$$\overline{w}(\mathbf{r},t) = \frac{1}{4}\text{Re}\left(\widehat{\mathbf{H}}_0\cdot\widehat{\mathbf{B}}_0^*+\widehat{\mathbf{E}}_0\cdot\widehat{\mathbf{D}}_0^*\right) \tag{4.209}$$

and for the **energy-current density** (4.45):

$$\overline{\mathbf{S}}(\mathbf{r},t) = \frac{1}{2}\text{Re}\left(\widehat{\mathbf{E}}_0\times\widehat{\mathbf{H}}_0^*\right)\ . \tag{4.210}$$

Let us assume in particular plane waves,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ ,$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ ,$$

so that we can apply (4.140) and (4.143):

$$\mathbf{B}_0 = \frac{1}{\omega}(\mathbf{k}\times\mathbf{E}_0) \implies |\mathbf{B}_0|^2 = \frac{1}{u^2}|\mathbf{E}_0|^2\ ,$$
$$\mathbf{E}_0 = -\frac{u^2}{\omega}(\mathbf{k}\times\mathbf{B}_0) \implies |\mathbf{E}_0|^2 = u^2|\mathbf{B}_0|^2\ .$$

This yields, e.g., for the **magnetic part of the energy density:**

$$\overline{w}_{\text{m}}(\mathbf{r},t) = \frac{1}{4}\text{Re}\left(\widehat{\mathbf{H}_0}\cdot\widehat{\mathbf{B}}_0^*\right) = \frac{1}{4\mu_{\text{r}}\mu_0}|\mathbf{B}_0|^2 = \frac{1}{4}\epsilon_{\text{r}}\epsilon_0|\mathbf{E}_0|^2\ . \tag{4.211}$$

For the **electric part of the energy density** we find the same expression:

$$\overline{w}_{\text{e}}(\mathbf{r},t) = \frac{1}{4}\text{Re}\left(\widehat{\mathbf{E}}_0\cdot\widehat{\mathbf{D}}_0^*\right) = \frac{1}{4}\epsilon_{\text{r}}\epsilon_0|\mathbf{E}_0|^2\ . \tag{4.212}$$

The time averages of the electric and magnetic energy densities in the plane wave are identical. For the total energy density we therefore get:

$$\overline{w}(\mathbf{r},t) = \frac{1}{2}\epsilon_r\epsilon_0|\mathbf{E}_0|^2 = \frac{1}{2\mu_r\mu_0}|\mathbf{B}_0|^2 \,. \tag{4.213}$$

Let us finally evaluate the **Poynting vector** for the plane wave:

$$\begin{aligned}\overline{\mathbf{S}}(\mathbf{r},t) &= \frac{1}{2\mu_r\mu_0}\mathrm{Re}\left(\widehat{\mathbf{E}}_0 \times \widehat{\mathbf{B}}_0^*\right) = \frac{1}{2\mu_r\mu_0\omega}\mathrm{Re}\left[\mathbf{E}_0 \times \left(\mathbf{k} \times \mathbf{E}_0^*\right)\right] \\ &= \frac{1}{2\omega\mu_r\mu_0}\mathrm{Re}\left(\mathbf{k}|\mathbf{E}_0|^2 - \mathbf{E}_0^* \underbrace{(\mathbf{E}_0 \cdot \mathbf{k})}_{=0}\right) .\end{aligned}$$

This means:

$$\overline{\mathbf{S}}(\mathbf{r},t) = \frac{1}{2}\sqrt{\frac{\epsilon_r\epsilon_0}{\mu_r\mu_0}}|\mathbf{E}_0|^2\frac{\mathbf{k}}{k} \,. \tag{4.214}$$

We can combine this expression with the energy density (4.213):

$$\overline{\mathbf{S}}(\mathbf{r},t) = u\,\overline{w}(\mathbf{r},t)\frac{\mathbf{k}}{k} \,. \tag{4.215}$$

Like for any *current density*, the energy-current density, too, can be written as a product of velocity and density of the *'flowing substance'*. The energy transport takes place in the direction of the propagation vector.

4.3.9 Wave Propagation in Electric Conductors

We have shown in Sect. 4.3.1 that the electromagnetic field quantities **E** and **B** solve the homogeneous wave equation (4.128) provided we presume at the start a homogeneous uncharged insulator as the medium. The existence of **electromagnetic waves** is therefore postulated by Maxwell's theory. One should remember that this statement resulted exclusively from the fact that, compared to the previously discussed stationary and quasistationary fields, now the displacement current $\dot{\mathbf{D}}$ appears in the Maxwell equations. Without this term one can not come to the wave equations for **E** and **B**.

Let us now extend the considerations of the last sections to

homogeneous, isotropic, neutral electric conductors ($\sigma \neq 0$)**.**

In this case we have to regard in the inhomogeneous Maxwell equation for $\mathbf{H}$ not only the displacement current $\dot{\mathbf{D}}$, but also the conduction current

$$\mathbf{j} = \sigma\,\mathbf{E}\ .$$

Hence, we now have to deal with the following set of Maxwell equations:

$$\begin{aligned} &\operatorname{div}\mathbf{E} = 0\ ; \qquad \operatorname{div}\mathbf{B} = 0\ ; \\ &\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}\ ;\ \operatorname{curl}\mathbf{B} = \mu_{\mathrm{r}}\mu_0\sigma\,\mathbf{E} + \frac{1}{u^2}\dot{\mathbf{E}}\ . \end{aligned} \tag{4.216}$$

We have formulated these differential equations again directly for the physically relevant fields $\mathbf{E}$ and $\mathbf{B}$. The wave velocity u was defined in (4.126). Strictly speaking, the homogeneous equation $\operatorname{div}\mathbf{E} = 0$ is not fully self-evident. Indeed we assume that the conductor is initially uncharged,

$$\rho(\mathbf{r}, t = 0) = 0\ ,$$

but that this remains valid for all times t must still be proved:

$$0 = \operatorname{div}\operatorname{curl}\mathbf{B} = \mu_{\mathrm{r}}\mu_0\sigma\operatorname{div}\mathbf{E} + \frac{1}{u^2}\operatorname{div}\dot{\mathbf{E}}\ .$$

If it were $\rho(\mathbf{r}, t \neq 0) \neq 0$, then it would hold:

$$\operatorname{div}\mathbf{E} = \frac{1}{\epsilon_{\mathrm{r}}\epsilon_0}\rho\ ,$$

and therewith:

$$\begin{aligned} 0 &= \frac{\mu_{\mathrm{r}}\mu_0}{\epsilon_{\mathrm{r}}\epsilon_0}\sigma\,\rho + \mu_{\mathrm{r}}\mu_0\dot{\rho} \\ \Longrightarrow\quad \dot{\rho} &= -\frac{1}{\tau}\rho; \quad \tau = \frac{1}{\sigma}\epsilon_{\mathrm{r}}\epsilon_0\ . \end{aligned}$$

The integration would finally lead to:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, t = 0)\ e^{-t/\tau}\ .$$

We see that, if the electric conductor is uncharged at the beginning, then it will remain so forever:

$$\rho(\mathbf{r}, t = 0) = 0 \implies \rho(\mathbf{r}, t) \equiv 0\ . \tag{4.217}$$

The Maxwell equations (4.216) are, because of the appearance of the conduction current, indeed a bit more complicated than those of the uncharged insulator (4.125). However, just as those for the uncharged insulator, in this case also they again build a system of coupled linear partial **homogeneous** differential equations of first order for **E** and **B**, which can, even in this case, be exactly decoupled:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\underbrace{(\operatorname{div}\mathbf{E})}_{=0} - \Delta\mathbf{E} = -\operatorname{curl}\dot{\mathbf{B}} = -\mu_r\mu_0\sigma\dot{\mathbf{E}} - \frac{1}{u^2}\ddot{\mathbf{E}}\,.$$

That yields for $\sigma \neq 0$, in a certain sense, as generalization of the homogeneous wave equation (4.128), the so-called

telegraph equation

$$\left[\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right) - \mu_r\mu_0\sigma\frac{\partial}{\partial t}\right]\mathbf{E}(\mathbf{r}, t) = \mathbf{0}\,, \tag{4.218}$$

which for $\sigma \to 0$ reduces to (4.128). In spite of the fact that the field equations (4.216) are asymmetric in **E** and **B**, nevertheless one finds that the magnetic induction $\mathbf{B}(\mathbf{r}, t)$, too, fulfills the telegraph equation:

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{B} &= \operatorname{grad}\underbrace{(\operatorname{div}\mathbf{B})}_{=0} - \Delta\mathbf{B} = \mu_r\mu_0\sigma\operatorname{curl}\mathbf{E} + \frac{1}{u^2}\operatorname{curl}\dot{\mathbf{E}} \\ &= -\mu_r\mu_0\sigma\dot{\mathbf{B}} - \frac{1}{u^2}\ddot{\mathbf{B}}\,.\end{aligned}$$

It follows therewith:

$$\left[\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right) - \mu_r\mu_0\sigma\frac{\partial}{\partial t}\right]\mathbf{B}(\mathbf{r}, t) = \mathbf{0}\,. \tag{4.219}$$

To solve the telegraph equation (4.218) we assume a temporally harmonic wave:

$$\mathbf{E}(\mathbf{r}, t) = \widehat{\mathbf{E}}_0(\mathbf{r})\, e^{-i\omega t}\,.$$

Insertion into (4.218) yields:

$$\left(\Delta + \frac{\omega^2}{u^2} + i\omega\mu_r\mu_0\sigma\right)\widehat{\mathbf{E}}_0(\mathbf{r}) = \mathbf{0}\,. \tag{4.220}$$

By formally introducing a **complex** dielectric constant (permittivity) $\overline{\epsilon}_r$ we can cast this differential equation into a form which is known to us from the insulators. We write:

$$\frac{\omega^2}{c^2}\mu_r\overline{\epsilon}_r \equiv \frac{\omega^2}{u^2} + i\omega\mu_r\mu_0\sigma$$
$$\Longrightarrow \overline{\epsilon}_r = \epsilon_r + i\frac{\mu_0\sigma\, c^2}{\omega}$$
$$\Longrightarrow \overline{\epsilon}_r = \epsilon_r + i\frac{\sigma}{\epsilon_0\omega} = \overline{\epsilon}_r(\omega)\ . \tag{4.221}$$

For $\sigma \to 0$ $\overline{\epsilon}_r$ equals the *normal* dielectric constant ϵ_r. Analogously, a complex wave velocity $\overline{u}$ can be defined:

$$\overline{u} = \frac{1}{\sqrt{\mu_r\overline{\epsilon}_r\mu_0\epsilon_0}} = \frac{c}{\sqrt{\overline{\epsilon}_r\mu_r}}\ . \tag{4.222}$$

With these definitions Eq. (4.220) formally takes again the structure of the homogeneous wave equation:

$$\left(\Delta + \frac{\omega^2}{\overline{u}^2}\right)\widehat{\mathbf{E}}_0(\mathbf{r}) = \mathbf{0}\ . \tag{4.223}$$

Hence, we can in principle adopt the extensively discussed theory of the solution of the homogeneous wave equation (4.128). We have to only replace in the result in each case ϵ_r by the complex $\overline{\epsilon}_r$. Let us further inspect, in the following, the consequences of this replacement.

The telegraph equation is obviously solved by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\ e^{i(\overline{\mathbf{k}}\cdot\mathbf{r} - \omega t)} \tag{4.224}$$

provided

$$\overline{\mathbf{k}} = \frac{\omega}{\overline{u}}\boldsymbol{\kappa}\ ; \quad \boldsymbol{\kappa} = \frac{\overline{\mathbf{k}}}{\overline{k}}\ , \tag{4.225}$$

where $\boldsymbol{\kappa}$ is the real unit vector in the propagation direction. Because of $\overline{u}$ the wave vector $\overline{\mathbf{k}}$ is also now of course complex.

In (4.127) we have introduced the **index of refraction** n of a medium via the equation

$$n = \sqrt{\epsilon_r\mu_r} \quad \textbf{(Maxwell's relation)}$$

which connects the optics with the theory of electromagnetic fields. We generalize this expression:

$$\sqrt{\mu_{\mathrm{r}}\overline{\epsilon}_{\mathrm{r}}} \equiv \overline{n} + i\gamma \ . \tag{4.226}$$

$\overline{n}, \gamma$ are thereby real quantities the meaning of which become clear by the following calculation:

$$\mu_{\mathrm{r}}\overline{\epsilon}_{\mathrm{r}} = \overline{n}^2 - \gamma^2 + 2i\,\gamma\,\overline{n} \ .$$

Inserting of (4.221) leads to:

$$n^2 + i\frac{\sigma}{\epsilon_0\omega}\mu_{\mathrm{r}} = \overline{n}^2 - \gamma^2 + 2i\,\gamma\overline{n} \ .$$

This equation must be fulfilled simultaneously for the real as well as for the imaginary part:

$$\begin{aligned} n^2 &= \overline{n}^2 - \gamma^2 \ , \\ \mu_{\mathrm{r}}\frac{\sigma}{\epsilon_0\omega} &= 2\gamma\,\overline{n} \ . \end{aligned}$$

We solve the second equation for γ and insert the result into the first equation:

$$\begin{aligned} n^2 = \overline{n}^2 - \frac{1}{\overline{n}^2}\left(\frac{n^2}{2}\,\frac{\sigma}{\epsilon_0\epsilon_{\mathrm{r}}\omega}\right)^2 &\implies \overline{n}^4 - n^2\overline{n}^2 = \frac{n^4}{4}\left(\frac{\sigma}{\epsilon_0\epsilon_{\mathrm{r}}\omega}\right)^2 \\ &\implies \overline{n}^2 = \frac{1}{2}n^2 \pm \sqrt{\frac{n^4}{4} + \frac{n^4}{4}\left(\frac{\sigma}{\epsilon_0\epsilon_{\mathrm{r}}\omega}\right)^2} \ . \end{aligned}$$

Since $\overline{n}$ has to be real, only the positive root can be correct:

$$\overline{n}^2 = \frac{1}{2}n^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon_0\epsilon_{\mathrm{r}}\omega}\right)^2}\,\right] \ . \tag{4.227}$$

We recognize

$$\overline{n} \underset{\sigma\to 0}{\longrightarrow} n$$

and can therefore interpret $\overline{n}$ as a **generalized index of refraction**.

For the quantity γ in the ansatz (4.226) it follows, because of

$$\gamma^2 = \overline{n}^2 - n^2 \ ,$$

immediately from (4.227):

$$\gamma^2 = \frac{1}{2}n^2\left[-1+\sqrt{1+\left(\frac{\sigma}{\epsilon_0\epsilon_r\omega}\right)^2}\,\right]. \tag{4.228}$$

As expected according to (4.226) it is

$$\gamma \underset{\sigma\to 0}{\longrightarrow} 0\,.$$

In contrast to $\overline{n}$, γ therefore has no direct analogue in insulators. The physical meaning of γ, however, becomes directly clear by inspecting the solution of the telegraph equation:

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0\; e^{i(\overline{\mathbf{k}}\cdot\mathbf{r}-\omega t)} = \mathbf{E}_0\, e^{i((\omega/\overline{u})\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}\\
&= \mathbf{E}_0\; e^{i[(\omega/c)(\overline{n}+i\gamma)(\boldsymbol{\kappa}\cdot\mathbf{r})-\omega t]}\\
&= \mathbf{E}_0\; e^{-\gamma(\omega/c)(\boldsymbol{\kappa}\cdot\mathbf{r})}\; e^{i[(\omega/c)\overline{n}(\boldsymbol{\kappa}\cdot\mathbf{r})-\omega t]}\,.
\end{aligned}$$

Without any loss of generality we identify the propagation direction with the z-direction ($\boldsymbol{\kappa} = \mathbf{e}_z$):

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\, e^{-(\gamma\omega/c)z}\; e^{i\omega[(\overline{n}/c)z-t]}\,. \tag{4.229}$$

So the solution has the form of a **damped** plane wave. The strength of the damping is lastly determined by γ:

$\boldsymbol{\gamma}$: **extinction coefficient**

The damping mediated by γ results finally from the generation of Joule heat in the electric conductor.

Discussion

(1) Penetration (Skin) Depth

The electromagnetic wave cannot penetrate arbitrarily far into the electric conductor. The distance $\Delta z = \delta$, after which the wave amplitude is damped down to the e-th fraction of its initial value, is denoted as the *penetration or skin depth*:

$$\delta = \frac{c}{\omega\gamma} = \frac{\lambda_0}{2\pi\gamma} \tag{4.230}$$

(λ_0: wave length in the vacuum ($c = \nu\lambda_0$)).

(2) Wave Number

Because of (4.225) the wave number $\overline{k}$ is complex:

$$\overline{k} = k_0 + i\,k_1 = \frac{\omega}{\overline{u}} = \frac{\omega}{c}\,(\overline{n} + i\gamma)\ , \tag{4.231}$$

where

$$k_0 = \frac{\omega}{c}\overline{n}\ ; \quad k_1 = \frac{\omega}{c}\gamma\ . \tag{4.232}$$

The solution of the telegraph equation (4.229) can therewith be written as:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\ e^{-k_1 z}\ e^{i(k_0 z - \omega t)}\ . \tag{4.233}$$

(3) Phase Velocity

From

$$k_0 z - \omega t \overset{!}{=} \text{const}$$

we find:

$$u_\mathrm{p} = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{c}{\overline{n}}\ . \tag{4.234}$$

The phase velocity in the conductor turns out to be smaller, because of $\overline{n} > n$, than that in the insulator.

(4) Wavelength

$$\overline{\lambda} = \frac{2\pi}{k_0} = \lambda\frac{n}{\overline{n}} < \lambda\ . \tag{4.235}$$

$\lambda = \frac{u}{\nu} = \frac{2\pi}{\omega} \cdot \frac{c}{n}$ is the wavelength in the corresponding insulator ($\sigma = 0$).

(5) Maxwell Equations

The solutions of the telegraph equation (4.218), (4.219),

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\ e^{i(\overline{\mathbf{k}}\cdot\mathbf{r} - \omega t)}\ ,$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0\ e^{i(\overline{\mathbf{k}}\cdot\mathbf{r} - \omega t)}\ ,$$

have to still fulfill the special couplings given by the Maxwell equations (4.216):

$$\begin{aligned} \operatorname{div}\mathbf{E} = 0 &\implies \boldsymbol{\kappa}\cdot\mathbf{E} = 0\ , \\ \operatorname{div}\mathbf{B} = 0 &\implies \boldsymbol{\kappa}\cdot\mathbf{B} = 0\ , \\ \operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}} &\implies \frac{1}{\overline{u}}\,\boldsymbol{\kappa}\times\mathbf{E} = \mathbf{B}\ . \end{aligned}$$

As in the insulator, $(\boldsymbol{\kappa}, \mathbf{E}, \mathbf{B})$ build, in this order, an orthogonal trihedron. The electromagnetic waves are even now **transverse**!

However, **E** and **B** are **no longer in (same) phase**! One realizes this as follows:

$$\mathbf{B} = \frac{1}{c}(\overline{n} + i\gamma)(\boldsymbol{\kappa}\times\mathbf{E})\ .$$

The polar representation of the complex number $(\overline{n} + i\gamma)$,

$$\begin{aligned} \overline{n} + i\gamma &= \sqrt{\overline{n}^2 + \gamma^2}\ e^{i\varphi}\ , \\ \tan\varphi &= \frac{\gamma}{\overline{n}} \end{aligned} \tag{4.236}$$

leads to

$$\mathbf{B} = \frac{1}{c}\sqrt{\overline{n}^2 + \gamma^2}(\boldsymbol{\kappa}\times\mathbf{E})\ e^{i\varphi}\ . \tag{4.237}$$

Hence, **B** and **E** are phase-shifted by the angle φ!

(6) Time-Averaged Energy-Current Density

For this quantity we have according to (4.210):

$$\begin{aligned} \overline{\mathbf{S}}(\mathbf{r}) &= \frac{1}{2\mu_{\mathrm{r}}\mu_0}\mathrm{Re}\left(\widehat{\mathbf{E}}_0(\mathbf{r})\times\widehat{\mathbf{B}}_0^*(\mathbf{r})\right) \\ &= \frac{1}{2\mu_{\mathrm{r}}\mu_0}\mathrm{Re}\left(\widehat{\mathbf{E}}_0(\mathbf{r})\times\frac{1}{\overline{u}^*}\left(\boldsymbol{\kappa}\times\widehat{\mathbf{E}}_0\right)^*\right) \\ &= \frac{1}{2\mu_{\mathrm{r}}\mu_0}\mathrm{Re}\frac{1}{\overline{u}^*}\left(\boldsymbol{\kappa}\,|\widehat{\mathbf{E}}_0(\mathbf{r})|^2 - \mathbf{E}_0^*\underbrace{(\boldsymbol{\kappa}\cdot\mathbf{E}_0)}_{=0}\right) \\ &= \frac{1}{2\mu_{\mathrm{r}}\mu_0}|\mathbf{E}_0|^2\ e^{-2k_1 z}\boldsymbol{\kappa}\frac{1}{c}\mathrm{Re}(\overline{n} - i\gamma)\ . \end{aligned}$$

That yields:

$$\overline{\mathbf{S}}(\mathbf{r}) = \frac{|\mathbf{E}_0|^2}{2\mu_\mathrm{r}\mu_0 u_\mathrm{p}} \; e^{-2\gamma(\omega/c)z}\boldsymbol{\kappa} \; . \tag{4.238}$$

$\overline{\mathbf{S}}$ decreases exponentially in the conductor. The reason is, as already mentioned, energy dissipation by the generation of Joule heat.

(7) Time-Averaged Energy Density

For the energy density we can use (4.209):

$$\overline{w}(\mathbf{r}) = \frac{1}{4}\mathrm{Re}\left(\widehat{\mathbf{H}}_0(\mathbf{r}) \cdot \widehat{\mathbf{B}}_0^*(\mathbf{r}) + \widehat{\mathbf{E}}_0(\mathbf{r}) \cdot \widehat{\mathbf{D}}_0^*(\mathbf{r})\right) \; .$$

The electric part is found as,

$$\overline{w}_\mathrm{e}(\mathbf{r}) = \frac{1}{4}\epsilon_\mathrm{r}\epsilon_0 |\widehat{\mathbf{E}}_0(\mathbf{r})|^2 = \frac{1}{4}\epsilon_\mathrm{r}\epsilon_0 |\mathbf{E}_0|^2 \; e^{-2k_1 z} \; , \tag{4.239}$$

while the magnetic part reads:

$$\overline{w}_\mathrm{m}(\mathbf{r}) = \frac{1}{4\mu_\mathrm{r}\mu_0} \, |\widehat{\mathbf{B}}_0(\mathbf{r})|^2 = \frac{1}{4\mu_\mathrm{r}\mu_0} \, \frac{1}{|\overline{\mathbf{u}}|^2} \, |\boldsymbol{\kappa} \times \widehat{\mathbf{E}}_0|^2 \; .$$

It follows therewith:

$$\overline{w}_\mathrm{m}(\mathbf{r}) = \frac{1}{4} \, \frac{\epsilon_0}{\mu_\mathrm{r}} (\overline{n}^2 + \gamma^2) |\mathbf{E}_0|^2 \; e^{-2k_1 z} \; . \tag{4.240}$$

Because of

$$\begin{aligned} \epsilon_\mathrm{r}\epsilon_0 + \frac{\epsilon_0}{\mu_\mathrm{r}}(\overline{n}^2 + \gamma^2) &= \epsilon_\mathrm{r}\epsilon_0 + \frac{\epsilon_0}{\mu_\mathrm{r}}(2\overline{n}^2 - n^2) = \epsilon_\mathrm{r}\epsilon_0 + 2\frac{\epsilon_0}{\mu_\mathrm{r}}\overline{n}^2 - \epsilon_0\epsilon_\mathrm{r} \\ &= 2\frac{\epsilon_0}{\mu_\mathrm{r}} \, \frac{c^2}{u_\mathrm{p}^2} = \frac{2}{\mu_\mathrm{r}\mu_0 u_\mathrm{p}^2} \end{aligned}$$

we get eventually as total energy density:

$$\overline{w}(\mathbf{r}) = \frac{|\mathbf{E}_0|^2}{2\mu_\mathrm{r}\mu_0 u_\mathrm{p}^2} \; e^{(-2\gamma\omega/c)z} \; . \tag{4.241}$$

The comparison with (4.238) yields, in analogy to (4.215), the connection between energy density and energy-current density:

$$\overline{\mathbf{S}}(\mathbf{r}) = u_{\mathrm{p}}\, \overline{w}(\mathbf{r})\, \boldsymbol{\kappa} \ . \tag{4.242}$$

4.3.10 Reflection and Refraction of Electromagnetic Waves at an Insulator

We want to discuss, as an important application of our hitherto developed theory, the reflection and refraction of electromagnetic waves at

plane interfaces in a dielectric.

The physical laws and systematics to be derived are in the last analysis consequences of

1. the general wave nature of the fields,
2. the special behavior of the fields at interfaces.

We consider at first which boundary conditions the electromagnetic field must obey at interfaces areas between two dielectric media.

(A) Field Behavior at Interfaces

We have investigated this so far only for the time-**independent** fields. We, however, use for the time-dependent terms also the same procedures as in Sects. 2.1.4 and 3.4.3. (Key-words: *Gauss-casket*, *Stokes-area*.)

The **div-equations** have formally not changed compared to the static case. Hence, we can directly take over (2.211) and (3.80) ($\mathbf{n}$ = normal of the interface):

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{\mathrm{F}} \ ; \quad \mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \ . \tag{4.243}$$

With the indexes 1 and 2 we mark the two adjacent media, σ_{F} is the **surface charge density**.

For the **curl-equations** we choose the *Stokes-area (Stokes-path)* as plotted in Fig. 4.49. The enclosed area is oriented such that its normal $\mathbf{t}$ lies tangentially to the interface coming perpendicularly out of the paper-plane:

$$d\mathbf{f} = df\, \mathbf{t} \ .$$

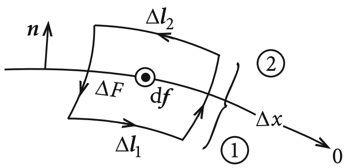


Fig. 4.49 Stokes-area for the determination of the behavior of the magnetic field at interfaces

Let $\mathbf{j}_F$ be the **surface-current density** which represents a current per unit-length on the interface. The Maxwell equation for curl **H** leads to:

$$\int\limits_{\Delta F} d\mathbf{f} \cdot \operatorname{curl} \mathbf{H} = \int\limits_{\Delta F} d\mathbf{f} \cdot \mathbf{j} + \frac{\partial}{\partial t} \int\limits_{\Delta F} d\mathbf{f} \cdot \mathbf{D} \,.$$

On the separation area **D** is finite, therefore the second summand vanishes for $\Delta x \to 0$:

$$\int\limits_{\Delta F} d\mathbf{f} \cdot \operatorname{curl} \mathbf{H} \quad \underset{\Delta x \to 0}{\longrightarrow} \quad \mathbf{j}_F \cdot \mathbf{t} \; \Delta l \,.$$

On the other hand, the Stokes theorem also works,

$$\int\limits_{\Delta F} d\mathbf{f} \cdot \operatorname{curl} \mathbf{H} = \int\limits_{\partial \Delta F} d\mathbf{r} \cdot \mathbf{H} \quad \underset{\Delta x \to 0}{\longrightarrow} \quad \mathbf{H}_2 \cdot \Delta \mathbf{l}_2 + \mathbf{H}_1 \cdot \Delta \mathbf{l}_1 \;,$$

with $\Delta \mathbf{l}_2 = (\mathbf{t} \times \mathbf{n}) \Delta l = -\Delta \mathbf{l}_1$. It follows therewith:

$$(\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_F \cdot \mathbf{t} \,. \tag{4.244}$$

t has to merely lie tangentially to the interface, but can have otherwise an arbitrary orientation. If we still exploit the cyclic invariance of the scalar triple product then we can formulate the boundary condition for **H**:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_F \,. \tag{4.245}$$

It follows analogously from $\operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}$:

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \,. \tag{4.246}$$

In this section we want to restrict ourselves to the case of **uncharged (neutral) insulators** thus presuming $\sigma_F = 0$ and $\mathbf{j}_F \equiv \mathbf{0}$. Then we have for the field behavior at interfaces the following **continuity conditions**:

$$
\begin{aligned}
(1)\ \mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1) &= 0\,,\\
(2)\ \mathbf{n}\cdot(\mathbf{D}_2-\mathbf{D}_1) &= 0\,,\\
(3)\ \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1) &= 0\,,\\
(4)\ \mathbf{n}\cdot(\mathbf{B}_2-\mathbf{B}_1) &= 0\,.
\end{aligned}
\tag{4.247}
$$

(B) Laws of Reflection and Refraction

We now want at first to formulate the problem in order to derive very generally, first relationships simply from the wave nature of the electromagnetic fields.

If an electromagnetic wave impinges at an interface, coming from medium 1, there it will be partially reflected and partially refracted (Fig. 4.50). Let us assume exclusively plane waves in the following.

incident:

$$
\begin{aligned}
\mathbf{E}_1 &= \mathbf{E}_{01}\ e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega_1 t)}\,,\\
\mathbf{B}_1 &= \frac{1}{\omega_1}\mathbf{k}_1\times\mathbf{E}_1 = \frac{1}{u_1}(\boldsymbol{\kappa}_1\times\mathbf{E}_1)\,.
\end{aligned}
\tag{4.248}
$$

The relation for $\mathbf{B}_1$ follows from (4.140); $\boldsymbol{\kappa}_1$ is the unit-vector in $\mathbf{k}_1$-direction and u_1 the wave velocity in the medium 1:

$$
u_1 = \frac{1}{\sqrt{\mu_r^{(1)}\mu_0\,\epsilon_r^{(1)}\epsilon_0}}\,.
\tag{4.249}
$$

reflected:

$$
\begin{aligned}
\mathbf{E}_{1r} &= \mathbf{E}_{01r}\ e^{i(\mathbf{k}_{1r}\cdot\mathbf{r}-\omega_{1r} t)}\,,\\
\mathbf{B}_{1r} &= \frac{1}{u_1}(\boldsymbol{\kappa}_{1r}\times\mathbf{E}_{1r})\,.
\end{aligned}
\tag{4.250}
$$

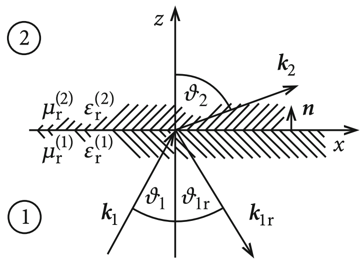


Fig. 4.50 Refraction and reflection of electromagnetic waves at interfaces

refracted:

$$\mathbf{E}_2 = \mathbf{E}_{02}\; e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega_2 t)}\;,$$

$$\mathbf{B}_2 = \frac{1}{u_2}(\boldsymbol{\kappa}_2 \times \mathbf{E}_2)\;, \tag{4.251}$$

$$u_2 = \frac{1}{\sqrt{\mu_r^{(2)}\mu_0\,\epsilon_r^{(2)}\epsilon_0}}\;. \tag{4.252}$$

Without loss of generality, we can assume that the interface represents the xy-plane of our system of coordinates and that the surface normal $\mathbf{n} = \mathbf{e}_z$ defines together with the *incident* wave vector $\mathbf{k}_1$ the xz-plane. As to the directions of $\boldsymbol{\kappa}_{1r}$ and $\boldsymbol{\kappa}_2$ we do not want to fix anything at first. Instead of this we only assume that the two planes spanned by the vector pairs ($\mathbf{n}$, $\boldsymbol{\kappa}_{1r}$) and ($\mathbf{n}$,$\boldsymbol{\kappa}_2$) enclose with the xz-plane the angle φ_{1r} and φ_2, respectively. We then have the unit vectors:

$$\begin{aligned}
\boldsymbol{\kappa}_1 &= \sin\vartheta_1\mathbf{e}_x + \cos\vartheta_1\mathbf{e}_z\;,\\
\boldsymbol{\kappa}_{1r} &= \sin\vartheta_{1r}\cos\varphi_{1r}\mathbf{e}_x + \sin\vartheta_{1r}\sin\varphi_{1r}\mathbf{e}_y - \cos\vartheta_{1r}\mathbf{e}_z\;,\\
\boldsymbol{\kappa}_2 &= \sin\vartheta_2\cos\varphi_2\mathbf{e}_x + \sin\vartheta_2\sin\varphi_2\mathbf{e}_y + \cos\vartheta_2\mathbf{e}_z\;.
\end{aligned}$$

The boundary conditions (4.247) have now to be fulfilled at any point of time and at any point of the separation area ($z = 0$). This, however, can be possible only if the phases of the three waves differ at the $z = 0$-plane at most by an integer factor of π:

$$(\mathbf{k}_1\cdot\mathbf{r}-\omega_1 t)_{z=0} \stackrel{!}{=} (\mathbf{k}_{1r}\cdot\mathbf{r}-\omega_{1r}t)_{z=0} + n\pi \stackrel{!}{=} (\mathbf{k}_2\cdot\mathbf{r}-\omega_2 t)_{z=0} + m\pi\;.$$

We choose in particular ($\mathbf{r} = \mathbf{0}$, $t = 0$) and get:

$$n = m = 0\;.$$

For ($\mathbf{r} = \mathbf{0}$, $t \neq 0$) we then have to conclude:

$$\omega_1 = \omega_{1r} = \omega_2 \equiv \omega\;. \tag{4.253}$$

There is no frequency change due to reflection and refraction at the (at rest, stationary) separation area. For $\mathbf{r} \neq 0$ it then remains to be fulfilled:

$$(\mathbf{k}_1\cdot\mathbf{r})_{z=0} \stackrel{!}{=} (\mathbf{k}_{1r}\cdot\mathbf{r})_{z=0} \stackrel{!}{=} (\mathbf{k}_2\cdot\mathbf{r})_{z=0}\;.$$

This means, for arbitrary x- and y-components of $\mathbf{r}$:

$$k_1 \sin\vartheta_1 = k_{1r} \sin\vartheta_{1r} \cos\varphi_{1r} = k_2 \sin\vartheta_2 \cos\varphi_2 \ ,$$
$$0 = k_{1r} \sin\vartheta_{1r} \sin\varphi_{1r} = k_2 \sin\vartheta_2 \sin\varphi_2 \ .$$

For $(\vartheta_{1r}, \vartheta_2 \neq 0)$ the second equation requires:

$$\varphi_{1r} = \varphi_2 = 0 \tag{4.254}$$

Note that $\varphi_{1r} = \pi$ or $\varphi_2 = \pi$ would lead to a contradiction in the first equation! We have to conclude that the wave vectors $\mathbf{k}_1$, $\mathbf{k}_{1r}$, $\mathbf{k}_2$ lie in one and the same plane, namely the

plane of incidence

which is fixed by the direction of incidence $\mathbf{k}_1$ and the normal of the interface $\mathbf{n}$.

From the above first equation it still remains:

$$k_1 \sin\vartheta_1 = k_{1r} \sin\vartheta_{1r} = k_2 \sin\vartheta_2 \ .$$

If we insert for the magnitudes of the wave vectors

$$k_1 = \frac{\omega}{u_1} = \frac{\omega}{c} n_1 = \frac{\omega}{c}\sqrt{\mu_r^{(1)}\epsilon_r^{(1)}} = k_{1r} \ ,$$
$$k_2 = \frac{\omega}{u_2} = \frac{\omega}{c} n_2 = \frac{\omega}{c}\sqrt{\mu_r^{(2)}\epsilon_r^{(2)}} \ . \tag{4.255}$$

then we find:

Law of reflection:

$$\vartheta_1 = \vartheta_{1r} \ . \tag{4.256}$$

Law of refraction (Snell's law):

$$\frac{\sin\vartheta_1}{\sin\vartheta_2} = \frac{k_2}{k_1} = \frac{n_2}{n_1} \ . \tag{4.257}$$

Medium 2 is called **optically denser** than medium 2 if

$$n_2 > n_1 \ .$$

Because of $0 \leq \vartheta_{1,2} \leq \pi/2$ it is then $\vartheta_1 > \vartheta_2$. Hence, the wave is refracted towards the vertical. Furthermore it is $u_1 > u_2$ and $\lambda_1 > \lambda_2$. On the other hand, if medium 2 is **optically rarer** than medium 1, i.e. $n_2 < n_1$, then the refraction bends away from the vertical. That has the important consequence that there does exist a *critical angle* $\vartheta_1 = \vartheta_t$, at which total reflexion ($\vartheta_2 = \pi/2$) takes place. According to (4.257) this critical angle ϑ_t is determined by

$$\sin\vartheta_t = \frac{n_2}{n_1}\,. \tag{4.258}$$

(see point G).

(C) Intensities at Reflection and Refraction

All the rules and laws derived so far resulted from very general considerations on the continuity of the fields at interfaces. However, they do not suffice when we want to get information also about the intensities of the reflected and refracted partial waves, which are determined by the square moduli of the field amplitudes.

We have shown previously that each elliptically polarized plane wave can be decomposed into two linearly polarized waves with their planes of polarization mutually perpendicular. We therefore discuss only the two special cases:

1. $\mathbf{E}_1$ linearly polarized perpendicular to the incidence plane,
2. $\mathbf{E}_1$ linearly polarized **within** the incidence plane.

We now derive statements from the continuity conditions (4.247), which we first rewrite with respect to the case of interest at present:

$$\mathbf{n}\times[\mathbf{E}_2-(\mathbf{E}_1+\mathbf{E}_{1r})]=0\,, \tag{4.259a}$$

$$\mathbf{n}\cdot\left[\epsilon_r^{(2)}\mathbf{E}_2-\epsilon_r^{(1)}(\mathbf{E}_1+\mathbf{E}_{1r})\right]=0\,, \tag{4.259b}$$

$$\mathbf{n}\times\left[\frac{1}{\mu_r^{(2)}}(\mathbf{k}_2\times\mathbf{E}_2)-\frac{1}{\mu_r^{(1)}}(\mathbf{k}_1\times\mathbf{E}_1+\mathbf{k}_{1r}\times\mathbf{E}_{1r})\right]=0\,, \tag{4.259c}$$

$$\mathbf{n}\cdot[(\mathbf{k}_2\times\mathbf{E}_2)-(\mathbf{k}_1\times\mathbf{E}_1+\mathbf{k}_{1r}\times\mathbf{E}_{1r})]=0\,. \tag{4.259d}$$

1. **$\mathbf{E}_1$ perpendicular to the plane of incidence**

   At first one should bear in mind that, as usual, the fields are chosen as complex quantities. That means that, in general, the amplitudes of course are also complex. In the following equations and in particular also in the Figs. 4.51, 4.52, and 4.53 the field quantities are therefore always to be understood as their *physical* real parts (optionally also the imaginary parts), without being explicitly so indicated.

   From the continuity of $\mathbf{E}$ at $z = 0$ follows that besides $\mathbf{E}_1$, $\mathbf{E}_{1r}$ and $\mathbf{E}_2$ are also linearly polarized, perpendicular to the incidence plane and parallel to the y-axis.

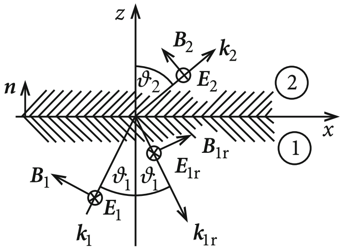


Fig. 4.51 Reflection and refraction of electromagnetic waves at interfaces, where the incident electric field vector is linearly polarized perpendicular to the plane of incidence

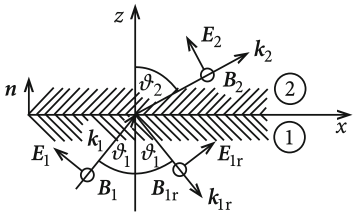


Fig. 4.52 Reflection and refraction of electromagnetic waves at interfaces, where the incident electric field vector is linearly polarized within the plane of incidence

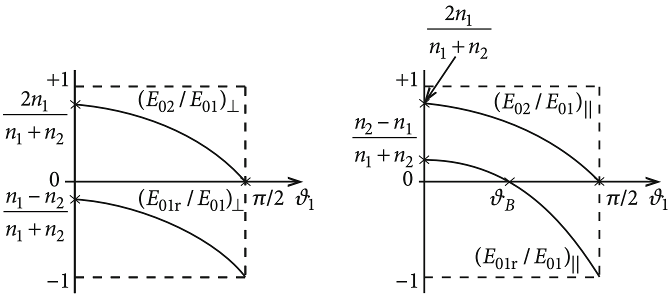


Fig. 4.53 Dependence of the ratio of the amplitudes of the refracted (reflected) and the incident field vector on the angle of incidence ϑ_1 for linear polarization perpendicular and parallel, respectively, to the plane of incidence

Equation (4.259b) is therefore trivially fulfilled. From (4.259a) firstly it follows:

$$\mathbf{E}_2 = \mathbf{E}_1 + \mathbf{E}_{1r} \, .$$

That holds for all points of the interface $z = 0$ and for arbitrary times. But since the phase factors of all the three fields (4.249)–(4.251) are at the interface the same, we can even conclude:

$$E_{02} - (E_{01} + E_{01r}) = 0 \,. \tag{4.260}$$

Equation (4.259d) leads together with the law of reflection (4.256) just the law of refraction (4.257) and vice versa, i.e. (4.259d) does not deliver any new information. It still remains (4.259c) to be evaluated:

$$\frac{1}{\mu_r^{(2)}}\left[\mathbf{k}_2\underbrace{(\mathbf{n}\cdot\mathbf{E}_2)}_{=0}-\mathbf{E}_2(\mathbf{n}\cdot\mathbf{k}_2)\right]-\frac{1}{\mu_r^{(1)}}\left[\mathbf{k}_1\underbrace{(\mathbf{n}\cdot\mathbf{E}_1)}_{=0}\right.$$

$$\left.-\mathbf{E}_1(\mathbf{n}\cdot\mathbf{k}_1)+\mathbf{k}_{1r}\underbrace{(\mathbf{n}\cdot\mathbf{E}_{1r})}_{=0}-\mathbf{E}_{1r}(\mathbf{n}\cdot\mathbf{k}_{1r})\right]$$

$$= -\frac{k_2}{\mu_r^{(2)}}\mathbf{E}_2\cos\vartheta_2 + \frac{k_1}{\mu_r^{(1)}}\mathbf{E}_1\cos\vartheta_1 - \frac{k_{1r}}{\mu_r^{(1)}}\mathbf{E}_{1r}\cos\vartheta_1 \stackrel{!}{=} 0 \,.$$

With (4.255) it further follows:

$$\sqrt{\frac{\epsilon_r^{(1)}}{\mu_r^{(1)}}}\,(E_{01} - E_{01r})\cos\vartheta_1 - \sqrt{\frac{\epsilon_r^{(2)}}{\mu_r^{(2)}}}\,E_{02}\cos\vartheta_2 = 0 \,. \tag{4.261}$$

We eliminate E_{01r} by use of (4.260):

$$2E_{01}\sqrt{\frac{\epsilon_r^{(1)}}{\mu_r^{(1)}}}\cos\vartheta_1 = E_{02}\left(\sqrt{\frac{\epsilon_r^{(1)}}{\mu_r^{(1)}}}\cos\vartheta_1 + \sqrt{\frac{\epsilon_r^{(2)}}{\mu_r^{(2)}}}\cos\vartheta_2\right) \,.$$

This yields eventually:

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \frac{2n_1\cos\vartheta_1}{n_1\cos\vartheta_1 + \frac{\mu_r^{(1)}}{\mu_r^{(2)}}\,n_2\cos\vartheta_2} \,. \tag{4.262}$$

With the law of refraction (4.257) we can express $\cos\vartheta_2$ still by the angle of incidence ϑ_1:

$$\cos\vartheta_2 = \sqrt{1-\sin^2\vartheta_2} = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\vartheta_1} \,.$$

It follows therewith:

$$\left(\frac{E_{02}}{E_{01}}\right)_\perp = \frac{2n_1 \cos\vartheta_1}{n_1 \cos\vartheta_1 + \frac{\mu_r^{(1)}}{\mu_r^{(2)}} \sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}} \ . \tag{4.263}$$

The ratio of the amplitudes of the refracted and the incident wave is herewith completely fixed by the angle of incidence ϑ_1 and the material constants $\epsilon_r^{(1,2)}$, $\mu_r^{(1,2)}$.

We still take from (4.260)

$$\left(\frac{E_{01r}}{E_{01}}\right)_\perp = \left(\frac{E_{02}}{E_{01}}\right)_\perp - 1 \ ,$$

so that with (4.263) the analogue formula for the reflected wave reads:

$$\left(\frac{E_{01r}}{E_{01}}\right)_\perp = \frac{n_1 \cos\vartheta_1 - \frac{\mu_r^{(1)}}{\mu_r^{(2)}} \sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}}{n_1 \cos\vartheta_1 + \frac{\mu_r^{(1)}}{\mu_r^{(2)}} \sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}} \ . \tag{4.264}$$

2. **$\mathbf{E}_1$ parallel to the plane of incidence**

Let us perform the analogous considerations for the case that the **E**-vectors are linearly polarized **within** the plane of incidence.

It follows from the continuity condition for D_n (4.259b):

$$\epsilon_r^{(2)} E_{02} \cos\left(\frac{\pi}{2} - \vartheta_2\right) - \epsilon_r^{(1)} \left[E_{01} \cos\left(\frac{\pi}{2} - \vartheta_1\right) + E_{01r} \cos\left(\frac{\pi}{2} - \vartheta_1\right)\right] = 0$$

$$\Longrightarrow \ \epsilon_r^{(2)} E_{02} \frac{n_1}{n_2} \sin\vartheta_1 - \epsilon_r^{(1)} \left(E_{01} + E_{01r}\right) \sin\vartheta_1 = 0$$

or

$$\epsilon_r^{(2)} E_{02} \frac{n_1}{n_2} = \epsilon_r^{(1)} \left(E_{01} + E_{01r}\right) \ . \tag{4.265}$$

The continuity condition (4.259a) leads to:

$$E_{02} \sin\left(\frac{\pi}{2} - \vartheta_2\right) - E_{01} \sin\left(\frac{\pi}{2} - \vartheta_1\right) + E_{01r} \sin\left(\frac{\pi}{2} - \vartheta_1\right) = 0 \ .$$

This means:

$$E_{02} \cos\vartheta_2 = \left(E_{01} - E_{01r}\right) \cos\vartheta_1 \ . \tag{4.266}$$

Equations (4.265) and (4.266) can be solved for E_{02}/E_{01} and E_{01r}/E_{01}, respectively:

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2n_1 n_2 \cos\vartheta_1}{\frac{\mu_r^{(1)}}{\mu_r^{(2)}} n_2^2 \cos\vartheta_1 + n_1\sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}} \,, \tag{4.267}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} = \frac{\frac{\mu_r^{(1)}}{\mu_r^{(2)}} n_2^2 \cos\vartheta_1 - n_1\sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}}{\frac{\mu_r^{(1)}}{\mu_r^{(2)}} n_2^2 \cos\vartheta_1 + n_1\sqrt{n_2^2 - n_1^2 \sin^2\vartheta_1}} \,. \tag{4.268}$$

(D) Fresnel Formulas

For the often encountered case that the media 1 and 2 have the same magnetic susceptibilities (see (3.74)),

$$\mu_r^{(1)} = \mu_r^{(2)} \,, \tag{4.269}$$

which also includes the important special case of the non-magnetizable bodies ($\mu_r^{(1)} = \mu_r^{(2)} = 1$), the general results ((4.263), (4.264), (4.267), (4.268)) simplify a bit more:

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2n_1 \cos\vartheta_1}{n_1 \cos\vartheta_1 + n_2 \cos\vartheta_2} \,, \tag{4.270}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\perp} = \frac{n_1 \cos\vartheta_1 - n_2 \cos\vartheta_2}{n_1 \cos\vartheta_1 + n_2 \cos\vartheta_2} \,, \tag{4.271}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2n_1 \cos\vartheta_1}{n_2 \cos\vartheta_1 + n_1 \cos\vartheta_2} \,, \tag{4.272}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} = \frac{n_2 \cos\vartheta_1 - n_1 \cos\vartheta_2}{n_2 \cos\vartheta_1 + n_1 \cos\vartheta_2} \,. \tag{4.273}$$

These relations can be further reformulated by means of the law of refraction and the addition theorems of trigonometric functions:

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{n_1}{n_2}\,\frac{2\sin\vartheta_1 \cos\vartheta_1}{\cos\vartheta_2 \sin\vartheta_1 + \sin\vartheta_2 \cos\vartheta_1}$$

$$\Longrightarrow \left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2\sin\vartheta_2 \cos\vartheta_1}{\sin(\vartheta_1 + \vartheta_2)} \,, \tag{4.274}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\perp} = \frac{\frac{\sin\vartheta_2}{\sin\vartheta_1}\cos\vartheta_1 - \cos\vartheta_2}{\frac{\sin\vartheta_2}{\sin\vartheta_1}\cos\vartheta_1 + \cos\vartheta_2}$$

$$\Longrightarrow \left(\frac{E_{01r}}{E_{01}}\right)_{\perp} = \frac{\sin(\vartheta_2 - \vartheta_1)}{\sin(\vartheta_2 + \vartheta_1)}\,, \tag{4.275}$$

$$\left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\sin\vartheta_2\cos\vartheta_1}{\sin\vartheta_1\cos\vartheta_1 + \sin\vartheta_2\cos\vartheta_2}\,,$$

$$\sin\vartheta_1\cos\vartheta_1 + \sin\vartheta_2\cos\vartheta_2 =$$
$$= (\sin\vartheta_1\cos\vartheta_2 + \sin\vartheta_2\cos\vartheta_1)\cdot(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2)$$

$$\Longrightarrow \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} = \frac{2\sin\vartheta_2\cos\vartheta_1}{\sin(\vartheta_1+\vartheta_2)\cos(\vartheta_1-\vartheta_2)}\,, \tag{4.276}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} = \frac{\frac{\sin\vartheta_1}{\sin\vartheta_2}\cos\vartheta_1 - \cos\vartheta_2}{\frac{\sin\vartheta_1}{\sin\vartheta_2}\cos\vartheta_1 + \cos\vartheta_2} = \frac{\sin(2\vartheta_1) - \sin(2\vartheta_2)}{\sin(2\vartheta_1) + \sin(2\vartheta_2)}$$
$$= \frac{\frac{2\tan\vartheta_1}{1+\tan^2\vartheta_1} - \frac{2\tan\vartheta_2}{1+\tan^2\vartheta_2}}{\frac{2\tan\vartheta_1}{1+\tan^2\vartheta_1} + \frac{2\tan\vartheta_2}{1+\tan^2\vartheta_2}} = \frac{(\tan\vartheta_1 - \tan\vartheta_2)(1-\tan\vartheta_1\tan\vartheta_2)}{(\tan\vartheta_1 + \tan\vartheta_2)(1+\tan\vartheta_1\tan\vartheta_2)}$$

$$\Longrightarrow \left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} = \frac{\tan(\vartheta_1-\vartheta_2)}{\tan(\vartheta_1+\vartheta_2)}\,. \tag{4.277}$$

Equations (4.274) to (4.277) are called the **Fresnel formulas**, named after their discoverer.

Let us inspect the case that **medium 2** is the **optically denser** medium, i.e.

$$n_2 > n_1 \iff \vartheta_1 > \vartheta_2\,.$$

1. For grazing incidence ($\vartheta_1 = \pi/2$) there is no refraction ($\mathbf{E}_2 \equiv 0$) (Fig. 4.53).
2. $(E_{01r}/E_{01})_{\perp} < 0$: The wave, which is polarized perpendicular to the plane of incidence, gets a phase jump of π.

$(E_{01r}/E_{01})_{\parallel} \geq 0$ as long as $\vartheta_1 + \vartheta_2 < \pi/2$. According to the directions of $\mathbf{E}_1$ and $\mathbf{E}_{1r}$, as chosen in Fig. 4.52, this means also for the parallel, reflected wave a phase jump of π.

All in all, the reflected wave thus performs for $\vartheta_1 + \vartheta_2 \leq \pi/2$ a phase jump of π. We will see that this gets importance in connection with interference phenomena for which the so-called optical path difference between two waves is decisive.

3. There exists a significant angle of incidence,

$$\vartheta_1 = \vartheta_B \; : \qquad \textbf{Brewster's angle} \; ,$$

at which

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} = 0 \; .$$

According to (4.273) this is exactly then the case when

$$n_2 \cos\vartheta_1 \overset{!}{=} n_1 \cos\vartheta_2 \; ,$$

i.e.

$$n_2 \cos\vartheta_B = n_1 \sqrt{1 - \sin^2\vartheta_2} = n_1 \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\vartheta_B}$$

$$\Longrightarrow \frac{n_2^2}{n_1^2} = \frac{1}{\cos^2\vartheta_B} - \frac{n_1^2}{n_2^2}\tan^2\vartheta = 1 + \tan^2\vartheta_B - \frac{n_1^2}{n_2^2}\tan^2\vartheta_B \; .$$

That means:

$$\tan\vartheta_B = \frac{n_2}{n_1} \; . \tag{4.278}$$

The reflected wave is then completely **linearly polarized** perpendicular to the plane of incidence.

(E) Perpendicular Incidence ($\vartheta_1 = \vartheta_2 = 0$)

It is now impossible to define a plane of incidence, the discrimination between *perpendicular* and *parallel* becomes meaningless. It follows from (4.270) to (4.273) for this special case:

$$\left(\frac{E_{02}}{E_{01}}\right)_{\perp} = \frac{2n_1}{n_1 + n_2} = \left(\frac{E_{02}}{E_{01}}\right)_{\parallel} , \tag{4.279}$$

$$\left(\frac{E_{01r}}{E_{01}}\right)_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} = -\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel} . \tag{4.280}$$

Notice and verify that the sign in (4.280) does not mean any contradiction!

(F) Energy Transport (Intensities!)

Incident, refracted, and reflected waves do transport energy. According to (4.214) we have for the corresponding energy-current densities:

$$\overline{\mathbf{S}} = \frac{1}{2}\sqrt{\frac{\epsilon_r \epsilon_0}{\mu_r \mu_0}}\, |\mathbf{E}_0|^2 \, \frac{\mathbf{k}}{k} .$$

One defines therewith

1. the **reflection coefficient (reflectance):**

$$R = \left| \frac{\overline{\mathbf{S}_{1r} \cdot \mathbf{n}}}{\overline{\mathbf{S}_1 \cdot \mathbf{n}}} \right| , \tag{4.281}$$

2. the **transmission coefficient:**

$$T = \left| \frac{\overline{\mathbf{S}_2 \cdot \mathbf{n}}}{\overline{\mathbf{S}_1 \cdot \mathbf{n}}} \right| . \tag{4.282}$$

Because of

$$\mathbf{k}_1 \cdot \mathbf{n} = k_1 \cos\vartheta_1 ,$$

$$\mathbf{k}_{1r} \cdot \mathbf{n} = k_{1r} \cos(\pi - \vartheta_{1r}) = -k_1 \cos\vartheta_1 ,$$

$$\mathbf{k}_2 \cdot \mathbf{n} = k_2 \cos\vartheta_2$$

that means for the here underlying case:

$$R = \left| \frac{E_{01r}}{E_{01}} \right|^2 , \tag{4.283}$$

$$T = \sqrt{\frac{\epsilon_r^{(2)} \mu_r^{(1)}}{\epsilon_r^{(1)} \mu_r^{(2)}}} \, \frac{\cos\vartheta_2}{\cos\vartheta_1} \left| \frac{E_{02}}{E_{01}} \right|^2 . \tag{4.284}$$

The energy-flow balance

$$T + R = 1 \tag{4.285}$$

should of course be fulfilled. Indeed, that can be shown. Multiply (4.260) by (4.261):

$$\left(E_{01}^{\perp}\right)^2 - \left(E_{01\mathrm{r}}^{\perp}\right)^2 = \sqrt{\frac{\epsilon_{\mathrm{r}}^{(2)}\mu_{\mathrm{r}}^{(1)}}{\epsilon_{\mathrm{r}}^{(1)}\mu_{\mathrm{r}}^{(2)}}}\,\frac{\cos\vartheta_2}{\cos\vartheta_1}\left(E_{02}^{\perp}\right)^2 .$$

Multiplication of (4.265) by (4.266) results in the analogous expression for the *parallel* components. If one then adds the two equations and takes into consideration that the orthogonality of the components leads to

$$\left(E_{0i}^{\perp}\right)^2 + \left(E_{0i}^{\parallel}\right)^2 = \left(E_{0i}\right)^2 ,$$

then it first follows:

$$\left(E_{01}\right)^2 - \left(E_{01\mathrm{r}}\right)^2 = \sqrt{\frac{\varepsilon_{\mathrm{r}}^{(2)}\mu_{\mathrm{r}}^{(1)}}{\varepsilon_{\mathrm{r}}^{(1)}\mu_{\mathrm{r}}^{(2)}}}\,\frac{\cos\vartheta_2}{\cos\vartheta_1}\left(E_{02}\right)^2 .$$

If we still remember that by the field terms always the real or the imaginary part of these in principle complex amplitudes are meant, then we can use the last equation once for the real part and once for the imaginary part in order to add the two expressions. That eventually yields, because of $(\mathrm{Re}E_{0i})^2 + (\mathrm{Im}E_{0i})^2 = |E_{0i}|^2$,

$$1 - \left|\frac{E_{01\mathrm{r}}}{E_{01}}\right|^2 = \sqrt{\frac{\varepsilon_{\mathrm{r}}^{(2)}\mu_{\mathrm{r}}^{(1)}}{\varepsilon_{\mathrm{r}}^{(1)}\mu_{\mathrm{r}}^{(2)}}}\,\frac{\cos\vartheta_2}{\cos\vartheta_1}\left|\frac{E_{02}}{E_{01}}\right|^2 ,$$

what proves the assertion (4.285).

(G) Total Internal Reflection

We had already argued with (4.258) from the Snellius's law of refraction that at the transition from the optically denser to the optically rarer medium,

$$n_1 > n_2 ,$$

there exists an angle of incidence $\vartheta_1 = \vartheta_t$ for which total internal reflection arises. The refracted wave propagates parallel to the interface. But what happens now for $\vartheta_1 > \vartheta_t$?

According to the law of refraction (4.257) we have to at first accept

$$\sin\vartheta_2 > 1 \ .$$

But then ϑ_2 can no longer be real. Since, on the other hand, we have anyway always calculated with complex fields, this fact should not create difficulties for our theory, in particular, the law of refraction should still retain its validity:

$$\sin\vartheta_2 = \frac{n_1}{n_2}\sin\vartheta_1 = \frac{\sin\vartheta_1}{\sin\vartheta_t} \ .$$

$\cos\vartheta_2$ is then purely imaginary:

$$\cos\vartheta_2 = i\sqrt{\left(\frac{\sin\vartheta_1}{\sin\vartheta_t}\right)^2 - 1} \ . \tag{4.286}$$

This we insert into the Fresnel formula (4.277):

$$\begin{aligned}\left(\frac{E_{01r}}{E_{01}}\right)_{\|} &= \frac{\sin\vartheta_1\cos\vartheta_1 - \sin\vartheta_2\cos\vartheta_2}{\sin\vartheta_1\cos\vartheta_1 + \sin\vartheta_2\cos\vartheta_2} \\ &= \frac{\cos\vartheta_1 - \frac{i}{\sin\vartheta_t}\sqrt{\left(\frac{\sin\vartheta_1}{\sin\vartheta_t}\right)^2 - 1}}{\cos\vartheta_1 + \frac{i}{\sin\vartheta_t}\sqrt{\left(\frac{\sin\vartheta_1}{\sin\vartheta_t}\right)^2 - 1}} \ .\end{aligned} \tag{4.287}$$

Numerator and denominator are complex conjugate numbers having therewith notably the same moduli:

$$\Longrightarrow \left(\frac{E_{01r}}{E_{01}}\right)_{\|} = \frac{\alpha\, e^{-i\varphi}}{\alpha\, e^{i\varphi}} = e^{-2i\varphi} \ , \tag{4.288}$$

$$\tan\varphi = \frac{1}{\sin^2\vartheta_t}\,\frac{\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}}{\cos\vartheta_1} \ . \tag{4.289}$$

The component which oscillates parallel to the plane of incidence thus experiences with the total reflection a phase shift by (-2φ). The amplitude E_{01r} is obviously complex.

Completely analogously one finds with (4.275) for the perpendicular component:

$$\left(\frac{E_{01r}}{E_{01}}\right)_\perp = \frac{\sin\vartheta_2\cos\vartheta_1 - \sin\vartheta_1\cos\vartheta_2}{\sin\vartheta_2\cos\vartheta_1 + \sin\vartheta_1\cos\vartheta_2} = \frac{\frac{\cos\vartheta_1}{\sin\vartheta_t} - \cos\vartheta_2}{\frac{\cos\vartheta_1}{\sin\vartheta_t} + \cos\vartheta_2}$$

$$= \frac{\cos\vartheta_1 - i\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}}{\cos\vartheta_1 + i\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}} = e^{-2i\psi}\ ,$$

$$\tan\psi = \frac{\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}}{\cos\vartheta_1}. \tag{4.290}$$

The phase angles φ and ψ for the two components are thus **not** the same, i.e. the two components of the reflected wave are phase-shifted relative to each other. If the incident wave is linearly polarized then the reflected wave now becomes elliptically polarized. The phase difference of the two components amounts to:

$$\delta = 2(\varphi - \psi)\ ,$$

$$\tan\frac{\delta}{2} = \tan(\varphi - \psi) = \frac{\tan\varphi - \tan\psi}{1 + \tan\varphi\tan\psi}$$

$$\Longrightarrow\ \tan\frac{\delta}{2} = \frac{\cos\vartheta_1\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}}{\sin^2\vartheta_1}\ . \tag{4.291}$$

The ratios of the amplitudes $(E_{01r}/E_{01})_\parallel$ and $(E_{01r}/E_{01})_\perp$ turn out to be complex numbers of the modulus 1, so that the name *total internal reflection* makes sense ((4.283) $\Longrightarrow R = 1$).

How do these ratios look like in the medium 2? Actually it should not happen anything there in the case of a real total reflection. According to (4.251) the factor

$$\exp(i\,\mathbf{k}_2\cdot\mathbf{r}) = \exp\left[i\,k_2(x\sin\vartheta_2 + z\cos\vartheta_2)\right]$$

$$= \exp\left[i\frac{k_2}{\sin\vartheta_t}\left(x\sin\vartheta_1 + iz\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_t}\right)\right]$$

$$= \exp\left[-k_2 z\sqrt{\left(\frac{\sin\vartheta_1}{\sin\vartheta_t}\right)^2 - 1}\,\right]\exp\left(i\,k_2\,x\frac{\sin\vartheta_1}{\sin\vartheta_t}\right)$$

is responsible for the propagation of the refracted wave. The wave is therefore exponentially damped in z-direction, fading therewith away for $\vartheta_1 > \vartheta_t$ very rapidly.

An energy flow into the medium 2 does not take place in the time average:

$$\overline{\mathbf{S}_2 \cdot \mathbf{n}} = \frac{1}{2}\sqrt{\frac{\epsilon_r^{(2)}\epsilon_0}{\mu_r^{(2)}\mu_0}}\,\mathrm{Re}\left(|E_{02}|^2 \mathbf{n}\cdot\frac{\mathbf{k}_2}{k_2}\right)$$
$$= \frac{1}{2}\sqrt{\frac{\epsilon_r^{(2)}\epsilon_0}{\mu_r^{(2)}\mu_0}}\,|E_{02}|^2 \mathrm{Re}(\cos\vartheta_2) = 0\,. \tag{4.292}$$

This definitely allows one now to speak for $\vartheta_1 > \vartheta_t$ of total internal reflection ((4.282) $\Longrightarrow T = 0$).

4.3.11 Interference and Diffraction

A decisive characteristic of the concept of *'wave'* is the

'ability for interference'

Naively formulated this is the feature that *'light can be destroyed by light'*! However, only the so-called **coherent (light)waves** are capable of doing that. Interfering wave trains have to have a fixed phase-relation during a time span $t \gg \tau = \frac{1}{\nu}$. According to the findings of atomic physics, light emission is due to atoms which are in principle independent of each other. The emission is carried out in form of wave trains of finite length. Hence, coherent light can **not** originate from two different sources. The single atom of course can not come into question, either. One needs instead *'indirect'* methods. An interesting possibility for the generation of coherent light is given by the classical

Fresnel's mirror experiment

In this experiment a splitting of the wave train takes place by reflection or refraction so that the resulting partial waves can then be brought into an interference with each other.

L_1 and L_2 (separation d) are the virtual images of the real light source L produced by two mirrors which are inclined relative to each other by the angle α (Fig. 4.54). The light beams B_1 und B_2 starting (virtually) at L_1 and L_2 are then surely coherent. Hence, they can interfere with each other. At the point P on the screen S the light beams enhance themselves or extinguish each other depending on whether the path difference $\Delta = \overline{PL_1} - \overline{PL_2}$ is an even or an odd multiple of half the wavelength $\frac{\lambda}{2}$. On the screen there appear **interference fringes** as hyperbolas since the hyperbola is defined by all points for which the difference of the distances from two fixed spots (L_1, L_2) is the same. The *bright* hyperbolas are running through the intersection points of the circles around L_1 and L_2, whose differences of radii amount to 0, λ, $2\lambda, \ldots$ since there the coherent waves coming from L_1, L_2 mutually reinforce. On

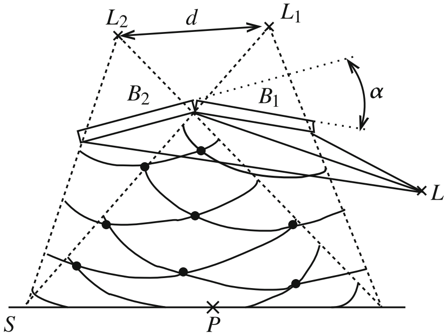


Fig. 4.54 Schematic set-up of the Fresnel's mirror experiment. B_1, B_2 are two mirrors inclined with respect to each other by the angle α; L_1, L_2 are the virtual images of the real light source L. S is a screen where the resulting interference fringes can be observed

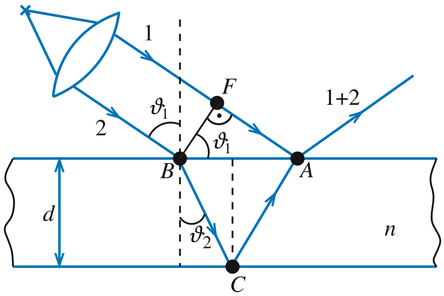


Fig. 4.55 Geometrical beam path for the reflection at two plane-parallel mirrors for the analysis of the 'interference of same inclination'

the other hand, **extinction** appears when the difference of the radii amounts to an odd multiple of $\frac{\lambda}{2}$ since then a wave trough meets a wave crest. On the screen dark and bright stripes alternate.

Another method to create coherent interfering light waves exploits the **reflection on two plane-parallel mirrors**.

The ray 1 impinges at A the plane-parallel layer (index of refraction n) and is partially reflected there. The ray 2 is at B partially refracted in direction to C where it is partially reflected, in order to interfere at A with ray 1. The optical path difference amounts to (Fig. 4.55)

$$\Delta = n(\overline{BC} + \overline{CA}) - \overline{FA} + \frac{\lambda}{2} \,. \tag{4.293}$$

The third term accommodates for the phase jump by π in connection with the reflection at the optically denser medium (see the Fresnel formulas (4.274) to (4.277)). Using further the law of refraction

$$n = \frac{\sin \vartheta_1}{\sin \vartheta_2} \,, \tag{4.294}$$

we get after simple geometrical considerations:

$$\Delta = 2d\sqrt{n^2 - \sin^2 \vartheta_1} + \frac{\lambda}{2} \,. \tag{4.295}$$

For a given thickness d of the layer the path difference Δ is determined exclusively by the **angle of inclination** ϑ_1. One therefore speaks of

interference of same inclination

$$\left.\begin{array}{lll}\Delta = z\lambda & \Longrightarrow & \text{constructive interference} \\ \Delta = (2z+1)\dfrac{\lambda}{2} & \Longrightarrow & \text{destructive interference}\end{array}\right. \qquad z = 0, 1, 2, \ldots$$

Both the reported examples of interference need for their analysis unavoidably the wave character of the light. This holds to the same extent also for the phenomenon of

diffraction

By diffraction we understand the deviation of the light from the straight-lined ray path which can *not* be interpreted as refraction or reflection. It is a phenomenon which is observed for *all* wave processes. Wave intensity can enter even the **geometrical shadow sector**. Well-known **examples** are the following:

1. **Pinhole**: Depending on the distance of the screen from the pinhole one observes in the center of the screen minima or maxima of the brightness (see Sect. 4.3.15),
2. **Airy disk**: In the center of the geometrical shadow region there is always a bright spot: **Poisson spot** (see Sect. 4.3.14),
3. **Halo of the moon**: The light flare around the moon originates from the diffraction on irregularly distributed water droplets in humid air (fog);
4. **Rainbow**: This arises as a consequence of refraction and reflection inside the raindrops which are illuminated by the sun which is behind the observer. The full explanation, though, relies again on a problem of diffraction,
5. **Umbrella**: The fine texture diffracts the light of a remote source whereby color pictures of a crossed grating may result,
6. **Acoustics**, **Sound**.

Diffraction phenomena are observed only when the linear dimensions of the diffracting barriers or holes are of the same order of magnitude as the wavelength of the light or smaller. In the optical region (small wavelengths) there are therefore not so many diffraction phenomena which belong to our *daily experience*. However, in **acoustics** with sound-wavelengths of the order of meters the diffraction plays a special role since it, in the first place, makes, e.g., hearing behind barriers possible. In a certain sense, sound can readily *circumvent barriers*. The fact that light is also a wave has been recognized therefore very much later than for sound.

The basis for the understanding of interference and diffraction is given by the **Huygens' principle**:

The ***henceforth*** *propagation of an arbitrarily given wavefront is determined if each point of the wavefront is treated as a source of a secondary spherical wave and takes then the envelope of all these coherent spherical waves as the 'new' wave front!*

In a homogeneous medium a surface parallel to the original wavefront arises in this way. According to **Kirchhoff** the Huygens'principle is in the last analysis a direct consequence of the Maxwell equations and their boundary conditions at interfaces and barriers. The exact verification, however, turns out to be rather complicated.

4.3.12 Kirchhoff's Formula

The vectorial character of the electromagnetic fields shall at first be disregarded. We consider instead the **scalar** quantity

$$E(\mathbf{r}, t) = E_0(\mathbf{r})\, e^{-i\omega t} \;, \tag{4.296}$$

where, in addition, only its space-dependence is actually interesting. One may think, e.g., of one of the two transverse components of the electric field. Only the intensity ($\sim|E|^2$) is decisive for the following. Because of the necessity of coherence, the waves discussed here must all have the same ω.

The field quantity is a solution of the homogeneous wave equation:

$$\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right) E = \left(\Delta + \frac{\omega^2}{u^2}\right) E_0\, e^{-i\omega t} = 0 \tag{4.297}$$

$$\frac{\omega}{u} = \frac{n\omega}{c} = \frac{2\pi}{\lambda} = k \tag{4.298}$$

Let E and E' be solutions of the wave equation with the same ω. As scalar fields they satisfy the **second Green identity** (1.68):

$$\int\limits_V d^3r(E\Delta E' - E'\Delta E) = \int\limits_{\partial V} (E\nabla E' - E'\nabla E) \cdot d\mathbf{f} \;. \tag{4.299}$$

The wave equation yields:

$$E\Delta E' - E'\Delta E = \left(-\frac{\omega^2}{u^2}\right)(EE' - E'E) = 0 \;. \tag{4.300}$$

Hence, it remains:

$$\int\limits_{\partial V} (E\nabla E' - E'\nabla E) \cdot d\mathbf{f} = 0 \;. \tag{4.301}$$

∂V is the surface of an arbitrarily given volume V. In Sect. 4.3.5 we have shown that besides the plane waves, the spherical waves also solve the wave equation. Let E' be such a solution with the above harmonic time-dependence. Let the spherical wave start from the point P, the origin of coordinates inside of V (Fig. 4.56)

$$E' \sim \frac{e^{ikr}}{r} \,. \tag{4.302}$$

It plays here at first only the role of a mathematical auxiliary quantity, in a certain sense as a *testing probe* by which we want to investigate the optical field E. It must therefore be only a solution of the wave equation and need not necessarily satisfy additionally the Maxwell equations (see point 6. in Sect. 4.3.5).

But we have to now take account of the fact that the spherical wave diverges for $r \to 0$. We therefore exclude this point by a small spherical volume V_i as sketched in Fig. 4.56. When applying the Green identity we remember that $d\mathbf{f}$ is always *outwardly* oriented. That means that on ∂V_i the vector $d\mathbf{f}$ therefore points to the center of the sphere:

$$0 = \left(\int\limits_{\partial V} + \int\limits_{\partial V_i}\right)\left(E\nabla\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\nabla E\right)\cdot d\mathbf{f} \,. \tag{4.303}$$

We consider the integral over the surface ∂V_i of the sphere and use the fact that E and ∇E remain continuous for $r \to 0$. The unit-vector $\mathbf{e}_\mathrm{r}$ has the radial direction seen from the center P of V_i being therewith antiparallel to $d\mathbf{f}$.

$$-\int\limits_{\partial V_i} \frac{e^{ikr}}{r}\nabla E(-\mathbf{e}_\mathrm{r})r^2 \sin\vartheta\, d\vartheta\, d\varphi \stackrel{r\to 0}{\longrightarrow} 0 \,. \tag{4.304}$$

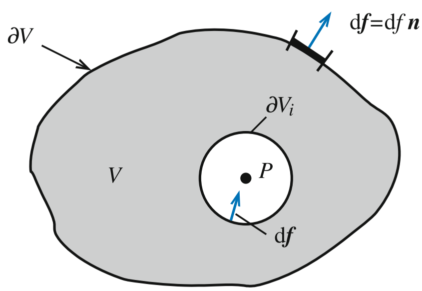


Fig. 4.56 Region of integration in (4.303) for the derivation of the Kirchhoff's formula (4.305)

The other term of the surface integral can be estimated as follows:

$$\nabla \frac{e^{ikr}}{r} = \left(-\frac{1}{r^2} + \frac{ik}{r}\right) e^{ikr} \mathbf{e}_\mathrm{r}$$

$$\Longrightarrow \quad \int\limits_{\partial V_i} E \nabla \frac{e^{ikr}}{r} \cdot d\mathbf{f} = \int\limits_{\partial V_i} E \left(-\frac{1}{r^2} + \frac{ik}{r}\right) e^{ikr} \mathbf{e}_\mathrm{r} \cdot d\mathbf{f}$$

$$= \int\limits_{\partial V_i} E(1 - ikr)\, e^{ikr} \sin\vartheta\, d\vartheta\, d\varphi$$

$$\xrightarrow{r \to 0} E(P) \cdot 4\pi \ .$$

Then follows the important **Kirchhoff's formula**

$$E(P) = \frac{1}{4\pi} \int\limits_{\partial V} \left(\frac{e^{ikr}}{r} \nabla E - E \nabla \frac{e^{ikr}}{r}\right) \cdot d\mathbf{f} \ . \tag{4.305}$$

The field at the point P is therewith represented by an integral over the surface enclosing an otherwise arbitrary volume V containing P. The right-hand side of the formula can be interpreted as the combined effect of spherical waves which start at the surface elements of the *boundary* and interfere at P to produce $E(P)$. This prefigures in a certain sense the Huygen's principle.

The choice of V is arbitrary, but in normal cases it is determined by the experimental arrangement. The investigation may, e.g., refer concretely to the diffraction of light by a small aperture in an otherwise opaque screen (Fig. 4.57):

$$\boldsymbol{\partial V}\textbf{: screen } \partial\widehat{V} \textbf{ plus aperture } \boldsymbol{\sigma}$$

The Kirchhoff's formula needs E and ∇E on $\partial\widehat{V}$ and σ. But this information is normally not available. In general one therefore uses the following **Kirchhoff approximation**, which is excellently confirmed by experience:

$$\begin{aligned} &\text{(a) } E, \nabla E \equiv 0 \textit{ on } \partial\widehat{V} \\ &\text{(b) } E, \nabla E \textit{ on } \sigma, \textit{ as if the screen were not present.} \end{aligned} \tag{4.306}$$

Let us use this approach now to calculate some special cases.

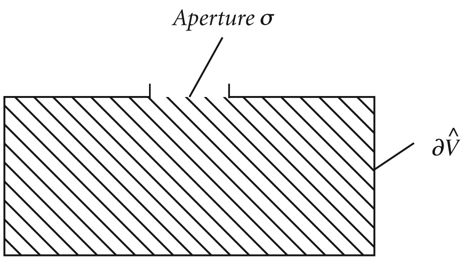


Fig. 4.57 Screen with an aperture to be used for the surface integral in (4.305)

4.3.13 Diffraction by a Screen with a Small Aperture

We presume a *point-like source of light Q as the primary stimulation* (Fig. 4.58). It generates on σ the field E:

$$E = A\frac{e^{ikr_0}}{r_0} \,. \tag{4.307}$$

This means:

$$\nabla_r E = \nabla_{|\mathbf{r}-\mathbf{x}|} E = \frac{d}{dr_0}\left(A\frac{e^{ikr_0}}{r_0}\right)\mathbf{e}_0 = A\left(-\frac{1}{r_0^2}+\frac{ik}{r_0}\right)e^{ikr_0}\mathbf{e}_0\,;\quad \mathbf{e}_0 = \frac{\mathbf{r}_0}{r_0}\,. \tag{4.308}$$

Insertion into the Kirchhoff's formula yields:

$$E(P) = \frac{A}{4\pi}\int\limits_{\sigma}\frac{e^{ik(r+r_0)}}{rr_0}\left\{\left(-\frac{1}{r_0}+ik\right)\cos(\mathbf{n},\mathbf{e}_0)-\left(-\frac{1}{r}+ik\right)\cos(\mathbf{n},\mathbf{e}_\mathrm{r})\right\}df\,. \tag{4.309}$$

Here we have already exploited the Kirchhoff approximation. In practically all cases of interest, it can be presumed

$$r, r_0 \gg \lambda \iff r^{-1}, r_0^{-1} \ll k = \frac{2\pi}{\lambda}$$

so that the imaginary part in the bracket in (4.309) will be dominant:

$$E(P) \approx \frac{i}{2\lambda}A\int\limits_{\sigma}\frac{e^{ik(r+r_0)}}{rr_0}\{\cos(\mathbf{n},\mathbf{e}_0)-\cos(\mathbf{n},\mathbf{e}_\mathrm{r})\}\,df\,. \tag{4.310}$$

According to this already a bit simplified formula, the light wave, which impinges on the aperture σ, propagates as if from each element df a spherical wave $\frac{e^{ikr}}{r}$ emanated, whose amplitude and phase are given by the impinging wave.

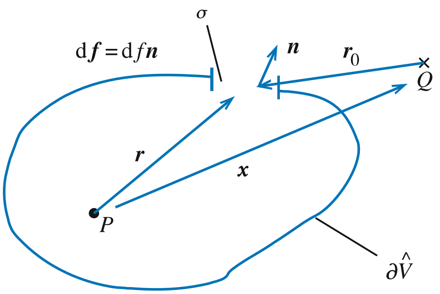


Fig. 4.58 Screen $\partial\widehat{V}$ and a small aperture σ in front of a point-like source of light Q

Let us agree upon a further simplification. Since

$$r, r_0 \gg \text{ linear dimension of the aperture } \sigma$$

it is obvious that

$$\frac{\cos(\mathbf{n}, \mathbf{e}_0) - \cos(\mathbf{n}, \mathbf{e}_r)}{r r_0}$$

will only slightly change over the parameters of σ. Hence, it should be allowed to replace the (variable) vectors $\mathbf{r}$ and $\mathbf{r}_0$ by the fixed vectors $\mathbf{R}$ and $\mathbf{R}_0$ if the latter point to P and Q, respectively, from any common characteristic point of σ, e.g. from the *'midpoint'* of the aperture (Fig. 4.59):

$$E(P) \approx \frac{i}{2\lambda} A \frac{\cos(\mathbf{n}, \mathbf{R}_0) - \cos(\mathbf{n}, \mathbf{R})}{R R_0} \int_\sigma e^{ik(r+r_0)} df \, . \tag{4.311}$$

This expression is symmetric in the source of light Q and the observer P. An interchange of both changes only the sign which has no influence on the intensity. One speaks of *'reciprocity'*!

The remaining task now is to calculate the **surface integral** in (4.311). We think of

a plane screen with a small aperture

Let the screen lie in the xy-plane. We choose the origin of coordinates within the aperture σ. For any point from σ it then holds:

$$\mathbf{r}' = (x', y', z' = 0) \, . \tag{4.312}$$

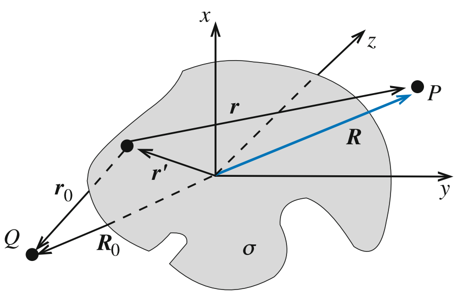


Fig. 4.59 Schematic representation of a plane screen with a small aperture σ for an illustration of the surface integral in (4.311). The directions of $\mathbf{R}$ and $\mathbf{R}_0$ are just opposite to those in Fig. 4.58 which merely leads to $E(P) \to -E(P)$ which, however, is meaningless for the intensities of actual interest

With

$$\mathbf{R} = (X, Y, Z) \; ; \quad \mathbf{R}_0 = (X_0, Y_0, Z_0) \tag{4.313}$$

and

$$\mathbf{r} = \mathbf{R} - \mathbf{r}' \; ; \quad \mathbf{r}_0 = \mathbf{R}_0 - \mathbf{r}' \tag{4.314}$$

it then follows:

$$r^2 = (X - x')^2 + (Y - y')^2 + Z^2 \tag{4.315}$$

$$r_0^2 = (X_0 - x')^2 + (Y_0 - y')^2 + Z_0^2 \; . \tag{4.316}$$

Because of the large distances $r, r_0, R, R_0 \gg r'$ we can terminate the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}\left(x^3\right) \tag{4.317}$$

after the first few terms in order to approximate:

$$\begin{aligned} r &= \left\{X^2 + Y^2 + Z^2 - 2Xx' - 2Yy' + x'^2 + y'^2\right\}^{1/2} \\ &= R\left\{1 - 2\frac{Xx'}{R^2} - 2\frac{Yy'}{R^2} + \frac{r'^2}{R^2}\right\}^{1/2} \\ &= R\left\{1 + \frac{1}{2}\left(-2\frac{Xx'}{R^2} - 2\frac{Yy'}{R^2} + \frac{r'^2}{R^2}\right) - \frac{1}{8}\left(-2\frac{Xx'}{R^2} - 2\frac{Yy'}{R^2} + \frac{r'^2}{R^2}\right)^2 + \ldots\right\} \\ &= R - \left(\frac{Xx'}{R} + \frac{Yy'}{R}\right) + \frac{1}{2}\frac{r'^2}{R} - \frac{1}{2R^3}\left(Xx' + Yy'\right)^2 + \mathcal{O}\left(\frac{r'^3}{R^2}\right) \; . \end{aligned} \tag{4.318}$$

Note that X and Y can be of the same order of magnitude as R, i.e. the fourth summand is of the order r'^2/R. Analogously one finds:

$$r_0 = R_0 - \frac{1}{R_0}\left(X_0 x' + Y_0 y'\right) + \frac{1}{2}\frac{r'^2}{R_0} - \frac{1}{2R_0^3}\left(X_0 x' + Y_0 y'\right)^2 + \mathcal{O}\left(\frac{r'^3}{R_0^2}\right) \; . \tag{4.319}$$

Finally we have found:

$$r + r_0 \approx R + R_0 + \varphi(x', y') \; . \tag{4.320}$$

$\varphi(x',y')$ is decisive for the phase:

$$\begin{aligned}\varphi(x',y') = &-x'\left(\frac{X}{R}+\frac{X_0}{R_0}\right)-y'\left(\frac{Y}{R}+\frac{Y_0}{R_0}\right)\\ &+\frac{1}{2}\left(x'^2+y'^2\right)\left(\frac{1}{R}+\frac{1}{R_0}\right)\\ &-\frac{1}{2R^3}\left(Xx'+Yy'\right)^2-\frac{1}{2R_0^3}\left(X_0x'+Y_0y'\right)^2 \ .\end{aligned} \tag{4.321}$$

The final result is the following version of the Kirchhoff's formula:

$$E(P) \approx \frac{i}{2\lambda}A\frac{\cos(\mathbf{n},\mathbf{R}_0)-\cos(\mathbf{n},\mathbf{R})}{R\cdot R_0}\,e^{ik(R+R_0)}\int\limits_{\sigma} df'\,e^{ik\varphi(x',y')} \ . \tag{4.322}$$

This formula, which will be later evaluated for special geometries, can serve for a reasonable classification of the diffraction phenomena:

(a) **Fraunhofer diffraction**
is given when the quadratic terms in $\varphi(x',y')$ can be neglected, i.e. when the diffracting aperture is penetrated by practically parallel rays:

$$R \to \infty\ ; \quad R_0 \to \infty\ . \tag{4.323}$$

That can be realized by putting the source of light at the focus of a lens.

(b) **Fresnel diffraction**
appears when at least one of the two distances R and R_0 is so small that terms in $\varphi(x',y')$, which are of the order r'^2/R and r'^2/R_0, respectively, cannot be neglected.

Let the diffraction-arrangements ∂V_1 and ∂V_2 be **complementary to each other**:

$$\partial\widehat{V}_1 \iff \sigma_2 \qquad \partial\widehat{V}_2 \iff \sigma_1\ . \tag{4.324}$$

That means, what is in the one arrangement the *screen* $\partial\widehat{V}_{1,2}$ represents in the other arrangement just the *aperture* $\sigma_{2,1}$. For such a scenario the **Babinet's principle** holds:

$$E_1(P)+E_2(P) = E_0(P)\ . \tag{4.325}$$

$E_0(P)$ refers to the undisturbed primary illumination at the point P in absence of the diffraction-screens; $E_1(P)$ and $E_2(P)$ are the corresponding field amplitudes for the two diffraction arrangements. The proof succeeds directly with the Kirchhoff's formula (4.305) by the use of the Kirchhoff approximation (4.306):

$$E_1(P) + E_2(P) = \frac{1}{4\pi}\left(\int\limits_{\sigma_1} d\mathbf{f} + \int\limits_{\sigma_2} d\mathbf{f}\right)\left(\frac{e^{ikr}}{r}\nabla E - E\nabla\frac{e^{ikr}}{r}\right) . \tag{4.326}$$

Note that in the Kirchhoff's formula P represents the origin of coordinates. With

$$\int\limits_{\sigma_1} d\mathbf{f} \ldots + \int\limits_{\sigma_2} d\mathbf{f} \ldots = \int\limits_{\partial V} d\mathbf{f} \ldots \tag{4.327}$$

and the Kirchhoff's formula the assertion (4.325) is confirmed.

4.3.14 Diffraction by the Circular Disc; Poisson Spot

We discuss an example of **Fresnel diffraction**:

object of diffraction $\partial\widehat{V}$ (*screen*) $\cong$ circular disc with radius a in the xy-plane

'aperture' σ $\cong$ the total xy-plane without the circular disc: $a < x^2 + y^2 < \infty$

We agree upon a special arrangement which does not essentially impair the physical information, but which strongly simplifies the mathematical evaluation:

The source of the light Q and the point of observation P lie on the vertical center line of the circular disc, and at equal distances from the disc (Fig. 4.60)!

$$\begin{aligned}
\alpha &= \sphericalangle(\mathbf{n}, \mathbf{r}) \\
\beta &= \sphericalangle(\mathbf{n}, \mathbf{r}_0) \\
\rho &= \rho_0 \\
r &= r_0 \ .
\end{aligned}$$

For symmetry reasons it must be $\gamma = \alpha$ (Fig. 4.60) and therewith:

$$\sphericalangle(\mathbf{n}, \mathbf{r}_0) = \pi - \sphericalangle(\mathbf{n}, \mathbf{r}) \ . \tag{4.328}$$

We use (4.310) to get the field $E(P)$ at the point P:

$$E(P) \simeq \frac{i}{2\lambda} A \int\limits_{\sigma} \frac{e^{ik(r+r_0)}}{rr_0} \left(\cos(\mathbf{n}, \mathbf{r}_0) - \cos(\mathbf{n}, \mathbf{r})\right) df \ . \tag{4.329}$$

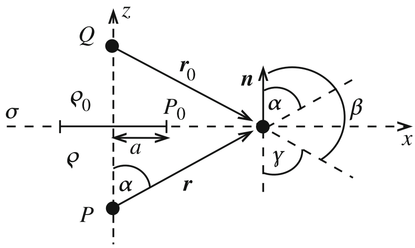


Fig. 4.60 Geometric relations for the determination of the diffraction by a small circular disc (radius a)

Because of $\cos(\mathbf{n}, \mathbf{r}_0) = -\cos(\mathbf{n}, \mathbf{r})$ this relation simplifies to:

$$E(P) = -\frac{i}{\lambda} A \int_\sigma \frac{e^{2ikr}}{r^2} \cos(\mathbf{n}, \mathbf{r}) df \; . \tag{4.330}$$

We calculate the surface integral:

$$df = 2\pi x dx \; . \tag{4.331}$$

With $r^2 = \rho^2 + x^2$ in addition it follows:

$$\frac{d}{dx} r^2 = 2x = 2r \frac{dr}{dx} \iff xdx = rdr \tag{4.332}$$

$$df = 2\pi r dr \tag{4.333}$$

$$\cos(\mathbf{n}, \mathbf{r}) = \rho / r \; . \tag{4.334}$$

With $2\pi/\lambda = k$ it then remains to be evaluated:

$$E(P) = -iA\rho k \int_{\sqrt{\rho^2+a^2}}^{\infty} dr \frac{e^{2ikr}}{r^2} \; . \tag{4.335}$$

Further evaluation uses integration by parts:

$$\begin{aligned} \int_{\sqrt{\rho^2+a^2}}^{\infty} dr \frac{e^{2ikr}}{r^2} &= \frac{1}{2ik} \left(\frac{e^{2ikr}}{r^2} \Bigg|_{\sqrt{\rho^2+a^2}}^{\infty} + 2 \int_{\sqrt{\rho^2+a^2}}^{\infty} \frac{e^{2ikr}}{r^3} dr \right) \\ &= \frac{1}{2ik} \left(\frac{e^{2ikr}}{r^2} \Bigg|_{\sqrt{\rho^2+a^2}}^{\infty} + \frac{1}{ik} \frac{e^{2ikr}}{r^3} \Bigg|_{\sqrt{\rho^2+a^2}}^{\infty} + \frac{3}{ik} \int_{\sqrt{\rho^2+a^2}}^{\infty} \frac{e^{2ikr}}{r^4} dr \right) \; . \end{aligned} \tag{4.336}$$

The expansion can be continued in this way. We now use the presumption

$$rk \gg 1 \,, \tag{4.337}$$

which led to the initial formula (4.310) for $E(P)$. It is possibly a bit problematic at the edge of the disc:

$$\int\limits_{\sqrt{\rho^2+a^2}}^{\infty} dr \frac{e^{2ikr}}{r^2} \simeq \frac{1}{2ik} \frac{e^{2ikr}}{r^2}\bigg|_{\sqrt{\rho^2+a^2}}^{\infty} \,. \tag{4.338}$$

The field strength $E(P)$ therewith reads:

$$E(P) \simeq \frac{\rho}{2} A \frac{e^{2ik\sqrt{\rho^2+a^2}}}{\rho^2 + a^2} \,. \tag{4.339}$$

The primary wave coming from Q has as spherical wave on the edge of the disc (at P_0) the form:

$$E(P_0) = A \frac{e^{ik\sqrt{\rho^2+a^2}}}{\sqrt{\rho^2 + a^2}} \,. \tag{4.340}$$

This can be used in the expression for $E(P)$ (4.339):

$$E(P) \simeq \frac{\rho}{2} \frac{e^{ik\sqrt{\rho^2+a^2}}}{\sqrt{\rho^2 + a^2}} E(P_0) \,. \tag{4.341}$$

The square of the absolute value provides the corresponding light intensity:

$$\frac{I(P)}{I(P_0)} = \frac{|E(P)|^2}{|E(P_0)|^2} = \frac{1}{4} \frac{\rho^2}{\rho^2 + a^2} \,. \tag{4.342}$$

This is a remarkable result for the **light intensity on the vertical center line** behind an opaque circular disc. Except for the immediate space behind the disc ($\rho \approx 0$) one observes on the center line, i.e. in the geometrical shadow region, always brightness (Fig. 4.61)! This phenomenon is called the

Poisson spot

It becomes even more astonishing when we discuss in the next section the complementary diffraction arrangement.

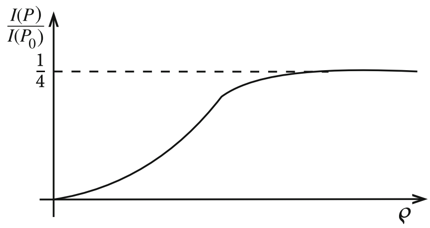


Fig. 4.61 Relative light intensity on the center line behind an opaque circular disc as function of the distance ρ

4.3.15 Diffraction by a Circular Aperture

Starting point now is a circular opening of radius a, and therewith, the *complementary* arrangement to that of the last section. Essentially, only the integration limits of the $E(P)$-integral (4.335) are actually to be changed:

$$E(P) = -iA\rho k \int\limits_{\rho}^{\sqrt{\rho^2+a^2}} dr \frac{e^{2ikr}}{r^2} \ . \tag{4.343}$$

The limits of integration follow from:

$$x = 0 \iff r = \rho \ ; \quad x = a \iff r = \sqrt{\rho^2 + a^2}$$

The same consideration as in the last section leads to:

$$E(P) \approx -\frac{\rho}{2} A \frac{e^{2ikr}}{r^2}\bigg|_{\rho}^{\sqrt{\rho^2+a^2}} \ . \tag{4.344}$$

Let us further estimate this expression a bit:

$$\frac{e^{2ikr}}{r^2}\bigg|_{\rho}^{\sqrt{\rho^2+a^2}} = \frac{e^{2ik\sqrt{\rho^2+a^2}}}{\rho^2 + a^2} - \frac{e^{2ik\rho}}{\rho^2} \tag{4.345}$$

$$= \frac{e^{2ik\rho}}{\rho^2}\left(\frac{\rho^2}{\rho^2 + a^2}\, e^{2ik\left(\sqrt{\rho^2+a^2}-\rho\right)} - 1\right) \ . \tag{4.346}$$

We assume

$$\rho^2 \gg a^2$$

which allows for the following simplifications

$$\frac{\rho^2}{\rho^2 + a^2} \longrightarrow 1 \tag{4.347}$$

$$\begin{aligned}\sqrt{\rho^2 + a^2} - \rho \;\longrightarrow\; & \rho\left(\sqrt{1 + a^2/\rho^2} - 1\right) \\ & \simeq \rho\left(1 + \frac{1}{2}a^2/\rho^2 - 1\right) = a^2/2\rho\,.\end{aligned} \tag{4.348}$$

It remains therewith for the field $E(P)$:

$$E(P) \simeq -\frac{A}{2\rho}\, e^{2ik\rho}\, e^{ik\frac{a^2}{2\rho}} \left(2i \sin\left(k\frac{a^2}{2\rho}\right)\right)\,. \tag{4.349}$$

We put this again in relation to the primary spherical wave on the edge of the disc at P_0,

$$E(P_0) = A\frac{e^{ik\sqrt{\rho^2+a^2}}}{\sqrt{\rho^2 + a^2}} \simeq \frac{A}{\rho}\, e^{ik(\rho + a^2/2\rho)}\,, \tag{4.350}$$

so that we can write:

$$E(P) \simeq -\,e^{ik\rho}\left(i \sin\left(k\frac{a^2}{2\rho}\right)\right) E(P_0)\,. \tag{4.351}$$

This gives us the intensity:

$$\frac{I(P)}{I(P_0)} = \frac{|E(P)|^2}{|E(P_0)|^2} = \sin^2\left(k\frac{a^2}{2\rho}\right) \tag{4.352}$$

Hence, as function of the distance ρ an infinite number of intensity maxima ($I(P) = I(P_0)$) and minima ($I(P) = 0$, destructive interference) appear, which tend to merge towards the aperture ($\rho \to 0$) (Fig. 4.62). Our above estimation, though, becomes questionable in this limit!

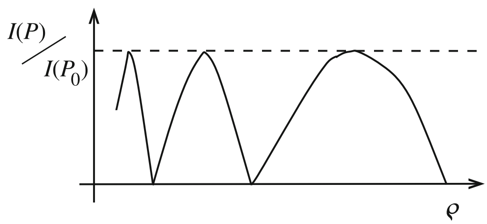


Fig. 4.62 Relative light intensity on the center line behind a circular aperture as function of the distance ρ

Note that we have found the in principle *'paradoxical result'* that for the circular disc (Sect. 4.3.14) it is **never** dark on the center line, while for the complementary circular aperture infinite dark spots appear as function of the vertical distance.

At the end, let us still check our results with respect to the *Babinet's principle* (4.325) which brings into relation the fields of the complementary diffraction arrangements treated in this and the preceding section, respectively.

$$E_1(P) + E_2(P) = -iA\rho k \int_\rho^\infty dr \frac{e^{2ikr}}{r^2} \,. \tag{4.353}$$

If there were no diffracting object then the field at P produced by the light source in Q should have, because of $\overline{QP} = 2\rho$, the following form:

$$E_0(P) = A\frac{e^{2ik\rho}}{2\rho} \,. \tag{4.354}$$

According to the Babinet's principle it should therefore be valid:

$$\frac{e^{2ik\rho}}{2\rho^2} \stackrel{!}{=} -ik \int_\rho^\infty dr \frac{e^{2ikr}}{r^2} \tag{4.355}$$

what can directly be shown with the estimations (4.338) and (4.344). The exact agreement, though, is more or less accidental. We further check (4.355) by differentiation with respect to ρ:

$$\left(-\frac{1}{\rho^3} + \frac{ik}{\rho^2}\right) e^{2ik\rho} \stackrel{!}{=} +ik\frac{e^{2ik\rho}}{\rho^2} \,. \tag{4.356}$$

In the interesting limit $k\rho \gg 1$, the first term in the bracket on the left-hand side can be neglected, whereby the equivalence is proven. The exact equality, however, is hindered by the various estimations we have exploited on the way to the final expression.

4.3.16 Diffraction by the Crystal Lattice

In this section we will discuss an important example of application from solid state physics. To do so we have to first compile some elementary terms which may be familiar, however, from basic lectures on experimental physics.

A crystal lattice consists of regularly and periodically arranged atoms (molecules), whose positions are defined by *'lattice vectors'*

$$\mathbf{R_n} = \sum_{i=1}^{3} n_i \mathbf{a}_i \; . \tag{4.357}$$

The three non-coplanar vectors $\mathbf{a}_i$ $(i = 1, 2, 3)$ are called *'primitive translations'*. With

$$\mathbf{n} = (n_1, n_2, n_3) \; ; \quad n_i \in \mathbb{Z}$$

they span the total (infinite, Bravais-) lattice. Since there is no restriction for the numbers n_i the (mathematical) Bravais lattice is thought to be infinitely large (no surface!). It can easily be seen that the choice of the primitive translations in general will not be unique. The two-dimensional lattice in Fig. 4.63 can be built up according to (4.357) by $(\mathbf{a}_1, \mathbf{a}_2)$, but also by $(\mathbf{a}_1, \mathbf{a}_3)$ or $(\mathbf{a}_2, \mathbf{a}_3)$. However, this ambiguity in the description of a crystalline solid will not have any effect on the following.

Many physical properties are better representable in the so-called *'reciprocal (dual) lattice'* than in the 'real' (direct) lattice. The corresponding *'reciprocal lattice vector'* has a similar structure as the 'real' lattice vector in Eq. (4.357):

$$\mathbf{K_p} = \sum_{i=1}^{3} p_i \mathbf{b}_i \; ; \quad \mathbf{p} = (p_1, p_2, p_3) \; ; \quad p_i \in \mathbb{Z} \; . \tag{4.358}$$

The *'primitive translations of the reciprocal lattice'* $\mathbf{b}_i$ are closely related to those of the real lattice and are defined by

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij} \quad \forall \, i, j \; . \tag{4.359}$$

We still need for the following the concept of the *'atomic lattice plane'*. By this one understands any plane in the crystal which is occupied by at least one lattice point. According to this rather general definition there are obviously infinitely many

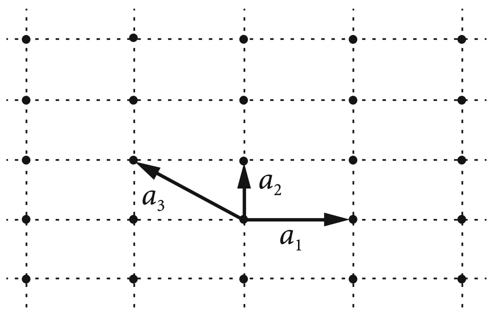


Fig. 4.63 Example of a two-dimensional Bravais-lattice

different lattice planes in a Bravais lattice. To a given lattice plane, e.g., innumerable *parallel* lattice planes exist. Together they build a *'family of lattice planes'*. The orientation of a lattice plane (family of lattice planes) is described by the so-called

Miller indexes (h, k, l) .

