MODULE - 1

* YOUNG'S DOUBLE SPLIT EXPERIMENT

→ When light passes through 2 slits, you get a light & dark pattern through interference.

→ Interference occurs when the size of the slit is approximately equal to the wavelength of the light emitted

→ Interference pattern can be seen even with 1 slit.

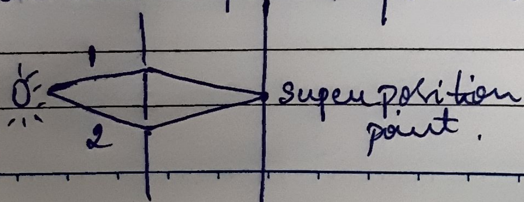
→ Even if 1 photon is released at a time the interference pattern can be seen

→ Interference is an intrinsic property of a photon.

→ Light acts as a wave in this experiment.

→ There are 2 states : either 1st or 2nd path and both path.

→ When the light passes through both slits superposition principle takes place



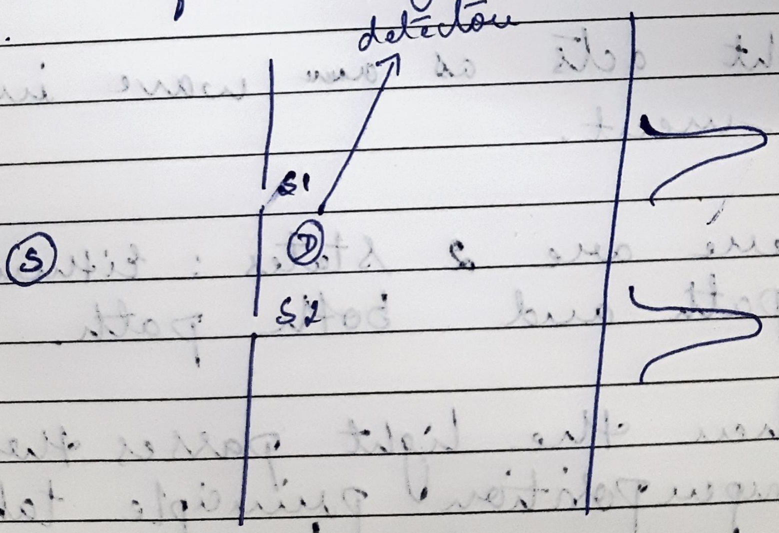
→ If light source is replaced by electron gun (Thermionic gun), then interference pattern can be seen.

This was proved by DeBroglie's particle nature of an electron.

→ The spin nature of an electron can be used as states in Quantum computing.

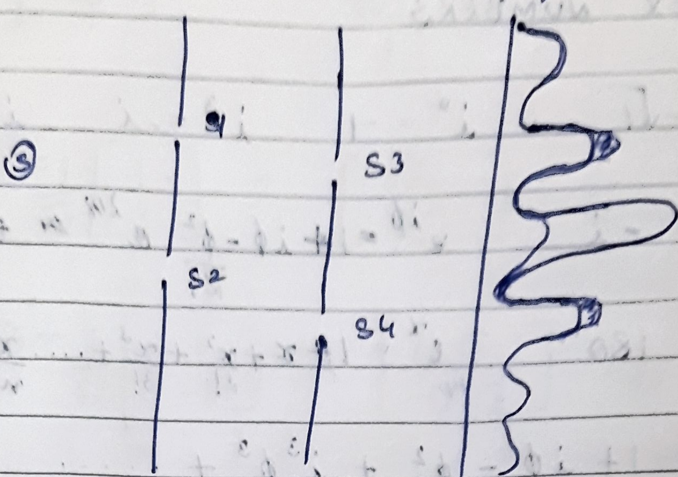
→ Even inside an atom, ground & excited states are there.

→ If the source is sound waves, then no interference can be seen because it is a longitudinal waves. whereas light, water, etc are transversal waves where the direction of propagation is ⊥ to the waves.



If a detector is placed b/w slits & it controls which slit each electron takes then superposition cannot happen so no interference pattern.

