1 MIG criterion

Descent proofs. XXX

Proposition 1. *Consider systems with state x and control u and an objective given by (??). Then, the user input given by (??) is a descent direction for all $t \in [t_o, t_f]$ and will decrease the cost if the MIG is negative for that $t \in [t_o, t_f]$.*

Proof. Using a first-order Taylor expansion, we write

$$J(v(t) + w(t)) \approx J(v(t)) + \frac{\partial J}{\partial u(t)}\Big|_{v(t)} \cdot w(t). \tag{1}$$

For objectives of the form in (??), we use the Gâteaux derivative to calculate the gradient of the cost with respect to the control

$$\frac{d}{d\epsilon} J(v(t) + \epsilon w(t))\Big|_{\epsilon=0} = \int_{t_o}^{t_f} \rho(t)^T B(t) \cdot w(t) \,\mathrm{d}t. \tag{2}$$

From (1) and (2), the first-order change in cost can be approximated with

$$\Delta J \approx \int_{t_o}^{t_f} \rho^T(t) B(t) \cdot w(t) \mathrm{d}t.$$

Equivalently, using (??), we can write

$$\Delta J \approx \int_{t_o}^{t_f} \frac{dJ}{d\lambda_+} \mathrm{d}t,$$

We examine the case of the equality, such that

$$\begin{aligned} \Delta J = 0 &\Leftrightarrow \rho^T(t) B(t) = 0 \ \forall \ t \in [t_o, t_f] \\ &\Leftrightarrow \frac{dJ_1}{d\lambda_+} = 0 \ \forall \ t \in [t_o, t_f]. \end{aligned}$$

If there exists $t \in [t_o, t_f]$ for which $dJ/d\lambda_+ < 0$, then $\Delta J < 0$.

The change in cost, to first order, will be zero if and only if $\rho^T(t)B(t) = 0$ for all $t \in [t_o, t_f]$, which is the condition for a minimum according to Pontryagin's Maximum Principle (for objectives of the form in (??)).

□