For this purpose one fixes the intersection points,

$$x_i\,\mathbf{a}_i \quad (i = 1, 2, 3)\ ,$$

of the considered plane with the axes defined by the primitive translations $\mathbf{a}_i$. Via

$$x_1^{-1} : x_2^{-1} : x_3^{-1} = h : k : l \tag{4.360}$$

one determines a triple of relatively prime (!) integers and speaks then of the

(h, k, l)**-plane**

of the crystal.

Assertion

The reciprocal lattice vector

$$\mathbf{K_p} = \sum_{j=1}^{3} p_j \mathbf{b}_j$$

is perpendicular to the (p_1, p_2, p_3)-plane of the direct lattice.

Proof The axis-intercepts of the (p_1, p_2, p_3)-plane in the real lattice (Fig. 4.64) are

$$\mathbf{A}_i = \frac{c}{p_i}\,\mathbf{a}_i \quad i = 1, 2, 3\ .$$

An arbitrary vector in the (p_1, p_2, p_3)-plane can then be represented as (Fig. 4.64):

$$\mathbf{R(p)} = \gamma_1(\mathbf{A}_1 - \mathbf{A}_2) + \gamma_2(\mathbf{A}_3 - \mathbf{A}_2).$$

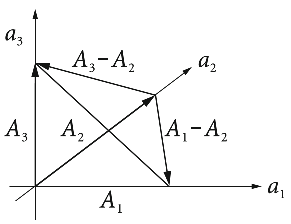


Fig. 4.64 Area element of the (p_1, p_2, p_3)-plane

That means:

$$\begin{aligned}
\mathbf{K_p}\cdot\mathbf{R}(\mathbf{p}) &= \sum_{j=1}^{3} p_j\mathbf{b}_j\left(\gamma_1\left(\frac{c}{p_1}\mathbf{a}_1-\frac{c}{p_2}\mathbf{a}_2\right)+\gamma_2\left(\frac{c}{p_3}\mathbf{a}_3-\frac{c}{p_2}\mathbf{a}_2\right)\right)\\
&= \sum_{j=1}^{3} 2\pi p_j\left(\gamma_1\left(\frac{c}{p_1}\delta_{j1}-\frac{c}{p_2}\delta_{j2}\right)+\gamma_2\left(\frac{c}{p_3}\delta_{j3}-\frac{c}{p_2}\delta_{j2}\right)\right)\\
&= 2\pi\left(\gamma_1(c-c)+\gamma_2(c-c)\right)\\
&= 0\,.
\end{aligned}$$

That was to be proven.

Assertion

The distance between adjacent (p_1, p_2, p_3)-planes is given by:

$$d(p_1,p_2,p_3)=\frac{2\pi}{\left|\mathbf{K_p}\right|} \tag{4.361}$$

Proof All lattice vectors whose projections on the $\mathbf{K_p}$-direction,

$$\mathbf{e}_K=\frac{\mathbf{K_p}}{\left|\mathbf{K_p}\right|}\perp(p_1,p_2,p_3)\text{-plane}\,,$$

amount to Δ define the (p_1, p_2, p_3)-plane at a distance Δ from the given origin of coordinates:

$$\Delta=\mathbf{e}_K\cdot\mathbf{R_n}=\frac{2\pi}{\left|\mathbf{K_p}\right|}\underbrace{(p_1n_1+p_2n_2+p_3n_3)}_{\min\mathbb{Z}}\equiv\Delta_m=\frac{2\pi}{\left|\mathbf{K_p}\right|}\,m\,.$$

We have therewith, as asserted, as distance between next adjacent planes:

$$\Delta_{m+1}-\Delta_m=\frac{2\pi}{\left|\mathbf{K_p}\right|}=d(p_1,p_2,p_3)\,.$$

Strictly speaking, however, it is of course still to be shown that to a given m there do exist three integers n_1, n_2, n_3 such that $p_1n_1 + p_2n_2 + p_3n_3 = m + 1$. The proof is left to the reader:

After these preparations we now consider a plane wave which impinges on a crystal lattice and is then scattered by all the lattice atoms. Let the plane wave have the amplitude

$$A(\mathbf{r},t)=A_0\;e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,,$$

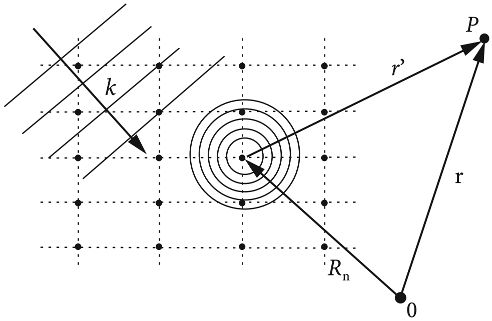


Fig. 4.65 Diffraction at a crystal lattice

where the time-dependence does not play any role here and is therefore ignored further. The amplitude at the respective lattice points then is:

$$A(\mathbf{R_n}) = A_0 \; e^{i\mathbf{k}\cdot\mathbf{R_n}} \; .$$

According to the Huygens's principle (Sect. 4.3.11) each lattice atom, which is reached by the plane wave, becomes the point of origin of a spherical wave (Fig. 4.65). This wave has at the point of observation P the amplitude

$$\left(A_0 \; e^{i\mathbf{k}\cdot\mathbf{R_n}}\right) \frac{e^{ikr'}}{r'} \; . \tag{4.362}$$

Thereby an elastic scattering is assumed (no absorption, ...):

$$|\mathbf{k}'| = |\mathbf{k}| = k \; .$$

We are interested in the intensity $I \propto |A|^2$ at the point of observation P. The position vector $\mathbf{r}$ determines, with respect to the origin of coordinates, the direction of the scattered wave:

$$\mathbf{k}' = k \, \frac{\mathbf{r}}{r} \; . \tag{4.363}$$

We assume that P is far outside the crystal while the origin of coordinates lies inside the crystal. Then we can use $r \gg R_\mathbf{n}$ in order to replace in the denominator of the expression (4.362) to a good approximation r' by r. Because of the oscillations of the exponential function, though, in the numerator one has to act more carefully:

$$\begin{aligned} r' &= \sqrt{(\mathbf{r}-\mathbf{R_n})^2} \\ &= \sqrt{r^2 + R_{\mathbf{n}}^2 - 2rR_{\mathbf{n}}\cos(\mathbf{r}, \mathbf{R_n})} \\ &= r\sqrt{1 + \left(\frac{R_{\mathbf{n}}}{r}\right)^2 - 2\frac{R_{\mathbf{n}}}{r}\cos(\mathbf{r}, \mathbf{R_n})} \\ &\approx r\left(1 - \frac{1}{2}2\frac{R_{\mathbf{n}}}{r}\cos(\mathbf{r}, \mathbf{R_n})\right) \\ &= r - R_{\mathbf{n}}\cos(\mathbf{r}, \mathbf{R_n})\ . \end{aligned}$$

This means:

$$k\,r' \approx k\,r - \mathbf{k}' \cdot \mathbf{R_n}\ . \tag{4.364}$$

The spherical wave starting at a lattice atom thus has at P the amplitude:

$$\left(A_0\, e^{i\mathbf{k}\cdot\mathbf{R_n}}\right)\frac{e^{ikr'}}{r'} \approx \left(A_0\, e^{i\mathbf{k}\cdot\mathbf{R_n}}\right)\frac{e^{ikr - i\mathbf{k}'\cdot\mathbf{R_n}}}{r} = A_0\frac{e^{ikr}}{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R_n}}\ . \tag{4.365}$$

The full amplitude results from the superposition of the contributions of all the lattice points, where the crystal consists in $\mathbf{a}_j$-direction by, say, N_j ($j = 1, 2, 3$) lattice sites:

$$\begin{aligned} A &= A_0\frac{e^{ikr}}{r}\prod_{j=1}^{3}\left\{\sum_{n_j=-N_j}^{N_j-1} e^{i(\mathbf{k}-\mathbf{k}')\cdot n_j\mathbf{a}_j}\right\} \\ &= A_0\frac{e^{ikr}}{r}\prod_{j=1}^{3}\left\{e^{-iN_j(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j}\sum_{n_j=0}^{2N_j-1} e^{i(\mathbf{k}-\mathbf{k}')\cdot n_j\mathbf{a}_j}\right\} \\ &= A_0\frac{e^{ikr}}{r}\prod_{j=1}^{3} e^{-iN_j(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j}\,\frac{e^{i2N_j(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j} - 1}{e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j} - 1} \\ &= A_0\frac{e^{ikr}}{r}\prod_{j=1}^{3}\frac{1}{e^{\frac{i}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j}}\,\frac{\sin\left(N_j(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j\right)}{\sin\left(\frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j\right)}\ . \end{aligned} \tag{4.366}$$

For the required intensity we need the square of the absolute value of the amplitude:

$$I(\mathbf{r}) = I_0 \frac{1}{r^2} \prod_{j=1}^{3} \left| \frac{\sin\left(N_j(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j\right)}{\sin\left(\frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j\right)} \right|^2 . \tag{4.367}$$

I_0 is the intensity of the incident wave.

The conditions for the maxima of intensity can be read off already from Eq. (4.366). Maxima appear obviously when all phase factors in the curly bracket of the first equation line are equal to 1. That, however, is exactly the case when the so-called

$$\textbf{Laue equations:} \qquad \mathbf{a}_j \cdot (\mathbf{k}-\mathbf{k}') = 2\pi z_j \; ; \quad z_j \in \mathbb{Z} \; . \tag{4.368}$$

are fulfilled. Because of

$$\lim_{x\to z\pi} \frac{\sin^2(Nx)}{\sin^2 x} = N^2$$

($x = \frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}_j$; $N = 2N_j$) it holds in this case for the intensity:

$$I_{\max}(\mathbf{r}) = I_0 \frac{1}{r^2} \prod_{j=1}^{3} (2N_j)^2 \; . \tag{4.369}$$

This is also directly readable on (4.366) if one puts the phase factors, as described, equal to 1:

The wave-number difference of the incident and the diffracted wave appears to be decisive for the intensity distribution:

$$\mathbf{K} = \mathbf{k} - \mathbf{k}' \implies \mathbf{a}_j \cdot \mathbf{K} = 2\pi z_j \; ; \quad z_j \in \mathbb{Z} \; . \tag{4.370}$$

One recognizes, with the definition equation (4.359) for the primitive translations of the reciprocal lattice, that $\mathbf{K}$ must be a **reciprocal lattice vector**:

$$\mathbf{K} = \sum_{j=1}^{3} z_j \mathbf{b}_j \; . \tag{4.371}$$

Let us try to find a physical interpretation of the Laue equations. Since the scattering is thought to be elastic it must hold:

$$k = k' = |\mathbf{k} - \mathbf{K}| \;\Leftrightarrow\; k^2 = k^2 - 2\mathbf{k}\cdot\mathbf{K} + K^2$$

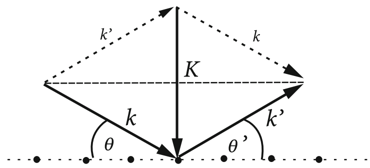


Fig. 4.66 Reflection at a net plane under consideration of the Laue equations

or with the unit-vector in **K**-direction $\mathbf{e}_K = \mathbf{K}/K$:

$$\mathbf{k} \cdot \mathbf{e}_K = \frac{1}{2} K \ . \tag{4.372}$$

Hence, the Laue equations are exactly fulfilled as soon as the projection of the incoming wave vector **k** on the direction of a reciprocal lattice vector is equal to the half of the length of this reciprocal lattice vector (Fig. 4.66). Such wave vectors define in the reciprocal lattice the so-called **'Bragg plane'**.

We draw out of the integral numbers z_j in Eqs. (4.368) and (4.370), respectively, the greatest common divisor $z_0 \in \mathbb{Z}$ obtaining then relatively prime integers p_1, p_2, p_3:

$$z_j = z_0 p_j \ ; \quad j = 1, 2, 3 \ .$$

That fixes the reciprocal lattice vector

$$\mathbf{K}_\mathbf{p} = \frac{1}{z_0} \mathbf{K} \stackrel{(4.371)}{=} \sum_{j=1}^{3} p_j \mathbf{b}_j$$

As proven above, $\mathbf{K}_\mathbf{p}$ and therewith also **K** stand perpendicular to the lattice plane with the Miller indexes p_1, p_2, p_3. The situation is represented in Fig. 4.66. Because of $k = k'$ one has obviously to conclude $\vartheta = \vartheta'$ which corresponds to the law of reflection (4.256). **k** and $\mathbf{k}'$ enclose the same angle with the Bragg plane. That means:

$$K = 2k \sin\vartheta = z_0 |\mathbf{K}_\mathbf{p}| = 2\pi \frac{z_0}{d} \ .$$

Here we have brought into play according to (4.361) the (p_1, p_2, p_3)-inter plane distance. Because of $k = 2\pi/\lambda$ the Laue equations are therewith equivalent to the so-called **'Bragg law'** (Fig. 4.67)

$$2d(p_1, p_2, p_3) \sin\vartheta = z_0 \lambda \ . \tag{4.373}$$

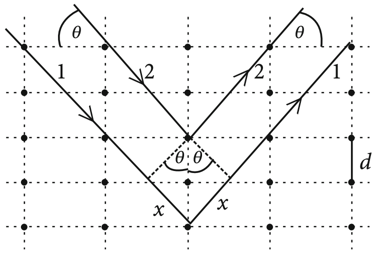


Fig. 4.67 Bragg reflection at a lattice-plane family

This can be seen as criterion for constructive interference of rays reflected at parallel lattice planes. The optical path difference of ray 1 and ray 2 in Fig. 4.67 amounts to

$$\Delta = 2x = 2d \sin \vartheta \ .$$

To lead to intensity maxima at constructive interference this optical path difference has to be an integral multiple of the wavelength of the incident radiation. But that is just the Bragg law (4.373)!

As to the eminently important impact of the Laue equations (4.368) or the Bragg law (4.373) on the investigation of the structure of solids the respective experimental textbook literature should be consulted.

4.3.17 The Transition from Wave Optics to 'Geometrical Optics'

The last sections dealt with typical phenomena of the wave optics, i.e. with electromagnetic waves like the waves of light, the properties of which are derivable in totality from the basic Maxwell equations of electrodynamics. The wave optics as superordinate theory possesses the limiting case of the *'geometrical optics'* which is dominated by the concept of *'light rays'* and leads under certain conditions to *'more illustrative'* and therewith to more easily interpretable results than the full theory. We therefore address the question: When does the wave nature of the light no longer play an essential role? When is the concept of the *'light rays'* useful and valid? How does one actually come from the wave optics to the geometrical optics?

It should be mentioned that a part of the following considerations were already used in another context in Sect. 3.6.2 of Vol. 2 of this course: Theoretical Physics, where we tried to understand, in analogy to the relationship between geometrical optics and wave optics, the classical mechanics as the *'geometrical-optical limiting case'* of the superordinate wave mechanics (quantum theory).

What do we understand now by light **rays**? For plane waves or spherical waves that appears to be clear. One defines

'light rays' as normals to the wave fronts

The latter are given as the solutions of the **scalar wave equation of optics** (4.128)

$$\Delta\psi - \frac{n^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0\,. \tag{4.374}$$

Thereby ψ stands, for instance, for a Cartesian component of the electric field $\mathbf{E}$ or the magnetic induction $\mathbf{B}$. The plane waves as well as the spherical waves have been found as solutions under the assumption of a homogeneous medium, i.e. its index of refraction is position-independent, except for interface regions:

$$n(\mathbf{r}) \equiv n = \text{const}\,.$$

In inhomogeneous media with space-dependent $n(\mathbf{r})$, though, the plane wave can represent still a solution at most 'locally', and one has to ask oneself whether the above 'ray-concept' remains useful. Above all, one has to realize that the validity of the wave equation in the form (4.374) is no longer guaranteed. Instead, we have to start with a position-dependent dielectric constant:

$$\varepsilon_r = \varepsilon_r(\mathbf{r})\;;\;\mu_r \approx 1 \;\curvearrowright\; n = \sqrt{\varepsilon_r\mu_r} \approx \sqrt{\varepsilon_r(\mathbf{r})} = n(\mathbf{r})\,. \tag{4.375}$$

Geometrical optics presumes that the scalar wave equation (4.374) is furthermore valid, at least as a good approximation!

That can of course be the case only under certain preconditions. To find these conditions we recall the derivation of the wave equation. Thereby it was used, inter alia:

$$\text{curl}\,\mathbf{D}(\mathbf{r}) = \varepsilon_0\varepsilon_r\,\text{curl}\,\mathbf{E}(\mathbf{r})\,.$$

That now becomes a bit more complicated:

$$\begin{aligned}
\text{curl}\,\mathbf{D}(\mathbf{r}) &= \text{curl}\,(\varepsilon_0\varepsilon_r(\mathbf{r})\mathbf{E}(\mathbf{r})) \approx \varepsilon_0\text{curl}\,(n^2(\mathbf{r})\mathbf{E}(\mathbf{r}))\\
&= \varepsilon_0\left(n^2(\mathbf{r})\text{curl}\,\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r})\times\nabla n^2(\mathbf{r})\right)\\
&= \varepsilon_0\left(n^2(\mathbf{r})\text{curl}\,\mathbf{E}(\mathbf{r}) - 2n(\mathbf{r})\mathbf{E}(\mathbf{r})\times\nabla n(\mathbf{r})\right)\,.
\end{aligned}$$

To keep the scalar wave equation at least approximatively valid it is to require:

$$|2n(\mathbf{r})\mathbf{E}(\mathbf{r})\times\nabla n(\mathbf{r})| \overset{!}{\ll} |n^2(\mathbf{r})\text{curl}\,\mathbf{E}(\mathbf{r})| \;\Leftrightarrow\; 2|\mathbf{E}(\mathbf{r})\times\nabla n(\mathbf{r})| \ll |n(\mathbf{r})\nabla\times\mathbf{E}(\mathbf{r})|\,.$$

That can be further estimated:

$$2|\mathbf{E}(\mathbf{r})|\,|\nabla n(\mathbf{r})| \ll n(\mathbf{r})|\nabla \times \mathbf{E}(\mathbf{r})| \propto n(\mathbf{r})|\mathbf{k} \times \mathbf{E}(\mathbf{r})| \propto n(\mathbf{r})|\mathbf{k}|\,|\mathbf{E}(\mathbf{r})|$$
$$\Leftrightarrow\ 2|\nabla n(\mathbf{r})| \ll |\mathbf{k}|n(\mathbf{r}) = \frac{2\pi}{\lambda} n(\mathbf{r})\ .$$

That results in the requirement on the

$$\textbf{'limiting case of geometrical optics':} \qquad |\nabla n(\mathbf{r})| \ll \frac{n(\mathbf{r})}{\lambda}\ . \tag{4.376}$$

The index of refraction $n(\mathbf{r})$ thus should not change significantly over a distance of the order of the wave length λ. That's why the geometrical optics is sometimes denoted as the *'$\lambda \to 0$'-limiting case* of the superordinate wave optics. The wave equation (4.374) then remains valid to a good approximation even if $n = n(\mathbf{r})$. That means that

$$\psi(\mathbf{r}, t) = \psi(\mathbf{r})\ e^{-i\omega t}$$

fulfills the equation

$$\Delta\psi(\mathbf{r}) + \frac{n^2(\mathbf{r})\,\omega^2}{c^2}\,\psi(\mathbf{r}) = 0\ ,$$
$$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}: \quad \text{wave number in the vacuum}\ ,$$
$$\Delta\psi(\mathbf{r}) + n^2(\mathbf{r})\,k_0^2\,\psi(\mathbf{r}) = 0\ . \tag{4.377}$$

We try the following ansatz of solution:

$$\psi(\mathbf{r}) = A(\mathbf{r})\ e^{ik_0 L(\mathbf{r})}\ . \tag{4.378}$$

$L(\mathbf{r})$ is denoted as *'eikonal'* (Greek: *'image'*) or *'optical light path'*. The areas $L(\mathbf{r}) =$ const define the areas of constant phase and therewith the *wavefronts*. According to the general presumption of geometrical optics the amplitude $A(\mathbf{r})$ is only 'weakly space-dependent'. With

$$\nabla\psi = (\nabla A + ik_0 A \nabla L)\ e^{ik_0 L(\mathbf{r})}\ ,$$
$$\Delta\psi = \left(\Delta A + ik_0 \Delta L A + 2ik_0 \nabla A \cdot \nabla L - k_0^2 A (\nabla L)^2\right)\ e^{ik_0 L(\mathbf{r})}$$

the wave equation leads to:

$$0 = \Delta A + ik_0 \Delta L A + 2ik_0 \nabla A \cdot \nabla L - k_0^2 A (\nabla L)^2 + n^2 k_0^2 A\ .$$

This equation must be separately valid for the real and imaginary parts:

$$A \Delta L + 2\nabla A \cdot \nabla L = 0\ , \tag{4.379}$$

$$-(\nabla L)^2 + n^2 - \frac{1}{k_0^2}\frac{\Delta A}{A} = 0\ . \tag{4.380}$$

In the case of geometrical optics we can assume:

$$A(\mathbf{r}) \text{ weakly space-dependent}\ ; \quad \lambda_0 \ll \text{changes in the medium}\ .$$

Hence, we can estimate:

$$\left|\frac{\Delta A}{A}\frac{\lambda_0^2}{4\pi^2}\right| \ll n^2\ .$$

That yields with Eq. (4.380) the

$$\textbf{eikonal-equation of geometrical optics:} \qquad (\nabla L(\mathbf{r}))^2 = n^2(\mathbf{r})\ . \tag{4.381}$$

It delivers the transition from the general wave optics (interference, diffraction) to the geometrical optics (rays, particle picture (photons)). One derives $L(\mathbf{r})$ from the eikonal-equation, in order to insert it then into the conditional equation (4.379) for the amplitude $A(\mathbf{r})$ therewith obtaining the complete (approximate) solution.

Where are the **light rays** now? In the last analysis, they are of course fixed by the direction of the energy transport, which on its part is determined by the electric field,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \widehat{\mathbf{E}}_0(\mathbf{r})\ e^{-i\omega t} \\ \widehat{\mathbf{E}}_0(\mathbf{r}) &= \mathbf{E}_0(\mathbf{r})\ e^{ik_0 L(\mathbf{r})}\ , \end{aligned} \tag{4.382}$$

and the magnetic induction,

$$\begin{aligned} \mathbf{B}(\mathbf{r},t) &= \widehat{\mathbf{B}}_0(\mathbf{r})\ e^{-i\omega t} \\ \widehat{\mathbf{B}}_0(\mathbf{r}) &= \mathbf{B}_0(\mathbf{r})\ e^{ik_0 L(\mathbf{r})}\ ,\ . \end{aligned} \tag{4.383}$$

In the range of validity of geometrical optics we can make use of an only weak space-dependence of the amplitudes:

$$\mathbf{E}_0(\mathbf{r}) \approx \mathbf{E}_0\ ; \quad \mathbf{B}_0(\mathbf{r}) \approx \mathbf{B}_0\ . \tag{4.384}$$

The fields are coupled with each other by the Maxwell equations, e.g.:

$$\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}} \curvearrowright \operatorname{curl}\widehat{\mathbf{E}}_0(\mathbf{r}) = i\omega\widehat{\mathbf{B}}_0(\mathbf{r})\ .$$

Because of (4.384) it is approximately:

$$\mathrm{curl}\,\widehat{\mathbf{E}}_0(\mathbf{r}) \approx ik_0(\nabla L(\mathbf{r}) \times \mathbf{E}_0)\; e^{ik_0 L(\mathbf{r})} \; .$$

It follows therewith:

$$\widehat{\mathbf{B}}_0(\mathbf{r}) = \frac{k_0}{\omega}\left(\nabla L(\mathbf{r}) \times \widehat{\mathbf{E}}_0(\mathbf{r})\right) \; .$$

We now apply the eikonal-equation (4.381):

$$\frac{k_0}{\omega} = \frac{1}{c} = \frac{1}{un} = \frac{1}{u}\frac{1}{|\nabla L|}$$

That yields:

$$\widehat{\mathbf{B}}_0(\mathbf{r}) = \frac{1}{u}\left(\mathbf{l}(\mathbf{r}) \times \widehat{\mathbf{E}}_0(\mathbf{r})\right) \; , \tag{4.385}$$

$$\mathbf{l}(\mathbf{r}) = \frac{\nabla L(\mathbf{r})}{|\nabla L(\mathbf{r})|} \; . \tag{4.386}$$

We still exploit another Maxwell equation:

$$\mathrm{div}\,\mathbf{D} = 0 \approx \varepsilon_0\varepsilon_{\mathrm{r}}(\mathbf{r})\mathrm{div}\,\mathbf{E} \;\curvearrowright\; \mathrm{div}\,\mathbf{E} = 0 \;\curvearrowright\; \nabla L(\mathbf{r}) \cdot \widehat{\mathbf{E}}_0(\mathbf{r}) = 0 \; .$$

This means:

$$\mathbf{l}(\mathbf{r}) \cdot \widehat{\mathbf{E}}_0(\mathbf{r}) = 0 \; . \tag{4.387}$$

We realize that in the range of validity of geometrical optics the vectors

$$\widehat{\mathbf{E}}_0(\mathbf{r}) \; , \quad \widehat{\mathbf{B}}_0(\mathbf{r}) \quad \text{and} \quad \mathbf{l}(\mathbf{r})$$

build to a good approximation a

local-orthogonal trihedron

with

$$\left|\widehat{\mathbf{B}}_0(\mathbf{r})\right|^2 = \frac{1}{u^2}\left|\widehat{\mathbf{E}}_0(\mathbf{r})\right|^2 \; . \tag{4.388}$$

We calculate therewith now the time-averaged energy density (4.209),

$$\begin{aligned}
\overline{w(\mathbf{r},t)} &= \frac{1}{4}\mathrm{Re}\left(\widehat{\mathbf{H}}_0(\mathbf{r}) \cdot \widehat{\mathbf{B}}_0^\star(\mathbf{r}) + \widehat{\mathbf{E}}_0(\mathbf{r}) \cdot \widehat{\mathbf{D}}_0^\star(\mathbf{r})\right) \\
&= \frac{1}{4}\frac{1}{\mu_{\mathrm{r}}\mu_0}\left|\widehat{\mathbf{B}}_0(\mathbf{r})\right|^2 + \frac{1}{4}\epsilon_0\epsilon_{\mathrm{r}}\left|\widehat{\mathbf{E}}_0(\mathbf{r})\right|^2 \\
&\overset{(4.388)}{=} \frac{1}{2}\epsilon_0\epsilon_{\mathrm{r}}\left|\widehat{\mathbf{E}}_0(\mathbf{r})\right|^2 \; ,
\end{aligned} \tag{4.389}$$

and the time-averaged energy-current density (4.210):

$$\begin{aligned}
\overline{S(\mathbf{r},t)} &= \frac{1}{2}\mathrm{Re}\left(\widehat{\mathbf{E}}_0(\mathbf{r}) \times \widehat{\mathbf{H}}_0^\star(\mathbf{r})\right) \\
&= \frac{1}{2\mu_0\mu_r}\mathrm{Re}\left(\widehat{\mathbf{E}}_0(\mathbf{r}) \times \widehat{\mathbf{B}}_0^\star(\mathbf{r})\right) \\
&\overset{(4.385)}{=} \frac{1}{2\mu_0\mu_r}\,\frac{1}{u}\mathrm{Re}\left(\widehat{\mathbf{E}}_0(\mathbf{r}) \times \left(\mathbf{l}(\mathbf{r}) \times \widehat{\mathbf{E}}_0^\star(\mathbf{r})\right)\right) \\
&\overset{(4.387)}{=} \frac{1}{2\mu_0\mu_r}\,\frac{u}{u^2}\,\left|\widehat{\mathbf{E}}_0(\mathbf{r})\right|^2 \mathbf{l}(\mathbf{r}) \\
&= \frac{1}{2}u\epsilon_0\epsilon_r\left|\widehat{\mathbf{E}}_0(\mathbf{r})\right|^2 \mathbf{l}(\mathbf{r}) \ .
\end{aligned} \tag{4.390}$$

With Eq. (4.389) we thus find in this case also the well-known relation (4.215) between energy-current density and energy density:

$$\overline{S(\mathbf{r},t)} = u\,\overline{w(\mathbf{r},t)}\,\mathbf{l}(\mathbf{r}) \tag{4.391}$$

Hence, the energy is flowing in the direction of the **'beam-vector'**

$$\mathbf{l}(\mathbf{r}) = \frac{\nabla L(\mathbf{r})}{|\nabla L(\mathbf{r})|} = \frac{\nabla L(\mathbf{r})}{n(\mathbf{r})} \ . \tag{4.392}$$

This vector stands perpendicular to the **'wave surfaces, wavefronts'** $L(\mathbf{r}) = \text{const}$ (Fig. 4.68). The wave surfaces and the rays build at each space point an orthogonal system (*law of Malus*).

At the end we want to discuss still a few simple applications:

- **Plane wave**

  As we know, the plane wave is the solution of the wave equation in media with

$$n = \text{const} \ .$$

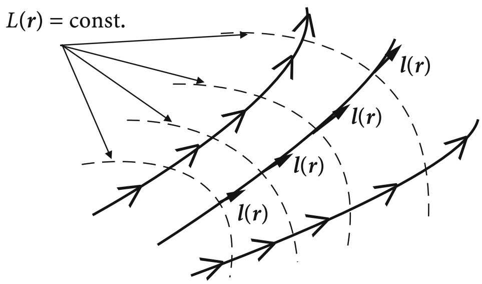


Fig. 4.68 Wavefronts and ray trajectories in the scope of geometrical optics

The precondition (4.376) for geometrical optics is trivially fulfilled and according to (4.378) it must hold:

$$e^{i\mathbf{k}\cdot\mathbf{r}} \stackrel{!}{=} e^{ik_0 L(\mathbf{r})} \;\Leftrightarrow\; \mathbf{k}\cdot\mathbf{r} \stackrel{!}{=} k_0 L(\mathbf{r}) = \frac{k}{n} L(\mathbf{r}) \; .$$

For the eikonal we thus have:

$$L(\mathbf{r}) = n\left(\frac{\mathbf{k}}{k}\cdot\mathbf{r}\right) \;\Rightarrow\; \nabla L(\mathbf{r}) = n\left(\frac{\mathbf{k}}{k}\right) \; .$$

The wavefronts $L(\mathbf{r}) = \text{const}$ are therefore planes and the direction of the ray is that of the propagation vector:

$$\mathbf{l}(\mathbf{r}) \equiv \mathbf{l} = \frac{\nabla L(\mathbf{r})}{n} = \frac{\mathbf{k}}{k} \; . \tag{4.393}$$

- **Spherical wave**
  We assume also here $n(\mathbf{r}) \approx n$. Then it is to require:

$$\frac{e^{ikr}}{r} \stackrel{!}{=} A(\mathbf{r})\; e^{ik_0 L(\mathbf{r})} \; .$$

For sufficiently large distances r, the amplitude $A(\mathbf{r}) \propto \frac{1}{r}$ is only weakly space-dependent so that:

$$kr \stackrel{!}{=} k_0 L(\mathbf{r}) = \frac{k}{n} L(\mathbf{r}) \;\Rightarrow\; \nabla L(\mathbf{r}) = n\mathbf{e}_r \; .$$

The wavefronts $L(\mathbf{r}) = \text{const}$ are now spherical surfaces with radial directions of the rays:

$$\mathbf{l}(\mathbf{r}) = \frac{\nabla L(\mathbf{r})}{n} = \mathbf{e}_r \tag{4.394}$$

- **Law of refraction**
  We consider a curved interface between two media 1 and 2 with different but otherwise constant indexes of refraction $n_1 \neq n_2$. In the range of validity of geometrical optics it is on both sides of the interface:

$$\text{curl}\left(n(\mathbf{r})\mathbf{l}(\mathbf{r})\right) = \text{curl}\left(\nabla L(\mathbf{r})\right) = 0$$

It follows with the Stokes theorem (1.60):

$$\int_F d\mathbf{f}\cdot\text{curl}\left(n(\mathbf{r})\mathbf{l}(\mathbf{r})\right) = \oint_{\partial F} d\mathbf{r}\cdot\left(n(\mathbf{r})\mathbf{l}(\mathbf{r})\right) = 0 \; .$$

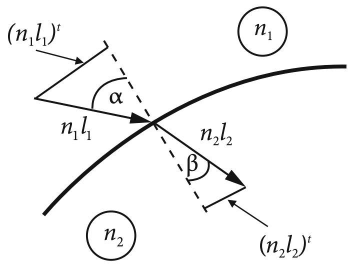


Fig. 4.69 To the law of refraction at curved interfaces

As for the discussion of the general field behavior at interfaces in Sect. 4.3.10 we can conclude from this that the tangential component of the vector $n(\mathbf{r})\mathbf{l}(\mathbf{r})$ is continuous at the interface (Fig. 4.69):

$$\left(n_1(\mathbf{r})\mathbf{l}_1(\mathbf{r})\right)^t = \left(n_2(\mathbf{r})\mathbf{l}_2(\mathbf{r})\right)^t \, .$$

This means:

$$n_1(\mathbf{r})|\mathbf{l}_1(\mathbf{r})| \sin\alpha = n_2(\mathbf{r})|\mathbf{l}_2(\mathbf{r})| \sin\beta \, .$$

Because of $|\mathbf{l}_1(\mathbf{r})| = |\mathbf{l}_2(\mathbf{r})| = 1$ we eventually get the **Snell's law of refraction** (4.257),

$$n_1 \sin\alpha = n_2 \sin\beta \, , \tag{4.395}$$

which therefore is valid, as a generalization of the derivation of (4.257), even for *curved* interfaces.

4.3.18 Exercises

Exercise 4.3.1

1. What is the equation of motion of a (point-like) particle with the charge q and the mass m in an electromagnetic field $(\mathbf{E}, \mathbf{B})$? (The emission of radiation by the moving charge is to be neglected.) Determine the temporal change of the energy W of the particle in the external field.
2. A circularly polarized monochromatic electromagnetic wave is described by the field

$$\mathbf{E}(\mathbf{r}, t) = E\,(\cos(kz - \omega t), \sin(kz - \omega t), 0) \, .$$

Calculate the corresponding magnetic induction $\mathbf{B}(\mathbf{r}, t)$. (The *underlying* medium is assumed to be linear, homogeneous,uncharged, and isolated, e.g. vacuum.)

3. The particle from 1. moves in the field from 2. Formulate the equation of motion!
4. Let the particle be at $t = 0$ at the origin of coordinates. How should the initial conditions for the velocity be chosen in order to keep the energy W of the particle constant?
5. Find the momentum $\mathbf{p}$ of the particle and verify that the direction of $\mathbf{p}_\perp = (p_x, p_y, 0)$ coincides at each point of time with the direction of $\mathbf{B}$.
6. Solve the equation of motion with the initial conditions from 4.
7. Which path in the xy-plane does the particle traverse?

Exercise 4.3.2 A transverse electromagnetic wave in an insulating uncharged medium ($\rho_\mathrm{f} = 0$, $j_\mathrm{f} = 0$, $\sigma = 0$) is

(a) linearly polarized,

$$\mathbf{E} = \mathbf{E}_0 \sin(kz - \omega t) \ ,$$

(b) circularly polarized,

$$\mathbf{E} = E_0 \left[\cos(kz - \omega t)\,\mathbf{e}_x + \sin(kz - \omega t)\,\mathbf{e}_y\right] \ ,$$

and propagates in the z-direction. Calculate

1. the magnetic induction $\mathbf{B}(\mathbf{r}, t)$,
2. the Poynting-vector $\mathbf{S}(\mathbf{r}, t)$,
3. the radiation pressure on a plane which is inclined by the angle ϑ relative to the propagation direction ($\mathbf{k} = k\,\mathbf{e}_z$).

Exercise 4.3.3 Consider a linear, homogeneous, uncharged insulator.

1. Formulate the Maxwell equations for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$?
2. Show that $\mathbf{B}$ fulfills the homogeneous wave equation.
3. The electric field strength $\mathbf{E}$ is given as a plane wave

$$\mathbf{E}(\mathbf{r}, t) = \frac{E_0}{5}\left(\mathbf{e}_x - 2\mathbf{e}_y\right) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad (\mathbf{k} = k\,\mathbf{e}_z) \ .$$

Calculate the magnetic induction $\mathbf{B}(\mathbf{r}, t)$ and determine its polarization.

4. The magnetic induction $\mathbf{B}$ is given as a plane wave of the type

$$\mathbf{B}(\mathbf{r}, t) = B_0 \cos(kz - \omega t)\,\mathbf{e}_x + B_0 \sin(kz - \omega t)\,\mathbf{e}_y \ ,$$

Calculate the electric field strength $\mathbf{E}(\mathbf{r}, t)$ and determine its polarization.

Exercise 4.3.4 Given is a linear, homogeneous, uncharged insulator.

1. The magnetic induction $\mathbf{B}(\mathbf{r}, t)$ is a plane wave of the form

$$\mathbf{B}(\mathbf{r}, t) = \widehat{B}_0(4\mathbf{e}_x - 3\mathbf{e}_y)\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad (\mathbf{k} = k\mathbf{e}_z, \widehat{B}_0 \text{ real}) \ .$$

Calculate the electric field strength $\mathbf{E}(\mathbf{r}, t)$ and investigate its polarization!

2. The electric field $\mathbf{E}(\mathbf{r}, t)$ is given by

$$\mathbf{E}(\mathbf{r}, t) = \alpha \mathbf{e}_x \cos(kz - \omega t + \varphi) - \beta \mathbf{e}_y \sin(kz - \omega t + \varphi)$$

(α, β real). Calculate the magnetic induction $\mathbf{B}(\mathbf{r}, t)$ and investigate its polarization!

Exercise 4.3.5 The magnetic induction in a linear, isotropic, uncharged insulator (ε_r, μ_r) is given by

$$\mathbf{B}(\mathbf{r}, t) = \widehat{\mathbf{B}}_0(\mathbf{r})\ e^{-i\omega t}\ ; \quad \widehat{\mathbf{B}}_0(\mathbf{r}) = (\alpha \mathbf{e}_x + i\gamma \mathbf{e}_y)\ e^{i\mathbf{k}\cdot\mathbf{r}} \quad (\alpha, \gamma \in \mathbb{R})\ .$$

Let $\mathbf{B}(\mathbf{r}, t)$ be a solution of the Maxwell equations.

1. Does $\mathbf{B}(\mathbf{r}, t)$ also solve the homogeneous wave equation?
2. Which relation does exist between k and ω?
3. Determine the direction of the wave vector $\mathbf{k}$!
4. Calculate the corresponding electric field

$$\mathbf{E}(\mathbf{r}, t) = \widehat{\mathbf{E}}_0(\mathbf{r})\ e^{-i\omega t}\ !$$

5. Express the time-averaged energy density $\overline{w(\mathbf{r}, t)}$ of the electromagnetic field as a function of α and γ!
6. Calculate the time-averaged energy-current density $\overline{\mathbf{S}(\mathbf{r}, t)}$ as a function of α and γ! Which relation does exist between $\overline{\mathbf{S}(\mathbf{r}, t)}$ and $\overline{w(\mathbf{r}, t)}$?

Exercise 4.3.6 Determine the Fourier-series of the following periodic functions:

1. $f(x) = f(x + 2\pi)$

$$f(x) = \begin{cases} -x \text{ for } -\pi \leq x \leq 0\ , \\ \ \ x \text{ for } 0 \leq x \leq \pi\ . \end{cases}$$

2. $f(x) = f(x + 2\pi)$

$$f(x) = \begin{cases} -1 \text{ for } -\pi \leq x \leq -\pi/2\ , \\ \ \ 1 \text{ for } -\pi/2 \leq x \leq \pi/2\ , \\ -1 \text{ for } \pi/2 \leq x \leq \pi\ . \end{cases}$$

3. $f(x) = f(x + 2\pi)$

$$f(x) = x^2 \quad -\pi \leq x \leq +\pi\ .$$

4. Verify with the result from 3. the following relations:

   (a)

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \; .$$

   (b)

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \; .$$

Exercise 4.3.7 Prove the relations (4.175) for the coefficients of the Fourier-series (4.174)!

Exercise 4.3.8 Calculate the Fourier transform of the δ-function ($a > 0, b > 0$)

$$\delta(x^2 + (b-a)x - ab) \; !$$

Check the result by back-transformation!

Exercise 4.3.9

1. Let $f(x)$ be continuous everywhere and at x_0 expandable in a Taylor series. Show that for

$$\text{(a)} \;\; \delta_l^{(1)}(x) = \frac{1}{\sqrt{2\pi l^2}} \exp\left(-\frac{x^2}{2l^2}\right)$$

$$\text{(b)} \;\; \delta_l^{(2)}(x) = \frac{\sin\left(\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2}\right)}$$

   it holds:

$$\lim_{l\to 0} \int\limits_{-\infty}^{+\infty} dx\, \delta_l^{(i)}(x - x_0) f(x) = f(x_0) \qquad i = 1, 2 \; .$$

2. Justify with 1(b) that the δ-function can be represented in the open interval $(-1, +1)$ by the sum:

$$\delta(x) = \frac{1}{2} \lim_{N\to\infty} \sum_{n=-N}^{+N} \exp(i\pi n x) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \exp(i\pi n x) \; .$$

3. Check for the orthonormalized system of functions,

$$\left\{\sqrt{\frac{2}{x_0}}\sin\left(\frac{n\pi}{x_0}x\right)\right\} \quad ; \quad n = 1, 2, 3, \ldots \; ; \quad x \in (0, x_0] \; ,$$

   the completeness relation:

$$\frac{2}{x_0}\sum_{n=0}^{\infty}\sin\left(\frac{n\pi}{x_0}x\right)\sin\left(\frac{n\pi}{x_0}x'\right) = \delta\left(x - x'\right) \; .$$

Exercise 4.3.10

1. Let $\tilde{f}_1(k), \tilde{f}_2(k)$ be the Fourier transforms of the functions $f_1(x), f_2(x)$:

$$\tilde{f}_{1,2}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\; e^{-ikx} f_{1,2}(x) \; .$$

   Find the Fourier transform $\tilde{g}(k)$ of the product

$$g(x) = f_1(x)f_2(x) \; !$$

2. Calculate the Fourier transforms of the functions
   (a) $f(x) = e^{-|x|}$;
   (b) $f(x) = e^{-x^2/(\Delta x^2)}$.
3. Show that for each square integrable function $f(x)$ the following relation (*Parseval*) is valid:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\tilde{f}(k)|^2 dk$$

Exercise 4.3.11

1. Calculate the Fourier transform $\tilde{f}(k)$ of the function

$$f(x) = x\; e^{-|x|}.$$

2. Use the result from (1) for the derivation of the formula:

$$\frac{\pi}{16} = \int_{-\infty}^{+\infty} \frac{k^2}{(1+k^2)^4}\, dk.$$

Exercise 4.3.12 Expand the spherical wave

$$\psi(\mathbf{r},t) = \frac{1}{r}\,e^{i(kr-\omega t)}$$

in plane waves. Assume for the evaluation that k possesses an arbitrarily small, positive imaginary part ($k \to k + i\,0^+$, *convergence generating factor*).

Exercise 4.3.13 Plane waves are solutions of the Maxwell equations in the vacuum:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;,$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\;.$$

1. A plane wave, linearly polarized in x-direction, propagates in the vacuum in the positive z-direction. At $z = 0$ it meets a region of infinite conductivity σ which extends over the full semi-infinite space $z \geq 0$. Calculate the wave field in the semi-infinite space $z \leq 0$.
2. Sketch the spatial course of the electric field strength $\mathbf{E}(\mathbf{r},t)$ and the magnetic induction $\mathbf{B}(\mathbf{r},t)$ for $t = 0$ and $t = \tau/4 = \pi/2\omega$.
3. Find the direction and the magnitude of the surface-current density in the boundary layer.
4. Calculate and discuss the energy density as well as the energy current of the electromagnetic wave.

Exercise 4.3.14 An electromagnetic wave is propagating in a conducting medium ($\sigma \neq 0$).

1. Find the dispersion law, i.e. the connection between the wave number k and the angular frequency ω of the plane wave in the form

$$k^2 = f(\omega)\;.$$

2. Consider for an electron gas with the particle density n_0 the motion of the electrons in the field $\mathbf{E} = \mathbf{E}_0\,e^{-i\omega t}$ neglecting collisions as well as the Lorentz force exerted by a magnetic field on the electron. Calculate the conductivity σ of the electron gas.
3. Calculate the critical frequency ω_p for the propagation of an electromagnetic wave in the electron gas $\left(k^2(\omega = \omega_\mathrm{p}) \overset{!}{=} 0\right)$ as well as the penetration depth for a low-frequency wave ($\omega \ll \omega_\mathrm{p}$).
4. In 2. the Lorentz force, executed from the magnetic field of the electromagnetic wave on the electron, has been neglected compared to the electric force. By use of the law of induction, find when this approximation is acceptable.
5. Discuss the *circular birefringence* of electromagnetic waves which propagate in a plasma during the presence of an external homogeneous magnetic induction

$\mathbf{B}_0$. Consider for this purpose circularly polarized waves, which propagate in direction of $\mathbf{B}_0$, and calculate the index of refraction by generalization of the parts 1. and 2. under the assumption that the precondition of part 4. is fulfilled.

Exercise 4.3.15 On a medium 3 ($\epsilon_r^{(3)}$; $\mu_r^{(3)} = 1$) a thin layer of a medium 2 ($\epsilon_r^{(2)}$; $\mu_r^{(2)} = 1$) is deposited. This layer is set-up such that a monochromatic plane wave coming perpendicularly from the medium 1 ($\epsilon_r^{(1)}$; $\mu_r^{(1)} = 1$) enters medium 3 **without** any reflection. All the involved media are insulators ($\sigma_f = 0, j_f = 0$). Calculate the index of refraction n_2 and the thickness d of this *'compensation layer'*.

Exercise 4.3.16 An electromagnetic wave is incident, coming from a medium 1, upon a plane interface to an 'optically rarer' medium 2 ($n_2 < n_1$: indexes of refraction). At the interface it splits into a reflected and a transmitted wave as represented in Fig. 4.50. For all the following considerations we use the notation as in Fig. 4.50 and in particular:

$$n_1 = 2\,; \quad n_2 = 1$$

1. Determine for the given arrangement the limiting angle of total reflexion ϑ_t!
2. Let the angle of incidence ϑ_1 be fixed by

$$\sin\vartheta_1 \stackrel{!}{=} \cos\vartheta_1 \ .$$

   Determine ϑ_1 as well as $\sin\vartheta_2$ and $\cos\vartheta_2$!
3. By applying the Fresnel formulas calculate

$$\left(\frac{E_{01r}}{E_{01}}\right)_\perp \quad \text{and} \quad \left(\frac{E_{01r}}{E_{01}}\right)_\parallel \ !$$

   Find the tangent of the relative phase shift δ of the two components! In this case, is it possible that the reflected wave comes out circularly polarized?
4. Calculate the coefficient of reflection R!

Exercise 4.3.17 An electromagnetic wave, coming from a medium 1, is incident upon a plane interface to an 'optically rarer' medium 2 ($n_2 < n_1$: indexes of refraction).

1. How large can the ratio n_2/n_1 be at the most in order to allow for a circularly polarized wave after total reflection?
2. For a given n_2/n_1, under which angle has the wave to strike the interface in order to be circularly polarized after total internal reflection?

Exercise 4.3.18 A plane wave (frequency ω) is incident on the (plane) interface between two media of different indexes of refraction n and n' (Fig. 4.70). At the

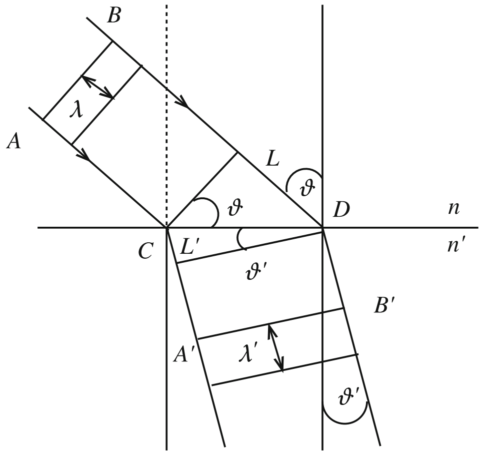


Fig. 4.70 Refraction of a plane wave at the interface between two media with different indexes of refraction

points A and B it has the same phase. Derive only by the requirement that the phase is also the same at the points A' and B' the law of refraction

$$\frac{\sin \vartheta}{\sin \vartheta'} = \frac{n'}{n} \; !$$

Exercise 4.3.19

1. Parallel light is incident on a screen which lies in the xy-plane and has a rectangular aperture σ (width $2A$, height $2B$). Discuss the intensity distribution of the diffraction pattern (Fraunhofer diffraction)!
2. How does the intensity distribution look like in the case of a slit ($B \gg A$)? Discuss in particular the perpendicular incidence of the light and derive conditions for the minima of diffraction!

4.4 Elements of Complex Analysis

For the further development of electrodynamics we need some auxiliary means (mathematical tools) of the

complex analysis,

i.e., of the theory of complex functions,

$$f(z) = u(z) + i\,v(z) = u(x, y) + i\,v(x, y) \; ; \quad i = \sqrt{-1} \; ,$$

of a complex variable

$$z = x + iy \quad x, y \in \mathbb{R} \; .$$

Each complex function is expressed by a pair of real functions u and v of two real variables x, y.

Remark The reader who is already familiar with complex analysis by a course in mathematics may either use this section as a kind of self-examination or simply skip it in order to proceed directly to Sect. 4.5

We already introduced the complex numbers in Sect. 2.3.5 of Vol. 1, where we discussed the classical mechanics, and applied them very often in the subsequent course on Theoretical Physics. We have seen that due to formal reasons it can be very useful to *extend* the physical, real quantities into the complex plane since many calculations can be performed essentially more elegantly in the region of the complex numbers. Simple complex-valued functions have proven to be of enormous value, e.g., as ansatze for the solution of linear differential equations.

Since the complex analysis is indispensable not only for the electrodynamics but also for many other fields of Theoretical Physics we want to compile here its most important definitions and propositions. It is clear, though, that in the framework of our compact representation much has to remain unproven being left as a matter of consulting appropriate special literature.

4.4.1 Sequences of Numbers

Definition 4.4.1 A point set M is called **neighborhood** of the point z_0 in the complex (number) plane if an $r_0 > 0$ exists such that all points within a circle around z_0 of the radius r_0 belong to M.

Definition 4.4.2 The sequence $\{z_n\}$ of complex numbers **converges** to $z_0 \in \mathbb{C}$,

$$\lim_{n\to\infty} z_n = z_0 \;,$$

if

1. in any neighborhood of z_0 lie *practically all* members of the sequence or
2. for each $\epsilon > 0$ there exists an $n_0(\epsilon)$ so that for all $n > n_0$

$$|z_n - z_0| < \epsilon \;.$$

As for real numbers one proves the following **rules**:

For

$$\lim_{n\to\infty} a_n = \alpha \;; \qquad \lim_{n\to\infty} b_n = \beta$$

one finds:

$$\begin{aligned}\lim_{n\to\infty}(a_n \pm b_n) &= \alpha \pm \beta \ ,\\ \lim_{n\to\infty} a_n b_n &= \alpha\,\beta \ ,\\ \lim_{n\to\infty}\frac{a_n}{b_n} &= \frac{\alpha}{\beta} \quad (\text{if } b_n \neq 0,\ \beta \neq 0)\ . \end{aligned} \tag{4.396}$$

4.4.2 Complex Functions

A complex function

$$w = f(z) = u(z) + i\,v(z) \tag{4.397}$$

represents a unique mapping of the complex z-plane onto the complex w-plane:

$$D \ni z \underset{f}{\longrightarrow} w \in W \ ,$$

D: complex domain of definition; W: complex co-domain.

The **continuity** of a complex function is defined analogous to the real function (Sect. 1.1.5, Vol. 1).

Definition 4.4.3 $f(z)$ is continuous at z_0 if *for all* $\epsilon > 0$ a $\delta > 0$ exists so that *for each* $z \in D$ with

$$|z - z_0| < \delta$$

holds:

$$|f(z) - f(z_0)| < \epsilon \tag{4.398}$$

Definition 4.4.4 If to each $\epsilon > 0$ a $\delta > 0$ exists so that for all $z, z' \in D$ with

$$|z - z'| < \delta$$

it follows:

$$|f(z) - f(z')| < \epsilon \ , \tag{4.399}$$

then $f(z)$ is called **uniformly continuous** on D.

Definition 4.4.5 The complex function $f(z)$ is **differentiable** at $z = z_0$ if the limiting value

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} \equiv f'(z_0) \equiv \left.\frac{df(z)}{dz}\right|_{z=z_0} \tag{4.400}$$

does exist for **each** sequence $z_n \to z_0$ being thereby independent of the special choice of the sequence.

All functions differentiable at z_0 are also continuous at z_0. The converse does not hold! The differentiability in the complex case presumes that the sequence of numbers $\{z_n\}$ can be led to z_0 from any direction within the complex plane. This is a more stringent criterion than that for the differentiability of a real function of two real variables x, y. So it is not sufficient to require for the differentiability of $f(z) = u(x, y) + i\,v(x, y)$ the differentiability of u and v, only. This can be seen as follows:

$$\begin{aligned} u + i\,v &= f(z) = f(x + iy) \\ \Longrightarrow \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= f'(z)\frac{\partial z}{\partial x} = f'(z)\,, \\ \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} &= f'(z)\frac{\partial z}{\partial y} = i f'(z)\,. \end{aligned}$$

If one multiplies the first differential equation by i and compares the real and imaginary parts of the left-hand sides then one recognizes the **Cauchy-Riemann differential equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{4.401}$$

Even the single functions are not completely arbitrary. If one differentiates the first equation with respect to x, the second with respect to y and adds then the two expressions together one gets:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv \Delta u = 0\,. \tag{4.402}$$

Similarly one finds:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \equiv \Delta v = 0\,. \tag{4.403}$$

Hence, real and imaginary part of a differentiable complex function $f(z)$ satisfy the two-dimensional Laplace equation.

It is easy to prove the following **rules of differentiation**:

$$(1)\quad (f_1+f_2)' = f_1' + f_2' \,,$$

$$(2)\quad (f_1 f_2)' = f_1 f_2' + f_1' f_2 \,, \tag{4.404}$$

$$(3)\quad \left(\frac{f_1}{f_2}\right)' = \frac{f_2 f_1' - f_1 f_2'}{f_2^2} \quad (f_2 \neq 0) \,,$$

$$(4)\quad h(z) = g\left(f(z)\right) \implies h'(z) = \frac{dg}{df}\,\frac{df}{dz} \quad (\text{ chain rule}). \tag{4.405}$$

Definition 4.4.6

1. By a **(complex) domain** G one understands an open point set in which each pair of points can be connected by a completely in G lying traverse line.
2. Let the complex variable z change along an arbitrary (!) closed path C_n in G. If the function $f(z)$ always takes the same value after z has returned on C_n to its initial point then $f(z)$ is called to be **unique** in the domain G.

Example of a Multiple-Valued (Ambiguous) Function

$$\begin{aligned} f(z) &= \sqrt{z} \,, \\ z &= x + iy = |z|\; e^{i\varphi} \\ \implies \sqrt{z} &= |z|^{1/2}\; e^{i\varphi/2} \,. \end{aligned}$$

If C is a closed path which circles once around the point $z = 0$, i.e. ($\varphi = \varphi_0 \to \varphi = \varphi_0 + 2\pi$) (Fig. 4.71), then $f(z)$ changes its sign after one circle because of $e^{i\pi} = -1$. $\sqrt{z}$ is therefore not unique but double-valued!

Definition 4.4.7 $f(z)$ is called **analytic (regular)** in a domain G of the z-plane if $f(z)$ is at all points $z \in G$ unique and differentiable.

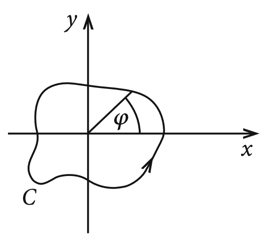


Fig. 4.71 Closed path in the complex plane circumscribing the point $z = 0$

The following **propositions** can be proven:

1. If the partial derivatives of the real functions $u(x, y)$, $v(x, y)$ with respect to the real variables x and y are continuous in G and fulfill the Cauchy-Riemann differential equations (4.401), then

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$$

   is analytic in G.
2. Let $f_1(z), f_2(z)$ be analytic in G then this holds also for

$$f_1 \pm f_2,\ f_1 f_2,\ f_1/f_2 \quad (f_2 \neq 0)\ .$$

3. Each in G analytic function has there analytic derivatives of arbitrarily (!) high order.

4.4.3 Integral Theorems

In the following, let $f(z)$ be a **continuous** function of the complex variable z in the domain G, z_0 and z^* two arbitrary points in G, and C a path from z_0 to z^* which lies completely within G (Fig. 4.72).

The complex **curvilinear (line) integral**

$$I = \int\limits_{\substack{z_0 \\ (C)}}^{z^*} f(z)dz$$

over the path C is then defined by

$$I = \lim_{n\to\infty} \sum_{\nu=0}^{n-1} f(\xi_\nu)\,(z_{\nu+1} - z_\nu) \tag{4.406}$$

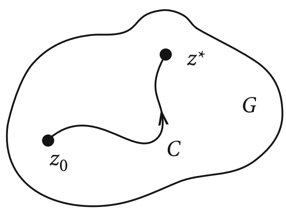
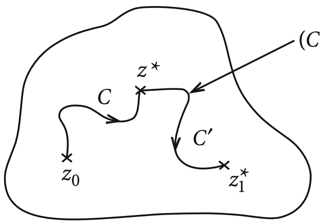


Fig. 4.72 Path C in a domain G

Fig. 4.73 Addition of two paths in a domain G



where the z_ν realize a decomposition of the path C: $z_\nu = z(t_\nu)$; $\alpha = t_0 < t_1 < \ldots < t_n = \beta$. The ξ_ν are intermediate points: $\xi_\nu = z(t_\nu^*)$;$t_\nu < t_\nu^* < t_{\nu+1}$.

Immediately from this definition some **simple integral theorems** follow which we list here in a symbolic form. The not explicitly written integrand is always $f(z)dz$:

1.

$$\underset{(C)}{\int_{z_0}^{z^*}} + \underset{(C')}{\int_{z^*}^{z_1^*}} = \underset{(C+C')}{\int_{z_0}^{z_1^*}} \,. \tag{4.407}$$

The path-notation $(C + C')$ means that one has to go at first from z_0 to z^* along C and then from z^* to z_1^* along C' (Fig. 4.73). Equivalent to (4.407) we have the statement:

2.

$$\underset{(C)}{\int_{z_0}^{z^*}} = \underset{(C_1)}{\int_{z_0}^{z_1^*}} + \underset{(C_2)}{\int_{z_1^*}^{z^*}} \,. \tag{4.408}$$

This holds when z_1^* is chosen in between z_0 and z^* on C whereby C decomposes into C_1 and C_2.

3.

$$\underset{(C)}{\int_{z_0}^{z^*}} = -\underset{(-C)}{\int_{z^*}^{z_0}} \,. \tag{4.409}$$

C and $(-C)$ denote the same paths which, however, are run through in opposite directions.

4.

$$\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz\,; \quad \alpha = \text{const} \in \mathbb{C}\,. \tag{4.410}$$

Constant factors can be drawn in front of the integral.

5.

$$\int_C (f_1(z) + f_2(z))\,dz = \int_C f_1(z)\,dz + \int_C f_2(z)\,dz\,. \tag{4.411}$$

Over a sum of **finitely** many functions can be integrated term by term.

The following formula can be important for estimations of complex curvilinear integrals:

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz| \le M\,L\,. \tag{4.412}$$

L is the length of the path C and M the maximum value of $|f(z)|$ on C. This relation, too, can be proven very easily with the definition (4.406).

For the formulation of the extremely important Cauchy's integral theorem we still need the

Definition 4.4.8 A domain G is called **simply connected** if each closed path that takes its course completely in G, without any double point, encloses exclusively points from the inside of G.

In other words, in a simply closed domain a closed path can always be contracted to a single point without leaving the domain anywhere.

Proposition 4.4.1 *Let $f(z)$ be an analytic function in a simply connected domain G and C a path remaining totally within G. Then the integral*

$$\int\limits_{\substack{z_0\\(C)}}^{z^*} f(z)dz$$

is dependent only on the end points z_0, z^*, **but not** *on any special course of C.*

Proof

$$\int_C f(z)dz = \int_C (u + i\,v)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \int_C \mathbf{p}_\mathrm{r} \cdot d\mathbf{r} + i\int_C \mathbf{p}_\mathrm{i} \cdot d\mathbf{r}\,,$$

where

$$\mathbf{p}_\mathrm{r} = (u, -v) \; ; \quad \mathbf{p}_\mathrm{i} = (v, u) \; ; \quad d\mathbf{r} = (dx, dy) \; .$$

The two line integrals are, as is well-known, path-independent as long as the curls of both the two-dimensional vectors $\mathbf{p}_\mathrm{r}, \mathbf{p}_\mathrm{i}$ vanish:

$$\operatorname{curl} \mathbf{p}_\mathrm{r} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & -v \end{vmatrix} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \; ,$$

$$\operatorname{curl} \mathbf{p}_\mathrm{i} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v & u \end{vmatrix} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \; .$$

These expressions correspond, however, just to the Cauchy-Riemann differential equations (4.401) being zero only when $f(z)$ is analytic in G.

Hence, we have an alternative formulation of the above proposition:

Proposition 4.4.2 ***Cauchy's integral theorem***
For all **closed** *paths which lie, together with the areas enclosed by them, totally within a simply connected domain G, where $f(z)$ is analytic, it holds:*

$$\oint_C f(z) dz = 0 \; . \tag{4.413}$$

This proposition represents the basis for all the further considerations on analytic functions. An important deduction is, e.g., the

Proposition 4.4.3 *Let C_1, C_2 be two closed paths where C_2 lies totally within the internal region of C_1 (Fig. 4.74). The ring-region defined by C_1 and C_2 belongs fully to a domain G in which the function $f(z)$ is analytic. Then:*

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz \; , \tag{4.414}$$

if C_1 and C_2 have the same direction of revolution, independently of whether or not the internal region of C_2 totally belongs to G.

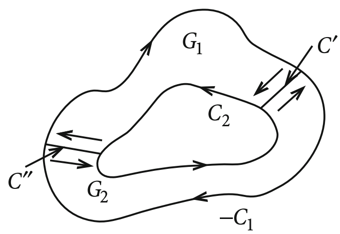


Fig. 4.74 Cutting of a ring-region by two auxiliary paths C' and C'' into two connected partial domains G_1 and G_2

Proof We cut the ring-region, as indicated in Fig. 4.74, at two positions by auxiliary paths C' and C''. That decomposes the ring-region into two simply connected domains G_1, G_2, in both of which $f(z)$ is analytic. Therefore, the preconditions for (4.413) are fulfilled. The contributions at the cut surfaces cancel each other because of (4.409). Hence:

$$\int\limits_{(-C_1)+C_2} f(z)dz = 0 \iff \int\limits_{C_2} f(z)dz = \int\limits_{C_1} f(z)dz \; .$$

Example of Application

$f(z) = 1/(z - z_0)$ is analytic everywhere except for the point z_0. We look for

$$I = \int\limits_C f(z)dz \; ,$$

where C shall be an arbitrary z_0 enclosing path. Let C' be a circle around z_0 with the radius ρ (Fig. 4.75):

$$C': \; z = z_0 + \rho \, e^{i\varphi} \; ; \quad 0 \le \varphi \le 2\pi \; .$$

Following the above proposition we replace for the calculation of I the path C by the path C':

$$I = \oint\limits_{C'} \frac{dz}{z - z_0} = \int\limits_0^{2\pi} d\varphi \frac{i\,\rho \, e^{i\varphi}}{\rho \, e^{i\varphi}} = i \int\limits_0^{2\pi} d\varphi = 2\pi \, i \; .$$

Hence, it holds for each path C that encloses z_0:

$$\oint\limits_C \frac{dz}{z - z_0} = 2\pi \, i \; . \tag{4.415}$$

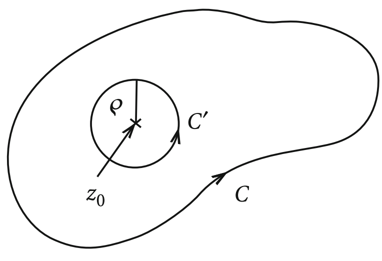


Fig. 4.75 Auxiliary construction for the integral over the complex function $f(z) = (z - z_0)^{-1}$ along an arbitrary path C which encloses z_0

As a further important conclusion from Cauchy's integral theorem (4.413) we prove

Proposition 4.4.4 ***Cauchy's integral formula***

Let $f(z)$ be analytic in the domain G. Then, for each closed path C, which lies completely within G without any double point, and for each point z_0 of the area enclosed by C:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} . \tag{4.416}$$

This is a really remarkable proposition, since it states that the values of the function f on the boundary of C are sufficient to fix the values of f for all points in the inside of C.

Proof

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad \text{with } F(z_0) = f'(z_0)$$

is analytic all over G so that because of (4.413):

$$\oint_C F(z)dz = 0 = \oint_C \frac{f(z)dz}{z - z_0} - f(z_0) \oint_C \frac{dz}{z - z_0} .$$

In the last step we have used (4.410). Equation (4.415) eventually leads to the proposition.

The inversion of the integral theorem (4.413) is known as **proposition of Morera**:

Proposition 4.4.5 ***(Morera)***

Let $f(z)$ be continuous in a simply-connected domain G. For ***each*** *closed path C which takes its course completely within G it holds*

$$\oint_C f(z)dz = 0 .$$

Then $f(z)$ is analytic in G.

Without proof we further present the ***integral formula for the derivatives****:*

Under the same preconditions as to (4.416) it holds for each analytic function $f(z)$:

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_C \frac{f(\xi)d\xi}{(\xi - z)^{n+1}} . \tag{4.417}$$

4.4.4 Series of Complex Functions

Definition 4.4.9 The series

$$\sum_{n=0}^{\infty} \alpha_n \; ; \quad \alpha_n \in \mathbb{C}$$

is called **convergent** if the sequence of partial sums

$$S_n = \sum_{\nu=0}^{n} \alpha_n$$

is convergent in the sense of Sect. 4.4.1; otherwise it is said **divergent**. One calls it **absolutely convergent** if

$$\sum_{n=0}^{\infty} |\alpha_n|$$

converges.

Definition 4.4.10

$\{f_n(z)\}$: Sequence of complex functions.
M: Set of all the points z which belong to the complex domains of definition of **all** f_n.

One denotes as **region of convergence** of the series the set M_C of all those z for which

$$\sum_{n=0}^{\infty} f_n(z)$$

converges.

Definition 4.4.11 One calls the series $\sum_{n=0}^{\infty} f_n(z)$ **uniformly convergent** in M if there exists for each $\epsilon > 0$ an $n_0(\epsilon) \in \mathbb{N}$, which only depends on ϵ but not on z, so that it follows for all $n \geq n_0, p \geq 1$ and all $z \in M$:

$$\left| f_{n+1}(z) + f_{n+2}(z) + \ldots + f_{n+p}(z) \right| < \epsilon \; .$$

To prove the uniform convergence one frequently uses the

comparison test (for series)

Let $\sum c_n$ be a convergent series with positive numbers $c_0, c_1, \ldots, c_n, \ldots$ which are such that for all z of the region of convergence of the series $\sum f_n(z)$ it can be estimated

$$|f_n(z)| \leq c_n \quad (n \in \mathbb{N}_0) \; .$$

Then the series $\sum f_n(z)$ is uniformly convergent.

Each series represents in its region of convergence M_C a certain function $F(z)$. Sometimes this is formulated also *'the other way round'* by saying that the function $F(z)$ can be expanded in M_C as such a series. For us it is above all interesting to know whether such a series represents an analytical function.

Let $\{f_n(z)\}$ be a series of functions which all are analytic in the same region G and for which the series

$$F(z) = \sum_{\nu=0}^{\infty} f_\nu(z)$$

converges uniformly in the inside of G. Then the following statements are valid:

1. $F(z)$ is continuous in G.
2. One can integrate term by term:

$$\int_C F(z)dz = \sum_{\nu=0}^{\infty} \int_C f_\nu(z)dz \,, \tag{4.418}$$

   C: path which lies totally within G.
3. $F(z)$ is analytic in G.
4. One can differentiate term by term:

$$F^{(n)}(z) = \sum_{\nu=0}^{\infty} f_\nu^{(n)}(z) \,. \tag{4.419}$$

Proof of 1

$$F(z) = S_n(z) + r_n(z) \,,$$
$$S_n(z) = \sum_{\nu=0}^{n} f_\nu(z) \,; \quad r_n(z) = \sum_{\nu=n+1}^{\infty} f_\nu(z) \,.$$

From the uniform convergence it follows: for each $\epsilon > 0$ there exists an $n_0(\epsilon)$ so that for $n \geq n_0$ one finds

$$|r_n(z)| < \frac{\epsilon}{3} \quad \text{(for all } z) \,.$$

$S_n(z)$ is a **finite** sum of continuous functions. Hence, it follows: For each $\epsilon > 0$ and all $z_0 \in G$ there exists a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ it holds:

$$|S_n(z) - S_n(z_0)| < \frac{\epsilon}{3}$$

Thus, let $\epsilon > 0$ be given and z_0 be an arbitrary point in G. Then there is always a $\delta > 0$ so that it holds for all $|z - z_0| < \delta$:

$$|F(z) - F(z_0)| \leq |S_n(z) - S_n(z_0)| + |r_n(z)| + |r_n(z_0)| < \epsilon \ .$$

Proof of 2 If $F(z)$ is continuous then $\int_C F(z)dz$ certainly exists. Because of (4.411) we can write

$$\int_C F(z)\,dz = \int_C S_n(z)dz + \int_C r_n(z)dz$$

and

$$\int_C S_n(z)dz = \sum_{\nu=0}^{n} \int_C f_\nu(z)dz \ .$$

Let L be the length of the path C, which shall be finite. Then there is for each $\epsilon > 0$ an $n_0(\epsilon)$ so that it follows for $n \geq n_0(\epsilon)$:

$$\left| \int_C r_n(z)\,dz \right| < \epsilon L \ .$$

Therewith it also holds:

$$\left| \int_C F(z)dz - \sum_{\nu=0}^{n} \int_C f_\nu(z)dz \right| < \epsilon L \ .$$

But that is just the statement of (4.418) since ϵ can be made arbitrarily small.

Proof of 3 For each totally in G proceeding, closed path C we have:

$$\oint_C f_\nu(z)dz = 0 \quad \text{for all } \nu \ .$$

According to 2. this also means:

$$\oint_C F(z)dz = 0 \ .$$

Hence, $F(z)$ is analytic in G (Morera theorem).

Proof of 4 $z_0 \in G$. According to 3. $F(z)$ fulfills the preconditions of the integral formula (4.417):

$$F^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{F(z)dz}{(z-z_0)^{n+1}} = \frac{n!}{2\pi\, i} \oint_C \frac{\sum\limits_{\nu=0}^{\infty} f_\nu(z)dz}{(z-z_0)^{n+1}} .$$

The proof 2. needed only the uniform convergence of the functions $f_\nu(z)$ on the path C. Let this be, e.g., a circle around z_0 then the uniform convergence is surely also valid for $f_\nu(z)(z-z_0)^{-n-1}$. It follows therewith:

$$F^{(n)}(z_0) = \sum_{\nu=0}^{\infty} \frac{n!}{2\pi\, i} \oint_C \frac{f_\nu(z)dz}{(z-z_0)^{n+1}} = \sum_{\nu=0}^{\infty} f_\nu^{(n)}(z_0) .$$

A special case of the series discussed so far is given by the **power series**

$$f_n(z) = \alpha_n(z-z_0)^n, \quad \alpha_n \in \mathbb{C}$$

The region of convergence M_{C} for a power series is always the inside of a circle around z_0, the so-called **circle of convergence**. It holds the

Cauchy-Hadamard theorem:
Three possibilities exist for the convergence of a power series:

1. The series converges only for $z = z_0$. Then it has the **radius of convergence** $R = 0$.
2. The series is absolutely convergent for all $z \iff R = \infty$.
3. The series converges absolutely for $|z-z_0| < R$ and diverges for $|z-z_0| > R$ with

$$R = \left(\overline{\lim_{n\to\infty}} \sqrt[n]{|\alpha_n|}\right)^{-1} , \tag{4.420}$$

$\overline{\lim}$: *limit superior*: Limiting value with the largest absolute value.

Proposition 4.4.6 *A power series converges uniformly in each circle which lies with a smaller radius concentrically within the circle of convergence.*

Proof Let us assume $R > 0$, $0 < \rho < R$ and $|z-z_0| \le \rho$. Then it is for all these z:

$$\left| \sum_{\nu=n+1}^{n+p} \alpha_\nu(z-z_0)^\nu \right| \le \sum_{\nu=n+1}^{n+p} |\alpha_\nu|\rho^\nu .$$

Since the point $z = z_0 + \rho$ lies within the circle of convergence $\sum |\alpha_\nu|\rho^\nu$ is by definition convergent. One can find thus to each $\epsilon > 0$ an $n_0(\epsilon)$ so that for all $n \geq n_0$ and all $p \geq 1$ it is

$$\sum_{\nu=n+1}^{n+p} |\alpha_\nu|\rho^\nu < \epsilon \ .$$

Just this characterizes the uniform convergence.

Therewith, we can repeat for power series the statements (4.418) and (4.419):

Proposition 4.4.7

1. *A power series is analytic everywhere inside the circle of convergence.*
2. *All derivatives have the same radius of convergence.*
3. *For all coefficients α_ν one finds:*

$$\alpha_\nu = \frac{f^{(\nu)}(z_0)}{\nu!} = \frac{1}{2\pi\, i} \oint\limits_C \frac{f(z)dz}{(z-z_0)^{\nu+1}} \ . \tag{4.421}$$

For C the same preconditions are needed as those in (4.416).

Proposition 4.4.8 ***Expansion theorem, Taylor expansion*** *Let $f(z)$ be analytic in G and $z_0 \in G$. Then there exists* **one and only one** *power series of the form*

$$\sum_{\nu=0}^{\infty} \alpha_\nu (z-z_0)^\nu$$

with α_ν from (4.421), which converges in each circle around z_0, that lies still fully in G, **and represents there** *the function $f(z)$. (Each analytic function can be therefore represented as a power series!)*

Proof Let K_R be a circle around $z_0 \in G$ with the radius R where K_R lies totally within G. $z \in K_R$ but not from the edge $\Longrightarrow$ $|z - z_0| = \rho < R$. Consider $\rho < \rho_1 < R$ and z^* as an arbitrary point of the circle K_{ρ_1}:

$$\frac{1}{z^* - z} = \frac{1}{(z^* - z_0) - (z - z_0)} = \frac{1}{z^* - z_0}\,\frac{1}{1 - \dfrac{z - z_0}{z^* - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z^* - z_0)^{n+1}} \ .$$

Because of

$$\left|\frac{z - z_0}{z^* - z_0}\right| = \frac{\rho}{\rho_1} < 1$$

the series is uniformly convergent according to the comparison test. This holds also for the series:

$$\frac{f(z^*)}{z^*-z} = \sum_{n=0}^{\infty} \frac{f(z^*)}{(z^*-z_0)^{n+1}} (z-z_0)^n \; .$$

It follows therewith ($\sum$ and $\oint$ are commutable!):

$$\begin{aligned} \frac{1}{2\pi\, i} \oint_{K_{\rho_1}} dz^* \frac{f(z^*)}{z^*-z} &= \sum_{n=0}^{\infty} \frac{1}{2\pi\, i} \oint_{K_{\rho_1}} \frac{f(z^*)}{(z^*-z_0)^{n+1}} (z-z_0)^n dz^* \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \; . \end{aligned}$$

That means eventually:

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n \; .$$

We accept the uniqueness of the expansion from the

Proposition 4.4.9 ***Identity theorem of power series***
Let us assume that the power series

$$\begin{aligned} F_\alpha(z) &= \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n \; , \\ F_\beta(z) &= \sum_{n=0}^{\infty} \beta_n (z-z_0)^n \end{aligned}$$

both have the same radius of convergence $R > 0$ *and that it holds*

$$F_\alpha(z) = F_\beta(z)$$

1. *in an arbitrarily small neighborhood of* z_0
   or
2. *for infinitely many points accumulating at* z_0.

Then $F_\alpha(z)$ *and* $F_\beta(z)$ *are identical!*

Proof

1. According to (4.421) we have:

$$\alpha_\nu = \frac{F_\alpha^{(\nu)}(z_0)}{\nu!} = \frac{1}{2\pi i} \oint_C \frac{F_\alpha(z)dz}{(z-z_0)^{\nu+1}} ,$$

$$\beta_\nu = \frac{1}{2\pi i} \oint_C \frac{F_\beta(z)dz}{(z-z_0)^{\nu+1}} ,$$

Let C be a circle which lies fully within the mentioned neighborhood. $\Longrightarrow$ for $z \in C$: $F_\alpha(z) = F_\beta(z) \Longrightarrow \alpha_\nu = \beta_\nu$ for all ν.

2. Proof by complete induction:

$\nu = 0$: $z \to z_0$ via the points for which $F_\alpha(z) = F_\beta(z)$. Power series are continuous $\Longrightarrow \alpha_0 = \beta_0$.
$\nu \Longrightarrow \nu + 1$: $\alpha_\mu = \beta_\mu$ for $\mu = 0, 1, 2 \ldots, \nu$.

Then it holds for infinitely many z:

$$\alpha_{\nu+1} + \alpha_{\nu+2}(z-z_0) + \ldots = \beta_{\nu+1} + \beta_{\nu+2}(z-z_0) + \ldots$$

Hence, we have with $z \to z_0$: $\alpha_{\nu+1} = \beta_{\nu+1}$, what was to be proved.

We now prove a theorem by which we can recognize the strong internal laws of the analytic functions as it has already been hinted by Cauchy's integral formula (4.416). Alone from the analyticity, which yet allows for a very big class of rather general functions, e.g. most of the functions needed in physical applications, a very intensive correlation between the function values can be suggested. If these are known for an arbitrarily small partial domain of the complex plane then they are already known for the total plane.

Proposition 4.4.10 ***Identity theorem for analytical functions***
Let $f_1(z), f_2(z)$ be analytic in G; $z_0 \in G$. It may hold

$$f_1(z) = f_2(z)$$

1. *in an arbitrarily small neighborhood of z_0, or*
2. *on an arbitrarily small piece of a path starting at z_0, or*
3. *in infinitely many points accumulating at z_0,*

then it is

$$f_1(z) \equiv f_2(z) \quad \textit{everywhere in } G\,.$$

Proof

1. Let f_1, f_2 be analytic functions which can be expanded as power series around z_0. These series converge at least in the maximal circle that still fits into G. Hence, according to the above proven identity theorem of power series these power series are identical within this circle because of 1. or 3., and therefore also $f_1(z)$ and $f_2(z)$.
2. Let now z^* be an **arbitrary** point in G. We now demonstrate that then it must also be $f_1(z^*) = f_2(z^*)$.
   (a) Connect z_0 and z^* by a path C. Let this have a minimal distance ρ to the margin of G.
   (b) Decompose the path C by the points
   $$z_0, z_1, z_2, \ldots, z_n = z^* \; ,$$
   such that the distances between neighboring points are in any case $< \rho$.
   (c) Draw around each point z_ν a circle K_ν which still just fits into G. The radii of these circles are then surely $\geq \rho$. Each circle thus certainly contains the center of the next circle (Fig. 4.76).
   (d) f_1, f_2 are analytic in each K_ν and therefore expandable as power series around z_ν. The identity is already proven for K_0.
   (e) $z_1 \in K_0 \implies f_1 = f_2$ is valid also at z_1 and in its neighborhood. Hence, the power series are identical in K_1, too.
   (f) In such a manner the procedure is continued via z_2 to $z_n = z^*$ (Fig. 4.76). Therewith, we can in particular conclude that the initial assertion is correct:
   $$f_1(z^*) = f_2(z^*)$$

Frequently one meets the situation that a given representation of a complex function, as for instance the Taylor expansion, converges only in a certain partial domain of the complex plane. But then it must not necessarily be excluded that the function is reasonably defined also outside of this domain, that only the special representation is no longer allowed. Sometimes one can extend then the domain of definition by use of the method of the

analytic continuation.

This is based on the just proven identity theorem for analytical functions.

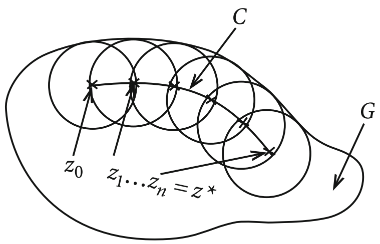


Fig. 4.76 Illustration of the procedure of the chains of circles

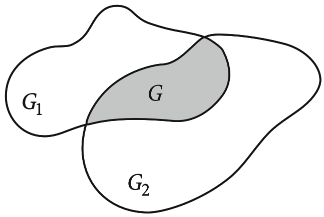


Fig. 4.77 Illustration for the analytic continuation of a complex function

Let G_1, G_2 be two domains which have the partial domain G in common (Fig. 4.77). Let $f_1(z)$ be an in G_1 analytic function. Then there does exist, according to the identity theorem, **no** or **one and only one** function $f_2(z)$, which is analytic in G_2, and for which one has

$$f_2(z) \equiv f_1(z) \quad \text{in } G$$

If such a function exists then it is said that $f_1(z)$ has been **analytically continued** beyond G_1 into the domain G_2. It is clear that also the converse viewpoint is valid. $f_2(z)$ is in G_1 the *analytic continuation* of $f_2(z)$.

According to the identity theorem $f_1(z)$ and $f_2(z)$ are completely entailing each other. They have to be understood as elements of one and the same function $F(z)$.

Example

$$G_1: \qquad \text{unit-circular area: } |z| < 1$$

$$f_1(z) = \sum_n z^n ,$$

$$G_2: \qquad \text{circular area } |z - i| < \sqrt{2}$$

$$f_2(z) = \sum_n \alpha_n (z - i)^n ,$$

$$\alpha_n = (1 - i)^{-(n+1)} .$$

Because of

$$|z - i| < \sqrt{2} \iff \left|\frac{z - i}{1 - i}\right| < 1$$

we see that $f_2(z)$ converges in G_2:

$$f_2(z) = \frac{1}{1 - i} \sum_n \left(\frac{z - i}{1 - i}\right)^n = \frac{1}{1 - i} \frac{1}{1 - \dfrac{z - i}{1 - i}} = \frac{1}{1 - z} .$$

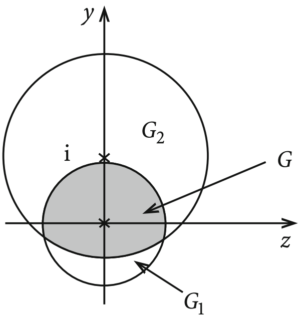


Fig. 4.78 Circular areas $|z| < 1$ and $|z - i| < \sqrt{2}$ in the complex plane

Because of $|z| < 1$ this holds also for $f_1(z)$ in G_1. In the *'overlap-domain'* G (Fig. 4.78) $f_1(z)$ and $f_2(z)$ are concurring. They represent in their circles of convergence G_1 and G_2, respectively, the function

$$F(z) = \frac{1}{1-z} \, .$$

This function is well-defined and analytic in the whole complex z-plane (except for $|z| = 1$). The above-given special power-series expansions, however, are valid only in G_1 and G_2, respectively.

4.4.5 Cauchy's Residue Theorem

So far we have investigated exclusively analytic functions. All points, at which a complex function is **not** analytic, are called

singular points

One distinguishes:

1. poles,
2. branching points,
3. essential singularities.

If $f(z)$ is analytic in the neighborhood of z_0, and if no statement about the analyticity at z_0 is possible, then one speaks of z_0 as an

isolated singularity .

If $(z-z_0)^n f(z)$ is, however, analytic at z_0 for any positive integer n then one says that $f(z)$ has at the point z_0 a **pole**. The smallest n, for which this statement is correct, is called the **order of the pole**.

A **branching point** of a function $f(z)$ is a point z_0, for which $f(z)$ after a revolution on a closed path C, which encloses z_0, does not come back to its initial value.

Essential singularities are all the other isolated singular points of a complex function $f(z)$.

Let $f(z)$ be analytic in a ring-like domain around z_0 (Fig. 4.79). Inside the smaller circle and outside the larger one the behavior of the function may be unknown. We therefore discuss $f(z)$ for z with

$$r_1 < |z - z_0| = \rho < r_2 \ .$$

$r_1, \ r_2$ are chosen such that $f(z)$ is analytic even on the boundary curves.

We decompose the ring-domain by two cuts C' und C'' into two simply connected domains, the boundary curves are run through in the mathematically positive sense (Fig. 4.79). Then it follows from Cauchy's integral formula (4.416):

$$f(z) = \frac{1}{2\pi\, i} \oint\limits_{C_2} \frac{f(\xi)d\xi}{\xi - z} - \frac{1}{2\pi\, i} \oint\limits_{C_1} \frac{f(\xi)d\xi}{\xi - z} \ .$$

First integral:

$$\frac{1}{\xi - z} \overset{(C_2)}{=} \frac{1}{\xi - z_0} \, \frac{1}{1 - \dfrac{z - z_0}{\xi - z_0}} \overset{(C_2)}{=} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} \ .$$

With

$$a_n = \frac{1}{2\pi\, i} \oint\limits_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}}$$

we thus have:

$$\frac{1}{2\pi\, i} \oint\limits_{C_2} \frac{f(\xi)d\xi}{\xi - z} = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ .$$

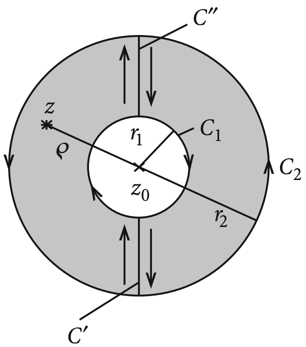


Fig. 4.79 Ring-like domain with auxiliary paths for the derivation of the Laurent expansion

Second integral:

$$\frac{1}{\xi - z} \overset{(C_1)}{=} -\frac{1}{z - z_0} \frac{1}{1 - \dfrac{\xi - z_0}{z - z_0}} \overset{(C_1)}{=} -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^n$$

$$\overset{(C_1)}{=} -\sum_{n=1}^{\infty} (\xi - z_0)^{n-1} (z - z_0)^{-n} \; .$$

If we now define

$$a_{-n} = \frac{1}{2\pi\, i} \oint_{C_1} \frac{f(\xi)d\xi}{(\xi - z_0)^{-n+1}} \; ,$$

then it is:

$$-\frac{1}{2\pi\, i} \oint_{C_1} \frac{f(\xi)d\xi}{\xi - z} = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \; .$$

As seen in Sect. 4.3.18 it is allowed to choose in the definitions of a_n and a_{-n} instead of C_1, C_2 also any other path C that encloses z_0 and lies completely within the ring-domain. Hence, we can define the coefficients very generally as follows,

$$a_n = \frac{1}{2\pi\, i} \oint_{C} \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}} \; , \tag{4.422}$$

where positive as well as negative n are permitted (cf. (4.421)).

Therewith we have derived for $f(z)$ the so-called **Laurent expansion**:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n \; . \tag{4.423}$$

One can show that this expansion is unique!

Of special interest is the case that z_0 is the only singular point of $f(z)$ inside the first circle. The Laurent expansion converges then for all

$$0 < |z - z_0| < r \; ,$$

where $r > 0$ is the distance to the nearest other singular point.

In case of a pole of p-th order the series starts at $n = -p$. The values a_n for $n < -p$ are then all equal zero. One calls

$$\sum_{n=-p}^{-1} a_n (z - z_0)^n\text{:} \quad \text{the } \textbf{principal part} \text{ of the function } f(z) \; .$$

The coefficient a_{-1} is of special importance:

$$a_{-1} = \mathrm{Res}f(z): \quad \textbf{residue} \text{ of } f(z) \text{ at the point } z_0 \, .$$

The comparison with (4.422) leads to the **Cauchy's residue theorem** which turns out to be a mighty auxiliary means for an effective calculation of integrals:

Let $f(z)$ be analytic in the neighborhood of z_0 and C a z_0 enclosing path. Then:

$$\mathrm{Res}_{z_0}f(z) = a_{-1} = \frac{1}{2\pi\, i}\oint_C f(z)dz \, . \tag{4.424}$$

One easily proves that for more than one, but finitely many, isolated singular points z_i lying in the internal domain of C the above formula is to be extended as follows:

$$\frac{1}{2\pi\, i}\oint_C f(z)dz = \sum_{i=1}^{N} \mathrm{Res}_{z_i}f(z) \, . \tag{4.425}$$

The residue of a pole of p-th order is often determined advisably according to the following formula:

$$\mathrm{Res}_{z_0}f(z) = \frac{1}{(p-1)!}\lim_{z\to z_0}\frac{d^{p-1}\left[(z-z_0)^p f(z)\right]}{dz^{p-1}} \, . \tag{4.426}$$

The residue theorem represents a mighty auxiliary means even for the calculation of **real** integrals which will be finally demonstrated by two examples.

Example 1

$$I = \int_{-\infty}^{+\infty}\frac{dx}{1+x^2} \, .$$

We choose the path C as indicated in Fig. 4.80 and integrate the function

$$f(z) = \frac{1}{1+z^2}$$

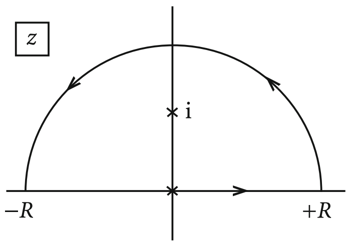


Fig. 4.80 Semi-circle with radius R in the upper complex half-plane

along C. The function

$$f(z) = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

possesses obviously two poles of first order but only one of them $z = i$ lies in the domain enclosed by C. The corresponding residue is found to be $1/2i$.

Hence, it follows:

$$\int_{\frown} \frac{dz}{1+z^2} = 2\pi\,\mathrm{i}\frac{1}{2\mathrm{i}} = \pi = \int_{-R}^{+R} \frac{dx}{1+x^2} + \int_{\frown} \frac{dz}{1+z^2} \ .$$

Estimation of the integral over the semi-circle:

$$\left|\int_{\frown} \frac{dz}{1+z^2}\right| \leq \frac{\frac{1}{2}\,2\pi R}{R^2-1} \underset{R\to\infty}{\longrightarrow} 0 \ .$$

For $R \to \infty$ the contribution on the semi-circle vanishes. Finally we have therefore found:

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi \ .$$

Example 2

$$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx \ .$$

This is an example for the oftentimes appearing case that a pole is located on the real axis. Such a pole we circumvent on a small semi-circle C_0 with the radius r (Fig. 4.81). Then it holds, for a start, according to the residue theorem (I_i: contributions on the respective partial paths C_i):

$$I_1 + I_0 + I_2 + I_3 = 2\pi\, i \sum_i \mathrm{Res}_{z_i} f(z) \ .$$

z_i are the singularities enclosed by the total path C.

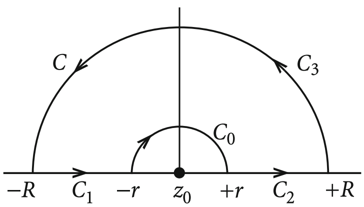


Fig. 4.81 Auxiliary paths for the investigation of an integrand with a pole on the real axis

Close to z_0 it holds (z_0: pole of first order):

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{a_{-1}}{z - z_0} + f_1(z) \ .$$

$f_1(z)$ is analytic in the neighborhood of z_0, i.e. continuous and therewith bounded:

$$|f_1(z)| \leq M \ .$$

As a consequence we find:

$$I_0 = \int\limits_{C_0} \frac{a_{-1} dz}{z - z_0} + \int\limits_{C_0} f_1(z)\, dz$$

$$\int\limits_{C_0} f_1(z)\, dz \leq M \pi r \underset{r \to 0}{\longrightarrow} 0 \ .$$

Thus we get for $r \to 0$:

$$I_0 = a_{-1} \int\limits_{C_0} \frac{dz}{z - z_0} = -a_{-1} \pi\, i \ . \tag{4.427}$$

The last step uses (4.415).

Let us come back to our example. We take

$$f(z) = \frac{e^{iz}}{z} \ .$$

This function has a pole of first order at $z = 0$ with the residue:

$$a_{-1} = \lim_{z \to 0} z f(z) = 1 \ .$$

Inside of C there is no pole:

$$0 = \oint\limits_C \frac{e^{iz} dz}{z} = (I_1 + I_2 + I_3)_{\substack{R \to \infty \\ r \to 0}} - i \pi \ ,$$

$$I_1 = \int\limits_{-R}^{-r} \frac{e^{ix}}{x}\, dx \ ; \quad I_2 = \int\limits_{r}^{R} \frac{e^{ix}}{x}\, dx \ .$$

That means:

$$(I_1 + I_2)_{\substack{R \to \infty \\ r \to 0}} = 2i \int\limits_0^{\infty} \frac{\sin x}{x} dx \ .$$

On the semi-circle it is $z = R(\cos\varphi + i\sin\varphi)$

$$\Longrightarrow \quad \frac{dz}{z} = \frac{-\sin\varphi + i\cos\varphi}{\cos\varphi + i\sin\varphi} d\varphi = +i\,d\varphi$$

$$\Longrightarrow \quad I_3 = i\int_0^{\pi} d\varphi\ e^{iR(\cos\varphi + i\sin\varphi)}$$

$$\Longrightarrow \quad |I_3| \leq \int_0^{\pi} d\varphi\ e^{-R\sin\varphi}$$

$$\Longrightarrow \quad \lim_{R\to\infty} I_3 = 0 \quad \Longrightarrow \quad 0 = 2i\int_0^{\infty} \frac{\sin x}{x} dx - i\pi\ .$$

From that we get the final result:

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi\ .$$

4.4.6 Exercises

Exercise 4.4.1 Verify the following representation of the step function:

$$\Theta(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dx \frac{e^{-ixt}}{x + i\,0^+}\ .$$

Exercise 4.4.2 Show as a reversal to Exercise 4.3.12 that a superposition of plane waves with the amplitudes

$$\tilde{\psi}(\overline{\mathbf{k}}, \overline{\boldsymbol{\omega}}) = \frac{2}{\overline{k}^2 - k^2} \delta(\overline{\omega} - \omega)$$

yields the spherical wave

$$\psi(\mathbf{r}, t) = \frac{1}{r} e^{i(kr - \omega t)}\ .$$

In this connection we presumed in Sect. 4.3.6 that k has an infinitesimally small, positive imaginary part ($k \to k + i\,0^+$).

4.5 Creation of Electromagnetic Waves

4.5.1 Inhomogeneous Wave Equation

So far we have discussed exclusively the propagation of electromagnetic waves, but completely excluded up to now how they are created. Electromagnetic waves are produced by time-dependent charge and current distributions. We had seen in Sect. 4.1.3 that the problem of calculating time-dependent fields from given current-charge distributions can be traced back to the solution of formally identical inhomogeneous wave equations for the electromagnetic potentials. In the Lorenz gauge the inhomogeneous differential equations (4.38) and (4.39) are valid:

$$\Box\, \mathbf{A}(\mathbf{r},t) = -\mu_{\mathrm{r}}\mu_0\, \mathbf{j}(\mathbf{r},t) \qquad \left(\Box \equiv \Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right) ;$$

$$\Box\, \varphi(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\epsilon_{\mathrm{r}}\epsilon_0} .$$

The solutions of the completely decoupled wave equations for φ and $\mathbf{A}$ still have to fulfill the Lorenz condition (4.37),

$$\operatorname{div}\mathbf{A} + \frac{1}{u^2}\dot{\varphi} = 0 ; \quad u^2 = \frac{1}{\epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0} . \tag{4.428}$$

The mathematical problem thus consists in solving the differential equation

$$\Box\, \psi(\mathbf{r},t) = -\sigma(\mathbf{r},t) , \tag{4.429}$$

where the source function $\sigma(\mathbf{r},t)$ is assumed to be known. As for the Poisson equation of the electrostatics, we approach the problem in such a way that we first solve (4.429) for a point charge,

$$\Box_{r,t} G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') , \tag{4.430}$$

in order to build up the complete solution then with the so found **Green's function**:

$$\psi(\mathbf{r},t) = \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}', t-t')\, \sigma(\mathbf{r}', t') . \tag{4.431}$$

To solve the differential equation we use the method of the Fourier transformation (Sect. 4.3.6):

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{(2\pi)^2}\int d^3k \int d\omega\, G(\mathbf{k},\omega)\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, e^{-i\omega(t-t')} \;,$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3k\; e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \;,$$

$$\delta(t-t') = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} d\omega\; e^{-i\omega(t-t')} \;.$$

Insertion into (4.430) yields:

$$\int d^3k \int d\omega\; e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, e^{-i\omega(t-t')} \left\{G(\mathbf{k},\omega)\left(-k^2+\frac{\omega^2}{u^2}\right)+\frac{1}{4\pi^2}\right\} = 0 \;.$$

By Fourier-reversal we get:

$$G(\mathbf{k},\omega)\left(k^2-\frac{\omega^2}{u^2}\right) = \frac{1}{4\pi^2} \;. \tag{4.432}$$

The general solution of this equation reads:

$$G(\mathbf{k},\omega) = G_0(\mathbf{k},\omega) + \{a_+(\mathbf{k})\,\delta(\omega+uk) + a_-(\mathbf{k})\delta(\omega-uk)\} \;, \tag{4.433}$$

$$G_0(\mathbf{k},\omega) = \frac{1}{4\pi^2}\,\frac{1}{k^2-\frac{\omega^2}{u^2}} \;. \tag{4.434}$$

The term in the curly bracket is the already known solution (4.199) of the homogeneous equation which always has to be added. We have extensively discussed this solution in Sect. 4.3.7 so that we can restrict our considerations here to $G_0(\mathbf{k},\omega)$:

$$G_0(\mathbf{r}-\mathbf{r}', t-t') = \frac{u^2}{(2\pi)^4}\int d^3k \int d\omega \frac{e^{i(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t'))}}{\omega_0^2-\omega^2} \;,$$

$$\omega_0 = uk \;. \tag{4.435}$$

We evaluate the ω-integral by complex integration (see Exercise 4.5.1). Because of

$$\frac{1}{\omega_0^2-\omega^2} = \frac{1}{2\omega_0}\left(\frac{1}{\omega+\omega_0}-\frac{1}{\omega-\omega_0}\right)$$

the integrand has two poles of first order at $\omega = \mp\omega_0$.

The Green's function G_0 describes a perturbation located at $\mathbf{r}$ at time t which has been *created* by a perturbation at the time t' at the place $\mathbf{r}'$. Due to **'causality-reasons'** we have to therefore require that G_0 is unequal zero only for $t - t' > 0$:

$$G_0(\mathbf{r} - \mathbf{r}', t - t') \implies G_0(\mathbf{r} - \mathbf{r}', t - t')\Theta(t - t') \; .$$

One calls this solution the

retarded Green's function

For the complex ω-integration the integration path is closed in the upper or lower half-plane on a semi-circle with the radius $R \to \infty$. The semi-circle has to be chosen such that its contribution to the integral vanishes. Because of the exponential function in the integrand that succeeds for $t-t' > 0$, if ω takes a negative imaginary part on the semi-circle, and for $t - t' < 0$, if ω has a positive imaginary part. Hence, the integration path has to be chosen as indicated in Fig. 4.82.

In order to guarantee the mentioned causality, we shift the path along the real axis infinitesimally into the upper half-plane, i.e.:

$$\int\limits_{-\infty}^{+\infty} d\omega \ldots \implies \int\limits_{-\infty+i0^+}^{+\infty+i0^+} d\omega \ldots$$

For $t - t' < 0$ **no** pole is then enclosed by the integration path so that the residue theorem (4.425) leads to

$$G_0^{\mathrm{ret}}(\mathbf{r} - \mathbf{r}', t - t') \equiv 0 \quad \text{for } t - t' < 0 \; . \tag{4.436}$$

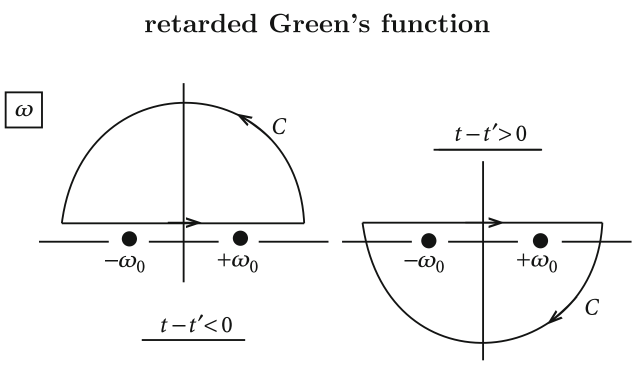


Fig. 4.82 Integration paths for determining the retarded Green's function

For $t - t' > 0$, however, both poles are circumvented by the integration path in the mathematically negative sense. For the residues one finds

$$a_{-1}(\pm\omega_0) = \lim_{\omega\to\pm\omega_0} (\omega \mp \omega_0) \frac{e^{-i\omega(t-t')}}{\omega_0^2 - \omega^2} = \mp \frac{1}{2\omega_0}\, e^{\mp i\omega_0(t-t')} \,,$$

so that the residue theorem (4.425) yields:

$$\int\limits_{-\infty+i0^+}^{+\infty+i0^+} d\omega \frac{1}{\omega_0^2 - \omega^2}\, e^{-i\omega(t-t')} = \frac{-2\pi i}{2\omega_0} \left(e^{i\omega_0(t-t')} - e^{-i\omega_0(t-t')} \right) \,.$$

This leads to the following intermediate result for the Green's function:

$$G_0^{\mathrm{ret}}(\mathbf{r}-\mathbf{r}', t-t') = \frac{-iu}{16\pi^3} \int \frac{d^3k}{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left(e^{iku(t-t')} - e^{-iku(t-t')} \right) \quad \text{for } t - t' > 0 \,.$$

On the right-hand side we recognize the function $D(\mathbf{r}-\mathbf{r}', t-t')$ which we introduced in (4.201) in connection with the homogeneous wave equation:

$$G_0^{\mathrm{ret}}(\mathbf{r}-\mathbf{r}', t-t') = u^2 D(\mathbf{r}-\mathbf{r}', t-t') \quad \text{for } t > t' \,.$$

We have already evaluated $D(\mathbf{r}-\mathbf{r}', t-t')$ explicitly in (4.203):

$$G_0^{\mathrm{ret}}(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\, \delta\left(\frac{|\mathbf{r}-\mathbf{r}'|}{u} - t + t' \right) \,. \tag{4.437}$$

This Green's function exhibits obviously a **causal** behavior. The signal that is observed at the time t at the space point $\mathbf{r}$ is caused by a perturbation at $\mathbf{r}'$, at a distance $|\mathbf{r}-\mathbf{r}'|$ from the point of observation, which acted at an earlier, the so-called

$$\textbf{retarded time} \quad t_{\mathrm{ret}} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{u} \,. \tag{4.438}$$

$|\mathbf{r}-\mathbf{r}'|/u$ is just the time the signal needs to come from $\mathbf{r}'$ to $\mathbf{r}$:

$$G_0^{\mathrm{ret}}(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta(t' - t_{\mathrm{ret}})}{4\pi|\mathbf{r}-\mathbf{r}'|} \,. \tag{4.439}$$

As a side note, notice that the quasi-stationary approximation (Sect. 4.2) just consists in the neglect of this retardation.