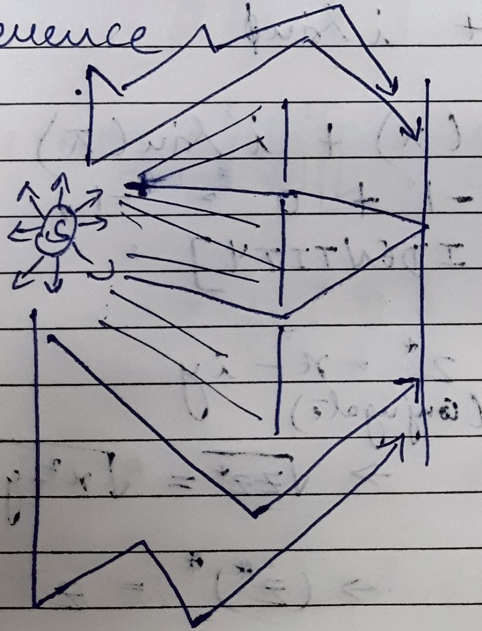
→



→ Moment you put a detector, pattern dissap

→ Interference can be seen as long as phase there are more than or equal to 2 slits, through which superposition occurs.

→ Light can take any path but are cancelled out in destructive interf-erence



DG Expt simulation

→ There is always error in quantum computing due to uncertainity principle.

* COMPLEX NUMBERS

→ $i = \sqrt{-1}$ $\quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$

→ $\frac{1}{i} = -i$ $\quad e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} + \frac{i^3\phi^3}{3!} + \ldots$

→ $\pi^c = 180°$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} + \frac{i^3\phi^3}{3!} + \ldots$$

$$e^{i\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \ldots\right) + \left(i\phi + \frac{i^3\phi^3}{3!} + \frac{i^5\phi^5}{5!} + \ldots\right)$$

$$e^{i\phi} = \cos\phi + i\left(\phi + \frac{i^2\phi^3}{3!} + \frac{i^4\phi^5}{5!} + \ldots\right)$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$* \; e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

$$-1 + 0 = -1$$

[EULER'S IDENTITY]

→ $z = x + iy$ $\quad \underset{\text{(conjugate)}}{z^*} = x - iy$

→ $zz^* = x^2 + y^2$ $\quad \rightarrow \sqrt{zz^*} = \sqrt{x^2+y^2} = |z|$

→ $|z^*| = \sqrt{zz^*}$ $\quad \rightarrow (z^*)^* = z$

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Q) $z = e^{i\phi}$ $\quad |z| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$

Q) $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$HA = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}(A+B)$ $\quad 2\times 2$, 2×1

$HB = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}(A-B)$

This is superposition principle.

$H(HA) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \cancel{\frac{1}{\sqrt{2}}}\ \cancel{A}\ \ A$

$H(HB) = \left(\frac{1}{\sqrt{2}}\right)^2\left[\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$

$= \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = B$

$(HH)A = H(HA)$ so $HH = I$

This is idempotent property

* TRANSPOSE [DAGGER]

→ $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A^\dagger = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$\dagger$ = dagger [works like Transpose]

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $B^\dagger = \begin{bmatrix} 0 & 1 \end{bmatrix}$

→ outer product

$AA^\dagger = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A^\dagger A = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

↳ Inner product

* Inner product always gives a scalar while outer product results in a matrix.

Q] $Y = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

$Y^* = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

$(Y^*)^\dagger = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} = Y$

$P(\theta)\, P^\dagger(\theta) = I$

if $MM^\dagger = I = M^\dagger M$
then M is a unitary matrix,
here $M^\dagger$ is called Hermition Congiugate

→ condition for unitary matrix.

Q] $P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$

$P^\dagger(\theta) = ?$

$P(\theta)^* = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$

$(P(\theta)^*)^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$ $P(\theta)^\dagger$

$P(\theta)\, P^\dagger(\theta)$

$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\Rightarrow P^{-1}(\theta) = P^\dagger(\theta)$

⇓

Unitary matrix.

Q] Which of the following matrices are unitary?

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow A^2 HH^\dagger = I = \frac{1}{\sqrt{2}}$ Normalization constant

$\sqrt{X} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ $\sqrt{X}^* = \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$ $\sqrt{X}^\dagger = \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

$$\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)(1-i)+(1-i)(1+i) & (1+i)(1+i)+(1-i)(1-i) \\ (1-i)(1-i)+(1+i)(1+i) & (1-i)(1+i)+(1+i)(1-i) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & [1+i^2+2i]+[1+i^2-2i] \\ [1+i^2-2i]+[1+i^2-2i] & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & \cancel{1-1}+2i \; \cancel{+1-1} -2i \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

