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(iii) Introduce 2 glass slabs after the 1^{st} BS along both dimensions. The operator for 2 glass slabs is given by

$$A(\psi_x, \psi_y) = \begin{bmatrix} e^{i\psi_x} & 0 \\ 0 & e^{i\psi_y} \end{bmatrix}$$

Calculate the probability of detection at D_1 & D_2 if $\psi_x = \frac{\pi}{2}$ & $\psi_y = \pi$

$$e^{i\psi_x} = 0 + i = i \qquad e^{i\psi_y} = -1$$

$$A = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$$

$$BS|x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + i x_2 \\ i x_1 + x_2 \end{bmatrix}$$

$$A \cdot BS|x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 + i x_2 \\ i x_1 + x_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -x_2 + i x_1 \\ -x_2 - i x_1 \end{bmatrix}$$

$$M \cdot A \cdot BS|x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -x_2 + i x_1 \\ -x_2 - i x_1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -x_2 - i x_1 \\ -x_2 + i x_1 \end{bmatrix}$$

$$|f\rangle = BS \cdot \pi \otimes BS\,|x\rangle = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} -x_2 - i x_1 \\ -x_2 + i x_1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -x_2 - i x_1 - i x_2 - x_1 \\ -i x_2 + x_1 - x_2 + i x_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -x_2(1+i) \ldots \\ -x_2(i+1) + \ldots \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (1+i)(-x_2 - x_1) \\ (1+i)(-x_2 + x_1) \end{bmatrix}$$

if $|x\rangle = |0\rangle$

$$= \frac{1}{2}\begin{bmatrix} (1+i)(-1) \\ (1+i)(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1-i \\ 1+i \end{bmatrix}$$

$$= \frac{1}{2}(1+i)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}(1+i)\left[\,|1\rangle - |0\rangle\,\right]$$

$$P(|f\rangle = |0\rangle) = |\langle 0|f\rangle|^2$$

$$= \left|\frac{1}{2}(1+i)\,\langle 0|1\rangle - \langle 0|0\rangle\right|^2$$

$$= \left|\frac{1+i}{2}(0-1)\right|^2 = \left[-\frac{1}{2}\sqrt{1^2+1^2}\right]^2 = \left[-\frac{1}{\sqrt{2}}\right]^2$$

$$= \frac{1}{2}$$

Both detectors detects the photons with equal probabilities.

(iv) given that $\psi_x - \psi_y = \pi$, calculate the probability of detection at D_1 & D_2. Also what happens when their relative phase different is $\frac{\pi}{4}$. $[\psi_x - \psi_y = \frac{\pi}{4}]$

$\psi_x = \frac{\pi}{2}$ $\psi_y = \frac{-\pi}{2}$ [assuming any value]

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$BS \cdot |0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$A \cdot BS|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$M \cdot A \cdot BS|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|f\rangle = BS \cdot M \cdot A \cdot BS|0\rangle = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 2i \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i|1\rangle$$

$$P(|f\rangle = |0\rangle) = |\langle 0|f\rangle|^2 = |i\langle 0|1\rangle|^2$$

$$= |i \times 0|^2 = 0$$ $P(|f\rangle = |1\rangle) = 1$

$\therefore$ Detected only at D_2.

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for $\psi_x - \psi_y = \frac{\pi}{4}$, $\psi_x = \frac{\pi}{4}$, $\psi_y = 0$

$$A = \begin{bmatrix} 1/\sqrt{2} + i\, 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1/\sqrt{2}(1+i) & 0 \\ 0 & 1 \end{bmatrix}$$

$$BS \cdot |0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$A \cdot BS|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2}(1+i) & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) \\ i \end{bmatrix}$$

$$M \cdot A \cdot BS|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1/\sqrt{2}(1+i) \end{bmatrix}$$

$$|f\rangle = BS \cdot M \cdot A \cdot BS|0\rangle = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1/\sqrt{2}(1+i) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} i + \frac{i}{\sqrt{2}}(1+i) \\ -1 + \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} i\left(1 + \frac{1}{\sqrt{2}}(1+i)\right) \\ \left(\frac{1}{\sqrt{2}}(1+i) - 1\right) \end{bmatrix}$$

$$P(|f\rangle = |0\rangle) = |\langle 0|f\rangle|^2$$

$$= \left| \frac{1}{2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i(1 + \frac{1}{\sqrt{2}}(1+i)) \\ (\frac{1}{\sqrt{2}}(1+i) - 1) \end{bmatrix} \right|^2$$

$$= \left| \frac{1}{2} \left[i\left(1 + \frac{1}{\sqrt{2}}(1+i)\right) \right] \right|^2$$

$$\left|\frac{1}{2}\left(i + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right|^2$$

$$\left|\frac{1}{2}\left(\frac{\sqrt{2}i + i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right|^2$$

$$= \left[\frac{1}{2\sqrt{2}} \left| i(\sqrt{2}+1) - 1\right|\right]^2$$

$$= \left[\frac{\sqrt{(\sqrt{2}+1)^2 + (-1)^2}}{2\sqrt{2}}\right]^2 = \frac{2+1+2\sqrt{2}+1}{8} = \frac{4+2\sqrt{2}}{8}$$

$$= \frac{2+\sqrt{2}}{4} \simeq 0.854$$

Q) The first qubit is in 1 state and the second qubit is in a state of superposition

$|\alpha_1\rangle = |1\rangle$ $\quad |\alpha_2\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

a) Calculate the total state of the system and ensure that it is normalized.

b) Phase gate is applied on 1st qubit & Y gate is applied on the 2nd qubit, what is the total state of the system after gate operations.