Had we shifted the integration path infinitesimally not into the upper but into the lower half-plane then we would have met the so-called **advanced** Green's function that differs from (4.439) by the fact that t_{ret} has to be replaced by

$$t_{\text{av}} = t + \frac{|\mathbf{r} - \mathbf{r}'|}{u} \,. \tag{4.440}$$

(One should check this explicitly!). In this case the principle of causality would be violated; not the past, as in (4.439), but the future would influence the present. Therefore, we discuss here, furtheron, the retarded solution which we insert into the general ansatz (4.431):

$$\psi(\mathbf{r}, t) = \int d^3r' \, \frac{\sigma(\mathbf{r}', t_{\text{ret}})}{4\pi |\mathbf{r} - \mathbf{r}'|} \,. \tag{4.441}$$

This means for the electromagnetic potentials:

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi \, \epsilon_0 \epsilon_{\text{r}}} \int d^3r' \, \frac{\rho(\mathbf{r}', t_{\text{ret}})}{|\mathbf{r} - \mathbf{r}'|} \,, \tag{4.442}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 \mu_{\text{r}}}{4\pi} \int d^3r' \, \frac{\mathbf{j}(\mathbf{r}', t_{\text{ret}})}{|\mathbf{r} - \mathbf{r}'|} \,. \tag{4.443}$$

The electromagnetic potentials have therewith formally the same structure as in the electrostatics and magnetostatics, respectively. Because of the retardation in the integrand the treatment of the integrals, however, turns out to be in general a rather awkward task.

With (4.442) and (4.443) the problem is completely solved, since from the potentials and the well-known relations

$$\mathbf{B} = \text{curl}\,\mathbf{A} \;; \quad \mathbf{E} = -\text{grad}\varphi - \dot{\mathbf{A}}$$

the magnetic induction $\mathbf{B}$ and the electric field $\mathbf{E}$ can be deduced in the whole space and for all times $t > t'$.

On can of course always add to the solutions presented here still the solution of the homogeneous wave equation that we suppressed after (4.434). The free solution can serve to fulfill the given boundary conditions.

At the end, we have to still show that the just found electromagnetic potentials indeed fulfill the Lorenz condition (4.428):

$$\begin{aligned}
\frac{1}{u^2}\frac{\partial}{\partial t}\varphi(\mathbf{r},t) &= \frac{1}{u^2} i \int d^3r'dt'\,\frac{\partial}{\partial t}G(\mathbf{r}-\mathbf{r}',t-t')\frac{\rho(\mathbf{r}',t')}{\epsilon_r\epsilon_0} \\
&= -\mu_r\mu_0 i \int d^3r'dt'\,\frac{\partial}{\partial t'}G(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r}',t') \\
&= \mu_r\mu_0 i \int d^3r'dt'\,G(\mathbf{r}-\mathbf{r}',t-t')\frac{\partial}{\partial t'}\rho(\mathbf{r}',t') \\
\text{(continuity equation)} &= -\mu_r\mu_0 i \int d^3r'dt'\,G(\mathbf{r}-\mathbf{r}',t-t')\operatorname{div}\mathbf{j}(\mathbf{r}',t') \\
(\operatorname{div}\mathbf{a}\varphi = \varphi\operatorname{div}\mathbf{a}+\mathbf{a}\cdot\nabla\varphi) &= \mu_r\mu_0 i \int d^3r'dt'\,\nabla_{r'}G(\mathbf{r}-\mathbf{r}',t-t')\cdot\mathbf{j}(\mathbf{r}',t') \\
&= -\mu_r\mu_0 i \int d^3r'dt'\,\nabla_{r}G(\mathbf{r}-\mathbf{r}',t-t')\cdot\mathbf{j}(\mathbf{r}',t') \\
&= -\operatorname{div}\mathbf{A}(\mathbf{r},t)
\end{aligned}$$

That was to be proven! For the transition from the second to the third line an integration by parts with respect to t' was performed. The integrated part does not contribute since the retarded time t_{ret} is finite so that the Green's function vanishes at $t' = \pm\infty$. At the transition from the fifth to the sixth line the integral $\int d^3r'\operatorname{div}(G\mathbf{j})$ could be changed into a surface integral with the aid of the Gauss theorem. That, however, vanishes since $\mathbf{j} \neq 0$ holds only in a finite, restricted space region.

4.5.2 Temporally Oscillating Sources

We consider a temporally oscillating system of charges and currents in a closed region of space and want to discuss for this situation the formal solutions (4.442) and (4.443) of the inhomogeneous wave equation. We start with a Fourier-decomposition with respect to the frequency

$$\begin{aligned}
\rho(\mathbf{r},t) &= \tfrac{1}{\sqrt{2\pi}}\int d\omega\,\rho_\omega(\mathbf{r})\,e^{-i\omega t}\,, \\
\mathbf{j}(\mathbf{r},t) &= \tfrac{1}{\sqrt{2\pi}}\int d\omega\,\mathbf{j}_\omega(\mathbf{r})\,e^{-i\omega t}\,.
\end{aligned}$$

Because of the linearity of the Maxwell equations we can restrict our considerations to a single Fourier component:

$$\begin{aligned}
\rho(\mathbf{r},t) &= \rho(\mathbf{r})\,e^{-i\omega t}\,, \\
\mathbf{j}(\mathbf{r},t) &= \mathbf{j}(\mathbf{r})\,e^{-i\omega t}\,.
\end{aligned} \tag{4.444}$$

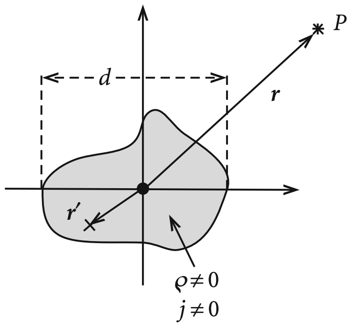


Fig. 4.83 System of coordinates for a spatially restricted, temporally oscillating system of charges and currents

$\rho(\mathbf{r})$, $\mathbf{j}(\mathbf{r})$ will in general be complex. They disappear outside a restricted space region (linear dimension d, Fig. 4.83). From the solutions derived with (4.444), we then obtain by linear combination with respect to ω, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

In the expression (4.443) for the vector potential we need:

$$\mathbf{j}(\mathbf{r}', t_{\text{ret}}) = \mathbf{j}(\mathbf{r}')\, e^{-i\omega t}\, e^{i(\omega/u)|\mathbf{r}-\mathbf{r}'|} \; . \tag{4.445}$$

That yields:

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r})\, e^{-i\omega t} \; , \tag{4.446}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\mu_{\text{r}}}{4\pi} \int d^3r' \, \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \mathbf{j}(\mathbf{r}') \; . \tag{4.447}$$

Via $k = \omega/u$ the vector potential $\mathbf{A}(\mathbf{r})$ is ω-dependent. It therefore oscillates with the same frequency as the source. When $\mathbf{r}$ lies in the space-region where no free currents and charges are present then everything is already fixed by $\mathbf{A}$. It is, e.g., not necessary to separately determine the scalar potential $\varphi(\mathbf{r}, t)$. That can be seen as follows: Outside the $(\rho \neq 0, \mathbf{j} \neq 0)$-region curl $\mathbf{H} = \dot{\mathbf{D}}$ and therewith:

$$\dot{\mathbf{E}} = u^2 \text{curl}\, \mathbf{B} = u^2 \text{curl}\,\text{curl}\, \mathbf{A}(\mathbf{r}, t) = u^2\, e^{-i\omega t} \text{curl}\,\text{curl}\, \mathbf{A}(\mathbf{r}) \; .$$

Hence, the electric field strength is already given by the vector potential:

$$\mathbf{E}(\mathbf{r}, t) = i \frac{u^2}{\omega}\, e^{-i\omega t} \text{curl}\,\text{curl}\, \mathbf{A}(\mathbf{r}) \; . \tag{4.448}$$

The basic formula (4.447) is in general not directly integrable. One is forced to accept approximations which, however, must be carefully defined since very often they are *reasonably justified* only in a very narrow region of the typical parameter space.

First simplifications arise from the assumption that the space-region, which contains charges and currents, is of linear dimensions d which are small compared to the wavelength λ of the electromagnetic radiation and small compared to the distance r to the point of observation P:

$$\textbf{small sources} \Longleftrightarrow \; d \ll \lambda, r \,. \tag{4.449}$$

The ratio λ/r can thereby be at first arbitrary. The following expansion suggests itself when $\mathbf{r}'$ lies in the ($\rho \neq 0$, $\mathbf{j} \neq 0$)-region:

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\,\mathbf{r}\cdot\mathbf{r}'} &\approx r\sqrt{1-\frac{2}{r}\,\mathbf{n}\cdot\mathbf{r}'} \quad \left(\mathbf{n} = \frac{\mathbf{r}}{r}\right) \\ &\approx r\left(1-\frac{1}{r}\mathbf{n}\cdot\mathbf{r}'\right) \,. \end{aligned}$$

If we neglect basically quadratic terms of r'^2 then it remains:

$$\begin{aligned} e^{ik|\mathbf{r}-\mathbf{r}'|} &\approx e^{ikr}\,e^{-ik\mathbf{n}\cdot\mathbf{r}'} \approx e^{ikr}(1-ik\,\mathbf{n}\cdot\mathbf{r}') \,, \\ |\mathbf{r}-\mathbf{r}'|^{-1} &\approx \frac{1}{r}\left(1+\frac{1}{r}\mathbf{n}\cdot\mathbf{r}'\right) \,. \end{aligned}$$

We combine these two terms:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{ikr}}{r}\left[1+(\mathbf{n}\cdot\mathbf{r}')\left(\frac{1}{r}-ik\right)\right] \,. \tag{4.450}$$

The vector potential therewith consists of some characteristic terms which we are going to investigate step by step in the next sections:

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0\mu_\mathrm{r}}{4\pi}\,\frac{e^{ikr}}{r}\int d^3r'\,\mathbf{j}(\mathbf{r}') + \frac{\mu_0\mu_\mathrm{r}}{4\pi}\left(\frac{1}{r}-ik\right)\frac{e^{ikr}}{r}\int d^3r'\,\mathbf{j}(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}') \,. \tag{4.451}$$

The first term corresponds to electric dipole radiation (Sect. 4.5.3), the second to magnetic dipole and electric quadrupole radiation (Sect. 4.5.4).

A further effective possibility for approximations is offered by a subdivision into so-called **zones**:

$d \ll r \ll \lambda$: *near zone (static zone),*
$d \ll r \sim \lambda$: *intermediate zone,*
$d \ll \lambda \ll r$: *far zone (radiation zone).*

This subdivision leads to heterogeneous estimations for the vector potential (4.447). One finds that the electromagnetic fields exhibit rather different behavior in the various zones:

(1) Radiation Zone

We exploit the expansion that led to (4.450) at first in the form:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r}\, e^{ikr}\, e^{-ik\,\mathbf{n}\cdot\mathbf{r}'} \ .$$

This yields the following simplified expression for the vector potential:

$$\mathbf{A}(\mathbf{r}) \approx \frac{e^{ikr}}{r}\left(\frac{\mu_0\mu_\mathrm{r}}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}')\; e^{-ik\,\mathbf{n}\cdot\mathbf{r}'}\right) \ . \tag{4.452}$$

The vector in the bracket is independent of r. In the radiation zone the vector potential behaves therefore like an outgoing spherical wave (4.169) with an angle-dependent coefficient.

If one still exploits $d \ll \lambda$, i.e. $kr' \ll 1$, then the series expansion of the exponential function in the integrand can be terminated after a finite number of terms. In the simplest case it remains:

$$\mathbf{A}(\mathbf{r}) \approx \frac{e^{ikr}}{r}\,\frac{\mu_0\mu_\mathrm{r}}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}') \ . \tag{4.453}$$

This is the first term in (4.451).

(2) Near Zone

In the near zone we have $k|\mathbf{r}-\mathbf{r}'| \ll 1$ so that to a good approximation the exponential function in the integrand of (4.447) can be put equal to 1. The vector potential is then, except for the harmonic time-dependence $e^{-i\omega t}$, identical to that of the magnetostatics. Retardation effects are completely suppressed.

4.5.3 Electric Dipole Radiation

Let us now go back to the expression (4.451) and investigate the first term in somewhat more detail:

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0\mu_\mathrm{r}}{4\pi}\,\frac{e^{ikr}}{r}\int d^3r'\,\mathbf{j}(\mathbf{r}') \ . \tag{4.454}$$

Let V be the volume of the $(\rho \neq 0, \mathbf{j} \neq 0)$-space region. For stationary (!) current densities the volume integral vanishes which we have proved as (3.40). But here that is no longer true. Let x_i' be a Cartesian component of $\mathbf{r}'$. Therewith:

$$\operatorname{div}(x_i'\mathbf{j}) = x_i' \operatorname{div}\mathbf{j} + \mathbf{j}\cdot\nabla x_i' = x_i' \operatorname{div}\mathbf{j} + j_i \ .$$

We use this to reformulate the volume integral:

$$\int d^3r'\, j_i(\mathbf{r}') = \int d^3r' \operatorname{div}(x_i'\mathbf{j}) - \int d^3r'\, x_i' \operatorname{div}\mathbf{j} \ .$$

Changing the first integral by use of the Gauss theorem into a surface integral one recognizes that this integral vanishes on an area which encloses the (finite) $\mathbf{j} \neq 0$-region:

$$\int d^3r'\, \mathbf{j}(\mathbf{r}') = -\int d^3r'\, \mathbf{r}' \operatorname{div}\mathbf{j} \ .$$

The continuity equation

$$\operatorname{div}\mathbf{j}(\mathbf{r},t) + \frac{\partial}{\partial t}\rho(\mathbf{r},t) = 0$$
$$\Longrightarrow \operatorname{div}\mathbf{j}(\mathbf{r}) - i\omega\rho(\mathbf{r}) = 0$$

allows for a further rearrangement:

$$\int d^3r'\, \mathbf{j}(\mathbf{r}') = -i\omega \int d^3r'\, \mathbf{r}'\rho(\mathbf{r}') \ .$$

On the right-hand side there appears the **electric dipole moment p** of the charge distribution ρ, already known from electrostatics (2.92),

$$\mathbf{p} = \int d^3r'\, \mathbf{r}'\rho(\mathbf{r}') \ , \tag{4.455}$$

by which the vector potential takes the following form:

$$\mathbf{A}_1(\mathbf{r}) = -i\omega \frac{\mu_0\mu_\mathrm{r}}{4\pi}\mathbf{p}\,\frac{e^{ikr}}{r} \ . \tag{4.456}$$

The nomenclature *electric dipole radiation* seems to be justified.

With $\mathbf{A}_1(\mathbf{r})$ we now calculate the electromagnetic fields. The relation

$$\operatorname{curl}(\mathbf{a}\varphi) = \varphi \operatorname{curl}\mathbf{a} - \mathbf{a}\times\nabla\varphi$$

at first yields, since $\mathbf{p}$ is independent of $\mathbf{r}$:

$$\operatorname{curl}\mathbf{A}_1(\mathbf{r}) = i\omega\frac{\mu_0\mu_\mathrm{r}}{4\pi}\mathbf{p}\times\left(\nabla\frac{e^{ikr}}{r}\right) ,$$

$$\nabla\frac{e^{ikr}}{r} = \mathbf{n}\frac{d}{dr}\frac{e^{ikr}}{r} = \mathbf{n}\,ik\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r} .$$

We thus get for the **magnetic induction**:

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0\mu_\mathrm{r}}{4\pi}u\,k^2\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)(\mathbf{n}\times\mathbf{p}) . \tag{4.457}$$

$\mathbf{B}_1$ is transversal to the position vector $\mathbf{r}$. If $\mathbf{p}$ defines the z-axis then the $\mathbf{B}$-field lines are concentric circles around the z-axis. The magnetic induction exhibits cylindrical symmetry. The calculation of the electric field turns out to be a bit more laborious. Starting point is (4.457):

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0\mu_\mathrm{r}}{4\pi}u\,k^2\Bigg\{&\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\operatorname{curl}(\mathbf{n}\times\mathbf{p})\\ &-(\mathbf{n}\times\mathbf{p})\times\left[\nabla\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right]\Bigg\} ,\end{aligned} \tag{4.458}$$

$$\begin{aligned}\nabla\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) &= \mathbf{n}\frac{e^{ikr}}{r}\left(ik-\frac{2}{r}+\frac{2}{ikr^2}\right) ,\\ (\mathbf{n}\times\mathbf{p})\times\nabla\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) &= ik\frac{e^{ikr}}{r}(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\\ &\quad+e^{ikr}\left(\frac{2}{r^2}-\frac{2}{ikr^3}\right)[\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}] ,\\ \operatorname{curl}(\mathbf{n}\times\mathbf{p}) &= (\mathbf{p}\cdot\nabla)\mathbf{n}-\underbrace{(\mathbf{n}\cdot\nabla)\mathbf{p}}_{=0}+\mathbf{n}\underbrace{\operatorname{div}\mathbf{p}}_{=0}-\mathbf{p}\operatorname{div}\mathbf{n} ,\\ \mathbf{p}\operatorname{div}\mathbf{n} &= \frac{2}{r}\mathbf{p} ,\\ (\mathbf{p}\cdot\nabla)\mathbf{n} &= \frac{1}{r}\mathbf{p}-\frac{1}{r}\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) .\end{aligned}$$

This we insert into (4.458):

$$\mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\frac{e^{ikr}}{r}\left\{k^2[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}]+\frac{1}{r}\left(\frac{1}{r}-ik\right)[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\right\} . \tag{4.459}$$

While $\mathbf{B}_1$ is polarized transversely to the radial unit-vector $\mathbf{n} = \mathbf{r}/r$, $\mathbf{E}_1$ has both longitudinal and transverse components. Let us still have a somewhat closer look at the fields in the various zones:

(1) Radiation Zone

Here it is $kr \gg 1$ and therewith:

$$\frac{k^2}{r} \gg \frac{k}{r^2} \gg \frac{1}{r^3} \; . \tag{4.460}$$

The electromagnetic fields now simplify as follows:

$$\mathbf{B}_1(\mathbf{r}) \simeq \frac{\mu_0\mu_\mathrm{r}}{4\pi} u k^2 \, \frac{e^{ikr}}{r} (\mathbf{n} \times \mathbf{p}) \; , \tag{4.461}$$

$$\mathbf{E}_1(\mathbf{r}) \simeq u(\mathbf{B}_1(\mathbf{r}) \times \mathbf{n}) \; . \tag{4.462}$$

Hence, in the radiation zone $\mathbf{E}_1(\mathbf{r})$, too, is transversal to $\mathbf{n}$. $\mathbf{E}_1$, $\mathbf{B}_1$ and $\mathbf{n}$ build locally an orthogonal trihedron (Fig. 4.84).

Typically the fields fall off in the radiation zone like spherical waves according to $1/r$:

$$|\mathbf{E}_1| \underset{r\to\infty}{\longrightarrow} \frac{1}{r} \; ,$$

$$|\mathbf{B}_1| \underset{r\to\infty}{\longrightarrow} \frac{1}{r} \; .$$

For the time-averaged **energy density** in these dipole fields Eq. (4.209) holds:

$$\overline{w}_1(\mathbf{r}) = \frac{1}{4}\left(\frac{1}{\mu_0\mu_\mathrm{r}}|\mathbf{B}_1|^2 + \epsilon_0\epsilon_\mathrm{r}|\mathbf{E}_1|^2\right) \; .$$

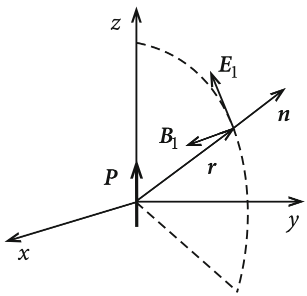


Fig. 4.84 Field directions of the dipole radiation in the radiation zone

With (4.462) we can write,

$$|\mathbf{E}_1|^2 = u^2 |\mathbf{B}_1|^2 \ ,$$

getting therewith

$$\overline{w}_1(\mathbf{r}) = \frac{1}{2\mu_0\mu_r} |\mathbf{B}_1|^2 \ .$$

The energy density of the electromagnetic dipole field in the radiation zone then results with (4.461) in:

$$\begin{aligned} \overline{w}_1(\mathbf{r}) &= \frac{1}{32\pi^2\epsilon_0\epsilon_r} \frac{(k^2 p)^2}{r^2} \sin^2\vartheta \ , \\ \vartheta &= \sphericalangle(\mathbf{n}, \mathbf{p}) \ . \end{aligned} \tag{4.463}$$

For the time-averaged **energy current density** we now apply (4.210):

$$\begin{aligned} \overline{\mathbf{S}}_1(\mathbf{r}) &= \frac{1}{2\mu_0\mu_r} \mathrm{Re}\left(\mathbf{E}_1(\mathbf{r}) \times \mathbf{B}_1^*(\mathbf{r})\right) \\ &= \frac{u}{2\mu_0\mu_r} \mathrm{Re}\left[(\mathbf{B}_1(\mathbf{r}) \times \mathbf{n}) \times \mathbf{B}_1^*(\mathbf{r})\right] \\ &= \frac{u}{2\mu_0\mu_r} \mathrm{Re}\left[-\mathbf{B}_1 \underbrace{(\mathbf{n} \cdot \mathbf{B}_1^*)}_{=0} + \mathbf{n} \, |\mathbf{B}_1|^2\right] \\ &= \mathbf{n} \frac{u}{2\mu_0\mu_r} \, |\mathbf{B}_1(\mathbf{r})|^2 \ . \end{aligned} \tag{4.464}$$

It obviously holds again

$$\overline{\mathbf{S}}_1(\mathbf{r}) = \mathbf{n} \, u \, \overline{w}_1(\mathbf{r})$$

and therewith

$$\overline{\mathbf{S}}_1(\mathbf{r}) = \frac{u}{32\pi^2\epsilon_0\epsilon_r} \frac{\left(k^2 p\right)^2}{r^2} \sin^2\vartheta \ \mathbf{n} \ . \tag{4.465}$$

The energy streams with the wave velocity u in the direction of the position vector,

$$\begin{aligned} d\mathbf{f} &= df \mathbf{n} \ , \\ df &= r^2 d\Omega \ , \end{aligned}$$

where $d\Omega = \sin\vartheta \, d\vartheta \, d\varphi$.

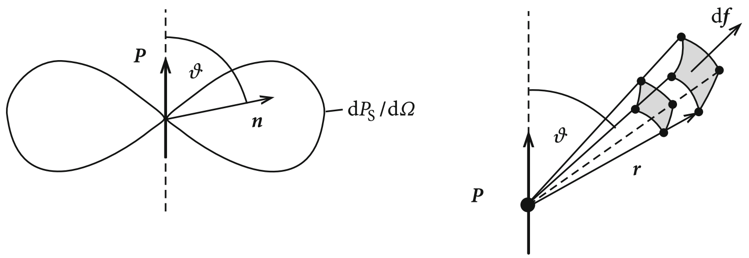


Fig. 4.85 Dipole characteristic for fixing the radiation power per solid-angle element

Often one discusses the radiated **power** per solid-angle element $d\Omega$:

$$dP_S^{(1)} = \overline{\mathbf{S}}_1(\mathbf{r}) \cdot d\mathbf{f}: \quad \text{radiation power through area the element } d\mathbf{f} \text{ at } \mathbf{r}$$

$$\Longrightarrow \quad \frac{dP_S^{(1)}}{d\Omega} = r^2\, \overline{\mathbf{S}}_1(\mathbf{r}) \cdot \mathbf{n} = \frac{u}{32\pi^2 \epsilon_0 \epsilon_r} k^4 p^2 \sin^2 \vartheta \ . \tag{4.466}$$

That is the typical **dipole characteristic** (Fig. 4.85). The dipole radiates strongest perpendicular to the dipole moment. No emission happens along the dipole axis. The characteristic is rotational-symmetric about the **p**-axis.

We get the total radiation power $P_S^{(1)}$ by integration over all solid angles:

$$\int d\Omega \sin^2 \vartheta = 2\pi \int_{-1}^{+1} d\cos\vartheta\,(1 - \cos^2 \vartheta) = 2\pi \left(\cos\vartheta - \frac{1}{3}\cos^3 \vartheta \right)\Bigg|_{-1}^{+1} = \frac{8\pi}{3}$$

$$\Longrightarrow \quad P_S^{(1)} = \frac{u}{12\pi\, \epsilon_0 \epsilon_r} k^4 p^2 \quad \left(k = \frac{\omega}{u} \right) . \tag{4.467}$$

The proportionality of the radiation power to the fourth power of the frequency and to the square of the dipole moment is the important aspect of this formula.

(2) Near Zone

For this region we have $kr \ll 1$ and therewith:

$$\frac{k^2}{r} \ll \frac{k}{r^2} \ll \frac{1}{r^3} \ . \tag{4.468}$$

With $e^{ikr} \approx 1$ the fields simplify now to:

$$\mathbf{E}_1(\mathbf{r}) \approx \frac{1}{4\pi\, \epsilon_0 \epsilon_r} \frac{3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}}{r^3} \ , \tag{4.469}$$

$$\mathbf{B}_1(\mathbf{r}) \approx \frac{\mu_0 \mu_r}{4\pi} u \frac{ik}{r^2} (\mathbf{n} \times \mathbf{p}) \ . \tag{4.470}$$

The electric field corresponds to the electrostatic dipole field (2.73), of course except for the harmonic time-dependence $e^{-i\omega t}$. Since we could assume in the near zone generally $\exp(ik|\mathbf{r}-\mathbf{r}'|) \approx 1$ **no** retardation effects appear in the near zone.

If one compares $|\mathbf{E}_1|$ with $|u\mathbf{B}_1|$, one realizes that because of $1/r^3 \gg k/r^2$ the electromagnetic field will be dominantly of electric character in the near zone.

4.5.4 Electric Quadrupole and Magnetic Dipole Radiation

We want to go a step forward in the expansion (4.451) of the vector potential and discuss the next higher term:

$$\mathbf{A}_2(\mathbf{r}) = \frac{\mu_0\mu_\mathrm{r}}{4\pi}\left(\frac{1}{r} - ik\right)\frac{e^{ikr}}{r}\int d^3r'\,\mathbf{j}(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')\ . \tag{4.471}$$

$\mathbf{A}_2$ can be decomposed into two characteristic terms. To see this we rewrite the integrand as follows:

$$\mathbf{n}\times(\mathbf{r}'\times\mathbf{j}) = \mathbf{r}'(\mathbf{n}\cdot\mathbf{j}) - \mathbf{j}(\mathbf{n}\cdot\mathbf{r}')$$

$$\Longrightarrow\quad (\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}(\mathbf{r}') = \frac{1}{2}(\mathbf{r}'\times\mathbf{j})\times\mathbf{n} + \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}(\mathbf{r}') + \big(\mathbf{n}\cdot\mathbf{j}(\mathbf{r}')\big)\,\mathbf{r}'\right]\ .$$

That can be written as:

$$\mathbf{A}_2(\mathbf{r}) = \mathbf{A}_2^{\mathrm{mD}}(\mathbf{r}) + \mathbf{A}_2^{\mathrm{eQ}}(\mathbf{r})\ . \tag{4.472}$$

The first term corresponds to **magnetic dipole radiation:**

$$\mathbf{A}_2^{\mathrm{mD}}(\mathbf{r}) = -\frac{\mu_0\mu_\mathrm{r}}{4\pi}\left(\frac{1}{r} - ik\right)\frac{e^{ikr}}{r}\left[\mathbf{n}\times\frac{1}{2}\int d^3r'\,\big(\mathbf{r}'\times\mathbf{j}(\mathbf{r}')\big)\right]\ .$$

In this expression we recognize the magnetic moment **m** defined in (3.43):

$$\mathbf{A}_2^{\mathrm{mD}}(\mathbf{r}) = ik\frac{\mu_0\mu_\mathrm{r}}{4\pi}\,\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n}\times\mathbf{m})\ . \tag{4.473}$$

The second term in (4.472) describes **electric quadrupole radiation:**

$$\mathbf{A}_2^{\mathrm{eQ}} = \frac{\mu_0\mu_\mathrm{r}}{8\pi}\,\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d^3r'\,\left[(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}(\mathbf{r}') + \big(\mathbf{n}\cdot\mathbf{j}(\mathbf{r}')\big)\,\mathbf{r}'\right]\ .$$

For a further rewriting we use

$$\int d^3r'\mathrm{div}\left[x'(\mathbf{n}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}')\right] = \int d^3r'\,x'\mathrm{div}\left[(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}(\mathbf{r}')\right] + \int d^3r'\,(\mathbf{n}\cdot\mathbf{r}')\,\mathbf{j}(\mathbf{r}')\cdot\nabla x'\ .$$

The left-hand side is zero since $\mathbf{j}$ is restricted to a finite space region (Gauss theorem!). Thus it follows:

$$\int d^3r' \, (\mathbf{n}\cdot\mathbf{r}')j_x(\mathbf{r}') = -\int d^3r' \, x' \Big[(\mathbf{n}\cdot\mathbf{r}')\mathrm{div}\,\mathbf{j}(\mathbf{r}') + \mathbf{j}(\mathbf{r}')\cdot \underbrace{\nabla_{r'}(\mathbf{n}\cdot\mathbf{r}')}_{\mathbf{n}}\Big] .$$

Corresponding relations exist also for the other components:

$$\begin{aligned}\int d^3r' \, \left[(\mathbf{n}\cdot\mathbf{r}') \, \mathbf{j}(\mathbf{r}') + \mathbf{r}' \left(\mathbf{n}\cdot\mathbf{j}(\mathbf{r}')\right)\right] &= -\int d^3r' \, \mathbf{r}'(\mathbf{n}\cdot\mathbf{r}')\mathrm{div}\,\mathbf{j}(\mathbf{r}') \\ &= -\int d^3r' \, \mathbf{r}'(\mathbf{n}\cdot\mathbf{r}') \left(i\omega\rho(\mathbf{r}')\right) .\end{aligned}$$

In the last step we have again applied the continuity equation:

$$\mathbf{A}_2^{\mathrm{eQ}}(\mathbf{r}) = -\frac{1}{2}uk^2\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\frac{\mu_0\mu_{\mathrm{r}}}{4\pi}\int d^3r' \, \mathbf{r}'(\mathbf{n}\cdot\mathbf{r}')\rho(\mathbf{r}') . \tag{4.474}$$

The integrand contains a moment of second order of the charge density ρ. Hence, it is a quadrupole term as the subsequent analysis will yet further clarify.

(1) Magnetic Dipole Radiation

For this type of radiation we can find the electromagnetic fields without explicit calculation, simply by analogy-observations on the electric dipole radiation in Sect. 4.5.3.

By comparison of (4.473) with (4.457) we recognize the following mapping:

$$\mathbf{A}_2^{\mathrm{mD}}(\mathbf{r}) \underset{m\leftrightarrow p}{\longleftrightarrow} \frac{i}{\omega}\mathbf{B}_1(\mathbf{r}) . \tag{4.475}$$

Because of

$$\mathbf{B}_2^{\mathrm{mD}}(\mathbf{r}) = \mathrm{curl}\,\mathbf{A}_2^{\mathrm{mD}}(\mathbf{r})$$

and because of (4.448)

$$\mathbf{E}_1(\mathbf{r}) = \frac{iu^2}{\omega}\mathrm{curl}\,\mathbf{B}_1(\mathbf{r}) = u^2\mathrm{curl}\left(\frac{i}{\omega}\mathbf{B}_1(\mathbf{r})\right)$$

we get the further mapping:

$$\mathbf{B}_2^{\mathrm{mD}}(\mathbf{r}) \underset{m\leftrightarrow p}{\longleftrightarrow} \frac{1}{u^2}\mathbf{E}_1(\mathbf{r}) . \tag{4.476}$$

We can therewith read off directly from (4.459) the magnetic induction of the magnetic dipole $\mathbf{m}$ belonging to the current density $\mathbf{j}$:

$$\mathbf{B}_2^{\mathrm{mD}}(\mathbf{r}) = \frac{\mu_0\mu_\mathrm{r}}{4\pi}\frac{e^{ikr}}{r}\left\{k^2\left[(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\right] + \frac{1}{r}\left(\frac{1}{r} - ik\right)\left[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}\right]\right\} . \tag{4.477}$$

From the law of induction

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r},t) = -\dot{\mathbf{B}}(\mathbf{r},t)$$

we find because of the assumed harmonic time-dependence:

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r}) = i\omega\mathbf{B}(\mathbf{r}) .$$

This holds in particular for the electric dipole radiation

$$-\frac{i}{\omega}\mathrm{curl}\,\mathbf{E}_1(\mathbf{r}) = \mathbf{B}_1(\mathbf{r}) .$$

The comparison with

$$\frac{iu^2}{\omega}\mathrm{curl}\,\mathbf{B}_2^{\mathrm{mD}}(\mathbf{r}) = \mathbf{E}_2^{\mathrm{mD}}(\mathbf{r})$$

delivers the last, still missing mapping:

$$\mathbf{E}_2^{\mathrm{mD}}(\mathbf{r}) \underset{m\leftrightarrow p}{\longleftrightarrow} -\mathbf{B}_1(\mathbf{r}) . \tag{4.478}$$

Eventually, it follows with (4.457):

$$\mathbf{E}_2^{\mathrm{mD}}(\mathbf{r}) = -\frac{1}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\frac{k^2}{u}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n}\times\mathbf{m}) . \tag{4.479}$$

All the previous statements about the electric dipole radiation can be transferred by use of the mappings (4.475), (4.476), (4.478) to the magnetic dipole radiation, if only one replaces the electric dipole moment ($\mathbf{p}$) by the magnetic one ($\mathbf{m}$). There are only minor differences. For instance, the electric field of the *electric* dipole radiation is polarized, according to (4.459), in the plane which is spanned by the vectors $\mathbf{n}$ and $\mathbf{p}$, while the electric field of the *magnetic* dipole radiation is oriented perpendicular to the plane defined by $\mathbf{n}$ and $\mathbf{m}$.

When calculating the energy-current density $\overline{\mathbf{S}}_2^{\mathrm{mD}}$ of the magnetic dipole radiation, one has to only replace $\mathbf{p}$ by $\mathbf{m}$ in the corresponding expression (4.465) for the electric dipole radiation. The mappings (4.476), (4.478) cause on the whole still a factor $1/u^2$. For the power emitted into the solid angle $d\Omega$ one thus finds as in (4.466):

$$\frac{dP_\mathrm{S}^{(2)}}{d\Omega_{\mathrm{mD}}} = \frac{1}{32\pi^2\epsilon_0\epsilon_\mathrm{r}}\frac{k^4m^2}{u}\sin^2\vartheta . \tag{4.480}$$

Hence, it can not be decided from the angular distribution only whether it is about electric or magnetic dipole radiation.

(2) Electric Quadrupole Radiation

We discuss now the term (4.474) for the vector potential which, as mentioned, represents electric quadrupole radiation. The integral is a vectorial quantity,

$$\mathbf{I}(\vartheta,\varphi) = \int d^3r'\,\mathbf{r}'(\mathbf{n}\cdot\mathbf{r}')\rho(\mathbf{r}') \equiv (I_1, I_2, I_3)\ ,$$

for the components of which we can write:

$$\begin{aligned} I_j(\vartheta,\varphi) &= \int d^3r'\,x'_j\left(\sum_{i=1}^{3} n_i x'_i\right)\rho(\mathbf{r}') \\ &= \frac{1}{3}\sum_{i=1}^{3} n_i \int d^3r'\,\left(3x'_j x'_i - r'^2\delta_{ij}\right)\rho(\mathbf{r}') + \frac{1}{3}n_j\int d^3r'\,r'^2\rho(\mathbf{r}')\ . \end{aligned}$$

This expression includes the *quadrupole tensor* (2.93):

$$\underline{\mathbf{Q}} = \left(Q_{ij}\right)_{i,j=1,2,3}\ ;\quad Q_{ij} = \int d^3r'\,\left(3x'_i x'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')\ .$$

We define

$$\mathbf{Q}(\mathbf{n}) = (Q_1(\mathbf{n}), Q_2(\mathbf{n}), Q_3(\mathbf{n})) \tag{4.481}$$

with

$$Q_i(\mathbf{n}) = \sum_{j=1}^{3} Q_{ij} n_j\ .$$

That means for $\mathbf{I}$:

$$\mathbf{I} = \frac{1}{3}\left(\mathbf{Q}(\mathbf{n}) + \mathbf{n}\int d^3r'\,r'^2\rho(\mathbf{r}')\right)\ .$$

The vector potential then reads:

$$\mathbf{A}_2^{\mathrm{eQ}}(\mathbf{r}) = -uk^2\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\frac{\mu_0\mu_{\mathrm{r}}}{24\pi}\left(\mathbf{Q}(\mathbf{n}) + \mathbf{n}\int d^3r'\,r'^2\rho(\mathbf{r}')\right)\ . \tag{4.482}$$

The electromagnetic fields to be derived from this formula are rather complicated. We therefore restrict ourselves here to an investigation of the **radiation zone**. For this the estimation (4.460) is applicable and therewith:

$$\mathbf{A}_2^{\mathrm{eQ}}(\mathbf{r}) \approx -uk^2 \frac{e^{ikr}}{r} \frac{\mu_0\mu_\mathrm{r}}{24\pi} \left(\mathbf{Q}(\mathbf{n}) + \mathbf{n} \int d^3r'\, r'^2 \rho(\mathbf{r}') \right) .$$

One finds

$$\begin{aligned} \operatorname{curl} \mathbf{n} &= 0 , \\ \operatorname{curl} \mathbf{Q}(\mathbf{n}) &= \mathcal{O}\left(\frac{1}{r}\right) , \\ \nabla \frac{e^{ikr}}{r} &= \mathbf{n}\, ik \frac{e^{ikr}}{r} \left[1 + \mathcal{O}\left(\frac{1}{r}\right) \right] , \end{aligned}$$

so that with the vector formula

$$\operatorname{curl}(\mathbf{a}\varphi) = \varphi \operatorname{curl} \mathbf{a} - \mathbf{a} \times \nabla\varphi$$

the magnetic induction of the electric quadrupole radiation can be written as:

$$\mathbf{B}_2^{\mathrm{eQ}}(\mathbf{r}) \approx ik\, \mathbf{n} \times \mathbf{A}_2^{\mathrm{eQ}}(\mathbf{r}) = -i \frac{\mu_0\mu_\mathrm{r}}{24\pi} uk^3 \frac{e^{ikr}}{r} (\mathbf{n} \times \mathbf{Q}(\mathbf{n})) \tag{4.483}$$

If one compares this result with the expression (4.461) for the magnetic induction $\mathbf{B}_1(\mathbf{r})$ of the electric dipole radiation then one realizes that, there, only the electric dipole moment $\mathbf{p}$ has to be replaced by $(-i(k/6)\mathbf{Q}(\mathbf{n}))$ in order to get (4.483). With the corresponding replacement we can therefore adopt also the expression (4.462) for the electric field:

$$\mathbf{E}_2^{\mathrm{eQ}}(\mathbf{r}) \approx u \left(\mathbf{B}_2^{\mathrm{eQ}}(\mathbf{r}) \times \mathbf{n} \right) \approx -i \frac{k^3}{24\pi\, \epsilon_0\epsilon_\mathrm{r}} \frac{e^{ikr}}{r} \left[(\mathbf{n} \times \mathbf{Q}(\mathbf{n})) \times \mathbf{n} \right] . \tag{4.484}$$

$\mathbf{E}_2^{\mathrm{eQ}}$, $\mathbf{B}_2^{\mathrm{eQ}}$, $\mathbf{n}$ build a local orthogonal trihedron.

The time-averaged **energy density**,

$$\overline{w}_2^{\mathrm{eQ}}(\mathbf{r}) = \frac{1}{2\mu_0\mu_\mathrm{r}} \left| \mathbf{B}_2^{\mathrm{eQ}} \right|^2 = \frac{1}{4\pi\, \epsilon_0\epsilon_\mathrm{r}} \frac{k^6}{288\pi} \frac{|\mathbf{n} \times \mathbf{Q}|^2}{r^2} , \tag{4.485}$$

belongs to this electric quadrupole radiation as well as the **energy current density**

$$\overline{\mathbf{S}}_2^{\mathrm{eQ}} = \mathbf{n}\, u\, \overline{w}_2^{\mathrm{eQ}} \tag{4.486}$$

and the **power** emitted per solid angle:

$$\left(\frac{dP_S^{(2)}}{d\Omega}\right)_{eQ} = \frac{1}{4\pi\,\epsilon_0\epsilon_r}\,\frac{uk^6}{288\pi}|\mathbf{n}\times\mathbf{Q}|^2\,. \tag{4.487}$$

The radiation pattern is for the general case rather complicated and can be presented in closed form only for simple geometries.

Example We consider an oscillating charge distribution with a quadrupole moment of the type (2.102) (*oscillating stretched quadrupole*):

$$Q_{ij} = 0 \quad \text{for } i \neq j\,,$$
$$Q_{33} = Q\,; \quad Q_{11} = Q_{22} = -\frac{1}{2}Q\,.$$

The quadrupole tensor is trace-less:

$$|\mathbf{n}\times\mathbf{Q}|^2 = (n_2Q_3 - n_3Q_2)^2 + (n_3Q_1 - n_1Q_3)^2 + (n_1Q_2 - n_2Q_1)^2\,.$$

In our example it is

$$Q_1 = -\frac{1}{2}Q\,n_1\,; \quad Q_2 = -\frac{1}{2}Q\,n_2\,; \quad Q_3 = Q\,n_3\,.$$

and therewith:

$$\begin{aligned}|\mathbf{n}\times\mathbf{Q}|^2 &= Q^2\left[\left(n_2n_3 + \frac{1}{2}n_3n_2\right)^2 + \left(\frac{1}{2}n_1n_3 + n_1n_3\right)^2 + \left(-\frac{1}{2}n_1n_2 + \frac{1}{2}n_1n_2\right)^2\right]\\ &= \frac{9}{4}Q^2\left[(n_2n_3)^2 + (n_1n_3)^2\right] = \frac{9}{4}Q^2\frac{z^2}{r^4}(y^2 + x^2) = \frac{9}{4}Q^2\cos^2\vartheta\,\sin^2\vartheta\,.\end{aligned}$$

It then follows:

$$\left(\frac{dP_S^{(2)}}{d\Omega}\right)_{eQ} \sim Q^2\cos^2\vartheta\,\sin^2\vartheta\,. \tag{4.488}$$

The radiation power is therefore maximal in the directions $\vartheta = \pi/4$ and $\vartheta = 3\pi/4$. It disappears for $\vartheta = 0, \pi/2$ and π (Fig. 4.86). It is rotational-symmetric with respect to the z-axis.

Final Remark

Higher expansions of the exact basic formula (4.447), beyond that performed in (4.451), are becoming more and more complicated and magnetic and electric contributions cannot be decoupled without further ado. Furthermore, we should not forget that all considerations performed here are strictly bound to the precondition (4.449) $d \ll \lambda, r$.

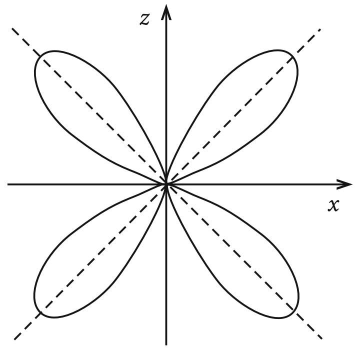


Fig. 4.86 Typical directionality of the electric quadrupole-radiation power

An exact, here not feasible multipole-expansion is in principle possible; however, mathematically not so simple, but in return independent of restrictions of any kind.

4.5.5 Radiation of Moving Point Charges

Finally, we want to discuss a special application of the retarded potentials (4.442) and (4.443). A point charge q which moves along a path $\mathbf{R}(t)$ with the (momentary) velocity $\mathbf{V}(t)$ creates a time variant electromagnetic field which we now be calculate. For that, we investigate the potentials belonging to the **charge density**

$$\rho(\mathbf{r}, t) = q\,\delta\,(\mathbf{r} - \mathbf{R}(t)) \tag{4.489}$$

and the **current density**

$$\mathbf{j}(\mathbf{r}, t) = q\,\mathbf{V}(t)\,\delta\,(\mathbf{r} - \mathbf{R}(t)) \ . \tag{4.490}$$

(1) Electromagnetic Potentials

We use for the potentials the expression (4.431) with the retarded Green's function (4.439):

$$\psi(\mathbf{r}, t) = \int d^3r' \int dt' \, \frac{\sigma(\mathbf{r}', t')}{4\pi|\mathbf{r} - \mathbf{r}'|}\,\delta\left(\frac{|\mathbf{r} - \mathbf{r}'|}{u} - t + t'\right) \ . \tag{4.491}$$

Here the following assignment is valid:

$$\sigma(\mathbf{r}', t') = \frac{\rho(\mathbf{r}', t')}{\epsilon_0\epsilon_\mathrm{r}} \iff \psi(\mathbf{r}, t) = \varphi(\mathbf{r}, t) \ ,$$
$$\sigma(\mathbf{r}', t') = \mu_0\mu_\mathrm{r}\,\mathbf{j}(\mathbf{r}', t') \iff \psi(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) \ .$$

The $\mathbf{r}'$-integration can be immediately performed because of (4.489) and (4.490), respectively:

$$\varphi(\mathbf{r}, t) = \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}} \int dt' \, \frac{\delta\left(\frac{1}{u}|\mathbf{r} - \mathbf{R}(t')| - t + t'\right)}{|\mathbf{r} - \mathbf{R}(t')|} \,, \tag{4.492}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0\mu_\mathrm{r}}{4\pi} q \int dt' \, \mathbf{V}(t') \, \frac{\delta\left(\frac{1}{u}|\mathbf{r} - \mathbf{R}(t')| - t + t'\right)}{|\mathbf{r} - \mathbf{R}(t')|} \,. \tag{4.493}$$

Since $\mathbf{R} = \mathbf{R}(t')$ the t'-integration is not so directly performable. We abbreviate

$$f(t') = \frac{1}{u}|\mathbf{r} - \mathbf{R}(t')| - t + t' \tag{4.494}$$

and exploit the property (1.10) of the δ-function:

$$\delta[f(t')] = \sum_{j=1}^{n} \frac{\delta(t' - t_j)}{\left|\left(\frac{df}{dt'}\right)_{t'=t_j}\right|} \,.$$

t_j are the simple zeros of the function $f(t')$.

$$\frac{df}{dt'} = 1 + \frac{1}{u} \frac{d}{dt'} |\mathbf{r} - \mathbf{R}(t')| = 1 - \frac{1}{u} \frac{(\mathbf{r} - \mathbf{R}(t')) \cdot \mathbf{V}(t')}{|\mathbf{r} - \mathbf{R}(t')|} \,. \tag{4.495}$$

Because of the unit-vector on the right-hand side we can estimate:

$$1 - \frac{V(t')}{u} \leq \frac{df}{dt'} \leq 1 + \frac{V(t')}{u} \,.$$

The particle-velocity V is in all cases smaller than the velocity of light u so that, because of

$$\frac{df}{dt'} > 0 \,,$$

$f(t')$ turns out to be a monotonically increasing function which can have at most one zero. In the case that there is no zero at all, one comes to the physically unrealistic situation $\varphi \equiv 0$, $\mathbf{A} \equiv 0$. We therefore can assume that $f(t')$ has exactly one zero $t' = t_\mathrm{ret}$ which comes out as the solution of the equation

$$t_\mathrm{ret}(\mathbf{r}, t) = t - \frac{1}{u} \left|\mathbf{r} - \mathbf{R}\left(t_\mathrm{ret}\right)\right| \,. \tag{4.496}$$

Therewith we are now able to perform the t'-integration in the expressions of the potentials:

$$\varphi(\mathbf{r},t) = \frac{q}{4\pi\,\epsilon_0\epsilon_r\left(|\mathbf{r}-\mathbf{R}(t_{\text{ret}})| - \frac{1}{u}\left(\mathbf{r}-\mathbf{R}(t_{\text{ret}})\right)\cdot\mathbf{V}(t_{\text{ret}})\right)}\,, \tag{4.497}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\mu_r q\,\mathbf{V}(t_{\text{ret}})}{4\pi\left(|\mathbf{r}-\mathbf{R}(t_{\text{ret}})| - \frac{1}{u}\left(\mathbf{r}-\mathbf{R}(t_{\text{ret}})\right)\cdot\mathbf{V}(t_{\text{ret}})\right)}\,. \tag{4.498}$$

These are the electromagnetic potentials of an arbitrarily moving particle which are called

Liénard-Wiechert potentials

They cannot b calculated easily because of the retardation (4.496) for complicated particle paths. t_{ret} accounts for the finite transit time of the electromagnetic wave from the momentary particle position $\mathbf{R}$ to the point of observation $\mathbf{r}$ (Fig. 4.87):

Retarded distance vector

$$\mathbf{D}_{\text{ret}}(\mathbf{r},t) = \mathbf{r} - \mathbf{R}(t_{\text{ret}})\,. \tag{4.499}$$

With the further definitions,

$$\mathbf{n}_{\text{ret}}(\mathbf{r},t) = \frac{\mathbf{D}_{\text{ret}}(\mathbf{r},t)}{D_{\text{ret}}(\mathbf{r},t)}\,, \tag{4.500}$$

$$\kappa_{\text{ret}}(\mathbf{r},t) = 1 - \frac{1}{u}\mathbf{n}_{\text{ret}}\cdot\mathbf{V}(t_{\text{ret}})\,,$$

the potentials can be written more compactly:

$$\varphi(\mathbf{r},t) = \frac{q}{4\pi\,\epsilon_0\epsilon_r D_{\text{ret}}\kappa_{\text{ret}}(\mathbf{r},t)}\,, \tag{4.501}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\mu_r q\,\mathbf{V}(t_{\text{ret}})}{4\pi\,D_{\text{ret}}\kappa_{\text{ret}}(\mathbf{r},t)}\,. \tag{4.502}$$

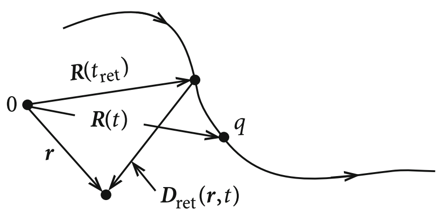


Fig. 4.87 Illustration of the retarded distance vector

(2) Special Cases

(a) **Point charge at rest:**

$$\mathbf{V} \equiv \mathbf{0} \iff \mathbf{R}(t) \equiv \mathbf{R}_0 \ .$$

Inserting these data into (4.497) and (4.498), respectively, leads to the result, which is well-known from electrostatics:

$$\varphi(\mathbf{r}, t) = \frac{q}{4\pi\,\epsilon_0\epsilon_r|\mathbf{r}-\mathbf{R}_0|} \ ; \quad \mathbf{A}(\mathbf{r}, t) \equiv 0 \ .$$

(b) **Uniformly moving point charge:**

$$\mathbf{V} \equiv \mathbf{v}_0 = \text{const} \ ; \quad \mathbf{R}(t) = \mathbf{R}_0 + \mathbf{v}_0 t \ .$$

At first we have to determine t_{ret} by the retardation condition (4.496) (Fig. 4.88):

$$\begin{aligned}
D_{\text{ret}}(\mathbf{r}, t) = u(t - t_{\text{ret}}) &= |\mathbf{r} - \mathbf{R}(t_{\text{ret}})| = |\mathbf{r} - \mathbf{R}_0 - \mathbf{v}_0 t_{\text{ret}}| \\
&= |\mathbf{r} - \mathbf{R}(t) + \mathbf{v}_0(t - t_{\text{ret}})| \ , \\
\mathbf{D}(\mathbf{r}, t) &= \mathbf{r} - \mathbf{R}(t) \qquad\qquad (4.503) \\
\implies (u^2 - v_0^2)(t - t_{\text{ret}})^2 &= D^2(\mathbf{r}, t) + 2\mathbf{D} \cdot \mathbf{v}_0(t - t_{\text{ret}}) \\
&= D^2(\mathbf{r}, t) + 2v_0 D(\mathbf{r}, t) \cos\alpha (t - t_{\text{ret}}) \\
\implies (t - t_{\text{ret}})^2 - 2\frac{D\,v_0 \cos\alpha}{u^2 - v_0^2}(t - t_{\text{ret}}) &= \frac{D^2}{u^2 - v_0^2} \\
\implies t - t_{\text{ret}} &= \frac{D\,v_0 \cos\alpha}{u^2 - v_0^2} \pm \sqrt{\frac{D^2}{u^2 - v_0^2} + \frac{D^2 v_0^2 \cos^2\alpha}{(u^2 - v_0^2)^2}} \ .
\end{aligned}$$

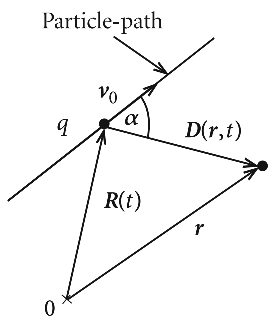


Fig. 4.88 For the calculation of the potentials of a uniformly moving particle

Since $u > v_0$ and $t > t_{\mathrm{ret}}$ must hold, only the positive sign can be valid, i.e.

$$t - t_{\mathrm{ret}} = \frac{D(\mathbf{r}, t)}{u^2 - v_0^2} \left(v_0 \cos\alpha + \sqrt{u^2 - v_0^2 \sin^2\alpha} \right) . \tag{4.504}$$

Therewith it further follows:

$$\begin{aligned} D_{\mathrm{ret}}(\mathbf{r}, t) - \frac{1}{u}\mathbf{D}_{\mathrm{ret}}(\mathbf{r}, t) \cdot \mathbf{V}(t_{\mathrm{ret}}) &= u(t - t_{\mathrm{ret}}) - \frac{1}{u}\mathbf{v}_0 \cdot (\mathbf{D}(\mathbf{r}, t) + \mathbf{v}_0(t - t_{\mathrm{ret}})) \\ &= \frac{1}{u}(u^2 - v_0^2)(t - t_{\mathrm{ret}}) - \frac{1}{u}\mathbf{v}_0 \cdot \mathbf{D}(\mathbf{r}, t) \\ &= \frac{1}{u}D(\mathbf{r}, t) \left(v_0 \cos\alpha + \sqrt{u^2 - v_0^2 \sin^2\alpha} - v_0 \cos\alpha \right) \\ &= |\mathbf{r} - \mathbf{R}(t)| \sqrt{1 - \frac{v_0^2}{u^2} \sin^2\alpha} \; . \end{aligned}$$

This means in (4.497):

$$\begin{aligned} \varphi(\mathbf{r}, t) &= \frac{q}{4\pi\,\epsilon_0 \epsilon_{\mathrm{r}} |\mathbf{r} - \mathbf{R}(t)|} \frac{1}{\sqrt{1 - \frac{v_0^2}{u^2} \sin^2\alpha}} \; , \\ \mathbf{A}(\mathbf{r}, t) &= \frac{1}{u^2}\mathbf{v}_0 \varphi(\mathbf{r}, t) \; . \end{aligned} \tag{4.505}$$

(3) Electromagnetic Fields

Analogous to (4.500) we define:

$$\begin{aligned} \mathbf{n}(\mathbf{r}, t) &= \frac{\mathbf{D}(\mathbf{r}, t)}{D(\mathbf{r}, t)} \; , \\ \kappa(\mathbf{r}, t) &= 1 - \frac{1}{u}\mathbf{n}(\mathbf{r}, t) \cdot \mathbf{V}(t) \; . \end{aligned} \tag{4.506}$$

Therewith follows, for instance:

$$\begin{aligned} \frac{\partial}{\partial t'} \delta \left(\frac{1}{u} D(\mathbf{r}, t') - t + t' \right) &= \left(1 + \frac{1}{u} \frac{\partial D}{\partial t'} \right) \delta'(\ldots) \\ &= \left(1 - \frac{1}{u}\mathbf{n}(\mathbf{r}, t') \cdot \mathbf{V}(t') \right) \delta'(\ldots) \\ &= \kappa(\mathbf{r}, t')\, \delta' \left(\frac{1}{u} D(\mathbf{r}, t') - t + t' \right) \; . \end{aligned} \tag{4.507}$$

$\delta'(\ldots)$ denotes the derivative of the δ-function with respect to the full argument:

$$\frac{\partial}{\partial t}\mathbf{n}(\mathbf{r},t) = -\frac{\dot{D}\,\mathbf{D}}{D^2} + \frac{\dot{\mathbf{D}}}{D} = -\frac{\dot{D}}{D}\mathbf{n} - \frac{\mathbf{V}}{D} = -\frac{1}{D}\left[-\left(\mathbf{n}\cdot\mathbf{V}(t)\right)\mathbf{n} + \mathbf{V}(t)\right]$$
$$= \frac{1}{D}\mathbf{n}\times(\mathbf{n}\times\mathbf{V})\,. \tag{4.508}$$

For the calculation of the **E**-field we conveniently use the original integral form (4.492):

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\nabla\varphi(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\\
&= \frac{-q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\int dt'\left(\nabla_r + \frac{\mathbf{V}(t')}{u^2}\frac{\partial}{\partial t}\right)\frac{\delta\left(\frac{1}{u}D(\mathbf{r},t') - t + t'\right)}{D(\mathbf{r},t')}\\
&= \frac{-q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\int dt'\left\{-\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')}\,\delta\left(\frac{1}{u}D(\mathbf{r},t') - t + t'\right)\right.\\
&\qquad\left.+\left(\frac{1}{u}\frac{\mathbf{n}(\mathbf{r},t')}{D(\mathbf{r},t')} - \frac{\mathbf{V}(t')}{u^2 D(\mathbf{r},t')}\right)\delta'\left(\frac{1}{u}D(\mathbf{r},t') - t + t'\right)\right\}\\
&\overset{(4.507)}{=} \frac{-q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\int dt'\left[-\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')} + \frac{1}{\kappa(\mathbf{r},t')}\left(\frac{\mathbf{n}(\mathbf{r},t')}{u\,D(\mathbf{r},t')} - \frac{\mathbf{V}(t')}{u^2 D(\mathbf{r},t')}\right)\frac{\partial}{\partial t'}\right]\\
&\qquad\cdot\delta\left(\frac{1}{u}D(\mathbf{r},t') - t + t'\right)\\
&\overset{\text{(int.b. parts)}}{=} \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\int dt'\left[\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')} + \left(\frac{\partial}{\partial t'}\frac{\mathbf{n}(\mathbf{r},t') - \mathbf{V}(t')/u}{u\,\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]\delta\left(\frac{1}{u}D(\mathbf{r},t') - t + t'\right).
\end{aligned}$$

Because of the δ-function the integrated part vanishes at the limits $t' = \pm\infty$. The t'-integration can now be performed as in (4.497):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\left[\frac{1}{\kappa(\mathbf{r},t')}\left(\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')} + \frac{1}{u}\frac{\partial}{\partial t'}\frac{\mathbf{n}(\mathbf{r},t') - \mathbf{V}(t')/u}{\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]_{t'=t_\mathrm{ret}}. \tag{4.509}$$

This we further rewrite with (4.508):

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}&\left[\frac{1}{\kappa(\mathbf{r},t')}\left(\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')} + \frac{\left(\mathbf{n}(\mathbf{r},t')\cdot\mathbf{V}(t')\right)\mathbf{n} - \mathbf{V}(t')}{u\,\kappa(\mathbf{r},t')D^2(\mathbf{r},t')}\right.\right.\\
&\left.\left.-\frac{\mathbf{n}(\mathbf{r},t') - \mathbf{V}(t')/u}{u\,\kappa^2(\mathbf{r},t')D^2(r,t')}\;\frac{\partial}{\partial t'}\left(\kappa(\mathbf{r},t')D(\mathbf{r},t')\right) - \frac{1}{u^2}\frac{\mathbf{a}(t')}{\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]_{t'=t_\mathrm{ret}}.
\end{aligned}$$

We have introduced here by

$$\mathbf{a}(t) = \frac{\partial}{\partial t}\mathbf{V}(t)$$

the **particle acceleration**. Eventually we still need:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\kappa(\mathbf{r},t)\,D(\mathbf{r},t)\right) &\overset{(4.403)}{=} \frac{\partial}{\partial t}\left(D(\mathbf{r},t)-\frac{1}{u}\mathbf{D}(\mathbf{r},t)\cdot\mathbf{V}(t)\right)\\
&= \dot{D}(\mathbf{r},t)-\frac{1}{u}\dot{\mathbf{D}}(\mathbf{r},t)\cdot\mathbf{V}(t)-\frac{1}{u}\mathbf{D}(\mathbf{r},t)\cdot\mathbf{a}(t)\\
&= -\mathbf{n}(\mathbf{r},t)\cdot\mathbf{V}(t)+\frac{1}{u}\mathbf{V}^2(t)-\frac{D(\mathbf{r},t)}{u}\left(\mathbf{n}(\mathbf{r},t)\cdot\mathbf{a}(t)\right)\ .
\end{aligned}
$$

This yields for the **E**-field:

$$
\begin{aligned}
&\mathbf{E}(\mathbf{r},t)\\
&= \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\left\{\frac{1}{\kappa^3(\mathbf{r},t')D^2(\mathbf{r},t')}\left[\frac{1}{u}\left(\mathbf{n}(\mathbf{r},t')-\frac{\mathbf{V}(t')}{u}\right)\left(\mathbf{n}(\mathbf{r},t')\cdot\mathbf{V}(t')-\frac{1}{u}V^2(t')\right)\right.\right.\\
&\qquad \left.+\mathbf{n}(\mathbf{r},t')\kappa^2(\mathbf{r},t')+\frac{1}{u}\kappa(\mathbf{r},t')\left(\mathbf{n}(\mathbf{r},t')(\mathbf{n}(\mathbf{r},t')\cdot\mathbf{V}(t'))-\mathbf{V}(t')\right)\right]\\
&\qquad \left.+\frac{1}{u^2\kappa^3(\mathbf{r},t')D(\mathbf{r},t')}\left[-\mathbf{a}(t')\kappa(\mathbf{r},t')+\left(\mathbf{n}(\mathbf{r},t')\cdot\mathbf{a}(t')\right)\left(\mathbf{n}(\mathbf{r},t')-\frac{\mathbf{V}(t')}{u}\right)\right]\right\}_{t'=t_\mathrm{ret}}\\
&= \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\left\{\frac{1}{\kappa^3D^2}\left(\mathbf{n}-\frac{\mathbf{V}}{u}\right)\left(\kappa+\frac{\mathbf{n}\cdot\mathbf{V}}{u}-\frac{V^2}{u^2}\right)\right.\\
&\qquad \left.+\frac{1}{u^2\kappa^3D}\left[-\mathbf{a}\left(1-\frac{1}{u}\,\mathbf{n}\cdot\mathbf{V}\right)+(\mathbf{n}\cdot\mathbf{a})\left(\mathbf{n}-\frac{\mathbf{V}}{u}\right)\right]\right\}_{t'=t_\mathrm{ret}}
\end{aligned}
$$

That leads to the final result for the electric field:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\,\frac{1}{\kappa_\mathrm{ret}^3(\mathbf{r},t)}&\left\{\frac{1}{D_\mathrm{ret}^2(\mathbf{r},t)}\left(\mathbf{n}_\mathrm{ret}(\mathbf{r},t)-\frac{\mathbf{V}(t_\mathrm{ret})}{u}\right)\left(1-\frac{V^2(t_\mathrm{ret})}{u^2}\right)\right.\\
&\left.+\frac{1}{u\,D_\mathrm{ret}(\mathbf{r},t)}\cdot\left[\mathbf{n}_\mathrm{ret}(\mathbf{r},t)\times\left(\left(\mathbf{n}_\mathrm{ret}(\mathbf{r},t)-\frac{\mathbf{V}(t_\mathrm{ret})}{u}\right)\times\frac{\mathbf{a}(t_\mathrm{ret})}{u}\right)\right]\right\}\ .
\end{aligned}
\tag{4.510}
$$

We still need the magnetic induction:

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \operatorname{curl}\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\mu_r q}{4\pi}\int dt'\,\nabla_r\times\left[\frac{\mathbf{V}(t')}{D(\mathbf{r},t')}\,\delta\left(\frac{1}{u}D(\mathbf{r},t')-t+t'\right)\right]\\
&= -\frac{\mu_0\mu_r q}{4\pi}\int dt'\,\mathbf{V}(t')\times\nabla_r\frac{\delta\left(\frac{1}{u}D(\mathbf{r},t')-t+t'\right)}{D(\mathbf{r},t')}\\
&= -\frac{\mu_0\mu_r q}{4\pi}\int dt'\,\mathbf{V}(t')\times\left[-\frac{\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')}\,\delta\left(\frac{1}{u}D(\mathbf{r},t')-t+t'\right)\right.\\
&\qquad\left.+\frac{\mathbf{n}(\mathbf{r},t')}{u\,D(\mathbf{r},t')}\delta'\left(\frac{1}{u}D(\mathbf{r},t')-t+t'\right)\right]\\
&= \frac{\mu_0\mu_r q}{4\pi}\int dt'\left[\frac{\mathbf{V}(t')\times\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')}+\left(\frac{1}{u}\frac{\partial}{\partial t'}\frac{\mathbf{V}(t')\times\mathbf{n}(\mathbf{r},t')}{\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]\\
&\qquad\cdot\delta\left(\frac{1}{u}D(\mathbf{r},t')-t+t'\right)\,.
\end{aligned}$$

In the last step first (4.507) was applied and then integration by parts was applied. Doing the t'-integration yields the intermediate result:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0\mu_r q}{4\pi}\left[\frac{1}{\kappa(\mathbf{r},t')}\left(\frac{\mathbf{V}(t')\times\mathbf{n}(\mathbf{r},t')}{D^2(\mathbf{r},t')}+\frac{1}{u}\frac{\partial}{\partial t'}\frac{\mathbf{V}(t')\times\mathbf{n}(\mathbf{r},t')}{\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]_{t'=t_{\text{ret}}}\,. \tag{4.511}$$

Let us try to derive a simple connection between the **E**- and the **B**-field:

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0\mu_r q}{4\pi}\left\{\frac{1}{\kappa}\left[\frac{\mathbf{V}\times\mathbf{n}}{D^2}+\frac{1}{u}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}}{\kappa D}\right)\times\mathbf{n}+\frac{1}{u}\frac{\mathbf{V}}{\kappa D}\times\left(\frac{\partial}{\partial t'}\mathbf{n}\right)\right]\right\}_{t'=t_{\text{ret}}}\\
&\overset{(4.508)}{=} \frac{\mu_0\mu_r q}{4\pi}\left\{\frac{1}{\kappa}\left[\frac{\mathbf{V}\times\mathbf{n}}{D^2}+\frac{1}{u}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}}{\kappa D}\right)\times\mathbf{n}+\frac{\mathbf{V}}{u\kappa D}\times\frac{1}{D}\left((\mathbf{n}\cdot\mathbf{V})\mathbf{n}-\mathbf{V}\right)\right]\right\}_{t'=t_{\text{ret}}}\\
&= \frac{\mu_0\mu_r q}{4\pi}\left\{\frac{1}{\kappa}\left[\frac{\mathbf{V}\times\mathbf{n}}{D^2}\left(1+\frac{\mathbf{V}\cdot\mathbf{n}}{u\kappa}\right)+\frac{1}{u}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}}{\kappa D}\right)\times\mathbf{n}\right]\right\}_{t'=t_{\text{ret}}}\,.
\end{aligned}$$

We use (4.506) in the form:

$$1+\frac{\mathbf{V}\cdot\mathbf{n}}{u\kappa} = \frac{1}{\kappa}\,.$$

The magnetic induction therewith reads:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0\mu_r q}{4\pi}\left\{\left[\frac{\mathbf{V}(t')}{D^2(\mathbf{r},t')\kappa^2(\mathbf{r},t')}+\frac{1}{u\kappa(\mathbf{r},t')}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}(t')}{\kappa(\mathbf{r},t')D(\mathbf{r},t')}\right)\right]\times\mathbf{n}(\mathbf{r},t')\right\}_{t'=t_{\text{ret}}}\,. \tag{4.512}$$

Otherwise it follows from (4.509):

$$\mathbf{n}\times\mathbf{E} = \frac{q}{4\pi\,\epsilon_0\epsilon_r}\left\{\frac{1}{\kappa\,u}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}(t')/u}{\kappa\,D}\right)\times\mathbf{n} + \frac{1}{\kappa\,u}\left[-\frac{1}{(\kappa\,D)^2}(\mathbf{n}\times\mathbf{n})\frac{\partial}{\partial t'}(\kappa\,D)\right]\right.$$
$$\left. + \frac{1}{\kappa\,u}\frac{1}{\kappa\,D}\mathbf{n}\times\left[\frac{1}{D}\left(\mathbf{n}(\mathbf{n}\cdot\mathbf{V})-\mathbf{V}\right)\right]\right\}$$
$$= \frac{q}{4\pi\,\epsilon_0\epsilon_r}\left[\frac{1}{\kappa\,u^2}\left(\frac{\partial}{\partial t'}\frac{\mathbf{V}(t')}{\kappa\,D}\right)\times\mathbf{n} + \frac{1}{\kappa^2 D^2 u}(\mathbf{V}\times\mathbf{n})\right]_{t'=t_{\mathrm{ret}}} .$$

The comparison with (4.512) yields:

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{u}(\mathbf{n}_{\mathrm{ret}}(\mathbf{r},t)\times\mathbf{E}(\mathbf{r},t)) . \tag{4.513}$$

The electromagnetic fields of the moving point charge q are with (4.510) and (4.513) completely determined.

(4) Poynting Vector

The electromagnetic fields decompose into two characteristic terms where one of them is **independent of the particle acceleration**:

$$\mathbf{E}_{(0)} = \frac{q}{4\pi\,\epsilon_0\epsilon_r}\,\frac{1}{\kappa_{\mathrm{ret}}^3}\,\frac{(\mathbf{n}_{\mathrm{ret}}-\boldsymbol{\beta}_{\mathrm{ret}})\left(1-\beta_{\mathrm{ret}}^2\right)}{D_{\mathrm{ret}}^2} ,$$
$$\mathbf{B}_{(0)} = \frac{\mu_0\mu_r q}{4\pi}\,\frac{1}{\kappa_{\mathrm{ret}}^3}\,\frac{\left(1-\beta_{\mathrm{ret}}^2\right)(\mathbf{V}(t_{\mathrm{ret}})\times\mathbf{n}_{\mathrm{ret}})}{D_{\mathrm{ret}}^2} . \tag{4.514}$$

Both the fields decrease at large distances according to the reciprocal square of the distance ($\sim 1/D_{\mathrm{ret}}^2$; $\sim 1/r^2$), behaving thus as the static (stationary) fields of point charges. In (4.514) we have applied the usual abbreviation:

$$\boldsymbol{\beta}_{\mathrm{ret}} = \frac{1}{u}\mathbf{V}(t_{\mathrm{ret}}) \tag{4.515}$$

For $\beta \ll 1$ one speaks of a **non-relativistic** and for $\beta \lesssim 1$ of a **relativistic** particle motion.

The second field term is essentially co-determined by the acceleration of the particle **a**:

$$\mathbf{E}_{(a)} = \frac{q}{4\pi\,\epsilon_0\epsilon_\mathrm{r}}\,\frac{1}{\kappa_\mathrm{ret}^3}\,\frac{\mathbf{n}_\mathrm{ret}\times[(\mathbf{n}_\mathrm{ret}-\boldsymbol{\beta}_\mathrm{ret})\times(\mathbf{a}_\mathrm{ret}/u)]}{u\,D_\mathrm{ret}}\,,$$

$$\mathbf{B}_{(a)} = \frac{\mu_0\mu_\mathrm{r}q}{4\pi}\,\frac{\mathbf{n}_\mathrm{ret}\times\{\mathbf{n}_\mathrm{ret}\times[(\mathbf{n}_\mathrm{ret}-\boldsymbol{\beta}_\mathrm{ret})\times(\mathbf{a}_\mathrm{ret}/u)]\}}{\kappa_\mathrm{ret}^3\,D_\mathrm{ret}}\,. \tag{4.516}$$

These field-contributions decrease for large distances as $1/D_\mathrm{ret}$, are therefore dominating in the radiation zone over those from (4.514).

Let us now discuss the **energy radiation** of the moving particle given by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0\mu_\mathrm{r}}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0\mu_\mathrm{r}u}\left[\mathbf{n}_\mathrm{ret}E^2 - (\mathbf{n}_\mathrm{ret}\cdot\mathbf{E})\,\mathbf{E}\right]\,. \tag{4.517}$$

When we insert the electric field given by (4.510) then we get because of (4.514) and (4.516), respectively, two different summands. These decay differently rapidly with increasing distance D_ret of the particle position **R** at the time t_ret from the point of observation **r**. At a sufficiently large distance (far field) we can restrict ourselves to the $(1/D_\mathrm{ret}^2)$-term, which results from (4.516). The $(1/D_\mathrm{ret}^3)$-summands namely do not contribute to the **energy radiation** since

$$\oint \mathbf{S}\cdot d\mathbf{f} \;\longrightarrow\; \oint \frac{1}{D_\mathrm{ret}^3}r^2 d\Omega \;\longrightarrow\; \oint \frac{1}{r}d\Omega \;\longrightarrow\; \frac{1}{r} \;\underset{r\to\infty}{\longrightarrow}\; 0\,.$$

These terms lead solely to a certain redistribution of the electromagnetic energy in the surroundings of the moving particle. Only the field energy that can flow up to the infinity leads to an actual loss of energy of the particle which is balanced by the kinetic energy. All the other contributions are bound to the neighborhood of the particle. For the energy radiation, thus, only (4.516) in (4.517) is interesting:

$$\mathbf{S} = \frac{q^2}{\mu_0\mu_\mathrm{r}u\,16\pi^2\epsilon_0^2\epsilon_\mathrm{r}^2}\mathbf{n}_\mathrm{ret}\frac{\{\mathbf{n}_\mathrm{ret}\times[(\mathbf{n}_\mathrm{ret}-\boldsymbol{\beta}_\mathrm{ret})\times(\mathbf{a}_\mathrm{ret}/u)]\}^2}{u^2\kappa_\mathrm{ret}^6\,D_\mathrm{ret}^2} + \mathcal{O}\left(\frac{1}{D^3}\right)$$

$$\Longrightarrow\; \mathbf{S} = \frac{q^2\mathbf{n}_\mathrm{ret}}{16\pi^2\epsilon_0\epsilon_\mathrm{r}u}\,\frac{\{\mathbf{n}_\mathrm{ret}\times[(\mathbf{n}_\mathrm{ret}-\boldsymbol{\beta}_\mathrm{ret})\times(\mathbf{a}_\mathrm{ret}/u)]\}^2}{\kappa_\mathrm{ret}^6\,D_\mathrm{ret}^2} + \mathcal{O}\left(\frac{1}{D^3}\right)\,. \tag{4.518}$$

The energy flux therefore has the direction from the particle position **R** at the time t_ret to the point of observation **r**. Furthermore, only **accelerated** particles ($\mathbf{a}\neq\mathbf{0}$) emit energy. A uniformly moving particle creates **E**- and **B**-fields, but does not lose energy by radiation.

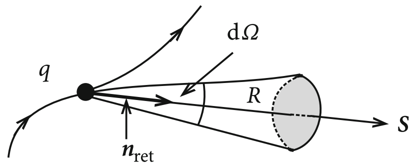


Fig. 4.89 For the calculation of the emitted energy per unit time of a moving charged particle

$(\mathbf{S} \cdot \mathbf{n}_{\text{ret}})\, D_{\text{ret}}^2$ is the energy emitted per time dt (at the point of observation) in the direction of $\mathbf{n}_{\text{ret}}$ in the solid angle $d\Omega$ (Fig. 4.89). It is even more interesting to look at the energy emitted per time dt_{ret} by the particle on its path:

$$\frac{dP_{\text{S}}}{d\Omega} = (\mathbf{S} \cdot \mathbf{n}_{\text{ret}})\, D_{\text{ret}}^2 \left(\frac{dt}{dt'}\right)_{t'=t_{\text{ret}}} .$$

Referred to (4.496) it is:

$$\left(\frac{dt}{dt'}\right)_{t'=t_{\text{ret}}} = \left(1 + \frac{1}{u}\frac{d}{dt'}D(\mathbf{r}, t')\right)_{t'=t_{\text{ret}}} = \left(1 - \frac{1}{u}\,\mathbf{n} \cdot \mathbf{V}(t')\right)_{t'=t_{\text{ret}}}$$
$$= \kappa_{\text{ret}}(\mathbf{r}, t) \ .$$

That yields the **radiation power:**

$$\frac{dP_{\text{S}}}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0\epsilon_{\text{r}}u}\, \frac{\{\mathbf{n}_{\text{ret}} \times [(\mathbf{n}_{\text{ret}} - \boldsymbol{\beta}_{\text{ret}}) \times (\mathbf{a}_{\text{ret}}/u)]\}^2}{(1 - \mathbf{n}_{\text{ret}} \cdot \boldsymbol{\beta}_{\text{ret}})^5} \ . \tag{4.519}$$

We add a short **discussion:**

(1) Non-relativistic

$$\beta_{\text{ret}} \ll 1 \ .$$

Then we can estimate as follows:

$$\frac{dP_{\text{S}}}{d\Omega} \approx \frac{\mu_0\mu_{\text{r}}q^2a_{\text{ret}}^2}{16\pi^2u}\sin^2\vartheta \ . \tag{4.520}$$

ϑ here is the angle between the acceleration $\mathbf{a}_{\text{ret}}$ and the radiation direction $\mathbf{n}_{\text{ret}}$. This type of radiation is realized in the X-ray apparatus. When electrons are decelerated in metals then that leads to an electromagnetic radiation which is known as **bremsstrahlung**.

(2) Relativistic

$$\beta_{\text{ret}} \lesssim 1 \ .$$

Let us in particular assume that the particle is accelerated and decelerated, respectively, in the direction of its motion, i.e.

$$\mathbf{a}_{\text{ret}} \uparrow\uparrow \boldsymbol{\beta}_{\text{ret}} \quad \text{or} \quad \mathbf{a}_{\text{ret}} \uparrow\downarrow \boldsymbol{\beta}_{\text{ret}} \ .$$

Then (4.519) becomes:

$$\frac{dP_{\text{S}}}{d\Omega} \approx \frac{\mu_0 \mu_{\text{r}} q^2 a_{\text{ret}}^2}{16\pi^2 u} \frac{\sin^2 \vartheta}{(1 - \beta_{\text{ret}} \cos \vartheta)^5} \ . \tag{4.521}$$

The space-direction of maximal emission is found by:

$$\frac{d}{d\cos\vartheta}\left(\frac{dP_{\text{S}}}{d\Omega}\right) \stackrel{!}{=} 0 \implies (\cos\vartheta)_{\max} = \frac{1}{3\beta_{\text{ret}}}\left(\sqrt{1 + 15\,\beta_{\text{ret}}^2} - 1\right) \ . \tag{4.522}$$

$\vartheta_{\max}$ decreases monotonically with increasing particle velocity (Fig. 4.90):

$\beta_{\text{ret}} \ll 1 \implies \vartheta_{\max} \approx \pi/2$: radiation maximal perpendicular to the forward direction,
$\beta_{\text{ret}} \lesssim 1 \implies \vartheta_{\max} \approx 0$: radiation mainly in the forward direction.

The radiation pattern is rotational-symmetric around the direction of the particle motion (Fig. 4.90).

For further details to the topic *'moving point charges'* the reader is referred to the advanced literature. Keywords:

1. limits of the Bohr atom model,
2. radiation damping,
3. synchrotron radiation.

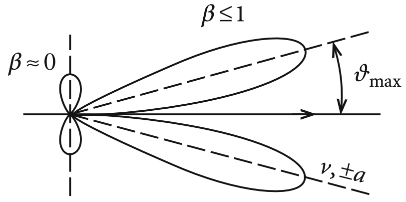


Fig. 4.90 Radiation pattern of a relativistic, charged particle

4.5.6 Exercises

Exercise 4.5.1 In Sect. 4.5.1 the inhomogeneous wave equation,

$$\Box\, \psi(\mathbf{r}, t) = -\sigma(\mathbf{r}, t)$$

was solved by means of the Green's function $G(\mathbf{r} - \mathbf{r}', t - t')$,

$$\Box_{r,t}\, G\left(\mathbf{r} - \mathbf{r}', t - t'\right) = -\delta\left(\mathbf{r} - \mathbf{r}'\right)\, \delta\left(t - t'\right) \ ,$$

and by applying complex integration. Alternatively, try to integrate the inhomogeneous wave equation directly!

1. Show at first that the known relation (1.69),

$$\Delta\, \frac{1}{r} = -4\pi\delta(\mathbf{r}) \ ,$$

   can be generalized to

$$\left(\Delta + k^2\right)\, \frac{e^{\pm ikr}}{r} = -4\pi\delta(\mathbf{r}) \ !$$

2. Solve the inhomogeneous wave equation by use of the ansatz:

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega\, \psi_\omega(\mathbf{r})\; e^{-i\omega t}$$

$$\sigma(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega\, \sigma_\omega(\mathbf{r})\; e^{-i\omega t}$$

Exercise 4.5.2 Consider the electric dipole radiation of a spatially restricted, temporally oscillating source. Demonstrate for $\mathbf{B}_1(\mathbf{r}, t)$ that in the vacuum the phase velocity is larger than c!

Exercise 4.5.3 In the infinitely extended yz-plane ($x = 0$) a spatially homogeneous current $I = I_0\, e^{i\omega t}$ (per unit-length in y-direction) is flowing in the z-direction. Calculate the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the whole space ($x \neq 0$)!

Exercise 4.5.4 Assume that the volumes V_1 and V_2 do not have any common space point. $\mathbf{j}_1$ and $\mathbf{j}_2$ are currents, restricted to V_1 and V_2, respectively, with the same harmonic time-dependence:

$$\mathbf{j}_i(\mathbf{r}, t) = \mathbf{j}_i(\mathbf{r})\, e^{-\omega t} \qquad (i = 1, 2)$$

Let $\mathbf{E}_i(\mathbf{r}, t)$ be the electric field caused by $\mathbf{j}_i(\mathbf{r}, t)$.

1. Show that it holds:

$$\mathbf{j}_1(\mathbf{r}, t) \cdot \mathbf{E}_2(\mathbf{r}, t) - \mathbf{j}_2(\mathbf{r}, t) \cdot \mathbf{E}_1(\mathbf{r}, t)$$
$$= \operatorname{div}(\mathbf{H}_1(\mathbf{r}, t) \times \mathbf{E}_2(\mathbf{r}, t)) - \operatorname{div}(\mathbf{H}_2(\mathbf{r}, t) \times \mathbf{E}_1(\mathbf{r}, t)) \; .$$

2. Prove with the electric dipole approximation for the radiation zone the so-called *'reciprocity theorem of the radiation theory'*

$$\int\limits_{V_1} \mathbf{j}_1 \cdot \mathbf{E}_2 d^3r = \int\limits_{V_2} \mathbf{j}_2 \cdot \mathbf{E}_1 d^3r \; .$$

3. Let the systems (volumes) $V_{1,2}$ have the dipole moments $\mathbf{p}_{1,2}$,

$$\mathbf{p}_{1,2} = \int\limits_{V_{1,2}} d^3r \mathbf{r} \rho_{1,2}(\mathbf{r}) \; .$$

Show that in the dipole approximation the result from part 2. can also be expressed in the form:

$$\mathbf{E}_2(\mathbf{r} \in V_1) \cdot \mathbf{p}_1 = \mathbf{E}_1(\mathbf{r} \in V_2) \cdot \mathbf{p}_2$$

Exercise 4.5.5 Let a monochromatic plane wave $(\mathbf{E}_\mathrm{i}, \mathbf{B}_\mathrm{i})$ impinge on a system whose linear dimensions are small compared to the wavelength of the radiation ($d \ll \lambda$). Let the surroundings of the scattering system be vacuum ($\mu_\mathrm{r} = \epsilon_\mathrm{r} = 1$). Assume the electric field $\mathbf{E}_\mathrm{i}$ to be linearly polarized in the direction $\boldsymbol{\eta}_\mathrm{i}$. The impinging field induces electric and magnetic multipoles in the system so that the system becomes the source of *scattered* radiation $(\mathbf{E}_\mathrm{s}, \mathbf{B}_\mathrm{s})$ (Fig. 4.91).

1. If one restricts the considerations to the electric dipole contribution, what are the fields $\mathbf{E}_\mathrm{s}$, $\mathbf{B}_\mathrm{s}$ in the so-called *radiation zone* ($kr \gg 1$) ?
2. Calculate the **differential cross section**:

$$\frac{d\sigma}{d\Omega}(\mathbf{n}_\mathrm{s}, \boldsymbol{\eta}_\mathrm{s}; \mathbf{n}_\mathrm{i}, \boldsymbol{\eta}_\mathrm{i}) = \frac{\text{scattered energy flux } (\mathbf{n}_\mathrm{s}, \boldsymbol{\eta}_\mathrm{s})}{d\Omega \cdot \text{incident energy-flux density } (\mathbf{n}_\mathrm{i}, \boldsymbol{\eta}_\mathrm{i})} \; .$$

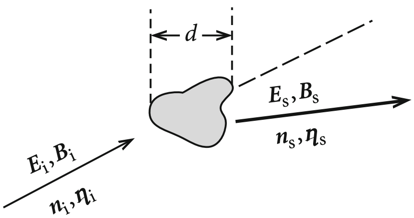


Fig. 4.91 Scattering of a monochromatic plane electromagnetic wave at a system with finite linear dimensions d

3. As a special case, the incident wave is scattered at a dielectric sphere ($\epsilon_r =$ const, $\mu_r = 1$) with the radius R. Calculate $d\sigma/d\Omega$. Which statement is possible about the polarization $\boldsymbol{\eta}_s$ of the scattered radiation?
4. Normally the incident electromagnetic wave is completely unpolarized, all directions of the polarization vector $\boldsymbol{\eta}_i$ are represented with the same weight. Calculate the **polarization** $P(\vartheta)$ of the scattered radiation:

$$P(\vartheta) = \frac{\left(\frac{d\sigma}{d\Omega}\right)_\perp - \left(\frac{d\sigma}{d\Omega}\right)_\parallel}{\left(\frac{d\sigma}{d\Omega}\right)_\perp + \left(\frac{d\sigma}{d\Omega}\right)_\parallel} \,.$$

$(d\sigma/d\Omega)_{\parallel\,(\perp)}$ is the scattering cross section for an incident wave which is linearly polarized within (perpendicular to) the **scattering plane**. By 'scattering plane' one understands the plane which is spanned by $\mathbf{n}_i$ and $\mathbf{n}_s$.

4.6 Self-Examination Questions

To Section 4.1

1. What is the physical essence of Faraday's law of induction? On which experimental observations is the law based?
2. What does one understand by Maxwell's supplement?
3. Explain the contradiction between the Ampère's law and the continuity equation in the case of time-dependent phenomena!
4. Quote the full set of Maxwell equations!
5. Why does it make sense to introduce the electromagnetic potentials φ and $\mathbf{A}$?
6. Which general gauge transformation is allowed for the electromagnetic potentials? Show that thereby the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ do not change.
7. What is understood by the Coulomb gauge? Which advantage is offered by its usage?
8. Which advantage is offered by the Lorenz gauge?
9. Which force does act on a point charge q in the electromagnetic field?
10. What is the work done by the electromagnetic field on a charge density $\rho(\mathbf{r}, t)$ that is restricted to a finite volume V?
11. What is the physical meaning of the Poynting vector? Which kind of continuity equation is fulfilled by it?
12. How is the energy density of the electromagnetic field defined?
13. Formulate the energy law of electrodynamics!
14. What do we understand by the field momentum? What is the momentum theorem of electrodynamics?
15. Define and interpret the Maxwell stress tensor!

To Section 4.2

1. What is meant by quasi-stationary approximation? Formulate the Maxwell equations in this approximation!
2. Explain the term *induction voltage*!
3. What is expressed by the Lenz's law?
4. Define self-inductance and mutual inductance!
5. What is the self-inductance of a long coil? Sketch its derivation!
6. Express the magnetic field energy of a system of current-carrying conductors by the inductance-coefficients! How does it look like in the case of a single conductor loop?
7. Which differential equation does the electric current I satisfy in a conductor loop consisting of a coil, a capacitor, and an ohmic resistance?
8. What is the meaning of the terms impedance, effective resistance, and reactance?
9. What does one understand by the effective values of the current and the voltage?
10. Derive and comment on the phase shifts between current and voltage as well as the time-averaged power of an alternating current circuit with either an ohmic resistance or a capacity or an inductance!
11. What does one understand by damping and eigenfrequency of the electric oscillation circuit?
12. Discuss the time-behavior of current and voltage in the electric oscillation circuit for weak, critical, and strong damping!
13. Which mechanical analogue to the electric oscillation circuit do you know?
14. How does the amplitude of the current I_0 in the series-resonant circuit depend on the frequency ω of the applied voltage? When does resonance appear?
15. How is the current in an RL-circuit built up after switching on a direct voltage? What happens after switching off? In this connection, what is understood by the characteristic time constant?

To Section 4.3

1. Under which conditions do the components of $\mathbf{E}$ and $\mathbf{B}$ fulfill the homogeneous wave equation? What is the wave equation?
2. Which structure does the general solution of the homogeneous wave equation have?
3. What is a plane wave? Define for it the terms phase velocity, wavelength, propagation vector, frequency, and period!
4. Which relation exists between phase velocity, wavelength, and frequency?
5. What is the solution of the homogeneous wave equation which simultaneously satisfies the Maxwell equations? Which connection does exist between $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$?
6. What is meant by linearly, circularly, and elliptically polarized plane waves?
7. When is a medium dispersive?
8. How and when do group and phase velocity differ?
9. What is a wave packet?

10. Are there other types of solution for the homogeneous wave equation, other than the plane waves?
11. Describe a spherical wave!
12. What do we understand by the Fourier series of a function $f(x)$?
13. How is the Fourier transform of the function $f(x)$ defined? List some of its most important properties!
14. What is expressed by the convolution theorem?
15. How can we find, by means of the Fourier transformation, the most general solution of the homogeneous wave equation?
16. How do energy density and energy-current density read for electromagnetic fields with harmonic time-dependences? What does in particular hold for plane waves?
17. How is the time-averaged energy density of a plane wave distributed over magnetic and electric portions?
18. Give the connection between the Poynting vector and the energy density (time-averaged) for the case of a plane wave!
19. Which differential equation does for a homogeneous, isotropic, uncharged, electric conductor ($\sigma \neq 0$) replace the homogeneous wave equation of an uncharged insulator?
20. By which kind of ansatz can the telegraph equation get the same structure as the homogeneous wave equation?
21. What determines the penetration depth of an electromagnetic wave into an electric conductor?
22. Is the phase velocity of the wave in a conductor larger or smaller than that in an insulator?
23. Which spatial dependence does the time-averaged energy-current density exhibit in an electric conductor?
24. What are the continuity conditions for the electromagnetic field at interfaces in uncharged insulators?
25. How do the laws of refraction and reflection read for electromagnetic waves at interfaces?
26. For which angle of incidence does total reflection appear?
27. Which physical facts are expressed by the Fresnel formulas?
28. How can reflection be used to create linearly polarized waves?
29. How are the coefficients of reflection and transmission defined?
30. What does happen to the electromagnetic wave when the angle of incidence is larger than the limiting angle of total reflection?
31. When and where do the phenomena *'interference'* and *'diffraction'* appear?
32. Sketch briefly the main steps necessary for the derivation of the Kirchhoff's formula!
33. Which simplifications lead to the Kirchhoff approximation? Try to make them plausible!
34. What is the difference between Fraunhofer- and Fresnel-diffraction?
35. In which kind of optical experiments does the *'Poisson spot'* appear? Why is its appearance astonishing?

36. Explain the Babinet's principle!
37. In which connection (experiment) are the Laue equations of great importance?
38. Illustrate the Bragg-law!
39. Define and interpret the limiting case of geometrical optics! What is its relationship to the general wave optics?
40. What is the eikonal equation of geometrical optics? Which preconditions are mandatory to come to this equation?

To Section 4.4

1. When does a sequence of complex numbers $\{z_n\}$ converge to $z_0 \in \mathbb{C}$?
2. When is a complex function denoted to be continuous at z_0? When is the function uniformly continuous?
3. How is the differentiability of a complex function defined? Which role do the Cauchy-Riemann differential equations play?
4. What is a domain G?
5. When is the function $f(z)$ analytic in a domain G?
6. How is the complex line integral defined?
7. When is a domain called simply connected?
8. Formulate the Cauchy's integral theorem!
9. What does the Cauchy's integral formula express? Which connection does exist with the proposition of Morera?
10. What is the region of convergence of a sequence of complex functions?
11. What is said by the Cauchy-Hadamard theorem about the convergence of a power series?
12. What is the content of the expansion theorem?
13. Formulate the identity theorems for power series and analytic functions!
14. Explain the principle of the analytic continuation!
15. What is understood by a pole of n-th order, by a branching point of a function $f(z)$?
16. Define and interpret the Laurent expansion of a function $f(z)$!
17. Let the function $f(z)$ have at z^* a pole of p-th order. How do you then calculate the residue of $f(z)$ at z^*?
18. Formulate Cauchy's residue theorem!

To Section 4.5

1. Sketch the way of solution for the inhomogeneous wave equation! How do effects of retardation become noticeable in the general solutions of the electromagnetic potentials?
2. What is understood in connection with electromagnetic radiation by the terms *near zone* and *radiation zone*?
3. How does the vector potential behave in the radiation zone?
4. How does the power per solid angle emitted by the electric dipole radiation depend in the radiation zone on the wavelength λ and on the dipole moment?

5. Are retardation effects observable in the near zone?
6. What are Liénard-Wiechert potentials?
7. When will a charged particle emit energy?
8. What is bremsstrahlung?

Appendix A
Solutions of the Exercises

Section 1.7

Solution 1.7.1 It is to show:

1. $\delta(x-a) = 0 \quad \forall x \neq a$,
2. $\displaystyle\int_{\alpha}^{\beta} dx\delta(x-a) = \begin{cases} 1 \text{ , if } \alpha < a < \beta \text{ ,} \\ 0 \quad \text{otherwise .} \end{cases}$

To 1. $x \neq a$:

$$\lim_{\eta \to 0^+} \frac{1}{\sqrt{\pi\eta}} e^{-(x-a)^2/\eta} = 0 \ .$$

To 2.

(a) $\alpha < a < \beta$:

$$F_\eta(a) \equiv \int\limits_{\alpha}^{\beta} \frac{1}{\sqrt{\pi\eta}} e^{-(x-a)^2/\eta} dx \ .$$

It follows with $y = (x-a)/\sqrt{\eta}$:

$$F_\eta(a) = \frac{1}{\sqrt{\pi}} \int\limits_{(\alpha-a)/\sqrt{\eta}}^{(\beta-a)/\sqrt{\eta}} dy \, e^{-y^2} \ .$$

W. Nolting, *Theoretical Physics 3*, DOI 10.1007/978-3-319-40168-3

That means:

$$\lim_{\eta\to 0+} F_\eta(a) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} dy\, e^{-y^2} = 1\ .$$

(b) $a < \alpha < \beta$:
$\alpha - a = \bar{\alpha} > 0;\ \beta - a = \bar{\beta} > 0$

$$F_\eta(a) = \frac{1}{\sqrt{\pi}} \int_{\bar{\alpha}/\sqrt{\eta}}^{\bar{\beta}/\sqrt{\eta}} dy\, e^{-y^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\bar{\beta}/\sqrt{\eta}} dy\, e^{-y^2} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\bar{\alpha}/\sqrt{\eta}} dy\, e^{-y^2}$$

$$\Rightarrow \lim_{\eta\to 0+} F_\eta(a) = 1 - 1 = 0\ .$$

(c) $\alpha < \beta < a$:
$a - \alpha = \alpha' > 0;\ a - \beta = \beta' > 0$

$$F_\eta(a) = \frac{1}{\sqrt{\pi}} \int_{-\alpha'/\sqrt{\eta}}^{-\beta'/\sqrt{\eta}} dy\, e^{-y^2} = \frac{1}{\sqrt{\pi}} \int_{-\alpha'/\sqrt{\eta}}^{+\infty} dy\, e^{-y^2} - \frac{1}{\sqrt{\pi}} \int_{-\beta'/\sqrt{\eta}}^{+\infty} dy\, e^{-y^2}$$

$$\Rightarrow \lim_{\eta\to 0+} F_\eta(a) = 1 - 1 = 0\ .$$

From 2(a), 2(b), 2(c) we get:

$$\lim_{\eta\to 0+} \int\limits_{\alpha}^{\beta} \frac{1}{\sqrt{\pi\eta}} \exp\left[-\frac{(x-a)^2}{\eta}\right] dx = \begin{cases} 1\ , & \text{if } \alpha < a < \beta\ , \\ 0 & \text{otherwise } (a \neq \alpha, \beta)\ . \end{cases}$$

It follows from 2(a) for the *edge points*:

$$\int\limits_{\alpha}^{\beta} dx\, \delta(x-a) = \frac{1}{2}\ , \quad \text{if } \alpha = a \text{ or } \beta = a\ .$$

Solution 1.7.2 Equation (1.7):

$$\delta(x-a) = \lim_{\eta\to 0+} \frac{1}{\pi} \frac{\eta}{\eta^2 + (x-a)^2}\ .$$

Therewith it follows:

$$\lim_{\eta\to 0^+} \operatorname{Im} \frac{1}{(x-a) \pm i\,\eta} = \lim_{\eta\to 0^+} \operatorname{Im} \frac{(x-a) \mp i\,\eta}{(x-a)^2+\eta^2} = \mp \lim_{\eta\to 0^+} \frac{\eta}{(x-a)^2+\eta^2}$$
$$= \mp\pi\,\delta(x-a)\ .$$

Solution 1.7.3

1.

$$\delta(g(x)) = \lim_{\eta\to 0^+} \frac{1}{\pi}\,\frac{\eta}{\eta^2+(g(x))^2} = 0 \quad \text{for } g(x) \neq 0\ .$$

On the other hand:

$$\sum_n \frac{1}{|g'(x_n)|}\,\delta(x-x_n) = 0 \quad \forall x \neq x_n\ , \qquad \text{i.e. for } g(x) \neq 0\ .$$

2.

$$\begin{aligned}
I \quad &\equiv \int\limits_\alpha^\beta dx\,\delta(g(x))f(x) = \sum_n^{\alpha<x_n<\beta} \int\limits_{x_n-\epsilon}^{x_n+\epsilon} dx\,\delta(g(x))f(x) \\
&= \sum_n^{\alpha<x_n<\beta} \int\limits_{x_n-\epsilon}^{x_n+\epsilon} dx\ \delta\left[\frac{g(x)-g(x_n)}{x-x_n}(x-x_n)\right] f(x) \quad (g(x_n)=0)\ , \\
\epsilon \to 0^+: \quad I &= \sum_n^{\alpha<x_n<\beta} \int\limits_{x_n-\epsilon}^{x_n+\epsilon} dx\,\delta\left(g'(x_n)(x-x_n)\right) f(x) \\
&= \int\limits_\alpha^\beta dx \sum_n \delta\left(g'(x_n)(x-x_n)\right) f(x)\ .
\end{aligned}$$

$g'(x_n) > 0$:

$$z = g'(x_n)\,x$$

$$\Longrightarrow\ I = \sum_n \int\limits_{\alpha\,g'(x_n)}^{\beta\,g'(x_n)} dz\,\frac{1}{g'(x_n)}\delta(z-z_n) f\left(\frac{z}{g'(x_n)}\right)$$

$$= \sum_n^{\alpha\, g'(x_n)<z_n<\beta\, g'(x_n)} \frac{1}{g'(x_n)} f\left(\frac{z_n}{g'(x_n)}\right) = \sum_n^{\alpha<x_n<\beta} \frac{1}{g'(x_n)} f(x_n)$$

$$= \int\limits_\alpha^\beta dx \sum_n \frac{1}{g'(x_n)} \delta(x-x_n) f(x) \; .$$

The comparison with the definition of I yields, since $f(x)$ is arbitrary:

$$\delta(g(x)) = \sum_n \frac{1}{g'(x_n)}\, \delta(x-x_n) \; .$$

$g'(x_n) < 0$:

$$I = -\sum_n \int\limits_{-\alpha|g'|}^{-\beta|g'|} dz\, \frac{1}{|g'(x_n)|}\, \delta(z-z_n) f\left(\frac{z}{g'(x_n)}\right)$$

$$= +\sum_n \int\limits_{-\beta|g'|}^{-\alpha|g'|} dz\, \frac{1}{|g'(x_n)|}\, \delta(z-z_n) f\left(\frac{z}{g'(x_n)}\right)$$

$$= \sum_n^{-\alpha|g'|>z_n>-\beta|g'|} \frac{1}{|g'(x_n)|} f\left(\frac{z_n}{g'(x_n)}\right)$$

$$= \sum_n^{\alpha<x_n<\beta} \frac{1}{|g'(x_n)|} f(x_n) = \int\limits_\alpha^\beta dx \sum_n \frac{1}{|g'(x_n)|}\, \delta(x-x_n) f(x) \; .$$

The comparison with the definition of I yields, since $f(x)$ is arbitrary:

$$\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|} \delta(x-x_n) \; .$$

Solution 1.7.4

1. $I = 9 - 15 + 6 = 0$.
2. $I = 0$.
3.

$$f(x) = x^2 - 3x + 2 = (x-2)(x-1) \; ,$$
$$\Longrightarrow \text{zeros: } x_1 = 2 \; , \quad x_2 = 1 \; .$$
$$f'(x) = 2x - 3 \;\Longrightarrow\; f'(x_1) = 1 = -f'(x_2)$$
$$\Longrightarrow\; \delta(x^2 - 3x + 2) = \delta(x-2) + \delta(x-1) \;\Longrightarrow\; I = 5 \; .$$

4.

$$I = -\int_0^{+\infty} dx\,(\ln x)'\delta(x-a) = -\frac{1}{a}\ .$$

5.

$$f(\vartheta) = \cos\vartheta - \cos\frac{\pi}{3} \Longrightarrow \text{zero: } \vartheta_1 = \frac{\pi}{3}\ ,$$

$$f'(\vartheta) = -\sin\vartheta \Longrightarrow f'(\vartheta_1) = -\sin\frac{\pi}{3} = -\frac{1}{2}\sqrt{3}\ ,$$

$$I = \int_0^{\pi} \frac{\sin^3\vartheta}{|\sin\vartheta_1|}\,\delta(\vartheta-\vartheta_1)\,d\vartheta = \sin^2\vartheta_1 = \frac{3}{4}\ .$$

Solution 1.7.5

$$\mathbf{r} = (x,y)\ ; \quad (\rho,\varphi)\ ,$$

$$\mathbf{r}_0 = (x_0,y_0)\ ; \quad (\rho_0,\varphi_0)$$

$$\delta(\mathbf{r}-\mathbf{r}_0) = 0 \quad \text{for } \mathbf{r} \neq \mathbf{r}_0\ , \tag{A.1}$$

$$\int_F df\,\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} 1\ , \text{ if } \mathbf{r}_0 \in F\ , \\ 0 \quad \text{otherwise}\ . \end{cases} \tag{A.2}$$

1. **Cartesian**
   ansatz: $\delta(\mathbf{r}-\mathbf{r}_0) = \alpha(x,y)\,\delta(x-x_0)\,\delta(y-y_0)$.
   Equation (A.1) obviously fulfilled.

$$\int_F df\,\delta(\mathbf{r}-\mathbf{r}_0) = \alpha(x_0,y_0)\iint_F dx\,dy\,\delta(x-x_0)\,\delta(y-y_0)$$

$$= \alpha(x_0,y_0)\begin{cases} 1\ , \text{ if } (x_0,y_0) \in F\ , \\ 0 \quad \text{otherwise}\ . \end{cases}$$

$$\Longrightarrow \alpha(x_0,y_0) = 1\ ,$$

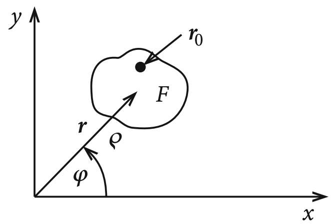
$$\text{i.e.,} \quad \delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\ .$$

2. **Plane polar coordinates** (Fig. A.1)

$$x = \rho\cos\varphi\ , \quad y = \rho\sin\varphi\ ,$$

$$df = dx\,dy = \frac{\partial(x,y)}{\partial(\rho,\varphi)}\,d\rho\,d\varphi = \rho\,d\rho\,d\varphi\ .$$

Fig. A.1



ansatz:

$$
\begin{aligned}
\delta(\mathbf{r}-\mathbf{r}_0) &= \beta(\rho,\varphi)\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\ ,\\
\iint\limits_F df\,\delta(\mathbf{r}-\mathbf{r}_0) &= \iint\limits_F \rho\,d\rho\,d\varphi\,\beta(\rho,\varphi)\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\\
&= \rho_0\beta(\rho_0,\varphi_0)\iint\limits_F d\rho\,d\varphi\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\\
&= \rho_0\beta(\rho_0,\varphi_0)\begin{cases} 1\ , \text{ if } (\rho_0,\varphi_0)\in F\ ,\\ 0 \quad \text{otherwise}\ .\end{cases}\\
&\Longrightarrow\ \beta = \frac{1}{\rho_0}\ ,\\
&\text{i.e., }\ \delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{\rho_0}\,\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\ .
\end{aligned}
$$

Solution 1.7.6 Equation (1.27):

$$
\varphi(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{j=1}^{3}x_j\frac{\partial}{\partial x_j}\right)^n\varphi(0) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{r}\cdot\nabla)^n\varphi(0) \equiv \exp(\mathbf{r}\cdot\nabla)\varphi(0)\ .
$$

1.

