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1) $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

$A^* = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$

$A^+ = (A^*)^T = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

$A = A^+$ so it is Hermatian.

2) $B = \begin{bmatrix} 0 & 1+i \\ 1-i & 3 \end{bmatrix}$

$B^* = \begin{bmatrix} 0 & 1-i \\ 1+i & 3 \end{bmatrix}$

$B^+ (B^*)^T = \begin{bmatrix} 0 & 1+i \\ 1-i & 3 \end{bmatrix}$

$B = B^+$ hence it is Hermatian.

3) $C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$C^* = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ $C^+ = (C^*)^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

$C \neq C^+$ Hence not Hermatian

4) $D = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ $D^{+} = (D^{*})^{T} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$D = D^{+}$ Hence it is Hermitian

5) $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $U^{+} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$UU^{+} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

Hence, it is unitary.

6) $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $V^{+} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$V^{+}V = \left(\frac{1}{\sqrt{2}}\right)^{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= I_{4}$

Hence, it is unitary.

7) $W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $W^+ = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$W^+ W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$W^+ \cdot W \neq I$

Hence, it is not unitary

8) $X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $X^+ = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$X^+ X = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \underline{I}$

Hence, its unitary

9) $A = \begin{bmatrix} 1 & -i \\ 2 & 3 \end{bmatrix}$ $A^+ = \begin{bmatrix} 1 & 2 \\ i & 3 \end{bmatrix}$

10) $M = \begin{bmatrix} 0 & 1+i \\ 2-i & 4 \end{bmatrix}$ $M^+ = \begin{bmatrix} 0 & 2+i \\ 1-i & 4 \end{bmatrix}$

11) $N = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $N^+ = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

12) $P = \begin{bmatrix} -1 & 2i \\ -2i & -1 \end{bmatrix}$ $P^+ = \begin{bmatrix} -1 & 2i \\ -2i & -1 \end{bmatrix}$

13) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$

14) $\langle 1| = [0 \quad 1]$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

15) $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\langle \psi| = \frac{1}{\sqrt{2}} [1 \quad -i]$

16) $\langle 0|1\rangle = [1 \; 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0] = 0$

17) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

18) $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -|1\rangle$

19) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle]$

20) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i|0\rangle$

21) $H|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

$P(H|0\rangle = |0\rangle) = |\langle 0| H|0\rangle\rangle|^2$

$= \left|\frac{1}{\sqrt{2}} \langle 0|0\rangle + \langle 0|1\rangle\right|^2 = \left|\frac{1}{\sqrt{2}} (1+0)\right|^2$

$= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

22) $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$P(H|1\rangle = |1\rangle) = |\langle 1|H|1\rangle|^2$

$= \left|\frac{1}{\sqrt{2}}(\langle 1|0\rangle - \langle 1|1\rangle)\right|^2 = \left|\frac{1}{\sqrt{2}}(0-1)\right|^2$

$= \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

23) $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$P(H|0\rangle = |0\rangle) = \frac{1}{2}$

24) $P(H|1\rangle = |1\rangle) = \frac{1}{2}$

25) $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

$P(|\psi\rangle = |1\rangle) = |\langle 1|\psi\rangle|^2$

$= \left|\frac{\sqrt{3}}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle\right|^2$

$= \left|\frac{\sqrt{3}}{2}(0) + \frac{1}{2}(1)\right|^2 = \frac{1}{4}$

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26) $X|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$ $P(X|0\rangle = |0\rangle) = |\langle 0|1\rangle|^2$
$= 0$

27) $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

$P(|\phi\rangle = |0\rangle) = \left|\frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{i}{\sqrt{2}}\langle 0|1\rangle\right|^2$

$= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

28) $|\psi\rangle = a|0\rangle + b|1\rangle$ $a = \frac{3}{5}$ $b = \frac{4}{5}$

$\langle\psi|\psi\rangle = \frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times \frac{4}{5} = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

