Notes on SfS - Orthographic Projection Model by Rouy and Tourin

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1 Viscosity Solutions

The classic orthographic Shape from Shading (SfS) model was developed by Rouy and Tourin, who presented the viscosity solution approach to solving the SfS Problem. The Model is described by a non-linear first order PDE, known as the Eikonal Equation (A Hamilton Jacobi Equation).

$$|\nabla u| = n(x) \tag{1}$$

where $n(x)$ is the source term. No solution exists in C^1, the proof of which is shown below.

Example 1 *Consider the following Boundary Value Problem*

$$\begin{aligned} |\nabla u| &= 1 \quad in\ \ \Omega = (0,1) & (2) \\ u &= 0 \quad on\ \ \partial\Omega = \{0,1\} & (3) \end{aligned}$$

Suppose if some $u \in C^1$, and as $u(0) = u(1) = 0$, by Rolle's Theorem $\exists\ \ x_0 \in (0,1)$ such that $u'(x_0) = 0$. This exactly contradicts (2), that $|u'| = 1$ on the entire domain. So $u \notin C^1$.

This is exactly why we are required to find the solutions which are in some sense, "weak" solutions to the problem [3]. For example, $u^+(x) = \frac{1}{2} - \left|\frac{1}{2} - x\right|$ is a solution to the problem discussed above. We see that $u \in C^0$ but $u \notin C^1$. But on the other hand, we see that this is not the only solution i.e., $u^-(x) = \left|\frac{1}{2} - x\right| - \frac{1}{2}$ is also a solution to the problem. So, the question of uniqueness arises.

We are now in need of a theory which addresses the issue of existence and uniqueness. P.L.Lions and M.Crandall, came up with a theory known as **viscosity solutions** [1] that describes a way to obtain a unique solution to the class of PDEs known as **Hamilton**

Jacobi Equations. This concept was obtained from the vanishing viscosity method, which involves adding an artificial viscosity term ϵu_{xx} to the PDE and letting $\epsilon \to 0$. With this Crandall and Lions were able to conclude that for problem (2) - (3), the unique solution is in fact

$$u(x) = u^+(x) = \frac{1}{2} - \left|\frac{1}{2} - x\right|. \tag{4}$$

To see why this is the case we define the following.

Definition 1 *Viscosity Sub- and Super- solution*

Consider the following Hamilton-Jacobi Equation

$$H(x, \nabla u) = 0. \tag{5}$$

1. *$u \in C^0$ is a viscosity sub-solution of the HJE; if for any $\phi \in C^1$, if $u - \phi$ attains the maximum at x_0, then*
$$H(x_0, \nabla \phi) \leq 0 \tag{6}$$

2. *$u \in C^0$ is a viscosity super-solution of the HJE; if for any $\phi \in C^1$, if $u - \phi$ attains the minimum at x_1, then*
$$H(x_1, \nabla \phi) \geq 0 \tag{7}$$

Any $u \in C^0$ is a viscosity solution if u is both a viscosity sub- and super- solution.

Now consider the Eikonal equation in 1D,

$$\begin{aligned} |u'(x)| &= 1 \quad x \in (-1, 1) & (8) \\ u &= 0 \quad x \in \{-1, 1\} & (9) \end{aligned}$$

The Hamiltoinan $H(x, p) = |p| - 1$ where $p = \nabla u$, is a convex function in p. We now show that the solution $u^+(x) = 1 - |x|$ is the viscosity solution to the problem.

Proof: Let $u - \phi$ attain maximum at x_0. In this case $x_0 = 0$.

This means,

$$\begin{aligned} u(x_0) - \phi(x_0) &\geq u(x) - \phi(x) \quad \forall x \in (-1, 1) & (10) \\ \implies \phi(x) - \phi(x_0) &\geq u(x) - u(x_0) & (11) \end{aligned}$$

If $x > x_0 = 0$, we have from (11)

$$\begin{aligned}
\frac{\phi(x)-\phi(x_0)}{x-x_0} &\geq \frac{u(x)-u(x_0)}{x-x_0} && (12)\\
\implies \lim_{x\to x_0}\frac{\phi(x)-\phi(x_0)}{x-x_0} &\geq \lim_{x\to x_0}\frac{u(x)-u(x_0)}{x-x_0} && (13)\\
\implies \lim_{x\to x_0}\frac{\phi(x)-\phi(x_0)}{x-x_0} &\geq \lim_{x\to x_0}\frac{1-x-(1)}{x} && (14)\\
\phi'(x) &\geq -1 && (15)
\end{aligned}$$

If $x < x_0 = 0$, we follow the same procedure to get,

$$\phi'(x) \leq 1 \tag{16}$$

Thus from (15) & (16), we get $|\phi'(x)| \leq 1$, which shows that u^+ is a sub-solution to the Eikonal Equation. One can verify that u^+ satisfies the super-solution condition as well and thus is a viscosity solution to the problem, which completes the proof.

