Advanced Algorithms for Programming Contests
Lecture 10. Lowest common ancestor

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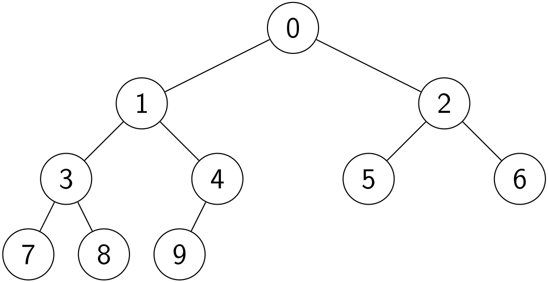
Overview

Lowest common ancestor
LCA binary lifting
LCA interval tree
Union find
LCA union find (Tarjan)

Lowest common ancestor

- Lowest common ancestor of two nodes v and w in a (rooted) tree is the node furthest away from the root having both v and w as a decendent.



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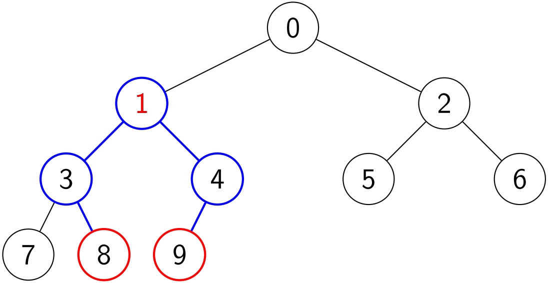

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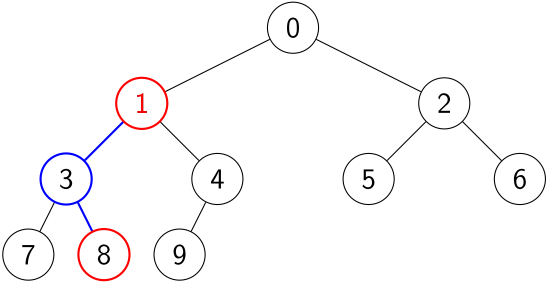

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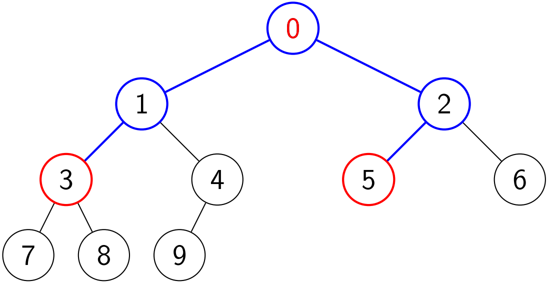


LCA binary lifting

Problem

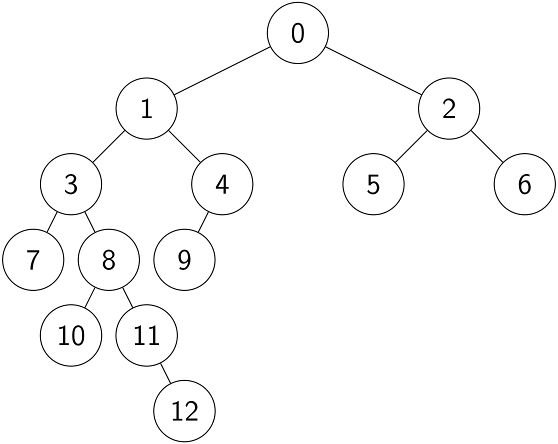
- Given a rooted tree, find LCAs of pairs of nodes.

Naive algorithm

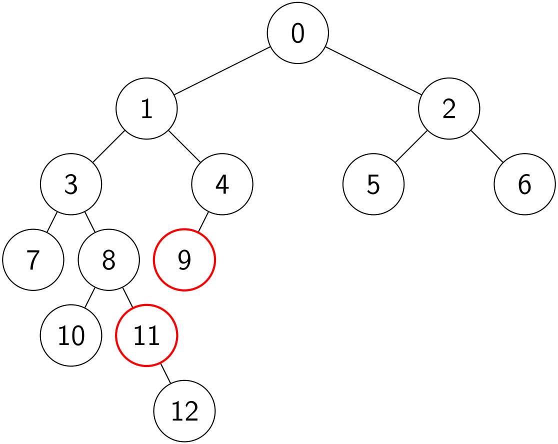
- Given v, w to find LCA.
- Check if v is ancestor of w
  - Yes $\rightarrow$ return v.
  - No $\rightarrow$ find LCA of ancestor of v and w.