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\,e^{i\,\mathbf{k}\cdot\mathbf{r}} &= i\,k_j\,e^{i\,\mathbf{k}\cdot\mathbf{r}}\ ,\\
\sum_{j=1}^{3}x_j\frac{\partial}{\partial x_j}\,e^{i\,\mathbf{k}\cdot\mathbf{r}} &= i\,(\mathbf{k}\cdot\mathbf{r})e^{i\,\mathbf{k}\cdot\mathbf{r}}\ ,\\
\left(\sum_j x_j\frac{\partial}{\partial x_j}\right)\varphi(0) &= i\,\mathbf{k}\cdot\mathbf{r}\ ,
\end{aligned}
$$

$$\left(\sum_j x_j \frac{\partial}{\partial x_j}\right)^n \varphi(0) = (i\,\mathbf{k}\cdot\mathbf{r})^n$$

$$\Longrightarrow \quad \varphi(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!}(i\,\mathbf{k}\cdot\mathbf{r})^n \ .$$

2.

$$\frac{\partial}{\partial x_j}|\mathbf{r}-\mathbf{r}_0| = \frac{\partial}{\partial x_j}\sqrt{\sum_{i=1}^{3}(x_i - x_{i0})^2} = \frac{x_j - x_{j0}}{|\mathbf{r}-\mathbf{r}_0|} \ .$$

$n = 0$:

$$\varphi_0 = r_0 \ .$$

$n = 1$:

$$\sum_j x_j \frac{\partial}{\partial x_j}\varphi(0) = \sum_j x_j \frac{(-x_{j0})}{r_0}$$

$$\Longrightarrow \quad \varphi_1 = -\frac{\mathbf{r}\cdot\mathbf{r}_0}{r_0} \ .$$

$n = 2$:

$$\begin{aligned}\sum_{j,k} x_j x_k \frac{\partial^2}{\partial x_k \partial x_j}|\mathbf{r}-\mathbf{r}_0| &= \sum_{j,k} x_j x_k \frac{\partial}{\partial x_k}\,\frac{x_j - x_{j0}}{|\mathbf{r}-\mathbf{r}_0|} \\ &= \sum_{j,k} x_j x_k \left[\frac{\delta_{jk}}{|\mathbf{r}-\mathbf{r}_0|} - \frac{(x_j - x_{j0})(x_k - x_{k0})}{|\mathbf{r}-\mathbf{r}_0|^3}\right]\end{aligned}$$

$$\Longrightarrow \quad \left(\sum_j x_j \frac{\partial}{\partial x_j}\right)^2 \varphi(0) = \frac{r^2}{r_0} - \frac{(\mathbf{r}\cdot\mathbf{r}_0)^2}{r_0^3}$$

$$\Longrightarrow \quad \varphi_2 = \frac{1}{2}\,\frac{1}{r_0^3}\left[r^2 r_0^2 - (\mathbf{r}\cdot\mathbf{r}_0)^2\right] \ .$$

$$\text{all in all:} \quad \varphi(\mathbf{r}) = r_0 - \frac{\mathbf{r}\cdot\mathbf{r}_0}{r_0} + \frac{r^2}{2r_0} - \frac{(\mathbf{r}\cdot\mathbf{r}_0)^2}{2r_0^3} + \cdots$$

Fig. A.2

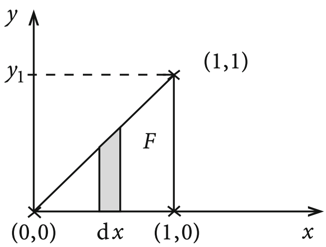


Fig. A.3

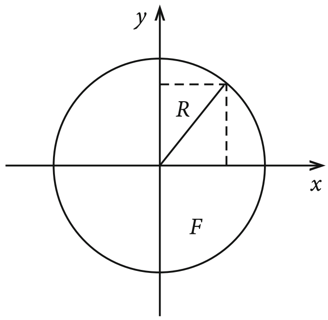
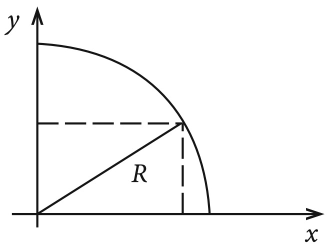


Fig. A.4



Solution 1.7.7 Multiple integrals (see Sect. 1.2.5, Vol. 1) (Figs. A.2, A.3, and A.4):

1.

$$I = \iint_F dx\,dy\, f(x,y) = \int_0^1 dx \int_0^x dy\, x^2 y^3$$

$$\Longrightarrow \quad I = \int_0^1 dx\, x^2 \left.\frac{y^4}{4}\right|_0^x = \frac{1}{4}\int_0^1 dx\, x^6 = \frac{1}{28}\,.$$

2.

$$I = \int_{-R}^{+R} dx \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} dy\, x^2 y^3 = \int_{-R}^{+R} dx\, x^2 \left(\frac{y^4}{4}\right)\Bigg|_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} = 0 \,.$$

3.

$$\begin{aligned} I &= \int_0^R dx\, x^2 \left(\frac{y^4}{4}\right)\Bigg|_0^{\sqrt{R^2-x^2}} \\ &= \frac{1}{4}\int_0^R dx\, x^2 \left(R^2-x^2\right)^2 = \frac{1}{4}\int_0^R dx \left(R^4x^2 - 2R^2x^4 + x^6\right) \\ &= \frac{1}{4}R^7\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{2}{105}R^7 \,. \end{aligned}$$

Solution 1.7.8

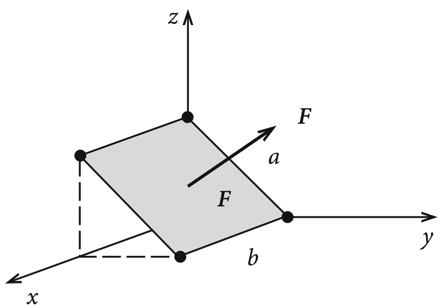
1. Parametric representation (Fig. A.5):

$$\begin{aligned} F &= \left\{\mathbf{r} = \left(x, y, z = -y + \frac{a}{\sqrt{2}}\right) \;;\quad 0 \le x \le b\,;\quad 0 \le y \le \frac{a}{\sqrt{2}}\right\} \\ &= F(x, y)\,. \end{aligned}$$

We get with Eq. (1.34):

$$\begin{aligned} d\mathbf{f} &= \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right) dx\, dy = (1, 0, 0) \times (0, 1, -1)\, dx\, dy \\ \Longrightarrow\quad d\mathbf{f} &= (0, 1, 1)\, dx\, dy \,. \end{aligned}$$

Fig. A.5



2. Total area:

$$\mathbf{F} = \iint d\mathbf{f} = (0,1,1)\int\limits_0^b dx \int\limits_0^{a/\sqrt{2}} dy$$

$$\Longrightarrow \mathbf{F} = \frac{a\,b}{\sqrt{2}}(0,1,1)\,;$$

$$\Longrightarrow |\mathbf{F}| = a\,b\,.$$

3. Flux:

$$\begin{aligned}\mathbf{a}\cdot d\mathbf{f} &= (2xy + 3z^2 - x^2)dx\,dy\\ &= \left(2xy - x^2 + 3y^2 - 3\sqrt{2}\,ay + \frac{3}{2}a^2\right)dx\,dy\,,\end{aligned}$$

$$\begin{aligned}\varphi_F(\mathbf{a}) &= \int\limits_F \mathbf{a}\cdot d\mathbf{f} = \int\limits_0^b dx \int\limits_0^{a/\sqrt{2}} dy \left(2xy - x^2 + 3y^2 - 3\sqrt{2}\,ay + \frac{3}{2}a^2\right)\\ &= \int\limits_0^b dx\left(x\frac{a^2}{2} - x^2\frac{a}{\sqrt{2}} + \frac{a^3}{2\sqrt{2}} - 3\sqrt{2}\frac{a^3}{4} + \frac{3a^3}{2\sqrt{2}}\right)\\ &= \frac{a^2b^2}{4} - \frac{a\,b^3}{3\sqrt{2}} + \frac{a^3b}{2\sqrt{2}}\\ \Longrightarrow\ \varphi_F(\mathbf{a}) &= \frac{a\,b}{\sqrt{2}}\left(\frac{1}{2}a^2 - \frac{1}{3}b^2 + \frac{1}{2\sqrt{2}}a\,b\right)\,.\end{aligned}$$

Solution 1.7.9 Area element of the surface of a sphere:

$$d\mathbf{f} = (R^2\sin\vartheta\; d\vartheta\; d\varphi)\,\mathbf{e}_\mathrm{r} \quad (\text{Eq.}\,(1.37))\,.$$

1.

$$\mathbf{a}(\mathbf{r}) = \frac{3}{r}\mathbf{e}_\mathrm{r}\,,$$

$$\varphi_1(\mathbf{a}) = \int\limits_{S_K}\mathbf{a}\cdot d\mathbf{f} = 3R\int\limits_0^\pi d\vartheta\int\limits_0^{2\pi} d\varphi\,\sin\vartheta\; = 12\pi\,R\,.$$

2.

$$\mathbf{a}(\mathbf{r}) = \frac{\mathbf{r}}{\sqrt{\alpha + r^2}} = \frac{r}{\sqrt{\alpha + r^2}}\,\mathbf{e}_{\mathrm{r}}\ ,$$

$$\varphi_2(\mathbf{a}) = 4\pi\,\frac{R^3}{\sqrt{\alpha + R^2}}\ .$$

3. Spherical coordinates:

$$\mathbf{e}_{\mathrm{r}} = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)\ ,$$

$$\mathbf{a}(\mathbf{r}) = (3r\cos\vartheta,\ r\sin\vartheta\cos\varphi,\ 2r\sin\vartheta\sin\varphi)\ ,$$

$$\varphi_3(\mathbf{a}) = R^3\int_0^{\pi} d\vartheta\int_0^{2\pi} d\varphi\,\sin\vartheta\,(3\sin\vartheta\cos\vartheta\cos\varphi$$

$$+\sin^2\vartheta\sin\varphi\cos\varphi + 2\sin\vartheta\cos\vartheta\sin\varphi)\ ,$$

$$\int_0^{2\pi} d\varphi\cos\varphi = \int_0^{2\pi} d\varphi\sin\varphi = 0\ ,$$

$$\int_0^{2\pi} d\varphi\sin\varphi\cos\varphi = \frac{1}{2}\sin^2\varphi\Big|_0^{2\pi} = 0$$

$$\Longrightarrow\ \varphi_3(\mathbf{a}) = 0\ .$$

Solution 1.7.10

1. **Vectorial surface elements**
Cylindrical coordinates (ρ, φ, z) are certainly appropriate.
We start with the **cylinder-barrel** $(\rho = R)$

$$x = R\cos\varphi\ ,\ y = R\sin\varphi\ ,\ z = z$$

$$\to \mathbf{r} = (R\cos\varphi, R\sin\varphi, z)\ , \quad -\frac{L}{2} \le z \le +\frac{L}{2}\ .$$

Parameters: φ, z

$$\begin{aligned} d\mathbf{f} &= \left(\frac{\partial\mathbf{r}}{\partial\varphi}\times\frac{\partial\mathbf{r}}{\partial z}\right)d\varphi\,dz \\ &= \left|\frac{\partial\mathbf{r}}{\partial\varphi}\right|\cdot\left|\frac{\partial\mathbf{r}}{\partial z}\right|\cdot(\mathbf{e}_\varphi\times\mathbf{e}_z)d\varphi\,dz \\ &= R\,\mathbf{e}_\rho d\varphi\,dz \\ &= Rd\varphi\,dz(\cos\varphi, \sin\varphi, 0)\ . \end{aligned}$$

It holds on the **front faces of the cylinder**:

$$x = \rho\cos\varphi\ ,\ y = \rho\sin\varphi\ ,\ z = \pm\frac{L}{2}$$

$$\rightarrow \mathbf{r} = \left(\rho\cos\varphi, \rho\sin\varphi, \pm\frac{L}{2}\right)\ .$$

Parameters: ρ, φ

$$\begin{aligned} d\mathbf{f} &= \pm\left(\frac{\partial\mathbf{r}}{\partial\rho}\times\frac{\partial\mathbf{r}}{\partial\varphi}\right) d\rho\, d\varphi \\ &= \pm\left|\frac{\partial\mathbf{r}}{\partial\rho}\right|\cdot\left|\frac{\partial\mathbf{r}}{\partial\varphi}\right|\cdot(\mathbf{e}_\rho\times\mathbf{e}_\varphi)d\rho\, d\varphi \\ &= \pm\rho\,\mathbf{e}_z d\rho\, d\varphi \\ &= \pm\rho\, d\rho\, d\varphi\,(0,0,1)\ . \end{aligned}$$

The plus sign holds for the ‘upper’, the minus sign for the ‘lower’ front face of the cylinder.

2. **Flux of the vector field through the surface of the cylinder**

$$\varphi_{\mathrm{Z}} = \int\limits_{S(Z)} \mathbf{E}\cdot d\mathbf{f}\ .$$

Contribution on the barrel:

$$\begin{aligned} \mathbf{E}\cdot d\mathbf{f} &= \alpha\,(R\cos\varphi, R\sin\varphi, z)\cdot(\cos\varphi, \sin\varphi, 0)\, R\, d\varphi\, dz \\ &= \alpha R^2\,(\cos^2\varphi + \sin^2\varphi)\, d\varphi\, dz \\ &= \alpha R^2\, d\varphi\, dz \end{aligned}$$

$$\rightarrow\ \varphi_{\mathrm{Z}}^{\mathrm{barrel}} = \alpha R^2 \int\limits_0^{2\pi} d\varphi \int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dz = 2\pi\alpha R^2 L\ .$$

Contribution on the front faces:

$$\begin{aligned} \mathbf{E}\cdot d\mathbf{f} = {} & \alpha\,\rho\, d\rho\, d\varphi\,\left(\rho\cos\varphi, \rho\sin\varphi, +\frac{L}{2}\right)\cdot(0,0,1) \\ & +\alpha\,\rho\, d\rho\, d\varphi\,\left(\rho\cos\varphi, \rho\sin\varphi, -\frac{L}{2}\right)\cdot(0,0,-1) \end{aligned}$$

$$= \alpha L\, \rho\, d\rho\, d\varphi$$

$$\rightarrow \varphi_{\mathrm{Z}}^{\mathrm{front}} = \alpha L \int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi = \pi \alpha R^2 L\,.$$

We have therewith as total flux (V: volume of the cylinder):

$$\varphi_{\mathrm{Z}} = \int_{S(Z)} \mathbf{E} \cdot d\mathbf{f} = \varphi_{\mathrm{Z}}^{\mathrm{barrel}} + \varphi_{\mathrm{Z}}^{\mathrm{front}} = 3\pi\alpha R^2 L = 3\alpha V\,.$$

3. **Gauss theorem**

$$\varphi_Z = \int_{S(Z)} \mathbf{E} \cdot d\mathbf{f} = \int_Z d^3 r \operatorname{div} \mathbf{E}\,.$$

With

$$\operatorname{div} \mathbf{E} = \alpha \operatorname{div} \mathbf{r} = 3\alpha$$

we have immediately recalculated the above result:

$$\varphi_{\mathrm{Z}} = 3\alpha \int_Z d^3 r = 3\,\alpha\, V\,.$$

Solution 1.7.11

1. **Sphere**
   Equation (1.36):

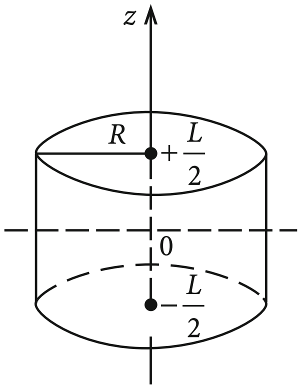
$$\begin{aligned} d\mathbf{f} &= (R^2 \sin\vartheta\, d\vartheta\, d\varphi)\, \mathbf{e}_{\mathrm{r}}\,, \\ \mathbf{a}(\mathbf{r}) &= \alpha\, r\, \mathbf{e}_{\mathrm{r}} \\ &\qquad \Longrightarrow \mathbf{a}(\mathbf{r}) \times d\mathbf{f} \sim \mathbf{e}_{\mathrm{r}} \times \mathbf{e}_{\mathrm{r}} = 0 \\ &\qquad \Longrightarrow \psi_\kappa \equiv 0\,. \end{aligned}$$

2. **Cylinder**
   barrel (1.37) (Fig. A.6):

$$d\mathbf{f} = (R\, d\varphi\, dz)\, \mathbf{e}_\rho\,.$$

Fig. A.6

$$d\mathbf{f} = (R\,d\varphi\,dz)\,\mathbf{e}_\rho \ .$$



front faces:

$$F_\pm = \{\mathbf{r} = (\rho\cos\varphi,\ \rho\sin\varphi,\ \pm L/2)\ ;\quad 0 \le \rho \le R\ ,\quad 0 \le \varphi \le 2\pi\}\ .$$

$$\begin{aligned} d\mathbf{f} &= \left(\frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \varphi}\right) d\rho\,d\varphi = (\cos\varphi,\ \sin\varphi,\ 0) \times (-\rho\sin\varphi,\ \rho\cos\varphi,\ 0)\,d\rho\,d\varphi \\ &= \rho\,d\rho\,d\varphi\,\mathbf{e}_z\ . \end{aligned}$$

We agreed that for closed surfaces the vector $d\mathbf{f}$ always points *outwards*:

$$\begin{aligned} \Longrightarrow\quad \psi_z &= \int_F \mathbf{a}(\mathbf{r}) \times d\mathbf{f} = \alpha \int_F (\rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z) \times d\mathbf{f} \\ &= \alpha \int_{\text{barrel}} (\rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z) \times (R\,d\varphi\,dz)\,\mathbf{e}_\rho \\ &\quad + \alpha \int_{\substack{\text{front face}\\ +L/2}} (\rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z) \times (\rho\,d\rho\,d\varphi)\,\mathbf{e}_z \\ &\quad - \alpha \int_{\substack{\text{front face}\\ -L/2}} (\rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z) \times (\rho\,d\rho\,d\varphi)\mathbf{e}_z \\ &= \alpha\,R \int_0^{2\pi} d\varphi \int_{-L/2}^{+L/2} dz\,z\,\mathbf{e}_\varphi + \alpha \int_0^R d\rho \int_0^{2\pi} d\varphi\,\rho^2(-\mathbf{e}_\varphi) \end{aligned}$$

$$-\alpha \int_0^R d\rho \int_0^{2\pi} d\varphi\, \rho^2(-\mathbf{e}_\varphi)\ ,$$

$$\int_0^{2\pi} d\varphi\, \mathbf{e}_\varphi = \int_0^{2\pi} d\varphi(-\sin\varphi,\ \cos\varphi,\ 0) = \mathbf{0}$$

$$\Longrightarrow\ \psi_z \equiv 0\ .$$

Solution 1.7.12

Charge:

$$Q = \int d^3r\, \rho(\mathbf{r}) = \rho_0 \int_{\text{sphere}} d^3r\ ,$$

$$d^3r = r^2 dr\ \sin\vartheta\ d\vartheta\ d\varphi\ ,$$

$$Q = \rho_0 \int_0^R r^2 dr \int_0^\pi \sin\vartheta\ d\vartheta \int_0^{2\pi} d\varphi = \rho_0 \frac{4\pi}{3} R^3\ .$$

Dipole moment:

$$\mathbf{p} = \rho_0 \int_0^R\int_0^\pi\int_0^{2\pi} r^2 dr\ \sin\vartheta\ d\vartheta\ d\varphi(r\sin\vartheta\cos\varphi,\ r\sin\vartheta\sin\varphi,\ r\cos\vartheta)\ ,$$

$$\int_0^{2\pi} d\varphi\cos\varphi = \int_0^{2\pi} d\varphi\sin\varphi = 0$$

$$\Longrightarrow\ \mathbf{p} = \rho_0 \int_0^R \int_0^\pi \int_0^{2\pi} r^3 dr\, d\vartheta\ d\varphi(0, 0, \sin\vartheta\cos\vartheta)$$

$$= 2\pi\ \rho_0 \frac{R^4}{4} \int_0^\pi d\vartheta \left(0, 0, \frac{1}{2}\frac{d}{d\vartheta}\sin^2\vartheta\right)$$

$$= \pi\ \rho_0 \frac{R^4}{4}\left(0, 0, \sin^2\vartheta\big|_0^\pi\right) = \mathbf{0}\ .$$

Solution 1.7.13

1. Gradient of a scalar product
   double vector product:

$$\boldsymbol{b}\times(\nabla\times\,\boldsymbol{a}) = \nabla(\boldsymbol{a}\,\cdot\boldsymbol{b}) - (\boldsymbol{b}\;\cdot\nabla)\,\boldsymbol{a}\;,$$

$$\boldsymbol{a}\times(\nabla\times\,\boldsymbol{b}) = \nabla(\boldsymbol{a}\,\cdot\boldsymbol{b}) - (\boldsymbol{a}\,\cdot\nabla)\,\boldsymbol{b}$$

$$\Rightarrow\;\boldsymbol{b}\times(\nabla\times\,\boldsymbol{a}) + \boldsymbol{a}\times(\nabla\times\,\boldsymbol{b}) = \nabla(\boldsymbol{a}\,\cdot\boldsymbol{b}) + \nabla\,(\boldsymbol{a}\cdot\boldsymbol{b}) - (\,\boldsymbol{b}\cdot\nabla)\,\boldsymbol{a} - (\,\boldsymbol{a}\cdot\nabla)\,\boldsymbol{b},$$

$$\textit{Produktregel}\text{: }\nabla\,(\boldsymbol{a}\cdot\boldsymbol{b}) = \nabla\,(\boldsymbol{a}\cdot\boldsymbol{b}) + \nabla\,(\boldsymbol{a}\cdot\boldsymbol{b}) \Rightarrow \text{q. e. d.}$$

2. Divergence of a vector product
   product rule:

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\;\cdot\;(\boldsymbol{a}\times\boldsymbol{b}) + \nabla\;\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\;\cdot\;(\boldsymbol{a}\times\boldsymbol{b}) - \nabla\,(\boldsymbol{b}\times\boldsymbol{a}).$$

Exploit now the cyclic invariance of the scalar triple product. Pay attention on which vector the operator ∇ has to act.

$$\nabla\;\cdot\;(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\,\boldsymbol{a}),$$

$$\nabla\;\cdot\;(\boldsymbol{b}\times\boldsymbol{a}) = \boldsymbol{a}\cdot(\nabla\times\,\boldsymbol{b}) \Rightarrow \text{q. e. d.}$$

3. Curl of a vector product
   product rule:

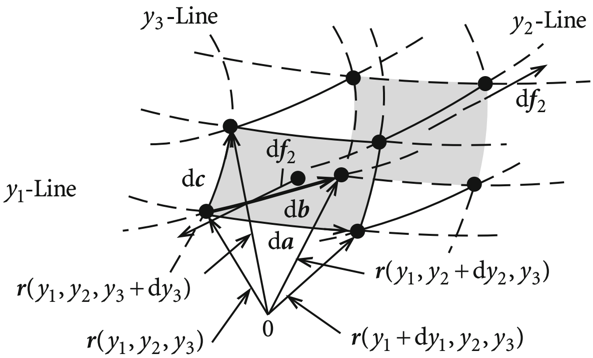
$$\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\times(\boldsymbol{a}\times\,\boldsymbol{b}) + \nabla\times(\boldsymbol{a}\times\,\boldsymbol{b}) = \nabla\times(\boldsymbol{a}\times\,\boldsymbol{b}) - \nabla\times(\boldsymbol{b}\times\,\boldsymbol{a}).$$

Double vector product (keep in mind the action of ∇*!):*

$$\nabla\times(\,\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{b}\,\cdot\nabla)\;\boldsymbol{a} - \boldsymbol{b}\,(\nabla\cdot\boldsymbol{a}),$$

$$\nabla\times(\,\boldsymbol{b}\times\boldsymbol{a}) = (\boldsymbol{a}\,\cdot\nabla)\;\boldsymbol{b} - \boldsymbol{a}\,(\nabla\;\cdot\boldsymbol{b}) \Rightarrow \text{q. e. d.}$$

Fig. A.7



Solution 1.7.14

1. (y_1, y_2, y_3) —curvilinear-orthogonal (Fig. A.7)
   Unit-vectors:

$$\mathbf{e}_{y_i} = \frac{\partial \mathbf{r}/\partial y_i}{|\partial \mathbf{r}/\partial y_i|} = \frac{1}{b_{y_i}} \frac{\partial \mathbf{r}}{\partial y_i} .$$

Differential parallelepiped, built by the coordinate lines:

$$\Delta V = d\mathbf{a} \cdot (d\mathbf{b} \times d\mathbf{c}) .$$

Taylor expansion:

$$d\mathbf{a} = \mathbf{r}(y_1 + dy_1, y_2, y_3) - \mathbf{r}(y_1, y_2, y_3) \approx \frac{\partial \mathbf{r}}{\partial y_1} dy_1 ,$$

$$d\mathbf{b} = \mathbf{r}(y_1, y_2 + dy_2, y_3) - \mathbf{r}(y_1, y_2, y_3) \approx \frac{\partial \mathbf{r}}{\partial y_2} dy_2 ,$$

$$d\mathbf{c} = \mathbf{r}(y_1, y_2, y_3 + dy_3) - \mathbf{r}(y_1, y_2, y_3) \approx \frac{\partial \mathbf{r}}{\partial y_3} dy_3 ,$$

therewith:

$$d\mathbf{a} = b_{y_1} dy_1 \mathbf{e}_{y_1} ; \quad d\mathbf{b} = b_{y_2} dy_2 \mathbf{e}_{y_2} ; \quad d\mathbf{c} = b_{y_3} dy_3 \mathbf{e}_{y_3} ; \quad \mathbf{e}_{y_1} \cdot \left(\mathbf{e}_{y_2} \times \mathbf{e}_{y_3}\right) = 1$$

$$\Longrightarrow \quad \Delta V = b_{y_1} b_{y_2} b_{y_3} dy_1 dy_2 dy_3 ,$$

$$\begin{aligned} d\bar{\mathbf{f}}_2 &= d\mathbf{a} \times d\mathbf{c}|_{(y_1,y_2,y_3)} = \left(\mathbf{e}_{y_1} \times \mathbf{e}_{y_3}\right) b_{y_1} b_{y_3} dy_1 dy_3 = -\mathbf{e}_{y_2} b_{y_1} b_{y_3} dy_1 dy_3 , \\ d\mathbf{f}_2 &= d\mathbf{c} \times d\mathbf{a}|_{(y_1,y_2+dy_2,y_3)} \\ &= \underbrace{\left(\mathbf{e}_{y_3} \times \mathbf{e}_{y_1}\right)}_{=\mathbf{e}_{y_2}} b_{y_1}(y_1, y_2 + dy_2, y_3)\, b_{y_3}(y_1, y_2 + dy_2, y_3)\, dy_1 dy_3 \end{aligned}$$

That leads to

$$
\begin{aligned}
\mathbf{E}\cdot d\mathbf{f}\big|_{\text{areas in } y_2\text{-direction}} &= dy_1 dy_3 \big(E_{y_2}(y_1, y_2+dy_2, y_3) \\
&\qquad \cdot b_{y_1}(y_1, y_2+dy_2, y_3) b_{y_3}(y_1, y_2+dy_2, y_3) \\
&\qquad - E_{y_2}(y_1, y_2, y_3) b_{y_1}(y_1, y_2, y_3)\, b_{y_3}(y_1, y_2, y_3)\big) \\
&= dy_1 dy_2 dy_3 \, \frac{\partial}{\partial y_2}\left(E_{y_2} b_{y_1} b_{y_3}\right) \ .
\end{aligned}
$$

Analogously one calculates the contributions on the other sides of the differential parallelepiped:

$$
\begin{aligned}
\lim_{\Delta V\to 0} \frac{1}{\Delta V} \oint \mathbf{E}\cdot d\mathbf{f} &= \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \left[\frac{\partial}{\partial y_1}\left(E_{y_1} b_{y_2} b_{y_3}\right) \right. \\
&\qquad \left. + \frac{\partial}{\partial y_2}\left(E_{y_2} b_{y_1} b_{y_3}\right) + \frac{\partial}{\partial y_3}\left(E_{y_3} b_{y_1} b_{y_2}\right) \right] \\
&= \operatorname{div} \mathbf{E} \ .
\end{aligned}
$$

(cf. with (1.378), Vol. 1.)

2. **Cylindrical coordinates** (ρ, φ, z)

$$
\begin{aligned}
x &= \rho\cos\varphi \ , \\
y &= \rho\sin\varphi \ , \\
z &= z \ ,
\end{aligned}
$$

$$
\frac{\partial \mathbf{r}}{\partial \rho} = (\cos\varphi,\ \sin\varphi,\ 0) \implies b_\rho = 1 \ ,
$$

$$
\frac{\partial \mathbf{r}}{\partial \varphi} = (-\rho\sin\varphi,\ \rho\cos\varphi,\ 0) \implies b_\varphi = \rho \ ,
$$

$$
\frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) \implies b_z = 1
$$

$$
\implies \operatorname{div}\mathbf{E} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho E_\rho) + \frac{\partial}{\partial \varphi}E_\varphi + \frac{\partial}{\partial z}(\rho E_z)\right]
$$

$$
\implies \operatorname{div}\mathbf{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}E_\varphi + \frac{\partial}{\partial z}E_z \ .
$$

3. **Spherical coordinates** (r, ϑ, φ)

$$
\begin{aligned}
x &= r\sin\vartheta\cos\varphi \ , \\
y &= r\sin\vartheta\sin\varphi \ ,
\end{aligned}
$$

$$
\begin{aligned}
z &= r\cos\vartheta \ , \\
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \implies b_r = 1 \ , \\
\frac{\partial \mathbf{r}}{\partial \vartheta} &= r(\cos\vartheta\cos\varphi,\ \cos\vartheta\sin\varphi,\ -\sin\vartheta) \implies b_\vartheta = r \ , \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= r(-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0) \implies b_\varphi = r\sin\vartheta
\end{aligned}
$$

$$
\implies \operatorname{div}\mathbf{E} = \frac{1}{r^2\sin\vartheta}\left[\frac{\partial}{\partial r}(r^2\sin\vartheta\, E_r) + \frac{\partial}{\partial\vartheta}(r\sin\vartheta\, E_\vartheta) + \frac{\partial}{\partial\varphi}(r\, E_\varphi)\right] \ ,
$$

$$
\implies \operatorname{div}\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, E_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}E_\varphi \ .
$$

Solution 1.7.15

1. Consider the hatched front side of the parallelepiped in Fig. A.7 (Exercise 1.7.14):

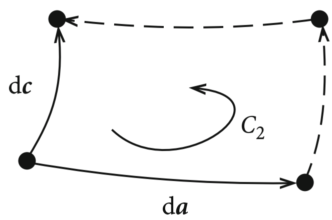
$$
\begin{aligned}
d\bar{f}_2 &= -\mathbf{e}_{y_2} b_{y_1} b_{y_3} dy_1 dy_3 \\
\implies |d\bar{f}_2| &= b_{y_1} b_{y_3} dy_1 dy_3 \ , \\
\oint_{C_2} \mathbf{a}\cdot d\mathbf{r} &= d\mathbf{a}\cdot\mathbf{a}|_{(y_1,y_2,y_3)} + d\mathbf{c}\cdot\mathbf{a}|_{(y_1+dy_1,y_2,y_3)} \\
&\quad - d\mathbf{a}\cdot\mathbf{a}|_{(y_1,y_2,y_3+dy_3)} - d\mathbf{c}\cdot\mathbf{a}|_{(y_1,y_2,y_3)} \ ,
\end{aligned}
$$

$d\mathbf{a}$ and $d\mathbf{c}$ as in Exercise 1.7.14 (Fig. A.8).

Therewith one calculates:

$$
\begin{aligned}
\mathbf{n}\cdot\operatorname{curl}\mathbf{a}(\mathbf{r}) = -\operatorname{curl}_{y_2}(\mathbf{a}(\mathbf{r})) &= \frac{1}{|d\bar{\mathbf{f}}_2|}\oint_{C_2}\mathbf{a}\cdot d\mathbf{r} \\
&= \frac{1}{b_{y_1} b_{y_3} dy_1 dy_3}\big[b_{y_1} dy_1 a_{y_1}(y_1,y_2,y_3) \\
&\qquad + b_{y_3}(y_1+dy_1,y_2,y_3)\, dy_3 a_{y_3}(y_1+dy_1,y_2,y_3)
\end{aligned}
$$

Fig. A.8



$$
\begin{aligned}
&\qquad -b_{y_1}(y_1, y_2, y_3 + dy_3)\, dy_1 a_{y_1}(y_1, y_2, y_3 + dy_3) \\
&-b_{y_3} dy_3 a_{y_3}(y_1, y_2, y_3)\big] \\
&= \frac{1}{b_{y_1} b_{y_3} dy_1 dy_3}\left[-dy_1 \frac{\partial}{\partial y_3}\left(b_{y_1} a_{y_1}\right) dy_3 + dy_3 \frac{\partial}{\partial y_1}\left(b_{y_3} a_{y_3}\right) dy_1\right] .
\end{aligned}
$$

Thus we have found:

$$
\mathrm{curl}_{y_2}\mathbf{a}(\mathbf{r}) = \frac{1}{b_{y_1} b_{y_3}}\left[\frac{\partial}{\partial y_3}\left(b_{y_1} a_{y_1}\right) - \frac{\partial}{\partial y_1}\left(b_{y_3} a_{y_3}\right)\right] .
$$

The same procedure leads to the other components:

$$
\begin{aligned}
\mathrm{curl}_{y_1}\mathbf{a}(\mathbf{r}) &= \frac{1}{b_{y_2} b_{y_3}}\left[\frac{\partial}{\partial y_2}\left(b_{y_3} a_{y_3}\right) - \frac{\partial}{\partial y_3}\left(b_{y_2} a_{y_2}\right)\right] , \\
\mathrm{curl}_{y_3}\mathbf{a}(\mathbf{r}) &= \frac{1}{b_{y_1} b_{y_2}}\left[\frac{\partial}{\partial y_1}\left(b_{y_2} a_{y_2}\right) - \frac{\partial}{\partial y_2}\left(b_{y_1} a_{y_1}\right)\right] .
\end{aligned}
$$

(cf. with (1.380), Vol. 1!)

2. **Cylindrical coordinates**
   It follows with $b_\rho = 1$, $b_\varphi = \rho$, $b_z = 1$:

$$
\begin{aligned}
\mathrm{curl}_\rho \mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial \varphi} a_z - \frac{\partial}{\partial z} a_\varphi , \\
\mathrm{curl}_\varphi \mathbf{a} &= \frac{\partial}{\partial z} a_\rho - \frac{\partial}{\partial \rho} a_z , \\
\mathrm{curl}_z \mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, a_\varphi) - \frac{1}{\rho}\frac{\partial}{\partial \varphi} a_\rho .
\end{aligned}
$$

3. **Spherical coordinates**
   It follows with $b_r = 1$, $b_\vartheta = r$, $b_\varphi = r\sin\vartheta$:

$$
\begin{aligned}
\mathrm{curl}_r \mathbf{a} &= \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(\sin\vartheta\, a_\varphi) - \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi} a_\vartheta , \\
\mathrm{curl}_\vartheta \mathbf{a} &= \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi} a_r - \frac{1}{r}\frac{\partial}{\partial r}(r\, a_\varphi) , \\
\mathrm{curl}_\varphi \mathbf{a} &= \frac{1}{r}\frac{\partial}{\partial r}(r\, a_\vartheta) - \frac{1}{r}\frac{\partial}{\partial \vartheta} a_r .
\end{aligned}
$$

Solution 1.7.16

1. Gradient in spherical coordinates ((1.395), Vol. 1):

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} + \mathbf{e}_\varphi \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi} \ .$$

(Generally: $\nabla = \sum_{i=1}^{3} \mathbf{e}_{y_i} b_{y_i}^{-1} \frac{\partial}{\partial y_i}$ (1.377), Vol. 1).
With $\boldsymbol{\alpha}$ as polar axis it is:

$$\boldsymbol{\alpha} \cdot \mathbf{r} = \alpha \, r \cos \vartheta \ ,$$

$$\operatorname{grad}(\boldsymbol{\alpha} \cdot \mathbf{r}) = \alpha (\cos \vartheta \, \mathbf{e}_r - \sin \vartheta \, \mathbf{e}_\vartheta) \ .$$

2. With the formulas from the Exercises 1.7.14 and 1.7.15 one finds:

$$\begin{aligned}
\operatorname{div} \mathbf{e}_r &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot 1) = \frac{2}{r} \ , \\
\operatorname{grad} \operatorname{div} \mathbf{e}_r &= -\frac{2}{r^2} \mathbf{e}_r \ , \\
\operatorname{curl} \mathbf{e}_r &= 0 \ , \\
\operatorname{div} \mathbf{e}_\varphi &= 0 \ , \\
\operatorname{curl} \mathbf{e}_\vartheta &= \mathbf{e}_\varphi \frac{1}{r} \frac{\partial}{\partial r} (r \cdot 1) = \frac{1}{r} \mathbf{e}_\varphi \ .
\end{aligned}$$

3. $\boldsymbol{\alpha}$ z-axis $\Longrightarrow$ $\boldsymbol{\alpha} = \alpha \, \mathbf{e}_z$, $\mathbf{r} = \rho \, \mathbf{e}_\rho + z \, \mathbf{e}_z$.
That leads to:

$$\boldsymbol{\alpha} \times \mathbf{r} = \alpha \rho \, \mathbf{e}_z \times \mathbf{e}_\rho = \alpha \rho \, \mathbf{e}_\varphi \quad \Longrightarrow \quad \text{with part 2. from Exercise 1.7.15}$$

$$\operatorname{curl}_z (\boldsymbol{\alpha} \times \mathbf{r}) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\alpha \, \rho^2) = 2\alpha \quad \Longrightarrow \quad \operatorname{curl}(\boldsymbol{\alpha} \times \mathbf{r}) = 2\alpha \, \mathbf{e}_z \ .$$

Solution 1.7.17 $\mathbf{F}(\mathbf{r})$ conservative $\Longleftrightarrow$ $\operatorname{curl} \mathbf{F}(\mathbf{r}) \equiv 0$.
With the special form (1.57) of the Gauss theorem,

$$\int_V \operatorname{curl} \mathbf{b} \, d^3 r = \oint_{S(V)} d\mathbf{f} \times \mathbf{b} \ ,$$

one finds immediately:

$$\oint_{S(V)} d\mathbf{f} \times \mathbf{F} \equiv 0 \ .$$

Solution 1.7.18

$$\begin{aligned}
\operatorname{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \quad \text{(Maxwell equation, law of induction)} , \\
\operatorname{div}\operatorname{curl}\mathbf{E} &= 0 \\
\Longrightarrow \; \operatorname{div}\frac{\partial}{\partial t}\mathbf{B} &= \frac{\partial}{\partial t}\operatorname{div}\mathbf{B} = 0 \;\Longrightarrow\; \operatorname{div}\mathbf{B} = \text{const} .
\end{aligned}$$

Exploit the presumption:

$$\begin{aligned}
&t = t_0: \quad \mathbf{B}(\mathbf{r}, t_0) \equiv 0 \\
&\Longrightarrow \; \operatorname{div}\mathbf{B}(\mathbf{r}, t_0) = 0 \Longrightarrow \; \operatorname{div}\mathbf{B}(\mathbf{r}, t) \equiv 0 .
\end{aligned}$$

Solution 1.7.19 It is to be calculated:

$$A = \oint_{\partial F} \mathbf{r} \times d\mathbf{r} .$$

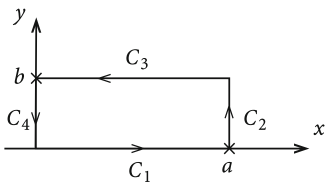
1. Direct calculation
   (a) Circle
   parametrization:

$$\begin{aligned}
\mathbf{r} &= (R\cos\varphi, R\sin\varphi, 0) \\
\frac{d\mathbf{r}}{d\varphi} &= (-R\sin\varphi, R\cos\varphi, 0) \\
\rightarrow \; \mathbf{r} \times \frac{d\mathbf{r}}{d\varphi} &= R^2\mathbf{e}_z .
\end{aligned}$$

   Therewith we calculate:

$$\begin{aligned}
A = \oint_{\partial F} \mathbf{r} \times d\mathbf{r} &= R^2\mathbf{e}_z \int_0^{2\pi} d\varphi \\
&= 2\pi R^2\mathbf{e}_z \\
&= 2\mathbf{F} .
\end{aligned}$$

Fig. A.9



(b) Rectangle (Fig. A.9)

$$\text{on } C_1: \quad \int_0^1 (a \cdot t, 0, 0) \times (a, 0, 0)dt = \int_0^1 0 \cdot dt = 0$$

$$\text{on } C_2: \quad \int_0^1 (a, b \cdot t, 0) \times (0, b, 0)dt = \int_0^1 ab \cdot \mathbf{e}_z dt = ab \cdot \mathbf{e}_z$$

$$\text{on } C_3: \quad \int_0^1 (a(1-t), b, 0) \times (-a, 0, 0)dt = \int_0^1 ab \cdot \mathbf{e}_z dt = ab \cdot \mathbf{e}_z$$

$$\text{on } C_4: \quad \int_0^1 (0, b(1-t), 0) \times (0, -b, 0)dt = \int_0^1 0 \cdot dt = 0$$

Therewith it follows eventually:

$$A = \oint_{\partial F} \mathbf{r} \times d\mathbf{r} = 2ab \cdot \mathbf{e}_z = 2\mathbf{F} \; .$$

2. Stokes theorem
   generally it holds (1.64):

$$\oint_{\partial F} d\mathbf{r} \cdots \equiv \int_F (d\mathbf{f} \times \nabla) \cdots \; .$$

That means here:

$$\oint_{\partial F} \mathbf{r} \times d\mathbf{r} = -\int_F (d\mathbf{f} \times \nabla) \times \mathbf{r} \; .$$

Rearrangement:

$$
\begin{aligned}
(d\mathbf{f} \times \nabla) \times \mathbf{r} &= -d\mathbf{f}(\nabla \cdot \mathbf{r}) - \nabla(\mathbf{r} \cdot d\mathbf{f}) \\
&= -3d\mathbf{f} + d\mathbf{f} = 2d\mathbf{f} \\
\rightarrow \oint\limits_{\partial F} \mathbf{r} \times d\mathbf{r} &= 2 \int\limits_{F} d\mathbf{f} = 2\mathbf{F} \; .
\end{aligned}
$$

We see that the result from part 1. is valid even for arbitrary areas. We have already come across the integral in connection with the area conservation principle ((2.251), Vol. 1).

Solution 1.7.20

1. Possible parametric representation ($u = x, \; v = y$):

$$
F = \left\{ \mathbf{r}(u, v) = \mathbf{r}\left(x, y, 6 - 3x - \frac{3}{2}y\right) \; ; \; 0 \le x \le 2 \; , \; 0 \le y \le 4 - 2x \right\} \; .
$$

Vectorial area element:

$$
\begin{aligned}
d\mathbf{f} &= \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv \; , \\
\frac{\partial \mathbf{r}}{\partial x} &= (1, 0, -3) \; ; \quad \frac{\partial \mathbf{r}}{\partial y} = \left(0, 1, -\frac{3}{2}\right) \; .
\end{aligned}
$$

This leads to

$$
\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \left(3, \frac{3}{2}, 1\right) \; ,
$$

and finally:

$$
d\mathbf{f} = \left(3, \frac{3}{2}, 1\right) dx\, dy \; .
$$

Surface normal:

$$
\mathbf{n} = \frac{1}{7}(6, 3, 2) \; ; \quad d\mathbf{f} = \frac{7}{2} dx\, dy\, \mathbf{n} \; .
$$

2.

$$\varphi = \int_F d\mathbf{f} \cdot \mathbf{a} = \frac{1}{2} \iint_F dx\, dy\, (6,3,2) \cdot (0,0,y) = \frac{1}{2} \int_0^2 dx \int_0^{4-2x} dy\, 2y$$

$$= \frac{1}{2} \int_0^2 dx\, (4-2x)^2 = \frac{1}{2} \int_0^2 dx\, (16 - 16x + 4x^2)$$

$$= \left(8x - 4x^2 + \frac{2}{3}x^3 \right)\Big|_0^2 = \frac{16}{3} \; .$$

3. The field $\mathbf{a}(\mathbf{r})$ is source-free!

$$\operatorname{div} \mathbf{a}(\mathbf{r}) = 0 \; .$$

The decomposition theorem (1.71) leads then to:

$$\mathbf{a}(\mathbf{r}) = \operatorname{curl} \boldsymbol{\beta}(\mathbf{r}) \; .$$

The choice of $\boldsymbol{\beta}$ is **not** unique, The gauge transformation

$$\boldsymbol{\beta}(\mathbf{r}) \mapsto \boldsymbol{\beta}(\mathbf{r}) + \operatorname{grad} \chi(\mathbf{r})$$

does not change the result because of

$$\operatorname{curl} \operatorname{grad} \chi(\mathbf{r}) \equiv 0 \; .$$

For $\boldsymbol{\beta}(\mathbf{r})$ it must be:

$$0 = \frac{\partial}{\partial y}\beta_z - \frac{\partial}{\partial z}\beta_y \; ,$$

$$0 = \frac{\partial}{\partial z}\beta_x - \frac{\partial}{\partial x}\beta_z \; ,$$

$$y = \frac{\partial}{\partial x}\beta_y - \frac{\partial}{\partial y}\beta_x \; .$$

A possible solution could then be:

$$\beta_x = \beta_z = 0 \; ; \quad \beta_y = xy \; ; \quad \boldsymbol{\beta}(\mathbf{r}) = (0, xy, 0)$$

4. Parametrization of the partial paths:
   C_1:

$$2x + y = 4 \ , \quad \mathbf{r} = \big(2(1-t), 4t, 0\big) \ ; \quad 0 \le t \le 1 \ ,$$
$$\frac{\partial \mathbf{r}}{\partial t} = (-2, 4, 0) \ .$$

C_2:

$$3y + 2z = 12 \ , \quad \mathbf{r} = \big(0, 4(1-t), 6t\big) \ ; \quad 0 \le t \le 1 \ ,$$
$$\frac{\partial \mathbf{r}}{\partial t} = (0, -4, 6) \ .$$

C_3:

$$3x + z = 6 \ , \quad \mathbf{r} = \big(2t, 0, 6(1-t)\big) \ ; \quad 0 \le t \le 1 \ ,$$
$$\frac{\partial \mathbf{r}}{\partial t} = (2, 0, -6) \ .$$

Flux of **a** through F:

$$\varphi = \int\limits_F \mathbf{a} \cdot d\mathbf{f} = \int\limits_F \operatorname{curl} \boldsymbol{\beta} \cdot d\mathbf{f} = \int\limits_{\partial F} \boldsymbol{\beta} \cdot d\mathbf{r}$$

$$\Longrightarrow \ \varphi = \int\limits_{(C_1)}^{1} dt \, \big(0, 2(1-t) \cdot 4t, 0\big) \cdot (-2, 4, 0)$$
$$+ \int\limits_{(C_2)}^{1} dt \, (0, 0 \cdot 4(1-t), 0) \cdot (0, -4, 6)$$
$$+ \int\limits_{(C_3)}^{1} dt \, (0, 2t \cdot 0, 0) \cdot (2, 0, -6)$$
$$= \int\limits_0^1 dt \ 32(t - t^2) = \left(16t^2 - \frac{32}{3} t^3\right)\Bigg|_0^1 = \frac{16}{3} \ .$$

The non-uniqueness of $\boldsymbol{\beta}$ **does not play any role** since

$$\int\limits_{\partial F} \operatorname{grad} \chi(\mathbf{r}) \cdot d\mathbf{r} = \int\limits_{\partial F} d\chi = 0 \ .$$

Solution 1.7.21 With

$$\begin{aligned} \boldsymbol{b} \cdot \operatorname{rot} \boldsymbol{a} &= \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) = \nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) \quad \text{(Spatprodukt)} \\ &= \nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) - \nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) \quad \text{(Produktregel)} \\ &= \operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) + \nabla \cdot (\boldsymbol{b} \times \boldsymbol{a}) \\ &= \operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{a} \cdot \operatorname{rot} \boldsymbol{b} \quad \text{(Spatprodukt)} \end{aligned}$$

follows:

$$\begin{aligned} \int_V d^3r\ \mathbf{b} \cdot \operatorname{curl}\mathbf{a} &= \int_V d^3r\ \operatorname{div}(\mathbf{a} \times \mathbf{b}) + \int_V d^3r\ \mathbf{a} \cdot \operatorname{curl}\mathbf{b} \\ &= \int_V d^3r\ \mathbf{a} \cdot \operatorname{curl}\mathbf{b} + \oint_{S(V)} d\mathbf{f} \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned}$$

(Gauss theorem).

Solution 1.7.22 Stokes theorem:

$$\oint_C \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} = \int_{F_C} \operatorname{curl}\mathbf{a}(\mathbf{r}) \cdot d\mathbf{f}\,,$$

$$\begin{aligned} \operatorname{curl}\mathbf{a}(\mathbf{r}) &= \big(xz, -yz, (x^2+y^2) + 2x^2 + (x^2+y^2) + 2y^2\big) \\ &= \big(xz, -yz, 4(x^2+y^2)\big)\,. \end{aligned}$$

Parameter representation of the area F_C (cylindrical coordinates):

$$\begin{aligned} F_C &= \big\{\mathbf{r} = (\rho\cos\varphi, \rho\sin\varphi, z=0)\,;\ 0 \le \rho \le R,\ 0 \le \varphi \le 2\pi\big\}\,, \\ \frac{\partial \mathbf{r}}{\partial \rho} &= (\cos\varphi, \sin\varphi, 0)\,; \quad \frac{\partial \mathbf{r}}{\partial \varphi} = \rho\,(-\sin\varphi, \cos\varphi, 0) \end{aligned}$$

$$\Longrightarrow\ d\mathbf{f} = \left(\frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \varphi}\right) d\rho\, d\varphi = \rho\, d\rho\, d\varphi\, \mathbf{e}_z$$

$$\Longrightarrow\ \operatorname{curl}\mathbf{a}(\mathbf{r}) \cdot d\mathbf{f} = 4\rho^3 d\rho\, d\varphi$$

$$\Longrightarrow\ \oint_C \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^R 4\rho^3 d\rho \int_0^{2\pi} d\varphi = 2\pi\, R^4\,.$$

Solution 1.7.23

1. $\mathbf{a}(\mathbf{r})$: gradient-field

$$\operatorname{div}\mathbf{a} = 2\ ; \quad \operatorname{curl}\mathbf{a} = 0\ .$$

2. $\mathbf{a}(\mathbf{r})$: gradient-field

$$\begin{aligned}\operatorname{div}\mathbf{a} &= 6\alpha - z^2 \sin yz - y^2 \sin yz \neq 0\ ,\\ \operatorname{curl}\mathbf{a} &= (\cos yz - yz\sin yz - \cos yz + yz\sin yz, 0-0, 0-0)\\ &= (0,0,0) = \mathbf{0}\ .\end{aligned}$$

3. $\mathbf{a}(\mathbf{r})$: curl-field

$$\begin{aligned}\operatorname{div}\mathbf{a} &= z - y + x - z + y - x = 0\ ,\\ \operatorname{curl}\mathbf{a} &= (z+y, x+z, y+x) \neq \mathbf{0}\ .\end{aligned}$$

4. Neither a pure gradient-field nor a pure curl-field:

$$\begin{aligned}\operatorname{div}\mathbf{a} &= 2xy + y \neq 0\\ \operatorname{curl}\mathbf{a} &= (z + 3z^2 \sin z^3, 0-0, 0-x^2) = (z + 3z^2\sin z^3, 0, -x^2) \neq \mathbf{0}\ .\end{aligned}$$

Solution 1.7.24 Green identity (1.66):

$$\int\limits_V \left[\varphi\,\Delta\psi + (\nabla\psi\cdot\nabla\varphi)\right]d^3r = \oint\limits_{S(V)} \varphi\frac{\partial\psi}{\partial n}df\ .$$

Poisson equation:

$$\Delta\,\varphi_{1,2}(\mathbf{r}) = f(\mathbf{r}) \quad \text{with } \varphi_1 = \varphi_2 \text{ on } S(V)\ .$$

For

$$\psi(\mathbf{r}) \equiv \varphi_1(\mathbf{r}) - \varphi_2(\mathbf{r})$$

we then have:

$$\Delta\,\psi \equiv 0 \quad \text{in } V\ .$$

Furthermore:

$$\psi \equiv 0 \quad \text{on } S(V)\ .$$

We choose in the Green identity $\varphi = \psi$ with ψ as given above:

$$\int_V [\psi \underbrace{\Delta\psi}_{=0} + (\nabla\psi)^2] d^3r = \oint_{S(V)} \underbrace{\psi}_{=0 \text{ on } S(V)} \frac{\partial\psi}{\partial n} df \ .$$

This means:

$$\int_V d^3r (\nabla\psi)^2 = 0 \implies \nabla\psi \equiv 0 \implies \psi = \text{const} \ .$$

Because of $\psi = 0$ on $S(V)$ it then holds:

$$\psi(\mathbf{r}) \equiv 0 \quad \text{in } V \text{ and therewith } \varphi_1(\mathbf{r}) \equiv \varphi_2(\mathbf{r}) \ .$$

Section 2.1.6

Solution 2.1.1

1. The sphere may carry the total charge Q:

$$Q = \int d^3r\, \rho(\mathbf{r}) = \rho_0 \int_0^R r^2 dr \int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi = \rho_0 \frac{4\pi}{3} R^3 \ ,$$

$$\rho(\mathbf{r}) = \begin{cases} \dfrac{Q}{(4\pi/3)R^3} \ , & \text{if } 0 \le r \le R \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

2. Total charge Q on the surface of a sphere:

ansatz:

$$\rho(\mathbf{r}) = \alpha(\vartheta, \varphi)\, \delta(r - R)$$

$$\implies Q = \int d^3r\, \rho(\mathbf{r}) = \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 dr\, \sin\vartheta\, d\vartheta\, d\varphi\, \alpha(\vartheta, \varphi)\, \delta(r - R) \ .$$

'homogeneous' means here $\alpha(\vartheta, \varphi) = \alpha$

$$\implies Q = R^2 \alpha\, 4\pi \implies \alpha = \frac{Q}{4\pi R^2} \implies \rho(\mathbf{r}) = \frac{Q}{4\pi R^2} \delta(r - R) \ .$$

Notice that $\delta(r-R)$ has the dimension 1/length!

3. Cylindrical coordinates $(\hat{\rho}, \varphi, z)$ are now useful:

$$\rho(\mathbf{r}) = \rho\left(\hat{\rho}, \varphi, z\right)$$

'infinitely thin' means

$$\rho(\mathbf{r}) = \gamma\left(\hat{\rho}, \varphi\right)\delta(z)$$

'homogeneously charged' means

$$\rho(\mathbf{r}) = \gamma\Theta\left(R-\hat{\rho}\right)\delta(z)$$

Total charge:

$$\begin{aligned}
Q &= \int d^3r\rho(\mathbf{r}) \\
&= 2\pi\gamma\int\limits_0^{\infty} d\hat{\rho}\,\hat{\rho}\int\limits_{-\infty}^{+\infty} dz\Theta\left(R-\hat{\rho}\right)\delta(z) \\
&= 2\pi\gamma\int\limits_0^{R} d\hat{\rho}\,\hat{\rho} \\
&= \pi\gamma R^2 \\
\rightarrow\ \gamma &= \frac{Q}{\pi R^2}
\end{aligned}$$

That gives finally:

$$\rho(\mathbf{r}) = \frac{Q}{\pi R^2}\Theta\left(R-\hat{\rho}\right)\ \delta(z)\ .$$

Solution 2.1.2 Charge density according to part 3. of Exercise 2.1.1 with cylindrical coordinates:

$$\rho(\mathbf{r}') = \sigma_0\,\delta(z')\,\Theta(R-\rho')\ .$$

With

$$\mathbf{r}' = \rho'\mathbf{e}_{\rho'} + z'\mathbf{e}_z$$

we get for the electrostatic potential on the z-axis:

$$
\begin{aligned}
4\pi\epsilon_0\varphi(z\mathbf{e}_z) &= \int d^3r'\,\frac{\rho(\mathbf{r}')}{\mid z\mathbf{e}_z - \mathbf{r}' \mid} \\
&= \sigma_0 \int\limits_0^{2\pi} d\varphi' \int\limits_0^{R} \rho' d\rho' \frac{1}{\mid z\mathbf{e}_z - \rho'\mathbf{e}_{\rho'} \mid} \\
&= 2\pi\sigma_0 \int\limits_0^{R} d\rho' \frac{\rho'}{\sqrt{z^2 + \rho'^2}} \\
&= 2\pi\sigma_0 \sqrt{z^2 + \rho'^2}\,\Big|_0^R \\
&= 2\pi\sigma_0 \left(\sqrt{z^2 + R^2} - |z|\right) .
\end{aligned}
$$

It follows therewith for the electrostatic potential:

$$
\varphi(z) = \frac{\sigma_0}{2\epsilon_0}|z| \left(\sqrt{1 + \left(\frac{R}{z}\right)^2} - 1\right) .
$$

From that one reads off the asymptotic behavior of the potential:

$$
\begin{aligned}
\varphi(z \to \pm\infty) &= \frac{\sigma_0}{2\epsilon_0}|z| \left(1 + \frac{1}{2}\left(\frac{R}{z}\right)^2 + \cdots - 1\right) \\
&= \frac{\sigma_0}{4\epsilon_0}\,\frac{R^2}{|z|} \;\to\; 0 .
\end{aligned}
$$

In addition it obviously holds:

$$
\varphi(z = 0) = \frac{\sigma_0}{2\epsilon_0} R .
$$

Electric field (normal component):

$$
\begin{aligned}
4\pi\epsilon_0\mathbf{E}(z\mathbf{e}_z) &= \int d^3r'\,\rho(\mathbf{r}')\,\frac{z\mathbf{e}_z - \mathbf{r}'}{\mid z\mathbf{e}_z - \mathbf{r}' \mid^3} \\
&= \sigma_0 \int\limits_0^{2\pi} d\varphi' \int\limits_0^{R} d\rho' \rho' \frac{z\mathbf{e}_z - \rho'\mathbf{e}_{\rho'}}{(z^2 + \rho'^2)^{\frac{3}{2}}} .
\end{aligned}
$$

Because of

$$\int_0^{2\pi} d\varphi' \mathbf{e}_{\rho'} = \int_0^{2\pi} d\varphi' (-\cos\varphi', \sin\varphi', 0) = (0, 0, 0)$$

it remains:

$$\begin{aligned}4\pi\epsilon_0 \mathbf{E}(z\mathbf{e}_z) &= \mathbf{e}_z \sigma_0 2\pi z \int_0^R d\rho' \frac{\rho'}{(z^2 + \rho'^2)^{\frac{3}{2}}} \\ &= \mathbf{e}_z \sigma_0 2\pi z \left(-\frac{1}{\sqrt{z^2 + R^2}} + \frac{1}{\sqrt{z^2}} \right) \\ &= \mathbf{e}_z \sigma_0 2\pi \left(\frac{z}{|z|} - \frac{z}{\sqrt{z^2 + R^2}} \right) .\end{aligned}$$

Therewith the electric field on the z-axis reads:

$$\mathbf{E}(z\mathbf{e}_z) = \mathbf{e}_z \frac{\sigma_0}{2\epsilon_0} \left(\frac{z}{|z|} - \frac{z}{\sqrt{z^2 + R^2}} \right) .$$

The direction of the field is clear from symmetry reasons. One sees that the field vanishes for $z \to \infty$. At $z = 0$ the field strength makes a jump

$$E(z = 0^+) - E(z = 0^-) = \frac{\sigma_0}{\epsilon_0} .$$

This corresponds to the general field behavior at interfaces (see (2.43)).

Solution 2.1.3

1. Total charge:

$$Q = \int d^3r\, \rho(\mathbf{r}) = \int_{R_\mathrm{i}}^{R_\mathrm{a}} r^2 dr \frac{\alpha}{r^2} \int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi = 4\pi\, \alpha (R_\mathrm{a} - R_\mathrm{i}) .$$

2. Total charge:

$$\begin{aligned}Q &= \int d^3r\, \rho(\mathbf{r}) = q - q \frac{\alpha^2}{4\pi} \int_0^\infty dr\, r^2 \int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi\, \frac{e^{-\alpha r}}{r} \\ &= q - q\,\alpha^2 \int_0^\infty dr\, r\, e^{-\alpha r} = q + q\,\alpha^2 \frac{d}{d\alpha} \int_0^\infty dr\, e^{-\alpha r}\end{aligned}$$

$$
\begin{aligned}
&= q + q\,\alpha^2 \frac{d}{d\alpha}\left(-\frac{1}{\alpha}e^{-\alpha\, r}\right)\Bigg|_0^\infty = q + q\,\alpha^2 \frac{d}{d\alpha}\,\frac{1}{\alpha} \\
&= q - q = 0\;.
\end{aligned}
$$

3. Total charge:

$$
\begin{aligned}
Q = \int d^3r\rho(\mathbf{r}) &= \sigma_0 \int\limits_0^\infty drr^2\delta(r-R)\int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\vartheta\,\cos\vartheta \\
&= 2\pi\sigma_0 R^2 \frac{1}{2}\cos^2\vartheta\Big|_{-1}^{+1} = 0\;.
\end{aligned}
$$

Dipole moment:

$$
\begin{aligned}
\mathbf{p} &= \int\limits_0^\infty r^2 dr \int\limits_0^\pi \sin\vartheta\, d\vartheta \int\limits_0^{2\pi} d\varphi\, \sigma_0 \cos\vartheta\, \delta(r-R)\, \mathbf{r} \\
&= \sigma_0 R^2 \int\limits_{-1}^{+1} d\cos\vartheta \int\limits_0^{2\pi} d\varphi\, \cos\vartheta\, R(\sin\vartheta\, \cos\varphi,\, \sin\vartheta\, \sin\varphi, \cos\vartheta) \\
&= 2\pi\,\sigma_0\, R^3 \int\limits_{-1}^{+1} d\cos\vartheta\,(0,0,\cos^2\vartheta) = \frac{4\pi}{3}\sigma_0 R^3 \mathbf{e}_z\;.
\end{aligned}
$$

Solution 2.1.4

1. The wire defines the z-axis. $\rho(\mathbf{r})$ is then surely independent of φ and z. We therefore choose as ansatz (cylindrical coordinates ρ', φ, z):

$$
\rho(\mathbf{r}) = \alpha(\rho')\,\delta(\rho')\;,
$$

(Z_l: cylinder of the height l, radius R, wire = axis)

$$
\Longrightarrow\ \kappa\, l = \int\limits_{Z_l} d^3r\,\rho(\mathbf{r}) = \int\limits_0^l dz \int\limits_0^{2\pi} d\varphi \int\limits_0^R \rho'\, d\rho'\, \rho(\mathbf{r}) = 2\pi\, l \int\limits_0^R \rho'\, d\rho'\, \alpha(\rho')\,\delta(\rho')\;.
$$

Only $\alpha(\rho') = a/\rho'$ does not lead to a contradiction:

$$\kappa\, l = 2\pi\, l\, a \int_0^R d\rho'\, \delta(\rho') \stackrel{(1.23)}{=} \pi\, l\, a$$

$$\Longrightarrow\ a = \frac{\kappa}{\pi}$$

$$\Longrightarrow\ \rho(\mathbf{r}) = \frac{\kappa}{\pi}\,\frac{\delta(\rho')}{\rho'}\ .$$

2. Electric field:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\,\epsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}') \\
&= \frac{\kappa}{4\pi\,\epsilon_0}\,\frac{1}{\pi}\int_0^\infty \rho'\,d\rho'\,\frac{1}{\rho'}\,\delta(\rho')\int_0^{2\pi} d\varphi' \int_{-\infty}^{+\infty} dz'\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}|^3} \\
&\stackrel{(1.23)}{=} \frac{\kappa}{4\pi\,\epsilon_0}\,\frac{1}{\pi}\,\frac{1}{2}\int_0^{2\pi} d\varphi' \int_{-\infty}^{+\infty} dz'\,\frac{\rho\,\mathbf{e}_\rho + (z-z')\,\mathbf{e}_z}{\left[\rho^2+(z-z')^2\right]^{3/2}} \\
&= \frac{\kappa}{4\pi\,\epsilon_0}\int_{-\infty}^{+\infty} dy\left[\frac{\rho\,\mathbf{e}_\rho}{(\rho^2+y^2)^{3/2}} - \underbrace{\frac{y\,\mathbf{e}_z}{(\rho^2+y^2)^{3/2}}}_{\mathbf{e}_z\frac{d}{dy}\frac{1}{\sqrt{\rho^2+y^2}}}\right] \\
&= \frac{\kappa\,\rho}{4\pi\,\epsilon_0}\mathbf{e}_\rho\underbrace{\int_{-\infty}^{+\infty} dy\,\frac{1}{(\rho^2+y^2)^{3/2}}}_{\frac{1}{\rho^2}\frac{y}{\sqrt{y^2+\rho^2}}\Big|_{-\infty}^{+\infty}=\frac{2}{\rho^2}}\ .
\end{aligned}$$

Therefrom it follows for the electric field strength,

$$\mathbf{E}(\mathbf{r}) = \frac{\kappa}{2\pi\,\epsilon_0\,\rho}\,\mathbf{e}_\rho\ ,$$

and for the potential:

$$\varphi(\mathbf{r}) = \frac{-\kappa}{2\pi\,\epsilon_0}\ln\rho + \text{const}\ .$$

Solution 2.1.5 Charge density:

$$\rho(\mathbf{r}) = \sigma\,\delta(z)\ .$$

Field strength:

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{4\pi\,\epsilon_0}\iint\limits_{-\infty}^{+\infty} dx'\,dy' \frac{(x-x', y-y', z)}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

$$= \frac{\sigma}{4\pi\,\epsilon_0} z\,\mathbf{e}_z \int\limits_{-\infty}^{+\infty} d\bar{x} \underbrace{\int\limits_{-\infty}^{+\infty} d\bar{y} \frac{1}{(\bar{x}^2+\bar{y}^2+z^2)^{3/2}}}_{\frac{1}{\bar{x}^2+z^2}\frac{\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2+z^2}}\Big|_{-\infty}^{+\infty}=\frac{2}{\bar{x}^2+z^2}}\ ,$$

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\pi\,\epsilon_0} z\,\mathbf{e}_z \int\limits_{-\infty}^{+\infty} d\bar{x} \frac{1}{\bar{x}^2+z^2}\ .$$

For $z \neq 0$ it holds for the integral:

$$\frac{1}{z}\arctan\frac{\bar{x}}{z}\Big|_{-\infty}^{+\infty} = \frac{1}{z}\frac{z}{|z|}\pi$$

$$\Longrightarrow\ \mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\epsilon_0}\frac{z}{|z|}\mathbf{e}_z\ ,$$

$$\varphi(\mathbf{r}) = -\frac{\sigma}{2\epsilon_0}|z| + \text{const}\ .$$

Solution 2.1.6

1. Potential of the dipole:

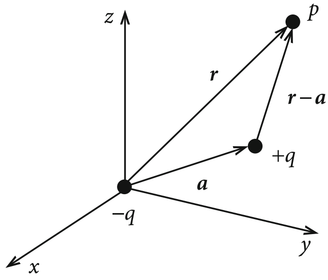
$$\varphi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\left\{\frac{-q}{r} + \frac{q}{|\mathbf{r}-\mathbf{a}|}\right\}\ .$$

Taylor expansion (1.32):

$$\frac{1}{|\mathbf{r}-\mathbf{a}|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{a}}{r^3} + \frac{1}{2}\frac{3(\mathbf{r}\cdot\mathbf{a})^2 - r^2a^2}{r^5} + \cdots$$

$$\Longrightarrow\ \varphi(\mathbf{r}) = \frac{q}{4\pi\,\epsilon_0}\left\{\frac{\mathbf{r}\cdot\mathbf{a}}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{a})^2 - r^2a^2}{2r^5} + \cdots\right\}\ .$$

Fig. A.10



Large distances: $r \gg a$.
Dipole moment: $\mathbf{p} = q\,\mathbf{a}$ (Fig. A.10):

$$\varphi(\mathbf{r}) \approx \frac{1}{4\pi\,\epsilon_0}\,\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}\,.$$

2. Polar axis $\uparrow\uparrow$ $\mathbf{a}$:
   Spherical coordinates: $\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\,\frac{\partial}{\partial\vartheta}, \frac{1}{r\sin\vartheta}\,\frac{\partial}{\partial\varphi}\right)$.

$$4\pi\,\epsilon_0\varphi(\mathbf{r}) = -\frac{q}{r} + \frac{q}{|\mathbf{r}-\mathbf{a}|}\,,$$

$$\mathbf{E}(\mathbf{r}) = -\nabla\,\varphi(\mathbf{r})\,,$$

$$-\nabla\left(-\frac{q}{r}\right) = -\frac{q}{r^2}\mathbf{e}_r\,,$$

$$|\mathbf{r}-\mathbf{a}| = \sqrt{r^2+a^2-2ra\,\cos\vartheta}\,,$$

$$\frac{\partial}{\partial r}|\mathbf{r}-\mathbf{a}| = \frac{r-a\,\cos\vartheta}{\sqrt{r^2+a^2-2ra\,\cos\vartheta}}\,,$$

$$\frac{\partial}{\partial\vartheta}|\mathbf{r}-\mathbf{a}| = \frac{r\,a\,\sin\vartheta}{\sqrt{r^2+a^2-2ra\,\cos\vartheta}}\,,$$

$$\frac{\partial}{\partial\varphi}|\mathbf{r}-\mathbf{a}| = 0$$

$$\Longrightarrow\quad \frac{\partial}{\partial r}\,\frac{1}{|\mathbf{r}-\mathbf{a}|} = -\frac{r-a\,\cos\vartheta}{(r^2+a^2-2ra\,\cos\vartheta)^{3/2}}\,,$$

$$\frac{1}{r}\,\frac{\partial}{\partial\vartheta}\,\frac{1}{|\mathbf{r}-\mathbf{a}|} = -\frac{a\,\sin\vartheta}{(r^2+a^2-2ra\,\cos\vartheta)^{3/2}}\,,$$

$$\frac{1}{r\,\sin\vartheta}\,\frac{\partial}{\partial\varphi}\,\frac{1}{|\mathbf{r}-\mathbf{a}|} = 0\,.$$

We have found therewith the components of the electric field:

$$4\pi\,\epsilon_0 E_r = -\frac{q}{r^2} + \frac{q(r - a\,\cos\vartheta)}{(r^2 + a^2 - 2r\,a\,\cos\vartheta)^{3/2}}\;,$$
$$4\pi\,\epsilon_0 E_\vartheta = \frac{q\,a\,\sin\vartheta}{(r^2 + a^2 - 2r\,a\,\cos\vartheta)^{3/2}}\;,$$
$$4\pi\,\epsilon_0 E_\varphi = 0\;.$$

Solution 2.1.7

$$\rho(\mathbf{r}) = \begin{cases} \dfrac{\alpha}{r^2} & \text{for } R_\mathrm{i} < r < R_\mathrm{a}\;, \\ 0 & \text{otherwise}\;. \end{cases}$$

Spherically-symmetric charge distribution:

$$\mathbf{E}(\mathbf{r}) = E_r(r,\vartheta,\varphi)\,\mathbf{e}_r + E_\vartheta(r,\vartheta,\varphi)\,\mathbf{e}_\vartheta + E_\varphi(r,\vartheta,\varphi)\,\mathbf{e}_\varphi = E_r(r)\,\mathbf{e}_r \quad \text{(justification?)}\;.$$

Gauss theorem:

$$\int\limits_V d^3r\,\mathrm{div}\;\mathbf{E}(\mathbf{r}) = \int\limits_{S(V)} d\mathbf{f}\cdot\mathbf{E}(\mathbf{r})\;.$$

Maxwell equation:

$$\int\limits_V d^3r\;\mathrm{div}\;\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0}\int\limits_V d^3r\,\rho(\mathbf{r})\;.$$

It follows:

$$\int\limits_{S(V)} d\mathbf{f}\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0}\int\limits_V d^3r\,\rho(\mathbf{r})\;,$$

V_r: concentric sphere with radius r

$$\Longrightarrow\;\; d\mathbf{f} = \mathbf{e}_r\,r^2\sin\vartheta\,d\vartheta\,d\varphi$$
$$\Longrightarrow\;\; 4\pi\,r^2 E_r(r) = \frac{1}{\epsilon_0}\int\limits_{V_r} d^3r'\rho(\mathbf{r}')\;.$$

(a) $0 \le r < R_\mathrm{i}$:

$$\rho(\mathbf{r}) \equiv 0 \;\Longrightarrow\; \mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r \equiv 0\;.$$

(b) $R_\mathrm{i} \le r \le R_\mathrm{a}$:

$$\int_V d^3r' \rho(\mathbf{r}') = 4\pi\,\alpha \int_{R_\mathrm{i}}^{r} r'^2 dr' \frac{1}{r'^2} = 4\pi\,\alpha(r - R_\mathrm{i})$$

$$\Longrightarrow \; E_r(r) = \frac{\alpha}{\epsilon_0 r^2}(r - R_\mathrm{i}) \; .$$

Using part (a) of Exercise 2.2.3 and the total charge:

$$Q = 4\pi\,\alpha(R_\mathrm{a} - R_\mathrm{i})$$

we get:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0 r^2}\,\frac{r - R_\mathrm{i}}{R_\mathrm{a} - R_\mathrm{i}}\mathbf{e}_r \; .$$

(c) $R_\mathrm{a} < r$:

$$\int_V d^3r' \rho(\mathbf{r}') = 4\pi\,\alpha \int_{R_\mathrm{i}}^{R_\mathrm{a}} r'^2 dr' \frac{1}{r'^2} = 4\pi\,\alpha(R_\mathrm{a} - R_\mathrm{i})$$

$$\Longrightarrow \; \mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0 r^2}\,\mathbf{e}_r \; .$$

This is the field of a point charge located at the origin of coordinates. Altogether we then have:

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0 r^2}\mathbf{e}_r \begin{cases} 0\,, & \text{for } r < R_\mathrm{i}\,, \\ \dfrac{r - R_\mathrm{i}}{R_\mathrm{a} - R_\mathrm{i}}\,, & \text{for } R_\mathrm{i} \le r \le R_\mathrm{a}\,, \\ 1\,, & \text{for } R_\mathrm{a} < r\,. \end{cases}$$

Electrostatic potential:

$$\mathbf{E} = -\nabla\varphi \; ; \quad E_r(r) = -\frac{\partial\varphi}{\partial r} \; ; \quad \varphi(\mathbf{r}) = \varphi(r) \; .$$

(c)

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0 r} + \text{const} \; .$$

const $= 0$, since $\varphi(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} 0$ (*physical boundary conditions.*)

(b)

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0(R_a - R_i)}\left(-\ln r - \frac{R_i}{r} + \text{const}\right)\,.$$

Continuity at $r = R_a$:

$$\varphi(r = R_a) = \frac{Q}{4\pi\,\epsilon_0(R_a - R_i)}\left(-\ln R_a - \frac{R_i}{R_a} + \text{const}\right)$$
$$\stackrel{!}{=} \frac{Q}{4\pi\,\epsilon_0 R_a}\,.$$

This holds only if

$$\text{const} = \ln R_a + 1\,.$$

Therewith the electrostatic potential reads:

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0(R_a - R_i)}\left(1 - \frac{R_i}{r} - \ln\frac{r}{R_a}\right)\,.$$

(a) $\varphi(\mathbf{r}) = \text{const} = \varphi(R_i)$
Continuity:

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0(R_a - R_i)}\ln\frac{R_a}{R_i}\,.$$

Thus the final result is:

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\,\epsilon_0}\begin{cases} \dfrac{\ln(R_a/R_i)}{R_a - R_i} & \text{for } 0 \le r \le R_i\,, \\ \dfrac{1 - R_i/r - \ln(r/R_a)}{R_a - R_i} & \text{for } R_i \le r \le R_a\,, \\ \dfrac{1}{r} & \text{for } R_a \le r\,. \end{cases}$$

Solution 2.1.8 We apply the *physical* Gauss theorem:

$$\int_{S(V)} \mathbf{E}\cdot d\mathbf{f} = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')\,d^3r'\,.$$

Charge density:

$$\rho(\mathbf{r}') = \rho(r') = \underbrace{\frac{e}{4\pi\,r'^2}\delta(r')}_{\substack{\text{point-like} \\ \text{nuclear charge } (Z=1)}} - \underbrace{\frac{e}{\pi\,a^3}e^{-(2r'/a)}}_{\substack{\text{electron in its} \\ \text{ground state}}}\,.$$

This is a spherically-symmetric charge distribution. Therefore the ansatz:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r \ .$$

We choose:
V_r: sphere with radius r, origin at the center of the sphere, $d\mathbf{f} = r^2 \sin\vartheta\, d\vartheta\, d\varphi\, \mathbf{e}_r$: area element on $S(V)$.
Then it holds:

$$4\pi\, r^2 E_r(r) = \frac{e}{\epsilon_0} - \frac{e}{\epsilon_0 \pi\, a^3}\, 4\pi \int_0^r dr' r'^2 e^{-(2r'/a)} \ ,$$

$$\begin{aligned}
\int_0^r dx\, x^2 e^{-\beta x} &= \frac{d^2}{d\beta^2} \int_0^r dx\, e^{-\beta x} = \frac{d^2}{d\beta^2} \left[-\frac{1}{\beta} \left(e^{-\beta r} - 1 \right) \right] \\
&= \frac{d}{d\beta} \left[\frac{1}{\beta^2} \left(e^{-\beta r} - 1 \right) + \frac{r}{\beta} e^{-\beta r} \right] \\
&= \left[-\frac{2}{\beta^3} \left(e^{-\beta r} - 1 \right) - \frac{2r}{\beta^2} e^{-\beta r} - \frac{r^2}{\beta} e^{-\beta r} \right] \\
&= \frac{2}{\beta^3} - e^{-\beta r} \left(\frac{2}{\beta^3} + \frac{2r}{\beta^2} + \frac{r^2}{\beta} \right) \ ,
\end{aligned}$$

$$\begin{aligned}
\int_0^r dr' r'^2 e^{-(2r'/a)} &= \frac{a^3}{4} - \frac{a}{2} e^{-(2r/a)} \left(\frac{a^2}{2} + a\, r + r^2 \right) \\
\Longrightarrow \quad 4\pi\, r^2 E_r(r) &= \frac{2e}{\epsilon_0 a^2} e^{-(2r/a)} \left(\frac{a^2}{2} + a\, r + r^2 \right) \\
&= \frac{e}{\epsilon_0} e^{-(2r/a)} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \ .
\end{aligned}$$

Therewith we have for the electric field:

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_r \frac{e}{4\pi\, \epsilon_0} e^{-(2r/a)} \left(\frac{1}{r^2} + \frac{2}{r\, a} + \frac{2}{a^2} \right) \ .$$

The potential we get by integration:

$$\varphi(\mathbf{r}) = \varphi(r) \quad \text{with} \quad E_r(r) = -\frac{d\varphi}{dr} \ .$$

From that it follows:

$$\varphi(r) = -\int\limits_{\infty}^{r} E_{r'} dr' \ ,$$

$$\int\limits_{\infty}^{r} dr' e^{-(2r'/a)} \frac{2}{a^2} = -\frac{1}{a} \ e^{-2r'/a}\Big|_{\infty}^{r} = -\frac{1}{a} e^{-(2r/a)} \ ,$$

$$\int\limits_{\infty}^{r} dr' \left(\frac{2}{r'a} + \frac{1}{r'^2}\right) e^{-(2r'/a)} = -\int\limits_{\infty}^{r} dr' \frac{d}{dr'} \left[\frac{1}{r'} e^{-(2r'/a)}\right] = -\frac{1}{r} e^{-(2r/a)} \ .$$

As the result we get a screened Coulomb potential:

$$\varphi(r) = \frac{e}{4\pi\,\epsilon_0} e^{-(2r/a)} \left(\frac{1}{r} + \frac{1}{a}\right) \ .$$

$r \ll a$:

$$\varphi(r) \approx \frac{e}{4\pi\,\epsilon_0 r}$$

(pure Coulomb potential of the nucleus).
$r \gg a$:

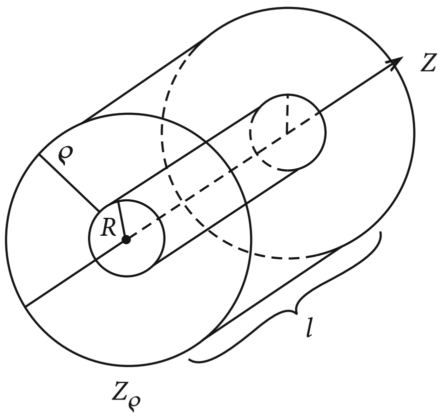
$$\varphi(r) \approx \frac{e}{4\pi\,\epsilon_0 a} e^{-(2r/a)} \ .$$

The total potential of the H-atom vanishes exponentially for large distances.

Solution 2.1.9 We choose cylindrical coordinates (Fig. A.11),

$$\rho, \varphi, z \ ,$$

Fig. A.11



and exploit the cylindrical symmetry of the problem:

$$\mathbf{E}(\mathbf{r}) = E(\rho)\,\mathbf{e}_\rho \ .$$

For the charge density we have:

$$\bar{\rho}(\mathbf{r}) = \begin{cases} \rho_0 & \text{for } \rho \le R \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Let Z_ρ be a cylinder with the length l, cylinder axis: z-axis, ρ: radius. Applying the physical Gauss theorem leads to:

$$\int\limits_{S(Z_\rho)} \mathbf{E} \cdot d\mathbf{f} = \frac{1}{\epsilon_0} \int\limits_{Z_\rho} \bar{\rho}(\mathbf{r}) d^3 r \ .$$

We calculate the various contributions separately:
Front faces:

$$\mathbf{E} \perp d\mathbf{f} \implies \text{no contribution to the flux} \ ,$$

Barrel (1.37):

$$\begin{aligned} d\mathbf{f} &= \rho\, d\varphi\, dz\, \mathbf{e}_\rho \\ \implies \mathbf{E} \cdot d\mathbf{f} &= \rho\, E_\rho(\rho)\, d\varphi\, dz \\ \implies \int\limits_{S(Z_\rho)} \mathbf{E} \cdot d\mathbf{f} &= 2\pi\, l\, \rho\, E_\rho(\rho) \ . \end{aligned}$$

$\rho \ge R$:

$$\int\limits_{Z_\rho} \bar{\rho}(\mathbf{r})\, d^3 r = \rho_0 2\pi \int\limits_0^R \rho'\, d\rho' \int\limits_0^l dz' = \rho_0 \pi\, R^2 l \ .$$

$\rho \le R$:

$$\int\limits_{Z_\rho} \bar{\rho}(\mathbf{r})\, d^3 r = \rho_0 2\pi \int\limits_0^\rho \rho' d\rho' \int\limits_0^l dz' = \rho_0 \pi\, \rho^2 l \ .$$

Finally this yields:

$$\mathbf{E}(\mathbf{r}) = \frac{\rho_0}{\epsilon_0}\mathbf{e}_\rho \begin{cases} \frac{1}{2}\rho\ , & \text{if } \rho \leq R\ , \\ \frac{1}{2}\frac{R^2}{\rho}\ , & \text{if } \rho \geq R\ . \end{cases}$$

The $(1/\rho)$-dependence for $\rho \geq R$ is typical.
Potential:

$$\mathbf{E} = -\left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z}\right)\varphi = E_\rho \mathbf{e}_\rho \implies \varphi = \varphi(\rho)\ ,$$

inside:

$$\varphi(\rho) = -\frac{\rho_0}{4\epsilon_0}\rho^2 + \varphi_0\ ,$$

outside:

$$\varphi(\rho) = -\frac{R^2\rho_0}{2\epsilon_0}\ln\rho + \varphi_1\ .$$

The choice of the reference point is still free, e.g.:

$$\varphi(\rho = R) \stackrel{!}{=} 0\ .$$

Then we have:

$$\varphi_0 = \frac{\rho_0}{4\epsilon_0}R^2\ ; \quad \varphi_1 = \frac{\rho_0 R^2}{2\epsilon_0}\ln R\ .$$

Thus it remains:

$$\varphi(\mathbf{r}) = \varphi(\rho) = \frac{\rho_0 R^2}{2\epsilon_0}\begin{cases} \frac{1}{2}\left(1 - \frac{\rho^2}{R^2}\right) & \text{for } \rho \leq R\ , \\ \ln\frac{R}{\rho} & \text{for } R \leq \rho\ . \end{cases}$$

Solution 2.1.10

1. **Charge density** according to Exercise 2.1.1:

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\,\delta(r - R)\ ,$$

   Q: total charge, R: radius of the sphere.

Electric field:
Spherically-symmetric charge distribution, therefore:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\mathbf{e}_r \ ,$$

V_r: concentric sphere with the radius r.

$$\int\limits_{S(V_r)} d\mathbf{f} \cdot \mathbf{E} = \int\limits_{S(V_r)} r^2 \sin\vartheta \, d\vartheta \, d\varphi \, E_r(r) \, \mathbf{e}_r \cdot \mathbf{e}_r = 4\pi \, r^2 E_r(r) \stackrel{!}{=} \frac{1}{\epsilon_0} \int\limits_{V_r} d^3r' \rho(\mathbf{r}')$$

$$= \begin{cases} \dfrac{Q}{\epsilon_0} \ , & \text{if } r > R \ , \\ 0 \ , & \text{if } r < R \ . \end{cases}$$

That leads to:

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_r \begin{cases} \dfrac{Q}{4\pi \, \epsilon_0} \dfrac{1}{r^2} \ , & \text{if } r > R \ , \\ 0 \ , & \text{if } r < R \ . \end{cases}$$

Energy density:

$$w(\mathbf{r}) = \begin{cases} \dfrac{Q^2}{32\pi^2 \epsilon_0} \dfrac{1}{r^4} \ , & \text{if } r > R \ , \\ 0 \ , & \text{if } r < R \ . \end{cases}$$

Total energy:

$$W = \frac{\epsilon_0}{2} \int d^3r |\mathbf{E}(\mathbf{r})|^2 = \frac{Q^2}{32\pi^2 \epsilon_0} 4\pi \int\limits_R^\infty dr \, r^2 \frac{1}{r^4} \implies W = \frac{Q^2}{8\pi \, \epsilon_0 R} \ .$$

2. **Electric field:**
We use Exercise 2.1.7:

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_r \frac{Q}{4\pi \, \epsilon_0 r^2} \begin{cases} 0 \ , & \text{if } r < R_1 \ , \\ \dfrac{r - R_1}{R_2 - R_1} \ , & \text{if } R_1 \le r \le R_2 \ , \\ 1 \ , & \text{if } r > R_2 \ , \end{cases}$$

$Q = 4\pi \, \alpha (R_2 - R_1)$.

Energy density:

$$w(\mathbf{r}) = \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \frac{Q^2}{32\pi^2\epsilon_0}\,\frac{1}{r^4}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ \left(\dfrac{r-R_1}{R_2-R_1}\right)^2, & \text{if } R_1 \le r \le R_2\,,\\ 1\,, & \text{if } R_2 < r\,. \end{cases}$$

Total energy:

$$\begin{aligned}
W &= \int d^3r\,w(\mathbf{r})\\
&= \frac{Q^2}{32\pi^2\epsilon_0}4\pi\left[\int\limits_{R_1}^{R_2} dr\,r^2\frac{1}{r^4}\,\frac{1}{(R_2-R_1)^2}\left(r^2-2rR_1+R_1^2\right)+\int\limits_{R_2}^{\infty} dr\,r^2\frac{1}{r^4}\right]\\
&= \frac{Q^2}{8\pi\,\epsilon_0}\left\{\frac{1}{R_2}+\frac{1}{(R_2-R_1)^2}\left[(R_2-R_1)-2R_1\ln\frac{R_2}{R_1}+R_1^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]\right\}\\
&= \frac{Q^2}{8\pi\,\epsilon_0}\left[\frac{1}{R_2}+\frac{1}{R_2}\,\frac{R_2+R_1}{R_2-R_1}-\frac{2R_1}{(R_2-R_1)^2}\ln\frac{R_2}{R_1}\right]\\
\Longrightarrow\quad W &= \frac{Q^2}{4\pi\,\epsilon_0}\left[(R_2-R_1)-R_1\ln\frac{R_2}{R_1}\right]\frac{1}{(R_2-R_1)^2}\,.
\end{aligned}$$

Section 2.2.9

Solution 2.2.1

1. Spherical capacitor:

$$Q_1 = Q\,;\quad Q_2 = -Q\,.$$

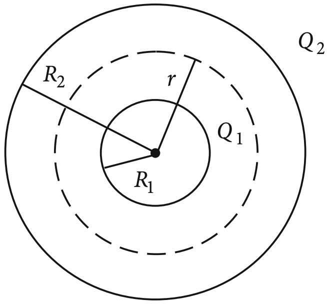
Charge density:

$$\rho(\mathbf{r}) = \frac{Q_1}{4\pi\,R_1^2}\,\delta(r-R_1)+\frac{Q_2}{4\pi\,R_2^2}\,\delta(r-R_2)\,.$$

Because of the spherically-symmetric charge distribution we can assume (Fig. A.12):

$$\mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r\,.$$

Fig. A.12



Let V_r be the volume of a sphere with the radius r. Then the electric field can be calculated as follows:

$$\int\limits_{S(V_r)} d\mathbf{f}\cdot\mathbf{E} = 4\pi\, r^2 E_r(r) = \frac{1}{\epsilon_0}\int\limits_{V_r} d^3r'\rho(r')$$

$$= \frac{4\pi}{\epsilon_0}\int\limits_0^r dr' r'^2 \left[\frac{Q_1}{4\pi\, R_1^2}\,\delta(r'-R_1) + \frac{Q_2}{4\pi\, R_2^2}\delta(r'-R_2)\right]$$

$$= \frac{1}{\epsilon_0}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ Q_1\,, & \text{if } R_1 < r < R_2\,,\\ Q_1+Q_2\,, & \text{if } R_2 < r\,.\end{cases}$$

We finally get:

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{e}_r}{4\pi\,\epsilon_0 r^2}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ Q_1\,, & \text{if } R_1 < r < R_2\,,\\ Q_1+Q_2\,, & \text{if } R_2 < r\,.\end{cases}$$

Energy density:

$$w(\mathbf{r}) = \frac{1}{32\pi^2\epsilon_0}\,\frac{1}{r^4}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ Q_1^2\,, & \text{if } R_1 < r < R_2\,,\\ (Q_1+Q_2)^2\,, & \text{if } R_2 < r\,,\end{cases}$$

Spherical capacitor: $Q_1 = Q$; $Q_1 + Q_2 = 0$.

Total energy:

$$W = \int d^3r\, w(\mathbf{r}) = \frac{1}{8\pi\,\epsilon_0}\left[Q_1^2\int\limits_{R_1}^{R_2} dr\,\frac{1}{r^2} + (Q_1+Q_2)^2\int\limits_{R_2}^{\infty} dr\,\frac{1}{r^2}\right]$$
$$= \frac{1}{8\pi\,\epsilon_0}\left\{Q_1^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) + (Q_1+Q_2)^2\frac{1}{R_2}\right\} .$$

Spherical capacitor:

$$W = \frac{Q^2}{8\pi\,\epsilon_0}\,\frac{R_2-R_1}{R_2R_1} .$$

2a. $Q_1 = Q, Q_2 = -Q/2$:

$$w(\mathbf{r}) = \frac{1}{32\pi^2\epsilon_0}\,\frac{1}{r^4}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ Q^2\,, & \text{if } R_1 < r < R_2\,,\\ \dfrac{Q^2}{4}\,, & \text{if } R_2 < r\,. \end{cases}$$

The energy density inside the spherical capacitor remains unchanged since the same fields are present there as in 1. But now there are still contributions of the exterior:

$$W = \frac{Q^2}{8\pi\,\epsilon_0}\left(\frac{R_2-R_1}{R_2R_1} + \frac{1}{4R_2}\right) .$$

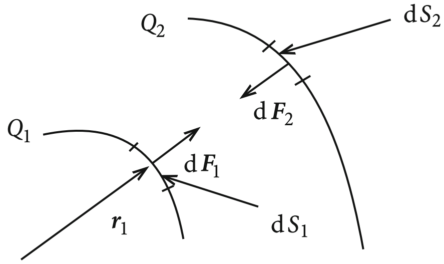
2b. $Q_1 = -Q/2; Q_2 = Q$:

$$w(\mathbf{r}) = \frac{1}{32\pi^2\epsilon_0}\,\frac{1}{r^4}\begin{cases} 0\,, & \text{if } r < R_1\,,\\ \dfrac{Q^2}{4}\,, & \text{if } R_1 < r < R_2\,,\\ \dfrac{Q^2}{4}\,, & \text{if } R_2 < r\,. \end{cases}$$

The energy density inside the spherical capacitor is smaller since the field there is weaker. Outside the capacitor everything is the same as in 2a.:

$$W = \frac{Q^2}{8\pi\,\epsilon_0}\left[\frac{1}{4}\,\frac{R_2-R_1}{R_2R_1} + \frac{1}{4}\,\frac{1}{R_2}\right] = \frac{Q^2}{32\pi\,\epsilon_0\,R_1} .$$

Fig. A.13



3.

dS_1: area element of the inner spherical shell ,

dS_2: area element of the outer spherical shell .

$$d\mathbf{F}_1 = dS_1 \frac{Q_1}{4\pi R_1^2} \mathbf{E}\left(\mathbf{r}_1^+\right) \ ,$$

$$d\mathbf{F}_2 = dS_2 \frac{Q_2}{4\pi R_2^2} \mathbf{E}\left(\mathbf{r}_2^-\right) \ .$$

This yields the pressure (Fig. A.13):

$$p_1 = \frac{dF_1}{dS_1} = \frac{Q_1^2}{16\pi^2 \epsilon_0 R_1^4} \ ,$$

$$p_2 = \frac{dF_2}{dS_2} = \frac{|Q_2 \, Q_1|}{16\pi^2 \epsilon_0 R_2^4} \ .$$

1. $Q_1 = Q, Q_2 = -Q$:

$$p_{1,2} = \frac{Q^2}{16\pi^2 \epsilon_0 \, R_{1,2}^4} \ .$$

2a. $Q_1 = Q, Q_2 = -Q/2$:

$$p_1 = \frac{Q^2}{16\pi^2 \epsilon_0 R_1^4} \ , \quad p_2 = \frac{Q^2}{32\pi^2 \epsilon_0 R_2^4} \ .$$

2b. $Q_1 = -Q/2, Q_2 = Q$:

$$p_1 = \frac{Q^2}{64\pi^2 \epsilon_0 R_1^4} \ , \quad p_2 = \frac{Q^2}{32\pi^2 \epsilon_0 R_2^4} \ .$$

Solution 2.2.2

1. For the single fields of the practically infinitely extended plates it holds according to (2.51) and Exercise 2.1.5, respectively:

$$\begin{aligned}\mathbf{E}_{Q_1} &= \frac{Q_1}{2\epsilon_0 F}\frac{x}{|x|}\mathbf{e}_x \\ \mathbf{E}_Q &= \frac{Q}{2\epsilon_0 F}\frac{x-x_0}{|x-x_0|}\mathbf{e}_x \\ \mathbf{E}_{Q_2} &= \frac{Q_2}{2\epsilon_0 F}\frac{x-d}{|x-d|}\mathbf{e}_x \ .\end{aligned}$$

   Therewith it results as the total field inside the capacitor, to the left of the Q-plate:

$$\mathbf{E}_l(x) = \frac{1}{2\epsilon_0 F}(Q_1 - Q - Q_2)\mathbf{e}_x$$

   and to the right of the Q-plate

$$\mathbf{E}_r(x) = \frac{1}{2\epsilon_0 F}(Q_1 + Q - Q_2)\mathbf{e}_x \ .$$

   Voltage at the capacitor:

$$\begin{aligned}U &= -\int_0^d E\,dx = -\left(E_l x_0 + E_r(d - x_0)\right) \\ &= \frac{-1}{2\epsilon_0 F}\left((Q_1 - Q_2)d - Q(2x_0 - d)\right) \ .\end{aligned}$$

2. Force on the Q-plate:

$$\begin{aligned}\mathbf{F} &= Q\left(\mathbf{E}_{Q_1}(x_0) + \mathbf{E}_{Q_2}(x_0)\right) \\ \rightarrow\ \mathbf{F} &= \frac{Q}{2\epsilon_0 F}(Q_1 - Q_2)\mathbf{e}_x \ .\end{aligned}$$

   Short circuit $U = 0$ means for Q_1 and Q_2:

$$Q_1 - Q_2 = Q\frac{2x_0 - d}{d} \ .$$

   The force becomes therewith now space-independent:

$$\mathbf{F} = \frac{Q^2}{2\epsilon_0 F d}(2x_0 - d)\mathbf{e}_x \ .$$

Thus it is:

$$x_0 = \frac{d}{2} \;\rightarrow\; \mathbf{F} = 0$$
$$x_0 > \frac{d}{2} \;\rightarrow\; \mathbf{F} \propto \mathbf{e}_x$$
$$x_0 < \frac{d}{2} \;\rightarrow\; \mathbf{F} \propto -\mathbf{e}_x \,.$$

3. Equation of motion in the presence of the voltage U:

$$M\ddot{x}_0 = \frac{Q}{2\epsilon_0 F}(Q_1 - Q_2) \,.$$

Short circuit $U = 0$ and with $y = x_0 - d/2$:

$$M\ddot{y} = \frac{Q^2}{\epsilon_0 F d} y \,.$$

It results therewith an exponential time-dependence.