It is easy to see that, by using the same steps ,that $u^-(x) = |x| - 1$ is **not a viscosity solution** if we choose the Hamiltonian $H(x,p) = |p| - 1$, but **is indeed a viscosity solution** for the Hamiltonian $H(x,p) = 1 - |p|$. Thus the viscosity solution is dependent on the nature of the Hamiltonian we choose, i.e., (concave / convex).

Let us consider the special case where the Eikonal Equation is,

$$|\nabla u| = 0 \tag{17}$$

This particular PDE presents us with a huge problem of **uniqueness**. This is evident from the sub-solution definition, that

$$\begin{aligned}
H(x, \nabla\phi) &\leq 0 && (18)\\
\implies |\nabla\phi| &\leq 0 && (19)
\end{aligned}$$

which cannot be satisfied for all cases, as $|\nabla\phi| \geq 0$. This particular case is a problem when we consider the numerical solution of the Shape from Shading Model proposed by Rouy and Tourin.

2 Shape from Shading

The model propsed by Rouy and Tourin[4] for an orthographic shape from shading model, with the light source vertically above the object, is given by

$$|\nabla u(x)| = \sqrt{\frac{1}{I(x)^2} - 1}, \quad x \in \Omega \subset \mathbb{R}^2 \tag{20}$$

Rouy and Tourin proposed to solve the PDE by modelling it as a transient problem and then solving till the steady state is reached, i.e.,

$$u_t + |\nabla u| = \sqrt{\frac{1}{I^2} - 1} \tag{21}$$

Sethian[5] proposed a variant of the Einguist Osher scheme to solve the Eikonal Equation. Instead, one can use any upwind scheme (like Godunov for ex.) to solve the transient equation.

Now, if $0 < I(x) < 1$, we solve the PDE by normally applying the boundary conditions and the scheme converges to the unique viscosity solution. In some cases, if $I(x) = 1$ locally, we have a problem of uniqueness, which was discussed above. Presenting an illustration with,

$$I(x,y) = \frac{1}{\sqrt{1 + (16y(1-y)(1-2x))^2 + (16x(1-x)(1-2y))^2}} \tag{22}$$

It is easy to see that **one of the solution** (the other possible one is obviously the negative of this curve) is given by $u(x,y) = 16x(1-x)y(1-y)$. This exact solution is shown in the Figure 1,

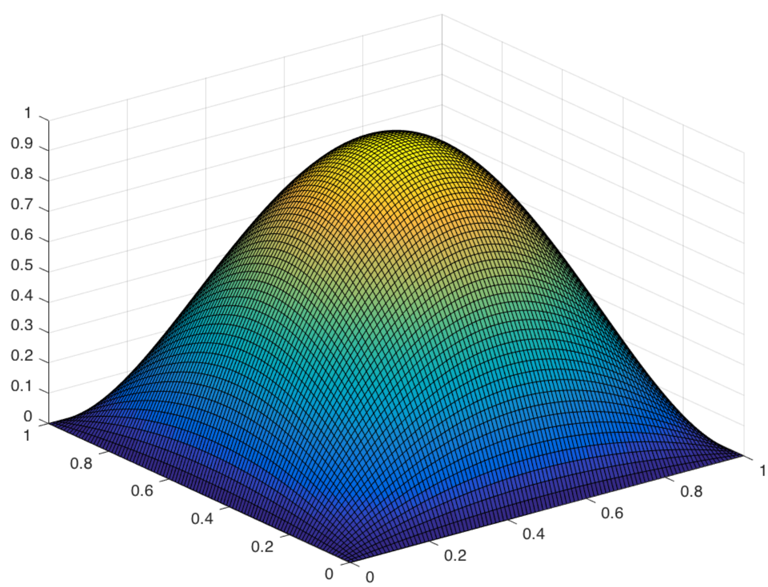
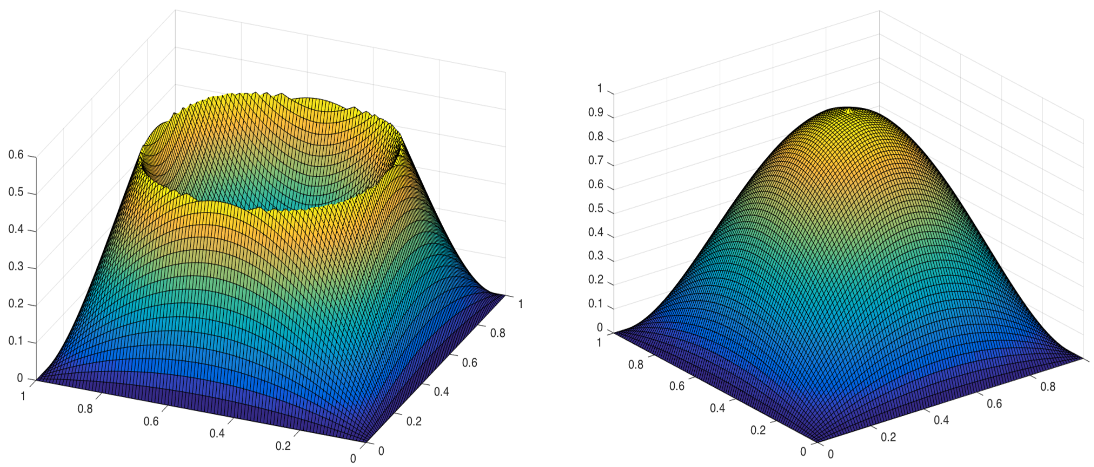


