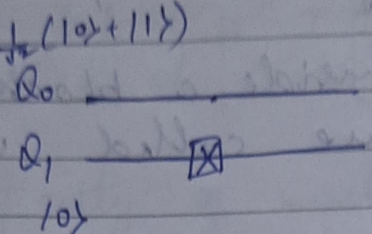
ALGORITHMS

★ NO CLONING THEOREM



$\} \; 1/\sqrt{2} \left[|00\rangle + |11\rangle \right]$

$$Q_0 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad Q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Q_0 \otimes Q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$CNOT \times Q_0 \otimes Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[|00\rangle + |11\rangle \right]$$

Now tensor product of Q_0 & Q_1

$$Q_0 \otimes Q_1 = \frac{1}{2} \left[|00\rangle + |01\rangle + |10\rangle + |11\rangle \right.$$

Even through theorem it feels like Q_1 copied Q_0 but it hasn't.

Assume ϕ is initial state & ψ is final state.

$$\Omega \underbrace{\left[|\phi\rangle \otimes \underbrace{|\psi\rangle}_{\text{ancillary}} \right]}_{\text{cloning operator (assumption)}} = \left[|\phi\rangle \otimes |\phi\rangle \right]$$

No cloning theorem states that no universal gate can copy the initial state.

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad |\phi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha^2 \\ \alpha\beta \\ \beta\alpha \\ \beta^2 \end{bmatrix}$$

$$|\phi\rangle \otimes |0\rangle \qquad = \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

$$= \begin{bmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{bmatrix} = \alpha|00\rangle + \beta$$

$$\Omega[|\phi\rangle \otimes |\psi\rangle] = [|\phi\rangle \otimes |\phi\rangle]$$

$$\alpha\Omega[|0\rangle \otimes |\psi\rangle]$$

$$= \alpha^2|00\rangle + \alpha\beta|01\rangle +$$

$$+\beta\Omega[|1\rangle \otimes |\psi\rangle] \qquad \beta\alpha|10\rangle + \beta^2|11\rangle$$

LHS: $\alpha\Omega[|0\rangle \otimes |\psi\rangle] = \alpha[|0\rangle \otimes |0\rangle] = \alpha|00\rangle$

$\beta\Omega[|1\rangle \otimes |\psi\rangle] = \beta[|1\rangle \otimes |1\rangle] = \beta|11\rangle$

$$\alpha|00\rangle + \beta|11\rangle \neq \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

except when $\alpha = 1$ and $\beta = 0$ or vice-versa;

Only when the gates are in pure state, you can clone or else not possible.

So, this proves that Ω (universal gate) cannot clone any arbitrary state.

Q) Prove that there is no universal NOT gate.

$U|\psi\rangle \neq X|\psi\rangle$

$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $X|\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

when $\alpha = \beta = 1/\sqrt{2}$

$$X\left[\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]\right] = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

* TELEPORTATION

Let us assume alice has state of infor stored in a qubit given by

$$|\psi_{a_1}\rangle = a_0 |0_{a_1}\rangle + a_1 |1_{a_1}\rangle$$

Consider a pair of entangled qubits in a bell state.

$$|\beta_{23}\rangle = \frac{1}{\sqrt{2}} \left[|0_2 0_3\rangle + |1_2 1_3\rangle \right]$$

So, give one qubit to Alice & other to Bob.

$$|\beta_{a_2 b_3}\rangle = \frac{1}{\sqrt{2}} \left[|0_{a_2} 0_{b_3}\rangle + |1_{a_2} 1_{b_3}\rangle \right]$$

$2 \rightarrow$ alice $\quad 3 \rightarrow$ Bob

$$|\psi_{123}\rangle = |\psi_{a_1}\rangle \otimes |\beta_{a_2 b_3}\rangle$$

$$= a_0 |0_{a_1}\rangle + a_1 |1_{a_1}\rangle$$

$$\otimes$$

$$\frac{1}{\sqrt{2}} \left[|0_{a_2} 0_{b_3}\rangle + |1_{a_2} 1_{b_3}\rangle \right]$$

$= \frac{1}{\sqrt{2}} \left[a_0 | 0_{a_1} 0_{a_2} 0_{b_3} \rangle + a_0 | 0_{a_1} 1_{a_2} 1_{b_3} \rangle + a_1 | 1_{a_1} 0_{a_2} 0_{b_3} \rangle + a_1 | 1_{a_1} 1_{a_2} 1_{b_3} \rangle \right]$

We then apply CNOT gate with 1st qubit as control & 2nd qubit as target, this entangles qubit 1 & 2 as qubit 1 is in a state of super--position. So, any changes made to qubit 1 will take place in qubit 2 also due to entangled state & qubit 2 is entangled to qubit 3 due to which the changes ~~place~~ takes place in qubit 3 also.