Solution 2.2.3

1. The potential energy of a dipole in the electric field amounts to (2.79):

$$V_{\mathrm{D}}(\mathbf{r}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}) \,.$$

The point charge creates the field

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\,\epsilon_0} \frac{\mathbf{r}}{r^3} \,.$$

This gives:

$$V_{\mathrm{D}}(\mathbf{r}) = -\frac{q}{4\pi\,\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} = \frac{q}{4\pi\,\epsilon_0} \mathbf{p} \cdot \nabla \frac{1}{r} \,.$$

2. The force on the dipole can be derived from the potential energy:

$$\mathbf{F}_{\mathrm{D}}(\mathbf{r}) = -\nabla\, V_{\mathrm{D}}(\mathbf{r}) = -\frac{q}{4\pi\,\epsilon_0} \nabla \left(\mathbf{p} \cdot \nabla \frac{1}{r} \right) \,.$$

We use the formula:

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} + (\mathbf{a} \cdot \nabla)\,\mathbf{b} + \mathbf{b} \times \mathrm{curl}\mathbf{a} + \mathbf{a} \times \mathrm{curl}\mathbf{b}$$

and obtain with $\mathbf{p} = \text{const}$:

$$\nabla\left(\mathbf{p}\cdot\nabla\frac{1}{r}\right) = (\mathbf{p}\cdot\nabla)\nabla\frac{1}{r} + \mathbf{p}\times\underbrace{\text{curl}\nabla\frac{1}{r}}_{=0} = -\sum_i p_i \frac{\partial}{\partial x_i}\frac{\mathbf{r}}{r^3}$$

$$= -\sum_i p_i\left(\frac{\mathbf{e}_i}{r^3} - 3\frac{\mathbf{r}}{r^4}\frac{x_i}{r}\right) = 3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p})}{r^5} - \frac{\mathbf{p}}{r^3}\ .$$

It follows therewith:

$$\mathbf{F}_{\mathrm{D}}(\mathbf{r}) = -\frac{q}{4\pi\,\epsilon_0}\,\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{p}) - \mathbf{p}\,r^2}{r^5}\ .$$

3. The field of the dipole at the position $\mathbf{0}$ of the point charge is according to (2.73):

$$\mathbf{E}_{\mathrm{D}}(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\,\frac{3(-\mathbf{r})[(-\mathbf{r})\cdot\mathbf{p}] - \mathbf{p}\,r^2}{r^5}\ .$$

From this a force results which the dipole exerts on the point charge:

$$\mathbf{F}_{\mathrm{p}}(\mathbf{r}) = q\,\mathbf{E}_{\mathrm{D}}(\mathbf{r}) = -\mathbf{F}_{\mathrm{D}}(\mathbf{r})\ .$$

The third Newtonian axiom is thus fulfilled.

Solution 2.2.4

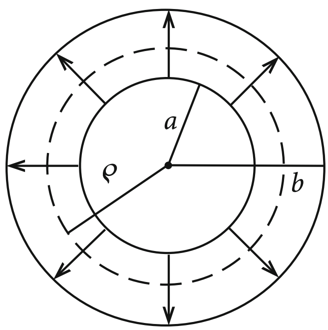
1. Cylindrical coordinates (Fig. A.14):

$$\rho, \varphi, z\ ,$$

Symmetry:

$$\mathbf{E}(\mathbf{r}) = E_\rho(\rho)\,\mathbf{e}_\rho\ ,$$

Fig. A.14



Z: cylinder, L: length, ρ: radius.

$$\int\limits_{S(Z)} d\mathbf{f}\cdot\mathbf{E} = E_\rho(\rho)\,2\pi\rho\,h \stackrel{!}{=} \frac{1}{\epsilon_0}\int\limits_Z d^3r'\rho(\mathbf{r}') = \frac{1}{\epsilon_0}\begin{cases} 0\,, & \text{if } \rho < a\,, \\ h\,\bar{q}\,, & \text{if } a < \rho < b\,, \\ 0\,, & \text{if } b < \rho\,, \end{cases}$$

$\bar{q}$: charge per unit length.
It follows:

$$E_\rho(\rho) = \frac{1}{2\pi\,\epsilon_0}\,\frac{\bar{q}}{\rho} \quad \text{(in the inside!)}\ .$$

Nabla-operator in cylindrical coordinates:

$$\nabla \equiv \left(\frac{\partial}{\partial\rho},\, \frac{1}{\rho}\,\frac{\partial}{\partial\varphi},\, \frac{\partial}{\partial z}\right)\ .$$

This yields via the potential,

$$\varphi(\mathbf{r}) = \frac{-\bar{q}}{2\pi\,\epsilon_0}\ln\rho + \text{const}\ ,$$

the voltage at the capacitor:

$$U = \varphi(a) - \varphi(b) = \frac{-\bar{q}}{2\pi\,\epsilon_0}\ln\frac{a}{b}\ .$$

This means for the charge per unit length:

$$\bar{q} = \frac{2\pi\,\epsilon_0 U}{\ln\,(b/a)}\ .$$

Therewith the electric field,

$$\mathbf{E}(\mathbf{r}) = \frac{U}{\ln\,(b/a)}\,\frac{1}{\rho}\,\mathbf{e}_\rho\ ,$$

as well as the scalar potential are determined:

$$\varphi(\mathbf{r}) = -\frac{U}{\ln\,(b/a)}\ln\rho + \text{const}\ .$$

Finally we get the capacity per unit length:

$$C = \frac{\bar{q}}{U} = \frac{2\pi\,\epsilon_0}{\ln\,(b/a)}\ .$$

2. Field at the inner cylinder:

$$E_\rho(\rho = a) = \frac{U}{a \ln(b/a)} \,,$$

$$\frac{d E_\rho(a)}{d a} = \frac{-U}{\big(a \ln(b/a)\big)^2} (\ln b - 1 - \ln a) \stackrel{!}{=} 0 \,.$$

Hence, the field becomes extremal at $a_0 = b\, e^{-1}$. Because of

$$\frac{d^2 E_\rho(a)}{d a^2} = \left[\frac{2U}{\big(a \ln(b/a)\big)^3} (\ln b - 1 - \ln a)^2 + \frac{U}{a\big(a \ln(b/a)\big)^2}\right]_{a=a_0}$$

$$= \frac{U}{b\, e^{-1}(b\, e^{-1})^2} = \frac{U\, e^3}{b^3} > 0$$

it is about a minimum.

Solution 2.2.5 The spherical symmetry of the problem entails a spherically-symmetric potential:

$$\Phi(\mathbf{r}) = \Phi(r) \,.$$

1. Laplace equation ($r \neq R_1, R_2$)

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{\vartheta,\varphi} \,,$$

$$\Delta\Phi = 0 \implies \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 0 \quad (r \neq 0)$$

$$\implies r^2 \frac{\partial}{\partial r}\Phi(r) = c$$

$$\implies \frac{\partial}{\partial r}\Phi(r) = \frac{c}{r^2}$$

$$\implies \Phi(\mathbf{r}) = \Phi(r) = \frac{\alpha}{r} + \beta \,.$$

1a. $r < R_1$
Regularity at the origin ($\Phi(0)$ finite)

$$\implies \alpha = 0 \quad \Phi_1(\mathbf{r}) = \beta = \Phi_1 \,.$$

1b. $r > R_2$

$$\Phi(r \longrightarrow \infty) = 0 \implies \beta = 0 ,$$
$$\Phi(r = R_2) = \Phi_2 = \frac{\alpha}{R_2}$$
$$\implies \alpha = R_2 \Phi_2$$
$$\implies \Phi_2(\mathbf{r}) = \Phi_2 \frac{R_2}{r} .$$

1c. $R_1 < r < R_2$
It must be

$$\frac{\alpha}{R_1} + \beta = \Phi_1 ; \quad \frac{\alpha}{R_2} + \beta = \Phi_2$$
$$\implies \alpha \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \Phi_1 - \Phi_2$$
$$\implies \alpha = \frac{(\Phi_1 - \Phi_2) R_1 R_2}{R_2 - R_1} ,$$
$$\beta = \Phi_1 - \frac{(\Phi_1 - \Phi_2) R_2}{R_2 - R_1} = \frac{R_2 \Phi_2 - R_1 \Phi_1}{R_2 - R_1} .$$

$\implies$ total potential

$$\Phi(\mathbf{r}) = \begin{cases} \Phi_1 & \text{if } r < R_1 , \\ \dfrac{(\Phi_1 - \Phi_2) R_1 R_2}{R_2 - R_1} \dfrac{1}{r} + \dfrac{R_2 \Phi_2 - R_1 \Phi_1}{R_2 - R_1} & \text{if } R_1 < r < R_2 , \\ \Phi_2 \dfrac{R_2}{r} & \text{if } r > R_2 . \end{cases}$$

2. Poisson equation:

$$\Delta \Phi = -\frac{1}{\epsilon_0} \rho(\mathbf{r}) ,$$
$$\rho(\mathbf{r}) = \frac{Q_1}{4\pi R_1^2} \delta(r - R_1) + \frac{Q_2}{4\pi R_2^2} \delta(r - R_2) ,$$
$$\implies \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = -\frac{1}{4\pi\epsilon_0} \left(\frac{Q_1}{R_1^2} \delta(r - R_1) + \frac{Q_2}{R_2^2} \delta(r - R_2) \right) r^2$$
$$= -\frac{1}{4\pi\epsilon_0} \left(Q_1 \delta(r - R_1) + Q_2 \delta(r - R_2) \right) .$$

We integrate over the interval

$$[R_1 - \epsilon, R_1 + \epsilon] \, ; \qquad \epsilon \ll R_2 - R_1, R_1$$

$$\Longrightarrow \; r^2 \frac{\partial \Phi}{\partial r}\bigg|_{R_1-\epsilon}^{R_1+\epsilon} = -\frac{Q_1}{4\pi\epsilon_0} \quad (\epsilon \to 0)$$

$$\Longrightarrow \; (R_1+\epsilon)^2 \frac{\partial \Phi}{\partial r}\bigg|_{R_1+\epsilon} - (R_1-\epsilon)^2 \frac{\partial \Phi}{\partial r}\bigg|_{R_1-\epsilon} = R_1^2 \frac{(\Phi_1 - \Phi_2) R_1 R_2}{R_2 - R_1} \left(-\frac{1}{R_1^2}\right) - \left(R_1^2 * 0\right)$$

$$= -\frac{Q_1}{4\pi\epsilon_0}$$

$$\Longrightarrow \; Q_1 = \frac{4\pi\epsilon_0 R_1 R_2 (\Phi_1 - \Phi_2)}{R_2 - R_1} \, .$$

We now integrate over $[R_2 - \epsilon, R_2 + \epsilon]$:

$$r^2 \frac{\partial \Phi}{\partial r}\bigg|_{R_2-\epsilon}^{R_2+\epsilon} = -\frac{Q_2}{4\pi\epsilon_0}$$

$$\Longrightarrow \; (R_2+\epsilon)^2 \frac{-\Phi_2 R_2}{(R_2+\epsilon)^2} - \quad (R_2-\epsilon)^2 \frac{(\Phi_1 - \Phi_2) R_1 R_2}{R_2 - R_1} \left(-\frac{1}{(R_2-\epsilon)^2}\right)$$

$$\stackrel{\epsilon \to 0}{\longrightarrow} \quad -\Phi_2 R_2 + \frac{(\Phi_1 - \Phi_2) R_1 R_2}{R_2 - R_1}$$

$$= \frac{-\Phi_2 R_2^2 + \Phi_1 R_1 R_2}{R_2 - R_1}$$

$$= \frac{R_2 (\Phi_1 R_1 - \Phi_2 R_2)}{R_2 - R_1}$$

$$= -\frac{Q_2}{4\pi\epsilon_0}$$

$$\Longrightarrow \; Q_2 = \frac{4\pi\epsilon_0 R_2 (\Phi_2 R_2 - \Phi_1 R_1)}{R_2 - R_1} \, .$$

Solution 2.2.6

$$Q_0 = C\,U_0 = 10^{-4}\,\frac{\mathrm{A\,s}}{\mathrm{V}}\,10^3\,\mathrm{V} = 10^{-1}\,\mathrm{A\,s} \, ,$$

$$W_0 = \frac{1}{2} C\,U_0^2 = \frac{1}{2} 10^{-4}\,\frac{\mathrm{A\,s}}{\mathrm{V}}\,10^6\,\mathrm{V}^2 = 50\,\mathrm{W\,s} = 50\,\mathrm{J} \, .$$

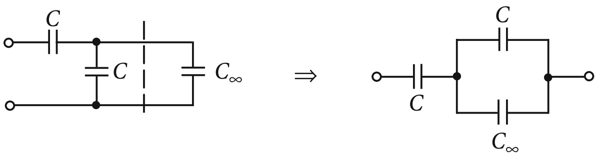


Fig. A.15

Connect in parallel:

$$Q_1 = Q_2 = \frac{1}{2}Q_0 ,$$

$$U_1 = U_2 = \frac{1}{2}U_0 = 500\,\mathrm{V} .$$

From that it follows:

$$W = W_1 + W_2 = \frac{1}{2}C\frac{U_0^2}{4} + \frac{1}{2}C\frac{U_0^2}{4} = \frac{W_0}{4} + \frac{W_0}{4} = \frac{1}{2}W_0 .$$

Paradox: half of the stored energy has *disappeared*! Where?

Solution 2.2.7 Figure A.15 shows the equivalent circuit diagram from which follows:

$$\begin{aligned}
&\frac{1}{C_\infty} = \frac{1}{C} + \frac{1}{C + C_\infty} \\
&\Longrightarrow \; C(C + C_\infty) = C_\infty(2C + C_\infty) \\
&\Longrightarrow \; 0 = C_\infty^2 + C\,C_\infty - C^2 = \left(C_\infty + \frac{1}{2}C\right)^2 - \frac{5}{4}C^2 \\
&\Longrightarrow \; C_\infty = C\left(-\frac{1}{2} + \sqrt{\frac{5}{4}}\right) = 0{,}618\,C .
\end{aligned}$$

We see that the capacity **does not** become infinite!

Solution 2.2.8 Preset charge density:

$$\rho(\mathbf{r}) = \sigma_0 \cos^2\theta\, \delta(r - R) .$$

1. Monopole:

$$q = \int d^3r\rho(\mathbf{r}) = \sigma_0 R^2 2\pi \int_{-1}^{+1} d\cos\theta\cos^2\theta$$
$$= \frac{4\pi}{3} R^2 \sigma_0 \ .$$

2. Dipole:

$$\mathbf{p} = \int d^3r\mathbf{r}\rho(\mathbf{r})$$
$$= \sigma_0 R^3 \int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi\cos^2\theta \ (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$
$$= 2\pi\sigma_0 R^3 \int_{-1}^{+1} d\cos\theta \ (0, 0, \cos^3\theta)$$
$$\rightarrow \ \mathbf{p} = 0 \,.$$

3. Quadrupole:

$$Q_{xx} = \int d^3r\rho(\mathbf{r})(3x^2 - r^2)$$
$$= \sigma_0 R^4 \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta\cos^2\theta \,(3\sin^2\theta\cos^2\varphi - 1) \ .$$

We use

$$\int_0^{2\pi} d\varphi\cos^2\varphi = \int_0^{2\pi} d\varphi\sin^2\varphi = \pi \ .$$

(Show it with integration by parts!). It follows then:

$$Q_{xx} = \pi\sigma_0 R^4 \int_{-1}^{+1} d\cos\theta \left(3\left(\cos^2\theta - \cos^4\theta\right) - 2\cos^2\theta\right)$$
$$= \pi\sigma_0 R^4 \left(\frac{2}{3} - \frac{6}{5}\right)$$

$$
\begin{aligned}
&= -\frac{8}{15}\pi\sigma_0 R^4 \\
Q_{yy} &= \int d^3r\rho(\mathbf{r})\left(3y^2 - r^2\right) \\
&= \sigma_0 R^4 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\theta\cos^2\theta\,\left(3\sin^2\theta\sin^2\varphi - 1\right) \\
&= Q_{xx} \\
Q_{zz} &= \int d^3r\rho(\mathbf{r})\left(3z^2 - r^2\right) \\
&= \sigma_0 R^4 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\theta\cos^2\theta\,\left(3\cos^2\theta - 1\right) \\
&= 2\pi\sigma_0 R^4\left(\frac{6}{5} - \frac{2}{3}\right) \\
&= \frac{16}{15}\pi\sigma_0 R^4 \quad \to \text{tracelessness of the tensor} \\
Q_{xy} &= \int d^3r\rho(\mathbf{r})(3xy) \\
&= 3\sigma_0 R^4 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\theta\sin^2\theta\cos\varphi\sin\varphi \\
&= 0 = Q_{yx}\ .
\end{aligned}
$$

The φ-integration makes the term to vanish. The same holds also for the following elements:

$$
\begin{aligned}
Q_{xz} &= \int d^3r\rho(\mathbf{r})(3xz) \\
&= 3\sigma_0 R^4 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\theta\sin\theta\cos\varphi\cos\theta \\
&= 0 = Q_{zx} \\
Q_{yz} &= \int d^3r\rho(\mathbf{r})(3yz) \\
&= 3\sigma_0 R^4 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\theta\sin\theta\sin\varphi\cos\theta \\
&= 0 = Q_{yz}\ .
\end{aligned}
$$

Quadrupole tensor:

$$\underline{\mathbf{Q}} = \frac{8}{15}\pi\sigma_0 R^4 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

4. Scalar potential and electric field:
We take for the scalar potential the expansion (2.94) getting then with the partial results 1. to 3.:

$$\begin{aligned} 4\pi\epsilon_0\varphi(\mathbf{r}) &= \frac{4\pi}{3}R^2\sigma_0\frac{1}{r} + \frac{1}{2r^5}\frac{8}{15}\pi\sigma_0 R^4(-x^2 - y^2 + 2z^2) + \cdots \\ &= \frac{4\pi}{3}R^2\sigma_0\left(\frac{1}{r} + \frac{1}{5}R^2\frac{1}{r^3}\left(2\cos^2\theta - \sin^2\theta\right) + \cdots\right) . \end{aligned}$$

This means

$$\varphi(\mathbf{r}) = \frac{\sigma_0 R^2}{3\epsilon_0}\left(\frac{1}{r} + \frac{1}{r^3}\frac{R^2}{5}\left(3\cos^2\theta - 1\right) + \cdots\right) .$$

From the scalar potential we derive with

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

the electric field:

$$\begin{aligned} E_r &= \frac{\sigma_0 R^2}{3\epsilon_0}\left(\frac{1}{r^2} + \frac{3}{r^4}\frac{R^2}{5}(3\cos^2\theta - 1) + \cdots\right) \\ E_\theta &= \frac{\sigma_0 R^2}{3\epsilon_0}\left(\frac{R^2}{5r^4}(-6\cos\theta\sin\theta) + \cdots\right) \\ E_\varphi &= 0 . \end{aligned}$$

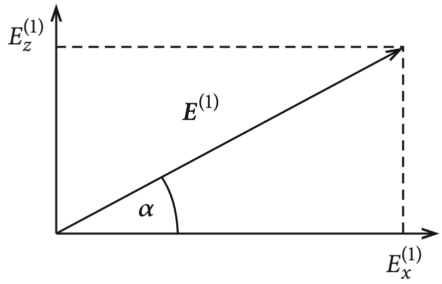
It is left therewith as final result:

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma_0 R^2}{3\epsilon_0}\left(\left(\frac{1}{r^2} + \frac{3}{r^4}\frac{R^2}{5}(3\cos^2\theta - 1) + \cdots\right)\mathbf{e}_r - \left(\frac{3R^2}{5r^4}\sin 2\theta + \cdots\right)\mathbf{e}_\theta\right) .$$

Solution 2.2.9 The dipole $\mathbf{p}_1 = p_1\mathbf{e}_z$ produces the potential

$$\varphi_1(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}_1}{r^3} = \frac{1}{4\pi\,\epsilon_0}\frac{p_1 z}{r^3}$$

Fig. A.16



and therewith the electric field:

$$E_x^{(1)} = -\frac{\partial \varphi_1}{\partial x} = \frac{3p_1}{4\pi\,\epsilon_0}\,\frac{xz}{r^5}\;,$$

$$E_y^{(1)} = -\frac{\partial \varphi_1}{\partial y} = \frac{3p_1}{4\pi\,\epsilon_0}\,\frac{yz}{r^5}\;,$$

$$E_z^{(1)} = -\frac{\partial \varphi_1}{\partial z} = \frac{p_1}{4\pi\,\epsilon_0}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\;.$$

The potential energy of the dipole $\mathbf{p}_2$ in the field of the other dipole $\mathbf{p}_1$ is then to be calculated from

$$V_{\mathrm{D}}^{(2)} = -\mathbf{p}_2 \cdot \mathbf{E}^{(1)}\;.$$

The potential energy becomes minimal for the direction we are looking for, i.e. $\mathbf{p}_2$ orients itself parallelly to $\mathbf{E}^{(1)}$ (Fig. A.16):

$$E_x^{(1)}(x_0, 0, z_0) = \frac{3p_1}{4\pi\,\epsilon_0}\,\frac{x_0 z_0}{r_0^5}\;,$$

$$E_y^{(1)}(x_0, 0, z_0) = 0\;,$$

$$E_z^{(1)}(x_0, 0, z_0) = \frac{p_1}{4\pi\,\epsilon_0}\,\frac{1}{r_0^5}\left(2z_0^2 - x_0^2\right)\;.$$

$$\tan\alpha = \frac{E_z^{(1)}(x_0, 0, z_0)}{E_x^{(1)}(x_0, 0, z_0)} = \frac{2z_0^2 - x_0^2}{3x_0 z_0}\;.$$

Solution 2.2.10 Charge density:

$$\rho(\mathbf{r}) = q\{\delta(x)\,\delta(z)[\delta(y-d) + \delta(y+d)] + \delta(x)\,\delta(y)[\delta(z-d) + \delta(z+d)]$$
$$-\delta(y)\,\delta(z)[\delta(x+d) + \delta(x+d/2) + \delta(x-d) + \delta(x-2d)]\}\;.$$

Dipole moment:

$$\mathbf{p} = \int d^3r\, \rho(\mathbf{r}) \cdot \mathbf{r} = q \begin{pmatrix} +d + d/2 - d - 2d \\ d - d \\ d - d \end{pmatrix}$$

$$\Longrightarrow \mathbf{p} = -q\, d \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix} .$$

Quadrupole tensor:

$$\begin{aligned}
Q_{ij} &= \int d^3r\, \rho(\mathbf{r}) \left(3x_i x_j - r^2 \delta_{ij}\right) , \\
Q_{ij} &= 0 \quad \forall i \neq j , \\
Q_{xx} &= \int d^3r\, \rho(\mathbf{r}) \left(2x^2 - y^2 - z^2\right) \\
&= q \left(-d^2 - d^2 - d^2 - d^2 - 2d^2 - 2d^2/4 - 2d^2 - 8d^2\right) \\
&= -q\, d^2 \left(16 + \frac{1}{2}\right) = -\frac{33}{2} q\, d^2 , \\
Q_{yy} &= \int d^3r\, \rho(\mathbf{r}) \left(2y^2 - x^2 - z^2\right) \\
&= q \left(2d^2 + 2d^2 - d^2 - d^2 + d^2 + d^2/4 + d^2 + 4d^2\right) \\
&= q\, d^2 \left(8 + \frac{1}{4}\right) = \frac{33}{4} q\, d^2 = -\frac{1}{2} Q_{xx} , \\
Q_{zz} &= \int d^3r\, \rho(\mathbf{r}) \left(2z^2 - x^2 - y^2\right) \\
&= q \left(-d^2 - d^2 + 2d^2 + 2d^2 + d^2 + d^2/4 + d^2 + 4d^2\right) \\
&= q\, d^2 \left(8 + \frac{1}{4}\right) = Q_{yy} .
\end{aligned}$$

It follows:

$$Q_{zz} = Q_{yy} = -\frac{1}{2} Q_{xx} = \frac{33}{4} q d^2 ,$$

The trace is zero as it must be!

Solution 2.2.11

1. Spherical coordinates: r, ϑ, φ,
   Axial symmetry: $\rho(\mathbf{r}) = \rho(r, \vartheta)$; $\partial\rho/\partial\varphi = 0$.

$$x = r \sin\vartheta \cos\varphi \; ,$$
$$y = r \sin\vartheta \sin\varphi \; ,$$
$$z = r \cos\vartheta \; .$$

$$Q_{xy} = \int d^3r\, \rho(\mathbf{r})(3xy) = 3\int\limits_0^\infty dr\, r^4 \int\limits_{-1}^{+1} d\cos\vartheta \; \sin^2\vartheta\; \rho(r,\vartheta) \underbrace{\int\limits_0^{2\pi} d\varphi \; \cos\varphi \; \sin\varphi}_{\frac{1}{2}\sin^2\varphi\big|_0^{2\pi}=0}$$
$$= 0 = Q_{yx} \; ,$$
$$Q_{xz} = 3\int\limits_0^\infty dr\, r^4 \int\limits_{-1}^{+1} d\cos\vartheta \; \sin\vartheta \; \cos\vartheta\; \rho(r,\vartheta) \underbrace{\int\limits_0^{2\pi} d\varphi \; \cos\varphi}_{=0}$$
$$= 0 = Q_{zx} \; ,$$
$$Q_{yz} = 3\int\limits_0^\infty dr\, r^4 \int\limits_{-1}^{+1} d\cos\vartheta \; \sin\vartheta \; \cos\vartheta\; \rho(r,\vartheta) \underbrace{\int\limits_0^{2\pi} d\varphi \; \sin\varphi}_{=0}$$
$$= 0 = Q_{zy} \; .$$

2.

$$Q_{xx} = \int d^3r\, \rho(\mathbf{r})(3x^2 - r^2) = \int d^3r\, \rho(\mathbf{r})(2x^2 - y^2 - z^2) \; ,$$
$$Q_{yy} = \int d^3r\, \rho(\mathbf{r})(2y^2 - x^2 - z^2) \; .$$

This can be combined:

$$Q_{xx} - Q_{yy} = 3\int d^3r\, \rho(\mathbf{r})(x^2 - y^2)$$
$$= 3\int\limits_0^\infty dr\, r^4 \int\limits_{-1}^{+1} d\cos\vartheta \; \sin^2\vartheta\; \rho(r,\vartheta) \int\limits_0^{2\pi} d\varphi \underbrace{(\cos^2\varphi - \sin^2\varphi)}_{\cos 2\varphi} \; ,$$

$$\frac{1}{2}\sin 2\varphi\bigg|_0^{2\pi} = 0 \; .$$

Thus we have found:

$$Q_{xx} = Q_{yy} \; .$$

It further follows from the tracelessness:

$$Q_{zz} = Q_0 = -(Q_{xx} + Q_{yy}) \; ,$$
$$Q_{xx} = Q_{yy} = -\frac{1}{2} Q_0 \; .$$

3.

$$4\pi\,\epsilon_0\,\varphi_Q\,(\mathbf{r}) = \frac{1}{2r^5} \sum_{i,j} Q_{ij} x_i x_j = \frac{Q_0}{2r^5} \left(z^2 - \frac{1}{2} x^2 - \frac{1}{2} y^2 \right)$$
$$= \frac{Q_0}{2r^3} \left(\cos^2\vartheta - \frac{1}{2} \sin^2\vartheta \right) = -\frac{Q_0}{4r^3} \left(1 - 3\cos^2\vartheta \right) \; .$$

This yields:

$$\varphi_Q(\mathbf{r}) = -\frac{Q_0}{16\pi\,\epsilon_0} \, \frac{1 - 3\,\cos^2\vartheta}{r^3} \; .$$

With the nabla-operator in spherical coordinates,

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \, \frac{1}{r} \frac{\partial}{\partial \vartheta}, \, \frac{1}{r\sin\vartheta} \frac{\partial}{\partial \varphi} \right) \; ,$$

one calculates:

$$\frac{\partial}{\partial r} \, \frac{1 - 3\,\cos^2\vartheta}{r^3} = -3\,\frac{1 - 3\,\cos^2\vartheta}{r^4} \; ,$$
$$\frac{1}{r} \frac{\partial}{\partial \vartheta} \, \frac{1 - 3\,\cos^2\vartheta}{r^3} = +\frac{3}{r^4}\, 2\,\cos\vartheta\,\sin\vartheta = \frac{3\,\sin 2\vartheta}{r^4} \; .$$

It results the electric field strength:

$$\mathbf{E}_Q(\mathbf{r}) = -\nabla\varphi_Q(\mathbf{r}) = -\frac{3\,Q_0}{16\pi\,\epsilon_0} \, \frac{1}{r^4} \left[(1 - 3\,\cos^2\vartheta)\,\mathbf{e}_r - \sin 2\vartheta\,\mathbf{e}_\vartheta \right] \; .$$

Solution 2.2.12 In a given system of coordinates the quadrupole moments Q_{ij} (2.93) are uniquely determined by the charge density ρ. A rotation of the system of coordinates ($\{x_i\} \to \{\hat{x}_i\}$) must not change the scalar quadrupole potential. According to (2.98) that means that the expression

$$\sum_{i,j} Q_{ij}\, x_i x_j = \sum_{i,j} \hat{Q}_{ij}\, \hat{x}_i \hat{x}_j$$

must be invariant. However, with a rotation of the system of coordinates the (Cartesian) components of the position vector will change ((1.310), Vol. 1):

$$\hat{x}_i = \sum_j d_{ij} x_j$$

The d_{ij} are the elements of the rotation matrix ((1.307), Vol. 1). But in order to keep the above double sum invariant, obviously the quadrupole components have to be co-transformed in a proper manner. If the 3×3-matrix $\mathbf{Q}$ is indeed a tensor of the second rank then each row and each column transforms with a rotation of the system of coordinates like the position vector:

$$\hat{Q}_{ij} = \sum_{l,m} d_{il} d_{jm}\, Q_{lm}$$

One can show therewith:

$$\begin{aligned}
\sum_{i,j} \hat{Q}_{ij}\, \hat{x}_i \hat{x}_j &= \sum_{i,j} \sum_{l,m} \sum_{s,t} d_{il} d_{jm} Q_{lm} d_{is} d_{jt} x_s x_t \\
&= \sum_{l,m} \sum_{s,t} \delta_{ls} \delta_{mt}\, Q_{lm} x_s x_t \\
&= \sum_{l,m} Q_{lm}\, x_l x_m
\end{aligned}$$

In the second line we have exploited the orthonormality relation ((1.316), Vol. 1) for the rows and the columns of the rotation matrix.

Section 2.3.9

Solution 2.3.1 We take over the preliminary studies in connection with the example of use *'point charge over a grounded metallic sphere'* of the method of image charges in Sect. 2.3.4, in particular the result (2.136) for the surface charge density σ. We use the notations from the Figs. 2.44 and 2.46. Because of the rotational symmetry of the surface charge density around the $\mathbf{e}_{r'}$-direction it is reasonable to choose this direction as polar axis. Then it remains to be calculated:

$$\mathbf{F} = -\mathbf{e}_{r'} \frac{1}{2\varepsilon_0} \int\limits_{\text{sphere}} df\, \sigma^2 \cos\vartheta = F\, \mathbf{e}_{r'} \ .$$

With (2.136) it is therefore to be evaluated:

$$F = -\frac{1}{2\varepsilon_0}R^2 \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\vartheta \, \frac{q^2}{16\pi^2 R^4} \, \frac{R^2}{r'^2} \, \frac{\left(1-\frac{R^2}{r'^2}\right)^2 \cos\vartheta}{\left(1+\frac{R^2}{r'^2}-2\frac{R}{r'}\cos\vartheta\right)^3}$$

$$= -\frac{q^2}{16\pi\varepsilon_0}\,\frac{1}{r'^2}\left(1-\frac{R^2}{r'^2}\right)^2 \cdot A$$

$$A \equiv \int\limits_{-1}^{+1} d\cos\vartheta \, \frac{\cos\vartheta}{\left(1+\frac{R^2}{r'^2}-2\frac{R}{r'}\cos\vartheta\right)^3} \, .$$

We take

$$c = -2\frac{R}{r'}\,; \quad d = 1+\frac{R^2}{r'^2}\,; \quad y = c\cos\vartheta + d \;\curvearrowright\; d\cos\vartheta = \frac{1}{c}dy$$

having then to evaluate:

$$A = \frac{1}{c}\int\limits_{d-c}^{d+c} dy \, \frac{\frac{1}{c}(y-d)}{y^3} = \frac{1}{c^2}\int\limits_{d-c}^{d+c} dy \frac{1}{y^2} - \frac{d}{c^2}\int\limits_{d-c}^{d+c} dy \frac{1}{y^3}$$

$$= \frac{1}{c^2}\left(-\frac{1}{y}+\frac{d}{2}\frac{1}{y^2}\right)_{d-c}^{d+c}$$

$$= \frac{1}{c^2}\left(-\frac{1}{d+c}+\frac{1}{d-c}+\frac{d}{2}\left(\frac{1}{(d+c)^2}-\frac{1}{(d-c)^2}\right)\right)$$

$$= \frac{1}{c^2}\left(\frac{d}{2}\frac{-4dc}{(d^2-c^2)^2}+\frac{2c}{d^2-c^2}\right) = \frac{-2c}{(d^2-c^2)^2} = \frac{-2c}{(d+c)^2(d-c)^2}$$

$$= 4\frac{R}{r'}\frac{1}{\left(1-\frac{R}{r'}\right)^4\left(1+\frac{R}{r'}\right)^4} = 4\frac{R}{r'}\frac{1}{\left(1-\frac{R^2}{r'^2}\right)^4} \, .$$

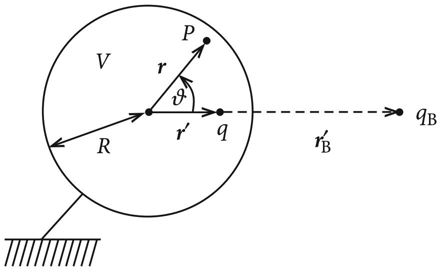
Insertion into the expression for F leads finally to:

$$F = -\frac{q^2\frac{R}{r'}}{4\pi\varepsilon_0}\,\frac{1}{r'^2}\,\frac{1}{\left(1-\frac{R^2}{r'^2}\right)^2} = \frac{1}{4\pi\varepsilon_0}\,\frac{q\left(-q\frac{R}{r'}\right)}{\left(r'-\frac{R^2}{r'}\right)^2} \stackrel{(2.131)}{=} \frac{1}{4\pi\varepsilon_0}\,\frac{q\cdot q_\mathrm{B}}{\left|\mathbf{r}'-\mathbf{r}'_\mathrm{B}\right|^2} \, .$$

That matches with (2.138)!

Solution 2.3.2 The *'interesting'* space region here is V: interior space of the hollow sphere,
boundary condition: $\varphi \equiv 0$ on $S(V)$ (Dirichlet).

Fig. A.17



The Poisson equation for $\mathbf{r} \in V$:

$$\Delta_r \varphi(\mathbf{r}) = -\frac{q}{\epsilon_0}\,\delta(\mathbf{r} - \mathbf{r}')$$

is solved by

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\,\epsilon_0}\,\frac{1}{|\mathbf{r} - \mathbf{r}'|} + f(\mathbf{r}, \mathbf{r}')$$

with $\Delta_r f(\mathbf{r}, \mathbf{r}') = 0$ in V.
Let $f(\mathbf{r}, \mathbf{r}')$ be the potential of an *image charge* located outside of V by which we can simulate the boundary conditions. From symmetry reasons it is to expect (Fig. A.17):

$$\text{image charge} = \text{point charge } q_B$$
$$\mathbf{r}_{B'} \uparrow\uparrow \mathbf{r}' \;\; (r_{B'} > R).$$

The ansatz

$$4\pi\,\epsilon_0 \varphi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}'|} + \frac{q_B}{|\mathbf{r} - \mathbf{r}_{B'}|}$$

fulfills in V the Poisson equation:

$$4\pi\,\epsilon_0 \varphi(\mathbf{r}) = \frac{q/r}{|\mathbf{e}_r - (r'/r)\mathbf{e}_{r'}|} + \frac{q_B/r_{B'}}{|(r/r_{B'})\,\mathbf{e}_r - \mathbf{e}_{r'}|}\ .$$

The boundary condition

$$\varphi(r = R) \overset{!}{=} 0$$

is satisfied if it holds:

$$0 = \frac{q}{R}\left(1 + \frac{r'^2}{R^2} - 2\frac{r'}{R}\,\mathbf{e}_r \cdot \mathbf{e}_{r'}\right)^{-1/2} + \frac{q_{\mathrm{B}}}{r_{\mathrm{B}'}}\left(\frac{R^2}{r_{\mathrm{B}}'^2} + 1 - 2\frac{R}{r_{\mathrm{B}'}}\,\mathbf{e}_r \cdot \mathbf{e}_{r'}\right)^{-1/2} .$$

This equation is solved by:

$$\begin{aligned}
\frac{q_{\mathrm{B}}}{r_{\mathrm{B}'}} &= -\frac{q}{R}\ ; \quad \frac{R}{r_{\mathrm{B}'}} = \frac{r'}{R} \\
\Longrightarrow\ r_{\mathrm{B}'} &= \frac{R^2}{r'} > R\ ; \quad q_{\mathrm{B}} = -q\frac{R}{r'} \\
\Longrightarrow\ \varphi(\mathbf{r}) &= \frac{q}{4\pi\,\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}\right) .
\end{aligned}$$

The solution fulfills in V the Poisson equation and on $S(V)$ Dirichlet-boundary conditions being therewith a unique solution.
We calculate the surface charge density:

$$\sigma = \epsilon_0\,\mathbf{n}\cdot(\underbrace{\mathbf{E}_a}_{=0} - \mathbf{E}_i) = \epsilon_0\mathbf{n}\cdot\nabla\varphi_i = \epsilon_0\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} ,$$

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{\partial}{\partial r}\left(r^2 + r'^2 - 2rr'\cos\vartheta\right)^{-1/2} = -\frac{r - r'\cos\vartheta}{|\mathbf{r}-\mathbf{r}'|^3} \\
\Longrightarrow\ \frac{\partial}{\partial r}\left.\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right|_{r=R} &= -\frac{R - r'\cos\vartheta}{(R^2 + r'^2 - 2Rr'\cos\vartheta)^{3/2}} , \\
\frac{\partial}{\partial r}\frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|} &= -\frac{r - (R^2/r')\cos\vartheta}{\left(r^2 + R^4/r'^2 - 2r(R^2/r')\cos\vartheta\right)^{3/2}}\,\frac{R}{r'} , \\
\frac{\partial}{\partial r}\left.\frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}\right|_{r=R} &= -\frac{R}{r'}\,\frac{R - (R^2/r')\cos\vartheta}{(R^3/r'^3)\left(r'^2 + R^2 - 2R\,r'\cos\vartheta\right)^{3/2}} \\
&= -\frac{r'^2/R - r'\cos\vartheta}{\left(r'^2 + R^2 - 2R\,r'\cos\vartheta\right)^{3/2}} .
\end{aligned}$$

Hence it follows:

$$\begin{aligned}
\sigma &= \frac{q}{4\pi}\,\frac{-R + r'\cos\vartheta + r'^2/R - r'\cos\vartheta}{r'^3\left(1 + R^2/r'^2 - 2(R/r')\cos\vartheta\right)^{3/2}} , \\
\sigma &= \frac{q}{4\pi\,R^2}\left(\frac{R}{r'}\right)\frac{1 - R^2/r'^2}{\left(1 + R^2/r'^2 - 2(R/r')\cos\vartheta\right)^{3/2}} \quad \text{(cf. (2.135))} .
\end{aligned}$$

We get the total induced charge by integration over the surface of the sphere:

$$\bar{q} = \frac{q}{2}\left(\frac{R}{r'}\right)\left(1 - \frac{R^2}{r'^2}\right)\underbrace{\int_{-1}^{+1} d\cos\vartheta \, \frac{d}{d\,\cos\vartheta}\frac{1}{\left(1 + R^2/r'^2 - 2(R/r')\cos\vartheta\right)^{1/2}}\left(\frac{r'}{R}\right)}_{\frac{r'}{R}\left(\frac{1}{|1-R/r'|}-\frac{1}{1+R/r'}\right)} .$$

This yields finally:

$$\bar{q} = -q$$

(cf. 2.137).

Solution 2.3.3

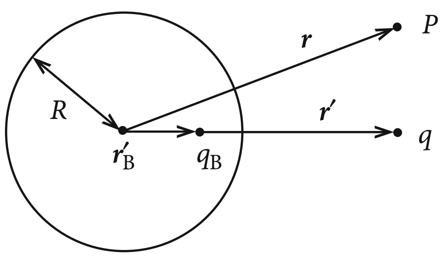
Without point charge:
Q disperses homogeneously over the metal surface. The action seen from outwards is so as if the total charge Q were concentrated in the center of the sphere (Fig. A.18).
With point charge:
q_B is needed as surface charge for the compliance of the boundary conditions. The rest $Q - q_B$ disperses homogeneously over the surface. We can therefore start with:

$$\varphi(\mathbf{r}) = \varphi_1(\mathbf{r}) + \varphi_2(\mathbf{r}) \, ,$$

$\varphi_1(\mathbf{r})$: as for the grounded metallic sphere (2.132),
$\varphi_2(\mathbf{r})$: potential of the point charge $Q - q_B$,

$$Q - q_B = Q + q\frac{R}{r'} \, ,$$

Fig. A.18



located in the center of the sphere:

$$4\pi\,\epsilon_0\varphi_1(\mathbf{r}) = q\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}\right) ,$$

$$4\pi\,\epsilon_0\varphi_2(\mathbf{r}) = \left(Q + q\frac{R}{r'}\right)\frac{1}{r} .$$

Force on the point charge:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 .$$

It follows with Eq. (2.138):

$$\mathbf{F}_1 = \mathbf{e}_{r'}\frac{-q^2 R/r'}{4\pi\,\epsilon_0\,(r' - R^2/r')^2} ,$$

$\mathbf{F}_1$ is always attractive!

$$\mathbf{F}_2 = \mathbf{e}_{r'}\frac{q\,(Q + q(R/r'))}{4\pi\,\epsilon_0 r'^2} .$$

When q and Q are of the same sign then

(a) large distances $\Longrightarrow$ repulsion; $\mathbf{F}_2$ dominates,
(b) $r' \overset{>}{\to} R \;\Longrightarrow$ attraction; F_1 dominates.

This result explains why the charges of the metal sphere do not leave the sphere in spite of their mutual electrostatic repulsion (*work function*). Energy must be provided independently of whether the charges Q and q are of the same or opposite sign (Fig. A.19).

Solution 2.3.4

1. Green's function: Solution of the Poisson equation for a point charge $q = 1$:

$$\Delta\, G = -\frac{1}{\epsilon_0}\delta(\mathbf{r}) ,$$

$$G(\mathbf{r}) = G(\rho,\varphi) \underset{\substack{\uparrow \\ \text{no boundary conditions}}}{=} G(\rho) .$$

Fig. A.19

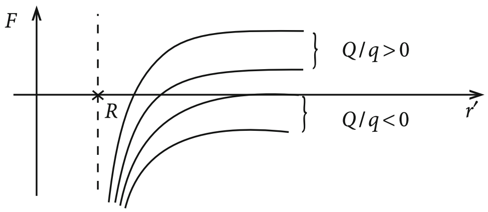
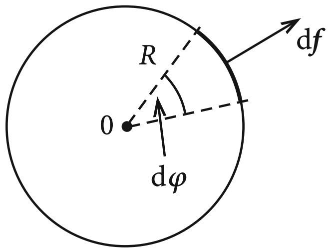


Fig. A.20



$\rho \neq 0$: Laplace equation:

$$0 = \Delta G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) \iff \rho\frac{\partial G}{\partial\rho} = C_1$$
$$\iff \frac{\partial G}{\partial\rho} = \frac{C_1}{\rho}$$
$$\iff G(\rho) = C_1 \ln C_2\rho \,.$$

Two-dimensional Gauss theorem to fix the constant C_1:

F_R: circular area around the origin, R: radius, $d\mathbf{f} = R\,d\varphi\,\mathbf{e}_\rho$: area element (Fig. A.20).

$$\int_{F_R} d\tau \operatorname{div}(\nabla G) = \int_{\partial F_R} d\mathbf{f}\cdot\nabla G \,,$$

$\nabla \equiv \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)$; $\nabla G = \frac{C_1}{\rho}\,\mathbf{e}_\rho$,

$$\int_{F_R} d\tau \operatorname{div}(\nabla G) = -\frac{1}{\epsilon_0}\int_{F_R} d\tau\,\delta(\mathbf{r}) = -\frac{1}{\epsilon_0} \,,$$

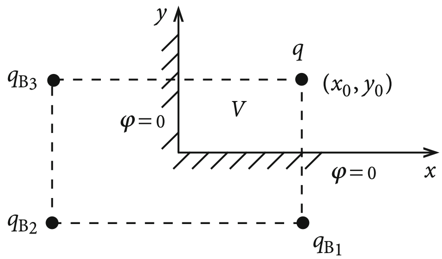
$$\int_{\partial F_R} d\mathbf{f}\cdot\nabla G = R\frac{C_1}{R}\int_0^{2\pi} d\varphi\;\mathbf{e}_\rho\cdot\mathbf{e}_\rho = 2\pi\,C_1$$

$$\Longrightarrow\; C_1 = -\frac{1}{2\pi\,\epsilon_0} \;\Longrightarrow\; G(\rho) = -\frac{1}{2\pi\,\epsilon_0}\ln C_2\rho = -\frac{1}{2\pi\,\epsilon_0}\ln C_2\sqrt{x^2+y^2} \,.$$

2. *'Interesting'* space region (Fig. A.21):

$$V = \{\mathbf{r} = (x, y)\;;\; x \geq 0,\, y \geq 0\} \,.$$

Fig. A.21



Boundary conditions are to be realized by image charges **outside** V!
q_{B_1} compensates q on $y = 0$:

$$\mathbf{r}_{B_1} = (x_0, -y_0) \; ; \quad q_{B_1} = -q \; ,$$

q_{B_3} compensates q on $x = 0$:

$$\mathbf{r}_{B_3} = (-x_0, y_0) \; ; \quad q_{B_3} = -q \; ,$$

q_{B_2} compensates q_{B_1} on $x = 0$ and q_{B_3} on $y = 0$:

$$\mathbf{r}_{B_2} = (-x_0, -y_0) \; ; \quad q_{B_2} = q \; .$$

That leads to:

$$\varphi(x,y) = -\frac{q}{2\pi\epsilon_0}\Big[\ln(C_2\sqrt{(x-x_0)^2+(y-y_0)^2}) - \ln(C_2\sqrt{(x-x_0)^2+(y+y_0)^2})$$
$$+ \ln(C_2\sqrt{(x+x_0)^2+(y+y_0)^2}) - \ln(C_2\sqrt{(x+x_0)^2+(y-y_0)^2})\Big]$$
$$\Longrightarrow \qquad \varphi(x,y) = -\frac{q}{4\pi\epsilon_0}\ln\frac{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}{[(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]} \; .$$

One should check:

(1) φ solves in V the Poisson equation $\Delta\varphi(x,y) = -(q/\epsilon_0)\,\delta(x-x_0)\,\delta(y-y_0)$,
(2) $\varphi(x=0,y) = \varphi(x,y=0) = 0$.

Solution 2.3.5 Plane polar coordinates ρ, φ are useful.
Laplace operator: $\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$. Separation ansatz:

$$\Phi(\rho,\varphi) = \Pi(\rho)\Theta(\varphi) \; ,$$

G is free of charge $\Longrightarrow$ Laplace equation: $\Delta\Phi = 0$:

$$0 = \Theta(\vartheta)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Pi}{\partial\rho}\right) + \frac{\Pi(\rho)}{\rho^2}\frac{\partial^2\Theta}{\partial\varphi^2} \; .$$

Multiply the equation with ρ^2/Φ:

$$0 = \frac{\rho}{\Pi(\rho)}\,\frac{d}{d\rho}\left(\rho\frac{d\Pi}{d\rho}\right) + \frac{1}{\Theta(\varphi)}\,\frac{d^2\Theta}{d\varphi^2} \implies \frac{\rho}{\Pi}\,\frac{d}{d\rho}\left(\rho\frac{d\Pi}{d\rho}\right) = \nu^2\;,$$

$$\frac{1}{\Theta}\,\frac{d^2\Theta}{d\varphi^2} = -\nu^2\;.$$

The case $\nu = 0$ can be excluded. We therefore can assume $\nu > 0$:

$$\begin{aligned}\Pi_\nu &= a_\nu\rho^\nu + b_\nu\rho^{-\nu}\;,\\ \Theta_\nu &= \bar{a}_\nu\sin(\nu\varphi) + \bar{b}_\nu\cos(\nu\varphi)\;.\end{aligned}$$

Boundary conditions:

$$\begin{aligned}\Phi(\rho,\varphi=0) = 0 &\implies \bar{b}_\nu = 0\;,\\ \Phi(\rho,\varphi=\alpha) = 0 &\implies \nu = \frac{n\pi}{\alpha}\;;\quad n\in\mathbb{N}\;.\\ \Phi \text{ regular at } \rho = 0 &\implies b_\nu = 0\;.\end{aligned}$$

This leads to the general solution:

$$\Phi(\rho,\varphi) = \sum_{n=1}^{\infty} c_n\rho^{n\pi/\alpha}\sin\left(\frac{n\pi}{\alpha}\varphi\right)\;.$$

With the orthogonality relation

$$\frac{2}{\alpha}\int\limits_0^\alpha d\varphi\,\sin\left(\frac{n\pi}{\alpha}\varphi\right)\sin\left(\frac{m\pi}{\alpha}\varphi\right) = \delta_{nm}$$

it follows from the last boundary condition:

$$\frac{2}{\alpha}\int\limits_0^\alpha d\varphi\,\Phi_0(\varphi)\sin\left(\frac{m\pi}{\alpha}\varphi\right) = \sum_{n=1}^{\infty} c_n R^{n\pi/\alpha}\delta_{nm} = c_m R^{m\pi/\alpha}$$

$$\implies c_n = R^{-(n\pi/\alpha)}\,\frac{2}{\alpha}\int\limits_0^\alpha d\varphi\,\Phi_0(\varphi)\sin\frac{n\pi}{\alpha}\varphi\;.$$

Solution 2.3.6 Except for the surface of the sphere the space is free of charge:

$$\Delta\varphi = 0\;.$$

The boundary conditions have azimuthal symmetry and therefore also the potential $\varphi(r,\vartheta,\varphi)=\varphi(r,\vartheta)$. General solution (see (2.165)):

$$\varphi(r,\vartheta)=\sum_{l=0}^{\infty}(2l+1)\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\vartheta)\ ,$$

φ_{i}: potential inside the sphere, φ_{a}: potential outside the sphere.
Regularity at the origin:

$$B_l^{(\mathrm{i})}=0$$
$$\Longrightarrow\ \varphi_{\mathrm{i}}(r,\vartheta)=\sum_{l=0}^{\infty}A_l^{(\mathrm{i})}(2l+1)\,r^l P_l(\cos\vartheta)\ .$$

Vanishing at infinity:

$$A_l^{(\mathrm{a})}=0$$
$$\Longrightarrow\ \varphi_{\mathrm{a}}(r,\vartheta)=\sum_{l=0}^{\infty}(2l+1)B_l^{(\mathrm{a})}r^{-(l+1)}P_l(\cos\vartheta)\ .$$

Continuity at $r=R$:

$$\varphi_{\mathrm{i}}(R,\vartheta)=\varphi_{\mathrm{a}}(R,\vartheta)$$
$$\Longrightarrow\ B_l^{(\mathrm{a})}=A_l^{(\mathrm{i})}R^{2l+1}\ .$$

Surface charge density:

$$\begin{aligned}\sigma(\vartheta)&=-\epsilon_0\left(\frac{\partial\varphi_{\mathrm{a}}}{\partial r}-\frac{\partial\varphi_{\mathrm{i}}}{\partial r}\right)_{r=R}\\&=-\epsilon_0\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\big[-(l+1)B_l^{(\mathrm{a})}R^{-l-2}-l\,A_l^{(\mathrm{i})}R^{l-1}\big]\end{aligned}$$

$$\begin{aligned}\Longrightarrow\ \sigma(\vartheta)&=\epsilon_0\sum_{l=0}^{\infty}(2l+1)^2A_l^{(\mathrm{i})}R^{l-1}P_l(\cos\vartheta)\\&\overset{!}{=}\sigma_0(3\cos^2\vartheta-1)=2\sigma_0P_2(\cos\vartheta)\ ,\end{aligned}$$

$$\begin{aligned}\int_{-1}^{+1}d\cos\vartheta\ \sigma(\vartheta)P_m(\cos\vartheta)&=2\sigma_0\int_{-1}^{+1}d\cos\vartheta\ \ P_2(\cos\vartheta)P_m(\cos\vartheta)\\&=2\sigma_0\frac{2}{2m+1}\delta_{m2}=\frac{4}{5}\sigma_0\delta_{m2}\ ,\end{aligned}$$

$$\int_{-1}^{+1} d\cos\vartheta\; \sigma(\vartheta) P_m(\cos\vartheta) = \epsilon_0 \sum_{l=0}^{\infty} (2l+1)^2 A_l^{(\mathrm{i})} R^{l-1} \underbrace{\int_{-1}^{+1} d\cos\vartheta\; P_m(\cos\vartheta) P_l(\cos\vartheta)}_{\frac{2}{2m+1}\delta_{lm}}$$

$$= 2\epsilon_0 (2m+1) A_m^{(\mathrm{i})} R^{m-1}$$

$$\Longrightarrow\; A_m^{(\mathrm{i})} = \frac{4}{5}\sigma_0 R^{1-m} \frac{1}{2\epsilon_0(2m+1)}\,\delta_{m2}$$

$$\Longrightarrow\; A_2^{(\mathrm{i})} = \frac{2\sigma_0}{25\epsilon_0 R}\;; \quad A_m^{(\mathrm{i})} = 0 \quad \text{for } m \neq 2\;.$$

Solution:

$$\varphi_{\mathrm{i}}(r,\vartheta) = \frac{2\sigma_0}{5\epsilon_0 R} r^2 P_2(\cos\vartheta)\;,$$

$$\varphi_{\mathrm{a}}(r,\vartheta) = \frac{2\sigma_0}{5\epsilon_0} R^4 \frac{P_2(\cos\vartheta)}{r^3}\;.$$

Solution 2.3.7 In the inside of the box the Laplace equation works:

$$\Delta\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z) = 0\;.$$

Separation ansatz:

$$\varphi(\mathbf{r}) = \varphi_1(x)\,\varphi_2(y)\,\varphi_3(z)\;.$$

Insertion into the Laplace equation and division by φ:

$$\underbrace{\frac{1}{\varphi_1}\frac{d^2\varphi_1}{dx^2}}_{\text{only dep. on } x} + \underbrace{\frac{1}{\varphi_2}\frac{d^2\varphi_2}{dy^2}}_{\text{only dep. on } y} + \underbrace{\frac{1}{\varphi_3}\frac{d^2\varphi_3}{dz^2}}_{\text{only dep. on } z} = 0$$

Hence it must hold:

$$\frac{1}{\varphi_1}\frac{d^2\varphi_1}{dx^2} = -\alpha^2$$

$$\frac{1}{\varphi_2}\frac{d^2\varphi_2}{dy^2} = -\beta^2$$

$$\frac{1}{\varphi_3}\frac{d^2\varphi_3}{dz^2} = \gamma^2 = (\alpha^2+\beta^2)\,.$$

This leads to the ansatz functions:

$$\begin{aligned}
\varphi_1(x) &= A_1 \sin(\alpha x) + B_1 \cos(\alpha x) \\
\varphi_2(y) &= A_2 \sin(\beta y) + B_2 \cos(\beta y) \\
\varphi_3(z) &= A_3 \sinh(\gamma z) + B_3 \cosh(\gamma z) \; .
\end{aligned}$$

Dirichlet-boundary conditions:

1.

$$\varphi(0, y, z) = \varphi(a, y, z) = 0 \; .$$

only fulfilled if:

$$B_1 = 0 \; ; \quad \alpha \to \alpha_n = \frac{n\pi}{a} \; .$$

2.

$$\varphi(x, 0, z) = \varphi(x, b, z) = 0$$

requires:

$$B_2 = 0 \; ; \quad \beta \to \beta_m = \frac{m\pi}{b} \; .$$

3. The third boundary condition

$$\varphi(x, y, 0) = \varphi(x, y, c) = \varphi_0$$

we fulfill by a trick. At first we seek a solution $\varphi^{(1)}(\mathbf{r})$ which satisfies the boundary conditions

$$\varphi^{(1)}(x, y, 0) = 0 \; ; \quad \varphi^{(1)}(x, y, c) = \varphi_0 \; .$$

In the next step we look for a potential $\varphi^{(2)}(\mathbf{r})$ with the boundary conditions:

$$\varphi^{(2)}(x, y, 0) = \varphi_0 \; ; \quad \varphi^{(2)}(x, y, c) = 0 \; .$$

Because of the linearity of the Laplace equation

$$\varphi(\mathbf{r}) = \varphi^{(1)}(\mathbf{r}) + \varphi^{(2)}(\mathbf{r})$$

is then obviously the (unique) solution with the required boundary conditions! In this sense we presume at first:

$$\varphi^{(1)}(x, y, 0) = 0 \; ; \quad \varphi^{(1)}(x, y, c) = \varphi_0$$

It follows immediately:

$$B_3 = 0 \; .$$

With

$$\gamma_{nm} = \pi \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

we come to an intermediate result:

$$\varphi^{(1)}(\mathbf{r}) = \sum_{n,m=1}^{\infty} c_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \; .$$

Still to fulfill:

$$\varphi^{(1)}(x, y, c) = \varphi_0 = \sum_{n,m=1}^{\infty} c_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \; .$$

Orthogonality relation (easy to check)

$$\int\limits_0^d dx \, \sin\left(\frac{n\pi}{d} x\right) \, \sin\left(\frac{m\pi}{d} x\right) = \frac{d}{2} \, \delta_{nm}$$

$$\begin{aligned} \curvearrowright \quad & \int\limits_0^a dx \, \sin\left(\frac{r\pi}{a} x\right) \int\limits_0^b dy \, \sin\left(\frac{s\pi}{b} y\right) \cdot \varphi_0 \\ & = \varphi_0 \frac{ab}{rs\pi^2} \left((-1)^r - 1\right) \left((-1)^s - 1\right) \\ & \overset{!}{=} \frac{ab}{4} \, c_{rs} \sinh\left(\gamma_{rs} c\right) \; . \end{aligned}$$

We see that only odd r and s contribute. With

$$\hat{\gamma}_{nm} = \pi \sqrt{\left(\frac{2n+1}{a}\right)^2 + \left(\frac{2m+1}{b}\right)^2}$$

it follows therewith as solution:

$$\varphi^{(1)}(\mathbf{r}) = \frac{16\varphi_0}{\pi^2} \sum_{n,m} \frac{1}{(2n+1)(2m+1)} \sin\left(\frac{2n+1}{a}x\right) \sin\left(\frac{2m+1}{b}y\right) \cdot \frac{\sinh(\hat{\gamma}_{nm}z)}{\sinh(\hat{\gamma}_{nm}c)} .$$

Now we still have to calculate $\varphi^{(2)}$ with the boundary conditions

$$\varphi^{(2)}(x,y,0) = \varphi_0 ; \quad \varphi^{(2)}(x,y,c) = 0 .$$

But that can be done by use of the same path of solution. We have simply to replace z by $c-z$:

$$\varphi^{(2)}(\mathbf{r}) = \frac{16\varphi_0}{\pi^2} \sum_{n,m} \frac{1}{(2n+1)(2m+1)} \sin\left(\frac{2n+1}{a}x\right) \sin\left(\frac{2m+1}{b}y\right) \cdot \frac{\sinh(\hat{\gamma}_{nm}(c-z))}{\sinh(\hat{\gamma}_{nm}c)} .$$

That leads to the complete solution:

$$\varphi(\mathbf{r}) = \frac{16\varphi_0}{\pi^2} \sum_{n,m} \frac{1}{(2n+1)(2m+1)} \sin\left(\frac{2n+1}{a}x\right) \sin\left(\frac{2m+1}{b}y\right) \cdot \frac{1}{\sinh(\hat{\gamma}_{nm}c)} \left[\sinh\left(\hat{\gamma}_{nm}z\right) + \sinh\left(\hat{\gamma}_{nm}(c-z)\right)\right] .$$

Solution 2.3.8 Legendre equation (2.151):

$$\frac{d}{dz}\left\{\left(1-z^2\right)\frac{d}{dz}\right\} P_l(z) + l(l+1)P_l(z) = 0 ; \quad l = 0,1,2,\ldots$$

This can be rearranged:

$$\left(1-z^2\right)\frac{d^2}{dz^2}P_l(z) - 2z\frac{d}{dz}P_l(z) + l(l+1)P_l(z) = 0 .$$

This is a differential equation of second order for which we need two linearly independent solutions.

1. We insert the ansatz

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

into the Legendre equation:

$$0 = \left(1 - z^2\right) \sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} - 2z \sum_{n=0}^{\infty} a_n n z^{n-1} + l(l+1) \sum_{n=0}^{\infty} a_n z^n$$

$$0 = \sum_{n=0}^{\infty} z^n \left\{ a_{n+2}(n+2)(n+1) - a_n n(n-1) - a_n 2n + l(l+1) a_n \right\} \; .$$

Comparison of the coefficients:

$$a_{n+2}(n+2)(n+1) - a_n n(n+1) + l(l+1) a_n = 0 \; .$$

That yields the recursion formula:

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} \, a_n \; .$$

One recognizes two linearly independent solutions determined by

$$(a_0 \neq 0, a_1 = 0) \quad \text{and} \quad (a_0 = 0, a_1 \neq 0) \; .$$

l even:

$$\begin{aligned} (a_0 \neq 0, a_1 = 0) &\curvearrowright \text{polynomial of } l\text{-th degree} \\ (a_0 = 0, a_1 \neq 0) &\curvearrowright \text{not terminating power series} \; . \end{aligned}$$

l odd:

$$\begin{aligned} (a_0 \neq 0, a_1 = 0) &\curvearrowright \text{not terminating power series} \\ (a_0 = 0, a_1 \neq 0) &\curvearrowright \text{polynomial of } l\text{-th degree} \; . \end{aligned}$$

2. $\mathbf{P_4(z)}$:
Choose

$$a_0 \neq 0 \; ; \quad a_1 = 0 \; .$$

That means at first:

$$a_1 = a_3 = a_5 = \ldots = 0 \; .$$

Furthermore:

$$a_2 = \frac{-4 \cdot 5}{2}\, a_0 = -10 a_0$$

$$a_4 = \frac{2 \cdot 3 - 4 \cdot 5}{4 \cdot 3}\, a_2 = \frac{-7}{6}(-10 a_0) = \frac{35}{3}\, a_0$$

$$a_6 = \frac{4 \cdot 5 - 4 \cdot 5}{6 \cdot 5}\, a_4 = 0 = a_8 = a_{10} = \ldots \; .$$

It is left therewith:

$$P_4(z) = a_0 \left(1 - 10z^2 + \frac{35}{3} z^4\right)$$

$$P_4(1) \stackrel{!}{=} 1 = a_0 \left(1 - 10 + \frac{35}{3}\right) = \frac{8}{3} a_0 \; .$$

This yields finally:

$$P_4(z) = \frac{3}{8} \left(1 - 10z^2 + \frac{35}{3} z^4\right) \; .$$

$\mathbf{P_5(z)}$:
Choose

$$a_0 = 0 \, ; \quad a_1 \neq 0 \; .$$

That means at first:

$$a_0 = a_2 = a_4 = \ldots = 0 \; .$$

Furthermore:

$$a_3 = \frac{1 \cdot 2 - 5 \cdot 6}{3 \cdot 2}\, a_1 = -\frac{14}{3} a_1$$

$$a_5 = \frac{3 \cdot 4 - 5 \cdot 6}{5 \cdot 4}\, a_3 = \frac{-18}{20}\left(-\frac{14}{3} a_1\right) = \frac{21}{5}\, a_1$$

$$a_7 = \frac{5 \cdot 6 - 5 \cdot 6}{5 \cdot 4}\, a_5 = 0 = a_9 = a_{11} = \ldots \; .$$

It is left therewith:

$$P_5(z) = a_1 \left(z - \frac{14}{3} z^3 + \frac{21}{5} z^5\right)$$

$$P_5(1) \stackrel{!}{=} 1 = a_1 \left(1 - \frac{14}{3} + \frac{21}{5}\right) = \frac{8}{15} a_1 \; .$$

This yields finally:

$$P_5(z) = \frac{15}{8}\left(z - \frac{14}{3}z^3 + \frac{21}{5}z^5\right) .$$

Solution 2.3.9

1. Ansatz because of the azimuthal symmetry:

$$\Phi = \sum_{l=0}^{\infty} R_l(r)P_l(\cos\vartheta) .$$

We use the Laplace operator as in (2.145) and take into consideration that the spherical harmonics $Y_{lm}(\vartheta, \varphi)$ are eigenfunctions of the operator $\Delta_{\vartheta,\varphi}$. That holds in particular for the $m = 0$-functions which, except for an unimportant factor, agree with the Legendre polynomials. Hence, it is also valid:

$$\Delta_{\vartheta,\varphi}\, P_l(\cos\vartheta) = -l(l+1)\, P_l(\cos\vartheta) .$$

Laplace equation:

$$\begin{aligned}
0 &= \Delta\Phi \\
&= \Delta \sum_{l=0}^{\infty} R_l(r)P_l(\cos\vartheta) \\
&= \sum_{l=0}^{\infty}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_l(r)P_l(\cos\vartheta) + \frac{1}{r^2}\Delta_{\vartheta,\varphi}\, R_l(r)P_l(\cos\vartheta)\right\} \\
&= \sum_{l=0}^{\infty}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2}\right)R_l(r)P_l(\cos\vartheta) .
\end{aligned}$$

$P_l(\cos\vartheta)$: complete orthogonal system. Therefore it must already be:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_l(r) - \frac{l(l+1)}{r^2}R_l(r) = 0 .$$

Ansatz:

$$R_l(r) = \frac{1}{r}\,u_l(r) .$$

It follows therewith:

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R_l(r) &= \frac{1}{r^2}\frac{d}{dr}\left(-u_l(r)+r\frac{du_l(r)}{dr}\right)\\ &= \frac{1}{r^2}\left(-\frac{du_l(r)}{dr}+\frac{du_l(r)}{dr}+r\frac{d^2u_l(r)}{dr^2}\right)\\ &= \frac{1}{r}\frac{d^2u_l(r)}{dr^2}\ .\end{aligned}$$

Thus it is to solve:

$$\frac{d^2u_l(r)}{dr^2}=\frac{l(l+1)}{r^2}\,u_l(r) \quad \curvearrowright \quad u_l(r)=\alpha_l\,r^{l+1}+\beta_l\,r^{-l}\ .$$

The radial function is therewith determined:

$$R_l(r)=\alpha_l\,r^l+\beta_l\,r^{-(l+1)}\ .$$

General solution for the potential:

$$\Phi(r,\vartheta)=\sum_{l=0}^{\infty}\left(\alpha_l\,r^l+\beta_l\,r^{-(l+1)}\right)\,P_l(\cos\vartheta)\ .$$

2. Regularity of the potential at the origin (center of the sphere with radius R):

$$\beta_l^{(i)}=0 \qquad \forall l\ .$$

It remains:

$$\Phi_{\mathrm{i}}(r,\vartheta)=\sum_{l=0}^{\infty}\alpha_l\,r^l\,P_l(\cos\vartheta)\ .$$

Grounding:

$$\Phi_{\mathrm{i}}(R,\vartheta)=0=\sum_{l=0}^{\infty}\alpha_l\,R^l\,P_l(\cos\vartheta)\ .$$

The orthogonality of the Legendre polynomials has the consequence:

$$\alpha_l=0\ \ \forall l \quad \curvearrowright \quad \Phi_{\mathrm{i}}\equiv 0\ .$$

3. Grounding means:

$$\Phi_a(R,\vartheta) = 0 = \sum_{l=0}^{\infty} \left(\alpha_l R^l + \beta_l R^{-(l+1)}\right) P_l(\cos\vartheta) \quad \curvearrowright \quad \beta_l = -\alpha_l R^{2l+1} \,.$$

The potential in the exterior space then takes the following form:

$$\Phi_a(r,\vartheta) = \sum_{l=0}^{\infty} \alpha_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\vartheta)\,.$$

Surface charge density:

$$\sigma = -\varepsilon_0 \left.\frac{\partial \Phi_a}{\partial r}\right|_{r=R} = -\varepsilon_0 \sum_{l=0}^{\infty} \alpha_l \left(lR^{l-1} + (l+1)R^{l-1}\right) P_l(\cos\vartheta)$$
$$\stackrel{!}{=} \varepsilon_0\sigma_0 \cos\vartheta = \varepsilon_0\sigma_0 P_1(\cos\vartheta)\,.$$

We exploit once more the orthogonality and find therewith:

$$-\alpha_1(1+2) = \sigma_0 \quad \curvearrowright \quad \alpha_1 = -\frac{1}{3}\sigma_0\,;\; \alpha_l = 0 \;\forall l \neq 1\,.$$

Potential in the exterior space:

$$\Phi_a(r,\vartheta) = -\frac{1}{3}\sigma_0 \left(r - \frac{R^3}{r^2}\right)\cos\vartheta\,.$$

Solution 2.3.10

1. Charge-free hollow sphere with azimuthal-symmetric surface charge density means that the potential is independent of the angle φ. Spherical coordinates are appropriate and the Legendre polynomials build a suitable ‘complete orthogonal system’. Therefore:

$$\sigma(\vartheta) = \sum_{l=0}^{\infty} \sigma_l P_l(\cos\vartheta)\,.$$

General solution of the Laplace equation (2.165):

$$\varphi(\mathbf{r}) \implies \varphi(r,\vartheta) = \sum_{l=0}^{\infty} (2l+1)\left\{A_l r^l + B_l r^{-(l+1)}\right\} P_l(\cos\vartheta)\,.$$

$\sigma(\vartheta)$ is given. With the orthogonality relation one finds:

$$\int_{-1}^{1} d\cos\vartheta\,\sigma(\vartheta)P_m(\cos\vartheta) = \sum_{l=0}^{\infty}\sigma_l \underbrace{\int_{-1}^{1} d\cos\vartheta\,P_l(\cos\vartheta)P_m(\cos\vartheta)}_{\frac{2}{(2m+1)}\delta_{lm}}$$

$$\Longrightarrow \quad \sigma_m = \frac{(2m+1)}{2}\int_{-1}^{1} d\cos\vartheta\,\sigma(\vartheta)P_m(\cos\vartheta)$$

$\Longrightarrow$ all coefficients σ_m are known. Useful decomposition:

$$\varphi_{\mathrm{i}}(r,\vartheta): \text{ potential inside the sphere.}$$

$$\varphi_{\mathrm{a}}(r,\vartheta): \text{ potential outside the sphere.}$$

Boundary conditions:

(a) φ_{i} regular for $r \to 0 \implies B_l^i \equiv 0$

$$\varphi_{\mathrm{i}}(r,\vartheta) = \sum_{l=0}^{\infty}(2l+1)A_l^i r^l P_l(\cos\vartheta)\ .$$

(b) $\varphi_{\mathrm{a}} \overset{r\to\infty}{\longrightarrow} 0 \implies A_l^a \equiv 0$

$$\varphi_{\mathrm{a}}(r,\vartheta) = \sum_{l=0}^{\infty}(2l+1)B_l^a r^{-(l+1)} P_l(\cos\vartheta)\ .$$

(c) Continuity at $r = R$:

$$\begin{aligned} \varphi_{\mathrm{i}}(R,\vartheta) &\overset{!}{=} \varphi_{\mathrm{i}}(R,\vartheta) \\ \text{orthogonality of the } P_l \implies B_l^a R^{-(l+1)} &= A_l^i R^l \\ \implies B_l^a &= A_l^i R^{2l+1}\ . \end{aligned}$$

(d) Surface charge density:

$$\begin{aligned} \sigma(\vartheta) &= -\epsilon_0\left(\frac{\partial\varphi_a}{\partial r} - \frac{\partial\varphi_i}{\partial r}\right)\Bigg|_{r=R} \\ &= -\epsilon_0\sum_{l=0}^{\infty}(2l+1)P_l(\cos\vartheta)\left\{-(l+1)B_l^a R^{-(l+2)} - lA_l^i R^{l-1}\right\} \\ &= +\epsilon_0\sum_{l=0}^{\infty}(2l+1)^2 P_l(\cos\vartheta)A_l^i R^{l-1}\ . \end{aligned}$$

Orthogonality of the Legendre polynomials:

$$\sigma_l = \epsilon_0 (2l+1)^2 A_l^i R^{l-1}$$
$$\Longrightarrow \; A_l^i = \frac{\sigma_l}{(2l+1)^2 \epsilon_0 R^{l-1}} \; .$$

The potential is therewith completely determined:

$$\varphi_{\mathrm{i}}(r,\vartheta) = \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{(2l+1)} \left(\frac{r}{R}\right)^l P_l(\cos\vartheta) \; ,$$
$$\varphi_{\mathrm{a}}(r,\vartheta) = \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{(2l+1)} \left(\frac{R}{r}\right)^{l+1} P_l(\cos\vartheta) \; .$$

2. Special case:

$$\begin{aligned} \sigma(\vartheta) &= \sigma_0 (2\cos^2\vartheta + \cos\vartheta - \sin^2\vartheta) \\ &= \sigma_0 (3\cos^2\vartheta - 1 + \cos\vartheta) \\ &= \sigma_0 (2P_2(\cos\vartheta) + P_1(\cos\vartheta)) \; . \end{aligned}$$

With the general relation from part 1. it can be calculated:

$$\begin{aligned} \sigma_m &= \frac{2m+1}{2}\sigma_0 \left(2\int_{-1}^{1} d\cos\vartheta P_2(\cos\vartheta) P_m(\cos\vartheta) + \int_{-1}^{1} d\cos\vartheta P_1(\cos\vartheta) P_m(\cos\vartheta) \right) \\ &= \frac{2m+1}{2}\sigma_0 \left(2\frac{2}{2m+1}\delta_{m2} + \frac{2}{2m+1}\delta_{m1} \right) \\ &= \sigma_0 (2\delta_{m2} + \delta_{m1}) \end{aligned}$$

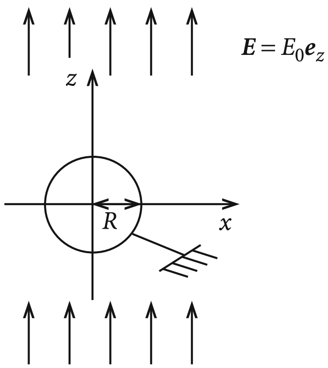
$\Longrightarrow$ Potential:

$$\varphi_{\mathrm{i}}(r,\vartheta) = \sigma_0 \frac{R}{\epsilon_0} \left\{ \frac{2}{5}\left(\frac{r}{R}\right)^2 P_2(\cos\vartheta) + \frac{1}{3}\left(\frac{r}{R}\right) P_1(\cos\vartheta) \right\} \; ,$$
$$\varphi_{\mathrm{a}}(r,\vartheta) = \sigma_0 \frac{R}{\epsilon_0} \left\{ \frac{2}{5}\left(\frac{R}{r}\right)^3 P_2(\cos\vartheta) + \frac{1}{3}\left(\frac{R}{r}\right)^2 P_1(\cos\vartheta) \right\} \; .$$

Solution 2.3.11 Azimuthal symmetry (Fig. A.22):

$$\varphi(r,\vartheta) = \sum_{l=0}^{\infty} (2l+1) \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\vartheta) \; .$$

Fig. A.22



Conducting grounded sphere:

$$\varphi(R,\vartheta) = 0 \implies B_l = -A_l R^{2l+1} \; .$$

Field in the inside of the sphere:

$$\begin{aligned}\text{regularity at } r = 0 &\implies B_l^{(\mathrm{i})} = 0 \quad \forall l \\ &\implies A_l^{(\mathrm{i})} = 0 \quad \forall l \\ &\implies \varphi \equiv 0 \quad \text{in the inside} \; .\end{aligned}$$

E-field asymptotically homogeneous:

$$\varphi \underset{r\to\infty}{\longrightarrow} -E_0 z = -E_0 r \cos\vartheta = -E_0 r \, P_1(\cos\vartheta) \; ,$$

$$\sum_{l=0}^{\infty}(2l+1)\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\vartheta) \underset{r\to\infty}{\longrightarrow} -E_0 r \, P_1(\cos\vartheta)$$

$$\implies A_1 = -\frac{1}{3}E_0 \; ; \quad A_l = 0 \quad \text{for } l \neq 1$$

$$\implies B_1 = -A_1 R^3 = +\frac{1}{3}E_0 R^3 \; .$$

Potential outside the sphere:

$$\varphi(r,\vartheta) = -E_0 R \left(\frac{r}{R} - \frac{R^2}{r^2}\right)\cos\vartheta \; .$$

Surface charge density:

$$\sigma = -\epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = 3\epsilon_0 E_0 \cos\vartheta \; .$$

Solution 2.3.12 Spherical coordinates

$$\begin{aligned}
\mathbf{r} &= r(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\ ,\\
Y_{10}(\vartheta,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos\vartheta\ ,\\
Y_{11}(\vartheta,\varphi) &= -\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{i\varphi}\ ,\\
Y_{1-1}(\vartheta,\varphi) &= \sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{-i\varphi}\\
\Longrightarrow\ \mathbf{r} &= r\sqrt{\frac{4\pi}{3}}\Bigg(-\frac{\sqrt{2}}{2}\left(Y_{11}(\vartheta,\varphi) - Y_{1-1}(\vartheta,\varphi)\right),\\
&\qquad -\frac{\sqrt{2}}{2i}\left(Y_{11}(\vartheta,\varphi) + Y_{1-1}(\vartheta,\varphi)\right), Y_{10}(\vartheta,\varphi)\Bigg)\ .
\end{aligned}$$

Similarly we get $\mathbf{r}'$, where, however, we still have to use:

$$Y^*_{lm}(\vartheta',\varphi') = (-1)^m Y_{l-m}(\vartheta',\varphi')$$

Therewith:

$$\begin{aligned}
\mathbf{r}' &= r'\sqrt{\frac{4\pi}{3}}\Bigg(\frac{\sqrt{2}}{2}\left(Y^*_{1-1}(\vartheta',\varphi') - Y^*_{11}(\vartheta',\varphi')\right),\\
&\qquad \frac{\sqrt{2}}{2i}\left(Y^*_{1-1}(\vartheta',\varphi') + Y^*_{11}(\vartheta',\varphi')\right), Y^*_{10}(\vartheta',\varphi')\Bigg)\\
\Longrightarrow\ \mathbf{r}\cdot\mathbf{r}' &= \frac{4\pi}{3}rr'\Bigg\{-\frac{1}{2}\{Y_{11}(\vartheta,\varphi)Y^*_{1-1}(\vartheta',\varphi') + Y_{1-1}(\vartheta,\varphi)Y^*_{11}(\vartheta',\varphi')\\
&\qquad -Y_{11}(\vartheta,\varphi)Y^*_{11}(\vartheta',\varphi') - Y_{1-1}(\vartheta,\varphi)Y^*_{1-1}(\vartheta',\varphi')\}\\
&\qquad \frac{1}{2}\{Y_{11}(\vartheta,\varphi)Y^*_{1-1}(\vartheta',\varphi') + Y_{1-1}(\vartheta,\varphi)Y^*_{11}(\vartheta',\varphi')\\
&\qquad +Y_{1-1}(\vartheta,\varphi)Y^*_{1-1}(\vartheta',\varphi') + Y_{11}(\vartheta,\varphi)Y^*_{11}(\vartheta',\varphi')\}\\
&\qquad +Y_{10}(\vartheta,\varphi)Y^*_{10}(\vartheta',\varphi')\Bigg\}\\
\Longrightarrow\ \mathbf{r}\cdot\mathbf{r}' &= \frac{4\pi}{3}rr'\big(Y_{11}(\vartheta,\varphi)Y^*_{11}(\vartheta',\varphi') + Y_{1-1}(\vartheta,\varphi)Y^*_{1-1}(\vartheta',\varphi')\\
&\qquad +Y_{10}(\vartheta,\varphi)Y^*_{10}(\vartheta',\varphi')\big)\\
\Longrightarrow\ \mathbf{r}\cdot\mathbf{r}' &= \frac{4\pi}{3}rr'\sum_{m=-1,0,1} Y^*_{1m}(\vartheta',\varphi')Y_{1m}(\vartheta,\varphi)\ .
\end{aligned}$$

Addition theorem (2.161);

$$\frac{2l+1}{4\pi} P_l(\cos\gamma) = \sum_{m=-l}^{+l} Y^*_{lm}(\vartheta', \varphi') Y_{lm}(\vartheta, \varphi) \; ,$$

$$\gamma = \sphericalangle \left(\vartheta'\varphi'; \vartheta\varphi\right) \; .$$

In our case here it is:

$$\gamma = \sphericalangle(\mathbf{r}, \mathbf{r}')$$

and therewith:

$$\begin{aligned}\mathbf{r} \cdot \mathbf{r}' &= rr' \cos\gamma = rr' P_1(\cos\gamma) \\ &= \frac{4\pi}{3} rr' \sum_{m=-1}^{+1} Y^*_{1m}(\vartheta', \varphi') Y_{1m}(\vartheta, \varphi)\end{aligned}$$

Solution 2.3.13 With

$$Y_{10}(\vartheta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\vartheta$$

the charge density is written:

$$\rho(\mathbf{r}) = \sigma_0 \cos\vartheta \, \delta(r-R) = \sqrt{\frac{4\pi}{3}} \sigma_0 \, Y_{10}(\vartheta, \varphi) \, \delta(r-R) \; .$$

In addition we apply (2.169) for $q = 1$:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r^l_<}{r^{l+1}_>} \, Y^*_{lm}(\vartheta', \varphi') \, Y_{lm}(\vartheta, \varphi) \; .$$

1. **Inside space**: $r < R$

$$r_< = r \; ; \quad r_> = r' \; .$$

Potential:

$$\begin{aligned}\Phi_{\mathrm{i}}(\mathbf{r}) = \frac{1}{\varepsilon_0} \sqrt{\frac{4\pi}{3}} \sigma_0 \int_0^\infty r'^2 dr' \sum_{l,m} \frac{1}{2l+1} \frac{r^l}{r'^{l+1}} \int_0^{2\pi} d\varphi' \int_{-1}^{+1} d\cos\vartheta' \\ \cdot Y_{10}(\vartheta', \varphi') Y^*_{lm}(\vartheta', \varphi') Y_{lm}(\vartheta, \varphi) \, \delta(r' - R)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon_0}\sqrt{\frac{4\pi}{3}}\sigma_0 \sum_{l,m} Y_{lm}(\vartheta,\varphi)\frac{1}{2l+1}\frac{r^l}{R^{l-1}} \\
&\quad \cdot \underbrace{\int_0^{2\pi} d\varphi' \int_{-1}^{+1} d\cos\vartheta'\, Y_{lm}^*(\vartheta',\varphi')Y_{10}(\vartheta',\varphi')}_{\delta_{l1}\delta_{m0}} \\
&= \frac{1}{\varepsilon_0}\sqrt{\frac{4\pi}{3}}\sigma_0\, Y_{10}(\vartheta,\varphi)\,\frac{1}{3}r \\
&= \frac{\sigma_0}{3\varepsilon_0}\, r\, \cos\vartheta \\
\curvearrowright \quad \Phi_\mathrm{i}(\mathbf{r}) &= \frac{\sigma_0}{3\varepsilon_0}\, z\ .
\end{aligned}
$$

To this belongs a homogeneous electric field in z-direction:

$$
\mathbf{E}_\mathrm{i}(\mathbf{r}) = -\frac{\sigma_0}{3\varepsilon_0}\mathbf{e}_z\ .
$$

2. **Outside space**: $r > R$

$$
r_< = r'\ ; \quad r_> = r\ .
$$

Potential:

$$
\begin{aligned}
\Phi_\mathrm{a}(\mathbf{r}) &= \frac{1}{\varepsilon_0}\sqrt{\frac{4\pi}{3}}\sigma_0 \int_0^\infty r'^2 dr'\,\delta(r'-R) \sum_{l,m}\frac{1}{2l+1}\frac{r'^l}{r^{l+1}} \\
&\quad \cdot \underbrace{\int_0^{2\pi} d\varphi' \int_{-1}^{+1} d\cos\vartheta'\, Y_{10}(\vartheta',\varphi')Y_{lm}^*(\vartheta',\varphi')}_{\delta_{l1}\delta_{m0}}\, Y_{lm}(\vartheta,\varphi) \\
&= \frac{1}{\varepsilon_0}\sqrt{\frac{4\pi}{3}}\sigma_0\,\frac{1}{3}\,Y_{10}(\vartheta,\varphi)\frac{R^3}{r^2} \\
&= \frac{\sigma_0}{3\varepsilon_0}\,\cos\vartheta\,\frac{R^3}{r^2}\ .
\end{aligned}
$$

That gives for the electric field:

$$\mathbf{E}_\mathrm{a} = -\nabla\Phi_\mathrm{a} = -\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta} + \mathbf{e}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\Phi_\mathrm{a}$$
$$= -\frac{\sigma_0}{3\varepsilon_0}\left(\mathbf{e}_r R^3\cos\vartheta\left(-\frac{2}{r^3}\right) + \mathbf{e}_\vartheta R^3\frac{1}{r^3}(-\sin\vartheta)\right)$$
$$\curvearrowright\quad \mathbf{E}_\mathrm{a} = \frac{\sigma_0 R^3}{3\varepsilon_0}\left(\frac{2\cos\vartheta}{r^3}\mathbf{e}_r + \frac{\sin\vartheta}{r^3}\mathbf{e}_\vartheta\right) .$$

With the dipole moment

$$\mathbf{p} = \frac{4\pi}{3}R^3\sigma_0\,\mathbf{e}_z$$

a pure dipole field is left (2.72):

$$E_\mathrm{a}^{(r)} = \frac{1}{4\pi\varepsilon_0}p\frac{2\cos\vartheta}{r^3}$$
$$E_\mathrm{a}^{(\vartheta)} = \frac{1}{4\pi\varepsilon_0}p\frac{\sin\vartheta}{r^3}$$
$$E_\mathrm{a}^{(\varphi)} = 0 .$$

Solution 2.3.14

1a. Equation (2.71):

$$4\pi\,\epsilon_0\varphi_\mathrm{D}(\mathbf{r}) = \frac{\mathbf{r}\cdot\mathbf{p}}{r^3} .$$

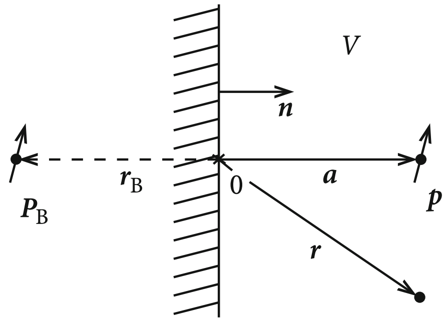
Equation (2.73):

$$4\pi\,\epsilon_0\mathbf{E}_\mathrm{D}(\mathbf{r}) = \frac{3(\mathbf{r}\cdot\mathbf{p})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3} .$$

1b.

$$4\pi\,\epsilon_0\varphi_\mathrm{D}(\mathbf{r}) = \frac{(\mathbf{r}-\mathbf{a})\cdot\mathbf{p}}{|\mathbf{r}-\mathbf{a}|^3} ,$$
$$4\pi\,\epsilon_0\mathbf{E}_\mathrm{D}(\mathbf{r}) = \frac{3[(\mathbf{r}-\mathbf{a})\cdot\mathbf{p}](\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^5} - \frac{\mathbf{p}}{|\mathbf{r}-\mathbf{a}|^3} .$$

Fig. A.23



2. Origin of coordinates on the metal surface, perpendicular in front of $\mathbf{p}$. On the metal surface it holds (Fig. A.23):

$$\mathbf{r} \cdot \mathbf{n} = 0 \, .$$

'Image dipole' $\mathbf{p}_B$ outside V

$$\Longrightarrow \quad 4\pi\,\epsilon_0\varphi(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{a}|^3}(\mathbf{r}-\mathbf{a})\cdot\mathbf{p} + \frac{1}{|\mathbf{r}-\mathbf{r}_B|^3}(\mathbf{r}-\mathbf{r}_B)\cdot\mathbf{p}_B \, .$$

Symmetry: $\mathbf{r}_B = -r_B\mathbf{n}$.
Metal surface:

$$|\mathbf{r}-\mathbf{a}| = \sqrt{r^2+a^2-2a\;\mathbf{r}\cdot\mathbf{n}} = \sqrt{r^2+a^2} \, ,$$

$$|\mathbf{r}-\mathbf{r}_B| = \sqrt{r^2+r_B^2+2r_B\;\mathbf{r}\cdot\mathbf{n}} = \sqrt{r^2+r_B^2} \, ,$$

$$4\pi\,\epsilon_0\varphi(\mathbf{r}) = \frac{\mathbf{r}\cdot\mathbf{p}-\mathbf{a}\cdot\mathbf{p}}{(r^2+a^2)^{3/2}} + \frac{\mathbf{r}\cdot\mathbf{p}_B-\mathbf{r}_B\cdot\mathbf{p}_B}{\left(r^2+r_B^2\right)^{3/2}} \overset{!}{=} 0$$

$$\Longrightarrow \quad r_B = a \, , \quad \text{i.e.} \quad \mathbf{r}_B = -\mathbf{a} = -a\,\mathbf{n} \, ,$$

$$\left.\begin{array}{r} \mathbf{p}_B\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n} \\ \mathbf{r}\cdot\mathbf{p} = -\mathbf{r}\cdot\mathbf{p}_B \end{array}\right\} \quad \begin{array}{l} \mathbf{p} = \mathbf{p}_\perp + \mathbf{p}_\parallel \, , \\ \mathbf{p}_B = \mathbf{p}_\perp - \mathbf{p}_\parallel \, . \end{array}$$

From that it follows the potential in V:

$$4\pi\,\epsilon_0\varphi(\mathbf{r}) = \mathbf{p}_\perp\left(\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} + \frac{(\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^3}\right) + (\mathbf{p}_\parallel\cdot\mathbf{r})\left(\frac{1}{|\mathbf{r}-\mathbf{a}|^3} - \frac{1}{|\mathbf{r}+\mathbf{a}|^3}\right) \, .$$

Since it holds on the metal surface

$$|\mathbf{r}-\mathbf{a}| = |\mathbf{r}+\mathbf{a}| = \sqrt{r^2+a^2} \quad \text{and} \quad \mathbf{p}_\perp\cdot\mathbf{r} = 0$$

the boundary condition

$$0 = \varphi(\mathbf{r})$$

on the metal surface is obviously fulfilled.