<u>Note</u>:- for a square matrix we take its conjugate first and then dagger of the matrix

normalization constant

* $C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad D = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$C^{+}C = 1$ $\quad D^{+}D = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$

$\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1+1 = 2$

* $(\sqrt{X})^2 = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$= \begin{bmatrix} 0+0 & 2+2 \\ 2+2 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}\left(\frac{1}{2}\right)^2$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ // $\Rightarrow (\sqrt{X})^2 = X$

Hermition $\Rightarrow A = A^{+}$

$= (A^{*})^{T}$

$= (A^{T})^{*}$

Q]

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad Y^{+}Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$Y^{*} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ $\quad = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix}$

$(Y^{*})^{T} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\quad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* State $\longrightarrow$ Momentum

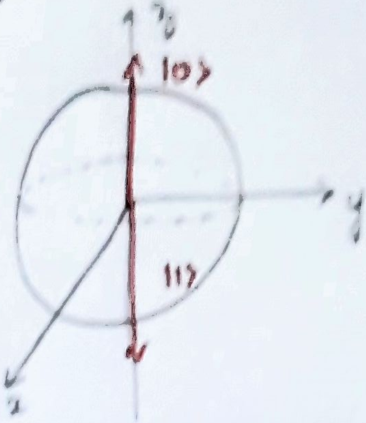
$|\psi\rangle$ Angular momentum
Spin
position
etc

$\uparrow |0\rangle$ $\downarrow |1\rangle$

$\downarrow$

$|0\rangle$ & $|1\rangle$ state
is what we are worried
about in Quantum computing

- Q → $|0\rangle$, $|1\rangle$ ⟶



$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

once the gates or operators applied only the direction of the state changes

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$ → X Gate rotate the Qubit rotate the x-axis 180°.

$\Rightarrow -|0\rangle = \boxed{e^{i\pi}}|0\rangle$ → phase

$P(\theta) \Rightarrow$ phase gate
↓
useful gate in QC but cannot be measured.

$|\alpha\rangle = e^{i\pi}|0\rangle$

$|\beta\rangle = |0\rangle$

$\langle\beta|\beta\rangle = 1$

$\langle\alpha|\alpha\rangle = ?$

$\langle\alpha| = e^{-i\pi}\langle 0|$

$\langle\alpha|\alpha\rangle = e^{i\pi - i\pi}\langle 0|0\rangle$

$= 1$

* HERMITIAN

$\rightarrow A = A^{\dagger}$ $A^{\dagger} = (A^{T})^{*}$ or $(A^{*})^{T}$

Q) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Y^{\dagger} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$Y^{*} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ $Y = Y^{\dagger}$ so yes

Q) $\sqrt{X} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$(\sqrt{X}^{*}) = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$ $(\sqrt{X})^{\dagger} = \frac{1}{2}\begin{bmatrix} 1-i & 1+ \\ 1+i & 1- \end{bmatrix}$

$\sqrt{X} \neq (\sqrt{X})^{\dagger}$ so not hermitian.

$\cancel{1+i+1-i-i^2+1+i-1-i^2}$

$(\sqrt{X})(\sqrt{X})^{\dagger} = \frac{1}{4}\begin{bmatrix} [illegible] \end{bmatrix} = I$

*NOTE : All Hermitian matrices are unitary and all unitary ~~[illegible]~~ need not be hermatian

NOTE : All quantum gates are unitary and need not be Hermatian.

*NOTE : $HA = \frac{1}{\sqrt{2}} \underbrace{(A+B)}_{\text{linear combination}}$ = Superposition

So H represents the state so it is both Hermitian & unitary.

Q) what happens when $\sqrt{X}$ operates on A

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \sqrt{X} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$\sqrt{X}A = \frac{1}{2}\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \alpha$$

$$\sqrt{X}\alpha = \frac{1}{4}\begin{bmatrix} 1+i^2+2i+1+i^2-2i \\ 1-i^2+1-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2(1+i^2) \\ 2(1-i^2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+i^2 \\ 1-i^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-1 \\ 1+1 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\sqrt{X}\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B.$$

So, this proves that even if $\sqrt{X}$ is not Hermitian, it still was able to convert A to B. So, it is fine as long as it is unitary.

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★ PAM DIRAC THEOREM

→ To differentiate b/w vectors & matrices vector matrices are called as kets.