Figure 1: Exact Solution

Now, $I(x = 0.5, y = 0.5) = 1$ and $u(x = 0.5, y = 0.5) = 1$. Not accounting for the value of $u(x, y)$ at places where $I(x, y) = 1$, can produce devastating effects as shown in Figure 2a. The data prescribed by the initial data is just inaccurately carried over by the scheme, if not handled properly. The correct result can be obtained by **manually setting** the value



(a) Wrong Result

(b) Physically Relevant Result

of $u(x, y)$ at places where $I(x_i, y_j) = I_{ij} = 1$. This is done by constructing an Index Set Q where,

$$Q_{ij} = \begin{Bmatrix} 1 & I_{ij} \neq 1 \\ 0 & I_{ij} = 1 \end{Bmatrix} \tag{23}$$

So to get the physically relevant solution, we set $u_{ij} = 1$ **where** $Q_{ij} = 0$**.** Shown in Figure 2b.

Prados and Faugeras[2], came up with a Generic Hamiltonian that combines both Orthographic and Perspective Projection Models. The Hamiltonian that they came up with, satisfies convexity, coercivity and regularity. Subsolution condition is satisfied as long as $0 < I(x) < 1$, and the uniqueness is lost as soon as $I(x) = 1$. This is a major challenge is that we are unable to correctly reconstruct the image, if we do not know what happens at places where $I(x) = 1$ - called the **singularity points**.

3 Results

An attempt to reconstruct the follwing classical **text vase** from the intensity image shown in Figure 3, was made using the Orthographic projection model. The result is shown in Figure 4. Homogeneous Neumann boundary conditions are employed on the boundaries.

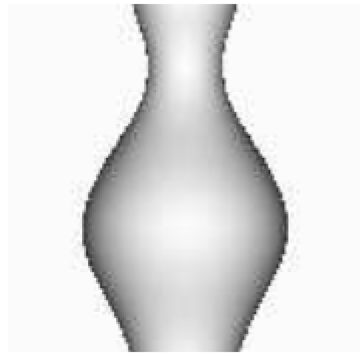


Figure 3: Test Vase

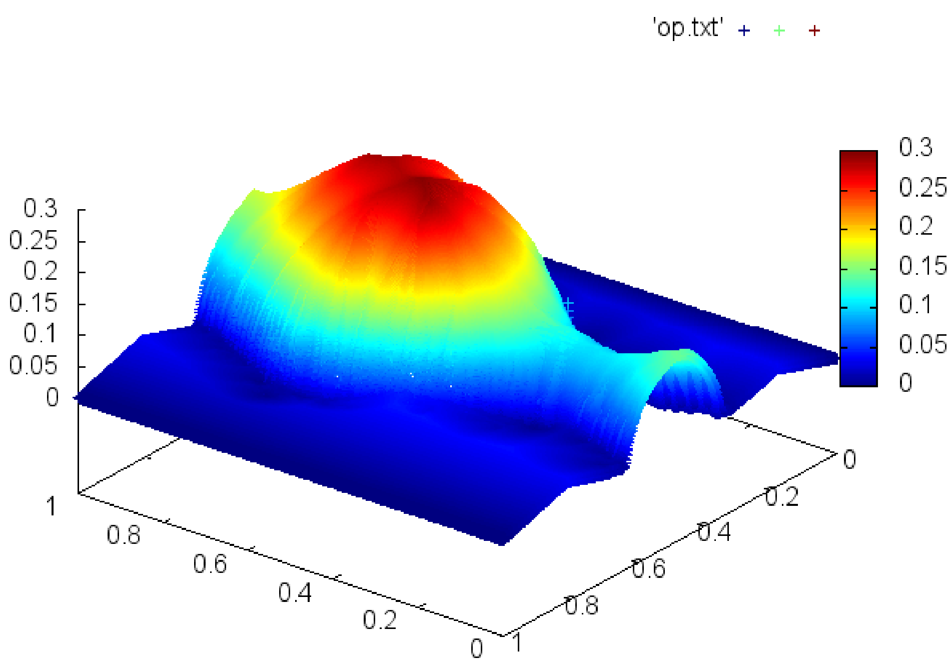


Figure 4: Reconstruction using Orthographic Projection

References

[1] Crandall, M. G., and Lions, P.-L. Viscosity solutions of hamilton-jacobi equations. *Trans. Amer. Math. Soc. 277 (1983), 1-42* (1983).

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