3. $\mathbf{E}_{\mathrm{i}} \equiv 0$ in the metal.
   in V:

$$\begin{aligned}4\pi\,\epsilon_0\mathbf{E}_{\mathrm{a}}(\mathbf{r}) = &\ \frac{3\left[(\mathbf{r}-\mathbf{a})\cdot\mathbf{p}_\perp\right](\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^5} - \frac{\mathbf{p}_\perp}{|\mathbf{r}-\mathbf{a}|^3}\\ &+\frac{3\left[(\mathbf{r}+\mathbf{a})\cdot\mathbf{p}_\perp\right](\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^5} - \frac{\mathbf{p}_\perp}{|\mathbf{r}+\mathbf{a}|^3}\\ &+\frac{3(\mathbf{r}\cdot\mathbf{p}_\parallel)(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^5} - \frac{\mathbf{p}_\parallel}{|\mathbf{r}-\mathbf{a}|^3} - \frac{3(\mathbf{r}\cdot\mathbf{p}_\parallel)(\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^5} + \frac{\mathbf{p}_\parallel}{|\mathbf{r}+\mathbf{a}|^3}\ .\end{aligned}$$

Surface charge density:

$$\begin{aligned}\sigma &= \epsilon_0\ \mathbf{E}_{\mathrm{a}}\cdot\mathbf{n}\big|_{\mathbf{r}\cdot\mathbf{n}=0}\\ &= \frac{1}{4\pi}\left[\frac{3(-\mathbf{a}\cdot\mathbf{p}_\perp)(-\mathbf{a}\cdot\mathbf{n})}{(r^2+a^2)^{5/2}} - \frac{\mathbf{p}_\perp\cdot\mathbf{n}}{(r^2+a^2)^{3/2}} + \frac{3(\mathbf{a}\cdot\mathbf{p}_\perp)(\mathbf{a}\cdot\mathbf{n})}{(r^2+a^2)^{5/2}}\right.\\ &\qquad\left. - \frac{\mathbf{p}_\perp\cdot\mathbf{n}}{(r^2+a^2)^{3/2}} + \frac{3(\mathbf{r}\cdot\mathbf{p}_\parallel)(-\mathbf{a}\cdot\mathbf{n})}{(r^2+a^2)^{5/2}} - \frac{3(\mathbf{r}\cdot\mathbf{p}_\parallel)(\mathbf{a}\cdot\mathbf{n})}{(r^2+a^2)^{5/2}}\right]\end{aligned}$$

$$\Longrightarrow\ \sigma = \frac{1}{4\pi}\,\frac{3a^2p_\perp - p_\perp(r^2+a^2) + 3a^2p_\perp - p_\perp(r^2+a^2) - 6a(\mathbf{r}\cdot\mathbf{p}_\parallel)}{(r^2+a^2)^{5/2}}$$

$$\Longrightarrow\ \sigma(r) = \frac{1}{4\pi}\,\frac{(4a^2-2r^2)\,p_\perp - 6a(\mathbf{r}\cdot\mathbf{p}_\parallel)}{(r^2+a^2)^{5/2}}\ ,\qquad \mathbf{r}\in\text{metal surface}\ .$$

4a. $p_\parallel = 0, p_\perp = p$:

$$\sigma = \frac{1}{2\pi}\,\frac{p(2a^2-r^2)}{(r^2+a^2)^{5/2}}\ ,$$

$$\sigma = 0 \quad \text{for } r = r_0 = \sqrt{2}a\ ,$$

$$\left.\begin{aligned}\sigma > 0 &\quad \text{for } r < r_0\ ,\\ \sigma < 0 &\quad \text{for } r > r_0\end{aligned}\right\}\ \begin{aligned}&\text{if } p > 0 \text{, i.e. dipole}\\ &\text{directed away from the metal}\ .\end{aligned}$$

4b. $p_\parallel = p, p_\perp = 0$:

$$\sigma = -\frac{1}{4\pi}\,\frac{6a\,(\mathbf{r}\cdot\mathbf{p})}{(r^2+a^2)^{5/2}}\ ,$$

$$\sigma > 0: \quad \mathbf{r}\cdot\mathbf{p} < 0\ ,$$

$$\sigma < 0: \quad \mathbf{r}\cdot\mathbf{p} > 0\ .$$

5. **To 4a.**:

Q_+: total charge within the circle with the radius $r_0 = \sqrt{2}a$ (Figs. A.24, A.25, and A.26):

$$Q_+ = \int_0^{r_0} \sigma(r) 2\pi \, r \, dr = p \int_0^{r_0} dr \, \frac{r(2a^2 - r^2)}{(r^2 + a^2)^{5/2}} \; ,$$

$$\cos\alpha = \frac{a}{(r^2 + a^2)^{1/2}} \; ; \quad \sin\alpha = \frac{r}{(r^2 + a^2)^{1/2}} \; ,$$

$$\tan\alpha = \frac{r}{a} \; ; \quad dr = \frac{a}{\cos^2\alpha} d\alpha$$

$$\Longrightarrow \; Q_+ = p \int_0^{r_0} dr \left(2 \frac{a^2}{r^2 + a^2} - \frac{r^2}{r^2 + a^2} \right) \frac{1}{r^2 + a^2} \, \frac{r}{(r^2 + a^2)^{1/2}}$$

Fig. A.24

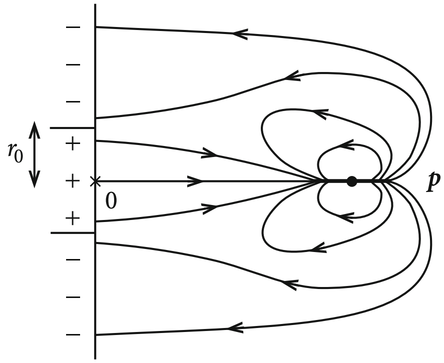


Fig. A.25

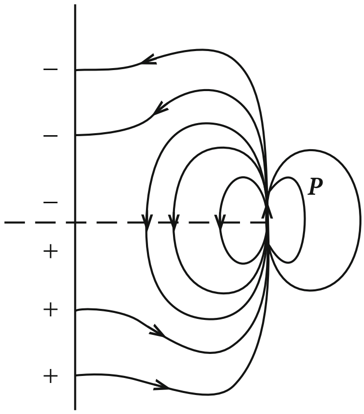
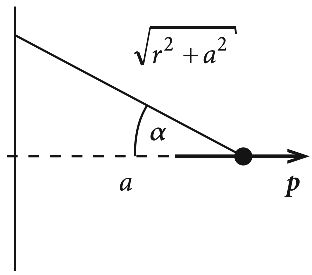


Fig. A.26



$$
\begin{aligned}
&= p \int_0^{\alpha_0} \frac{a}{\cos^2\alpha} d\alpha (2\cos^2\alpha - \sin^2\alpha) \frac{\cos^2\alpha}{a^2} \sin\alpha \\
&= \frac{p}{a} \int_1^{\cos(\alpha_0)} d\cos\alpha (1 - 3\cos^2\alpha) = \frac{p}{a} (\cos\alpha - \cos^3\alpha)\Big|_1^{\cos\alpha_0} \\
&= \frac{p}{a} \cos\alpha_0 \sin^2\alpha_0 = \frac{p}{a} \frac{a}{(r_0^2 + a^2)^{1/2}} \frac{r_0^2}{(r_0^2 + a^2)} \\
&= \frac{p}{a} \frac{2a^3}{(3a^2)^{3/2}} \\
\Longrightarrow\ Q_+ &= \frac{2}{3\sqrt{3}} \frac{p}{a} \,.
\end{aligned}
$$

Analogously:

$$
\begin{aligned}
Q_- &= \int_{r_0}^{\infty} \sigma(r) 2\pi\, r\, dr = \frac{p}{a} \left(\cos\alpha - \cos^3\alpha\right)\Big|_{\cos\alpha_0}^{\cos(\pi/2)=0} = -Q_+ \\
&\Longrightarrow \text{ total charge} = 0 \,.
\end{aligned}
$$

To 4b.:
Q: Absolute value of the positive charge in the lower half and the negative charge in the upper half of Fig. A.25. The two absolute values are equal because of symmetry reasons; the total charge is therefore also zero. We choose $\mathbf{r}\cdot\mathbf{p} > 0$;

$\mathbf{p}_\parallel$: polar axis,

$$Q = \int_0^\infty r\,dr \int_{-\pi/2}^{+\pi/2} d\beta\,\sigma(r,\beta)\ ,$$

$$\mathbf{p}_\parallel \cdot \mathbf{r} = r\,p\,\cos\beta\ , \qquad \int_{-\pi/2}^{+\pi/2} \cos\beta\,d\beta = 2\ .$$

$$Q = \frac{3ap}{\pi}\int_0^\infty dr \frac{r^2}{(r^2+a^2)^{5/2}} = \frac{3ap}{\pi}\int_0^\infty \underbrace{dr}_{\frac{a\,d\alpha}{\cos^2\alpha}}\ \underbrace{\frac{r^2}{r^2+a^2}}_{\sin^2\alpha}\ \underbrace{\frac{1}{(r^2+a^2)^{3/2}}}_{\frac{\cos^3\alpha}{a^3}}$$

$$= \frac{3p}{\pi a}\int_0^{\pi/2} \underbrace{d\alpha\ \cos\alpha}_{d\ \sin\alpha}\ \sin^2\alpha = \frac{p}{\pi a}\sin^3\alpha\Big|_0^1$$

$$\implies\ Q = \frac{p}{\pi a}\ .$$

Solution 2.3.15 Laplace equation in two dimensions:

$$\Delta\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi = 0\ .$$

Separation ansatz:

$$\varphi(x,y) = f(x)\,g(y)\ .$$

Insertion into the Laplace equation:

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{\text{only dependent on } x} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{\text{only dependent on } y} = 0$$

$$\implies\ \frac{1}{f}\frac{d^2f}{dx^2} = \alpha^2 = -\frac{1}{g}\frac{d^2g}{dy^2}\ .$$

Structure of the solution:

$$f(x) = a\,e^{\alpha x} + b\,e^{-\alpha x}\ ,$$
$$g(y) = \bar{a}\,\cos(\alpha y) + \bar{b}\,\sin(\alpha y)\ .$$

Boundary conditions:

$$\begin{aligned}
\varphi(x=0,y)=0 &\implies b=-a\,,\\
\varphi(x,y=0)=0 &\implies \bar{a}=0\,,\\
\varphi(x,y=y_0)=0 &\implies \alpha\to\alpha_n=\frac{n\pi}{y_0}\,;\quad n\in\mathbb{N}\,.
\end{aligned}$$

Intermediate result:

$$\varphi(x,y)=\sum_n c_n\sin\left(\frac{n\pi}{y_0}y\right)\sinh\left(\frac{n\pi}{y_0}x\right)\,.$$

Further boundary condition:

$$\varphi_0=\varphi(x=x_0,y)=\sum_n c_n\sin\left(\frac{n\pi}{y_0}y\right)\sinh\left(\frac{n\pi}{y_0}x_0\right)\overset{!}{=}\sin\left(\frac{\pi}{y_0}y\right)\,.$$

Orthogonality relation:

$$\begin{aligned}
&\frac{2}{y_0}\int_0^{y_0}\sin\left(\frac{m\pi}{y_0}y\right)\sin\left(\frac{\pi}{y_0}y\right)dy=\delta_{m1}\\
&\qquad=\sum_n c_n\sinh\left(\frac{n\pi}{y_0}x_0\right)\frac{2}{y_0}\int_0^{y_0}\sin\left(\frac{n\pi}{y_0}y\right)\sin\left(\frac{m\pi}{y_0}y\right)dy\\
&\qquad=\sum_n c_n\sinh\left(\frac{n\pi}{y_0}x_0\right)\delta_{nm}\\
&\implies c_m=\frac{\delta_{m1}}{\sinh\left(\frac{m\pi}{y_0}x_0\right)}\,.
\end{aligned}$$

Solution:

$$\varphi(x,y)=\frac{\sinh\left(\frac{\pi}{y_0}x\right)}{\sinh\left(\frac{\pi}{y_0}x_0\right)}\sin\left(\frac{\pi}{y_0}y\right)\,.$$

Solution 2.3.16

1. Cylindrical coordinates: ρ,φ,z,
   ρ: distance from the center of the wire.
   Symmetries:

$$\mathbf{E}(\mathbf{r})=E(\rho)\,\mathbf{e}_\rho\,.$$

Gauss theorem:
V_ρ: Cylinder with radius ρ and height L; concentric around the wire:

$$\int_{V_\rho} d^3r \operatorname{div} \mathbf{E} = \frac{1}{\epsilon_0} q(V_\rho) = \frac{1}{\epsilon_0} \lambda L = \int_{S(V_\rho)} d\mathbf{f} \cdot \mathbf{E} = E(\rho) 2\pi \rho L \ .$$

Electric field:

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{\rho} \mathbf{e}_\rho \ .$$

Potential:

$$\varphi(\mathbf{r}) = -\frac{\lambda}{2\pi \epsilon_0} \ln \rho \ .$$

2. *'Image wire'*:
To the left of the plate with distance $(-x_0)$, parallel to the plate, charge per length $(-\lambda)$.
Potential:

$$\begin{aligned} \text{wire} \quad &\Longrightarrow \quad \varphi_{\mathrm{W}}(\mathbf{r}) = -\frac{\lambda}{2\pi \epsilon_0} \ln \sqrt{(x - x_0)^2 + y^2} \ , \\ \text{image wire} \quad &\Longrightarrow \quad \varphi_{\mathrm{IW}}(\mathbf{r}) = +\frac{\lambda}{2\pi \epsilon_0} \ln \sqrt{(x + x_0)^2 + y^2} \ . \end{aligned}$$

Total potential:

$$\varphi(\mathbf{r}) = \frac{\lambda}{2\pi \epsilon_0} \ln \sqrt{\frac{(x + x_0)^2 + y^2}{(x - x_0)^2 + y^2}} \ .$$

Boundary condition:

$$\varphi(x = 0, y, z) = \frac{\lambda}{2\pi \epsilon_0} \ln 1 = 0 \ .$$

3. Induced surface charge density:

$$\begin{aligned} \sigma &= -\epsilon_0 \left. \frac{\partial \varphi}{\partial x} \right|_{x=0} \ , \\ \frac{\partial \varphi}{\partial x} &= \frac{\lambda}{2\pi \epsilon_0} \frac{1}{2} \left[\frac{2(x + x_0)}{(x + x_0)^2 + y^2} - \frac{2(x - x_0)}{(x - x_0)^2 + y^2} \right] \ , \\ \left. \frac{\partial \varphi}{\partial x} \right|_{x=0} &= \frac{\lambda}{\pi \epsilon_0} \frac{x_0}{x_0^2 + y^2} \quad \Longrightarrow \quad \sigma = -\frac{\lambda}{\pi} \frac{x_0}{x_0^2 + y^2} \ . \end{aligned}$$

Section 2.4.4

Solution 2.4.1

1. Charge density 'electron plus nucleus':

$$\rho(\mathbf{r}) = e\,\delta(\mathbf{r}) - \frac{e}{\pi\, a^3}\ \exp\left(-\frac{2r}{a}\right) \quad (\textbf{without}\ \text{field})\ ,$$

$$\rho_{\mathrm{E}}(\mathbf{r}) = e\,\delta(\mathbf{r}) - \frac{e}{\pi\, a^3}\ \exp\left(-\frac{2|\mathbf{r}-\mathbf{r}_0|}{a}\right) \quad (\textbf{with}\ \text{field})\ .$$

Dipole moment:

$$\begin{aligned}
\mathbf{p} &= \int d^3r\,\mathbf{r}\,\rho_{\mathrm{E}}(\mathbf{r}) = \int d^3r'(\mathbf{r}'+\mathbf{r}_0)\rho_{\mathrm{E}}(\mathbf{r}'+\mathbf{r}_0) \\
&= \mathbf{r}_0 \underbrace{\int d^3r'\rho_{\mathrm{E}}(\mathbf{r}'+\mathbf{r}_0)}_{(\mathrm{I})} + \underbrace{\int d^3r'\mathbf{r}'\rho_{\mathrm{E}}(\mathbf{r}'+\mathbf{r}_0)}_{(\mathrm{II})}\ ,
\end{aligned}$$

$$\begin{aligned}
(\mathrm{I}) &= \left(e - \frac{4e}{a^3}\underbrace{\int\limits_0^\infty dr'r'^2 e^{-2r'/a}}_{=a^3/4}\right) \\
&= 0\ ; \quad \text{understandable because the total charge vanishes.} \\
(\mathrm{II}) &= -e\,\mathbf{r}_0 - \frac{e}{\pi\, a^3}\int d^3r'\mathbf{r}'\,e^{-2r'/a} \\
&= -e\,\mathbf{r}_0 - \frac{e}{\pi\, a^3}\int\limits_0^\infty dr'r'^3 e^{-2r'/a}\int\limits_0^{2\pi} d\varphi' \int\limits_{-1}^{+1} d\cos\vartheta' \begin{pmatrix}\sin\vartheta'\cos\varphi' \\ \sin\vartheta'\sin\varphi' \\ \cos\vartheta'\end{pmatrix} \\
&= -e\,\mathbf{r}_0 - \mathbf{0} \implies \mathbf{p} = -e\,\mathbf{r}_0\ .
\end{aligned}$$

2. Restoring force:

$$\mathbf{F}_{\mathrm{R}} = e\underset{\substack{\uparrow \\ \text{field of the electron}}}{\mathbf{E}_{\mathrm{e}}}(\mathbf{r}=0)\ .$$

At first: electron at the origin

$$\rho_e(\mathbf{r}) = -\frac{e}{\pi\, a^3}\ \exp\left(-\frac{2r}{a}\right)\ .$$

Field to be calculated by use of the Gauss theorem:

$$4\pi\, r^2 E(\mathbf{r}) = -\frac{4e}{\epsilon_0 a^3} \underbrace{\int_0^r dr' r'^2 e^{-2r'/a}}_{\frac{a^3}{4}-\frac{a}{2}e^{-2r/a}\left(\frac{a^2}{2}+a\,r+r^2\right)}$$

$$\Longrightarrow \;\; \mathbf{E}(\mathbf{r}) = -\frac{e}{4\pi\,\epsilon_0}\left[\frac{1}{r^2} - \frac{e^{-2r/a}}{r^2}\left(1+\frac{2r}{a}+\frac{2r^2}{a^2}\right)\right]\mathbf{e}_r \,.$$

Now: electron at the position $\mathbf{r}_0$

$$\Longrightarrow \;\; \mathbf{E}_e(\mathbf{r}) = -\frac{e}{4\pi\,\epsilon_0}\,\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\left[1-e^{(-2|\mathbf{r}-\mathbf{r}_0|)/a}\left(1+\frac{2|\mathbf{r}-\mathbf{r}_0|}{a}+\frac{2|\mathbf{r}-\mathbf{r}_0|^2}{a^2}\right)\right] .$$

Restoring force:

$$\begin{aligned}\mathbf{F}_\mathrm{R} &= \frac{e^2}{4\pi\,\epsilon_0 r_0^2}\,\mathbf{e}_{r_0}\left[1-e^{-2r_0/a}\left(1+\frac{2r_0}{a}+\frac{2r_0^2}{a^2}\right)\right]\\ &\approx \frac{e^2}{3\pi\,\epsilon_0 a^3}\mathbf{r}_0 = -\frac{e}{3\pi\,\epsilon_0 a^3}\;\mathbf{p}\;,\end{aligned}$$

This estimation is correct since it holds for $r_0 \ll a$:

$$\begin{aligned}&\left[1-e^{-2r_0/a}\left(1+\frac{2r_0}{a}+\frac{2r_0^2}{a^2}\right)\right]\\ &\quad = 1-\left(1-\frac{2r_0}{a}+\frac{2r_0^2}{a^2}-\frac{4}{3}\,\frac{r_0^3}{a^3}+\cdots\right)\left(1+\frac{2r_0}{a}+\frac{2r_0^2}{a^2}\right)\\ &\quad = 1-\left(1+\frac{2r_0}{a}+\frac{2r_0^2}{a^2}\right)+\left(\frac{2r_0}{a}+\frac{4r_0^2}{a^2}+\frac{4r_0^3}{a^3}\right)\\ &\qquad -\left(\frac{2r_0^2}{a^2}+\frac{4r_0^3}{a^3}+\frac{4r_0^4}{a^4}\right)+\left(\frac{4}{3}\,\frac{r_0^3}{a^3}+\frac{8}{3}\,\frac{r_0^4}{a^4}+\frac{8}{3}\,\frac{r_0^5}{a^5}\right)+\cdots\\ &\quad = \frac{4}{3}\,\frac{r_0^3}{a^3}+\mathcal{O}\left(\frac{r_0^4}{a^4}\right) .\end{aligned}$$

Equilibrium condition:

$$e\,\mathbf{E}_0 \stackrel{!}{=} -\mathbf{F}_\mathrm{R} = \frac{e}{3\pi\,\epsilon_0\,a^3}\mathbf{p} \;\Longrightarrow\; \mathbf{p} = 3\pi\,\epsilon_0\,a^3\mathbf{E}_0 \;.$$

3.

$$n = \frac{N}{V}$$

is so small that the electron-clouds to first approximation do not *disturb* each other. It follows then:

Polarization:

$$\mathbf{P} = 3\pi\,\epsilon_0\, n\, a^3\, \mathbf{E}_0 \ ,$$

Electric field:

$$\mathbf{E} = \mathbf{E}_0 - \frac{1}{\epsilon_0}\mathbf{P} = (1 - 3\pi\, n\, a^3)\mathbf{E}_0 \ ,$$

Dielectric displacement:

$$\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon_0\mathbf{E}_0 = \epsilon_0\mathbf{E} + \mathbf{P} \ ,$$

Relative dielectric constant:

$$\epsilon_r = \frac{1}{1 - 3\pi\, n\, a^3} \ .$$

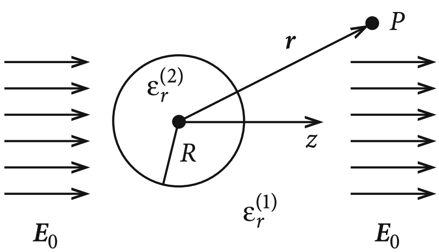
Solution 2.4.2 Since no **free** charges are present the **Laplace** equation is to be solved:

$$\Delta\,\varphi = 0 \ .$$

(a) Azimuthal symmetry (Fig. A.27):

$$\varphi(r, \vartheta) = \sum_{l=0}^{\infty}(2l+1)\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\vartheta) \quad \text{(see (2.165))} \ .$$

Fig. A.27



(b) Regularity at $r = 0$:

$$\varphi_\mathrm{i}(r, \vartheta) = \sum_{l=0}^{\infty}(2l+1)A_l r^l P_l(\cos\vartheta) \; .$$

(c) Asymptotically homogeneous field:

$$\varphi_\mathrm{a}(r, \vartheta) \underset{r\to\infty}{\longrightarrow} -E_0 z = -E_0 r \,\cos\vartheta = -E_0 r\, P_1(\cos\vartheta)$$

$$\Longrightarrow \; \varphi_\mathrm{a}(r, \vartheta) = -E_0 r\, P_1(\cos\vartheta) + \sum_{l=0}^{\infty}(2l+1)B_l r^{-(l+1)} P_l(\cos\vartheta) \; .$$

(d) Continuity at $r = R$:

$$\begin{aligned}
\varphi_\mathrm{i}(r = R, \vartheta) &\overset{!}{=} \varphi_\mathrm{a}(r = R, \vartheta) \\
\Longrightarrow \; A_0 &= \frac{B_0}{R} \; , \\
3A_1 R &= -E_0 R + \frac{3B_1}{R^2} \; , \\
A_l &= \frac{B_l}{R^{2l+1}} \quad \text{for } l \geq 2 \; .
\end{aligned}$$

(e) D_n continuous:

$$\epsilon_\mathrm{r}^{(2)}\left(\frac{\partial\varphi_\mathrm{i}}{\partial r}\right)_{r=R} = \epsilon_\mathrm{r}^{(1)}\left(\frac{\partial\varphi_\mathrm{a}}{\partial r}\right)_{r=R} \; ,$$

$$\epsilon_\mathrm{r}^{(2)}\sum_l l(2l+1)A_l R^{l-1}P_l(\cos\vartheta)$$

$$= -E_0\epsilon_\mathrm{r}^{(1)}P_1(\cos\vartheta) - \epsilon_\mathrm{r}^{(1)}\sum_l (l+1)(2l+1)B_l R^{-(l+2)}P_l(\cos\vartheta) \; .$$

Comparison of the coefficients (orthogonal functions!):

$$\begin{aligned}
0 &= B_0 \; , \\
3\epsilon_\mathrm{r}^{(2)}A_1 &= -E_0\,\epsilon_\mathrm{r}^{(1)} - 6\epsilon_\mathrm{r}^{(1)}B_1\frac{1}{R^3} \; , \\
\epsilon_\mathrm{r}^{(2)}l(2l+1)A_l &= -\epsilon_\mathrm{r}^{(1)}(l+1)(2l+1)B_l\frac{1}{R^{2l+1}} \quad \text{for } l \geq 2 \; .
\end{aligned}$$

Comparison with (d):

$$A_l = B_l = 0 \quad \text{for } l \neq 1 \ ,$$

$$A_1 = -E_0 \frac{\epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} \ ,$$

$$B_1 = \frac{1}{3} R^3 E_0 \frac{\epsilon_r^{(2)} - \epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} \ .$$

(f) Solution:

$$\varphi_i(\mathbf{r}) = -\frac{3\epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} E_0 r \cos\vartheta \ ,$$

$$\varphi_a(\mathbf{r}) = -E_0 r \cos\vartheta + E_0 R^3 \frac{\epsilon_r^{(2)} - \epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} \frac{\cos\vartheta}{r^2} \ .$$

(g) Electric field:
Inside:

$$\mathbf{E}_i = \frac{3\epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} E_0 \mathbf{e}_z \ .$$

Inside the sphere the resulting electric field is parallel to the external field $\mathbf{E}_0$ in z-direction. According to (2.196) the polarization of the sphere is given by:

$$\mathbf{P} = (\epsilon_r^{(2)} - 1)\epsilon_0 \mathbf{E}_i = \frac{3\epsilon_r^{(1)}(\epsilon_r^{(2)} - 1)}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} \epsilon_0 \mathbf{E}_0 \ .$$

Outside:
Take

$$\mathbf{p} = 4\pi\,\epsilon_0 R^3 E_0 \frac{\epsilon_r^{(2)} - \epsilon_r^{(1)}}{2\epsilon_r^{(1)} + \epsilon_r^{(2)}} \ \mathbf{e}_z \ .$$

Then we get:

$$\varphi_a(\mathbf{r}) = -\mathbf{E}_0 \cdot \mathbf{r} + \frac{1}{4\pi\,\epsilon_0} \underbrace{\frac{\mathbf{p} \cdot \mathbf{r}}{r^3}}_{\text{dipole potential (2.71)}} \ .$$

From that it follows as in (2.73):

$$\mathbf{E}_a = \mathbf{E}_0 + \frac{1}{4\pi\,\epsilon_0} \left[\frac{3(\mathbf{r} \cdot \mathbf{p})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3} \right] \ .$$

Fig. A.28

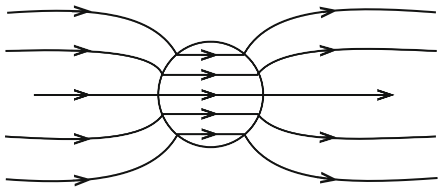
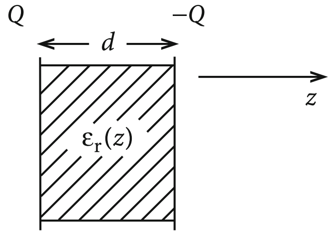


Fig. A.29



The external homogeneous field $\mathbf{E}_0$ is thus superimposed by the field of a dipole $\mathbf{p}$ which is located at the center of the sphere and is oriented in z-direction (Fig. A.28).

Solution 2.4.3 The dielectric displacement $\mathbf{D}$ has only a z-component, determined by the *true, free* excess charges, therefore:

$$D_z = \frac{Q}{F} .$$

Furthermore it holds:

$$D_z = \epsilon_0 \epsilon_r(z) E_z(z) .$$

It follows therewith for the space-dependent electric field:

$$E_z(z) = \frac{Q}{F\,\epsilon_r(z)\epsilon_0} .$$

Potential difference between the plates by integration (Fig. A.29):

$$U = \varphi(z=0) - \varphi(z=d) = \frac{Q}{F\,\epsilon_0} \int_0^d \frac{dz}{\epsilon_r(z)} .$$

Hence, the capacity is:

$$C = \frac{Q}{U} = \frac{F\,\epsilon_0}{\int_0^d \frac{dz}{\epsilon_r(z)}} = \frac{C_0}{\frac{1}{d}\int_0^d \frac{dz}{\epsilon_r(z)}} ,$$

$C_0 = \epsilon_0 F/d$: capacity of the plane-parallel capacitor in the vacuum.

Special case: Dielectric material consisting of two layers with thicknesses d_1, d_2 and permittivities $\epsilon_r^{(1)}$, $\epsilon_r^{(2)}$:

$$\int_0^d \frac{dz}{\epsilon_r(z)} = \frac{d_1}{\epsilon_r^{(1)}} + \frac{d_2}{\epsilon_r^{(2)}} = \frac{d_1\epsilon_r^{(2)} + d_2\epsilon_r^{(1)}}{\epsilon_r^{(1)}\epsilon_r^{(2)}} .$$

From that we get the capacity:

$$C = \frac{\epsilon_0\epsilon_r^{(1)}\epsilon_r^{(2)}F}{d_1\epsilon_r^{(2)} + d_2\,\epsilon_r^{(1)}} = C_0\frac{\epsilon_r^{(1)}\epsilon_r^{(2)}d}{d_1\,\epsilon_r^{(2)} + d_2\,\epsilon_r^{(1)}} .$$

Solution 2.4.4

1.

$$D_{\mathrm{I}} = \epsilon_r\epsilon_0 E_{\mathrm{I}} ; \qquad D_{\mathrm{II}} = \epsilon_0 E_{\mathrm{II}} ;$$

$$\mathbf{D}_{\mathrm{I,II}} = D_{\mathrm{I,II}}\ \mathbf{e}_z ; \qquad \mathbf{E}_{\mathrm{I,II}} = E_{\mathrm{I,II}}\ \mathbf{e}_z .$$

2.

$$E_{\mathrm{I}} = E_{\mathrm{II}} = E ,$$

because of curl$\mathbf{E} = 0$ the tangential component does **not** change,

$$D_{\mathrm{I}} = \epsilon_r D_{\mathrm{II}} .$$

3.

$$D_{\mathrm{I}} = \sigma_{\mathrm{I}} ; \quad D_{\mathrm{II}} = \sigma_{\mathrm{II}} \quad \text{because of } \operatorname{div}\mathbf{D} = \rho .$$

4.

$$Q = \sigma_{\mathrm{I}}F_{\mathrm{I}} + \sigma_{\mathrm{II}}F_{\mathrm{II}} = D_{\mathrm{I}}F_{\mathrm{I}} + D_{\mathrm{II}}F_{\mathrm{II}} = \epsilon_0 E(\epsilon_r F_{\mathrm{I}} + F_{\mathrm{II}})$$
$$= \epsilon_0 E\, b\,[\epsilon_r x + (a - x)] = \epsilon_0 E\, b\,[a + (\epsilon_r - 1)x] .$$

It follows therewith:

$$\mathbf{E} = \frac{Q}{\epsilon_0 b\,[a + (\epsilon_r - 1)x]}\,\mathbf{e}_z ,$$

$$\mathbf{D}_{\mathrm{I}} = \epsilon_r\epsilon_0\mathbf{E} ; \quad \mathbf{D}_{\mathrm{II}} = \epsilon_0\mathbf{E} .$$

5.

$$
\begin{aligned}
W &= \frac{1}{2}\int d^3r\,\mathbf{E}\cdot\mathbf{D} = \frac{1}{2}\left(E\,D_{\mathrm{I}}F_{\mathrm{I}}d + E\,D_{\mathrm{II}}F_{\mathrm{II}}d\right)\\
&= \frac{1}{2}E\,d\,\epsilon_0 E\left(\epsilon_{\mathrm{r}}F_{\mathrm{I}} + F_{\mathrm{II}}\right) = \frac{1}{2}\epsilon_0 E^2 d\,b\left[a + (\epsilon_{\mathrm{r}}-1)x\right]\\
&= \frac{\epsilon_0}{2}\,\frac{Q^2 d\,b}{\epsilon_0^2 b^2\left[a + (\epsilon_{\mathrm{r}}-1)x\right]}\ .
\end{aligned}
$$

This yields:

$$
W = \frac{1}{2}\,\frac{d\,Q^2}{\epsilon_0 b\left(a + (\epsilon_{\mathrm{r}}-1)x\right)}\ .
$$

6.

$$
\begin{aligned}
\mathbf{F} &= F\,\mathbf{e}_x\ ,\\
F &= -\frac{dW}{dx} = \frac{1}{2}\,\frac{d\,Q^2(\epsilon_{\mathrm{r}}-1)}{\epsilon_0 b\left[a + (\epsilon_{\mathrm{r}}-1)x\right]^2} \geq 0\ .
\end{aligned}
$$

The dielectric is pulled into the capacitor!

Solution 2.4.5

1. Macroscopic potential (formula after (2.185)):

$$
4\pi\varepsilon_0\,\varphi(\mathbf{r}) = \int d^3r'\left\{\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}')\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right\}\ .
$$

Excess charge density:

$$
\rho(\mathbf{r}') \equiv 0\ .
$$

It is left:

$$
4\pi\varepsilon_0\,\varphi(\mathbf{r}) = \mathbf{P}_0\cdot\int\limits_{r'\leq R} d^3r'\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\mathbf{P}_0\cdot\nabla_r\int\limits_{r'\leq R} d^3r'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\ .
$$

For the calculation of the integral we choose the direction of $\mathbf{r}$ as polar axis:

$$
\int\limits_{r'\leq R} d^3r'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 2\pi\int\limits_0^R dr'\,r'^2\int\limits_{-1}^{+1} d\cos\vartheta\;\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\vartheta}}
$$

$$
\begin{aligned}
&= -\frac{2\pi}{r} \int\limits_0^R dr'\, r' \; \sqrt{r^2 + r'^2 - 2rr' \cos\vartheta}\,\Big|_{-1}^{+1} \\
&= \frac{2\pi}{r} \int\limits_0^R dr'\, r' \; \left(\left|r + r'\right| - \left|r - r'\right|\right) \\
&= \frac{2\pi}{r} \begin{cases} 2\int\limits_0^r dr' r'^2 + 2\int\limits_r^R dr'\, r' r & (r < R) \\ 2\int\limits_0^R dr'\, r'^2 & (r > R) \end{cases} \\
&= \frac{4\pi}{r} \begin{cases} \frac{r^3}{3} + \frac{r}{2}\left(R^2 - r^2\right) & (r < R) \\ \frac{1}{3}R^3 & (r > R) \end{cases} \\
&= 4\pi \begin{cases} \frac{1}{2}R^2 - \frac{1}{6}r^2 & (r < R) \\ \frac{1}{3r}R^3 & (r > R)\;. \end{cases}
\end{aligned}
$$

Thus it holds

$$
4\pi\varepsilon_0\,\varphi(\mathbf{r}) = -\mathbf{P}_0 \cdot \nabla_r\, 4\pi \begin{cases} \frac{1}{2}R^2 - \frac{1}{6}r^2 & (r < R) \\ \frac{1}{3r}R^3 & (r > R)\;. \end{cases}
$$

It follows therewith the potential in the inside and the outside space:

$$
\begin{aligned}
\varphi(\mathbf{r}) &= \frac{1}{\varepsilon_0}\mathbf{P}_0 \cdot \mathbf{e}_r \begin{cases} \frac{1}{3}r & (r < R) \\ \frac{1}{3r^2}R^3 & (r > R) \end{cases} \\
&= \frac{1}{3\varepsilon_0} \begin{cases} \mathbf{P}_0 \cdot \mathbf{r} & (r < R) \\ \mathbf{P}_0 \cdot \mathbf{r}\, \frac{R^3}{r^3} & (r > R)\;. \end{cases}
\end{aligned}
$$

Dipole moment of the sphere:

$$
\mathbf{p} = \frac{4\pi}{3} R^3\, \mathbf{P}_0\;.
$$

Finally we can write therewith:

$$
\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \begin{cases} \frac{4\pi}{3}\mathbf{P}_0 \cdot \mathbf{r} & (r < R) \\ \frac{1}{r^3}\,\mathbf{p} \cdot \mathbf{r} & (r > R)\;. \end{cases}
$$

We see that outside the sphere the normal dipole potential (2.96) appears.

2. Electric field strength:

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) \ .$$

It follows with the scalar potential from part 1.:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{1}{3\varepsilon_0} P_0 \mathbf{e}_z & (r < R) \\[2ex] \dfrac{3\mathbf{e}_r(\mathbf{e}_r \cdot \mathbf{p}) - \mathbf{p}}{4\pi\varepsilon_0 r^3} & (r > R) \ . \end{cases}$$

In the inside of the sphere the field lines start at the positive polarization charges on the surface of the upper hemisphere and go straight-lined to the corresponding negative polarization charges on the lower hemisphere (Fig. A.30). In the outside region it is a pure dipole field.

3. Polarization charge density:

$$\rho_\mathrm{P}(\mathbf{r}) = -\mathrm{div}\,\mathbf{P} \ .$$

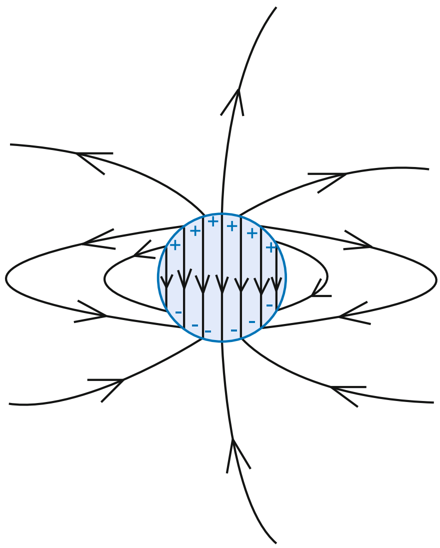
Thus it is

$$\rho_\mathrm{P}(\mathbf{r}) \equiv 0 \quad \text{for} \quad r > R \quad \text{and} \quad r < R \ .$$

This means:

$$\rho_\mathrm{P}(\mathbf{r}) = \sigma_\mathrm{P}(\mathbf{r})\,\delta(r - R) \ .$$

Fig. A.30



σ_{P} is the surface charge density. As in Fig. 2.12 we position around the surface of the sphere a *'Gauss-casket'* and let the width Δx of the front surface approach zero. Then it must hold:

$$\int_{\Delta V} d^3r \mathrm{div}\,\mathbf{P} = \int_{S(\Delta V)} d\mathbf{f}\cdot\mathbf{P} \quad\longrightarrow\quad \Delta F\,\mathbf{n}\cdot(\mathbf{P}_{\mathrm{a}}-\mathbf{P}_{\mathrm{i}}) \ .$$

$\mathbf{n} = \mathbf{e}_r$ is the normal-unit vector on the surface of the sphere. The indexes a and i mark, respectively, the outside space and the inside space of the sphere. In addition we get:

$$\int_{\Delta V} d^3r \mathrm{div}\,\mathbf{P} = -\int_{\Delta V} d^3r\,\rho_{\mathrm{P}}(\mathbf{r}) = -\Delta F\,\sigma_{\mathrm{P}}(\mathbf{r}) \ .$$

The comparison leads to:

$$\mathbf{n}\cdot(\mathbf{P}_{\mathrm{a}}-\mathbf{P}_{\mathrm{i}}) = -\sigma_{\mathrm{P}}(\mathbf{r}) \, .$$

Furthermore, it holds in the present special case:

$$\mathbf{P}_{\mathrm{a}} \equiv 0\,; \ \ \mathbf{P}_{\mathrm{i}} = \mathbf{P}_0 \ .$$

It results as polarization charge density:

$$\rho_{\mathrm{P}}(\mathbf{r}) = \sigma_{\mathrm{P}}(\mathbf{r})\,\delta(r-R) = \mathbf{P}_0\cdot\mathbf{e}_r\,\delta(r-R) = P_0\cos\vartheta\,\delta(r-R) \ .$$

As already mentioned above, positive charges appear on the upper spherical interface and negative charges on the lower spherical interface.

4. As a realization of the above discussed model situation one could imagine two spheres, homogeneously charged with $+q$ and $-q$, the centers of which are shifted against each other by $\mathbf{a}$ where $a \ll R$. q and a are thereby to be chosen so that:

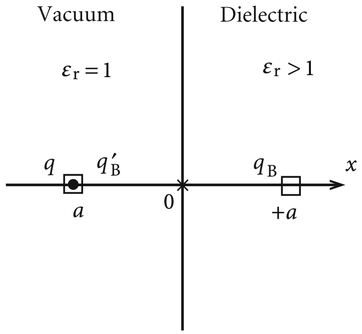
$$p = qa = \left(\frac{4\pi}{3}R^3\right)\,P_0 \, .$$

Solution 2.4.6 For the solution in the left half-space ($x < 0$) we introduce in the right half-space at (Fig. A.31)

$$\hat{\mathbf{r}}_0 = +a\,\mathbf{e}_x$$

an image charge q_{B} which does not *'disturb'* the corresponding Poisson equation of the left half-space. For the solution in the right half-space($x > 0$) we place an image

Fig. A.31



charge q'_B at

$$\mathbf{r}_0 = -a\,\mathbf{e}_x$$

which replaces there the real charge q. The system of point charges leads then to the following potentials:

$$4\pi\varepsilon_0\,\varphi(\mathbf{r}) = \begin{cases} q\,\dfrac{1}{|\mathbf{r}+a\mathbf{e}_x|} + q_B\,\dfrac{1}{|\mathbf{r}-a\mathbf{e}_x|} & x<0 \\[2ex] q'_B\,\dfrac{1}{|\mathbf{r}+a\mathbf{e}_x|} & x>0\,. \end{cases}$$

That gives us the ansatz for the electrostatic field:

$$4\pi\varepsilon_0\,\mathbf{E}(\mathbf{r}) = \begin{cases} q\,\dfrac{\mathbf{r}+a\mathbf{e}_x}{|\mathbf{r}+a\mathbf{e}_x|^3} + q_B\,\dfrac{\mathbf{r}-a\mathbf{e}_x}{|\mathbf{r}-a\mathbf{e}_x|^3} & x<0 \\[2ex] q'_B\,\dfrac{\mathbf{r}+a\mathbf{e}_x}{|\mathbf{r}+a\mathbf{e}_x|^3} & x>0\,. \end{cases}$$

We determine the image charges by the continuity conditions for the electrostatic field at the

$$\text{interface:}\quad \{\mathbf{r}(x=0,y,z); -\infty \le y,z \le +\infty\}\,.$$

Tangential component (2.213):

$$\mathbf{E}_{lt}(x=0) = \mathbf{E}_{rt}(x=0)\,.$$

Normal component ((2.213), uncharged interface):

$$\mathbf{E}_{ln}(x=0) = \frac{\varepsilon_r}{1}\,\mathbf{E}_{rn}(x=0)\,.$$

For points in the interface it is:

$$|\mathbf{r} + a\,\mathbf{e}_x| = |\mathbf{r} - a\,\mathbf{e}_x| = \sqrt{a^2 + y^2 + z^2} = b(y, z)\ .$$

Normal component at the interface:

$$\mathbf{E} \cdot \mathbf{e}_x|_{x=0}\ .$$

Because of $\mathbf{e}_x \cdot \mathbf{r}|_{x=0} = 0$ it must hold:

$$q\,\frac{a}{b^3} + q_\mathrm{B}\,\frac{-a}{b^3} \stackrel{!}{=} \varepsilon_\mathrm{r} q'_\mathrm{B}\,\frac{a}{b^3}\ .$$

That yields:

$$q - q_\mathrm{B} = \varepsilon_\mathrm{r}\, q'_\mathrm{B}\ . \tag{A.3}$$

Tangential component at the interface:

$$\mathbf{E} \cdot \mathbf{e}_z|_{x=0}\ .$$

Now it must be:

$$q\,\frac{z}{b^3} + q_\mathrm{B}\,\frac{z}{b^3} \stackrel{!}{=} q'_\mathrm{B}\,\frac{z}{b^3}\ .$$

That yields:

$$q + q_\mathrm{B} = q'_\mathrm{B}\ . \tag{A.4}$$

Combination of (A.3) and (A.4):

$$q'_\mathrm{B} = q\,\frac{2}{\varepsilon_\mathrm{r} + 1}$$

$$q_\mathrm{B} = q\,\frac{1 - \varepsilon_\mathrm{r}}{1 + \varepsilon_\mathrm{r}}\ .$$

The electrostatic field is therewith completely determined.

Polarization (2.196):

$$\mathbf{P} = \chi_\mathrm{e}\varepsilon_0\,\mathbf{E}\ ; \quad \chi_\mathrm{e} = \varepsilon_\mathrm{r} - 1\ .$$

Vacuum: $\varepsilon_r = 1$. Hence it remains:

$$\mathbf{P}(\mathbf{r}) = \begin{cases} 0 & x < 0 \\ \dfrac{q}{2\pi}\,\dfrac{\varepsilon_r - 1}{\varepsilon_r + 1}\,\dfrac{\mathbf{r} + a\mathbf{e}_x}{|\mathbf{r} + a\mathbf{e}_x|^3} & x > 0\,. \end{cases}$$

The vacuum is of course not polarizable.
Polarization charge density (2.189):

$$\operatorname{div}\mathbf{P}(\mathbf{r}) = -\rho_P(\mathbf{r})\,.$$

We have in our case for all $\mathbf{r}$ in the *right* half-space:

$$\begin{aligned} \operatorname{div}\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} &= -\operatorname{div}\cdot\operatorname{grad}\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \\ &= 4\pi\,\delta\,(\mathbf{r} - \mathbf{r}_0) = 0\,. \end{aligned}$$

This is correct since $\mathbf{r}_0$ lies in the *left* half-space. Furthermore we used (1.69). It follows therewith for all $\mathbf{r}$ with $x < 0$ and $x > 0$:

$$\rho_P(\mathbf{r}) \equiv 0\,.$$

Induced charges can exist only in the interface:

$$\rho_P(\mathbf{r}) = \sigma_P\,\delta(x)\,.$$

Surface charge density (2.193):

$$\sigma_P = -\,\mathbf{e}_x \cdot \mathbf{P}|_{x=0}\,.$$

That means here:

$$\sigma_P = -\frac{q}{2\pi}\,\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\,\frac{a}{(a^2 + y^2 + z^2)^{\frac{3}{2}}}\,.$$

For $\varepsilon_r = 1$ ($\curvearrowright$ $q_B = 0, q'_B = q$) the right half-space is also vacuum. Polarization charges are impossible. For $\varepsilon_r \longrightarrow \infty$ ($\curvearrowright$ $q'_B = 0, q_B = -q$) it results the known problem of a point charge in front of a metallic plate (2.128).

Section 3.2.4

Solution 3.2.1 We place around the conductor a (fictitious) cylinder with the radius $\rho > R$. $\Longrightarrow$ cylindrical coordinates ρ, φ, z surely appropriate, therefore the following ansatz:

$$\mathbf{B} = B_\rho(\rho, \varphi, z)\mathbf{e}_\rho + B_\varphi(\rho, \varphi, z)\mathbf{e}_\varphi + B_z(\rho, \varphi, z)\mathbf{e}_z \ .$$

We exploit the symmetry (Fig. A.32):

$$\begin{aligned} \text{infinitely long wire} &\Longrightarrow B_z \equiv 0, \ \mathbf{B} \neq \mathbf{B}(z) \ , \\ \text{rotational symmetry} &\Longrightarrow \mathbf{B} \neq \mathbf{B}(\varphi) \ . \end{aligned}$$

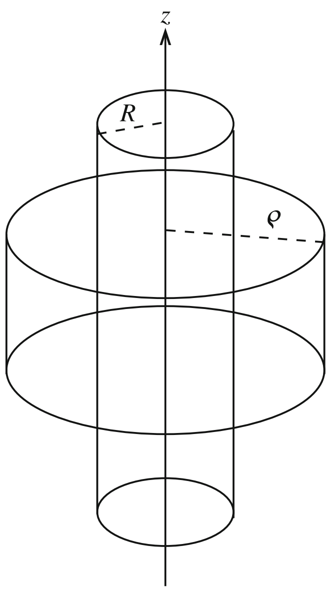
$\Longrightarrow$ new ansatz:

$$\mathbf{B} = B_\rho(\rho)\mathbf{e}_\rho + B_\varphi(\rho)\mathbf{e}_\varphi \ .$$

Z_ρ: cylinder of the length L, radius ρ, conducting wire as axis.
Maxwell equation:

$$\int\limits_{S(Z_\rho)} \mathbf{B} \cdot d\mathbf{f} \stackrel{!}{=} 0 \ .$$

Fig. A.32



Front surface: $d\mathbf{f} \sim \mathbf{e}_z \implies$ no contribution.
Barrel: $d\mathbf{f} \sim \mathbf{e}_\rho$.
$\implies \ 0 = \int_{S(Z_\rho)} \mathbf{B} \cdot d\mathbf{f} = B_\rho(\rho) 2\pi\rho L$
$\implies \ B_\rho(\rho) = 0$
$\implies \ \mathbf{B} = B_\varphi(\rho)\mathbf{e}_\varphi$.
Maxwell equation (F_ρ: front side of the fictitious cylinder):

$$\int\limits_{\partial F_\rho} d\mathbf{r} \cdot \mathbf{B} = \mu_0 \int\limits_{F_\rho} d\mathbf{f} \cdot \mathbf{j} = \mu_0 I \begin{cases} 1 & \text{if } \rho > R\,, \\ \dfrac{\rho^2}{R^2} & \text{if } \rho \leq R\,, \end{cases}$$

$$d\mathbf{r} = d\rho \mathbf{e}_\rho + \rho d\varphi \mathbf{e}_\varphi + dz \mathbf{e}_z \implies d\mathbf{r} \cdot \mathbf{B} = \rho d\varphi B_\varphi(\rho)$$

$$\implies \int\limits_{\partial F} d\mathbf{r} \cdot \mathbf{B} = \rho B_\varphi(\rho) \int\limits_0^{2\pi} d\varphi = 2\pi\rho B_\varphi(\rho)\,,$$

$$\implies B_\varphi(\rho) = \frac{\mu_0 I}{2\pi} \begin{cases} \dfrac{1}{\rho} & \text{if } \rho > R\,, \\ \dfrac{\rho}{R^2} & \text{if } \rho \leq R\,. \end{cases}$$

Solution 3.2.2

$$\mathbf{F} = \oint\limits_C (\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})) d^3r = I \oint\limits_C d\mathbf{r} \times \mathbf{B}(\mathbf{r})\,.$$

$\mathbf{B}(\mathbf{r})$ is effected by $\mathbf{j}$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint\limits_C d^3r' \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mu_0}{4\pi} I \oint\limits_C d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\,.$$

'Self-force':

$$\mathbf{F} = \frac{\mu_0}{4\pi} I^2 \oint\limits_C \oint\limits_C d\mathbf{r} \times \left(d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

$$= \frac{\mu_0}{4\pi} I^2 \oint\limits_C \oint\limits_C \left(d\mathbf{r}' \left(d\mathbf{r} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} (d\mathbf{r}' \cdot d\mathbf{r}) \right)\,.$$

For the first summand we get:

$$\oint_C d\mathbf{r} \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\oint_C d\mathbf{r} \cdot \nabla_r \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$\overset{\text{Stokes}}{=} -\int_{F_c} d\mathbf{f} \cdot \underbrace{\operatorname{curl}\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{0} = 0 \; .$$

It remains:

$$\mathbf{F} = -\frac{\mu_0}{4\pi} I^2 \oint_C \oint_C d\mathbf{r} \cdot d\mathbf{r}' \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \; .$$

When interchanging the variables $\mathbf{r}$ and $\mathbf{r}'$ the force $\mathbf{F}$ turns to $-\mathbf{F}$. That is possible only if

$$\mathbf{F} = 0 \; .$$

Conclusion:
The current-loop does not exert any force on itself!

Solution 3.2.3

$$\mathbf{j} = j(r,\vartheta)\mathbf{e}_\varphi$$

1.

$$\begin{aligned} \mathbf{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0) \; , \\ \mathbf{j} &= j(r,\vartheta)(-\sin\varphi, \cos\varphi, 0) \\ \Longrightarrow \; j_x + i j_y &= j(-\sin\varphi + i\cos\varphi) = i j(r,\vartheta)e^{i\varphi} \; . \end{aligned}$$

From

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

it follows

$$\begin{aligned}
A_z(\mathbf{r}) &= 0 \; , \\
A_x + iA_y &= \frac{\mu_0}{4\pi} \int d^3r' \, \frac{j_x + ij_y}{|\mathbf{r} - \mathbf{r}'|} \\
&= i\frac{\mu_0}{4\pi} \int d^3r' \, \frac{j(r', \vartheta')}{|\mathbf{r} - \mathbf{r}'|} e^{i\varphi'} \; .
\end{aligned}$$

According to (2.169),

$$\begin{aligned}
\frac{1}{|\mathbf{r} - \mathbf{r}'|} &= 4\pi \sum_{m,l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\vartheta', \varphi') Y_{lm}(\vartheta, \varphi) \; , \\
r_> &= \max(r, r') \; , \\
r_< &= \min(r, r') \; , \\
Y_{lm}(\vartheta, \varphi) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\vartheta) e^{im\varphi} \; .
\end{aligned}$$

Insertion into $A_x + iA_y \implies \varphi'$-integration lets survive only the $m = 1$-term:

$$A_x + iA_y = iA(r, \vartheta) e^{i\varphi} \; .$$

$e^{i\varphi}$ stems from $Y_{l1}(\vartheta, \varphi)$:

$$A(r, \vartheta) = \frac{\mu_0}{4\pi} \int d^3r' \, j(r', \vartheta') \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{(l-1)!}{(l+1)!} P_l^1(\cos\vartheta') \, P_l^1(\cos\vartheta) \; .$$

With this $A(r, \vartheta)$ it holds for the vector potential:

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= A(r, \vartheta) \mathbf{e}_\varphi = (Ax, Ay, Az) \\
&= (-A(r, \vartheta) \sin\varphi, A(r, \vartheta) \cos\varphi, 0) \; .
\end{aligned}$$

2.

$$\Delta \, \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{j}(\mathbf{r}) \qquad (*)$$

valid component by component:

$$\Delta(A_x + iA_y) = -\mu_0 (j_x + ij_y) \; ,$$

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{r^2 \sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \; .$$

performing the φ-differentiation $\implies$

$$\left(\Delta - \frac{1}{r^2 \sin^2 \vartheta}\right) A(r, \vartheta) = -\mu_0 j(r, \vartheta) \; .$$

Notice, to divide $(*)$ simply by $\mathbf{e}_\varphi$ is not allowed, leads to the wrong result!

Solution 3.2.4 Cylindrical coordinates ρ, φ, z obviously useful. Because of

$$\mathbf{j} = j\mathbf{e}_z$$

we find:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = A(\rho, \varphi, z)\mathbf{e}_z \; .$$

Due to symmetry reasons there cannot be neither a φ- nor a z-dependence:

$$\mathbf{A}(\mathbf{r}) = A(\rho)\mathbf{e}_z$$

Nabla operator:

$$\begin{aligned} \nabla &= \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z} \; , \\ \implies \mathbf{B} &= \mathrm{curl}\mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \sim \mathbf{e}_\rho \times \mathbf{e}_z = -\mathbf{e}_\varphi \\ \implies \mathbf{B} &= B(\rho)\mathbf{e}_\varphi \; . \end{aligned}$$

Circle with the radius ρ around the cylinder axis:

$$d\mathbf{r} = \rho d\varphi \mathbf{e}_\varphi \; ; \quad d\mathbf{f} = \rho d\rho d\varphi \mathbf{e}_z \; ,$$

$$\begin{aligned} \implies \int\limits_{\partial F_\rho} d\mathbf{r} \cdot \mathbf{B} &= 2\pi\rho B(\rho) \\ &= \int\limits_{F_\rho} d\mathbf{f} \cdot \mathrm{curl}\mathbf{B} = \int_{F_\rho} d\mathbf{f} \cdot \mu_0 \mathbf{j} \\ &= \mu_0 I(F_\rho) \; , \end{aligned}$$

$$I(F_\rho) = \begin{cases} 0 & \text{if } \rho \le R_1 \; , \\ I \dfrac{\rho^2 - R_1^2}{R_2^2 - R_1^2} & \text{if } R_1 \le \rho \le R_2 \; , \\ I & \text{if } \rho \ge R_2 \; , \end{cases}$$

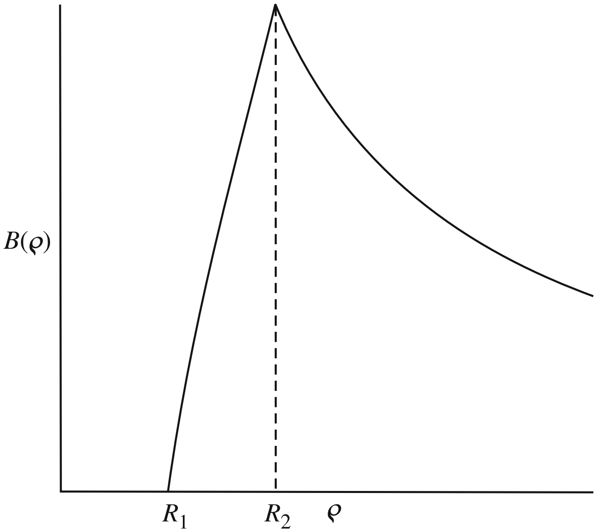


Fig. A.33

$$\Longrightarrow \quad B(\rho) = \frac{\mu_0 I}{2\pi} \begin{cases} 0 & \text{if } \rho \leq R_1 \ , \\ \dfrac{\rho - \frac{R_1^2}{\rho}}{R_2^2 - R_1^2} & \text{if } R_1 \leq \rho \leq R_2 \ , \\ \dfrac{1}{\rho} & \text{if } \rho \geq R_2 \ . \end{cases}$$

Solution 3.2.5 The same symmetry considerations as in Exercise 3.2.1 lead to the ansatz (Fig. A.33):

$$\mathbf{B} = B_\varphi(\rho)\mathbf{e}_\varphi \ .$$

F_ρ: Front surface (radius ρ) of a coaxial cylinder.

Maxwell equation:

$$\int\limits_{\partial F_\rho} d\mathbf{r} \cdot \mathbf{B} = 2\pi\rho B_\varphi(\rho) \stackrel{!}{=} \mu_0 \int\limits_{F_\rho} d\mathbf{f} \cdot \mathbf{j} \ .$$

(Ampère's law)

$$\int_{F_\rho} d\mathbf{f}\cdot\mathbf{j} = \begin{cases} I_1 \dfrac{\rho^2}{\rho_1^2}\ ; & 0 \le \rho \le \rho_1\ , \\ I_1\ ; & \rho_1 \le \rho \le \rho_2\ , \\ I_1 + I_2 \dfrac{\rho^2-\rho_2^2}{\rho_3^2-\rho_2^2}\ ; & \rho_2 \le \rho \le \rho_3\ , \\ I_1 + I_2\ ; & \rho_3 \le \rho\ . \end{cases}$$

The **B**-field is therewith already determined:

$$B_\varphi(\rho) = \frac{\mu_0}{2\pi} \begin{cases} \dfrac{I_1}{\rho_1^2}\rho\ ; & 0 \le \rho \le \rho_1\ , \\ I_1 \dfrac{1}{\rho}\ ; & \rho_1 \le \rho \le \rho_2\ , \\ \left(I_1 + I_2 \dfrac{\rho^2-\rho_2^2}{\rho_3^2-\rho_2^2}\right)\dfrac{1}{\rho}\ ; & \rho_2 \le \rho \le \rho_3\ , \\ (I_1 + I_2)\dfrac{1}{\rho}\ ; & \rho_3 \le \rho\ . \end{cases}$$

Special case: $I_1 = -I_2 = I$ (Fig. A.34)

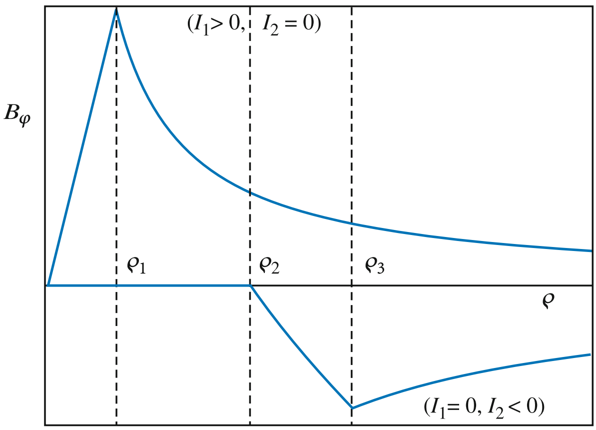


Fig. A.34

$$B_\varphi(\rho) = \frac{\mu_0 I}{2\pi} \begin{cases} \dfrac{1}{\rho_1^2}\rho\ ; & 0 \le \rho \le \rho_1\ , \\ \dfrac{1}{\rho}\ ; & \rho_1 \le \rho \le \rho_2 \\ \dfrac{\rho_3^2 - \rho^2}{\rho_3^2 - \rho_2^2}\dfrac{1}{\rho}\ ; & \rho_2 \le \rho \le \rho_3\ , \\ 0\ ; & \rho_3 \le \rho\ . \end{cases}$$

Solution 3.2.6

1. Vector potential (Fig. A.35):

$$\mathbf{A} = \frac{1}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ .$$

The symmetry suggests cylindrical coordinates (ρ, φ, z). Because of $\mathbf{j} = j\,\mathbf{e}_z$ and again from symmetry reasons the vector potential cannot exhibit any z- and φ-dependence:

$$\mathbf{A} = A(\rho)\,\mathbf{e}_z\ .$$

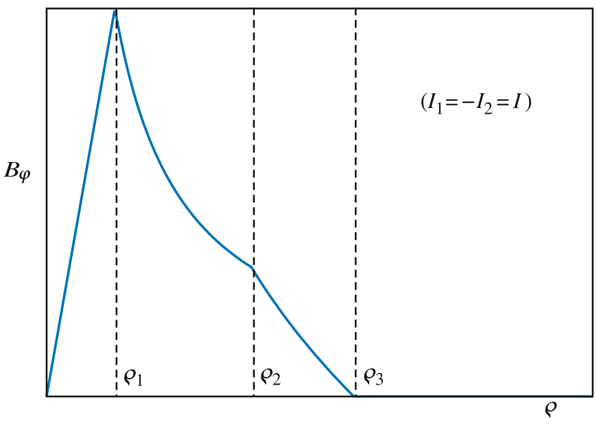


Fig. A.35

It is to be calculated:

$$\begin{aligned}
\mathbf{B} &= \mathrm{curl}\mathbf{A}\ . \\
\mathrm{curl}\mathbf{A} &= \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y, \frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z, \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) \\
&= \left(\frac{\partial}{\partial y}A_z, -\frac{\partial}{\partial x}A_z, 0\right) \\
&= \frac{\partial}{\partial \rho}A(\rho)\left(\frac{y}{\rho}, -\frac{x}{\rho}, 0\right) \\
&= \frac{\partial}{\partial \rho}A(\rho)\,(\sin\varphi, -\cos\varphi, 0) \\
&= \frac{\partial}{\partial \rho}A(\rho)\,\mathbf{e}_\varphi\ .
\end{aligned}$$

Thus it must be valid:

$$\mathbf{B} = B(\rho)\,\mathbf{e}_\varphi\ .$$

Let C be a concentric circle around the conductor with the radius ρ (see Fig. A.36). Then it follows with the Stokes theorem and the Maxwell equation (3.31):

$$\begin{aligned}
\int_{F_C} \mathrm{curl}\mathbf{B}\cdot d\mathbf{f} &= \int_C \mathbf{B}\cdot d\mathbf{r} = B(\rho)\,2\pi\rho \\
&= \mu_0 \int_{F_C} \mathbf{j}\cdot d\mathbf{f} = \mu_0 j \begin{cases} \pi\rho^2\colon\ 0 \le \rho \le R \\ \pi R^2\colon\ R \le \rho\ . \end{cases}
\end{aligned}$$

Fig. A.36

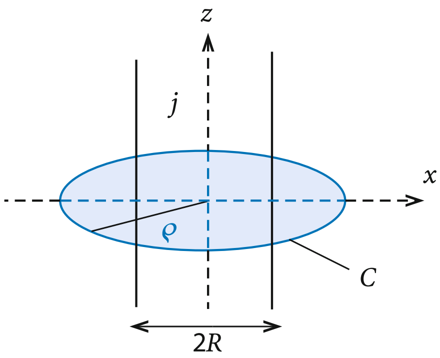
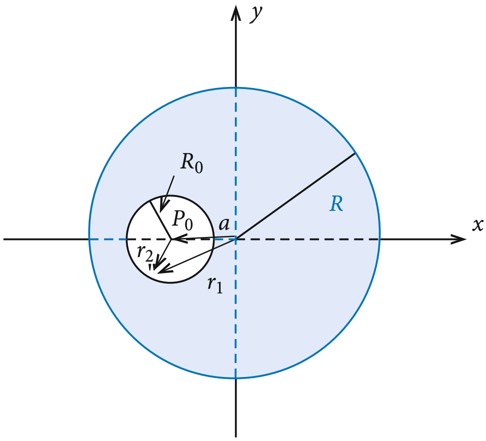


Fig. A.37



That leads eventually to:

$$\mathbf{B} = \mathbf{e}_\varphi \, \frac{1}{2}\mu_0 j \begin{cases} \rho: & 0 \leq \rho \leq R \\ \frac{R^2}{\rho}: & R \leq \rho \ . \end{cases}$$

2. We can presume (see Fig. A.37):

$$\mathbf{a} = (-a, 0, 0) \ .$$

We conclude from the linearity of the Maxwell equations that the field inside the drill hole is composed by the field of the total cylinder through which flows homogeneously $\mathbf{j}$ (see part 1.) and the drilled cylinder of radius R_0 through which flows $-\mathbf{j}$ homogeneously:

$$\mathbf{B} = B_1(\rho_1)\,\mathbf{e}_\varphi^{(1)} + B_2(\rho_2)\,\mathbf{e}_\varphi^{(2)} \ .$$

$\mathbf{e}_\varphi^{(2)}$, ρ_2 refer to the origin P_0 of the cylinder drill hole:

$$\mathbf{e}_\varphi^{(1)} = \frac{1}{\rho_1}\,(-y_1, x_1, 0)$$

$$\mathbf{e}_\varphi^{(2)} = \frac{1}{\rho_2}\,(-y_2, x_2, 0) = \frac{1}{\rho_2}\,(-y_1, x_1 + a, 0) \ .$$

This yields the field inside the cylinder-drill hole ($\rho_2 \leq R_0$; $\rho_1 \leq R$):

$$\begin{aligned}\mathbf{B} &= \frac{1}{2}\mu_0 j\,(-y_1, x_1, 0) - \frac{1}{2}\mu_0 j\,(-y_1, x_1 + a, 0) \\ &= \frac{1}{2}\mu_0 j\,(0, -a, 0) = -\frac{1}{2}\mu_0 j a\,\mathbf{e}_y\ .\end{aligned}$$

Hence, the **B**-field of the cylinder-drill hole is homogeneous and oriented in ($\mathbf{j} \times \mathbf{a}$)-direction:

$$\mathbf{B} = \frac{1}{2}\mu_0\,(\mathbf{j} \times \mathbf{a})\ .$$

Section 3.3.3

Solution 3.3.1 Because of the special form of the current density

$$\mathbf{j}(\mathbf{r}) = I\delta(\rho - R)\delta(z)\mathbf{e}_\varphi$$

from symmetry reasons the vector potential will have the structure

$$\mathbf{A}(\mathbf{r}) = A(\rho, z)\mathbf{e}_\varphi$$

with the unit-vector:

$$\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$$

The y-component is then:

$$\begin{aligned}A_y &= A(\rho, z)\cos(\varphi) \\ &= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{I\delta(\rho' - R)\delta(z')\cos\varphi'}{|\mathbf{r} - \mathbf{r}'|}\ .\end{aligned}$$

The evaluation for $\varphi = 0$ is sufficient:

$$\begin{aligned}\mathbf{r} &= (x, y, z) = (\rho, 0, z) \\ \mathbf{r}' &= (x', y', z') = (R\cos\varphi', R\sin\varphi', 0)\ ,\end{aligned}$$

$$\begin{aligned}A_y \longrightarrow A(\rho, z) &= \frac{\mu_0}{4\pi}\int\limits_0^\infty \rho' d\rho' \int\limits_0^{2\pi} d\varphi' \int\limits_{-\infty}^{+\infty} dz'\,\frac{I\delta(\rho' - R)\delta(z')\cos\varphi'}{\sqrt{(R\cos\varphi' - \rho)^2 + R^2\sin^2\varphi' + z^2}} \\ &= \frac{\mu_0}{4\pi}IR\int\limits_0^{2\pi} d\varphi'\cos\varphi'\,\frac{1}{\sqrt{R^2 + \rho^2 - 2\rho R\cos\varphi' + z^2}}\ .\end{aligned}$$

Elliptic integral, not solvable elementarily! Therefore we restrict our discussion to two limiting cases:

(a) $\rho \ll R$
In this limit it can be estimated:

$$\frac{1}{\sqrt{R^2+\rho^2-2\rho R\cos\varphi'+z^2}} = \frac{1}{\sqrt{R^2+z^2}}\left(1+\frac{\rho^2-2\rho R\cos\varphi'}{R^2+z^2}\right)^{-1/2}$$
$$\simeq \frac{1}{\sqrt{R^2+z^2}}\left(1-\frac{\rho^2-2\rho R\cos\varphi'}{2(R^2+z^2)}\right) .$$

The first summand does not contribute. For the second summand the relation

$$\int_0^{2\pi} d\varphi' \cos^2\varphi' = \pi$$

leads to

$$A(\rho,z) \simeq \frac{1}{4}\mu_0 I R^2 \frac{\rho}{(R^2+z^2)^{3/2}} .$$

From that we derive in the next step the *components of the magnetic induction*. We need the *curl* in cylindrical coordinates (see Vol. 1, (1.380)):

$$\text{curl}\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\varphi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial z \\ 0 & \rho A(\rho,z) & 0 \end{vmatrix} = \frac{1}{\rho}\left(-\frac{\partial}{\partial z}\rho A(\rho,z), 0, \frac{\partial}{\partial\rho}\rho A(\rho,z)\right) .$$

It follows therewith:

$$B_\rho = \frac{1}{4}\mu_0 I R^2 \frac{3z\rho}{(R^2+z^2)^{\frac{5}{2}}} ,$$
$$B_\varphi = 0 ,$$
$$B_z = \frac{1}{4}\mu_0 I R^2 \frac{2}{(R^2+z^2)^{\frac{3}{2}}} .$$

(b) $\rho \gg R$

$$\frac{1}{\sqrt{R^2+\rho^2-2\rho R\cos\varphi'+z^2}} = \frac{1}{\sqrt{\rho^2+z^2}}\left(1+\frac{R^2-2\rho R\cos\varphi'}{\rho^2+z^2}\right)^{-\frac{1}{2}}$$
$$\simeq \frac{1}{\sqrt{\rho^2+z^2}}\left(1-\frac{R^2}{2(\rho^2+z^2)}\right)+\frac{\rho R\cos\varphi'}{(\rho^2+z^2)^{\frac{3}{2}}} \; .$$

The first summand does not contribute to $A(\rho, z)$ in contrast to the second:

$$A(\rho,z) \approx \frac{1}{4}\mu_0 I R^2 \frac{\rho}{(\rho^2+z^2)^{\frac{3}{2}}} \; .$$

Cylindrical coordinates

$$\begin{aligned} \mathbf{r} &= \rho\mathbf{e}_\rho + z\mathbf{e}_z \\ &\Longrightarrow \; (\rho^2+z^2)^{\frac{3}{2}} = r^3 \; , \\ \mathbf{e}_z\times\mathbf{r} &= \rho\mathbf{e}_z\times\mathbf{e}_\rho = \rho\mathbf{e}_\varphi \; . \end{aligned}$$

We get therewith the vector potential,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \; ,$$

of a magnetic dipole of a circular conductor loop (see (3.46)):

$$\mathbf{m} = \pi R^2 I\mathbf{e}_z \; .$$

From that it results as to (3.45) the magnetic induction in the form well-known for the dipole field:

$$\mathbf{B}(\mathbf{r}) = \mathrm{curl}\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r})}{r^5}-\frac{\mathbf{m}}{r^3}\right) \; .$$

Solution 3.3.2

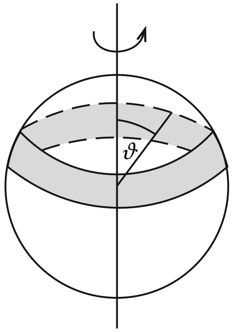
1. Charge density (Fig. A.38):

$$\rho(\mathbf{r}) = \frac{q}{4\pi R^2}\delta(r-R) \; .$$

Current density:

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) = \rho(\mathbf{r})[\boldsymbol{\omega}\times\mathbf{r}] \; .$$

Fig. A.38



Surface:

$$\begin{aligned}
\mathbf{r} &= R\,\mathbf{e}_r = R(\sin\vartheta\ \cos\varphi,\ \sin\vartheta\ \sin\varphi,\ \cos\vartheta)\ ,\\
\boldsymbol{\omega} &= \omega\,\mathbf{e}_z = \omega(0,0,1)\\
\Longrightarrow\quad \mathbf{e}_z\times\mathbf{e}_r &= (-\sin\vartheta\ \sin\varphi,\ \sin\vartheta\ \cos\varphi,\ 0)\\
&= \sin\vartheta(-\sin\varphi,\ \cos\varphi, 0) = \sin\vartheta\ \mathbf{e}_\varphi\ .
\end{aligned}$$

From that we find the current density:

$$\mathbf{j}(\mathbf{r}) = \frac{q\,\omega}{4\pi\,R}\sin\vartheta\ \delta(r-R)\,\mathbf{e}_\varphi\ .$$

2. Magnetic moment
   Definition:

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\int(\mathbf{r}\times\mathbf{j}(\mathbf{r}))\,d^3r\ ,\\
\left(\mathbf{e}_r\times e_\varphi\right) &= (-\cos\vartheta\ \cos\varphi,\ -\cos\vartheta\ \sin\varphi,\ \sin\vartheta) = -\mathbf{e}_\vartheta\\
\Longrightarrow\quad \mathbf{m} &= \frac{q\omega}{8\pi R}\int\limits_0^\infty dr\,r^3\delta(r-R)\int\limits_0^{2\pi}d\varphi\int\limits_{-1}^{+1}d\cos\vartheta\ (-\sin\vartheta\ \mathbf{e}_\vartheta)\\
&= \frac{1}{4}q\,\omega\,R^2\int\limits_{-1}^{+1}\underbrace{d\cos\vartheta(1-\cos^2\vartheta)}_{2-\frac{2}{3}}(0,0,1)\\
\Longrightarrow\quad \mathbf{m} &= \frac{1}{3}q\,\omega\,R^2\mathbf{e}_z = \frac{1}{3}q\,R^2\,\boldsymbol{\omega}\ .
\end{aligned}$$

3. Vector potential
   Definition:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ ,$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\frac{q}{4\pi R^2}\boldsymbol{\omega}\times\int d^3r'\delta(r'-R)\frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\ .$$

Polar axis $\uparrow\uparrow$ $\mathbf{r}$:

$$\mathbf{r} = r(0,0,1)\ ,$$

$$\mathbf{r}' = r'(\sin\vartheta'\cos\varphi',\ \sin\vartheta'\sin\varphi',\ \cos\vartheta')\ .$$

From that it follows:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 q}{8\pi R^2}\boldsymbol{\omega}\times\int\limits_0^\infty dr'r'^3\delta(r'-R)\int\limits_{-1}^{+1}dx\frac{x(0,0,1)}{\sqrt{r^2+r'^2-2rr'x}}$$

$$= \frac{\mu_0 qR}{8\pi}(\boldsymbol{\omega}\times\mathbf{e}_r)\int\limits_{-1}^{+1}dx\frac{x}{\sqrt{r^2+R^2-2rRx}}\ ,$$

$$I = \int\limits_{-1}^{+1}dx\frac{x}{\sqrt{r^2+R^2-2rRx}} = -\frac{1}{rR}x\ \sqrt{r^2+R^2-2rRx}\Big|_{-1}^{+1}$$

$$+\frac{1}{rR}\int\limits_{-1}^{+1}dx\sqrt{r^2+R^2-2rRx}$$

$$= -\frac{1}{rR}\left(|r-R|+|r+R|\right)+\frac{1}{rR}\left(-\frac{2}{3}\,\frac{1}{2rR}\right)\left(r^2+R^2-2rRx\right)^{3/2}\Big|_{-1}^{+1}$$

$$= -\frac{1}{rR}(|r-R|+|r+R|)-\frac{1}{3r^2R^2}\left(|r-R|^3-|r+R|^3\right)\ .$$

$r > R$:

$$I = -\frac{1}{rR}(r-R+r+R)$$

$$-\frac{1}{3r^2R^2}\left(r^3-3r^2R+3rR^2-R^3-r^3-3r^2R-3rR^2-R^3\right)$$

$$= -\frac{2}{R}-\frac{1}{3r^2R^2}\left(-6r^2R-2R^3\right) = +\frac{2R}{3r^2}\ .$$

$r < R$:

$$I = -\frac{2}{r} - \frac{1}{3r^2R^2}\left(-6r\,R^2 - 2r^3\right) = +2\frac{r}{3R^2} \ .$$

It results therewith for the vector potential:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \mu_0 \dfrac{q\,R^2}{12\pi\,r^2}(\boldsymbol{\omega} \times \mathbf{e}_r) \ , \text{ if } r > R \ , \\ \\ \mu_0 \dfrac{q\,r}{12\pi\,R}(\boldsymbol{\omega} \times \mathbf{e}_r) \ , \ \text{ if } r < R \ . \end{cases}$$

In the outside space it thus holds:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\frac{\mathbf{m} \times \mathbf{r}}{r^3} \ .$$

Finally we get:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\frac{3\mathbf{e}_r(\mathbf{e}_r \cdot \mathbf{m}) - \mathbf{m}}{r^3} \ .$$

Solution 3.3.3 Magnetic moment according to (3.43):

$$\mathbf{m} = \frac{1}{2}\int d^3r\,(\mathbf{r} \times \mathbf{j}(\mathbf{r})) \ .$$

Charge density of the hollow sphere:

$$\rho(\mathbf{r}) = \sigma_0 \cos\vartheta\,\delta(r - R) \ .$$

Current density:

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \ .$$

1. Translational motion

$$\mathbf{v} = v\,\mathbf{e}_x \ .$$

Spherical coordinates:

$$\mathbf{r} = r\left(\sin\vartheta\,\cos\varphi\,\mathbf{e}_x + \sin\vartheta\,\sin\varphi\,\mathbf{e}_y + \cos\vartheta\,\mathbf{e}_z\right) \ .$$

It is then:

$$\mathbf{r} \times \mathbf{v} = rv \left(-\sin\vartheta \sin\varphi \, \mathbf{e}_z + \cos\vartheta \, \mathbf{e}_y\right) \ .$$

This we use for the above expression of the magnetic moment:

$$\mathbf{m} = \frac{1}{2}\sigma_0 R^3 v \int\limits_{-1}^{+1} d\cos\vartheta \int\limits_{0}^{2\pi} d\varphi \, \cos\vartheta \left(-\sin\vartheta \sin\varphi \, \mathbf{e}_z + \cos\vartheta \, \mathbf{e}_y\right) \ .$$

The first summand vanishes because of the φ-integration. It remains:

$$\mathbf{m} = \pi\sigma_0 R^3 v \int\limits_{-1}^{+1} d\cos\vartheta \, \cos^2\vartheta \, \mathbf{e}_y \ .$$

The magnetic moment is therewith orthogonal to the symmetry axis of the charge distribution ($\propto \mathbf{e}_z$) as well as to the translational velocity ($\propto \mathbf{e}_x$):

$$\mathbf{m} = \frac{2\pi}{3} R^3 \sigma_0 v \, \mathbf{e}_y \ .$$

2.

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}\big|_{r=R} \ .$$

Current density:

$$\mathbf{j}(\mathbf{r}) = \sigma_0 \cos\vartheta \, \delta(r-R) \, (\boldsymbol{\omega} \times \mathbf{r}) \ .$$

Magnetic moment:

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\sigma_0 \int d^3r \, \cos\vartheta \, \delta(r-R) \, (\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})) \\ &= \frac{1}{2}\sigma_0 \int d^3r \, \cos\vartheta \, \delta(r-R) \left(\boldsymbol{\omega} \, r^2 - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})\right) \ .\end{aligned}$$

- $\boldsymbol{\omega} \propto \mathbf{e}_z$ (parallel to the symmetry axis of the charge)

Then it holds:

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\sigma_0 R^4 \int\limits_{-1}^{+1} d\cos\vartheta \int\limits_0^{2\pi} d\varphi\ \cos\vartheta\,\omega\,(\mathbf{e}_z - \cos\vartheta\,\mathbf{e}_r)\\ &= \pi\sigma_0 R^4\omega\mathbf{e}_z \int\limits_{-1}^{+1} d\cos\vartheta\,\left(\cos\vartheta - \cos^3\vartheta\right)\\ &= 0\ .\end{aligned}$$

Here it was again the φ-integration which let the x- and y-components of $\mathbf{e}_r$ vanish.

- $\boldsymbol{\omega} \propto \mathbf{e}_x$ (orthogonal to the symmetry axis of the charge)
  Then it is to be calculated:

$$\mathbf{m} = \frac{1}{2}\sigma_0 R^4 \int\limits_{-1}^{+1} d\cos\vartheta \int\limits_0^{2\pi} d\varphi\ \cos\vartheta\,\omega\,(\mathbf{e}_x - \mathbf{e}_r(\mathbf{e}_r\cdot\mathbf{e}_x))\ .$$

For the bracket we find:

$$\begin{aligned}\mathbf{e}_x - \mathbf{e}_r(\mathbf{e}_r\cdot\mathbf{e}_x) = &\ \mathbf{e}_x\ \left(1 - \sin^2\vartheta\cos^2\varphi\right)\\ &+ \mathbf{e}_y\ (-\sin\vartheta\sin\varphi\sin\vartheta\cos\varphi)\\ &+ \mathbf{e}_z\,(-\cos\vartheta\sin\vartheta\cos\varphi)\ .\end{aligned}$$

The φ-integration provides now the vanishing of the y- and z-components of the moment. Hence it remains:

$$\mathbf{m} = \frac{1}{2}\sigma_0 R^4\omega\mathbf{e}_x \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\vartheta\,\left(\cos\vartheta - \cos^2\varphi\left(\cos\vartheta - \cos^3\vartheta\right)\right)\ .$$

One recognizes that the integral over $\cos\vartheta$ yields zero. Thus in this case also:

$$\mathbf{m} = 0\ .$$

The same result one finds of course also for the rotation around the y-axis. So the magnetic moment vanishes for rotations around each of the three Cartesian axes, therefore also for rotations around arbitrary axes through the center of the hollow sphere!

Solution 3.3.4

$$\mathbf{m} = \frac{1}{2}\int (\mathbf{r} \times \mathbf{j}(\mathbf{r}))d^3r \ .$$

1. Solid sphere:
   Charge density:

$$\rho(\mathbf{r}) = \frac{Q}{\frac{4\pi}{3}R^3}\Theta(r-R) \ .$$

   Current density:

$$\begin{aligned}
\mathbf{j}(\mathbf{r}) &= \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \\
&= \rho(\mathbf{r})(\boldsymbol{\omega} \times \mathbf{r}) \ , \\
\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega} r^2 - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}) \ ,
\end{aligned}$$

$$\boldsymbol{\omega} \uparrow\uparrow \text{polar axis:} \quad \boldsymbol{\omega} = \omega \mathbf{e}_z \ ,$$

$$\begin{aligned}
\mathbf{m} &= \frac{3Q}{8\pi R^3}\int \Theta(r-R)(\boldsymbol{\omega} r^2 - \mathbf{r} r\omega \cos\vartheta)d^3r \\
&= \frac{3Q}{8\pi R^3}\left(\boldsymbol{\omega}\, 4\pi\int_0^R r^4 dr - \omega\int r^2\Theta(r-R)\cos\vartheta\,(\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, \cos\vartheta)d^3r\right).
\end{aligned}$$

   x- and y-components of the second integral obviously do not contribute. Hence, it remains:

$$\begin{aligned}
\mathbf{m} &= \frac{3Q}{8\pi R^3}\boldsymbol{\omega}\left(4\pi\frac{R^5}{5} - 2\pi\int_0^R r^4 dr \int_{-1}^{1} d\cos\vartheta\cos^2\vartheta\right) \\
&= \frac{3QR^2}{40\pi}\boldsymbol{\omega}\underbrace{\left(4\pi - 2\pi\cdot\frac{2}{3}\right)}_{\frac{2}{3}\cdot 4\pi}
\end{aligned}$$

$$\Longrightarrow \ \mathbf{m} = \frac{1}{5}QR^2\boldsymbol{\omega} \ .$$

2. Inhomogeneously charged hollow sphere:
   As in 1.,

$$
\begin{aligned}
\mathbf{j}(\mathbf{r}) &= \sigma_0 \delta(r-R)\cos^2\vartheta\,(\boldsymbol{\omega}\times\mathbf{r})\ ,\\
\Longrightarrow\quad \mathbf{m} &= \frac{1}{2}\sigma_0\int d^3r\,\delta(r-R)\cos^2\vartheta\,(\boldsymbol{\omega}r^2-\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega}))\\
&= \frac{1}{2}\sigma_0R^4\int\limits_0^{2\pi}d\varphi\int\limits_{-1}^{+1}d\cos\vartheta\,\cos^2\vartheta\,(\boldsymbol{\omega}-\omega\cos\vartheta\,\mathbf{e}_r)\\
&= \frac{1}{2}\sigma_0R^4 2\pi\int\limits_{-1}^{+1}d\cos\vartheta\,\cos^2\vartheta\,(\boldsymbol{\omega}-\omega\cos\vartheta\,(0,0,\cos\vartheta))\\
&= \pi\sigma_0R^4\boldsymbol{\omega}\int\limits_{-1}^{+1}d\cos\vartheta\,\underbrace{(\cos^2\vartheta-\cos^4\vartheta)}_{\frac{2}{3}-\frac{2}{5}=\frac{4}{15}}\\
\Longrightarrow\quad \mathbf{m} &= \frac{4\pi}{15}\sigma_0R^4\boldsymbol{\omega}\ .
\end{aligned}
$$

Total charge:

$$
\begin{aligned}
Q &= \int d^3r\,\sigma_0\delta(r-R)\cos^2\vartheta\\
&= 2\pi\sigma_0R^2\int\limits_{-1}^{+1}d\cos\vartheta\,\cos^2\vartheta\\
&= \frac{4\pi}{3}\sigma_0R^2\\
\Longrightarrow\quad \sigma_0 &= \frac{3Q}{4\pi}\frac{1}{R^2}\\
\Longrightarrow\quad \mathbf{m} &= \frac{1}{5}QR^2\boldsymbol{\omega}\quad \text{as in 1.}
\end{aligned}
$$

Solution 3.3.5

1. Biot-Savart law (3.23):

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}
$$

with

$$\mathbf{r} = (0,0,z)$$
$$\mathbf{r}' = (R\cos\varphi, R\sin\varphi, z')\ .$$

Number of turns on dz': $\frac{n}{L}dz'$
Current density of one winding (q: cross section of the conductor)

$$\mathbf{j}(\mathbf{r}') \to \frac{I}{q}\mathbf{e}_\varphi\ ; \quad d^3r' = qRd\varphi\ .$$

Principle of superposition:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{n}{L}\int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dz'\ \frac{I}{q}qR\int\limits_0^{2\pi}\mathbf{e}_\varphi \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d\varphi\ ,$$

$$\begin{aligned}
\mathbf{e}_\varphi \times (\mathbf{r}-\mathbf{r}') &= (-\sin\varphi, \cos\varphi, 0) \times (-R\cos\varphi, -R\sin\varphi, z-z') \\
&= \big((z-z')\cos\varphi, (z-z')\sin\varphi, R\big) \\
|\mathbf{r}-\mathbf{r}'|^3 &= \big(R^2 + (z-z')^2\big)^{3/2}\ ,
\end{aligned}$$

$$\int\limits_0^{2\pi}\cos\varphi d\varphi = \int\limits_0^{2\pi}\sin\varphi d\varphi = 0\ .$$

For points $\mathbf{r}$ away from the axis the integral is not elementarily solvable! For points on the z-axis one finds:

$$\begin{aligned}
\mathbf{B}(z) &= \frac{\mu_0 n I R^2}{2L}\int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dz' \frac{1}{\left(R^2 + (z-z')^2\right)^{3/2}}\,\mathbf{e}_z \\
&= \frac{\mu_0 n I R^2}{2L}\,\mathbf{e}_z\, \frac{-(z-z')}{R^2\sqrt{R^2+(z-z')^2}}\Bigg|_{-\frac{L}{2}}^{+\frac{L}{2}} \\
&= \mu_0\frac{nI}{2L}\mathbf{e}_z\left(\frac{z+\frac{L}{2}}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \frac{z-\frac{L}{2}}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right).
\end{aligned}$$

Especially:

$$B_z(0) = \mu_0 \frac{nI}{\sqrt{4R^2 + L^2}}$$

$$B_z\left(\pm\frac{L}{2}\right) = \mu_0 \frac{nI}{\sqrt{4R^2 + 4L^2}} \ .$$

2. Inside the coil $\left(|z| < \frac{L}{2}\right)$:
$L \gg R$:

$$B_z \approx \mu_0 \frac{n}{L} I \ .$$

$L \ll R$:

$$B_z \approx \mu_0 \frac{n}{2R} I \ .$$

Outside the coil:
$|z| \gg L, R$:

$$B_z \approx 0 \ .$$

$|z| \gg L \gg R$:

$$\begin{aligned}
B_z &= \pm\mu_0 \frac{nI}{2L}\left[\left(1 + \frac{R^2}{(z + L/2)^2}\right)^{-\frac{1}{2}} - \left(1 + \frac{R^2}{(z - L/2)^2}\right)^{-\frac{1}{2}}\right] \\
&\simeq \pm\mu_0 \frac{nI}{2L}\left[1 - \frac{1}{2}\left(\frac{2R}{2z + L}\right)^2 - 1 + \frac{1}{2}\left(\frac{2R}{2z - L}\right)^2\right] \\
&= \pm\mu_0 \frac{nIR^2}{L}\left[\frac{1}{(2z - L)^2} - \frac{1}{(2z + L)^2}\right] \\
&= \pm\mu_0 \frac{nIR^2}{4Lz^2}\left[\left(1 - \frac{L}{2z}\right)^{-2} - \left(1 + \frac{L}{2z}\right)^{-2}\right] \\
&\simeq \pm\mu_0 \frac{nIR^2}{4Lz^2}\left(1 + \frac{L}{z} - 1 + \frac{L}{z}\right) \\
\Longrightarrow \quad B_z &\approx \pm\mu_0 \frac{nIR^2}{2z^3} \ .
\end{aligned}$$

3. Magnetic moment

$$\mathbf{m} = \frac{1}{2}\int \left(\mathbf{r}' \times \mathbf{j}(\mathbf{r}')\right) d^3r' \ .$$

As in 1.:

$$\mathbf{m} = \frac{n}{2L}\int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dz' \frac{I}{q} qR \int\limits_{0}^{2\pi} (\mathbf{r}' \times \mathbf{e}_\varphi) d\varphi \ ,$$

$$\begin{aligned}
\mathbf{r}' \times \mathbf{e}_\varphi &= (R\cos\varphi, R\sin\varphi, z') \times (-\sin\varphi, \cos\varphi, 0) \\
&= (\underbrace{-z'\cos\varphi, -z'\sin\varphi}_{\text{no contribution to } \mathbf{m}}, R)
\end{aligned}$$

Therewith:

$$\mathbf{m} = \frac{nIR^2}{2L}\mathbf{e}_z \int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dz' \int\limits_{0}^{2\pi} d\varphi = nI(\pi R^2)\mathbf{e}_z \ .$$

4. Dipole field (3.45):

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{r}\cdot\mathbf{m})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right) \ .$$

On the coil axis ($\mathbf{r} = z\mathbf{e}_z$) and with the result from 3.:

$$\mathbf{B} = \mu_0 \frac{nIR^2}{2|z|^3}\mathbf{e}_z \ .$$

Section 3.4.5

Solution 3.4.1 Potential energy of a magnetic dipole $\mathbf{m}$ in the external field $\mathbf{B}$,

$$V = -\mathbf{m}\cdot\mathbf{B}_0 \ .$$

Equilibrium if V minimal $\Longrightarrow$

$$\mathbf{m} \uparrow\uparrow \mathbf{B}_0 \ , \quad \text{thus in } x\text{-direction}$$

Current-carrying wire $\Longrightarrow$

$$\mathbf{B}_1(\mathbf{r}) = \mu_0 \frac{I\mathbf{e}_\varphi}{2\pi\rho} \quad \text{(see (3.22))}$$

$$\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \ .$$

Total field

$$\mathbf{B}(\mathbf{r}) = B_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_0 \frac{I}{2\pi(x^2+y^2)} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

at the point $\mathbf{x}_0 = (x_0, 0, 0)$:

$$\mathbf{B}(\mathbf{x}_0) = B_0\mathbf{e}_x + \mu_0 \frac{I}{2\pi x_0}\mathbf{e}_y \ .$$

The dipole orients itself parallel to $\mathbf{B}$, thus enclosing with the x-axis the angle α:

$$\tan\alpha = \mu_0 \frac{I}{2\pi x_0 B_0} \ .$$

Small angle:

$$\tan\alpha \simeq \alpha = \mu_0 \frac{I}{2\pi x_0 B_0} \quad \text{(possibility of current measurement!)} \ .$$

Solution 3.4.2

1. Inside and outside the sphere: $\mathbf{j} \equiv 0$
   $\Longrightarrow$ Maxwell equation of the magnetostatics:

$$\text{curl}\mathbf{H} = 0$$

   $\Longrightarrow$ **H** is a gradient field:

$$\mathbf{H} = -\nabla\varphi_m \ .$$

   With

$$0 = \text{div}\,\mathbf{B} = \mu_0\,\text{div}\,(\mathbf{M} + \mathbf{H})$$

   it follows:

$$\Delta\varphi_m = \text{div}\,\mathbf{M} \ .$$

Solution according to (3.90):

$$\varphi_m(\mathbf{r}) = -\frac{1}{4\pi}\nabla_r \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ .$$

Homogeneously magnetized sphere, i.e. $\mathbf{M} = M_0\mathbf{e}_z$:

$$\varphi_m(\mathbf{r}) = -\frac{M_0}{4\pi}\frac{d}{dz}\int\limits_{V_K} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \ .$$

We choose $\mathbf{r}$ parallel to the polar axis:

$$\begin{aligned}
\int\limits_{V_K} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= 2\pi \int\limits_0^R r'^2 dr' \int\limits_{-1}^{+1} d\cos\vartheta' \frac{1}{(r^2+r'^2-2rr'\cos\vartheta')^{1/2}} \\
&= -\frac{2\pi}{r}\int\limits_0^R r'dr' \left(|r-r'|-|r+r'|\right) \\
&= \frac{4\pi}{r}\int\limits_0^R r'^2 dr' \quad (r > r') \\
&= \frac{4\pi}{3}R^3 \cdot \frac{1}{r} \ .
\end{aligned}$$

In addition:

$$\begin{aligned}
&\frac{d}{dz}\frac{1}{r} = -\frac{1}{r^2}\frac{z}{r} = -\frac{\cos\vartheta}{r^2} \ , \\
\Longrightarrow \quad &\varphi_m(\mathbf{r}) = \frac{1}{3}M_0R^3 \cdot \frac{\cos\vartheta}{r^2} \ .
\end{aligned}$$

Total moment of the sphere:

$$\mathbf{m} = \int d^3r'\mathbf{M}(\mathbf{r}') = \frac{4\pi}{3}R^3 \cdot M_0\mathbf{e}_z \ .$$

$\Longrightarrow$ dipole potential:

$$\varphi_m(\mathbf{r}) = \frac{1}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3} \ .$$

This first part was already calculated as an example of use after (3.94).

2. Calculation of $\mathbf{H}$ outside the sphere as previously done for the electrostatic dipole field:

$$
\begin{aligned}
\nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{b}\cdot\nabla)\,\mathbf{a} + (\mathbf{a}\cdot\nabla)\,\mathbf{b} + \mathbf{b}\times\operatorname{curl}\mathbf{a} + \mathbf{a}\times\operatorname{curl}\mathbf{b}\ , \\
\Longrightarrow\ \mathbf{H} &= -\nabla\varphi_m = -\frac{1}{4\pi}\nabla\left(\mathbf{m}\cdot\frac{\mathbf{r}}{r^3}\right) \\
&= \frac{1}{4\pi}\nabla\left(\mathbf{m}\cdot\nabla\frac{1}{r}\right) \\
&= \frac{1}{4\pi}\left((\mathbf{m}\cdot\nabla)\,\nabla\frac{1}{r}\right)\left(\operatorname{curl}\left(\nabla\frac{1}{r}\right)=0\right) \\
&= -\frac{1}{4\pi}\,(\mathbf{m}\cdot\nabla)\,\frac{\mathbf{r}}{r^3} \\
&= -\frac{1}{4\pi}m\frac{d}{dz}\frac{\mathbf{r}}{r^3} \\
&= -\frac{1}{4\pi}m\left(\frac{1}{r^3}\mathbf{e}_z - \mathbf{r}\frac{3}{r^4}\cdot\frac{z}{r}\right)\ .
\end{aligned}
$$

Therewith it holds:

$$
\mathbf{H} = \frac{1}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3}\right) \quad \text{for} \quad r > R
$$

Typical dipole field!
Inside the sphere we have:

$$
\begin{aligned}
\mathbf{M} &= \chi_m\mathbf{H} \quad \text{(isotropic, linear medium)}\ , \\
\Longrightarrow\ \mathbf{H} &= \frac{M_0}{\chi_M}\mathbf{e}_z \quad \text{for} \quad r < R\ .
\end{aligned}
$$

3. Surface current density:
Cylindrical symmetry:

$$
|\mathbf{j}| = \alpha(\underset{\substack{\uparrow \\ \text{no } \varphi\text{-dependence}}}{\vartheta})\delta(r-R)\ .
$$

Plausible:

$$
\mathbf{j} \sim \mathbf{e}_\varphi\ .
$$

Test:
For the magnetic moment of the sphere it must be $\mathbf{m} \sim \mathbf{e}_z$:

$$\begin{aligned}
\mathbf{m} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j}(\mathbf{r}))\\
&= \frac{1}{2}\int\limits_0^\infty dr\cdot r^3\delta(r-R)\int\limits_{-1}^{+1} d\cos\vartheta\;\alpha(\vartheta)\int\limits_0^{2\pi} d\varphi\,\left(\mathbf{e}_r\times\mathbf{e}_\varphi\right)\\
&= \frac{1}{2}R^3\int\limits_{-1}^{+1} d\cos\vartheta\;\alpha(\vartheta)\int\limits_0^{2\pi} d\varphi\qquad\underbrace{(-\mathbf{e}_\vartheta)}_{(-\cos\vartheta\cos\varphi,-\cos\vartheta\sin\varphi,\sin\vartheta)}\\
&= \pi R^3\int\limits_{-1}^{+1} d\cos\vartheta\;\alpha(\vartheta)\sin\vartheta\cdot(0,0,1)\\
&\sim \mathbf{e}_z\quad\text{(was to be proven)}\\
&\Longrightarrow\;\mathbf{j}(\mathbf{r}) = \alpha(\vartheta)\delta(r-R)\mathbf{e}_\varphi\;.
\end{aligned}$$

We determine $\alpha(\vartheta)$ from the boundary conditions of the fields at the surface of the sphere (Fig. A.39).

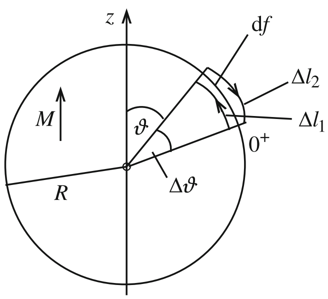
$\operatorname{div}\mathbf{B} = 0 \implies$ with the *Gauss-casket* one finds for the normal components (see (3.80)):

$$B_{2n} - B_{1n} = 0$$

Here:

$$B_r(R+0^+) - B_r(R-0^+) = 0\;.$$

Fig. A.39



Tangential components:

$$\mathbf{n} = \mathbf{e}_r \; ; \quad \mathbf{t} = \mathbf{e}_\varphi \implies \mathbf{t} \times \mathbf{n} = \mathbf{e}_\vartheta \; ,$$

$$\Delta \mathbf{l}_1 = -\Delta \mathbf{l}_2 = \Delta l (\mathbf{n} \times \mathbf{t}) = R \Delta \vartheta \, (-\mathbf{e}_\vartheta) \; .$$

Element of the *'Stokes area'*:

$$d\mathbf{f} = r dr d\vartheta \mathbf{e}_\varphi$$

$$\implies \int_{\Delta F} d\mathbf{f} \cdot \mathbf{j} = \alpha(\vartheta) \Delta \vartheta \int_{R-0^+}^{R+0^+} r dr \cdot \delta(r - R)$$

$$= \alpha(\vartheta) R \Delta \vartheta \; .$$

On the other hand:

$$\begin{aligned}
\int_{\Delta F} d\mathbf{f} \cdot \mathbf{j} &= \int_{\partial \Delta F} d\mathbf{r} \cdot \mathbf{H} \\
&= \mathbf{H}(R + 0^+) \cdot \Delta \mathbf{l}_2 + \mathbf{H}(R - 0^+) \cdot \Delta \mathbf{l}_1 \\
&= R \Delta \vartheta \left(H_\vartheta (R + 0^+) - H_\vartheta (R - 0+) \right) \; , \\
\implies & H_\vartheta (R + 0^+) - H_\vartheta (R - 0^+) = \alpha(\vartheta) \; .
\end{aligned}$$

Field components from part 1.:

$r > R$:

$$\varphi_m(\mathbf{r}) = \frac{m}{4\pi} \frac{\cos \vartheta}{r^2}$$

$$\implies H_\vartheta = -\frac{1}{r} \frac{\partial}{\partial \vartheta} \varphi_m = \frac{m}{4\pi} \frac{\sin \vartheta}{r^3} \; .$$

$r < R$:

$$\mathbf{H} = H_0 \mathbf{e}_z \; ; \quad H_0 = \frac{M_0}{\chi_m}$$

$$\mathbf{e}_z \cdot \mathbf{e}_\vartheta = -\sin \vartheta$$

$$\implies H_\vartheta = \frac{M_0}{\chi_m} (-\sin \vartheta) \; .$$

In conclusion:

$$\begin{aligned} H_\vartheta(R+0^+) - H_\vartheta(R-0^+) &= \frac{m}{4\pi}\frac{\sin\vartheta}{R^3} + \frac{M_0}{\chi_m}\sin\vartheta \\ &= \frac{1}{3}R^3 M_0 \frac{\sin\vartheta}{R^3} + \frac{M_0}{\chi_m}\sin\vartheta \\ &= M_0\sin\vartheta\left(\frac{1}{3} + \frac{1}{\chi_m}\right) , \\ \Longrightarrow \quad \alpha(\vartheta) &= \frac{3+\chi_m}{3\chi_m} M_0 \sin\vartheta \ . \end{aligned}$$

Solution 3.4.3

1. Maxwell equations of the magnetostatics:

$$\mathrm{curl}\mathbf{H} = \mathbf{j} \ ; \quad \mathrm{div}\,\mathbf{B} = 0 \ .$$

In regions G, in which $\mathbf{j} = 0$, it holds:

$$\mathrm{curl}\mathbf{H} = 0 \ ,$$

so that because of curl grad$\varphi_\mathrm{m} = 0$ a scalar magnetic potential can be introduced:

$$\mathbf{H} = -\mathrm{grad}\varphi_\mathrm{m}$$

2. Equations (3.33) and (3.85):

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_\mathrm{r}\mu_0}{4\pi}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ ,$$

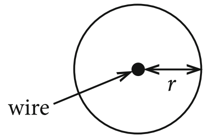
$\mathbf{j}(\mathbf{r}') = j(\mathbf{r}')\,\mathbf{e}_z$ (cylindrical coordinates!). It follows:

$$\mathbf{A}(\mathbf{r}) = A_z(r,\ \varphi,\ z)\,\mathbf{e}_z \ .$$

Symmetry:

$$\begin{aligned} & A_z(r,\varphi,z) = A_z(r) \\ \Longrightarrow \quad & \mathrm{curl}\mathbf{A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \mathbf{e}_\varphi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial\varphi}\right) \\ & \qquad = -\frac{\partial A_z}{\partial r}\mathbf{e}_\varphi = \mu_\mathrm{r}\mu_0\mathbf{H} \quad \text{(see (1.380), Vol. 1)} \\ & \Longrightarrow \quad \mathbf{H} = H(r)\,\mathbf{e}_\varphi \ . \end{aligned}$$

Fig. A.40



F_r: circular area $\perp$ wire, radius r (Fig. A.40).
It follows:

$$I = \int_{F_r} \mathbf{j} \cdot d\mathbf{f} = \int_{F_r} \operatorname{curl}\mathbf{H} \cdot d\mathbf{f} = \int_{\partial F_r} \mathbf{H} \cdot d\mathbf{r} = H(r)\, 2\pi r \,.$$

That yields as magnetic field if the plate is absent:

$$\mathbf{H}(\mathbf{r}) = \frac{I}{2\pi\, r}\, \mathbf{e}_\varphi \,.$$

Cylindrical coordinates:

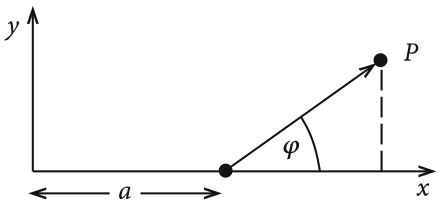
$$\begin{aligned}
&\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\right) \,, \\
&\mathbf{H} = -\nabla \varphi_{\mathrm{m}} = -\mathbf{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}\varphi_{\mathrm{m}} \overset{!}{=} \frac{I}{2\pi\, r}\, \mathbf{e}_\varphi \\
&\Longrightarrow \quad \frac{\partial}{\partial \varphi}\varphi_{\mathrm{m}} = -\frac{I}{2\pi} \quad (r \neq 0) \\
&\Longrightarrow \quad \varphi_{\mathrm{m}} = -\frac{I}{2\pi}\varphi + \mathrm{const} \,.
\end{aligned}$$

Area perpendicular to the wire:

$$\begin{aligned}
\tan\varphi &= \frac{y}{x-a} \\
\Longrightarrow \quad \varphi &= \arctan\frac{y}{x-a} \\
\Longrightarrow \quad \varphi_{\mathrm{m}} &= -\frac{I}{2\pi}\arctan\frac{y}{x-a} \,,
\end{aligned}$$

where an additive constant has been put to zero (Fig. A.41).