$|A\rangle$ = symbol.   Inner prod = $A^{+}A = \langle A|A\rangle$ (bra, ket)

$A^{+} = \langle A|$

Outer prod = $AA^{+} = |A\rangle\langle A|$

★ OPERATORS

X, Y, H, $\sqrt{X}$ are operators represented as $\hat{X}, \hat{Y}, \hat{H}$, etc.

So, operators operate on kets to change them.

★ Tensor Product

$$|A\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \end{bmatrix} \quad |B\rangle = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \end{bmatrix}$$

$$|A\rangle \otimes |B\rangle = |A\rangle|B\rangle = |AB\rangle$$

All represents tensor product.

$$|AB\rangle = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_i B \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_i b_i \\ a_1 b_2 \\ a_2 b_2 \\ \vdots \\ a_i b_2 \\ a_i b_i \end{bmatrix}$$

when there are multiple qubits, so to handle all of them simultaneously we do tensor product.

Tensor product expands a ket.

$|A\rangle = 2\times 1 \quad |B\rangle = 2\times 1 \quad |AB\rangle = 8\times 1$

Q) $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|AB\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |AA\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$|BA\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |BB\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

* A & B are basis vectors

$|A\rangle = |0\rangle \quad |B\rangle = |1\rangle$

A qubit can be in 3 states

$|0\rangle$ or $|1\rangle$ or both.

* CNOT is 4×4 gate operating on 2 qubits [4×1]

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$CNOT \times |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

$$CNOT \times |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

$$CNOT \times |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

$$CNOT \times |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$$

→ [controlled NOT gate]

CNOT flips 2nd qubit when 1st qubit is 1. So, 2nd qubit is target & 1st qubit is controll

$X|0\rangle = |1\rangle$ $\quad X|1\rangle = |0\rangle$ $\quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So X is a not gate

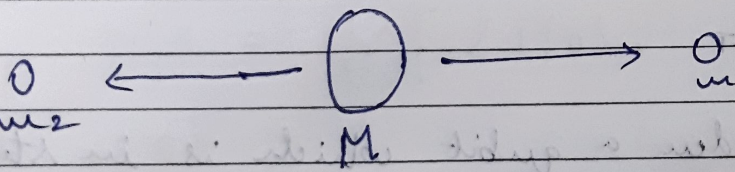
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

This is superposition

★ ENTANGLEMENT

If a mass M is split into 2 parts m_1, m_2 then both parts fly in opposite directions with same velocity due to conservation of momentum & energy (kinetic).



$$m_1 v_1 + m_2 v_2 = MV$$
$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} MV^2$$

So, Entanglement states that if one path is known then the other path can also be known.

But it doesn't violate uncertainty principle. So, this implies that if one part (m₁)'s of momentum is being measured with high accuracy then the other part's (m₂) distance cannot be measured accurately & vice-versa.

This phenomenon was called as spooky action at a distance by Einstein.

★ STATE ($|\psi\rangle$)

→ A state has momentum, angular momentum, spin & position.

$\uparrow - |0\rangle \qquad \downarrow - |1\rangle$

★ QUBIT

→ Consider a qubit which is in state ψ_0. We initialize it to 0.

→ Act the Hadamard gate on the state ψ_0.

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H|\psi_0\rangle = |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

→ Until you measure it, it can be in both states but not while measuring.

→ To measure if the system is in 0 state, we do inner product.

$$\langle 0|\psi_1\rangle = \langle 0| \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle 0|0\rangle + \langle 0|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 + 0\right] = \frac{1}{\sqrt{2}}$$

→ To check if system is in 1 state:

$$\langle 1|\psi_1\rangle = \langle 1| \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle 1|0\rangle + \langle 1|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[0 + 1\right] = \frac{1}{\sqrt{2}}$$

This is called probability amplitude. It tells you something but does have physical significance.

Probability is given by taking the magnitude & square it.

$$P(|\psi_1\rangle = |1\rangle) = |\langle 1|\psi_1\rangle|^2 = \frac{1}{2}$$

$$P(|\psi_1\rangle = |0\rangle) = |\langle 0|\psi_1\rangle|^2 = \frac{1}{2}$$

Probability that a system is in one, is 50%.

