Numerical solution of Perspective Shape from Shading

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1 Mathematical Model

The shape from shading problems are typically described by Hamilton Jacobi Equations. Prados et.al established a mathematical model and a numerical scheme to solve the Shape from Shading problem that converges to the unique viscosity solution. Refer[1].

The Hamilton Jacobi Equation that governs the Shape from Shading problem, proposed by Prados[1] is given by,

$$-e^{-2v} + J(\mathbf{x})\sqrt{f^2|\nabla v|^2 + (\mathbf{x}.\nabla v)^2 + Q(\mathbf{x})^2} = 0 \tag{1}$$

where f is the focal length, $v = \ln u$

$$J(\mathbf{x}) = \frac{I(\mathbf{x})f^2}{Q(\mathbf{x})} \tag{2}$$

$$Q(\mathbf{x}) = \sqrt{\frac{f^2}{\mathbf{x}^2 + f^2}} \tag{3}$$

$I(\mathbf{x})$ is the intensity image to be given as input, $\mathbf{x} \in \mathbb{R}^2$, and $u : \mathbb{R}^2 \to \mathbb{R}$ is the depth of the reconstructed image.

2 Numerical Scheme

We used a Godunov Type upwinded Numerical scheme to solve the PDE with Dirichlet Boundary conditions. The description of the scheme is given below.

We first add a transient term u_t to the PDE given in 1

$$v_t + \left(-e^{-2v} + J(\mathbf{x})\sqrt{f^2|\nabla v|^2 + (\mathbf{x}.\nabla v)^2 + Q(\mathbf{x})^2}\right) = 0 \tag{4}$$

and solve for $v(x,t)$ till we reach a steady state $\|U^{n+1} - U^n\| < \epsilon$. We replace the derivatives in $|\nabla v|^2$ term by the respective upwinded finite difference formulae as shown below.

$$\frac{\partial u}{\partial x} \approx D^x_{upwind} = \max\left\{\left|\max\left\{\frac{u_{i,j} - u_{i-1,j}}{h}, 0\right\}\right|, \left|\min\left\{\frac{u_{i+1,j} - u_{i,j}}{h}, 0\right\}\right|\right\} \tag{5}$$

$$\frac{\partial u}{\partial y} \approx D^y_{upwind} = \max\left\{\left|\max\left\{\frac{u_{i,j} - u_{i,j-1}}{h}, 0\right\}\right|, \left|\min\left\{\frac{u_{i,j+1} - u_{i,j}}{h}, 0\right\}\right|\right\} \tag{6}$$

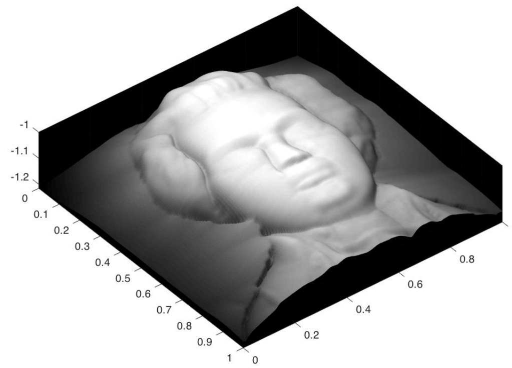
For the derivatives in $(\mathbf{x}.\nabla v)^2$ term, we use the same finite difference that was chosen in the previous step. Let us call that D^x and D^y. The final explicit numerical scheme that we use is given by,

$$v_{i,j}^{n+1} = v_{i,j}^{n} - \Delta t \left(-e^{-2v_{i,j}} + J(x_i, y_j) \sqrt{f^2 \left((D^x_{upwind})^2 + (D^y_{upwind})^2 \right) + (x_i D^x + y_j D^y)^2 + Q(x_i, y_j)^2} \right) \tag{7}$$

3 Results

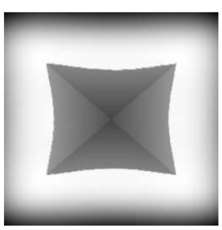
We ran our algorithm for a few standard test cases and the results are presented below.