Fig. A.41



3. **Boundary-value problem** for the arrangement **with** plate
   (a) $\Delta\varphi_\mathrm{m} = 0$ for $r \neq 0$.
   (b) Continuity conditions for the fields:

$$H_\mathrm{t} \text{ continuous} \iff \left.\frac{\partial \varphi_\mathrm{m}}{\partial y}\right|_{x=0-} = \left.\frac{\partial \varphi_\mathrm{m}}{\partial y}\right|_{x=0+} ,$$

$$B_\mathrm{n} \text{ continuous} \iff \mu_\mathrm{r}^{(1)} \left.\frac{\partial \varphi_\mathrm{m}}{\partial x}\right|_{x=0-} = \mu_\mathrm{r}^{(2)} \left.\frac{\partial \varphi_\mathrm{m}}{\partial x}\right|_{x=0+} .$$

4. **Image currents**
   Region 2:

$$\varphi_\mathrm{m}^{(2)} = -\frac{I}{2\pi} \arctan \frac{y}{x-a} - \frac{I_1}{2\pi} \arctan \frac{y}{x+a} .$$

Region 1:

$$\varphi_\mathrm{m}^{(1)} = -\frac{I_2}{2\pi} \arctan \frac{y}{x-a} .$$

Magnetic field strength
Region 2:

$$\begin{aligned} H_x^{(2)} &= -\frac{\partial}{\partial x}\varphi_\mathrm{m}^{(2)} \\ &= +\frac{I}{2\pi} \frac{1}{1+\left(\frac{y}{x-a}\right)^2} \left[-\frac{y}{(x-a)^2}\right] + \frac{I_1}{2\pi} \frac{1}{1+\left(\frac{y}{x+a}\right)^2} \left[-\frac{y}{(x+a)^2}\right] \\ &= \frac{1}{2\pi} \frac{I}{(x-a)^2+y^2}(-y) + \frac{1}{2\pi} \frac{I_1}{(x+a)^2+y^2}(-y) , \\ H_y^{(2)} &= -\frac{\partial}{\partial y}\varphi_\mathrm{m}^{(2)} = \frac{I}{2\pi} \frac{1}{1+\left(\frac{y}{x-a}\right)^2} \frac{1}{(x-a)} + \frac{I_1}{2\pi} \frac{1}{1+\left(\frac{y}{x+a}\right)^2} \frac{1}{(x+a)} \end{aligned}$$

$$= \frac{1}{2\pi} \frac{I}{(x-a)^2 + y^2} (x-a) + \frac{1}{2\pi} \frac{I_1}{(x+a)^2 + y^2} (x+a) ,$$

$$H_z^{(2)} = -\frac{\partial}{\partial z} \varphi_{\mathrm{m}}^{(2)} = 0 ,$$

Thus:

$$\mathbf{H}^{(2)} = \frac{1}{2\pi} \frac{I}{(x-a)^2 + y^2} (-y, x-a, 0) + \frac{1}{2\pi} \frac{I_1}{(x+a)^2 + y^2} (-y, x+a, 0) .$$

Region 1:
Analogously:

$$\mathbf{H}^{(1)} = \frac{1}{2\pi} \frac{I_2}{(x-a)^2 + y^2} (-y, x-a, 0) ,$$

$$\mathbf{B}^{(1)} = \mu_{\mathrm{r}}^{(1)} \mu_0 \mathbf{H}^{(1)} ; \quad \mathbf{B}^{(2)} = \mu_{\mathrm{r}}^{(2)} \mu_0 \mathbf{H}^{(2)} .$$

5. I_1, I_2 from the boundary conditions of the fields:

$$\begin{aligned}
H_{\mathrm{t}} \text{ continuous } &\iff H_y^{(1)}(x=0) = H_y^{(2)}(x=0) \\
&\iff \frac{-a\,I_2}{a^2+y^2} = \frac{-a\,I}{a^2+y^2} + \frac{a\,I_1}{a^2+y^2} \\
&\iff I_2 = I - I_1 , \\
B_{\mathrm{n}} \text{ continuous } &\iff \mu_{\mathrm{r}}^{(1)} H_x^{(1)}(x=0) = \mu_{\mathrm{r}}^{(2)} H_x^{(2)}(x=0) \\
&\iff \mu_{\mathrm{r}}^{(1)} \frac{-yI_2}{a^2+y^2} = \mu_{\mathrm{r}}^{(2)} \frac{-yI}{a^2+y^2} + \mu_{\mathrm{r}}^{(2)} \frac{-yI_1}{a^2+y^2} \\
&\iff \mu_{\mathrm{r}}^{(1)} I_2 = \mu_{\mathrm{r}}^{(2)} (I + I_1) ,
\end{aligned}$$

Thus:

$$\begin{aligned}
I_1 &= \frac{\mu_{\mathrm{r}}^{(1)}}{\mu_{\mathrm{r}}^{(2)}} I_2 - I \\
\Longrightarrow \quad I_2 &= 2I - \frac{\mu_{\mathrm{r}}^{(1)}}{\mu_{\mathrm{r}}^{(2)}} I_2 \\
\Longrightarrow \quad I_2 &= \frac{2\mu_{\mathrm{r}}^{(2)}}{\mu_{\mathrm{r}}^{(1)} + \mu_{\mathrm{r}}^{(2)}} I ; \quad I_1 = \frac{\mu_{\mathrm{r}}^{(1)} - \mu_{\mathrm{r}}^{(2)}}{\mu_{\mathrm{r}}^{(1)} + \mu_{\mathrm{r}}^{(2)}} I .
\end{aligned}$$

6. According to (3.24):

$$\mathbf{F} = \int (\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}))\; d^3r .$$

From that we get the force density

$$\mathbf{f} = \mathbf{j} \times \mathbf{B}$$

and the force per length:

$$\hat{\mathbf{f}} = \mathbf{I} \times \mathbf{B} \; .$$

Field of I_1 at the position of the wire (without plate!):

$$H_x^{(2)}(I_1) = \frac{1}{2\pi}\,\frac{-y\,I_1}{(x+a)^2+y^2} \underset{\substack{\text{wire}\\(x=a,y=0)}}{\longrightarrow} 0 \; ,$$

$$H_y^{(2)}(I_1) = \frac{1}{2\pi}\,\frac{(x+a)I_1}{(x+a)^2+y^2} \underset{\text{wire}}{\longrightarrow} \frac{I_1}{2\pi}\,\frac{1}{2a} \; ,$$

$$H_z^{(2)}(I_1) \equiv 0$$

$$\Longrightarrow \; \mathbf{B}^{(I_1)}(x=a, y=0) = \mu_0\mu_{\mathrm{r}}^{(2)}\,\frac{I_1}{4\pi\,a}\,\mathbf{e}_y \; ,$$

$$\mathbf{I} = I\,\mathbf{e}_z$$

$$\Longrightarrow \; \hat{\mathbf{f}} = -\frac{I^2}{4\pi\,a}\,\frac{\mu_0\mu_{\mathrm{r}}^{(2)}\left(\mu_{\mathrm{r}}^{(1)}-\mu_{\mathrm{r}}^{(2)}\right)}{\mu_{\mathrm{r}}^{(1)}+\mu_{\mathrm{r}}^{(2)}}\,\mathbf{e}_x \; .$$

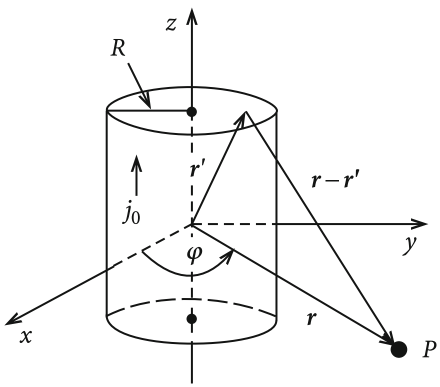
Solution 3.4.4

(a) **Current density**
Cylindrical coordinates: ρ, φ, z (Fig. A.42).

$$\mathbf{j}(\mathbf{r}) = j_0(\rho)\,\mathbf{e}_z \; ,$$

$$j_0(\rho) = \frac{I}{\pi\,R^2}\Theta(R-\rho) \; .$$

Fig. A.42



(b) **Vector potential**
General solution:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Longrightarrow \quad \mathbf{A}(\mathbf{r}) \sim \mathbf{e}_z \quad \Longrightarrow \quad A_\rho = A_\varphi = 0\ .$$

$$A_z = A_z(\rho, \varphi, z)$$

$$\text{cylindrical symmetry} \quad \Longrightarrow \quad A_z = A_z(\rho, z)\ ,$$

$$\textit{infinitely long} \quad \Longrightarrow \quad A_z = A_z(\rho)\ .$$

(c) **Poisson equation**
According to (3.37) we start at:

$$\Delta\,\mathbf{A} = -\mu_0\,\mathbf{j}\ ,$$

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\ .$$

Thus it is to be solved:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}A_z(\rho)\right) = -\mu_0 j_0(\rho)\ .$$

Outside ($\rho > R$):

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}A_z(\rho)\right) = 0$$

$$\Longleftrightarrow \quad \rho\frac{\partial}{\partial\rho}A_z(\rho) = c$$

$$\Longleftrightarrow \quad \frac{\partial}{\partial\rho}A_z(\rho) = \frac{c}{\rho}$$

$$\Longrightarrow \quad A_z(\rho) = c\ln\rho + A_z^{(0)}\ .$$

Inside ($\rho < R$):

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}A_z(\rho)\right) = -\mu_0\frac{I}{\pi R^2}\rho$$

$$\Longleftrightarrow \quad \rho\frac{\partial}{\partial\rho}A_z(\rho) = -\mu_0\frac{I}{2\pi R^2}\rho^2 + c_1$$

$$\Longleftrightarrow \frac{\partial}{\partial\rho}A_z(\rho) = -\mu_0 \frac{I}{2\pi R^2}\rho + \frac{c_1}{\rho}$$

$$\Longleftrightarrow A_z(\rho) = -\mu_0 \frac{I}{4\pi R^2}\rho^2 + c_1 \ln\rho + c_2 \ .$$

Without loss of generality: $c_2 = 0$,
Regularity at the origin: $c_1 = 0$

$$\Longrightarrow A_z(\rho) = -\mu_0 \frac{I}{4\pi R^2}\rho^2 \ ,$$

Continuity at $\rho = R$:

$$c \ln R + A_z^{(0)} = -\mu_0 \frac{I}{4\pi}$$

$$\Longrightarrow \mathbf{A}(\mathbf{r}) = A_z(\rho)\,\mathbf{e}_z \ ,$$

$$A_z(\rho) = \begin{cases} -\mu_0 \dfrac{I}{4\pi R^2}\rho^2 \ , & \text{if } \rho \leq R \ , \\[2ex] c \ln \dfrac{\rho}{R} - \mu_0 \dfrac{I}{4\pi} \ , & \text{if } \rho \geq R \ . \end{cases}$$

(d) **Magnetic field** ($\mu_{\mathbf{r}} = 1$)

$$\mu_0\mathbf{H} = \mathrm{curl}\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_z - \frac{\partial}{\partial z}A_\varphi\right)\mathbf{e}_\rho + \left(\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial\rho}A_z\right)\mathbf{e}_\varphi$$

$$+ \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{1}{\rho}\frac{\partial}{\partial\varphi}A_\rho\right)\mathbf{e}_z$$

$$\mathbf{H} = -\frac{1}{\mu_0}\frac{\partial}{\partial\rho}A_z(\rho)\mathbf{e}_\varphi = H_\varphi(\rho)\,\mathbf{e}_\varphi \ ,$$

$$H_\varphi(\rho) = \begin{cases} \dfrac{I}{2\pi R^2}\rho \ , & \text{if } \rho \leq R \ , \\[2ex] -\dfrac{c}{\mu_0\rho} \ , & \text{if } \rho \geq R \ . \end{cases}$$

Continuity at $\rho = R$:

$$-\frac{c}{\mu_0} = \frac{I}{2\pi}$$

$$\Longrightarrow \mathbf{H} = H_\varphi(\rho)\,\mathbf{e}_\varphi \ ,$$

$$H_\varphi(\rho) = \frac{I}{2\pi}\begin{cases} \dfrac{\rho}{R^2}\,, & \text{if } \rho \le R\,, \\ \dfrac{1}{\rho}\,, & \text{if } \rho \ge R\,. \end{cases}$$

(e) **Test by the Stokes theorem**

K_ρ: Circle with the radius $\rho \perp \mathbf{e}_z$:

$$\oint\limits_{K_\rho} \mathbf{H}\cdot d\mathbf{r} = \begin{cases} H_\varphi(\rho)\, 2\pi\, \rho \quad (d\mathbf{r} \uparrow\uparrow \mathbf{H}) & \text{(direct)} \\ \int\limits_{F_{K_\rho}} d\mathbf{f}\cdot \mathrm{curl}\mathbf{H} = \int\limits_{F_{K_\rho}} d\mathbf{f}\cdot \mathbf{j}(\mathbf{r}) & \text{(Stokes)} \end{cases}$$

$$= 2\pi \frac{I}{\pi R^2}\int\limits_0^\rho d\rho' \rho' \Theta(R-\rho')$$

$$= \frac{2I}{R^2}\begin{cases} \dfrac{R^2}{2}\,, & \text{if } \rho \ge R\,, \\ \dfrac{\rho^2}{2}\,, & \text{if } \rho \le R\,. \end{cases}$$

From this it follows:

$$H_\varphi(\rho) = \begin{cases} \dfrac{I}{2\pi}\dfrac{1}{\rho}\,, & \text{if } \rho \ge R\,, \\ \dfrac{I}{2\pi}\dfrac{\rho}{R^2}\,, & \text{if } \rho \le R \end{cases}$$

Solution 3.4.5

1.

$$\mathbf{M}(\mathbf{r}) = M(r)\,\mathbf{e}_r\,.$$

We write

$$m(r) = \frac{M(r)}{r}$$

and calculate

$$\mathrm{curl}\mathbf{M}(\mathbf{r}) = \mathrm{curl}(m(r)\,\mathbf{r})\,.$$

With

$$\frac{\partial}{\partial x_i} m(r) = \frac{\partial m}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\partial m}{\partial r}\frac{x_i}{r}$$

we find for the curl of the magnetization:

$$\mathrm{curl}\mathbf{M}(\mathbf{r}) = \frac{\partial m}{\partial r}\left(\frac{x_2}{r}x_3 - \frac{x_3}{r}x_2, \frac{x_3}{r}x_1 - \frac{x_1}{r}x_3, \frac{x_1}{r}x_2 - \frac{x_2}{r}x_1\right) = 0\ .$$

This is valid in the whole space, i.e. inside as well as outside the spherical shell; outside of course trivially because of $\mathbf{M} \equiv 0$. Since in addition there do not flow any currents ($\mathbf{j} \equiv 0$) it holds likewise in the whole space:

$$\mathrm{curl}\mathbf{H} = 0\ .$$

Therewith it follows also:

$$\mathrm{curl}\mathbf{B} = \mathrm{curl}\left(\mu_0(\mathbf{H} + \mathbf{M})\right) = 0\ .$$

Furthermore it is always valid: $\mathrm{div}\,\mathbf{B} = 0$. According to the decomposition theorem (1.71) the magnetic induction thus vanishes in the whole space:

$$\mathbf{B}(\mathbf{r}) \equiv 0\ .$$

That means for the magnetic field:

$$\mathbf{H}(\mathbf{r}) = -\mathbf{M}(\mathbf{r})\ .$$

It is therefore unequal zero only within the spherical shell and opposes the magnetization.

2.

$$\mathbf{M}(\mathbf{r}) = \widehat{M}(\rho)\,\mathbf{e}_\varphi\ .$$

Nabla operator in cylindrical coordinates (Vol. 1, (1.388)):

$$\nabla \equiv \mathbf{e}_\rho\,\frac{\partial}{\partial\rho} + \mathbf{e}_\varphi\,\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \mathbf{e}_z\,\frac{\partial}{\partial z}\ .$$

One easily calculates therewith the divergence of the magnetization:

$$\mathrm{div}\,\mathbf{M}(\mathbf{r}) = \nabla\cdot\mathbf{M}(\mathbf{r}) = \frac{1}{\rho}\frac{\partial}{\partial\varphi}\widehat{M}(\rho) = 0\ .$$

Since it is always $\mathrm{div}\,\mathbf{B} = 0$ it follows further

$$\mathrm{div}\,\mathbf{H}(\mathbf{r}) = \mathrm{div}\left(\frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r})\right) = 0\ .$$

Because of the absence of currents, in addition, curl$\mathbf{H} = 0$ so that in line with the decomposition theorem (1.71) the magnetic field vanishes in the whole space:

$$\mathbf{H}(\mathbf{r}) \equiv 0 \, .$$

The magnetic induction is now unequal zero only in the region of the hollow cylinder:

$$\mathbf{B}(\mathbf{r}) = \mu_0 \left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{M}(\mathbf{r}) \, .$$

Section 4.1.6

Solution 4.1.1 **General**: Σ: Lorentz force on the charge q:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \, .$$

Σ': $\mathbf{r}' = \mathbf{r} - \mathbf{R}$; $\mathbf{R} = \mathbf{v}_0 t$. Herefrom it follows:

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \text{:} \quad \text{particle velocity in } \sum{}' \, .$$

Lorentz force:

$$\mathbf{F}' = q(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') = q[\mathbf{E}' + (\mathbf{v} - \mathbf{v}_0) \times \mathbf{B}'] \, .$$

Σ, Σ': inertial systems $\Longleftrightarrow$ $\mathbf{F} = \mathbf{F}'$. Herefrom we find:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{E}' + (\mathbf{v} - \mathbf{v}_0) \times \mathbf{B}' \, .$$

Especially: Σ: particle at rest, i.e. $\mathbf{v} = 0$

$$\Longrightarrow \quad \mathbf{E}' = \mathbf{E} + \mathbf{v}_0 \times \mathbf{B}' \, .$$

According to the presumption: $\mathbf{v}_0 \uparrow\uparrow \mathbf{E} \quad \Longleftrightarrow \quad \mathbf{v}_0 = \alpha \, \mathbf{E}$

$$\Longrightarrow \quad \mathbf{E}' = \mathbf{E} + \alpha \, \mathbf{E} \times \mathbf{B}' \, .$$

Thus it follows for the component of $\mathbf{E}'$ in the direction of $\mathbf{E}$:

$$\frac{\mathbf{E}' \cdot \mathbf{E}}{E} = \frac{E^2}{E} = E \, .$$

Solution 4.1.2

1. Electric field:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\nabla\varphi(\mathbf{r},t) - \dot{\mathbf{A}}(\mathbf{r},t)\\ &= -\dot{\mathbf{A}}(\mathbf{r},t)\\ &= +2\alpha c(x-ct)\mathbf{e}_z \ .\end{aligned}$$

2. Magnetic induction:

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \mathrm{curl}\mathbf{A}(\mathbf{r},t)\\ &= \left(0, -\frac{\partial}{\partial x}\alpha(x-ct)^2, 0\right)\\ &= -2\alpha(x-ct)\mathbf{e}_y \ .\end{aligned}$$

3. Field-energy density (vacuum):

$$\begin{aligned}w(\mathbf{r},t) &= \frac{1}{2}\left[\frac{1}{\mu_0}\mathbf{B}^2(\mathbf{r},t) + \varepsilon_0\mathbf{E}^2(\mathbf{r},t)\right]\\ &= \frac{1}{2\mu_0}\left[\mathbf{B}^2(\mathbf{r},t) + \frac{1}{c^2}\mathbf{E}^2(\mathbf{r},t)\right]\\ &= \frac{4\alpha^2}{\mu_0}(x-ct)^2 \ .\end{aligned}$$

4. Poynting vector (vacuum):

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\\ &= \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\\ &= \frac{4\alpha^2}{\mu_0}c\,(x-ct)^2\,\mathbf{e}_x\\ &= c\,w(\mathbf{r},t)\,\mathbf{e}_x \ .\end{aligned}$$

Solution 4.1.3

1. Generally it holds:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\nabla\varphi(\mathbf{r},t) - \dot{\mathbf{A}}(\mathbf{r},t) \ ,\\ \mathbf{B}(\mathbf{r},t) &= \mathrm{curl}\mathbf{A}(\mathbf{r},t) \ .\end{aligned}$$

Let us use:

$$\square \frac{\partial}{\partial t} \ldots = \frac{\partial}{\partial t} \square \ \ldots \ ,$$
$$\square \, \nabla \ldots = \nabla \square \ \ldots \ ,$$
$$\square \, \mathrm{curl} \ldots = \mathrm{curl} \square \ \ldots \ ,$$

We then obtain:

$$\square \, \mathbf{E}(\mathbf{r}, t) = -\nabla \underbrace{\square \, \varphi(\mathbf{r}, t)}_{=0} - \frac{\partial}{\partial t} \underbrace{\square \, \mathbf{A}(\mathbf{r}, t)}_{=0} = 0 \ ,$$
$$\square \, \mathbf{B}(\mathbf{r}, t) = \mathrm{curl} \underbrace{\square \, \mathbf{A}(\mathbf{r}, t)}_{=0} = 0 \ .$$

2.

$$\frac{\partial^2}{\partial x^2} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = -k_x^2 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \ .$$

Analogously the other components:

$$\Delta \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = -k^2 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \ ,$$
$$\frac{\partial^2}{\partial t^2} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) = -\omega^2 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$$
$$\Longrightarrow \ \square \, \mathbf{E}(\mathbf{r}, t) = \left(k^2 - \frac{\omega^2}{c^2} \right) \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \equiv 0 \ ,$$
$$\square \, \mathbf{B}(\mathbf{r}, t) = \left(k^2 - \frac{\omega^2}{c^2} \right) \mathbf{B}(\mathbf{r}, t) \equiv 0$$
$$\Longrightarrow \ \omega = \pm c |\mathbf{k}| \ .$$

No charges:

$$\mathrm{div}\, \mathbf{E} \equiv 0 \ = \ -\cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \left\{ E_0^x k_x + E_0^y k_y + E_0^z k_z \right\}$$
$$\Longrightarrow \ \mathbf{E}_0 \cdot \mathbf{k} = 0 \ ; \quad \mathbf{E}_0 \perp \mathbf{k} \ .$$

Analogously:

$$\mathrm{div}\, \mathbf{B} \equiv 0 \ \Longrightarrow \ \mathbf{B}_0 \cdot \mathbf{k} = 0 \ ; \quad \mathbf{B}_0 \perp \mathbf{k} \ .$$

Furthermore:

$$
\begin{aligned}
&\mathrm{curl}\mathbf{E} = -\dot{\mathbf{B}} \\
\iff\ & -\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\left[\mathbf{e}_x\left(k_y E_0^z - k_z E_0^y\right) + \mathbf{e}_y\left(k_z E_0^x - k_x E_0^z\right) + \mathbf{e}_z\left(k_x E_0^y - k_y E_0^x\right)\right] \\
&\qquad = -\omega\,\mathbf{B}_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) \\
\iff\ & \mathbf{k}\times\mathbf{E}_0 = \omega\,\mathbf{B}_0\ ; \quad \mathbf{B}_0 \perp \mathbf{E}_0\ .
\end{aligned}
$$

3. Energy-flux density $\simeq$ Poynting vector:

$$
\begin{aligned}
&\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t) \\
&\Longrightarrow\ \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\mathbf{E}_0\times\mathbf{B}_0\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\ , \\
&\mathbf{E}_0\times\mathbf{B}_0 = \frac{1}{\omega}\,\mathbf{E}_0\times(\mathbf{k}\times\mathbf{E}_0) = \frac{1}{\omega}\,\mathbf{k}\,E_0^2 - \frac{1}{\omega}\,\mathbf{E}_0(\mathbf{E}_0\cdot\mathbf{k}) = \frac{1}{\omega}\,E_0^2\,\mathbf{k} \\
&\Longrightarrow\ \mathbf{S} = \frac{1}{\omega\mu_0}\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)E_0^2\,\mathbf{k} \\
&\Longrightarrow\ \mathbf{S}_{\parallel} = S,\ S_{\perp} = 0\ ; \quad \text{energy flux only in the } \mathbf{k}\text{-direction.}
\end{aligned}
$$

4. Field-energy density:

$$
w(\mathbf{r},t) = \frac{1}{2}(\mathbf{E}(\mathbf{r},t)\cdot\mathbf{D}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t))\ .
$$

Here:

$$
\begin{aligned}
w(\mathbf{r},t) &= \frac{1}{2}\epsilon_0\mathbf{E}^2(\mathbf{r},t) + \frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r},t) \\
&= \frac{1}{2}\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\left(\epsilon_0 E_0^2 + \frac{1}{\mu_0}B_0^2\right)\ , \\
B_0^2 &= \frac{1}{\omega^2}k^2E_0^2 = \frac{1}{c^2}E_0^2 = \mu_0\epsilon_0 E_0^2 \\
\Longrightarrow\ w(\mathbf{r},t) &= \epsilon_0 E_0^2\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t) = \frac{1}{\mu_0}B_0^2\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\ .
\end{aligned}
$$

Solution 4.1.4 Wave equations of the electromagnetic field in the vacuum:

$$
\begin{aligned}
&\mathrm{curl}\mathbf{H} = \mathbf{j} + \dot{\mathbf{D}} \\
\Longrightarrow\ &\mathrm{curl}\mathbf{B} = \mu_0\mathbf{j} + \mu_0\epsilon_0\dot{\mathbf{E}} = \mu_0\mathbf{j} + \frac{1}{c^2}\dot{\mathbf{E}}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\text{curlcurl}\mathbf{B} &= \text{graddiv}\,\mathbf{B} - \Delta\mathbf{B} = -\Delta\mathbf{B} \\
&= \mu_0 \text{curl}\mathbf{j} + \frac{1}{c^2}\text{curl}\dot{\mathbf{E}}\ , \\
\frac{\partial}{\partial t}\text{curl}\mathbf{E} &= -\frac{\partial^2}{\partial t^2}\mathbf{B} \\
\Longrightarrow \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} &= \Box\,\mathbf{B} = -\mu_0 \text{curl}\mathbf{j} = \lambda_2(\mathbf{r}, t)\ .
\end{aligned}
$$

In the same manner one proceeds:

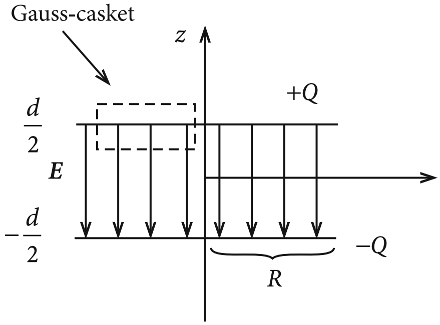
$$
\begin{aligned}
\text{curl}\mathbf{E} &= -\dot{\mathbf{B}} = -\mu_0 \dot{\mathbf{H}} \\
\Longrightarrow \text{curlcurl}\mathbf{E} &= \text{graddiv}\,\mathbf{E} - \Delta\mathbf{E} \\
&= -\mu_0 \text{curl}\dot{\mathbf{H}}\ , \\
\text{curl}\dot{\mathbf{H}} &= \frac{\partial}{\partial t}\text{curl}\mathbf{H} = \dot{\mathbf{j}} + \epsilon_0 \ddot{\mathbf{E}}\ , \\
\text{div}\,\mathbf{E} &= \frac{1}{\epsilon_0}\text{div}\,\mathbf{D} = \frac{1}{\epsilon_0}\rho \\
\Longrightarrow \frac{1}{\epsilon_0}\nabla\rho + \mu_0 \frac{\partial}{\partial t}\mathbf{j} &= \Delta\mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{E} \\
\Longrightarrow \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} &= \Box\,\mathbf{E} = \frac{1}{\epsilon_0}\nabla\rho + \mu_0 \frac{\partial}{\partial t}\mathbf{j} = \lambda_1(\mathbf{r}, t)\ .
\end{aligned}
$$

Solution 4.1.5

1. We use (2.211) (Fig. A.43):

$$
\begin{aligned}
\sigma &= \mathbf{D}^{(a)} \cdot \mathbf{e}_z - \mathbf{D}^{(i)} \cdot \mathbf{e}_z\ , \\
\mathbf{D}^{(a)} &= \mathbf{0}\ ,
\end{aligned}
$$

Fig. A.43



$$\mathbf{D}^{(\mathrm{i})} = (0, 0, -D)$$
$$\Longrightarrow \; D = \pm\sigma\left(\pm\frac{d}{2}\right) = \frac{Q}{\pi R^2}\,.$$

Electric field:

$$\mathbf{E} = E(z)\mathbf{e}_z\,; \quad E(z) = \frac{-D}{\epsilon_0\epsilon_{\mathrm{r}}(z)} = \frac{-1}{\epsilon_0\epsilon_{\mathrm{r}}(z)}\,\frac{Q}{\pi R^2}\,.$$

Voltage:

$$\begin{aligned}
U &= -\int\limits_{-d/2}^{+d/2} E(z)dz = \frac{Q}{\epsilon_0\pi R^2}\int\limits_{-d/2}^{+d/2}\frac{dz}{\epsilon_1 + (1/2)\Delta\epsilon\,(1 + 2z/d)} \\
&= \frac{Q}{\epsilon_0\pi R^2}\,\frac{d}{\Delta\epsilon}\ln\left[\epsilon_1 + \frac{1}{2}\Delta\epsilon\left(1 + 2\frac{z}{d}\right)\right]\Bigg|_{-d/2}^{+d/2} \\
&= \frac{Q}{\epsilon_0\pi R^2}\,\frac{d}{\Delta\epsilon}\ln\frac{\epsilon_1 + \Delta\epsilon}{\epsilon_1}\,.
\end{aligned}$$

Capacity:

$$C = \frac{\epsilon_0\pi R^2}{d}\,\frac{\Delta\epsilon}{\ln\left(1 + \Delta\epsilon/\epsilon_1\right)}\,.$$

Density of the charges bound in the dielectric:
Polarization:

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \left(1 - \frac{1}{\epsilon_{\mathrm{r}}(z)}\right)\mathbf{D}\,.$$

Polarization-charge density (2.189):

$$\rho_{\mathrm{p}} = -\mathrm{div}\,\mathbf{P}\,.$$

Therefrom we get the surface density of the bound charges:

$$\begin{aligned}
\sigma_{\mathrm{p}}\left(\pm\frac{d}{2}\right) &= \mp P\left(\pm\frac{d}{2}\right) = \mp\frac{Q}{\pi R^2}\left(1 - \frac{1}{\epsilon_{\mathrm{r}}\left(\pm d/2\right)}\right) \\
\Longrightarrow \; \sigma_{\mathrm{p}}\left(+\frac{d}{2}\right) &= -\frac{Q}{\pi R^2}\left(1 - \frac{1}{\epsilon_1 + \Delta\epsilon}\right)\,, \\
\sigma_{\mathrm{p}}\left(-\frac{d}{2}\right) &= +\frac{Q}{\pi R^2}\left(1 - \frac{1}{\epsilon_1}\right)\,.
\end{aligned}$$

σ_p compensates partly the actual surface charge on the plates so that the field between the plates is weakened.

Volume density:

$$\begin{aligned}
\rho_\mathrm{p} &= \mathrm{div}\,(-\mathbf{P}) = -\mathrm{div}\left[\left(1-\frac{1}{\epsilon_\mathrm{r}(z)}\right)\mathbf{D}\right] = -\frac{Q}{\pi R^2}\frac{d}{dz}\frac{1}{\epsilon_\mathrm{r}(z)} \\
&\Longrightarrow \rho_\mathrm{p} = 0\,, \quad \text{if } \epsilon_\mathrm{r} \neq \epsilon_\mathrm{r}(z)\,, \\
\rho_\mathrm{p} &= \frac{Q}{\pi R^2}\,\frac{\Delta\epsilon}{d\left[\epsilon_1 + (1/2)\Delta\epsilon\,(1+2z/d)\right]^2}\,.
\end{aligned}$$

2. Equation (4.53):

$$\frac{d}{dt}\left(\mathbf{p}_V^{(\mathrm{mech})} + \mathbf{p}_V^{(\mathrm{field})}\right) = \sum_{i=1}^{3}\mathbf{e}_i\int_V d^3r\sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij}\,,$$

$$T_{ij} = \epsilon_\mathrm{r}\epsilon_0 E_i E_j + \frac{1}{\mu_\mathrm{r}\mu_0}B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_\mathrm{r}\epsilon_0 E^2 + \frac{1}{\mu_\mathrm{r}\mu_0}B^2\right)\,.$$

In our exercise it is

$$\mathbf{B} \equiv 0\,; \quad \mathbf{E} \equiv (0, 0, E(z))$$

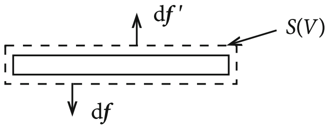
within the capacitor. It follows therewith:

$$\begin{aligned}
\overline{\mathbf{T}} &= \frac{1}{2}\epsilon_\mathrm{r}(z)\epsilon_0 E^2(z)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \left(\frac{Q}{\pi R^2}\right)^2\frac{1}{2\epsilon_\mathrm{r}(z)\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Force density:

$$\begin{aligned}
\mathbf{f}^{(\mathrm{total})} &= \sum_{i=1}^{3}\mathbf{e}_i\sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij} = \sum_{i=1}^{3}\mathbf{e}_i\frac{\partial}{\partial z}\underset{\substack{\uparrow \\ \text{only dependent on } z}}{T_{iz}}\,, \\
\mathbf{f}^{(\mathrm{total})} &= \left(0, 0, \frac{\partial}{\partial z}T_{zz}\right)\,, \\
f_z^{(\mathrm{total})} &= \frac{\partial}{\partial z}T_{zz} = -\frac{1}{2\epsilon_0}\left(\frac{Q}{\pi R^2}\right)^2\frac{\Delta\epsilon}{d\left[\epsilon_1 + (1/2)\Delta\epsilon\,(1+2z/d)\right]^2}\,.
\end{aligned}$$

Fig. A.44



Force on the plates of the capacitor (Fig. A.44):

Force components (4.54):

$$F_i = \int_{S(V)} d\mathbf{f} \cdot \mathbf{T}_i = \int_{S(V)} d\mathbf{f} \cdot \sum_j T_{ij}\, \mathbf{e}_j = \int_{S(V)} df \sum_j T_{ij} n_j \, .$$

$T_{ij} \neq 0$ only in the inside of the capacitor:

$$\mathbf{n} = (0, 0, -1) \quad \text{upper plate}\, ,$$
$$\mathbf{n} = (0, 0, +1) \quad \text{lower plate}\, .$$

Force on the upper plate:

$$F_z\left(+\frac{d}{2}\right) = -\pi R^2 T_{zz}\left(+\frac{d}{2}\right) = -\frac{Q^2}{\pi R^2}\,\frac{1}{2\epsilon_0(\epsilon_1 + \Delta\epsilon)} \, .$$

Force on the lower plate:

$$F_z\left(-\frac{d}{2}\right) = +\pi R^2\, T_{zz}\left(-\frac{d}{2}\right) = \frac{Q}{\pi R^2}\,\frac{1}{2\epsilon_0 \epsilon_1} \, .$$

Because of the space-dependence of the dielectric constant $\epsilon_{\mathrm{r}} = \epsilon_{\mathrm{r}}(z)$ the forces on the two plates of the capacitor are different!

Solution 4.1.6

1. The rotation does not change the homogeneous charge density of the sphere:

$$\rho(\mathbf{r}, t) \equiv \rho(\mathbf{r}) = \begin{cases} q/\frac{4\pi}{3}R^3 \, , & \text{if } r \leq R \, , \\ 0 \, , & \text{if } r > R \, . \end{cases}$$

The **electric field**, too, is therewith time-independent and identical to that of the homogeneously charged sphere *'at rest'*. This field has been calculated as an example in Sect. 2.1.3:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\,\mathbf{e}_r \begin{cases} r/R^3 \, , & \text{if } r \leq R \, , \\ 1/r^2 \, , & \text{if } r > R \, . \end{cases}$$

2. Let the axis of rotation coincide with the z-axis and the center of the sphere with the origin of coordinates. Then the point of the sphere at $\mathbf{r}$ $(r < R)$ has the velocity

$$\mathbf{v}(\mathbf{r}) = \omega\, \mathbf{e}_z \times \mathbf{r} \; .$$

The time-independent charge density leads to a time-independent **current density**:

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\, \mathbf{v}(\mathbf{r}) = \frac{3q}{4\pi R^3}\, \omega(\mathbf{e}_z \times \mathbf{r})\Theta(R-r) \; .$$

With

$$\begin{aligned} \mathbf{e}_z \times \mathbf{e}_r &= (0,0,1) \times (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \\ &= (-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0) \\ &= \sin\vartheta\, \mathbf{e}_\varphi \end{aligned}$$

it is also valid:

$$\mathbf{j}(\mathbf{r}) = \frac{3q\omega}{4\pi R^3}\, r\sin\vartheta\, \Theta(R-r)\, \mathbf{e}_\varphi \; .$$

Cartesian components of the current density:

$$\begin{aligned} j_x(\mathbf{r}) &= -\frac{3q\omega}{4\pi R^3}\, r\, \Theta(R-r)\, \sin\vartheta\sin\varphi \; , \\ j_y(\mathbf{r}) &= \frac{3q\omega}{4\pi R^3}\, r\, \Theta(R-r)\, \sin\vartheta\cos\varphi \; , \\ j_z(\mathbf{r}) &= 0 \; . \end{aligned}$$

For the next part of this exercise a further transformation is advisable. With the spherical harmonic

$$Y_{11}(\vartheta,\varphi) = -\sqrt{\frac{3}{8\pi}}\, \sin\vartheta\, e^{i\varphi}$$

it can obviously be written:

$$j_x + ij_y = -i\sqrt{\frac{8\pi}{3}}\, \frac{3q\omega}{4\pi R^3}\, r\, \Theta(R-r)\, Y_{11}(\vartheta,\varphi) \; .$$

3. With **j** also the **vector potential** will be time-independent:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \equiv \mathbf{A}(\mathbf{r}) \, .$$

Because of $j_z = 0$, $A_z = 0$. For the calculation of the combination

$$\begin{aligned} A_x + iA_y &= \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{j_x(\mathbf{r}') + i j_y(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= -i\frac{\mu_0}{4\pi}\sqrt{\frac{8\pi}{3}}\,\frac{3q\omega}{4\pi R^3} \int d^3 r' \, \frac{r'\,\Theta(R-r')}{|\mathbf{r}-\mathbf{r}'|}\, Y_{11}(\vartheta',\varphi') \end{aligned}$$

we apply in the integrand the formula (2.169) ($r_< = \min(r,r')$; $r_> = \max(r,r')$),

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{r_>} \sum_{l,m} \frac{1}{2l+1} \left(\frac{r_<}{r_>}\right)^l Y^\star_{lm}\left(\vartheta',\varphi'\right) Y_{lm}(\vartheta,\varphi) \, ,$$

in order to exploit then the orthonormality of the spherical harmonics:

$$\begin{aligned} A_x + iA_y &= -i\frac{\mu_0}{4\pi}\sqrt{\frac{8\pi}{3}}\,\frac{3q\omega}{4\pi R^3}\, 4\pi \sum_{l,m} \frac{1}{2l+1} Y_{lm}(\vartheta,\varphi) \\ &\quad \cdot \int d^3 r' \, r' \, \Theta(R-r') \frac{r_<^l}{r_>^{l+1}} Y_{11}\left(\vartheta',\varphi'\right) \, Y^\star_{lm}\left(\vartheta',\varphi'\right) \\ &= -i\mu_0 \sqrt{\frac{8\pi}{3}}\,\frac{3q\omega}{4\pi R^3} \sum_{l,m} \frac{1}{2l+1} Y_{lm}(\vartheta,\varphi) \\ &\quad \cdot \int_0^R dr' \, r'^3 \frac{r_<^l}{r_>^{l+1}} \underbrace{\int_0^{2\pi} d\varphi' \int_{-1}^{+1} d\cos\vartheta' \, Y_{11}\left(\vartheta',\varphi'\right) \, Y^\star_{lm}\left(\vartheta',\varphi'\right)}_{\delta_{1l}\delta_{1m}} \\ &= -i\mu_0 \sqrt{\frac{8\pi}{3}}\,\frac{3q\omega}{4\pi R^3}\,\frac{1}{3} Y_{11}(\vartheta,\varphi) \int_0^R dr' \, r'^3 \frac{r_<}{r_>^2} \, . \end{aligned}$$

For the remaining integral

$$f(r,R) = \int_0^R dr' \, r'^3 \frac{r_<}{r_>^2}$$

we have to distinguish whether the point of observation $\mathbf{r}$ lies inside or outside the sphere:

$$f(r,R) = \begin{cases} \int\limits_0^R dr' \frac{r'^4}{r^2} = \frac{R^5}{5r^2} \ , & \text{if } r \geq R \\ \frac{1}{r^2} \int\limits_0^r dr' \, r'^4 + r \int\limits_r^R dr' \, r' = \frac{r}{2}\left(R^2 - \frac{3}{5}r^2\right) \ , & \text{if } r \leq R \ . \end{cases}$$

It follows therewith:

$$A_x + iA_y = i\mu_0 \frac{q\omega}{4\pi R^3} \sin\vartheta \, e^{i\varphi} f(r,R) \ .$$

In detail the following is thus valid for the Cartesian components of the vector potential:

$$\begin{aligned} A_x(r,\vartheta,\varphi) &= -\mu_0 \frac{q\omega}{4\pi R^3} \sin\vartheta \, \sin\varphi f(r,R) \ , \\ A_y(r,\vartheta,\varphi) &= \mu_0 \frac{q\omega}{4\pi R^3} \sin\vartheta \, \cos\varphi f(r,R) \ , \\ A_z(r,\vartheta,\varphi) &= 0 \ . \end{aligned}$$

With

$$-\sin\varphi \, \mathbf{e}_x + \cos\varphi \, \mathbf{e}_y = \mathbf{e}_\varphi$$

the vector potential can eventually be compressed to:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= A_\varphi(r,\vartheta) \, \mathbf{e}_\varphi \ , \\ A_\varphi(r,\vartheta) &= \mu_0 \frac{q\omega}{4\pi R^3} \sin\vartheta \, f(r,R) \ . \end{aligned}$$

4. According to (1.380) from Vol. 1 the curl reads in arbitrary curvilinear coordinates y_1, y_2, y_3:

$$\operatorname{curl}\mathbf{A} = \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \begin{vmatrix} b_{y_1}\mathbf{e}_{y_1} & b_{y_2}\mathbf{e}_{y_2} & b_{y_3}\mathbf{e}_{y_3} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ b_{y_1}A_{y_1} & b_{y_2}A_{y_2} & b_{y_3}A_{y_3} \end{vmatrix} \ .$$

Thereby it is ((1.370), Vol. 1):

$$b_{y_i} = \left| \frac{\partial \mathbf{r}}{\partial y_i} \right| \ .$$

In the case of spherical coordinates ($b_r = 1$; $b_\vartheta = r$; $b_\varphi = r\sin\vartheta$) it remains to be evaluated for the **magnetic induction**:

$$\begin{aligned}\mathbf{B} = \mathrm{curl}\mathbf{A} &= \frac{1}{r^2\sin\vartheta}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\vartheta & r\sin\vartheta\,\mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & r\sin\vartheta A_\varphi\end{vmatrix} \\ &= B_r\,\mathbf{e}_r + B_\vartheta\,\mathbf{e}_\vartheta \ .\end{aligned}$$

Radial component:

$$\begin{aligned}B_r &= \frac{1}{r^2\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\big(r\sin\vartheta A_\varphi(r,\vartheta)\big)\right) \\ &= \frac{1}{r^2\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\big(r\sin\vartheta\mu_0\,\frac{q\omega}{4\pi R^3}\,\sin\vartheta f(r,R)\big)\right) \\ &= \mu_0\,\frac{q\omega}{4\pi R^3}\frac{f(r,R)}{r\sin\vartheta}\big(2\sin\vartheta\cos\vartheta\big) \ .\end{aligned}$$

With $f(r,R)$ from part 3. one gets:

$$B_r(r,\vartheta) = \mu_0\,\frac{q\omega\cos\vartheta}{4\pi R^3}\begin{cases}\dfrac{2R^5}{5r^3} \ , & \text{if } r \geq R \ , \\ R^2 - \dfrac{3}{5}r^2 \ , & \text{if } r \leq R \ .\end{cases}$$

Angular component:

$$\begin{aligned}B_\vartheta &= -\frac{1}{r\sin\vartheta}\,\frac{\partial}{\partial r}\big(r\sin\vartheta\,A_\varphi(r,\vartheta)\big) \\ &= -\frac{1}{r\sin\vartheta}\,\frac{\partial}{\partial r}\left(r\sin\vartheta\,\mu_0\,\frac{q\omega}{4\pi R^3}\,\sin\vartheta f(r,R)\right) \\ &= -\mu_0\,\frac{q\omega}{4\pi R^3}\,\frac{\sin\vartheta}{r}\,\frac{\partial}{\partial r}\big(rf(r,R)\big) \\ &= -\mu_0\,\frac{q\omega}{4\pi R^3}\,\frac{\sin\vartheta}{r}\,\frac{\partial}{\partial r}\begin{cases}\dfrac{R^5}{5r} \ , & \text{if } r \geq R \\ \dfrac{r^2}{2}\left(R^2 - \dfrac{3}{5}r^2\right) \ , & \text{if } r \leq R\end{cases} \\ &= -\mu_0\,\frac{q\omega}{4\pi R^3}\,\frac{\sin\vartheta}{r}\begin{cases}-\dfrac{R^5}{5r^2} \ , & \text{if } r \geq R \\ rR^2 - \dfrac{12}{10}r^3 \ , & \text{if } r \leq R \ .\end{cases}\end{aligned}$$

We have therewith found for the ϑ-component of the magnetic induction:

$$B_\vartheta(r,\vartheta) = \mu_0 \frac{q\omega \sin\vartheta}{4\pi R^3} \begin{cases} \dfrac{R^5}{5r^3}\,, & \text{if } r \geq R \\ -R^2 + \dfrac{6}{5}r^2\,, & \text{if } r \leq R\,. \end{cases}$$

5. **Field-momentum density:**

$$\begin{aligned} \bar{\mathbf{p}}_{\text{field}} &= \mathbf{D} \times \mathbf{B} = \varepsilon_0\, E_r \mathbf{e}_r \times \left(B_r \mathbf{e}_r + B_\vartheta \mathbf{e}_\vartheta\right) \\ &= \varepsilon_0 E(r) B_\vartheta(r,\vartheta)\, \mathbf{e}_\varphi = \bar{p}_{\text{Feld}}\, \mathbf{e}_\varphi\ . \end{aligned}$$

Thereby we can use the results from the parts 1. and 4.:

$$\bar{p}_{\text{field}}(r,\vartheta) = \mu_0 \left(\frac{q}{4\pi}\right)^2 \frac{\omega \sin\vartheta}{R^3} \begin{cases} \dfrac{R^5}{5r^5}\,, & \text{if } r \geq R \\ \dfrac{6}{5}\dfrac{r^3}{R^3} - \dfrac{r}{R}\,, & \text{if } r \leq R\,. \end{cases}$$

6. **Field-angular momentum:**

$$\mathbf{L}_{\text{field}} = \int d^3r(\mathbf{r} \times \bar{\mathbf{p}}_{\text{field}}) = -\int d^3r\, r\, \bar{p}_{\text{field}}(r,\vartheta)\, \mathbf{e}_\vartheta$$

Because of

$$\mathbf{e}_\vartheta = (\cos\vartheta \cos\varphi, \cos\vartheta \sin\varphi, -\sin\vartheta)$$

the φ-integration,

$$\int_0^{2\pi} d\varphi\, \cos\varphi(\sin\varphi) = 0\ \ ,$$

lets the x- and y- components of the angular momentum vanish:

$$\mathbf{L}_{\text{field}} = L\,\mathbf{e}_z$$

It remains to be calculated:

$$L = 2\pi \int_0^\infty r^2 dr \int_{-1}^{+1} d\cos\vartheta\ r \sin\vartheta\ p_{\text{field}}(r,\vartheta)$$

$$= \mu_0 \frac{q^2\omega}{8\pi R^3} \int_{-1}^{+1} d\cos\vartheta \, (1-\cos^2\vartheta) \left(\int_0^R r^3 dr \left(\frac{6}{5}\frac{r^3}{R^3} - \frac{r}{R} \right) + \int_R^\infty r^3 dr \frac{R^5}{5r^5} \right)$$

$$= \mu_0 \frac{q^2\omega}{8\pi R^3} \left(2 - \frac{2}{3} \right) \left(\frac{6}{5}\frac{R^4}{7} - \frac{R^4}{5} + \frac{R^4}{5} \right)$$

Therewith a relatively simple result arises eventually for the angular momentum of the electromagnetic field:

$$\mathbf{L}_{\text{field}} = \mu_0 \frac{q^2 R}{35\pi} \, \omega \, \mathbf{e}_z \; .$$

It thus has the same direction as the angular velocity of the rotating homogeneously charged sphere!

Section 4.2.7

Solution 4.2.1

1. **Current density** (cylindrical coordinates)

$$\mathbf{j}(\mathbf{r}) = j(\rho) \, \mathbf{e}_z \; ,$$
$$j(\rho) = j_\mathrm{i} \delta(\rho - R_\mathrm{i}) + j_\mathrm{a} \, \delta(\rho - R_\mathrm{a}) \; ,$$

K_R: circle with the radius R perpendicular to the z-axis.
$R_\mathrm{i} < R < R_\mathrm{a}$:

$$I = \int_{K_R} \mathbf{j}(\mathbf{r}) \cdot d\mathbf{f} = 2\pi \int_{K_R} d\rho \, \rho \, j_\mathrm{i} \delta(\rho - R_\mathrm{i}) = 2\pi \, R_\mathrm{i} \, j_\mathrm{i}$$

$$\Longrightarrow \; j_\mathrm{i} = \frac{I}{2\pi \, R_\mathrm{i}} \; .$$

$R_\mathrm{a} < R$:

$$0 = \int_{K_R} \mathbf{j}(\mathbf{r}) \cdot d\mathbf{f} = 2\pi (R_\mathrm{i} j_\mathrm{i} + R_\mathrm{a} j_\mathrm{a}) = I + 2\pi \, R_\mathrm{a} j_\mathrm{a}$$

$$\Longrightarrow \; j_\mathrm{a} = -\frac{I}{2\pi \, R_\mathrm{a}}$$

$$\Longrightarrow \; \mathbf{j}(\mathbf{r}) = \frac{I}{2\pi\rho} \big(\delta(\rho - R_\mathrm{i}) - \delta(\rho - R_\mathrm{a}) \big) \, \mathbf{e}_z \; .$$

Quasi-stationary approximation:

$$\mathrm{curl}\mathbf{B} \approx \mu_\mathrm{r}\mu_0 \mathbf{j} \iff \oint\limits_C \mathbf{B} \cdot d\mathbf{r} \approx \mu_\mathrm{r}\mu_0 \int\limits_{F_c} \mathbf{j} \cdot d\mathbf{f} \,.$$

From symmetry reasons:

$$\mathbf{B} = B(\rho)\,\mathbf{e}_\varphi \,,$$

$$\oint\limits_{K_\rho} \mathbf{B} \cdot d\mathbf{r} = B(\rho) 2\pi\,\rho = \begin{cases} 0\,, & \text{if } \rho < R_\mathrm{i}\,, \\ \mu_\mathrm{r}\mu_0 I\,, & \text{if } R_\mathrm{i} < \rho < R_\mathrm{a}\,, \\ 0\,, & \text{if } R_\mathrm{a} < \rho\,, \end{cases}$$

$$\Longrightarrow \quad B(\rho) = \begin{cases} \mu_\mathrm{r}\mu_0 \dfrac{I}{2\pi\,\rho}\,, & \text{if } R_\mathrm{i} < \rho < R_\mathrm{a}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

2. **Magnetic flux**
   Appears only between the inner and the outer conductor (Fig. A.45). There it is

$$\Phi_c = \int\limits_{F_c} \mathbf{B} \cdot d\mathbf{f} = \int\limits_{R_\mathrm{i}}^{R_\mathrm{a}} d\rho \int dz\, B(\rho) = l_c \mu_\mathrm{r}\mu_0 \frac{I}{2\pi} \int\limits_{R_\mathrm{i}}^{R_\mathrm{a}} d\rho \frac{1}{\rho}$$

$$= l_c \mu_\mathrm{r}\mu_0 \frac{I}{2\pi} \ln \frac{R_\mathrm{a}}{R_\mathrm{i}} \,.$$

It follows herefrom:

The magnetic flux per unit-length that penetrates the space $R_\mathrm{i} < \rho < R_\mathrm{a}$ amounts to:

$$\Phi = \frac{\Phi_c}{l_c} = \mu_\mathrm{r}\,\mu_0 \frac{\ln(R_\mathrm{a}/R_\mathrm{i})}{2\pi} I \,.$$

Fig. A.45

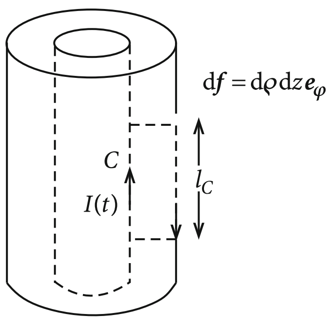
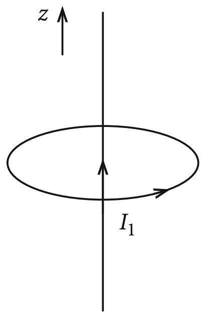


Fig. A.46



Result:
Self-inductance per unit-length of the hollow conductor:

$$L = \mu_r \mu_0 \frac{\ln(R_a/R_i)}{2\pi} \ .$$

Solution 4.2.2

1. **Magnetic field of the wire**
Quasi-stationary approximation:

$$\text{curl}\mathbf{H} \approx \mathbf{j} \ .$$

Cylindrical coordinates (ρ, φ, z).
Symmetry (Fig. A.46) $\Longrightarrow$ ansatz:

$$\mathbf{H}_1 = H(\rho)\, \mathbf{e}_\varphi^{(1)} \ .$$

K_ρ: circle in a plane perpendicular to the wire with the radius ρ:

$$\int\limits_{K_\rho} \text{curl}\mathbf{H} \cdot d\mathbf{f} \ = \int\limits_{\partial K_\rho} \mathbf{H} \cdot d\mathbf{r} = 2\pi\, \rho H(\rho) \approx \int\limits_{K_\rho} \mathbf{j} \cdot d\mathbf{f} = I_1$$

$$\Longrightarrow \ \mathbf{H}_1 = \frac{I_1}{2\pi\, \rho}\, \mathbf{e}_\varphi^{(1)} \ .$$

Flux through the conductor loop
Area element: $d\mathbf{f}_2 = -dx\,dy\,\mathbf{e}_z$. There it is obviously $\mathbf{e}_\varphi^{(1)} = \mathbf{e}_z$; $\rho = y$.
We therewith get for the magnetic flux:

$$\Phi_{21} = \int \mathbf{B}_1\, d\mathbf{f}_2 = -\mu_0 \frac{I_1}{2\pi} \int_0^a dx \int_d^{d+b} \frac{dy}{y} = -\mu_0 \frac{I_1}{2\pi} a \ln\left(1 + \frac{b}{d}\right) = L_{21} I_1$$

$$\Longrightarrow L_{21} = -\mu_0 \frac{a}{2\pi} \ln\left(1 + \frac{b}{d}\right) .$$

2. **Magnetic interaction energy** (4.72):

$$L_{12} = L_{21}$$
$$\Longrightarrow W_\mathrm{m} = L_{21} I_1 I_2 .$$

Change Δd of the distance d where $I_1, I_2 = \mathrm{const}$:

$$dW_\mathrm{m} = I_1 I_2 \Delta L_{21} = -\mu_0 \frac{a}{2\pi} I_1 I_2 \frac{-b/d^2}{1 + b/d} \Delta d = I_1 I_2 \frac{\mu_0 a\, b}{2\pi\, d(d+b)} \Delta d ,$$
$$dW_\mathrm{mech} = -dW_\mathrm{m} = -F_y \Delta d$$
$$\Longrightarrow F_y = I_1 I_2 \frac{\mu_0 a\, b}{2\pi\, d(d+b)} .$$

3.

$$I_1(t) = I_0\left(1 - e^{-\alpha t}\right) \Longrightarrow \dot{I}_1(t) = \alpha I_0\, e^{-\alpha t} .$$

That means:

$$U_\mathrm{ind}(t) = -L_{21} \dot{I}_1(t) = \mu_0 \frac{a}{2\pi} \ln\left(1 + \frac{b}{a}\right) \alpha I_0\, e^{-\alpha t} .$$

Solution 4.2.3

1. **Switching-on process**
Differential equation to be solved:

$$L\dot{I}(t) + R(t) I(t) = U .$$

This reads for $0 \le t \le \tau$:

$$L\dot{I}(t) + R_0 \tau \frac{I(t)}{t} = U .$$

Plausible ansatz:

$$I(t) = \alpha\, t\ .$$

This leads to:

$$L\alpha + R_0\tau\,\alpha = U \implies \alpha = \frac{U}{L + R_0\tau}\ .$$

It therefore holds:

$$I(t) = \frac{U}{L + R_0\tau}t \quad 0 \le t \le \tau\ .$$

For $t \ge \tau$ it is $R(t) \equiv R_0$.

It is therefore to be solved:

$$\begin{aligned}
&L\dot{I}(t) + R_0 I(t) = U \\
&\implies L\dot{I}(t) + R_0\left(I(t) - \frac{U}{R_0}\right) = 0 \\
&\implies L\frac{d}{dt}\left(I(t) - \frac{U}{R_0}\right) + R_0\left(I(t) - \frac{U}{R_0}\right) = 0\ .
\end{aligned}$$

Solution:

$$I(t) - \frac{U}{R_0} = \left(I(\tau) - \frac{U}{R_0}\right)\exp\left[-\frac{R_0}{L}(t-\tau)\right]\ .$$

Continuity of $I(t)$:

$$\begin{aligned}
I(\tau) &= \frac{U\tau}{L + R_0\tau} = \frac{U}{R_0}\,\frac{\tau}{L/R_0 + \tau}\ , \\
I(\tau) - \frac{U}{R_0} &= \frac{U}{R_0}\left(\frac{\tau}{L/R_0 + \tau} - 1\right) = \frac{U}{R_0}\,\frac{-L/R_0}{L/R_0 + \tau} \\
\implies I(t) &= \frac{U}{R_0}\left\{1 - \frac{L/R_0}{L/R_0 + \tau}\exp\left[-\frac{R_0}{L}(t-\tau)\right]\right\} \quad \tau \le t\ .
\end{aligned}$$

The final value U/R_0 is only exponentially achieved.

Characteristic time constant of the switching-on process: L/R_0

$\implies$ '*quick*' switching-on: $\tau \ll L/R_0 \implies I(\tau) \ll U/R_0$,
'*slow*' switching-on: $\tau \gg L/R_0 \implies I(\tau) \approx U/R_0$.

2. **Switching-off process**

$$L\dot{I}(t) + R(t)I(t) = U \;.$$

This is an inhomogeneous differential equation of the first order!
For $0 \leq t < \tau$ we define:

$$\alpha = \frac{R_0\tau}{L} \;.$$

Homogeneous differential equation:

$$\begin{aligned}
\dot{I}(t) + \frac{\alpha}{\tau - t}I = 0 \;\Longrightarrow\; & \frac{dotI}{I} = -\frac{\alpha}{\tau - t} \;, \\
\Longrightarrow\; & \frac{d}{dt}\ln I = \frac{d}{dt}\ln(\tau - t)^{\alpha} \\
\Longrightarrow\; & I_{\mathrm{hom}}(t) = c(\tau - t)^{\alpha} \;.
\end{aligned}$$

Special solution:
Ansatz: $I_{\mathrm{S}}(t) = \beta(\tau - t)$.
Insertion:

$$\begin{aligned}
-\beta L + \frac{R_0\tau}{\tau - t}\beta(\tau - t) = U \;\Longrightarrow\; & \beta = \frac{U}{L(\alpha - 1)} \quad (\alpha \neq 1) \\
\Longrightarrow\; & I_{\mathrm{S}}(t) = \frac{U}{L(\alpha - 1)}(\tau - t) \;.
\end{aligned}$$

General solution of the inhomogeneous differential equation:

$$I(t) = c(\tau - t)^{\alpha} + \frac{U}{L(\alpha - 1)}(\tau - t) \;.$$

Boundary condition:

$$\begin{aligned}
I(0) = \frac{U}{R_0} &= c\,\tau^{\alpha} + \frac{U\,\tau}{L(\alpha - 1)} \\
\Longrightarrow\; c &= \frac{U}{R_0}\tau^{-\alpha} - \frac{U\,\tau^{1-\alpha}}{L(\alpha - 1)} \\
\Longrightarrow\; I(t) &= \frac{U}{R_0}\left(1 - \frac{t}{\tau}\right)^{\alpha} - \frac{U\,\tau}{L(\alpha - 1)}\left(1 - \frac{t}{\tau}\right)^{\alpha} + \frac{U\,\tau}{L(\alpha - 1)}\left(1 - \frac{t}{\tau}\right) \\
&= U\left(1 - \frac{t}{\tau}\right)^{\alpha}\left(\frac{1}{R_0} - \frac{\tau}{R_0\tau - L}\right) + \frac{U\,\tau}{\alpha\,L}\,\frac{\alpha}{\alpha - 1}\left(1 - \frac{t}{\tau}\right) \\
&= \frac{U}{R_0}\left(1 - \frac{t}{\tau}\right)^{\alpha}\frac{-1}{\alpha - 1} + \frac{U}{R_0}\,\frac{\alpha}{\alpha - 1}\left(1 - \frac{t}{\tau}\right) \;.
\end{aligned}$$

That means:

$$I(t) = \frac{U}{R_0} \, \frac{\alpha \, (1 - t/\tau) - (1 - t/\tau)^\alpha}{\alpha - 1} \; .$$

Special cases:

$$I(t = 0) = \frac{U}{R_0} \; ; \quad I(t = \tau) = 0 \; .$$

Solution 4.2.4

1. $t > t_0$:

$$\begin{aligned} U_0 &= U_C + U_R \; , \\ I &= \dot{Q} = C\,\dot{U}_C \\ \Longrightarrow \; U_0 &= U_C + RC\,\dot{U}_C \; . \end{aligned}$$

General solution of the homogeneous differential equation:

$$\begin{aligned} &\dot{U}_C + \frac{1}{RC} U_C = 0 \\ \Longrightarrow \; &U_C^{(\mathrm{hom})}(t) = A\, e^{-t/RC} \; . \end{aligned}$$

Special solution:

$$U_C = U_0 \quad \text{(after the settling phase)} \; .$$

General solution of the inhomogeneous differential equation:

$$U_C(t) = U_0 + A\, e^{-t/RC} \; .$$

Initial conditions:

$$U_C(t = t_0) = 0 \; \Longrightarrow \; A = -U_0\, e^{t_0/RC} \; .$$

Solution:

$$\begin{aligned} U_C(t) &= U_0 (1 - e^{-(t - t_0)/RC}) \; , \\ I(t) &= C\,\dot{U}_C(t) = \frac{U_0}{R} e^{-(t - t_0)/RC} \; , \\ U_R(t) &= R\,I(t) = U_0\, e^{-(t - t_0)/RC} \; . \end{aligned}$$

2.

$$
\begin{aligned}
&t > t_1: \quad 0 = U_\mathrm{C} + RC\dot{U}_C \;, \\
&t = t_1: \quad U_0 = U_C \\
&\Longrightarrow \; U_C(t) = A\, e^{-t/RC} \;; \quad U_0 = A\, e^{-t_1/RC} \;.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
U_C(t) &= U_0\, e^{-(t-t_1)/RC} \;, \\
I(t) &= -\frac{U_0}{R}\, e^{-(t-t_1)/RC} \;, \\
U_R(t) &= -U_0\, e^{-(t-t_1)/RC} \;.
\end{aligned}
$$

Solution 4.2.5 Voltage balances:

- left:

$$
U_e(t) - \frac{1}{C_1}\int^t I_1\, dt' + \frac{1}{C_1}\int^t I_2\, dt' + U_{L_1} = 0 \;.
$$

- right:

$$
-\left(\frac{1}{C_{21}} + \frac{1}{C_1} + \frac{1}{C_{22}}\right)\int^t I_2\, dt' + \frac{1}{C_1}\int^t I_1\, dt' + U_{L_2} = 0 \;.
$$

- Inductances

$$
U_{L_1} = -L_1\dot{I}_1 \;; \quad U_{L_2} = -L_2\dot{I}_2 \;.
$$

For abbreviation we define,

$$
\frac{1}{C_2} \equiv \frac{1}{C_{21}} + \frac{1}{C_1} + \frac{1}{C_{22}} \;,
$$

and differentiate the voltage-equations with respect to the time:

$$
\begin{aligned}
\frac{1}{C_1}I_1 - \frac{1}{C_1}I_2 + L_1\ddot{I}_1 &= \dot{U}_e \;, \\
-\frac{1}{C_2}I_2 + \frac{1}{C_1}I_1 - L_2\ddot{I}_2 &= 0 \;.
\end{aligned}
$$

For the calculation of the eigenfrequencies it must be taken $U_e = 0$! With the ansatz for the then homogeneous system of equations,

$$I_1 = I_{10}\, e^{i\omega t}\ ; \quad I_2 = I_{20}\, e^{i\omega t}\ ,$$

it remains to solve:

$$\left(\frac{1}{C_1} - \omega^2 L_1\right) I_{10} - \frac{1}{C_1} I_{20} = 0\ ,$$
$$\frac{1}{C_1} I_{10} + \left(-\frac{1}{C_2} + \omega^2 L_2\right) I_{20} = 0\ .$$

A non-trivial solution requires a vanishing secular determinant:

$$\left(\frac{1}{C_1} - \omega^2 L_1\right)\left(-\frac{1}{C_2} + \omega^2 L_2\right) + \frac{1}{C_1^2} = 0$$
$$\curvearrowright\ \left(1 - \omega^2 C_1 L_1\right)\left(1 - \omega^2 C_2 L_2\right) = \frac{C_2}{C_1}\ .$$

With the eigenfrequencies of the **uncoupled** circuits,

$$\omega_1^2 = \frac{1}{L_1 C_1}\ ; \quad \omega_2^2 = \frac{1}{L_2 C_2}$$

it must eventually be found the solutions for

$$\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right) = \frac{C_2}{C_1}\omega_1^2\omega_2^2\ .$$

These are then the **eigenfrequencies of the capacitively coupled circuits**:

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\frac{1}{4}\left(\omega_1^2 - \omega_2^2\right)^2 + \frac{C_2}{C_1}\omega_1^2\omega_2^2}\ .$$

Solution 4.2.6

1. **n**: Unit vector perpendicular to the area of the wire:

$$d\mathbf{f} = df\,\mathbf{n}\ ,$$
$$\sphericalangle(\mathbf{n}, \mathbf{B}) = \varphi(t) = \omega(t - t_0)\ ,$$
$$U_{\text{ind}} = -\frac{\partial}{\partial t}\Phi\ ,$$

$$\Phi = \int\limits_{\text{ring}} d\mathbf{f} \cdot \mathbf{B} \;=\; \int\limits_{\text{ring}} df\, \mathbf{n} \cdot \mathbf{B} = B\, \cos[\omega(t-t_0)] \int\limits_{\text{ring}} df \;= B\,\pi\, R^2\, \cos[\omega(t-t_0)]$$

$$\Longrightarrow\; U_{\text{ind}} = B\,\pi\, R^2\,\omega\, \sin[\omega(t-t_0)]\;.$$

2.

$$U_{\text{ind}} = \oint\limits_{\text{Ring}} \mathbf{E} \cdot d\mathbf{r} = \frac{1}{\sigma} \oint\limits_{\text{ring}} \mathbf{j} \cdot d\mathbf{r}\;, \quad \mathbf{j} \uparrow\uparrow d\mathbf{r}\;,$$

$$U_{\text{ind}} = \frac{1}{\sigma} \oint\limits_{\text{ring}} j\, dr = \frac{I}{\sigma\, A} \oint\limits_{\text{ring}} dr = \frac{2\pi\, R}{\sigma\, A} I$$

$$\Longrightarrow\; I(t) = \frac{1}{2} \sigma\, B\, A\, R\, \omega\, \sin\big(\omega(t-t_0)\big)\;.$$

Solution 4.2.7

1. Which part of the rectangular conductor loop is covered by the magnetic induction (Fig. A.47)?
   $0 \le vt < d$:

$$F(t) = a_2 vt$$

$d \le vt < a_1$:

$$F(t) = a_2 d$$

$a_1 \le vt < a_1 + d$:

$$F(t) = a_2(a_1 + d - vt)$$

Fig. A.47

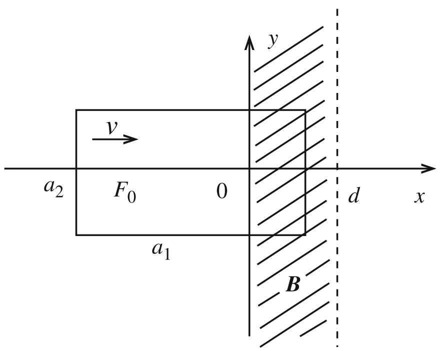
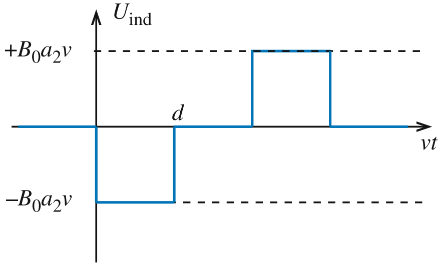


Fig. A.48



$\Longrightarrow$ magnetic flux through the conductor loop:

$$\Phi(t) = \int_{F_0} \mathbf{B} \cdot d\mathbf{f} = \int_{F(t)} \mathbf{B} \cdot d\mathbf{f}$$

$$= B_0 a_2 \; , \begin{cases} vt \; , & \text{if } 0 \leq vt < d \; , \\ d \; , & \text{if } d \leq vt < a_1 \; , \\ a_1 + d - vt \; , & \text{if } a_1 \leq vt < a_1 + d \; , \\ 0 \; , & \text{otherwise} \; . \end{cases}$$

$\Longrightarrow$ induced voltage (Fig. A.48):

$$U_{\text{ind}} = -\dot{\Phi}$$

$$= -B_0 a_2 v \cdot \begin{cases} +1 \; , & \text{if } 0 \leq vt < d \; , \\ 0 \; , & \text{if } d \leq vt < a_1 \; , \\ -1 \; , & \text{if } a_1 \leq vt < a_1 + d \; , \\ 0 \; , & \text{otherwise} \; . \end{cases}$$

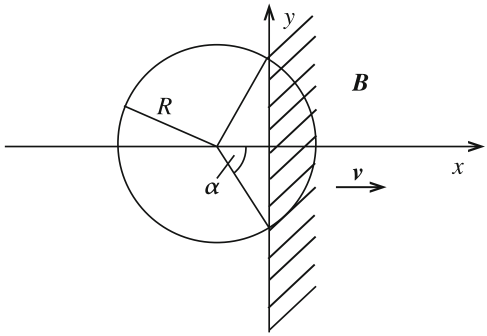
2. Finite overlap of the conductor loop with the homogeneous magnetic induction $\mathbf{B} = B_0 \mathbf{e}_z$ (Fig. A.49) for

$$0 < vt \leq 2R \; .$$

The area covered by the field is the difference from the segment of the circle

$$\Delta_\alpha = \alpha R^2$$

Fig. A.49



and the triangle

$$\begin{aligned} \Delta_\Delta &= \frac{1}{2}(R\cos\alpha)(2R\sin\alpha) \\ &= R^2\cos\alpha\sin\alpha \\ \Longrightarrow \quad F(t) &= R^2(\alpha - \cos\alpha\sin\alpha)\ . \end{aligned}$$

The angle α is time-dependent:

$$\alpha = \alpha(t)\ ; \quad \cos\alpha(t) = \frac{R - vt}{R}\ , \Longrightarrow \dot{\alpha}(t)\sin\alpha(t) = \frac{v}{R}\ .$$

In addition:

$$F(t) = \pi R^2 = \text{const}\ , \quad \text{if } vt \geq 2R\ .$$

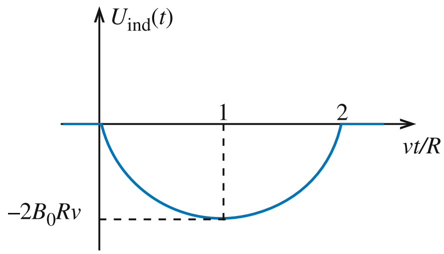
Magnetic flux:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{f} = B_0 F(t)\ .$$

Induced voltage:

$$\begin{aligned} U_{\text{ind}} &= -\dot{\Phi} = -R^2(1 + \sin^2\alpha - \cos^2\alpha)\dot{\alpha}(t)B_0 \\ &= -2R^2\sin^2\alpha \cdot \dot{\alpha}(t)B_0 \\ &= -2R^2\sin\alpha \cdot \frac{v}{R}B_0 \\ \sin\alpha &= \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \frac{1}{R^2}(R^2 - 2Rvt + v^2t^2)} \\ \Longrightarrow \quad U_{\text{ind}} &= -2B_0 Rv\sqrt{\frac{vt}{R}\left(2 - \frac{vt}{R}\right)}\ . \end{aligned}$$

Fig. A.50



One should realize that it is just about a circle equation (Fig. A.50):

$$x = \frac{U_{\mathrm{ind}}}{2B_0Rv}$$

$$y = \frac{vt}{R}$$

$$\Longrightarrow \; x^2 = 2y - y^2 = -(y-1)^2 + 1$$

$$\Longrightarrow \; x^2 + (y-1)^2 = 1 \; .$$

Center at

$$(x, y) = (0, 1) \; .$$

Solution 4.2.8

1. Quasi-stationary approximation

$$\begin{aligned} \mathrm{curl}\mathbf{E} &= -\dot{\mathbf{B}} \; ; \; \mathrm{curl}\mathbf{H} = \mathbf{j} \; , \\ \mathrm{div}\,\mathbf{D} &= \rho \; ; \quad \mathrm{div}\,\mathbf{B} \; = 0 \; . \end{aligned}$$

Coulomb-gauge ($\mathrm{div}\,\mathbf{A} = 0$)

$$\begin{aligned} \Longrightarrow \quad \Delta\varphi(\mathbf{r}, t) &= -\frac{1}{\epsilon_0}\rho(\mathbf{r}, t) = -\frac{1}{\epsilon_0}\rho(r) \\ \Delta\mathbf{A}(\mathbf{r}, t) &= -\mu_0\mathbf{j}(\mathbf{r}, t) \; , \\ \Longrightarrow \quad \varphi(\mathbf{r}, t) \equiv \varphi(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0 r} \quad (r > R) \; . \end{aligned}$$

With respect to the vector potential it is the same problem as in Exercise 3.3.1, however now with a time-dependent magnetic moment!

$$\mathbf{m}(t) = \frac{1}{3} q R^2 \boldsymbol{\omega}(t)$$
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{\mathbf{m}(t) \times \mathbf{r}}{r^3}$$

Electric field

$$\mathbf{E} = \underbrace{\mathbf{E}_0}_{\substack{\text{before the} \\ \text{deceleration } (\gamma=0)}} + \mathbf{E}_1$$
$$\mathbf{E}_0 = -\nabla\varphi = \frac{q}{4\pi\epsilon_0 r^3}\mathbf{r}$$

Induced excess field: $\mathbf{E}_1 = -\dot{\mathbf{A}}$

$$\dot{\boldsymbol{\omega}}(t) = -\gamma \boldsymbol{\omega}(t) \quad (t > 0)$$
$$\Longrightarrow \mathbf{E}_1 = -\frac{\mu_0}{4\pi} \cdot \frac{1}{3} q R^2 \frac{\dot{\boldsymbol{\omega}}(t) \times \mathbf{r}}{r^3} ,$$
$$\mathbf{E}_1(\mathbf{r}, t) = \frac{\mu_0 \gamma q R^2}{12\pi r^3} (\boldsymbol{\omega}(t) \times \mathbf{r}) \quad (t > 0) .$$

2.

$$\Longrightarrow \frac{|\mathbf{E}_1|}{|\mathbf{E}_0|} \leq \frac{\epsilon_0 \mu_0}{3} \gamma R^2 |\boldsymbol{\omega}| \overset{!}{\ll} 1 ,$$
$$\gamma \omega_0 R^2 \ll 3 \cdot c^2 \quad (c\text{: velocity of light}) .$$

3. Emitted energy per unit-time (4.48)

$$\int_{S(V)} d\mathbf{f} \cdot \mathbf{S}(\mathbf{r}, t) = \dot{W}_S .$$

$S(V)$: Surface of a sphere with the radius R^+
 $\mathbf{S}(\mathbf{r}, t)$: Poynting vector

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) ,$$
$$d\mathbf{f} = df \mathbf{e}_r$$
$$\Longrightarrow d\mathbf{f} \cdot \mathbf{S} = df \mathbf{e}_r \cdot (\mathbf{E}_1 \times \mathbf{H}) , \quad \text{since} \quad E_0 \sim \mathbf{e}_r .$$

Emission thus only as a consequence of the deceleration.

It holds as in Exercise 3.3.1:

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{4\pi}\left(\frac{3(\mathbf{r}\cdot\mathbf{m}(t))\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right)$$

$$\Longrightarrow \mathbf{E}_1 \times \mathbf{H} = \frac{\mu_0\gamma q R^2}{48\pi^2 r^6}\left((\boldsymbol{\omega}\times\mathbf{r})\times(3(\mathbf{m}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{m})\right)$$

$$= \frac{\mu_0\gamma q^2 R^4}{144\pi^2 r^6}\underbrace{\left((\boldsymbol{\omega}\times\mathbf{r})\times(3(\boldsymbol{\omega}\cdot\mathbf{e}_r)\mathbf{e}_r - \boldsymbol{\omega})\right)}_{-2(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}+\mathbf{r}(3(\boldsymbol{\omega}\cdot\mathbf{e}_r)^2-\omega^2)}$$

$$\Longrightarrow \mathbf{e}_r\cdot(\mathbf{E}_1\times\mathbf{H}) = r\left((\boldsymbol{\omega}\cdot\mathbf{e}_r)^2 - \omega^2\right)\mu_o\gamma\left(\frac{qR^2}{12\pi r^3}\right)^2$$

$$= \mu_0\gamma\left(\frac{qR^2}{12\pi r^3}\right)^2 r\omega^2(\cos^2\vartheta - 1)\,.$$

$\Longrightarrow$ emitted energy per unit-time:

$$\dot{W}_S(t) = \mu_0\gamma\left(\frac{qR^2\omega}{12\pi R^3}\right)^2 R^3 2\pi \underbrace{\int_{-1}^{+1} d\cos\vartheta\,(\cos^2\vartheta - 1)}_{\frac{2}{3}-2=-\frac{4}{3}}$$

$$= -\mu_0\gamma\frac{8\pi}{3}\left(\frac{q\omega}{12\pi R}\right)^2 R^3$$

$$\Longrightarrow \dot{W}_S(t) = -\mu_0\gamma\frac{q^2 R}{54\pi}\omega^2 \quad (t>0)\,.$$

4.

$$W_S = \int_0^\infty \dot{W}_S(t)dt$$

$$= -\mu_0\gamma\frac{q^2 R}{54\pi}\omega_0^2\int_0^\infty dt e^{-2\gamma t}$$

$$= -\mu_0\frac{q^2 R}{108\pi}\omega_0^2$$

independent of γ, proportional to the square of the initial velocity.

Section 4.3.18

Solution 4.3.1

1. Lorentz force:

$$\mathbf{F} = q\big[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\big] \ .$$

Equation of motion:

$$m\ddot{\mathbf{r}} = q\big[\mathbf{E} + (\dot{\mathbf{r}} \times \mathbf{B})\big] \ .$$

Temporal change of the particle energy:

$$\dot{W} = \mathbf{v} \cdot \mathbf{F} = q\, \mathbf{v} \cdot \mathbf{E} \ .$$

2. Maxwell equations ($\rho_f = 0,\ \mathbf{j}_f = 0,\ \sigma = 0$):

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0 \ ; \quad \operatorname{div} \mathbf{B} = 0 \ ; \\ \operatorname{curl}\mathbf{E} &= -\dot{\mathbf{B}} \ ; \operatorname{curl}\mathbf{B} = \frac{1}{u^2}\dot{\mathbf{E}} \ , \end{aligned}$$

where $u = 1/\sqrt{\epsilon_{\mathrm{r}}\epsilon_0\mu_{\mathrm{r}}\mu_0}$.

$$\begin{aligned} \operatorname{curl}\mathbf{E} &= \mathbf{e}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{e}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{e}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\ &= -\frac{\partial E_y}{\partial z}\mathbf{e}_x + \frac{\partial E_x}{\partial z}\mathbf{e}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{e}_z \\ &= -k\, E\big(\cos(kz - \omega t), \sin(kz - \omega t), 0\big) = -k\,\mathbf{E} \implies \dot{\mathbf{B}} = k\,\mathbf{E} \ . \end{aligned}$$

This means:

$$\mathbf{B} = k\, E\left(-\frac{1}{\omega}\sin(kz - \omega t),\ \frac{1}{\omega}\cos(kz - \omega t), 0\right) \ .$$

Magnetic induction:

$$\begin{aligned} \mathbf{B} &= -\frac{1}{u}(E_y, -E_x, 0) = \frac{1}{u}\mathbf{e}_z \times \mathbf{E} \ , \\ \mathbf{B} &= \frac{1}{\omega}\mathbf{k} \times \mathbf{E} \ . \end{aligned}$$

3. Equation of motion:

$$m\,\ddot{\mathbf{r}} = q\big[\mathbf{E} + (\dot{\mathbf{r}} \times \mathbf{B})\big] = q\left\{\mathbf{E} + \frac{1}{u}\big[\dot{\mathbf{r}} \times (\mathbf{e}_z \times \mathbf{E})\big]\right\}$$

$$= q\left[\mathbf{E} + \frac{1}{u}\,\mathbf{e}_z(\dot{\mathbf{r}} \cdot \mathbf{E}) - \frac{1}{u}\mathbf{E}(\dot{\mathbf{r}} \cdot \mathbf{e}_z)\right] .$$

Components:

$$m\,\ddot{x} = q\,E_x\left(1 - \frac{\dot{z}}{u}\right) = q\,E\left(1 - \frac{\dot{z}}{u}\right)\cos(kz - \omega t) \;,$$

$$m\,\ddot{y} = q\,E_y\left(1 - \frac{\dot{z}}{u}\right) = q\,E\left(1 - \frac{\dot{z}}{u}\right)\sin(kz - \omega t) \;,$$

$$m\,\ddot{z} = \frac{q}{u}(\dot{\mathbf{r}} \cdot \mathbf{E}) \;.$$

4. Requirement:

$$\dot{W} = 0 \quad \text{fulfilled if} \quad \mathbf{v} \cdot \mathbf{E} = 0 \quad \text{valid for all times} \;.$$

This means:

$$\ddot{z} = 0 \iff \dot{z} = \text{const} = v_0 \iff z(t) = v_0\,t \quad (z(0) = 0) \;.$$

Equations of motion:

$$m\ddot{x} = qE\left(1 - \frac{v_0}{u}\right)\cos\left(\omega\left(\frac{v_0}{u} - 1\right)t\right)$$

$$m\ddot{y} = qE\left(1 - \frac{v_0}{u}\right)\sin\left(\omega\left(\frac{v_0}{u} - 1\right)t\right) \;.$$

Integration:

$$\dot{x}(t) = -\frac{qE}{m\omega}\sin\left(\omega\left(\frac{v_0}{u} - 1\right)t\right) + \dot{x}_0$$

$$\dot{y}(t) = +\frac{qE}{m\omega}\cos\left(\omega\left(\frac{v_0}{u} - 1\right)t\right) + \dot{y}_0 \;.$$

$\dot{x}_0,\ \dot{y}_0$ still unknown:

$$0 \stackrel{!}{=} \mathbf{v} \cdot \mathbf{E} = \dot{x}(t)\,E_x + \dot{y}(t)\,E_y$$

$$= \frac{qE^2}{m\omega}(-\sin(kz - \omega t)\cos(kz - \omega t) + \cos(kz - \omega t)\sin(kz - \omega t))$$

$$\begin{aligned}&+E_x\dot{x}_0 + E_y\dot{y}_0\\ &= E_x\dot{x}_0 + E_y\dot{y}_0\ .\end{aligned}$$

That must hold for arbitrary times t and all space-points $\mathbf{r}$. The initial conditions are therefore to be chosen such that

$$\dot{x}_0 = \dot{y}_0 = 0\ .$$

With $z(t=0) = 0$ that fixes the initial velocity:

$$\dot{\mathbf{r}}(t=0) = \left(0, \frac{qE}{m\omega}, v_0\right)\ .$$

5.

$$\mathbf{p} = \left(-\frac{qE}{\omega}\sin(kz-\omega t)\,,\ \frac{qE}{\omega}\cos(kz-\omega t)\,,\ mv_0\right)$$

Comparison with part 2.:

$$\mathbf{p}_\perp = \left(p_x, p_y, 0\right) = \frac{q}{k}\,\mathbf{B}\ .$$

6. Solution of the equation of motion:

$$x(t) = -\frac{qE}{m\omega}\left(-\frac{1}{\omega\left(\frac{v_0}{u}-1\right)}\cos\left(\omega\left(\frac{v_0}{u}-1\right)t\right)\right) + x_0$$

$$x(t=0) = 0 = \frac{qE}{m\omega^2}\,\frac{1}{\frac{v_0}{u}-1} + x_0\ .$$

x-component:

$$x(t) = \frac{qE}{m\omega^2}\,\frac{u}{v_0-u}\left(\cos\left(\frac{v_0-u}{u}\,\omega t\right) - 1\right)\ .$$

y-component:

$$y(t) = \frac{qE}{m\omega}\,\frac{1}{\omega\left(\frac{v_0}{u}-1\right)}\sin\left(\omega\left(\frac{v_0}{u}-1\right)t\right) + y_0$$

$$y(t=0) = 0 = y_0\ .$$

It remains:

$$y(t) = \frac{qE}{m\omega^2}\,\frac{u}{v_0-u}\sin\left(\frac{v_0-u}{u}\,\omega t\right)\ .$$

z-component:

$$z(t) = v_0 t \; .$$

7. With the abbreviation

$$R = -\frac{qE}{m\omega^2}\,\frac{u}{v_0 - u} \qquad (u > v_0)$$

we find:

$$(x(t) - R)^2 + (y(t))^2 = R^2 \; .$$

Hence, the path is a circle in the xy-plane with the radius R and its center at $(R, 0)$.

Solution 4.3.2

1. **Magnetic induction**

$$\mathrm{curl}\mathbf{E} = -\dot{\mathbf{B}} \; .$$

(a)

$$\begin{aligned}
\mathrm{curl}\mathbf{E} &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\
&= \left(-\frac{\partial}{\partial z}E_y, \frac{\partial}{\partial z}E_x, 0\right) = k(-E_{0y}, E_{0x}, 0)\cos(kz - \omega t) \\
\Longrightarrow \dot{\mathbf{B}} &= (E_{0y}, -E_{0x}, 0)k\,\cos(kz - \omega t) \\
\Longrightarrow \mathbf{B} &= \frac{k}{\omega}(-E_{0y}, E_{0x}, 0)\sin(kz - \omega t) = \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}) \; .
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{curl}\mathbf{E} &= \left(-\frac{\partial}{\partial z}E_y, \frac{\partial}{\partial z}E_x, 0\right) = -E_0 k\big[\cos(kz - \omega t)\,\mathbf{e}_x + \sin(kz - \omega t)\,\mathbf{e}_y\big] \\
\Longrightarrow \dot{\mathbf{B}} &= E_0 k\big[\cos(kz - \omega t)\,\mathbf{e}_x + \sin(kz - \omega t)\,\mathbf{e}_y\big] \\
\Longrightarrow \mathbf{B} &= E_0\frac{k}{\omega}\big[-\sin(kz - \omega t)\mathbf{e}_x + \cos(kz - \omega t)\,\mathbf{e}_y\big] \\
&= \frac{k}{\omega}(-E_y, E_x, 0) = \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}) \; .
\end{aligned}$$

2. **Poynting vector**

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu_r\mu_0}\mathbf{E}\times\mathbf{B} \quad (\textit{energy-current density}) .$$

It holds for both the cases (a) and (b):

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) &= \frac{1}{\mu_r\mu_0}\frac{1}{\omega}\mathbf{E}\times(\mathbf{k}\times\mathbf{E}) \\ &= \frac{1}{\omega\mu_r\mu_0}(\mathbf{k}\mathbf{E}^2 - \mathbf{E}\underbrace{(\mathbf{E}\cdot\mathbf{k})}_{=0}) = \frac{1}{u\mu_r\mu_0}\mathbf{E}^2\mathbf{e}_z\end{aligned}$$

That means:

(a)

$$\mathbf{S}(\mathbf{r},t) = \sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_r\mu_0}}\,E_0^2\,\sin^2(kz-\omega t)\mathbf{e}_z .$$

(b)

$$\mathbf{S}(\mathbf{r},t) = \sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_r\mu_0}}\,E_0^2\,\mathbf{e}_z .$$

3. **Radiation pressure**

Radiation pressure $\sim$ momentum transfer to the area $\sim$ normal component $(\mathbf{n}\cdot\mathbf{F})$ of the force $\mathbf{F}$ exerted on the unit area.

Density of the field momentum:

$$\hat{\mathbf{p}}_{\text{field}} = \mathbf{D}\times\mathbf{B} = \epsilon_r\mu_r\epsilon_0\mu_0\,\mathbf{S} = \frac{1}{u^2}\mathbf{S} ,$$

u: phase velocity of the electromagnetic wave.

All wave fronts in the inclined cylinder in Fig. A.51, whose volume is

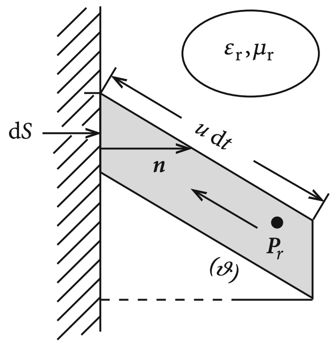
$$\Delta V = u\,dt\,\cos\vartheta\,dS ,$$

reach within the time dt the area-element dS. The plane shall be totally absorbing. The waves create a field momentum $\hat{\mathbf{p}}_{\text{Feld}}\Delta V$ on dS in dt.

Force $=$ momentum per time:

$$\mathbf{F} = \hat{\mathbf{p}}_{\text{field}}\;u\,\cos\vartheta\,dS .$$

Fig. A.51



Radiation pressure:

$$p_S = \frac{\mathbf{n}\cdot\mathbf{F}}{dS} = u\,\cos\vartheta\;\mathbf{n}\cdot\hat{\mathbf{p}}_{\text{Feld}} = \frac{\cos\vartheta}{u}\mathbf{n}\cdot\mathbf{S}\;.$$

Solution:

(a)

$$p_S = \frac{1}{u}\,|\mathbf{S}|\cos^2\vartheta = \varepsilon_r\varepsilon_0 E_0^2 \sin^2\omega t\,\cos^2\vartheta \quad (\text{wall at } z = 0)\;.$$

(b)

$$p_S = \frac{1}{u}\,|\mathbf{S}|\cos^2\vartheta = \varepsilon_r\varepsilon_0 E_0^2\cos^2\vartheta\;.$$

Solution 4.3.3

1. Linear, homogeneous: $\mathbf{B} = \mu_r\mu_0\mathbf{H}$; $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$.
   Uncharged insulator: $\rho_f \equiv 0$, $\mathbf{j}_f \equiv 0$, $\sigma = 0$.
   Maxwell equations:

$$\begin{aligned}&\operatorname{div}\mathbf{E} = 0\;, \quad \operatorname{div}\mathbf{B} = 0\;,\\ &\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}\;,\; \operatorname{curl}\mathbf{B} = \epsilon_r\epsilon_0\mu_r\mu_0\dot{\mathbf{E}} = \tfrac{1}{u^2}\dot{\mathbf{E}}\;.\end{aligned}$$

2.

$$\operatorname{curl}\operatorname{curl}\mathbf{B} = \operatorname{grad}(\underbrace{\operatorname{div}\mathbf{B}}_{=0}) - \Delta\,\mathbf{B} = \frac{1}{u^2}\operatorname{curl}\dot{\mathbf{E}} = -\frac{1}{u^2}\ddot{\mathbf{B}}\;,$$

$$\Box\,\mathbf{B} = 0\;, \quad \text{where } \Box = \Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\;.$$

3.

$$
\begin{aligned}
&\mathrm{curl}\mathbf{E} = -\dot{\mathbf{B}} \\
&\Longrightarrow\ i\,\mathbf{k}\times\mathbf{E} = i\,\omega\,\mathbf{B} \\
&\Longrightarrow\ \mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} = \frac{k}{\omega}\frac{E_0}{5}(\mathbf{e}_z\times\mathbf{e}_x - 2\mathbf{e}_z\times\mathbf{e}_y)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ .
\end{aligned}
$$

$\mathbf{B}$ is linearly polarized:

$$
\mathbf{B} = \frac{E_0 k}{5\omega}(2\mathbf{e}_x + \mathbf{e}_y)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}
$$

4.

$$
\begin{aligned}
\mathrm{curl}\mathbf{B} &= \frac{1}{u^2}\dot{\mathbf{E}} = \mathbf{e}_x\big[-B_0 k\ \cos(kz-\omega t)\big] + \mathbf{e}_y\big[-B_0 k\ \sin(kz-\omega t)\big] \\
&\Longrightarrow\ \dot{\mathbf{E}} = -u\,\omega\,B_0\big[\mathbf{e}_x\ \cos(kz-\omega t) + \mathbf{e}_y\ \sin(kz-\omega t)\big] \\
&\Longrightarrow\ \mathbf{E} = u\,B_0\big[\mathbf{e}_x\ \sin(kz-\omega t) - \mathbf{e}_y\ \cos(kz-\omega t)\big]\ ,
\end{aligned}
$$

i.e. $\mathbf{E}$ is circularly polarized.

Solution 4.3.4

1. For

$$
\mathbf{E} = \mathbf{E}_0 e^{i(kz-\omega t)}\ ,\quad \mathbf{B} = \mathbf{B}_0 e^{i(kz-\omega t)}
$$

it holds in a linear uncharged insulator (Maxwell equation):

$$
\mathrm{curl}\mathbf{H} = \mathbf{j} + \dot{\mathbf{D}}\ \Longrightarrow\ \mathrm{curl}\mathbf{B} = \frac{1}{u^2}\dot{\mathbf{E}}\ .
$$

This is equivalent to

$$
\mathbf{k}\times\mathbf{B}_0 = -\frac{\omega}{u^2}\mathbf{E}_0\ .
$$

Because of

$$
\mathbf{B}_0 = \widehat{B}_0(4\mathbf{e}_x - 3\mathbf{e}_y)
$$

it follows

$$
\mathbf{k}\times\mathbf{B}_0 = k\widehat{B}_0(4\mathbf{e}_y + 3\mathbf{e}_x) = -\frac{k}{u}\mathbf{E}_0\ .
$$

Electric field

$$\mathbf{E} = -u\widehat{B_0}(4\mathbf{e}_y + 3\mathbf{e}_x)e^{i(kz-\omega t)} \ .$$

E (as well as **B**) is linearly polarized since

$$\tan\alpha = \frac{E_y}{E_x} = \text{const} \ .$$

2. Maxwell equation

$$\text{curl}\mathbf{E} = -\dot{\mathbf{B}} \ .$$

We therefore need

$$\text{curl}\mathbf{E} = \mathbf{e}_x(\beta k\cos(kz-\omega t+\varphi)) + \mathbf{e}_y(\alpha(-k\sin(kz-\omega t+\varphi)) \ .$$

Time-integration yields for the magnetic induction **B** with $\omega = ku$:

$$\mathbf{B} = \frac{1}{u}\left(\beta\sin(kz-\omega t+\varphi)\mathbf{e}_x + \alpha\cos(kz-\omega t+\varphi)\mathbf{e}_y\right) \ .$$

One recognizes:

$$\left(\frac{B_x}{\beta}\right)^2 + \left(\frac{B_y}{\alpha}\right)^2 = \frac{1}{u^2} \ .$$

$\longrightarrow$ elliptically polarized! Semiaxes: β/u and α/u.
Special cases:
$\alpha = \beta \ \longrightarrow$ circularly polarized!

Solution 4.3.5

1. Each solution of the Maxwell equations is automatically also a solution of the homogeneous wave equation. One knows that the reversal does not hold.
2. The wave equation

$$\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = 0$$

is solved by

$$\left(-k^2 + \frac{\omega^2}{u^2}\right) = 0 \ \curvearrowright \ k = \frac{\omega}{u} \ .$$

3.

$$\operatorname{div}\mathbf{B} = 0 \;\Leftrightarrow\; \mathbf{k}\cdot\mathbf{B} = 0 \;\Rightarrow\; \alpha k_x + i\gamma k_y = 0 \;\Rightarrow\; k_x = 0\,,\; k_y = 0\,.$$

Thus it is:

$$\mathbf{k} = k\,\mathbf{e}_z\,.$$

4.

$$\operatorname{curl}\mathbf{B} = \mu_r\mu_0\dot{\mathbf{D}} = \frac{1}{u^2}\dot{\mathbf{E}} = -i\omega\frac{1}{u^2}\mathbf{E} \stackrel{!}{=} i\,(\mathbf{k}\times\mathbf{B})\,.$$

It follows therewith:

$$\mathbf{E} = -\frac{u^2}{\omega}(\mathbf{k}\times\mathbf{B}) = -u(\mathbf{e}_z\times\mathbf{B}) = -u(\alpha\mathbf{e}_y - i\gamma\mathbf{e}_x)\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,,$$
$$\hat{\mathbf{E}}_0(\mathbf{r}) = u(i\gamma\mathbf{e}_x - \alpha\mathbf{e}_y)\,e^{i\,\mathbf{k}\cdot\mathbf{r}}\,.$$

5. Time-averaged energy density:

$$\overline{w(\mathbf{r},t)} = \frac{1}{4}\mathrm{Re}\left(\hat{\mathbf{E}}_0\cdot\hat{\mathbf{D}}_0^{\star} + \hat{\mathbf{H}}_0\cdot\hat{\mathbf{B}}_0^{\star}\right) = \frac{1}{4}\mathrm{Re}\left(\varepsilon_r\varepsilon_0\left|\hat{\mathbf{E}}_0\right|^2 + \frac{1}{\mu_r\mu_0}\left|\hat{\mathbf{B}}_0\right|^2\right)\,.$$

With

$$\left|\hat{\mathbf{B}}_0\right|^2 = (\alpha^2+\gamma^2)\,;\quad \left|\hat{\mathbf{E}}_0\right|^2 = u^2\,(\alpha^2+\gamma^2)$$

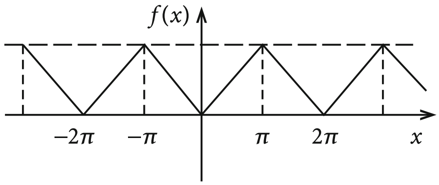
it results:

$$\overline{w(\mathbf{r},t)} = \frac{1}{2}\frac{1}{\mu_r\mu_0}\,(\alpha^2+\gamma^2)\,.$$

6. Time-averaged energy-current density

$$\begin{aligned}\overline{S(\mathbf{r},t)} &= \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}}_0(\mathbf{r})\times\hat{\mathbf{H}}_0^{\star}(\mathbf{r})\right)\\ &= \frac{u}{2\mu_r\mu_0}\,\mathrm{Re}\big((i\gamma\mathbf{e}_x - \alpha\mathbf{e}_y)\times(\alpha\mathbf{e}_x - i\gamma\mathbf{e}_y)\big)\\ &= \frac{u}{2\mu_r\mu_0}\,\mathrm{Re}(\gamma^2\mathbf{e}_z - \alpha^2(-\mathbf{e}_z))\,.\end{aligned}$$

Fig. A.52



Thus it is

$$\overline{S(\mathbf{r},t)} = \frac{u}{2\mu_r\mu_0}\,(\alpha^2+\gamma^2)\,\mathbf{e}_z = u\,\overline{w(\mathbf{r},t)}\,\mathbf{e}_z\,.$$

Solution 4.3.6

1. (Figure A.52)

$$f(x) = \begin{cases} -x\colon\ -\pi \le x \le 0\,, \\ +x\colon\ 0 \le x \le \pi\ . \end{cases}$$

General Fourier series:

$$f(x) = f(x+2a), \quad \text{square-integrable in } [-a,a]$$

$$\Longrightarrow\ f(x) = f_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\,\pi}{a}x\right) + b_n\sin\left(\frac{n\,\pi}{a}x\right)\right]\ .$$

Here:

$$a = \pi\ ,$$

$$f(x)\ \text{even}\ \Longrightarrow\ b_n = 0 \quad \forall n\ .$$

$$f_0 = \frac{1}{2a}\int_{-a}^{+a} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{2\pi}\left(\int_0^{\pi} x\,dx + \int_{-\pi}^{0}(-x)dx\right)$$

$$\Longrightarrow\ f_0 = \frac{\pi}{2}\ ,$$

$$a_n = \frac{1}{a}\int_{-a}^{+a} f(x)\,\cos\left(\frac{n\,\pi}{a}x\right)dx = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,\cos(n\,x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,\cos(nx)dx - \frac{1}{\pi}\int_{-\pi}^{0} x\,\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\,\cos(nx)dx$$

$$\Longrightarrow \; a_n = \frac{2}{n\pi} x \sin(nx)\Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin(nx) dx$$

$$= \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\pi} = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} \dfrac{-4}{n^2\pi}, & \text{if } n \text{ odd,} \\ 0, & \text{if } n \text{ even .} \end{cases}$$

Fourier series:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)x]}{(2k+1)^2} \; .$$

2. (Figure A.53)

$$f(x) \text{ even} \Longrightarrow b_n = 0 \quad \forall n \; .$$

$$f_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, dx = \frac{1}{2\pi}\left[(-x)\big|_{-\pi}^{-\pi/2} + (x)\big|_{-\pi/2}^{+\pi/2} + (-x)\big|_{\pi/2}^{\pi} \right] = 0 \; ,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\, dx$$

$$= \frac{1}{n\pi}\left[-\sin(nx)\big|_{-\pi}^{-\pi/2} + \sin(nx)\big|_{-\pi/2}^{+\pi/2} - \sin(nx)\big|_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + 2\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) \right] = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) \; .$$

Fig. A.53

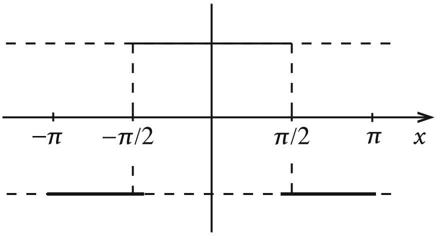
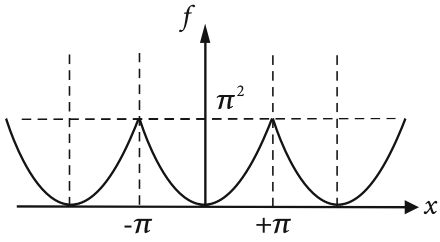


Fig. A.54



Fourier series:

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \cos(nx) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left[(2k+1)x\right] .$$

3. (Figure A.54)

$$f(x) = f(-x) \;\Rightarrow\; b_n = 0 \quad \forall n .$$

It then remains to be calculated:

$$f_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} x^2 dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left(\frac{1}{n} x^2 \underbrace{\sin nx \Big|_{-\pi}^{+\pi}}_{=0} - \frac{2}{n} \int_{-\pi}^{+\pi} x \sin nx \, dx \right)$$

$$= \frac{1}{\pi} \left(\frac{2}{n^2} \underbrace{x \cos nx \Big|_{-\pi}^{+\pi}}_{2\pi(-1)^n} - \frac{2}{n^2} \int_{-\pi}^{+\pi} \cos nx \, dx \right)$$

$$= \frac{4}{n^2} (-1)^n - \frac{4}{\pi n^3} \underbrace{\sin nx \Big|_{-\pi}^{+\pi}}_{=0} .$$

Therewith the Fourier series reads:

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx .$$

4. (a) Because of $f(0) = 0$ it must be:

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \; .$$

(b) For $x = \pi$ we have $f(\pi) = \pi^2$ and $\cos(n\pi) = (-1)^n$. The Fourier series from 3. yields therewith for this special case:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \; .$$

Solution 4.3.7 The addition theorems of the trigonometric functions ((1.60), (1.61) in Vol. 1) are useful:

$$\begin{aligned}
\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) &= \frac{1}{2}\left(\cos\left((n+m)\frac{\pi}{a}x\right) + \cos\left((n-m)\frac{\pi}{a}x\right)\right) \; , \\
\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) &= \frac{1}{2}\left(\cos\left((n-m)\frac{\pi}{a}x\right) - \cos\left((n+m)\frac{\pi}{a}x\right)\right) \; , \\
\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) &= \frac{1}{2}\left(\sin\left((n+m)\frac{\pi}{a}x\right) + \sin\left((n-m)\frac{\pi}{a}x\right)\right) \; .
\end{aligned}$$

In detail it follows therewith for $\boldsymbol{n \neq m}$:

•

$$\begin{aligned}
\int_{-a}^{+a} dx \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) &= \frac{1}{2}\int_{-a}^{+a} dx \left(\cos\left((n+m)\frac{\pi}{a}x\right)\right. \\
&\qquad \left. + \cos\left((n-m)\frac{\pi}{a}x\right)\right) \\
&= \frac{1}{2}\left(\frac{a}{(n+m)\pi}\sin\left((n+m)\frac{\pi}{a}x\right)\right. \\
&\qquad \left. + \frac{a}{(n-m)\pi}\sin\left((n-m)\frac{\pi}{a}x\right)\right)\Bigg|_{-a}^{+a} \\
&= 0 \; .
\end{aligned}$$

- $$\int_{-a}^{+a} dx \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) = \frac{1}{2}\int_{-a}^{+a} dx \left(\cos\left((n-m)\frac{\pi}{a}x\right)\right.$$
$$\left. - \cos\left((n+m)\frac{\pi}{a}x\right)\right)$$
$$= \frac{1}{2}\left(\frac{a}{(n-m)\pi}\sin\left((n-m)\frac{\pi}{a}x\right)\right.$$
$$\left.\left. - \frac{a}{(n+m)\pi}\sin\left((n+m)\frac{\pi}{a}x\right)\right)\right|_{-a}^{+a} .$$
$$= 0$$

- $$\int_{-a}^{+a} dx \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) = 0 .$$

  Here one can already exploit that the integrand is an odd function of x.