$P(|\psi\rangle = |1\rangle) = |\langle 1|\psi\rangle|^2 = \left|\frac{3}{5}\langle 1|0\rangle + \frac{4}{5}\langle 1|1\rangle\right|^2$

$= \left|\frac{3}{5}(0) + \frac{4}{5}(1)\right|^2 = \frac{16}{25}$

29) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $H|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$

$H|+\rangle = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

$P(|H\rangle \to |0\rangle) = 1$

30) $P(|x\rangle = |1\rangle) = \left| \frac{1}{2} \langle 1|0\rangle + \frac{\sqrt{3}}{2} \langle 1|1\rangle \right|^2$

$= \left| \frac{1}{2}(0) + \frac{\sqrt{3}}{2}(1) \right|^2 = \frac{3}{4}$

$P(|x\rangle = |0\rangle) = \frac{1}{4}$

31) $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

a) $M^{+} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad M^{+}M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$M^{+}M \neq I$ Hence, not normalised.

$U = M(M^{+}M)^{-1/2}$

$(M^{+}M)^{-1} = \frac{1}{4-0} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$(M^{+}M)^{-1/2} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$M(M^{+}M)^{-1/2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

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b) $|\psi\rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $|\psi| = \sqrt{4+1} = \sqrt{5}$

normalized $|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

c) $M|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

d) $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|\alpha\rangle = Z \cdot M|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle$

e) $P(|\alpha\rangle = |0\rangle) = |\langle 0|\alpha\rangle|^2$

$= \left|\frac{2}{\sqrt{5}}\langle 0|0\rangle - \frac{1}{\sqrt{5}}\langle 0|1\rangle\right|^2 = \frac{4}{5} = 0.8$

$P(|\alpha\rangle = |1\rangle) = \frac{1}{5} = 0.2$

32) $M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $M^{+} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$M^{+}M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \neq I$

col 1 $= \sqrt{0+1} = 1$ col 2 $= \sqrt{4+0} = 2$

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $|\psi\rangle = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\quad |\psi| = \sqrt{9+16} = \sqrt{25} = 5$

normalized $|\psi\rangle = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

c) $M|\psi\rangle = \frac{1}{5}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

d) $Y|M|\psi\rangle = \frac{1}{5}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3i \\ 4i \end{bmatrix} = |\alpha\rangle = \frac{-3i}{5}|0\rangle + \frac{4i}{5}|1\rangle$

e) $P(|\alpha\rangle = |0\rangle) = |\langle 0|\alpha\rangle|^2$ $\quad P(|\alpha\rangle = |1\rangle) = \frac{16}{25}$

$= \left| \frac{-3i}{5} \right|^2 = \frac{9}{25}$

33) $M = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad M^+ = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad M^+M = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$
$\neq I$

a) $col1 = \sqrt{9+0} = 3$ $\quad col2 = \sqrt{16+0} = \sqrt{16} = 4$

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

b) $|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ c) $M'|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d) $|\alpha\rangle = Z \cdot M''|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{5}}|0\rangle - \frac{2}{\sqrt{5}}|1\rangle$

e) $P(|\alpha\rangle = |0\rangle) = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5}$ $\quad P(|\alpha\rangle = |1\rangle) = \frac{4}{5}$

34) $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $M^{+} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $M^{+}M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

$col1 = \sqrt{1+0} = \sqrt{\cancel{5}\,1} = 1$ $col2 = \sqrt{4+1} = \sqrt{5}$

$M'' = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $col1 \cdot col2 \neq 0$

To make it orthogonal

35) a) $M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ $col1 = \sqrt{4} = 2$ $col2 = \sqrt{1} = 1$ $M' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $|\psi\rangle = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ $|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

c) $M'|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

d) $|\alpha\rangle = Z\,M'|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ -1 \end{bmatrix} = \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

e) $P(|\alpha\rangle = |0\rangle) = |\langle 0|\alpha\rangle|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$P(|\alpha\rangle = |1\rangle) = \frac{1}{2}$