→ Consider a qubit in state $|\psi_0\rangle$ initialised to $|1\rangle$

Act a hadamard gate to get $|\psi_1\rangle$

what is the probability that $|\psi_1\rangle$ is in zero state.

$$|\psi_1\rangle = H|\psi_0\rangle = H|1\rangle =$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$$\langle 1|\psi_1\rangle = \langle 1|\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle 1|0\rangle - \langle 1|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[0 + 1\right] = \frac{+1}{\sqrt{2}}$$

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$P(|\psi_1\rangle = |1\rangle) = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

Q) Consider a system whose state $|A\rangle$ is given as $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$. Check if $|A\rangle$ is normalised.
Act the ~~...~~ X gate on $|A\rangle$ to obtain the state $|B\rangle$.
Calculate the probability that the system is in state 1.

if $A^{\dagger} \cdot A^{*} = I$ then it is normalized

$$A = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{\sqrt{3}}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\sqrt{3}/2 \end{bmatrix}$$

$$A^{\dagger} = (A^T)^{*} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$A^{\dagger} \cdot A^{*} = \cancel{\begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{bmatrix}} \begin{bmatrix} 1/4 + 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \end{bmatrix} = I$$

$$H|A\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1}{2}+\frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \end{bmatrix}$$

$$|B\rangle = X|A\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\langle 1|B\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \end{bmatrix}$$

$$P(|B\rangle = |1\rangle) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

Q) $|X\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

$$|B\rangle = X|\alpha\rangle$$

Q) Consider a state α given by $|\alpha\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$

(i) Normalize the given state to get $|x\rangle$

(ii) This normalized state is rotated about the y-axis by the application of y gate, followed by application of x gate and finally the hadamard gate.

(iii) Calculate the probabilities of obtaining 0 and 1 after the application of each gate.

$$\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{9}{16}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4}$$

$$|x\rangle = \frac{\frac{1}{4}}{\frac{\sqrt{10}}{4}}|0\rangle + \frac{\frac{3}{4}}{\frac{\sqrt{10}}{4}}|1\rangle = \frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad |\alpha\rangle = \frac{1}{4}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

$$|\beta\rangle = Y|x\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} -3i/\sqrt{10} \\ i/\sqrt{10} \end{bmatrix} = \frac{i}{\sqrt{10}}|1\rangle - \frac{3i}{\sqrt{10}}|0\rangle$$

$$|\gamma\rangle = X|\beta\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3i}{\sqrt{10}} \\ \frac{i}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{\sqrt{10}} \\ -\frac{3i}{\sqrt{10}} \end{bmatrix} = \frac{i}{\sqrt{10}}|0\rangle - \frac{3i}{\sqrt{10}}|1\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|\delta\rangle = H|\gamma\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{10}} \\ -\frac{3i}{\sqrt{10}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{i}{\sqrt{10}} & -\frac{3i}{\sqrt{10}} \\ \frac{i}{\sqrt{10}} & \frac{3i}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{2i}{\sqrt{10}} \\ \frac{4i}{\sqrt{10}} \end{bmatrix}$$

$$P(|\beta\rangle = |0\rangle) = |\langle 0|\beta\rangle|^2$$

$$|\langle 0| \frac{i}{\sqrt{10}}|1\rangle - \frac{3i}{\sqrt{10}}|0\rangle \rangle|^2$$

$$= |\frac{i}{\sqrt{10}} \langle 0|1\rangle - \frac{3i}{\sqrt{10}} \langle 0|0\rangle|^2$$

$$= |\frac{i}{\sqrt{10}} \times 0 - \frac{3i}{\sqrt{10}} \times 1|^2 = |\frac{-3i}{\sqrt{10}}|^2$$

$$= \frac{9}{10}$$

$$P(|\beta\rangle = |1\rangle) = |\langle 1|\beta\rangle|^2$$

$$\left|\langle 1| \left(\frac{i}{\sqrt{10}}|1\rangle - \frac{3i}{\sqrt{10}}|0\rangle \right)\right|^2$$

$$\left| \frac{i}{\sqrt{10}} \langle 1|1\rangle - \frac{3i}{\sqrt{10}} \langle 1|0\rangle \right|^2$$

$$= \left| \frac{i}{\sqrt{10}} \times 1 - \frac{3i}{\sqrt{10}} \times 0 \right|^2 = \left|\frac{i}{\sqrt{10}}\right|^2$$

$$= +\frac{1}{10}$$

$$P(|\gamma\rangle = |0\rangle) = |\langle 0|\gamma\rangle|^2$$

$$= \left|\langle 0| \frac{i}{\sqrt{10}}|0\rangle - \frac{3i}{\sqrt{10}}|1\rangle \rangle\right|^2$$