Complexity: $O(N)$

LCA binary lifting



LCA binary lifting



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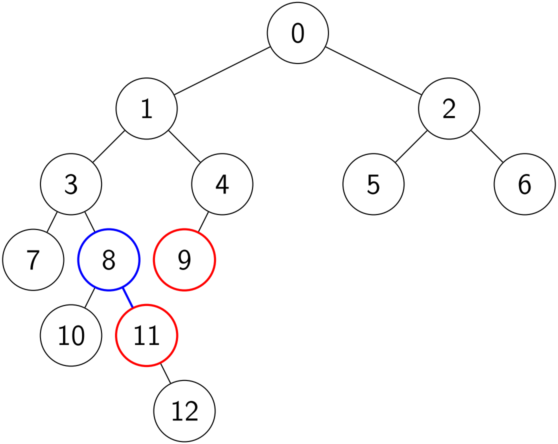

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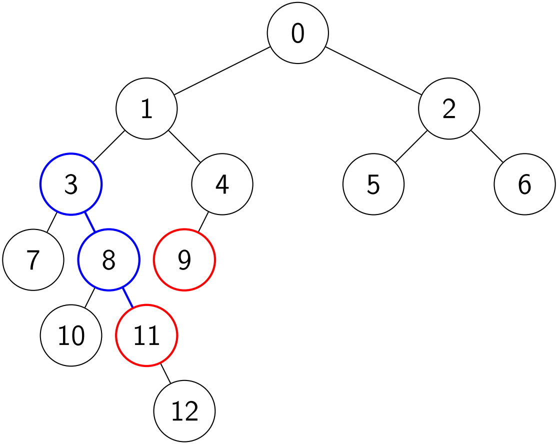
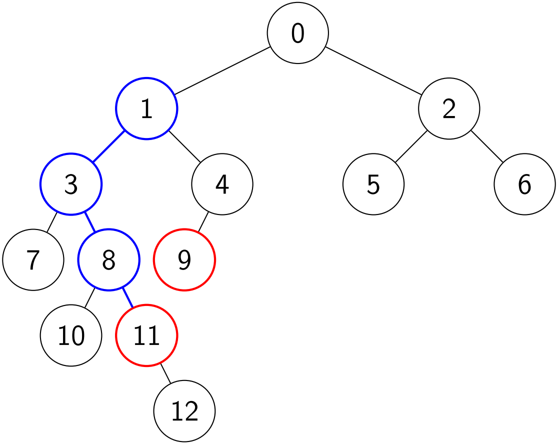
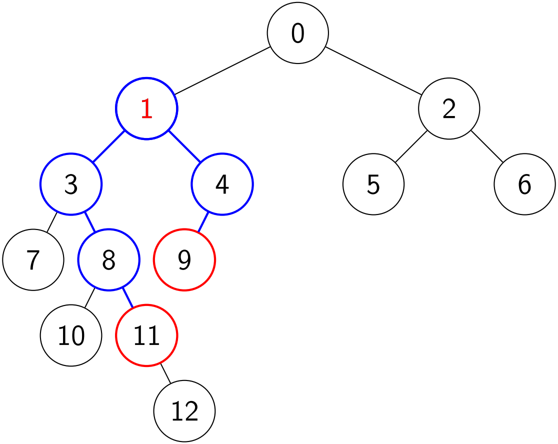


LCA binary lifting



LCA binary lifting



LCA binary lifting

Idea

- Precalculate 2^k-th ancestor for any node.
- For every query find LCA by going up these ancestor links.

Algorithm

- $up[l][i] = (2^l)$-th ancestor of i (or root if it isn't that deep)
- Recursive: $up[l][i] = up[l-1][up[l-1][v]]$.
- Find $lca(a, b)$:
  - If a is ancestor of b or b of a $\Rightarrow$ trivial.
  - Find $\max\{l \mid up[l][a] \text{ not anc. of } b\}$.
  - Find $lca(up[l][a], b)$ recursively.