$P(\theta)|q_1\rangle \otimes Y|q_2\rangle$

$P(\theta) \otimes Y \cdot |q_1\rangle \otimes |q_2\rangle$

Total state $= |\alpha_1\rangle \otimes |\alpha_2\rangle$

$$|\alpha\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\cancel{|\alpha\rangle\langle\alpha| = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \end{bmatrix}}$$

&

$$\langle\alpha|\alpha\rangle = \begin{bmatrix} 0 & 0 & \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

Hence, normalized.

b) $P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$

$$P(\theta)|\alpha_1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{i\phi} \end{bmatrix}$$

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$Y|\alpha_2\rangle = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i \\ 3i \end{bmatrix} = \frac{i}{2}\begin{bmatrix} - \\ 3 \end{bmatrix}$$

$$P(\theta)|\alpha_1\rangle \otimes Y|\alpha_2\rangle = \begin{bmatrix} 0 \\ e^{i\phi} \end{bmatrix} \otimes \frac{i}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

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$$= \begin{bmatrix} 0 \\ 0 \\ \frac{-ie^{i\phi}}{2} \\ \frac{3ie^{i\phi}}{2} \end{bmatrix} = 0\langle 00\rangle + 0\langle 01\rangle + \frac{-e^{i\phi}}{2}\langle 10\rangle + \frac{3e^{i\phi}}{2}\langle 11\rangle$$

$\hookrightarrow a\langle 00\rangle + b\langle 01\rangle + c\langle 10\rangle + d\langle 11\rangle$

So if $a\langle 00\rangle + b\langle 01\rangle + c\langle 10\rangle + d\langle 11\rangle$

$= (\alpha_1\langle 00\rangle + \beta_1\langle 01\rangle) \otimes$

$(\alpha_2\langle 00\rangle + \beta_2\langle 11\rangle)$

So $a = \alpha_1\alpha_2$, $b = \alpha_1\beta_2$, $c = \beta_1\alpha_2$ &

$d = \beta_1\beta_2$

$\alpha_1\alpha_2 = 0 \qquad \alpha_1\beta_2 = 0$

$\alpha_1 = 0 \qquad \alpha_1\beta_2 = 0$

$\beta_1\alpha_2 = \frac{-e^{i\phi}}{2} \qquad \beta_1\beta_2 = \frac{3e^{i\phi}}{2}$

$\beta_1\alpha_2 = \frac{-ie^{i\phi}}{2} \qquad = \frac{\alpha_2}{\beta_2} = \frac{-1}{3}$

$\beta_1\beta_2 = \frac{3ie^{i\phi}}{2}$

$\alpha_1 = 0 \quad \alpha_2 = i/2 \quad \beta_1 = e^{i\phi} \quad \beta_2 = \sqrt{3}i/2$

ideal if both are same state.

Q) Consider a 2 qubit state both initialised to $|0\rangle$. On the first qubit, we apply Hadamard gate followed by application of CNOT gate with q_1 as the source and q_2 as the target. What is the final state of system?

$$\text{Initial state } \langle 00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}$$

$$H \otimes I \langle 00\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$ 4x4 4x1

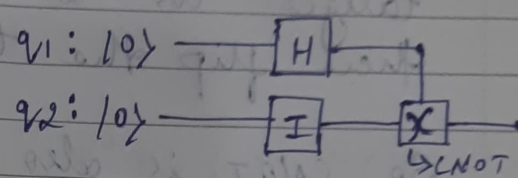
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4×1
4×4
4×1

$$\text{CNOT} \cdot H \otimes I \langle 00 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



$a = 1 \quad b = 0 \quad c = 0 \quad d = 1$

$\alpha_1 \alpha_2 = 1 \quad \alpha_1 \beta_2 = 0 \quad \beta_1 \alpha_2 = 0 \quad \beta_1 \beta_2 = 1$

Cannot be split into tensor products. Such states are called entangled states.

But you can represent it as

$$\frac{1}{\sqrt{2}} \left[\langle 00 \rangle + \langle 11 \rangle \right].$$

Q) Consider a 2 qubit system which is in an entangled state whose vector is $|\psi\rangle = \frac{1}{\sqrt{2}}[|01\rangle + i|10\rangle]$. Calculate the final state of system if CNOT is acted on the state.

For CNOT gate, it should be acted upon 2 qubits, if control is 0 then leave target as it is else if it is 1 then flip the target.

So, CNOT is also called entangler & detangler [It can entangle distangled states & vice-versa].

CNOT on $|\psi\rangle = \frac{1}{\sqrt{2}}[|01\rangle + i|11\rangle]$

$= \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + i|1\rangle \otimes |1\rangle]$

$= \frac{1}{\sqrt{2}}[|0\rangle \otimes i|1\rangle] \otimes |1\rangle$