$$= \left| \frac{i}{\sqrt{10}} \langle 0|0\rangle - \frac{3i}{\sqrt{10}} \langle 0|1\rangle \right|^2$$

$$= \left|\frac{i}{\sqrt{10}}\right|^2 = \frac{1}{10}$$

$$P(|\gamma\rangle = |1\rangle) = \frac{9}{10}$$

$$P(|\delta\rangle = |0\rangle) = |\langle 0|\delta\rangle|^2$$

$$= \left| \langle 0| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{-2i}{\sqrt{10}} \\ \frac{4i}{\sqrt{10}} \end{bmatrix} \right|$$

$$= \left| \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{-2i}{\sqrt{10}} \\ \frac{4i}{\sqrt{10}} \; 0 \end{bmatrix} \right|^2 = \left(\frac{1}{\sqrt{2}} \times \frac{-2i}{\sqrt{10}} \right)^2$$

$$= \frac{1}{5}$$

$$P(|\delta\rangle = |1\rangle) = 1 - \frac{1}{5} = \frac{4}{5}$$

Q) Prove that $R_y\left(\frac{\pi}{2}\right) R_z(\pi) e^{i\pi/2} = H$

where $R_y(\theta) = \left(\cos\frac{\theta}{2}\right) I + i \left(\sin\frac{\theta}{2}\right) Y$

$R_z(\theta) = \left(\cos\frac{\theta}{2}\right) I - i \left(\sin\frac{\theta}{2}\right) Z$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\left[I - iY\right]$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$R_z(\pi) = -iZ = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -i \\ -i & i \end{bmatrix}$$

$$e^{i\frac{\pi}{2}} = i$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -i^2 & -i^2 \\ -i^2 & i^2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

Hence, proved

Q) i) Consider a gate which is obtained from the product of Z gate and $\sqrt{Y}$ gate. Consider another gate which is obtained by the product of $\sqrt{Z}$, $\sqrt{Y}$ and $\sqrt{Z}$ gates. Obtain their matrix representation.

(ii) What happens when you apply these gates on a qubit whose state is 0.

$$\sqrt{Y} = \left(\frac{1+i}{2}\right)\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$g_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\frac{1+i}{2}\right)$$

$$= \left(\frac{1+i}{2}\right)\begin{bmatrix} 1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$g_B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\left(\frac{1+i}{2}\right)$$

$$= \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}\left(\frac{1+i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$= \frac{(1+i)}{2}\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$$

*[Multiply gates in opposite manner]

[illegible]

$$|\alpha\rangle = g_A|0\rangle = \left(\frac{1+i}{2}\right)\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= \left(\frac{1+i}{2}\right)\begin{bmatrix}1\\ 1\end{bmatrix} = \left(\frac{1+i}{2}\right)[|0\rangle + |1\rangle]$$

$$|\beta\rangle = g_B|0\rangle = \left(\frac{1+i}{2}\right)\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= \left(\frac{1+i}{2}\right)\begin{bmatrix}1\\ i\end{bmatrix} = \left(\frac{1+i}{2}\right)[|0\rangle + i|1\rangle]$$

$$P(|\alpha\rangle = |0\rangle) = |\langle 0|\alpha\rangle|^2$$

$$= \left|\langle 0|\left(\frac{1+i}{2}\right)(|0\rangle + |1\rangle)\right|^2$$

$$= \left|\frac{1+i}{2}\right|^2 = \left(\frac{\sqrt{1+1}}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

$$= \frac{2}{4} = \frac{1}{2}$$

The matrices which produce results similar to a hadamard gate is called a pseudo Haddamard gates. eg: $\left(\frac{1+i}{2}\right)\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

Q) Prove that the unitary operator for the hadamard gate can be written as

$$H = \frac{1}{\sqrt{2}} \left[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1| \right]$$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\langle 0| = [1 \ 0]$