Complexity

- Precalc, Memory $O(N \log(N))$
- Query $O(\log(N))$

LCA binary lifting

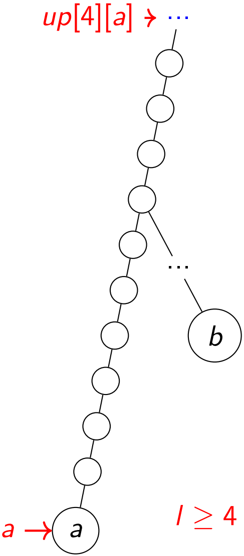
```
const int MAXN, MAXLOG; // 1 << (MAXLOG-1) >= MAXN
int n, root;
vector<int> edges[MAXN];
int tin[MAXN], tout[MAXN], timer = 1;
int up[MAXLOG][MAXN];

void dfs(int v = root, int p = root) {
  tin[v] = timer++;
  up[0][v] = p;
  for (int l = 1; l < MAXLOG; l++)
    up[l][v] = up[l-1][up[l-1][v]];
  for (int w : edges[v]) {
    if (w != p)
      dfs(w, v);
  }
  tout[v] = timer++;
}
```

LCA binary lifting

```
bool upper(int a, int b) {
  return tin[a] <= tin[b] && tout[a] >= tout[b];
}

int lca(int a, int b) {
  if (upper(a, b)) return a;
  if (upper(b, a)) return b;
  for (int l = MAXLOG - 1; l >= 0; l--)
    if (!upper(up[l][a], b))
      a = up[l][a];
  return up[0][a];
}
```



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}
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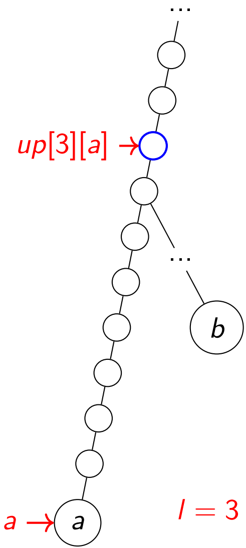

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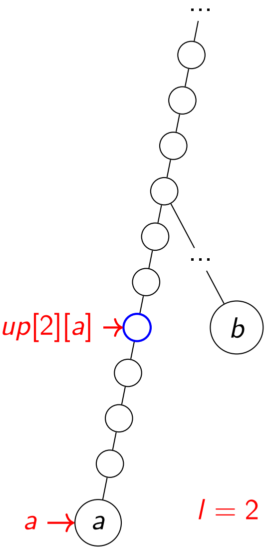

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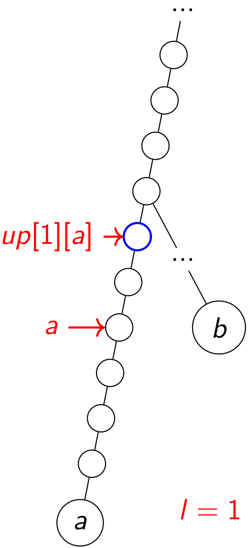

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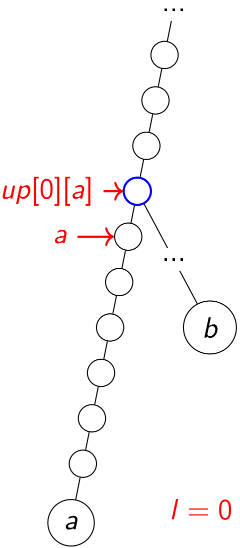

LCA interval tree

Problem

- Calculate LCA efficiently in linear memory.

Idea

- The $lca(a, b)$ is the vertex closest to root of those visited by dfs between first entering a and first entering b.
- Use interval tree to find it in $O(\log(N))$.

Algorithm

- Run dfs for visiting order and *depth* calculation.
- Build interval tree for $\text{minarg}_v\, depth(v)$ on visit vector.
- Answer queries $lca(a, b) = \text{minarg}([firstVisit[a], firstVisit[b]])$.

LCA interval tree

```
int n, root;
vector<int> edges[MAXN];

vector<int> visit;
int firstVisit[MAXN];
int depth[MAXN];

void lca_dfs(int v = root, int d = 1) {
  firstVisit[v] = visit.size();
  visit.push_back(v);
  depth[v] = d;
  for (int w : edges[v]) {
    if (depth[w] != 0) continue;
    lca_dfs(w, d+1);
    visit.push_back(v);
  }
}
```

LCA interval tree

```
int tree[8*MAXN];
void lca_build_tree(int v = 1, int tl = 0, int tr = visit.size()-1) {
  if (tl == tr)
    tree[v] = visit[tl];
  else {
    int tm = (tl + tr) / 2;
    lca_build_tree(2*v, tl, tm);
    lca_build_tree(2*v+1, tm+1, tr);
    if (depth[tree[2*v]] < depth[tree[2*v+1]])
      tree[v] = tree[2*v];
    else
      tree[v] = tree[2*v+1];
  }
}

void lca_prepare() {
  lca_dfs();
  lca_build_tree();
}
```