For $\boldsymbol{n = m \neq 0}$ we find:

- $$\frac{1}{a}\int_{-a}^{+a} dx \cos^2\left(\frac{n\pi}{a}x\right) = \frac{1}{2a}\int_{-a}^{+a} dx \left(1 + \cos\left(\frac{2n\pi}{a}x\right)\right)$$
$$= \frac{1}{2a}\left(2a + \frac{a}{2\pi n}\underbrace{\sin\left(\frac{2n\pi}{a}x\right)\Big|_{-a}^{+a}}_{=0}\right) = 1 .$$

- $$\frac{1}{a}\int_{-a}^{+a} dx \sin^2\left(\frac{n\pi}{a}x\right) = \frac{1}{a}\int_{-a}^{+a} dx \left(1 - \cos^2\left(\frac{n\pi}{a}x\right)\right) = 2 - 1 = 1 .$$

- $$\frac{1}{a}\int_{-a}^{+a} dx \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right) = 0 .$$

This follows, as above, because of the odd integrand.

Summary:

$$\frac{1}{a}\int\limits_{-a}^{+a} dx \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) = \delta_{nm} \quad (n, m = 1, 2, \ldots)\ ,$$

$$\frac{1}{a}\int\limits_{-a}^{+a} dx \cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) = \delta_{nm}\ ,$$

$$\frac{1}{a}\int\limits_{-a}^{+a} dx \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) = 0\ .$$

The relations (4.175) are now easily demonstrated:

1.

$$\begin{aligned}\frac{1}{2a}\int\limits_{-a}^{+a} dx f(x) &= \frac{1}{2a}\left[\int\limits_{-a}^{+a} f_0\, dx + \sum_{m=1}^{\infty}\left(a_m\int\limits_{-a}^{+a}\cos\left(\frac{m\pi}{a}x\right)dx + b_m\int\limits_{-a}^{+a}\sin\left(\frac{m\pi}{a}x\right)dx\right)\right]\\ &= f_0 + \sum_{m=1}^{\infty}\left(\frac{a_m}{2m\pi}\underbrace{\sin\left(\frac{m\pi}{a}x\right)\Big|_{-a}^{+a}}_{=0} - \frac{b_m}{2m\pi}\underbrace{\cos\left(\frac{m\pi}{a}x\right)\Big|_{-a}^{+a}}_{=0}\right)\\ &= f_0\ .\end{aligned}$$

2.

$$\begin{aligned}\frac{1}{a}\int\limits_{-a}^{+a} dx f(x)\cos\left(\frac{n\pi}{a}x\right) &= \frac{1}{a}\int\limits_{-a}^{+a} dx f_0 \cos\left(\frac{n\pi}{a}x\right)\\ &\quad + \sum_{m=1}^{\infty}\left(a_m\frac{1}{a}\int\limits_{-a}^{+a} dx \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right)\right.\\ &\quad \left. + b_m\frac{1}{a}\int\limits_{-a}^{+a} dx \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right)\right)\\ &= \frac{f_0}{n\pi}\underbrace{\sin\left(\frac{n\pi}{a}x\right)\Big|_{-a}^{+a}}_{=0} + \sum_{m=1}^{\infty}(a_m\delta_{nm} + 0)\\ &= a_n\ .\end{aligned}$$

3.

$$
\begin{aligned}
\frac{1}{a}\int\limits_{-a}^{+a} dx f(x)\sin\left(\frac{n\pi}{a}x\right) &= \frac{1}{a}\int\limits_{-a}^{+a} dx f_0 \sin\left(\frac{n\pi}{a}x\right) \\
&\quad + \sum_{m=1}^{\infty}\left(a_m \frac{1}{a}\int\limits_{-a}^{+a} dx \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\right. \\
&\quad \left. + b_m \frac{1}{a}\int\limits_{-a}^{+a} dx \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\right) \\
&= -\frac{f_0}{n\pi}\underbrace{\cos\left(\frac{n\pi}{a}x\right)\Big|_{-a}^{+a}}_{=0} + \sum_{m=1}^{\infty}(0 + b_m\delta_{nm}) \\
&= b_n \; .
\end{aligned}
$$

The relations in (4.175) are therewith proven.

Solution 4.3.8

$$
f(x) \equiv x^2 + (b-a)x - ab \; .
$$

Zeros: $f(x_i) \overset{!}{=} 0$:

$$
\begin{aligned}
x^2 + (b-a)x &= ab \\
\curvearrowright \; \left(x + \frac{1}{2}(b-a)\right)^2 &= \frac{1}{4}(b-a)^2 + ab = \frac{1}{4}(b+a)^2 \\
\curvearrowright \; x_{1,2} &= -\frac{1}{2}(b-a) \pm \frac{1}{2}(b+a) \\
\curvearrowright \; x_1 &= a \; ; \quad x_2 = -b \; .
\end{aligned}
$$

It follows with $f'(x) = 2x + (b-a)$:

$$
f'(x_1) = a + b \; ; \quad f'(x_2) = -(a+b) \; .
$$

Therewith:

$$
\delta(f(x)) = \sum_{i=1}^{2} \frac{1}{|f'(x_i)|}\,\delta(x - x_i) = \frac{1}{a+b}\left(\delta(x-a) + \delta(x+b)\right) \; .
$$

Fourier transform:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx\, \delta(f(x))\, e^{ikx} = \frac{1}{\sqrt{2\pi}(a+b)} \left(e^{ika} + e^{-ikb}\right) \,.$$

Checkup:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dk \left(\frac{1}{\sqrt{2\pi}(a+b)} \left(e^{ika} + e^{-ikb}\right)\right) e^{-ikx} \\
&= \frac{1}{a+b} \left(\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dk\, e^{-ik(x-a)} + \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dk e^{-ik(x+b)} \right) \\
&= \frac{1}{a+b} \left(\delta(x-a) + \delta(x+b)\right)
\end{aligned}$$

That was to be proven!

Solution 4.3.9

1. (a) The Taylor expansion of $f(x)$ around x_0 is inserted into the following integral:

$$\begin{aligned}
\int\limits_{-\infty}^{+\infty} dx \delta_l^{(1)}(x-x_0) f(x) &= \sum_{r=0}^{\infty} \int\limits_{-\infty}^{+\infty} dx\, \delta_l^{(1)}(x-x_0)\, \frac{f^{(r)}(x_0)}{r!} (x-x_0)^r \\
&= \sum_{r=0}^{\infty} \frac{f^{(r)}(x_0)}{r!} \int\limits_{-\infty}^{+\infty} du\, \delta_l^{(1)}(u)\, u^r \\
&= \sum_{r=0}^{\infty} \frac{f^{(2r)}(x_0)}{(2r)!} \int\limits_{-\infty}^{+\infty} du\, \delta_l^{(1)}(u)\, u^{2r} \,.
\end{aligned}$$

In the last step we have used that $\delta_l^{(1)}(u)$ is an even function of u so that only the even powers of u contribute to the integral. One shows, e.g. by full induction, that the integral can be evaluated as:

$$\int\limits_{-\infty}^{+\infty} dx\, x^{2n} e{-}\alpha x^2 = \sqrt{\frac{\pi}{\alpha}}\, \frac{(2n+1)!!}{(2n+1)(2\alpha)^n}$$

Therewith we have:

$$\int\limits_{-\infty}^{+\infty} dx\delta_l^{(1)}(x-x_0)f(x) = \sum_{r=0}^{\infty} \frac{f^{(2r)}(x_0)}{(2r+1)!}(2r+1)!!\, l^{2r} \; .$$

For $l \to 0$ only the $r = 0$-summand survives:

$$\lim_{l\to 0} \int\limits_{-\infty}^{+\infty} dx\, \delta_l^{(1)}(x-x_0)f(x) = f(x_0) \tag{A.5}$$

(b)

$$\begin{aligned}
&\lim_{l\to 0} \int\limits_{-\infty}^{+\infty} dx\, \delta_l^{(2)}(x-x_0)f(x) \\
&= \lim_{l\to 0} \int\limits_{-\infty}^{+\infty} dx\, \frac{\sin\left(\frac{\pi}{l}(x-x_0)\right)}{2\sin\left(\frac{\pi}{2}(x-x_0)\right)} f(x) \\
&= \lim_{l\to 0} \int\limits_{-\infty}^{+\infty} du\, \frac{l}{2}\frac{\sin(\pi u)}{\sin\left(\frac{1}{2}l\pi u\right)} f(x_0+lu) \qquad (u = 1/l(x-x_0)) \\
&= \int\limits_{-\infty}^{+\infty} du\, \frac{\sin(\pi u)}{\pi u} f(x_0) \\
&= f(x_0)\,\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} dy\, \frac{\sin y}{y} \; .
\end{aligned}$$

The left-over integral can be solved with Cauchy's residue theorem (see second example in Sect. 4.4.5):

$$\int\limits_{-\infty}^{+\infty} dy\, \frac{\sin y}{y} = \pi \; .$$

Hence:

$$\lim_{l\to 0} \int\limits_{-\infty}^{+\infty} dx\, \delta_l^{(2)}(x-x_0)f(x) = f(x_0) \; . \tag{A.6}$$

That is the definition equation for the δ-function:

$$\lim_{l \to 0} \delta_l^{(2)}(x - x_0) = \delta(x - x_0) \; .$$

2. With the l'Hospital's rule one easily finds:

$$\delta_l^{(2)}(x = 0) = \frac{1}{l} \;\curvearrowright\; \lim_{l \to 0} \delta_l^{(2)}(x = 0) \to \infty \; .$$

In contrast, $\lim_{l \to 0} \delta_l^{(2)}(x)$ remains undetermined for $x \neq 0$ (oscillatory behavior!). However, Eq. (A.6) is sufficient for the definition of the δ-function!

Let us now consider for $x \in (-1, +1)$:

$$\begin{aligned}
\sum_{n=-N}^{+N} e^{i\pi n x} &= \sum_{n=0}^{N} \left(e^{i\pi n x} + e^{-i\pi n x} \right) - 1 \\
&= \frac{1 - e^{i\pi(N+1)x}}{1 - e^{i\pi x}} + \frac{1 - e^{-i\pi(N+1)x}}{1 - e^{-i\pi x}} - 1 \\
&= \frac{2 - \left(e^{i\pi x} + e^{-i\pi x} \right) - \left(e^{i\pi(N+1)x} + e^{-i\pi(N+1)x} \right)}{2 - \left(e^{i\pi x} + e^{-i\pi x} \right)} \\
&\quad + \frac{e^{i\pi N x} + e^{-i\pi N x}}{2 - \left(e^{i\pi x} - e^{-i\pi x} \right)} - 1 \\
&= \frac{\cos(N\pi x) - \cos\left((N+1)\pi x\right)}{1 - \cos(\pi x)} \; .
\end{aligned}$$

We still use ((1.60), (1.61) in Vol. 1):

$$\begin{aligned}
1 - \cos(\pi x) &= 2 \sin^2 \frac{\pi x}{2} \\
\cos(N\pi x) - \cos\left((N+1)\pi x\right) &= \cos\left(\left(\left(N + \frac{1}{2} \right) - \frac{1}{2} \right) \pi x \right) \\
&\quad - \cos\left(\left(\left(N + \frac{1}{2} \right) + \frac{1}{2} \right) \pi x \right) \\
&= 2 \sin\left(\left(N + \frac{1}{2} \right) \pi x \right) \sin\left(\frac{\pi}{2} x \right) \; .
\end{aligned}$$

This means:

$$\sum_{n=-N}^{+N} e^{i\pi n x} = \frac{\sin\left(\left(N + \frac{1}{2} \right) \pi x \right)}{\sin\left(\frac{\pi}{2} x \right)} = 2\delta_l^{(2)}(x) \quad \text{with} \quad l = \frac{1}{N + \frac{1}{2}} \; .$$

Conclusion:

$$\lim_{N\to\infty}\frac{1}{2}\sum_{n=-N}^{+N} e^{i\pi nx} = \lim_{l\to 0}\delta_l^{(2)}(x) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} e^{i\pi nx}\ .$$

All expressions are periodic with respect to transformations

$$x \longrightarrow x + 2z \quad \text{with } z = \pm 1, \pm 2, \pm 3, \dots\ .$$

One obtains therefore the corresponding behavior also in the other intervals. It remains therewith:

$$\delta(x) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} e^{i\pi nx}\ .$$

3. For $x, x' \in (0, x_0)$ we calculate:

$$\begin{aligned}
&\frac{2}{x_0}\sum_{n=0}^{\infty}\sin\left(\frac{n\pi}{x_0}x\right)\,\sin\left(\frac{n\pi}{x_0}x'\right)\\
&= \frac{2}{x_0}\frac{1}{(2i)^2}\sum_{n=0}^{\infty}\left(e^{i\frac{n\pi}{x_0}x} - e^{-i\frac{n\pi}{x_0}x}\right)\left(e^{i\frac{n\pi}{x_0}x'} - e^{-i\frac{n\pi}{x_0}x'}\right)\\
&= -\frac{1}{2x_0}\sum_{n=0}^{\infty}\left(e^{i\frac{n\pi}{x_0}(x+x')} - e^{-i\frac{n\pi}{x_0}(x-x')} - e^{i\frac{n\pi}{x_0}(x-x')} + e^{-i\frac{n\pi}{x_0}(x+x')}\right)\\
&= \frac{1}{2x_0}\sum_{n=-\infty}^{+\infty}\left(e^{i\frac{n\pi}{x_0}(x-x')} - e^{i\frac{n\pi}{x_0}(x+x')}\right)\\
&= \frac{1}{x_0}\left[\delta\left(\frac{x-x'}{x_0}\right) - \delta\left(\frac{x+x'}{x_0}\right)\right]\\
&= \delta(x-x') - \delta(\underbrace{x+x'}_{>0})\\
&= \delta(x-x')\ .
\end{aligned}$$

That was to be proven!

Solution 4.3.10

1.

$$\bar{g}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx\, e^{-ikx} f_1(x) f_2(x)$$

$$= \frac{1}{(2\pi)^{3/2}} \int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} dk_1 \int\limits_{-\infty}^{+\infty} dk_2 \tilde{f}_1(k_1) \tilde{f}_2(k_2)\, e^{-ikx} e^{i(k_1+k_2)x}$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} dk_1\, dk_2 \tilde{f}_1(k_1) \tilde{f}_2(k_2) \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dx\, e^{-i(k-k_1-k_2)x} \, .$$

δ-function (4.189):

$$\delta(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} dx\, e^{-ikx} \, .$$

It follows:

$$\bar{g}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} dk_1\, dk_2 \tilde{f}_1(k_1) \tilde{f}_2(k_2)\, \delta(k - k_1 - k_2) \, ,$$

$$\bar{g}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dk_1 \tilde{f}_1(k_1) \tilde{f}_2(k - k_1) \, .$$

2a. $f(x) = e^{-|x|}$:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx\, e^{-|x|} e^{-ikx}, \quad e^{-|x|} \text{ even} \, .$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} dx\, e^{-|x|} \cos kx = \frac{2}{\sqrt{2\pi}} I \, ,$$

$$I = \int\limits_{0}^{\infty} dx\, e^{-|x|} \cos kx = \int\limits_{0}^{\infty} dx\, e^{-x} \cos kx$$

$$
\begin{aligned}
&= \frac{1}{k} e^{-x} \sin kx \Big|_0^{\infty} + \frac{1}{k} \int_0^{\infty} dx\, e^{-x} \sin kx \\
&= 0 - \frac{1}{k^2} \cos kx\; e^{-x}|_0^{\infty} - \frac{1}{k^2} I \\
&\Longrightarrow I\left(1 + \frac{1}{k^2}\right) = \frac{1}{k^2} \Longrightarrow I = \frac{1}{1+k^2} \\
&\Longrightarrow \tilde{f}(k) = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} \qquad \text{(Lorentz curve] .}
\end{aligned}
$$

2b. $f(x) = \exp\left(-x^2/\Delta x^2\right)$:

$$
\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-(x^2/\Delta x^2)} e^{-ikx} , \\
\frac{x^2}{\Delta x^2} + ikx &= \left(\frac{x}{\Delta x} + \frac{i}{2} k \Delta x\right)^2 + \frac{1}{4} k^2 \Delta x^2 , \\
y &= \frac{x}{\Delta x} + \frac{i}{2} k \Delta x \Longrightarrow dy = \frac{dx}{\Delta x} \\
&\Longrightarrow \tilde{f}(k) = \frac{\Delta x}{\sqrt{2\pi}} e^{-(1/4)k^2 \Delta x^2} \int_{-\infty + i\ldots}^{+\infty + i\ldots} dy\, e^{-y^2} \\
&\Longrightarrow \tilde{f}(k) = \frac{\Delta x}{\sqrt{2}} e^{-(1/4)k^2 \Delta x^2} \quad \textit{also gaussian-like} .
\end{aligned}
$$

3. Proof by insertion:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} dx |f(x)|^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dk \tilde{f}^*(k) e^{-ikx} \int_{-\infty}^{+\infty} dk' \tilde{f}(k') e^{ik'x} \\
&= \iint_{-\infty}^{+\infty} dk\, dk'\, \tilde{f}^*(k) \tilde{f}(k')\, \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, e^{i(k'-k)x}}_{\delta(k'-k)} \\
&= \int_{-\infty}^{+\infty} dk\, |\tilde{f}(k)|^2 .
\end{aligned}
$$

Solution 4.3.11

1. $f(x)$ is an odd function of x. The Fourier transform therefore reads:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} x\, e^{-|x|}\, e^{-ikx} dx = -i\sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} x\, e^{-x} \sin kx\, dx\ .$$

We calculate:

$$\begin{aligned}
\widehat{I} &\equiv \int\limits_0^\infty x\, e^{-x} \sin kx\, dx \\
&= -\frac{1}{k} \cos kx (x\, e^{-x})\Big|_0^\infty + \frac{1}{k} \int\limits_0^\infty \cos kx (1-x) e^{-x}\, dx \\
&= \frac{1}{k} \underbrace{\int\limits_0^\infty dx e^{-x}\ \cos x\, dx}_{\equiv I} - \frac{1}{k} \int\limits_0^\infty dx x\, e^{-x}\ \cos x\, dx\ .
\end{aligned}$$

I was calculated in part 2a. of Exercise 4.3.10:

$$I \equiv \int\limits_0^\infty dx\, e^{-x}\ \cos x\, dx = \frac{1}{1+k^2}\ .$$

Therewith:

$$\begin{aligned}
\widehat{I} &= \frac{1}{k} I - \frac{1}{k^2} \sin kx\, (x\, e^{-x})\Big|_0^\infty + \frac{1}{k^2} \int\limits_0^\infty \sin kx\, (1-x)\, e^{-x} dx \\
\curvearrowright\ \widehat{I}(1+\frac{1}{k^2}) &= \frac{1}{k} I + \frac{1}{k^2} \int\limits_0^\infty \sin kx\, e^{-x} dx \\
&= \frac{1}{k} I - \frac{1}{k^3} \cos kx\, e^{-x}\Big|_0^\infty + \frac{1}{k^3} \int\limits_0^\infty \cos kx\, (-e^{-x})\, dx \\
&= \frac{1}{k}\left(1-\frac{1}{k^2}\right) I + \frac{1}{k^3} \\
\curvearrowright\ \widehat{I}(1+k^2) &= \frac{1}{k}\frac{k^2-1}{k^2+1} + \frac{1}{k} = \frac{2k}{1+k^2}\ .
\end{aligned}$$

The Fourier transform is therewith determined:

$$\tilde{f}(k) = -i\sqrt{\frac{2}{\pi}}\,\frac{2k}{(1+k^2)^2}\ .$$

2. We use the so-called *Parseval-relation* from part (3) of Exercise 4.3.10:

$$\int\limits_{-\infty}^{+\infty} dk\,|\tilde{f}(k)|^2 \overset{!}{=} \int\limits_{-\infty}^{+\infty} dx\,|f(x)|^2\ .$$

That means with the functions from part (1):

$$\frac{8}{\pi}\int\limits_{-\infty}^{+\infty} dk\,\frac{k^2}{(1+k^2)^4} = \int\limits_{-\infty}^{+\infty} dx\,x^2\,e^{-2|x|} = 2\int\limits_{0}^{\infty} dx\,x^2\,e^{-2x}$$

$$\curvearrowright \int\limits_{-\infty}^{+\infty} dk\,\frac{k^2}{(1+k^2)^4} = \frac{\pi}{4}\int\limits_{0}^{\infty} dx\,x^2\,e^{-2x}$$

$$= \frac{\pi}{4}\left(\frac{d^2}{d\alpha^2}\int\limits_{0}^{\infty} dx\,e^{-\alpha x}\right)_{\alpha=2}$$

$$= \frac{\pi}{4}\left(\frac{d^2}{d\alpha^2}\,\frac{1}{\alpha}\right)_{\alpha=2} = \frac{\pi}{16}\ .$$

That was to be proven.

Solution 4.3.12

$$\tilde{\Psi}(\bar{\mathbf{k}},\bar{\omega}) = \frac{1}{\left(\sqrt{2\pi}\right)^4}\int d^3r\int\limits_{-\infty}^{+\infty} dt\,e^{-i(\bar{\mathbf{k}}\cdot\mathbf{r}-\bar{\omega}t)}\,\frac{e^{i(kr-\omega t)}}{r}$$

$$= \underbrace{\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} dt\,e^{i(\bar{\omega}-\omega)t}}_{\delta(\bar{\omega}-\omega)\ \text{(see (4.189))}}\ \frac{1}{2\pi}\int d^3r\,\frac{1}{r}\,e^{i(kr-\bar{\mathbf{k}}\cdot\mathbf{r})}$$

$$= \delta(\bar{\omega}-\omega)\,\widehat{\Psi}(\bar{\mathbf{k}})\ ,$$

$$\widehat{\Psi}(\bar{\mathbf{k}}) = \frac{1}{2\pi}\int d^3r\,\frac{1}{r}\,e^{i(kr-\bar{\mathbf{k}}\cdot\mathbf{r})}\ ,$$

Spherical coordinates ($\bar{\mathbf{k}}$: polar axis),

$$\Longrightarrow \quad \widehat{\Psi}(\bar{\mathbf{k}}) = \int_0^\infty dr\, r \int_{-1}^{+1} dx\, e^{i(kr - \bar{k}rx)}$$

$$= \int_0^\infty dr\, r\, e^{ikr} \frac{i}{\bar{k}r}\left(e^{-i\bar{k}r} - e^{i\bar{k}r}\right) = \frac{i}{\bar{k}} \int_0^\infty dr \left[e^{i(k-\bar{k})r} - e^{i(k+\bar{k})r}\right]$$

$$= \frac{i}{\bar{k}}\left[\frac{1}{i(k-\bar{k})}\, e^{i(k-\bar{k})r}\Big|_0^\infty - \frac{1}{i(k+\bar{k})}\, e^{i(k+\bar{k})r}\Big|_0^\infty\right] .$$

The equation is actually not defined at the upper bound, therefore introduction of a *convergence generating factor*:

$$k \longrightarrow k + i\,0^+ ,$$

i.e., the spherical wave is arbitrarily weakly exponentially damped.

$$\Longrightarrow \quad \widehat{\Psi}(\bar{\mathbf{k}}) = \frac{1}{\bar{k}}\left(\frac{1}{k+\bar{k}} - \frac{1}{k-\bar{k}}\right) = \frac{2}{\bar{k}^2 - k^2}$$

$$\Longrightarrow \quad \tilde{\Psi}(\bar{\mathbf{k}}, \bar{\omega}) = \frac{2}{\bar{k}^2 - k^2}\, \delta(\bar{\omega} - \omega) .$$

Expansion of the spherical wave in plane waves:

$$\Psi(\mathbf{r}, t) = \frac{1}{r} e^{i(kr - \omega t)} = \frac{1}{2\pi^2} \int d^3\bar{k}\, \frac{e^{i(\bar{\mathbf{k}}\cdot\mathbf{r} - \omega t)}}{\bar{k}^2 - k^2} .$$

Solution 4.3.13

1. Propagation in z-direction:

$$\mathbf{k} = \pm k\, \mathbf{e}_z .$$

   Linearly polarized in x-direction:

$$\mathbf{E}_0 = E_0\, \mathbf{e}_x .$$

   Maxwell equations:

$$\Longrightarrow \quad \mathbf{B}_0 = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}_0 = \pm \frac{1}{c} E_0\, \mathbf{e}_y \qquad \text{(vacuum!)} .$$

In the semi-infinite space $z \geq 0$:

$$\sigma = \infty \implies \text{extinction coefficient} \, ,$$

$$(4.228): \quad \gamma^2 = \frac{1}{2} n^2 \left[-1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon_0 \epsilon_r \omega} \right)^2} \right] \underset{\sigma \to \infty}{\longrightarrow} \infty \, .$$

The wave cannot penetrate the region $z \geq 0$, i.e. total reflection.

Continuity condition:

$$\mathbf{n} \times (\mathbf{E}_> - \mathbf{E}_<)|_{z=0} = 0 \quad (\mathbf{n} = \mathbf{e}_z) \, ,$$

$$\mathbf{E}_> \equiv 0 \, ; \quad \mathbf{E} \sim \mathbf{e}_x \implies \quad \mathbf{E} = 0 \quad \text{at } z = 0 \, .$$

Ansatz:

$$\mathbf{E} = \mathbf{e}_x \left(E_0 e^{ikz} + \widehat{E}_0 e^{-ikz} \right) e^{-i\omega t} \, ,$$

$$\mathbf{B} = \frac{1}{c} \mathbf{e}_y \left(E_0 e^{ikz} - \widehat{E}_0 e^{-ikz} \right) e^{-i\omega t} \, ,$$

$$\mathbf{E} = 0 \quad \text{at } z = 0 \implies \widehat{E}_0 = -E_0 \, .$$

This yields **standing waves**:

$$\mathbf{E}(\mathbf{r}, t) = 2i E_0 \sin(kz) e^{-i\omega t} \, \mathbf{e}_x \, ,$$

$$\mathbf{B}(\mathbf{r}, t) = 2 \frac{E_0}{c} \cos(kz) e^{-i\omega t} \, \mathbf{e}_y \, .$$

The fields are real:

$$\mathrm{Re}\mathbf{E}(\mathbf{r}, t) = 2E_0 \sin(kz) \sin(\omega t) \, \mathbf{e}_x \, ,$$

$$\mathrm{Re}\mathbf{B}(\mathbf{r}, t) = 2 \frac{E_0}{c} \cos(kz) \cos(\omega t) \, \mathbf{e}_y \, .$$

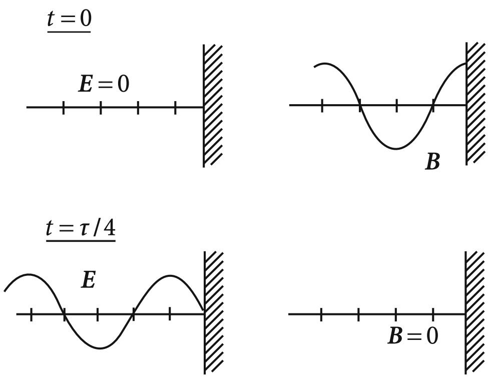
They are spatially as well as temporally phase-shifted by $\pi/2$!

2. see Fig. A.55!
3. Boundary condition:

$$\mathbf{n} \times (\mathbf{H}_> - \mathbf{H}_<)|_{z=0} = \mathbf{j}_F \, ;$$

$\mathbf{H}_> \equiv 0$ because $\sigma = \infty$.

Fig. A.55



From this it follows:

$$\mathbf{j}_F = -\frac{1}{\mu_0}\,\mathbf{e}_z \times \mathbf{B}(z=0) = -2E_0\sqrt{\frac{\epsilon_0}{\mu_0}}\cos\omega t\ (\mathbf{e}_z \times \mathbf{e}_y)$$

$$\Longrightarrow\ \mathbf{j}_F = 2E_0\sqrt{\frac{\epsilon_0}{\mu_0}}\cos\omega t\,\mathbf{e}_x\ ,$$

Alternating current in x-direction!

4. **Energy density:**

$$\begin{aligned}
w(\mathbf{r},t) &= \frac{1}{2}(\mathrm{Re}\mathbf{H}\cdot\mathrm{Re}\mathbf{B} + \mathrm{Re}\mathbf{E}\cdot\mathrm{Re}\mathbf{D})\\
&= \frac{1}{2}\left[\frac{1}{\mu_0}(\mathrm{Re}\mathbf{B})^2 + \epsilon_0(\mathrm{Re}\mathbf{E})^2\right]\\
&= 2E_0^2\epsilon_0(\sin^2 kz\,\sin^2\omega t + \cos^2 kz\,\cos^2\omega t)\\
&= 2E_0^2\epsilon_0\left[\frac{1}{2}(1-\cos 2kz)\sin^2\omega t + \frac{1}{2}(1+\cos 2kz)\cos^2\omega t\right]\\
&= \epsilon_0E_0^2\left[1+\cos 2kz(\cos^2\omega t - \sin^2\omega t)\right]\ ,\\
w(\mathbf{r},t) &= \epsilon_0E_0^2(1+\cos 2kz\,\cos 2\omega t)\ .
\end{aligned}$$

Time-averaged:

$$\bar{w}(\mathbf{r}) = \frac{1}{\tau}\int_0^\tau dt\,w(\mathbf{r},t) = \epsilon_0E_0^2\ .$$

The energy density has a spatial period of $\Delta z = \pi/k = \lambda/2$ and oscillates temporally with 2ω around the mean value $\epsilon_0E_0^2$.

Energy-current density:

$$\mathbf{S}(\mathbf{r},t) = \mathrm{Re}\mathbf{E}(\mathbf{r},t) \times \mathrm{Re}\mathbf{H}(\mathbf{r},t) = 4E_0^2\sqrt{\frac{\epsilon_0}{\mu_0}}\sin kz\,\cos kz\,\sin\omega t\,\cos\omega t\,\mathbf{e}_z$$

$$\Longrightarrow\ \mathbf{S}(\mathbf{r},t) = E_0^2\sqrt{\frac{\epsilon_0}{\mu_0}}\sin 2kz\,\sin 2\omega t\ .$$

Time-averaged:

$$\bar{\mathbf{S}}(\mathbf{r},t) \equiv 0 \quad (\textit{standing wave})\ .$$

Solution 4.3.14

1. Telegraph equation (4.218):

$$\left[\left(\Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right) - \mu_{\mathrm{r}}\mu_0\sigma\frac{\partial}{\partial t}\right]\mathbf{E}(\mathbf{r},t) = 0\ .$$

Ansatz:

$$\mathbf{E}(\mathbf{r},t) \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\Longrightarrow\ -k^2 + \frac{1}{u^2}\omega^2 + i\,\mu_{\mathrm{r}}\,\mu_0\sigma\,\omega = 0$$

$$\Longrightarrow\ k^2 = \frac{\omega^2}{u^2} + i\,\mu_{\mathrm{r}}\mu_0\sigma\,\omega\ .$$

2. Single electron: mass m, charge $-e$
   Equation of motion:

$$\mathbf{m}\,\dot{\mathbf{v}} = -e\,\hat{\mathbf{E}}_0\,e^{-i\omega t}$$

$$\Longrightarrow\ m\,\mathbf{v}(t) = \frac{e\,\hat{\mathbf{E}}_0}{i\omega}e^{-i\omega t} + \text{const}\ .$$

From this we get as current density:

$$\mathbf{j} = -e\,n_0\,\mathbf{v} = \frac{i\,e^2 n_0}{m\,\omega}\mathbf{E} + \text{const}\ .$$

Ohm's law:

$$\mathbf{j} = \mathbf{0}\text{ for }\mathbf{E} = \mathbf{0} \quad\Longrightarrow\quad \text{const} = 0\ .$$

$$\mathbf{j} = i\frac{e^2 n_0}{m\,\omega}\mathbf{E} \quad\Longrightarrow\quad \sigma = i\frac{e^2 n_0}{m\,\omega}\ .$$

3. σ imaginary since **E** was considered as a complex quantity:

$$\begin{aligned}
k^2 &= \frac{\omega^2}{u^2} - \frac{\mu_r\mu_0\, e^2 n_0}{m}\,, \\
k^2(\omega_p) \stackrel{!}{=} 0 &\iff \omega_p^2 \mu_r\mu_0\epsilon_r\epsilon_0 - \frac{\mu_r\mu_0\, e^2 n_0}{m} = 0 \\
&\implies \omega_p^2 = \frac{n_0\, e^2}{\epsilon_r\epsilon_0 m}\,. \\
k^2 \geq 0 \text{ for } \omega \geq \omega_p\,,\ \omega \ll \omega_p &\implies k^2 \approx -\mu_r\mu_0 \frac{e^2 n_0}{m} = (i\bar{\mathbf{k}})^2 \\
&\implies \bar{k}^2 = \mu_r\mu_0 \frac{e^2 n_0}{m} \\
&\implies \mathbf{E}(\mathbf{r}, t) \sim e^{-\bar{k}z - i\omega t} \quad (\mathbf{k} \parallel z\text{-axis})\,.
\end{aligned}$$

Penetration depth $\bar{k}\,\delta = 1$:

$$\implies \delta = \sqrt{\frac{m}{\mu_r\mu_0 e^2 n_0}} = \frac{u}{\omega_p}\,,$$

u: wave velocity in the electron gas.

4. Actual total force:

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\,, \\
\mathbf{F}_1 &= -e\,\mathbf{E}\,, \\
\mathbf{F}_2 &= -e\,\mathbf{v} \times \mathbf{B} \\
&\implies |\mathbf{F}_2| \ll |\mathbf{F}_1|\,, \quad \text{if } v|\mathbf{B}| \ll |\mathbf{E}|\,.
\end{aligned}$$

One obtains from 2.:

$$v = \frac{e}{m\,\omega}|\mathbf{E}|\,.$$

The law of induction yields:

$$\begin{aligned}
&\mathrm{curl}\mathbf{E} = -\dot{\mathbf{B}} \implies \mathbf{B} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E}\,, \quad |\mathbf{B}| = \frac{1}{\omega}k|\mathbf{E}| \\
&\implies v|\mathbf{B}| = \frac{e}{m\,\omega}|\mathbf{E}|\,\frac{1}{\omega}k|\mathbf{E}| \stackrel{!}{\ll} k\,|\mathbf{E}| \\
&\implies |\mathbf{E}| \ll \frac{m\,\omega^2}{e\,k} = \frac{m\,\omega^2}{e\sqrt{\frac{\omega^2}{u^2} - \frac{\mu_r\mu_0 e^2 n_0}{m}}} = \frac{m\,\omega}{e\sqrt{\epsilon_r\epsilon_0\mu_r\mu_0}\sqrt{1 - \frac{e^2 n_0}{\omega^2 m \epsilon_r\epsilon_0}}}\,.
\end{aligned}$$

5. Index of refraction: $k = (\omega/c)n$

$$\Longrightarrow \; n = c\frac{k}{\omega} \; \Longrightarrow \; n^2 = \frac{k^2}{\epsilon_0\mu_0\omega^2} \; .$$

We seek after the (k^2, ω^2)-relation for the case that the external field $\mathbf{B}_0$ is absent, as a generalization of 1.:

$$m\,\dot{\mathbf{v}} = -e\,\mathbf{E}(t) - e\,\mathbf{v}\times\mathbf{B}_0 \quad (\mathbf{B}_0 = B_0\,\mathbf{e}_z) \; .$$

Circularly polarized wave, **complex** ansatz (cf. (4.150)):

$$\mathbf{E}(t) = \widehat{E}_0(\mathbf{r})(\mathbf{e}_x \pm i\,\mathbf{e}_y)e^{-i\omega t} \; ,$$
$$\mathbf{j} = \sigma\,\mathbf{E} \sim \mathbf{v} \; \Longrightarrow \; \mathbf{v} \sim \mathbf{E} \; .$$

Therefore the following approach:

$$\mathbf{v}(\mathbf{r}, t) = v_\pm(\mathbf{r})(\mathbf{e}_x \pm i\,\mathbf{e}_y)e^{-i\omega t}$$
$$\Longrightarrow \; -i\,m\,\omega\,v_\pm(\mathbf{r})(\mathbf{e}_x \pm i\,\mathbf{e}_y) = -e\,\widehat{E}_0(\mathbf{r})(\mathbf{e}_x \pm i\,\mathbf{e}_y) - e\,B_0(-\mathbf{e}_y \pm i\,\mathbf{e}_x)v_\pm(\mathbf{r})$$
$$\Longrightarrow \; v_\pm(\mathbf{r})\left\{-i\,m\,\omega \pm i\,e\,B_0\right\}(\mathbf{e}_x \pm i\,\mathbf{e}_y) = -e\,\widehat{E}_0(\mathbf{r})(\mathbf{e}_x \pm i\,\mathbf{e}_y)$$
$$\Longrightarrow \; v_\pm(\mathbf{r}) = \frac{-i\,e\,\widehat{E}_0(\mathbf{r})}{m\,\omega \mp e\,B_0} = \frac{-i\,e\,\widehat{E}_0(\mathbf{r})}{m(\omega \mp \omega_\mathrm{c})} \; ,$$

$\omega_\mathrm{c} = e\,B_0/m$: cyclotron frequency.

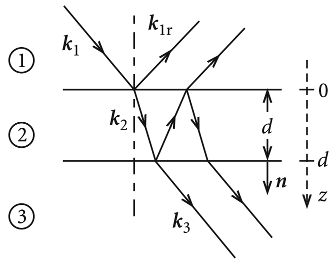
We now proceed as in 2.:

$$\mathbf{j} = -e\,n_0\mathbf{v} = -e\,n_0\frac{v_\pm(r)}{\widehat{E}_0(\mathbf{r})}\mathbf{E}(\mathbf{r}, t) \stackrel{!}{=} \sigma\,\mathbf{E}$$
$$\Longrightarrow \; \sigma = \frac{i\,e^2 n_0}{m(\omega \mp \omega_\mathrm{c})} \; .$$

As in 1. it follows from the telegraph equation:

$$k^2 = \frac{\omega^2}{u^2} + i\,\mu_\mathrm{r}\mu_0\sigma\,\omega = \mu_\mathrm{r}\mu_0\epsilon_\mathrm{r}\epsilon_0\omega^2\left[1 - \frac{e^2 n_0}{m\,\epsilon_\mathrm{r}\epsilon_0\omega(\omega \mp \omega_\mathrm{c})}\right] \; .$$

Fig. A.56



Plasma:

$$k^2 = \mu_r \mu_0 \epsilon_r \epsilon_0 \omega^2 \left(1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}\right) ,$$

$$k_\pm^2 = \frac{\omega^2}{u^2}\left(1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}\right)$$

$$\Longrightarrow \quad n_\pm^2 = \epsilon_r \mu_r \left(1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}\right) .$$

$n_+ \neq n_-$ for $B_0 \neq 0$, i.e. the index of refraction is different in the plasma for right and left circularly polarized waves. This is the reason for the *circular birefringence.*

Solution 4.3.15 Assume for all the three media (Fig. A.56)

$$\mu_r^{(1)} = \mu_r^{(2)} = \mu_r^{(3)} = 1 .$$

Then it holds for the indexes of refraction:

$$n_i = \sqrt{\epsilon_r^{(i)}} ; \quad i = 1, 2, 3 .$$

Interfaces: xy-plane. That means:

Normal of the interfaces: $\mathbf{n} = \mathbf{e}_z$,

Incidence plane: $(\mathbf{n}, \mathbf{k}_1)$-plane.

In general:

The incident wave $\mathbf{E}_1$ can be decomposed into two linearly polarized waves, one perpendicular and the other parallel to the incidence plane.

Here:

Perpendicular incidence $\Longrightarrow$ incidence plane not definable; distinction between *parallel* and *perpendicular* meaningless.

Therefore w.l.o.g.: $\mathbf{E}_1 = E_1 \, \mathbf{e}_x$.

Medium 1:
Incident wave:

$$
\begin{aligned}
\mathbf{E}_1 &= E_{01}\, e^{i(k_1 z-\omega t)}\, \mathbf{e}_x \ , \\
\mathbf{B}_1 &= \frac{1}{u_1}(\boldsymbol{\kappa}_1 \times \mathbf{E}_1) = \frac{n_1}{c} E_{01}\, e^{i(k_1 z-\omega t)}\, \mathbf{e}_y \ \Longrightarrow\ \boldsymbol{\kappa}_1 = \frac{\mathbf{k}_1}{k_1} = \mathbf{e}_z \ .
\end{aligned}
$$

Reflected wave:

$$
\begin{aligned}
\mathbf{E}_{1r} &= E_{01r}\, e^{i(-k_1 z-\omega t)}\, \mathbf{e}_x \quad \left(k_{1r} = \frac{\omega}{c} n_1 = k_1\right) \ , \\
\mathbf{B}_{1r} &= \frac{1}{u_1}\Big(\underbrace{\boldsymbol{\kappa}_{1r}}_{-\mathbf{e}_z} \times \mathbf{E}_{1r}\Big) = -\frac{n_1}{c} E_{01r}\, e^{-ik_1 z - i\omega t}\, \mathbf{e}_y \ .
\end{aligned}
$$

The sum of both contributions yields the total field in the medium 1.
Medium 2:

$$
\begin{aligned}
\mathbf{E}_2 &= E_{02}\, e^{i(k_2 z-\omega t)}\, \mathbf{e}_x \ , \\
\mathbf{B}_2 &= \frac{n_2}{c} E_{02}\, e^{i(k_2 z-\omega t)}\, \mathbf{e}_y \ , \\
\mathbf{E}_{2r} &= E_{02r}\, e^{-ik_2 z - i\omega t}\, \mathbf{e}_x \ , \\
\mathbf{B}_{2r} &= -\frac{n_2}{c}\, E_{02r}\, e^{-ik_2 z - i\omega t}\, \mathbf{e}_y \ .
\end{aligned}
$$

Medium 3:
Here only a refracted wave:

$$
\begin{aligned}
\mathbf{E}_3 &= E_{03}\, e^{i(k_3 z-\omega t)} \mathbf{e}_x \ , \\
\mathbf{B}_3 &= \frac{n_3}{c} E_{03}\, e^{i(k_3 z-\omega t)}\, \mathbf{e}_y \ .
\end{aligned}
$$

Boundary conditions:
Tangential components of **E** and **H** continuous at the interfaces.

$z = 0$:

$$
\begin{aligned}
E_{01} + E_{01r} &= E_{02} + E_{02r} \ , \\
n_1(E_{01} - E_{01r}) &= n_2(E_{02} - E_{02r}) \ .
\end{aligned}
$$

$z = d$:

$$
\begin{aligned}
E_{02}\, e^{ik_2 d} + E_{02r}\, e^{-ik_2 d} &= E_{03}\, e^{ik_3 d} \ , \\
n_2\left(E_{02}\, e^{ik_2 d} - E_{02r}\, e^{-ik_2 d}\right) &= n_3 E_{03}\, e^{ik_3 d} \ .
\end{aligned}
$$

Compensation layer such that

$$E_{01r} \stackrel{!}{=} 0 \; .$$

For this purpose it remains to be solved:

$z = 0$:

$$\begin{aligned} E_{01} - E_{02} - E_{02r} &= 0 \; , \\ n_1 E_{01} - n_2 E_{02} + n_2 E_{02r} &= 0 \\ \curvearrowright \;\; (n_1 - n_2)E_{02} + (n_1 + n_2)E_{02r} &= 0 \; . \end{aligned}$$

$z = d$

$$(n_2 - n_3)E_{02}\, e^{ik_2 d} - (n_2 + n_3)E_{02r}\, e^{-ik_2 d} = 0 \; .$$

The determinant of the coefficients of the homogeneous system of equations for E_{02} and E_{02r} must vanish:

$$\begin{aligned} &(n_2 - n_3)e^{ik_2 d}(n_2 + n_1) = (n_2 - n_1)(n_2 + n_3)e^{-ik_2 d} \\ &\Longrightarrow \;\; e^{2ik_2 d} = \frac{(n_2 - n_1)(n_2 + n_3)}{(n_2 + n_1)(n_2 - n_3)} \; . \end{aligned}$$

The right-hand side is real, therefore the left-hand side must also be real.

(a) $e^{2ik_2 d} = 1$

$$\Longleftrightarrow \;\; k_2 d = m\pi \;\; \Longleftrightarrow \;\; d = \frac{m\lambda_2}{2}$$

and

$$\begin{aligned} &(n_2 + n_1)(n_2 - n_3) \stackrel{!}{=} (n_2 - n_1)(n_2 + n_3) \\ &\Longleftrightarrow \;\; n_2^2 + n_1 n_2 - n_2 n_3 - n_1 n_3 \stackrel{!}{=} n_2^2 + n_2 n_3 - n_1 n_3 - n_1 n_2 \\ &\Longleftrightarrow \;\; 2n_1 n_2 \stackrel{!}{=} 2n_2 n_3 \\ &\Longleftrightarrow \;\; n_1 = n_3 \quad \text{not the interesting case!} \end{aligned}$$

(b) $e^{2ik_2 d} = -1$

$$\Longleftrightarrow \;\; k_2 d = \frac{2m+1}{2}\pi \;\; \Longleftrightarrow \;\; d = (2m+1)\frac{\lambda_2}{4}$$

and

$$
\begin{aligned}
& n_2^2 + n_1 n_2 - n_2 n_3 - n_1 n_3 = -n_2^2 - n_2 n_3 + n_1 n_3 + n_1 n_2 \\
\iff & 2n_2^2 = 2n_1 n_3 \\
\iff & n_2^2 = n_1 n_3 \; .
\end{aligned}
$$

Law of refraction:

$$
\frac{k_2}{k_1} = \frac{n_2}{n_1} = \frac{\lambda_1}{\lambda_2} \; .
$$

From that it follows for the **compensation layer**:

$$
n_2 = \sqrt{n_1 n_3} \; ; \quad d = (2m+1)\frac{n_1}{n_2}\frac{\lambda_1}{4} \; .
$$

Solution 4.3.16

1. According to (4.258) one finds the angle of total reflection:

$$
\sin \vartheta_g = \frac{n_2}{n_1} = \frac{1}{2} \;\Rightarrow\; \vartheta = 30^\circ \; .
$$

2. From the requirement

$$
\sin \vartheta_1 \overset{!}{=} \cos \vartheta_1
$$

we derive

$$
2\sin^2 \vartheta_1 = \sin^2 \vartheta_1 + \cos^2 \vartheta_2 = 1 \;\Rightarrow\; \sin \vartheta_1 = \cos \vartheta_1 = \frac{1}{\sqrt{2}} \;\Rightarrow\; \vartheta_1 = 45^\circ \; .
$$

Law of refraction (4.257):

$$
\begin{aligned}
\sin \vartheta_2 &= \frac{n_1}{n_2} \sin \vartheta_1 = \frac{2}{\sqrt{2}} = \sqrt{2} > 1 \; , \\
\cos \vartheta_2 &= \sqrt{1 - \sin^2 \vartheta_2} = \sqrt{-1} = i \; .
\end{aligned}
$$

3. Fresnel formula (4.271):

$$
\left(\frac{E_{01\mathrm{r}}}{E_{01}}\right)_\perp = \frac{n_1 \cos \vartheta_1 - n_2 \cos \vartheta_2}{n_1 \cos \vartheta_1 + n_2 \cos \vartheta_2} = \frac{\sqrt{2} - i}{\sqrt{2} + i} = \frac{|a| e^{-i\psi}}{|a| e^{i\psi}} = e^{-2i\psi} \; .
$$

Thereby:

$$\tan\psi = \frac{1}{\sqrt{2}} .$$

Fresnel formula (4.273):

$$\left(\frac{E_{01\mathrm{r}}}{E_{01}}\right)_{\parallel} = \frac{n_2\cos\vartheta_1 - n_1\cos\vartheta_2}{n_2\cos\vartheta_1 + n_1\cos\vartheta_2} = \frac{\frac{1}{\sqrt{2}} - 2i}{\frac{1}{\sqrt{2}} + 2i} = \frac{|b|e^{-i\varphi}}{|b|e^{i\varphi}} = e^{-2i\varphi} .$$

Thereby:

$$\tan\varphi = \frac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2} .$$

Relative phase shift:

$$\delta = 2(\varphi - \psi) .$$

That means:

$$\begin{aligned}
\tan\frac{\delta}{2} &= \tan(\varphi - \psi) = \frac{\tan\varphi - \tan\psi}{1 + \tan\varphi\tan\psi} \\
&= \frac{2\sqrt{2} - \frac{1}{\sqrt{2}}}{1+2} = \frac{1}{3}\frac{1}{\sqrt{2}}(4-1) \\
\Rightarrow \quad \tan\frac{\delta}{2} &= \frac{1}{\sqrt{2}} .
\end{aligned}$$

If the reflected wave were circularly polarized then it would have to be $\delta = \pi/2$ and therewith $\tan\frac{\delta}{2} = \tan\frac{\pi}{4} = 1$. This is not the case. The reflected wave is thus **not** circularly polarized!

4. Reflection coefficient (4.283):

$$R = \left|\frac{E_{01\mathrm{r}}}{E_{01}}\right|^2 .$$

Part 3. leads to:

$$\left|\frac{E_{01\mathrm{r}}}{E_{01}}\right|^2_{\perp(\parallel)} = 1 .$$

That means:

$$\begin{aligned}|E_{01\mathrm{r}}|^2 &= \left|E_{01\mathrm{r}}^{\perp}\right|^2 + \left|E_{01\mathrm{r}}^{\parallel}\right|^2 \\ &= \left|E_{01}^{\perp}\right|^2 + \left|E_{01}^{\parallel}\right|^2 = |E_{01}|^2 \ .\end{aligned}$$

We have to conclude

$$R = 1 \quad (\text{‘total reflection’}) \ .$$

Solution 4.3.17 ϑ_g: critical angle of total reflection

$$\vartheta_1 = \vartheta_\mathrm{g} \implies \vartheta_2 = \frac{\pi}{2} \ .$$

Law of refraction:

$$\implies \sin\vartheta_\mathrm{g} = \frac{n_2}{n_1} \quad (n_2 < n_1) \ .$$

For $\vartheta_1 > \vartheta_\mathrm{g}$ the linearly polarized components of the **reflected** wave, parallel and perpendicular to the incidence plane, are phase shifted relative to each other by δ. According to (4.291) δ follows from:

$$\tan\frac{\delta}{2} = \frac{\cos\vartheta_1\sqrt{\sin^2\vartheta_1 - \sin^2\vartheta_\mathrm{g}}}{\sin^2\vartheta_1} \ .$$

1. Total reflection (4.288):

$$\left|\left(\frac{E_{01r}}{E_{01}}\right)_{\perp}\right| = \left|\left(\frac{E_{01r}}{E_{01}}\right)_{\parallel}\right| = 1$$

‘Circularly polarized’ means:

(a)

$$\left|E_{01r}^{\perp}\right| = \left|E_{01r}^{\parallel}\right| \ .$$

(b)

$$\delta = \frac{\pi}{2} \ .$$

(a) is fulfilled if it holds for the incident wave

$$\left|E_{01}^{\perp}\right| = \left|E_{01}^{\|}\right| \;,$$

which we want to presume here. The sought-after condition for the ratio n_2/n_1 follows namely directly from the requirement (b), i.e.:

$$\tan\frac{\delta}{2} \stackrel{!}{=} 1 \;.$$

This means:

$$\begin{aligned}
1 &= \frac{\cos^2\vartheta_1}{\sin^4\vartheta_1}(\sin^2\vartheta_1 - \sin^2\vartheta_{\mathrm{g}}) \\
\Longleftrightarrow\quad \sin^2\vartheta_{\mathrm{g}} &= \left(\frac{n_2}{n_1}\right)^2 = \sin^2\vartheta_1 - \frac{\sin^4\vartheta_1}{\cos^2\vartheta_1} \;.
\end{aligned}$$

We look for the maximal ratio n_2/n_1 for which this equation still has a solution. That can be considered as an extreme-value problem:

$$\begin{aligned}
y &= x - \frac{x^2}{1-x} \\
\Longrightarrow\quad \frac{dy}{dx} &= 1 - \frac{x^2}{(1-x)^2} - \frac{2x}{1-x} = \frac{1-2x+x^2-x^2-2x+2x^2}{(1-x)^2} \\
&= \frac{1-4x+2x^2}{(1-x)^2} \stackrel{!}{=} 0 \\
\Longleftrightarrow\quad x_0^2 - 2x_0 &= -\frac{1}{2} \;, \\
(x_0-1)^2 &= \frac{1}{2} \;\Longrightarrow\; x_0^{\pm} = 1 \pm \frac{1}{\sqrt{2}} \;.
\end{aligned}$$

Because of $x_0 \leq 1$ only

$$x_0 = 1 - \frac{1}{\sqrt{2}}$$

makes sense!

It follows herefrom:

$$y_{\max} = 3 - 2\sqrt{2} \Longrightarrow \left(\frac{n_2}{n_1}\right)^2_{\max} \approx 0{,}18 \;.$$

2.

$$\tan\frac{\delta}{2} \stackrel{!}{=} 1 \quad\Longrightarrow\quad \left(\frac{n_2}{n_1}\right)^2 \stackrel{1.}{=} \frac{\sin^2\vartheta_1(1-2\sin^2\vartheta_1)}{1-\sin^2\vartheta_1}$$

$$\Longleftrightarrow\quad \left(\frac{n_2}{n_1}\right)^2 = -2\sin^4\vartheta_1 + \sin^2\vartheta_1\left[1+\left(\frac{n_2}{n_1}\right)^2\right]$$

$$\Longrightarrow\quad \left\{\sin^2\vartheta_1 - \frac{1}{4}\left[1+\left(\frac{n_2}{n_1}\right)^2\right]\right\}^2 = -\frac{1}{2}\left(\frac{n_2}{n_1}\right)^2 + \frac{1}{16}\left[1+\left(\frac{n_2}{n_1}\right)^2\right]^2$$

$$\Longrightarrow\quad \sin^2\vartheta_1 = \frac{1}{4}\left[1+\left(\frac{n_2}{n_1}\right)^2 \pm \sqrt{1-6\left(\frac{n_2}{n_1}\right)^2+\left(\frac{n_2}{n_1}\right)^4}\,\right].$$

Solution 4.3.18 In order to guarantee that the plane wave

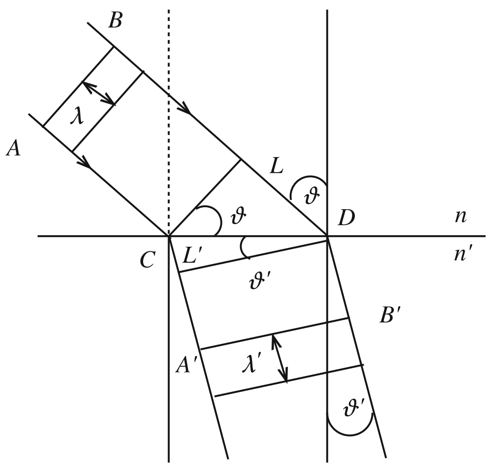
$$\sim e^{i(kr-\omega t)}$$

is also at A' and B' in phase the phase difference on the pathways L and L' must be the same (Fig. A.57):

$$kL = \frac{2\pi}{\lambda}L = L\frac{n\omega}{c}\,,$$

$$k'L' = \frac{2\pi}{\lambda'}L' = L'\frac{n'\omega}{c}\,.$$

Fig. A.57



Requirement:

$$kL \stackrel{!}{=} k'L' \;\Leftrightarrow\; Ln \stackrel{!}{=} L'n' \;.$$

Consider in Fig. A.57 now the two triangles in which L and L' are involved with the common hypotenuse $\overline{CD}$:

$$\sin\vartheta = \frac{L}{\overline{\overline{CD}}} \;; \quad \sin\vartheta' = \frac{L'}{\overline{\overline{CD}}}$$
$$\Rightarrow \frac{L}{L'} = \frac{\sin\vartheta}{\sin\vartheta'} \;.$$

This is to be equated with the above requirement:

$$\frac{\sin\vartheta}{\sin\vartheta'} = \frac{n'}{n}$$

That was to be shown!

Solution 4.3.19

1. We take over the notation from Fig. A.58:

$\mathbf{n} = \mathbf{e}_z$:	surface normal of the aperture σ ,
$\mathbf{r}' = (x', y', 0) \in \sigma$	
Q:	source: $\mathbf{R}_0 = (X_0, Y_0, Z_0)$,
P:	observer: $\mathbf{R} = (X, Y, Z)$,
$k = \frac{2\pi}{\lambda}$; λ:	wavelength of the light .

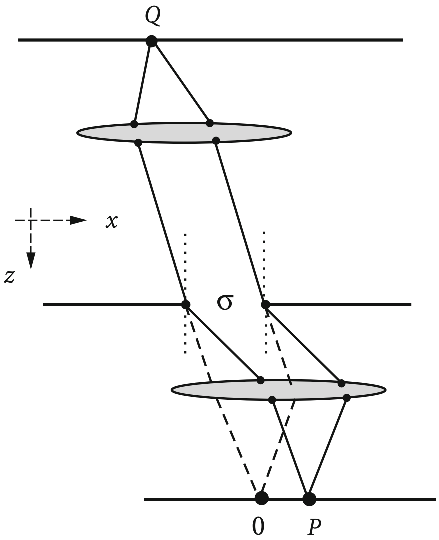
For the field vector at the point P Kirchhoff's formula applies (4.322):

$$E(P) = \frac{i}{2\lambda}\widehat{A}\,\frac{\cos(\mathbf{n}, \mathbf{R}_0) - \cos(\mathbf{n}, \mathbf{R})}{RR_0}\, e^{ik(R+R_0)} \int_\sigma df' \,\exp(ik\varphi(x', y')) \;.$$

$\widehat{A}$ is the amplitude of the incident wave. In the arrangement sketched in Fig. A.58 parallel light is realized by the fact that the source of light Q and the observer P are located in the focal planes of lenses. The preconditions of Fraunhofer diffraction ($R \to \infty$, $R_0 \to \infty$) are therewith fulfilled and it holds according to (4.321):

$$\varphi(x', y') = -x'\left(\frac{X}{R} + \frac{X_0}{R_0}\right) - y'\left(\frac{Y}{R} + \frac{Y_0}{R_0}\right) \equiv -x'(\alpha - \alpha_0) - y'(\beta - \beta_0) \;.$$

Fig. A.58



Here we have introduced for abbreviation the direction cosines (see Fig 4.59)

$$\alpha = \frac{X}{R} \; ; \quad \alpha_0 = \frac{-X_0}{R_0} \; ; \quad \beta = \frac{Y}{R} \; ; \quad \beta_0 = \frac{-Y_0}{R_0} \; .$$

It is then yet to be calculated:

$$E(P) \approx Cke^{ik(R+R_0)} \int_{-A}^{+A} dx' \, e^{-ik(\alpha-\alpha_0)x'} \int_{-B}^{+B} dy' \, e^{-ik(\beta-\beta_0)y'} \; .$$

C is here an unimportant complex quantity being essentially fixed by the amplitude and the phase of the incident wave. The integrations are easily done:

$$\begin{aligned} E(P) &= Cke^{ik(R+R_0)} \left\{ \frac{-2i}{-ik(\alpha-\alpha_0)} \sin(k(\alpha-\alpha_0)A) \right\} \\ &\quad \cdot \left\{ \frac{-2i}{-ik(\beta-\beta_0)} \sin(k(\beta-\beta_0)B) \right\} \\ &= 4AB \, Cke^{ik(R+R_0)} \left\{ \frac{\sin x_A}{x_A} \right\} \left\{ \frac{\sin y_B}{y_B} \right\} \; . \end{aligned}$$

Here it was abbreviated:

$$x_A = k(\alpha-\alpha_0)\,A \; ; \quad y_B = k(\beta-\beta_0)\,B \; .$$

It results therewith for the required **intensity**:

$$I(P) \propto |E(P)|^2 \propto \left\{\frac{\sin x_A}{x_A}\right\}^2 \left\{\frac{\sin y_B}{y_B}\right\}^2 .$$

Discussion:

We consider at first only the diffraction pattern in x-direction.

- $$\lim_{x_A \to 0} \frac{\sin x_A}{x_A} = \lim_{x_A \to 0} \frac{\cos x_A}{1} = 1 .$$

  $x_A = 0$ defines the *'principal maximum'* of the intensity distribution.
- *'Zeros'*:

  $$x_A = k(\alpha - \alpha_0)A \stackrel{!}{=} n\pi \;\Leftrightarrow\; (\alpha - \alpha_0)A \stackrel{!}{=} n\frac{\lambda}{2} \; ; \quad n \in \mathbb{Z} .$$

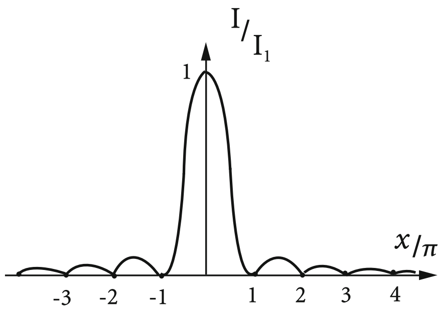
- *'Submaxima'*:

  $$\frac{d}{dx_A}\frac{\sin x_A}{x_A} = \frac{\cos x_A}{x_A} - \frac{\sin x_A}{x_A^2} \stackrel{!}{=} 0 \;\Rightarrow\; \tan x_A \stackrel{!}{=} x_A .$$

  The heights of the submaxima can be calculated numerically: 0,047 $\to$ 0,017 $\to$ ....The principal maximum thus strongly dominates. With a deviation of less than 5 % the area under the I/I_0-curve is restricted to the principal maximum (Fig. A.59).
- Distance between the principal maximum and the first zero:

  $$x_A \stackrel{!}{=} \pi \;\Leftrightarrow\; \alpha - \alpha_0 = \frac{\lambda}{2A} .$$

Fig. A.59



The angular distance thus becomes larger with decreasing width of the aperture. Obviously diffraction phenomena are observable only if the linear dimensions of the diffracting object are comparable with the wavelength λ.

The same considerations can be applied for the y-direction. The diffraction image is then a cross-shaped pattern.

2. **Diffraction by a slit:**

$$B \gg A \ .$$

The diffraction image contracts more and more parallel to the y-axis so that light intensity is practically no longer present outside of $\beta = \beta_0$ (center of the diffraction pattern):

$$\lim_{B\to\infty} \left(\frac{\sin(k(\beta - \beta_0)B)}{k(\beta - \beta_0)B} \right)^2 = 0 \quad (\beta \neq \beta_0) \ .$$

Finite intensities exist therefore only on the x-axis, in fact exactly as discussed in the first part of this exercise:

$$\frac{I}{I_0} = \left(\frac{\sin(k(\alpha - \alpha_0)A)}{k(\alpha - \alpha_0)A} \right)^2 \ .$$

Notice, however, that, because of previous assumptions ('small aperture'), strictly speaking, $B \to \infty$ is of course not realizable, but probably $B \gg A$.

In the case of perpendicular incidence ($\mathbf{R}_0 \propto \mathbf{e}_z$) one has $\alpha_0 = \cos(\mathbf{e}_x, \mathbf{e}_z) = \cos\frac{\pi}{2} = 0$ and $\alpha = \cos(\mathbf{e}_x, \mathbf{R}) = \cos\alpha^\star$. For the angle of emergence $\hat{\alpha}$ (angle between $\mathbf{R}$ and the normal $\mathbf{e}_z$) it holds $\hat{\alpha} = \frac{\pi}{2} - \alpha^\star$ and therewith:

$$\alpha = \sin\hat{\alpha} \ .$$

The minimum condition for the single split then reads according to part 1.:

$$\alpha\, k A \overset{!}{=} n\,\pi \ \Leftrightarrow \ \sin\hat{\alpha}_n \frac{2\pi}{\lambda} A \overset{!}{=} n\,\pi \ .$$

This means:

$$2A\ \sin\hat{\alpha}_n = n\,\lambda \ ; \quad n \in \mathbb{Z} \ .$$

Section 4.4.6

Solution 4.4.1

(a) $t > 0$:

We close the semi-circle in the lower half-plane (Fig. A.60) which then does not contribute to the integral for $R \to \infty$:

$$\Rightarrow \int\limits_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{e}^{-\mathrm{i}xt}}{x + \mathrm{i}\,0^+} = \int\limits_{\cup} \mathrm{d}z \, \frac{\mathrm{e}^{-\mathrm{i}zt}}{z + \mathrm{i}\,0^+} \, .$$

A pole of first order at $z_0 = -i\,0^+$.

Residue:

$$\operatorname*{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) \, \frac{e^{-izt}}{z + i\,0^+} = 1 \, .$$

Residue theorem:

$$\int\limits_{\cup} \mathrm{d}z \frac{\mathrm{e}^{-\mathrm{i}zt}}{z + \mathrm{i}0^+} = \underset{\uparrow}{-}2\pi\mathrm{i} \cdot 1 \, .$$

Pole is traversed in
mathematically
negative direction

Thus it follows:

$$\Theta(t) = \frac{i}{2\pi}(-2\pi\, i) = 1 \quad \text{for } t > 0 \, .$$

(b) $t < 0$:

We now close the semi-circle in the upper half-plane (Fig. A.61). But then there is no pole inside:

$$\Longrightarrow \; \Theta(t) = 0 \quad \text{for } t < 0 \, .$$

Fig. A.60

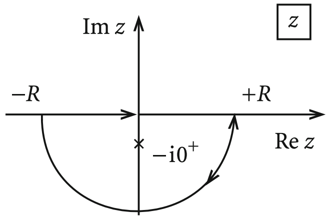
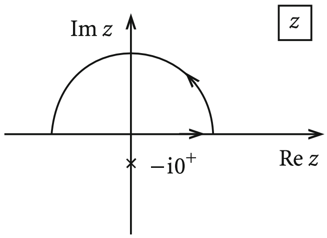


Fig. A.61



Solution 4.4.2

$$\Psi(\mathbf{r},t) = \frac{1}{4\pi^2}\int d^3\bar{k}\int\limits_{-\infty}^{+\infty} d\bar{\omega}\, e^{i(\bar{\mathbf{k}}\cdot\mathbf{r}-\bar{\omega}t)}\tilde{\Psi}(\bar{\mathbf{k}},\bar{\omega}) = \frac{1}{2\pi^2}\int d^3\bar{k}\,\frac{e^{i\bar{\mathbf{k}}\cdot\mathbf{r}}}{\bar{k}^2-k^2}\,e^{-i\omega t}\ ,$$

$\bar{\mathbf{k}}$-integration with spherical coordinates,
$\mathbf{r} =$ polar axis, $\vartheta = \sphericalangle(\bar{\mathbf{k}},\mathbf{r})$, $d^3\bar{k} = \bar{k}^2 d\bar{k}\cdot\sin\vartheta\, d\vartheta\, d\varphi$.

$$\Psi(\mathbf{r},t) = e^{-i\omega t}\frac{1}{\pi}\int\limits_0^\infty \bar{k}^2\, d\bar{k}\,\frac{1}{\bar{k}^2-k^2}\int\limits_{-1}^{+1} d\cos\vartheta\, e^{i\bar{k}r\cos\vartheta}$$
$$= \frac{e^{-i\omega t}}{i\pi r}\int\limits_0^\infty d\bar{k}\,\frac{\bar{k}}{\bar{k}^2-k^2}\left(e^{i\bar{k}r}-e^{-i\bar{k}r}\right)\ .$$

The integrand is an even function of $\bar{k}$

$$\Longrightarrow\ \int\limits_0^\infty d\bar{k}\,\frac{\bar{k}}{\bar{k}^2-k^2}\left(e^{i\bar{k}r}-e^{-i\bar{k}r}\right) = \frac{1}{2}\int\limits_{-\infty}^{+\infty} d\bar{k}\,\frac{\bar{k}}{\bar{k}^2-k^2}\left(e^{i\bar{k}r}-e^{-i\bar{k}r}\right)\ .$$

Furthermore:

$$\frac{\bar{k}}{\bar{k}^2-k^2} = \frac{1}{2}\left(\frac{1}{\bar{k}-k}+\frac{1}{\bar{k}+k}\right)$$
$$\Longrightarrow\ \Psi(\mathbf{r},t) = \frac{e^{-i\omega t}}{r}\,\frac{1}{4\pi\, i}\,(I_+ - I_-)\ ,$$
$$I_\pm = \int\limits_{-\infty}^{+\infty} d\bar{k}\left(\frac{1}{\bar{k}-k}+\frac{1}{\bar{k}+k}\right)e^{\pm i\bar{k}r}\ .$$

I_+:

Fig. A.62

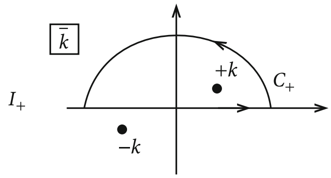
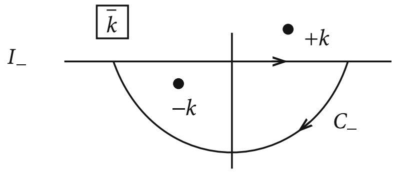


Fig. A.63



No contribution on the semi-circle (Fig. A.62). Inside only one pole of first order: $\bar{k}_+ = +(k + i\,0^+)$.

$$\mathrm{Res}_{k_+}\left[\left(\frac{1}{\bar{k}-k} + \frac{1}{\bar{k}+k}\right) e^{i\bar{k}r}\right] = e^{ikr}$$
$$\Longrightarrow \quad I_+ = 2\pi\, i\, e^{ikr} \;.$$

I_-:
The semi-circle when closed in the lower half-plane does not contribute (Fig. A.63). In the inside of the path C_- there is now the pole:

$$\bar{k}_- = -(k + i\,0^+) \;,$$
$$\mathrm{Res}_{k_-}\left[\left(\frac{1}{\bar{k}-k} + \frac{1}{\bar{k}+k}\right) e^{-i\bar{k}r}\right] = e^{ikr}$$
$$\Longrightarrow \quad I_- = -2\pi\, i\, e^{ikr} \;.$$

That means all in all:

$$\Psi(\mathbf{r}, t) = \frac{1}{r}\, e^{i(kr - \omega t)}$$

That was to be proven!

Section 4.5.6

Solution 4.5.1

1. We have to prove the assertion:

$$\left(\Delta + k^2\right) \frac{e^{\pm ikr}}{r} = -4\pi\delta(\mathbf{r}) \; .$$

Thereby (1.69),

$$\Delta \frac{1}{r} = -4\pi\delta(\mathbf{r}) \; ,$$

can be used.

(a) $\mathbf{r} \neq \mathbf{0}$
Laplace operator in spherical coordinates:

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \Delta_{\vartheta,\varphi} \; .$$

Therewith one finds

$$\Delta \frac{e^{\pm ikr}}{r} = \frac{1}{r}\frac{\partial^2}{\partial r^2} e^{\pm ikr} = -k^2 \frac{1}{r} e^{\pm ikr} \; .$$

Hence, the first condition for the δ-function is fulfilled:

$$\left(\Delta + k^2\right) \frac{e^{\pm ikr}}{r} = 0 \quad \text{for} \quad r \neq 0 \; .$$

(b) The second condition needs some more effort. If $f(\mathbf{r})$ is an arbitrary function then it holds at first with the result from part (a):

$$\int d^3r f(\mathbf{r}) \left(\Delta + k^2\right) \frac{e^{\pm ikr}}{r} = \lim_{\varepsilon \to 0} \int\limits_{K_\varepsilon} d^3r f(\mathbf{r}) \left(\Delta + k^2\right) \frac{e^{\pm ikr}}{r} \; .$$

Here K_ε shall be a sphere with its center at $\mathbf{r} = 0$ and the radius ε. We estimate:

$$\begin{aligned} \Delta \frac{e^{\pm ikr}}{r} &= \Delta \frac{1}{r}\left(1 \pm ikr - \frac{1}{2}k^2r^2 + \frac{1}{6}(\pm ikr)^3 + \frac{1}{24}(\pm ikr)^4 + \sum_{n=5}^{\infty} \frac{1}{n!}(\pm ikr)^n\right) \\ &= \Delta \frac{1}{r} + \Delta\left(\pm ik - \frac{1}{2}k^2 r + \frac{1}{6}(\pm ik)^3 r^2 + \frac{1}{24}(\pm ik)^4 r^3 + \sum_{n=5}^{\infty} \frac{1}{n!}(\pm ik)^n r^{n-1}\right) \end{aligned}$$

$$
\begin{aligned}
&= \Delta \frac{1}{r} + \frac{1}{r} \frac{\partial^2}{\partial r^2} r(\dots) \\
&= \Delta \frac{1}{r} + \frac{1}{r}\left(-k^2 + (\pm ik)^3 r + \frac{1}{2}(\pm ik)^4 r^2 + \sum_{n=5}^{\infty} \frac{1}{n!} n(n-1)(\pm ik)^n r^{n-2}\right) \\
&= \Delta \frac{1}{r} - \frac{k^2}{r} \mp ik^3 + \frac{1}{2} k^4 r + \mathcal{O}(r^2) \ .
\end{aligned}
$$

On the other hand:

$$
k^2 \, \frac{e^{\pm ikr}}{r} = \frac{k^2}{r} \pm ik^3 - \frac{1}{2} k^4 r + \mathcal{O}\left(r^2\right) \ .
$$

That justifies the estimation:

$$
\left(\Delta + k^2\right) \, \frac{e \pm ikr}{r} = \Delta \, \frac{1}{r} + \mathcal{O}\left(r^2\right) \ .
$$

Hence we can write:

$$
\begin{aligned}
\int d^3 r f(\mathbf{r}) \left(\Delta + k^2\right) \, \frac{e^{\pm ikr}}{r} &= \lim_{\varepsilon \to 0} \int\limits_{K_\varepsilon} d^3 r f(\mathbf{r}) \left(\Delta + k^2\right) \, \frac{e^{\pm ikr}}{r} \\
&= \lim_{\varepsilon \to 0} \int\limits_{K_\varepsilon} d^3 r f(\mathbf{r}) \left(\Delta \frac{1}{r} + \mathcal{O}(\varepsilon^2)\right) \\
&= \lim_{\varepsilon \to 0} \int\limits_{K_\varepsilon} d^3 r f(\mathbf{r}) \left(-4\pi \delta(\mathbf{r})\right) \\
&= -4\pi f(0)
\end{aligned}
$$

With the results from parts (a) and (b) the assertion is proven!

2. It is of course also valid:

$$
\left(\Delta + k^2\right) \, \frac{\exp\left(\pm ik \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} = -4\pi \delta(\mathbf{r} - \mathbf{r}') \ .
$$

We insert the given ansatz into the wave equation:

$$
\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega \, \left(\Delta_r + \frac{\omega^2}{u^2}\right) \, \psi_\omega(\mathbf{r}) \, e^{-i\omega t} = -\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega \, \sigma_\omega(\mathbf{r}) \, e^{-i\omega t} \ .
$$

Fourier-reversal:

$$\left(\Delta_r + \frac{\omega^2}{u^2}\right) \psi_\omega(\mathbf{r}) = \left(\Delta_r + k^2\right) \psi_\omega(\mathbf{r}) = -\sigma_\omega(\mathbf{r}) \; .$$

One recognizes immediately with part 1. the solution of this differential equation:

$$\psi_\omega(\mathbf{r}) = \int d^3r' \, \sigma_\omega(\mathbf{r}') \frac{\exp\left(\pm ik\,|\mathbf{r}-\mathbf{r}'|\right)}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \; .$$

Insertion into the ansatz for $\psi(\mathbf{r}, t)$:

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega \left(\int d^3r' \, \sigma_\omega(\mathbf{r}') \frac{\exp\left(\pm ik\,|\mathbf{r}-\mathbf{r}'|\right)}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \right) e^{-i\omega t} \; .$$

With $k = \omega/u$ it follows eventually:

$$\psi(\mathbf{r}, t) = \int d^3r' \, \frac{\sigma\left(\mathbf{r}', t \mp \frac{1}{u}\,|\mathbf{r}-\mathbf{r}'|\right)}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \; .$$

$\frac{1}{u}\,|\mathbf{r}-\mathbf{r}'|$ is the time which the signal needs to come from $\mathbf{r}'$, where it is created, to $\mathbf{r}$, where it is observed. Causality requires the minus sign. The signal is created at the point $\mathbf{r}'$ at the *'retarded time'*

$$t_{\mathrm{ret}} = t - \frac{1}{u}\left|\mathbf{r}-\mathbf{r}'\right| \; .$$

That is the result (4.438).

Solution 4.5.2 According to (4.457) we have for the magnetic induction of the electric dipole radiation:

$$\mathbf{B}_1(\mathbf{r}, t) = \frac{\mu_0\mu_{\mathrm{r}}}{4\pi} uk^2 \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n}\times\mathbf{p})e{-}i\omega t = \mathbf{B}_{10}(\mathbf{r})e^{i(kr-\omega t)} \; .$$

Thereby $\mathbf{n} = \frac{\mathbf{r}}{r}$ and $k = \frac{\omega}{u}$. The amplitude $\mathbf{B}_{10}(\mathbf{r})$ is complex:

$$\begin{aligned}
\mathbf{B}_{10}(\mathbf{r}) &= \frac{\mu_0\mu_{\mathrm{r}}}{4\pi} uk^2 \left(\frac{1}{r} + \frac{i}{kr^2}\right)(\mathbf{n}\times\mathbf{p}) \\
&= \frac{\mu_0\mu_{\mathrm{r}}}{4\pi} uk^2 (\mathbf{n}\times\mathbf{p}) \sqrt{\frac{1}{r^2} + \frac{1}{k^2r^4}}\, e^{i\psi} \; , \\
\tan(\psi) &= \frac{1/kr^2}{1/r} = \frac{1}{kr} = \frac{u}{\omega r} \; .
\end{aligned}$$

It holds therewith for the magnetic induction:

$$\mathbf{B}_1(\mathbf{r},t) \propto \exp\left(-i\omega\left(t - \frac{r}{u} - \frac{\psi}{\omega}\right)\right) .$$

The phase velocity $\dot{r}$ is the velocity of an observer who is moving such that

$$\varphi(\mathbf{r},t) \equiv \omega\left(t - \frac{r}{u} - \frac{\psi}{\omega}\right) = \text{const} .$$

Differentiating with respect to the time:

$$\begin{aligned}
0 &= 1 - \frac{\dot{r}}{u} - \frac{1}{\omega}\frac{d}{dt}\arctan\frac{u}{\omega r} \\
&= 1 - \frac{\dot{r}}{u} - \frac{1}{\omega}\frac{1}{1+\frac{u^2}{\omega^2 r^2}}\left(-\frac{u}{\omega r^2}\right)\dot{r} , \\
1 &= \dot{r}\left(\frac{1}{u} - \frac{u}{\omega^2 r^2}\frac{1}{1+\frac{u^2}{\omega^2 r^2}}\right) , \\
1 &= \dot{r}\frac{1}{u}\left(1 - \frac{1}{\frac{\omega^2 r^2}{u^2}+1}\right) .
\end{aligned}$$

That yields for the phase velocity

$$\dot{r} = \frac{u}{1 - \frac{1}{1+\left(\frac{\omega r}{u}\right)^2}} > u .$$

In the case of the assumed vacuum: $u = c$:

$$\dot{r} > c .$$

No contradiction to the Special Relativity since no information is transported by the phase velocity!

Solution 4.5.3 Current density:

$$\mathbf{j}(x,t) = I_0 e^{i\omega t}\delta(x)\mathbf{e}_z .$$

Because of

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\mu_r}{4\pi}\int d^3r'\frac{\mathbf{j}(\mathbf{r}',t_{\text{ret}})}{|\mathbf{r}-\mathbf{r}'|}$$

it must be:

$$\mathbf{A} \propto \mathbf{e}_z \; .$$

No charge density, therefore:

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0\epsilon_\mathrm{r}} \int d^3r' \frac{\rho(\mathbf{r}',t_\mathrm{ret})}{|\mathbf{r}-\mathbf{r}'|} = 0 \; .$$

Therewith it follows for the electric field:

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}} = -\dot{\mathbf{A}} \;\longrightarrow\; \mathbf{E} \propto \mathbf{e}_z \; .$$

From symmetry reasons $\mathbf{E}$ cannot exhibit a y- or z-dependence. This is then valid for $\mathbf{A}$, too:

$$\mathbf{A}(\mathbf{r},t) = A(x,t)\mathbf{e}_z \; .$$

The inhomogeneous wave equation simplifies therewith to

$$\frac{\partial^2 A}{\partial x^2} - \frac{1}{u^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0\mu_\mathrm{r} I_0 e^{i\omega t}\delta(x) \; .$$

Solution for $x \neq 0$:

$$A(x,t) = \begin{cases} Be^{i\omega(t-\frac{x}{u})} & \text{for } x > 0 \quad \text{(to the right propagating wave)} \; , \\ Ce^{i\omega(t+\frac{x}{u})} & \text{for } x < 0 \quad \text{(to the left propagating wave)} \; . \end{cases}$$

The symmetry requires $B = C$. For the determination of the constant B the inhomogeneous wave equation is integrated over x from -0^+ to $+0^+$:

$$\left.\frac{\partial A}{\partial x}\right|_{+0^+} - \left.\frac{\partial A}{\partial x}\right|_{-0^+} - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\int\limits_{-0^+}^{+0^+} A dx = -\mu_0\mu_\mathrm{r} I_0 e^{i\omega t} \; .$$

The integral on the left-hand side vanishes and the equation simplifies to

$$-\frac{i\omega}{u}\left(Be^{i\omega t} + Ce^{i\omega t}\right) = -\mu_0\mu_\mathrm{r} I_0 e^{i\omega t} \; .$$

It remains:

$$B = C = \frac{\mu_0\mu_\mathrm{r}}{2i\omega} u I_0$$

The vector potential is therewith determined:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\mu_\mathrm{r}}{2i\omega}uI_0\mathbf{e}_z \begin{cases} e^{i\omega(t-\frac{x}{u})} & \text{for } x > 0\ , \\ e^{i\omega(t+\frac{x}{u})} & \text{for } x < 0\ . \end{cases}$$

Because of

$$\mathbf{E}(\mathbf{r},t) = -\dot{\mathbf{A}}(\mathbf{r},t)$$

and

$$\mathbf{B}(\mathbf{r},t) = \mathrm{curl}\mathbf{A}(\mathbf{r},t) = \mathbf{e}_x\frac{\partial}{\partial y}A_z - \mathbf{e}_y\frac{\partial}{\partial x}A_z = -\mathbf{e}_y\frac{\partial}{\partial x}A_z$$

it follows eventually:

Electric field strength

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{2}\mu_0\mu_\mathrm{r}uI_0\exp\left(i\omega\left(t-\frac{|x|}{u}\right)\right)\mathbf{e}_z\ .$$

Magnetic induction

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{2}\mu_0\mu_\mathrm{r}I_0\mathbf{e}_y \begin{cases} \exp\left(i\omega\left(t-\frac{x}{u}\right)\right) & \text{for } x > 0\ , \\ -\exp\left(i\omega\left(t+\frac{x}{u}\right)\right) & \text{for } x < 0\ . \end{cases}$$

Solution 4.5.4

1. Let us start with the Maxwell equations:

$$\begin{aligned} \mathrm{curl}\mathbf{E}_1 &= -\dot{\mathbf{B}}_1 && \text{(A.7)} \\ \mathrm{curl}\mathbf{H}_1 &= \mathbf{j}_1 + \dot{\mathbf{D}}_1 && \text{(A.8)} \\ \mathrm{curl}\mathbf{E}_2 &= -\dot{\mathbf{B}}_2 && \text{(A.9)} \\ \mathrm{curl}\mathbf{H}_2 &= \mathbf{j}_2 + \dot{\mathbf{D}}_2\ . && \text{(A.10)} \end{aligned}$$

With (A.8) and (A.10) and the harmonic time-dependence one gets at first:

$$\begin{aligned} \mathbf{j}_1\cdot\mathbf{E}_2 - \mathbf{j}_2\cdot\mathbf{E}_1 &= \mathbf{E}_2\cdot\mathrm{curl}\mathbf{H}_1 - \mathbf{E}_2\cdot\dot{\mathbf{D}}_1 - \mathbf{E}_1\cdot\mathrm{curl}\mathbf{H}_2 + \mathbf{E}_1\cdot\dot{\mathbf{D}}_2 \\ &= \mathbf{E}_2\cdot\mathrm{curl}\mathbf{H}_1 - \mathbf{E}_1\cdot\mathrm{curl}\mathbf{H}_2 + i\omega\epsilon_0\epsilon_\mathrm{r}\mathbf{E}_2\cdot\mathbf{E}_1 - i\omega\epsilon_0\epsilon_\mathrm{r}\mathbf{E}_1\cdot\mathbf{E}_2 \\ &= \mathbf{E}_2\cdot\mathrm{curl}\mathbf{H}_1 - \mathbf{E}_1\cdot\mathrm{curl}\mathbf{H}_2 && \text{(A.11)} \end{aligned}$$

It follows from (A.7) and (A.9):

$$\begin{aligned}0 &= \mathbf{H}_2 \cdot \text{curl}\mathbf{E}_1 + \mathbf{H}_2 \cdot \dot{\mathbf{B}}_1 - \mathbf{H}_1 \cdot \text{curl}\mathbf{E}_2 - \mathbf{H}_1 \cdot \dot{\mathbf{B}}_2 \\ &= \mathbf{H}_2 \cdot \text{curl}\mathbf{E}_1 - \mathbf{H}_1 \cdot \text{curl}\mathbf{E}_2 - i\omega\mu_r\mu_0\mathbf{H}_2 \cdot \mathbf{H}_1 + i\omega\mu_0\mu_r\mathbf{H}_1 \cdot \mathbf{H}_2 \\ &= \mathbf{H}_2 \cdot \text{curl}\mathbf{E}_1 - \mathbf{H}_1 \cdot \text{curl}\mathbf{E}_2 \; . \end{aligned} \tag{A.12}$$

We add together (A.11) and (A.12):

$$\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1 = \mathbf{E}_2 \cdot \text{curl}\mathbf{H}_1 - \mathbf{E}_1 \cdot \text{curl}\mathbf{H}_2 + \mathbf{H}_2 \cdot \text{curl}\mathbf{E}_1 - \mathbf{H}_1 \cdot \text{curl}\mathbf{E}_2$$

With

$$\text{div}\,(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \text{curl}\mathbf{a} - \mathbf{a} \cdot \text{curl}\mathbf{b}$$

this proves the assertion:

$$\begin{aligned}\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1 &= (\mathbf{E}_2 \cdot \text{curl}\mathbf{H}_1 - \mathbf{H}_1 \cdot \text{curl}\mathbf{E}_2) - (\mathbf{E}_1 \cdot \text{curl}\mathbf{H}_2 - \mathbf{H}_2 \cdot \text{curl}\mathbf{E}_1) \\ &= \text{div}\,(\mathbf{H}_1 \times \mathbf{E}_2) - \text{div}\,(\mathbf{H}_2 \times \mathbf{E}_1) \; .\end{aligned}$$

2. Dipole approximation, radiation zone ($d \ll \lambda \ll r$) $\rightarrow$ according to (4.461) and (4.462):

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &\approx \frac{\mu_0\mu_r}{4\pi} u k^2 \frac{e^{ikr}}{r} (\mathbf{n} \times \mathbf{p}) \qquad \left(\mathbf{n} = \frac{\mathbf{r}}{r}\right) \; , \\ \mathbf{E}(\mathbf{r}) &\approx u(\mathbf{B}(\mathbf{r}) \times \mathbf{n}) \; .\end{aligned}$$

Dipole moment:

$$\mathbf{p} = \int d^3r'\mathbf{r}'\rho(\mathbf{r}')$$

$$\Rightarrow \;\; \mathbf{E} \times \mathbf{n} = u(\mathbf{B} \times \mathbf{n}) \times \mathbf{n} = -u \left(\mathbf{B}n^2 - \mathbf{n}\underbrace{(\mathbf{B} \cdot \mathbf{n})}_{=0}\right)$$

$$\Rightarrow \;\; \mathbf{B} = \frac{1}{u}\mathbf{n} \times \mathbf{E} \; ,$$

i.e. here:

$$\mathbf{B}_i = \frac{1}{u}\mathbf{n} \times \mathbf{E}_i \; ; \quad i = 1, 2 \; .$$

That is used to further evaluate part 1.:

$$\int_V \operatorname{div}(\mathbf{H}_i \times \mathbf{E}_j) d^3r = \frac{1}{u}\frac{1}{\mu_0\mu_r}\int_V \operatorname{div}((\mathbf{n}\times\mathbf{E}_i)\times\mathbf{E}_j) d^3r$$
$$= \frac{1}{u}\frac{1}{\mu_0\mu_r}\int_V \operatorname{div}\left(-\mathbf{n}(\mathbf{E}_i\cdot\mathbf{E}_j) + \mathbf{E}_i \underbrace{(\mathbf{n}\cdot\mathbf{E}_j)}_{=0}\right) d^3r$$
$$= -\frac{1}{u}\frac{1}{\mu_0\mu_r}\int_{\partial V} (\mathbf{E}_i\cdot\mathbf{E}_j)(\mathbf{n}\cdot d\mathbf{f}) \; ,$$

This leads to:

$$\int_V d^3r(\operatorname{div}(\mathbf{H}_1\times\mathbf{E}_2) - \operatorname{div}(\mathbf{H}_2\times\mathbf{E}_1)) = -\frac{1}{u}\frac{1}{\mu_0\mu_r}\int_{\partial V}(\mathbf{n}\cdot d\mathbf{f})(\mathbf{E}_1\cdot\mathbf{E}_2 - \mathbf{E}_2\cdot\mathbf{E}_1)$$
$$= 0 \; .$$

With the result from part 1. we thus have:

$$\int_V d^3r(\mathbf{j}_1\cdot\mathbf{E}_2 - \mathbf{j}_2\cdot\mathbf{E}_1) = 0$$

$\mathbf{j}_1 \neq 0$ only in V_1; $\mathbf{j}_2 \neq 0$ only in V_2 with $V_1 \cap V_2 = \emptyset$

$$\Rightarrow \int_{V_1} d^3r\mathbf{j}_1\cdot\mathbf{E}_2 = \int_{V_2} d^3r\mathbf{j}_2\cdot\mathbf{E}_1 \; .$$

3. According to the ansatz for the dipole approximation the linear dimensions of the source are small compared to the *'other'* distances. We can therefore assume to a good approximation that the macroscopic fields $\mathbf{E}_1$, $\mathbf{E}_2$ are practically constant over the respective space-regions V_2, V_1:

$$\mathbf{E}_2(\mathbf{R}_1)\cdot\int_{V_1} d^3r\mathbf{j}_1 = \mathbf{E}_1(\mathbf{R}_2)\cdot\int_{V_2} d^3r\mathbf{j}_2 \; . \tag{A.13}$$

$\mathbf{R}_1$, $\mathbf{R}_2$ are, e.g., the *'midpoints'* of the volumes V_1, V_2.

By-calculation:

$$\begin{aligned}
\int_V d^3r j_i &= \sum_j \int_V j_j \delta_{ij} d^3r \quad (i\text{: Cartesian components}) \\
&= \sum_j \int_V j_j \frac{\partial x_i}{\partial x_j} d^3r \\
&= \sum_j \int_V \left(\frac{\partial}{\partial x_j}(j_j x_i) - x_i \frac{\partial}{\partial x_j} j_j \right) d^3r \\
&= \underbrace{\int_V \operatorname{div}(\mathbf{j}x_i) d^3r}_{=\int_{\partial V} d\mathbf{f}\cdot(\mathbf{j}x_i) \overset{V\to\infty}{\longrightarrow} 0} - \int_V x_i \operatorname{div}\mathbf{j} d^3r \,.
\end{aligned}$$

The first term vanishes since $\mathbf{j}$ is unequal zero only in a finite space region. Therewith:

$$\int_V d^3r j_i = -\int_V x_i \operatorname{div}\mathbf{j} d^3r \,.$$

Continuity equation:

$$\begin{aligned}
\int_V d^3r j_i &= \int_V x_i \dot{\rho} d^3r = -i\omega \int_V x_i \rho d^3r \\
&= -i\omega p_i \quad (p_i\text{: } i\text{-th component of the dipole moment}) \\
\Rightarrow \int_V d^3r \mathbf{j} &= -i\omega \mathbf{p} \,.
\end{aligned}$$

That is inserted into (A.13):

$$-i\omega \mathbf{E}_2 \cdot \mathbf{p}_1 = -i\omega \mathbf{E}_1 \cdot \mathbf{p}_2$$

That was to be shown!

Solution 4.5.5

1. Radiation zone: $d \ll \lambda \ll r$

$$\mathbf{B}_S(\mathbf{r}) \approx \frac{\mu_0}{4\pi} c\, k^2 \frac{e^{ikr}}{r} (\mathbf{n}_S \times \mathbf{p}) \;,$$
$$\mathbf{E}_S(\mathbf{r}) \approx c(\mathbf{B}_S(\mathbf{r}) \times \mathbf{n}_S) \;,$$

$(\mathbf{E}_S, \mathbf{B}_S, \mathbf{n}_S)$: orthogonal trihedron, $\mathbf{p} = \int d^3r'\, \mathbf{r}'\, \rho(\mathbf{r}')$: electric dipole moment.
Time-dependencies:

$$\mathbf{B}_S(\mathbf{r}, t) = \mathbf{B}_S(\mathbf{r}) e^{-i\omega t} \;, \ldots$$

2. Incident wave

$$\mathbf{E}_i = \boldsymbol{\eta}_i E_0\, e^{ik\,\mathbf{n}_i \cdot \mathbf{r}} \;,$$
$$\mathbf{B}_i = \frac{1}{c} (\mathbf{n}_i \times \mathbf{E}_i) \;.$$

The scattering cross section has the dimension of a plane:

$$\frac{d\sigma}{d\Omega}(\mathbf{n}_S, \boldsymbol{\eta}_S; \mathbf{n}_i, \boldsymbol{\eta}_i) = \frac{\left(\mathbf{n}_S \cdot \overline{\mathbf{S}_S(\mathbf{n}_S, \boldsymbol{\eta}_S)}\right) r^2 d\Omega}{d\Omega\, \mathbf{n}_i \cdot \overline{\mathbf{S}_i(\mathbf{n}_i, \boldsymbol{\eta}_i)}} \;.$$

We use:

$$\overline{\mathbf{S}} = \frac{1}{2\mu_0} \operatorname{Re}(\mathbf{E} \times \mathbf{B}^*) = \frac{1}{2\mu_0 c} \operatorname{Re}\left(\mathbf{E} \times (\mathbf{n} \times \mathbf{E}^*)\right)$$
$$= \frac{1}{2\mu_0 c}\left[\operatorname{Re}(\mathbf{n}|\mathbf{E}|^2) - \operatorname{Re}(\mathbf{E}^*(\underbrace{\mathbf{E} \cdot \mathbf{n}}_{=0}))\right] = \mathbf{n} \frac{|\mathbf{E}|^2}{2\mu_0 c} \;.$$

Therewith it holds in particular:

$$\overline{\mathbf{S}_S(\mathbf{n}_S, \boldsymbol{\eta}_S)} = \mathbf{n}_S \frac{|\boldsymbol{\eta}_S \cdot \mathbf{E}_S|^2}{2\mu_0 c} \;,$$
$$\overline{\mathbf{S}_i(\mathbf{n}_i, \boldsymbol{\eta}_i)} = \mathbf{n}_i \frac{|\boldsymbol{\eta}_i \cdot \mathbf{E}_i|^2}{2\mu_0 c} = \mathbf{n}_i |\mathbf{E}_0|^2 \frac{1}{2\mu_0 c} \;,$$
$$\boldsymbol{\eta}_S \cdot \mathbf{E}_S = \frac{k^2}{4\pi\, \epsilon_0} \frac{e^{ikr}}{r} \boldsymbol{\eta}_S \cdot \left[(\mathbf{n}_S \times \mathbf{p}) \times \mathbf{n}_S\right]$$
$$= \frac{k^2}{4\pi\, \epsilon_0} \frac{e^{ikr}}{r} \left[(\boldsymbol{\eta}_S \cdot \mathbf{p}) - (\boldsymbol{\eta}_S \cdot \mathbf{n}_S)(\mathbf{p} \cdot \mathbf{n}_S)\right] .$$

In the radiation zone the fields are transversally polarized, therefore:

$$\boldsymbol{\eta}_S \cdot \mathbf{n}_S = 0$$

$$\Longrightarrow \ |\boldsymbol{\eta}_S \cdot \mathbf{E}_S|^2 = \frac{k^4}{16\pi^2\epsilon_0^2} \frac{1}{r^2} |\boldsymbol{\eta}_S \cdot \mathbf{p}|^2$$

$$\Longrightarrow \ \frac{d\sigma}{d\Omega}(\mathbf{n}_S, \boldsymbol{\eta}_S; \mathbf{n}_i, \boldsymbol{\eta}_i) = \frac{1}{16\pi^2\epsilon_0^2} \frac{k^4}{|E_0|^2} (\boldsymbol{\eta}_S \cdot \mathbf{p})^2 .$$

The dependence on $(\mathbf{n}_i, \boldsymbol{\eta}_i)$ is of course 'hidden' implicitly in the induced dipole moment $\mathbf{p}$.

Rayleigh law:

$$\frac{d\sigma}{d\Omega} \sim k^4 \sim \lambda^{-4}$$

(blue sky, afterglow).

3.

$$\begin{aligned} &\lambda \gg d\text{: field } \mathbf{E}_i \text{ inside the sphere almost homogeneous} ,\\ &\tau \sim \lambda\text{: } \textit{quasistationary} . \end{aligned}$$

The result from Exercise 2.4.2 can be adopted:

$$\mathbf{p} = 4\pi\,\epsilon_0 R^3 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \mathbf{E}_i$$

$$\Longrightarrow \ \frac{d\sigma}{d\Omega} = k^4 R^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 (\boldsymbol{\eta}_S \cdot \boldsymbol{\eta}_i)^2 .$$

Polarization:

$$\boldsymbol{\eta}_S \sim [(\mathbf{n}_S \times \mathbf{p}) \times \mathbf{n}_S] \sim [(\mathbf{n}_S \times \boldsymbol{\eta}_i) \times \mathbf{n}_S] = \boldsymbol{\eta}_i - \mathbf{n}_S(\mathbf{n}_S \cdot \boldsymbol{\eta}_i) .$$

The scattered wave is in the plane spanned by $\boldsymbol{\eta}_i$ and $\mathbf{n}_S$ linearly polarized, perpendicular to $\mathbf{n}_S$!

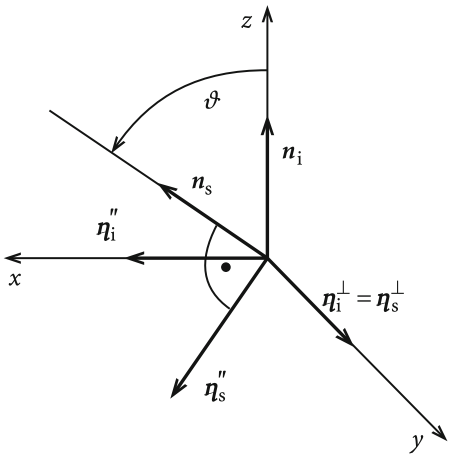
4. Figure A.64 explains itself by the preceding partial results:

$$(\boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_S)_\parallel = \cos\vartheta ,$$

$$(\boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_S)_\perp = 1$$

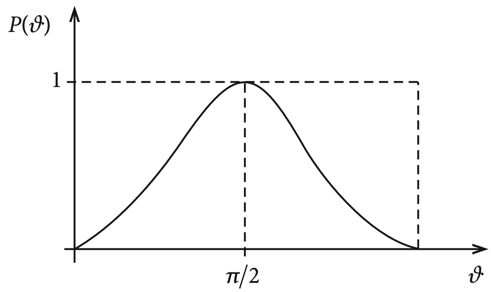
$$\Longrightarrow \ P(\vartheta) = \frac{1 - \cos^2\vartheta}{1 + \cos^2\vartheta} .$$

Fig. A.64



4.

Fig. A.65



P has its maximum at $\vartheta = \pi/2$ (Fig. A.65). In this direction the unpolarized incident radiation has become a completely linearly polarized wave.

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