$\langle 1| = [0 \ 1]$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

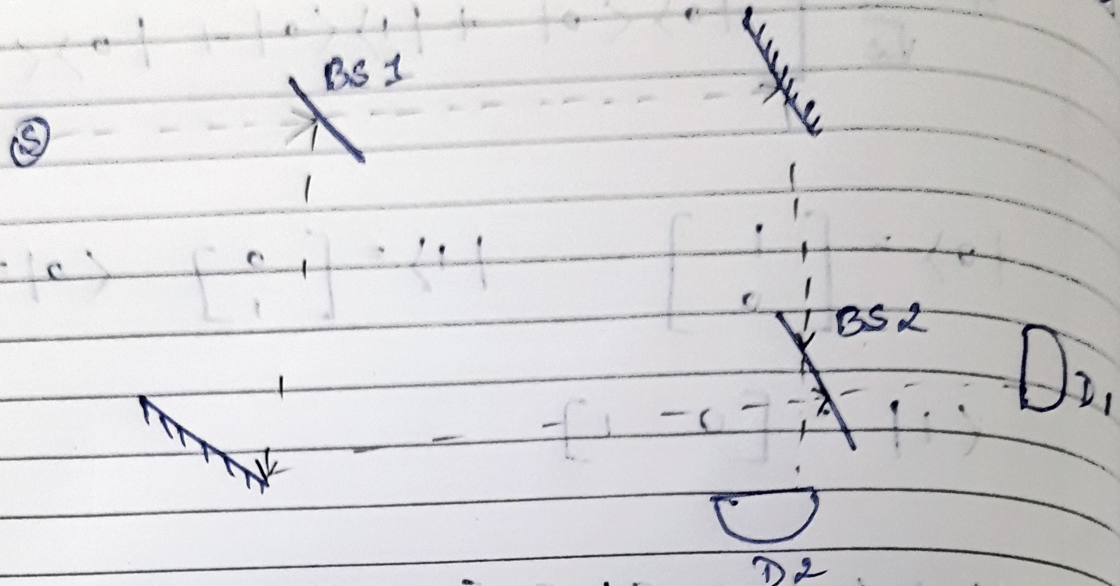
$$|1\rangle\langle 0| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \ 1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

Q) The Mach Zender Interferen consists of a photon source, 2 beam splitter mirrors, 2 mirrors & 2 detectors



(i) If a photon is incident on the given setup along the x direction, what is the probability of detecting at D1 & D2.

The operators representing a beam splitter and a mirror are given by

$$BS = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|x\rangle_{\text{photon}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$BS|x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + i x_2 \\ i x_1 + x_2 \end{bmatrix}$$

$$\langle M|(BS|x\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} x_2 + i x_1 \\ x_1 + i x_2 \end{bmatrix}$$

$$\langle BS|M|\rangle BS|x\rangle = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} x_2 + i x_1 \\ x_1 + i x_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \cancel{x_2} + i x_1 + i x_1 + i^2 \cancel{x_2} \\ x_2 i + i^2 \cancel{x_1} + \cancel{x_1} + i x_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 i x_1 \\ 2 i x_2 \end{bmatrix} = i\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = i|x\rangle$$

$$= e^{i\pi/2}|x\rangle = |f\rangle$$

$$P(|f\rangle = |x\rangle) = |\langle x|f\rangle|^2$$

$$= i\begin{bmatrix} \cancel{x_1} & x_2 \\ & \cancel{x_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = i[x_1^2 + x_2^2]$$

$$= |e^{i\pi/2}\langle x|x\rangle|^2 = |e^{i\pi/2}|^2 = 1$$

so photon can only be detected by D_1

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(ii) Assume that the photon was incident on y, does the probability of detection change?

~~BS\|y\}~~

$BS \cdot |M\rangle \cdot BS \cdot |y\rangle$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}|y\rangle$

$\cancel{= \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix}}$

$= \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}$

$= i I_{4} = |f\rangle$

$P(|f\rangle = |y\rangle) = |\langle y|f\rangle|^2$

$= \left| [y_1 \; y_2]\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \right|^2 = \left| [iy_1 \; iy_2 \right.$

$= 1_4$

So only detected at D2.

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(iii) Introduce 2 glass slabs after the 1st BS along both dimensions. The operator for 2 glass slabs is given by

$$A(\psi_x, \psi_y) = \begin{bmatrix} e^{i\psi_x} & 0 \\ 0 & e^{i\psi_y} \end{bmatrix}$$

Calculate the probability of detection at D_1 & D_2 if $\psi_x = \frac{\pi}{2}$ & $\psi_y = \pi$

$$e^{i\psi_x} = 0 + i = i \qquad e^{i\psi_y} = -1$$

$$A = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$$

$$BS|x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + i x_2 \\ i x_1 + x_2 \end{bmatrix}.$$

$$A \cdot BS|x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 + i x_2 \\ i x_1 + x_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -x_2 + i x_1 \\ -x_2 - i x_1 \end{bmatrix}$$