(a) The input file for Mozart

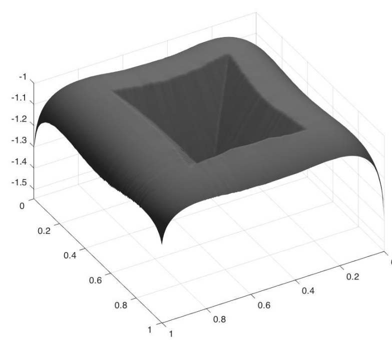


(b) Reconstructed Image

Figure 1: Mozart Test. Courtesy [1]



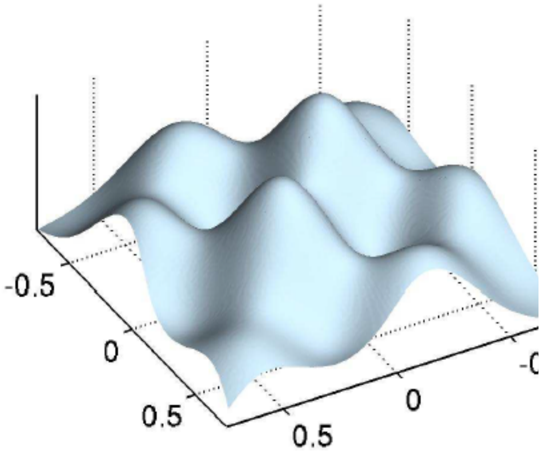
(a) The input file for Trough



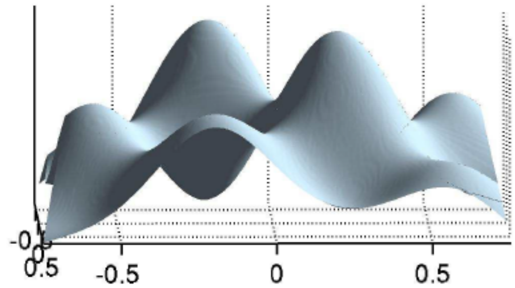
(b) Reconstructed Image

Figure 2: Trough Test. Courtesy [1]

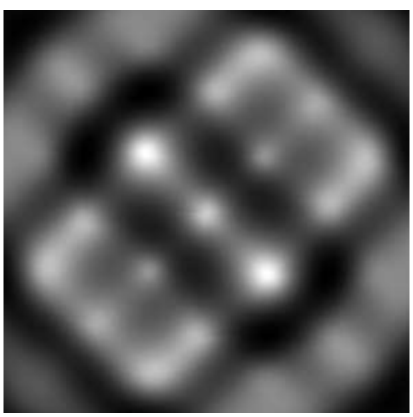
However, for some particular examples, we could not accurately capture the Troughs. We instead get Peaks in place of Troughs, despite including the brighting attenuation term $\frac{1}{r^2}$. See Figure below.



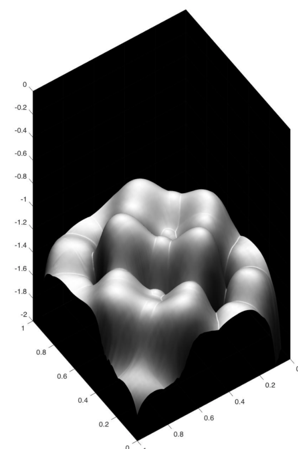
(a) Ground Truth



(b) Ground Truth



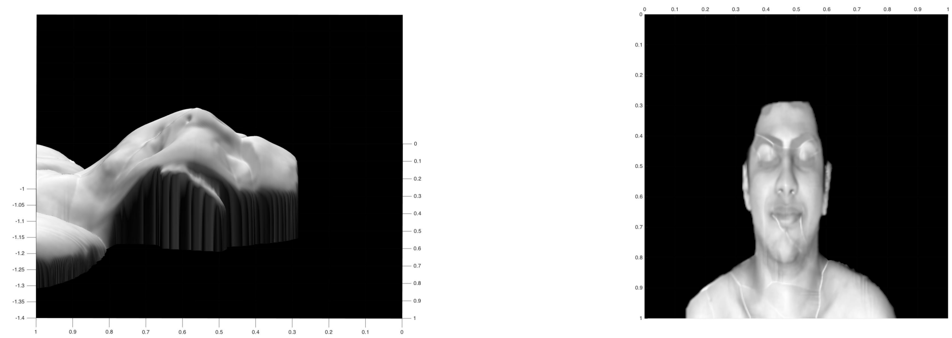
(c) Input Image



(d) Wrongly Reconstructed Image

Figure 3: Wrong Reconstruction. “Ground Truth” Courtesy [1]

For real life images, the reconstruction yields distorted and elongated 3D shapes, because of incorrectly procured input image intensities. The figure below shows the condition.



(a) Incorrect Reconstruction

(b) The top view of the reconstruction

Figure 4: Incorrect Reconstruction

References

[1] Emmanuel Prados, Olivier Faugeras, F. C. Shape from shading: A well posed problem? *RR-5297 INRIA* (2004).