LCA interval tree

```
int lca_get_tree(int l, int r, int v=1, int tl=0, int tr=visit.size()-1) {
  if (l == tl && r == tr)
    return tree[v];
  int tm = (tl + tr) / 2;
  if (r <= tm)
    return lca_get_tree(l, r, 2*v, tl, tm);
  if (l > tm)
    return lca_get_tree(l, r, 2*v+1, tm+1, tr);
  int lmin = lca_get_tree(l, tm, 2*v, tl, tm);
  int rmin = lca_get_tree(tm+1, r, 2*v+1, tm+1, tr);
  return depth[lmin] < depth[rmin] ? lmin : rmin;
}

int lca(int a, int b) {
  int l = min(firstVisit[a], firstVisit[b]);
  int r = max(firstVisit[a], firstVisit[b]);
  return lca_get_tree(l, r);
}
```

Union find

Problem

- Starting with n different elements in n sets $\{i\}$.
- Perform operations:
  - $union(a, b)$ - merge the sets containing a and b.
  - $find(a)$ - return a unique representative of the set containing a.

Idea

- Keep a tree for every set.
- Implement $find(a)$ as the root of tree containing a.
- Implement $union(a, b)$ as setting the parent of $find(a)$ to $find(b)$ if they are not the same.

Naive approach

```
void make_set(int v) {
  parent[v] = v;
}

int find_set(int v) {
  if (v == parent[v])
    return v;
  return find_set(parent[v]);
}

void union_sets(int a, int b) {
  a = find_set(a);
  b = find_set(b);
  if (a != b)
    parent[b] = a;
}
```

Path compression

Idea

- Use lazy dynamics on *find* to optimize.

```
int find_set(int v) {
  if (v == parent[v])
    return v;
  return parent[v] = find_set(parent[v]);
}
```

Complexity on average $O(\log(n))$

Union by rank

Idea

- Consider *rank* of trees to always keep them balanced.

```
void make_set(int v) {
  parent[v] = v;
  rank[v] = 0;
}

void union_sets(int a, int b) {
  a = find_set(a);
  b = find_set(b);
  if (a != b) {
    if (rank[a] < rank[b])
      swap(a, b);
    parent[b] = a;
    if (rank[a] == rank[b])
      ++rank[a];
  }
}
```

Complexity on average $O(\log(n))$

Final realisation

```
void make_set(int v) {
  parent[v] = v; rank[v] = 0;
}

int find_set(int v) {
  return (v == parent[v]) ? v : parent[v] = find_set (parent[v]);
}

void union_sets(int a, int b) {
  a = find_set(a); b = find_set(b);
  if (a != b) {
    if (rank[a] < rank[b])
      swap(a, b);
    parent[b] = a;
    if (rank[a] == rank[b])
      ++rank[a];
  }
}
```

Complexity on average $O(\log^*(n))$

LCA union find (Tarjan)

Problem

- Find LCAs of a given set of pairs of vertices.

Idea

- The LCA of two vertices a, b is the lowest vertex, that has a and b in different subtrees.
- Use dfs, merge completed subtrees with their direct ancestor using union-find.
- Remember highest vertex of merged subtree.
- LCA of current vertex v and vertex w visited before is ancestor of completed subtree containing w.

LCA union find (Tarjan)

```cpp
const int MAXN = 100000;
vector<int> edges[MAXN], requests[MAXN];
int dsu[MAXN], ancestor[MAXN];
bool used[MAXN];

int find_set(int v) {
  return (v == dsu[v]) ? v : (dsu[v] = find_set(dsu[v]));
}

void union_sets(int a, int b, int new_ancestor) {
  a = find_set(a), b = find_set(b);
  if (rand() & 1)
    swap(a, b);
  dsu[a] = b;
  ancestor[b] = new_ancestor;
}
```

LCA union find (Tarjan)

```
void dfs (int v) {
  dsu[v] = v;
  ancestor[v] = v;
  used[v] = true;
  for (int w : edges[v])
    if (!used[w]) {
      dfs(w);
      union_sets(v, w, v);
    }
  for (int w : requests[v])
    if (used[w])
      printf("%d %d -> %d\n", v, w, ancestor[find_set(w)]);
}
```

Do your homework!