$\text{CNOT}_{12} | \psi_{123} \rangle = \frac{1}{\sqrt{2}} \left[a_0 | 0_{a_1} 0_{a_2} 0_{b_3} \rangle + a_0 | 0_{a_1} 1_{a_2} 1_{a_3} \rangle + a_1 | 1_{a_1} 1_{a_2} 0_{b_3} \rangle + a_1 | 1_{a_1} 0_{a_2} 1_{b_3} \rangle \right]$

Applying Hadamard gate on qubit 1

$H | \psi_{a_1} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 + a_1 \\ a_0 - a_1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \left[\frac{(a_0 + a_1)}{\sqrt{2}} | 0_{a_1} \rangle + \frac{(a_0 - a_1)}{\sqrt{2}} | 1_{a_1} \rangle \right]$

$$H \cdot CNOT |\psi_{123}\rangle = \frac{1}{\sqrt{2}}\Big[\frac{a_0}{\sqrt{2}}\big(|0_{a_1}\rangle + |1_{a_1}\rangle\big) \otimes \big(|0_{a_2} 0_{b_3}\rangle + |1_{a_2} 0_{b_3}\rangle\big) + \frac{a_1}{\sqrt{2}}\big(|0_{a_1}\rangle - |1_{a_1}\rangle\big) \otimes \big(|1_{a_2} 0_{b_3}\rangle + |0_{a_2} 1_{b_3}\rangle\big)\Big]$$

$$|\psi^4_{123}\rangle = \frac{1}{2}\Big[|\overset{\text{alice's}}{0_{a_1} 0_{a_2}}\rangle \otimes \big(a_0|\overset{\text{bob's}}{0_{b_3}}\rangle + a_1|1_{b_3}\rangle\big) + |0_{a_1} 1_{a_2}\rangle \otimes \big(a_0|1_{b_3}\rangle + a_1|0_{b_3}\rangle\big) + |1_{a_1} 0_{a_2}\rangle \otimes \big(a_0|0_{b_3}\rangle - a_1|1_{b_3}\rangle\big) + |1_{a_1} 1_{a_2}\rangle \otimes \big(a_0|1_{b_3}\rangle - a_1|0_{b_3}\rangle\big)\Big]$$

Step 5 - Alice makes measurements on her pair of qubits.

$[\,|0_{a_1} 0_{a_2}\rangle$ or $|0_{a_1} 1_{a_2}\rangle$ or $|1_{a_1} 0_{a_2}\rangle$ or $|1_{a_1} 1_{a_2}\rangle\,]$

Step 6 - She relays the information to bob classically as to which state her qubit is in.

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Step-7 - Based on the information given by Alice, Bob performs the following gate operations,

Alice's qubit state		Bob's operation
$\vert 0_{a_1} 0_{a_2} \rangle$	[already in Alice's state]	Identity
$\vert 0_{a_1} 1_{a_2} \rangle$	[kets are swapped so apply X]	X
$\vert 1_{a_1} 0_{a_2} \rangle$	[imaginary coeff is -ve so apply Z]	Z
$\vert 1_{a_1} 1_{a_2} \rangle$	[both 2nd & 3rd so apply Z & X]	ZX

Our ultimate goal is to convert Bob's qubit to Alice's initial qubits

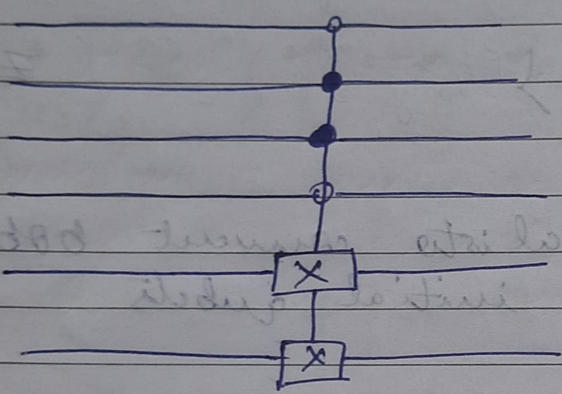
* Grover's Algorithm

→ TC : $O(\sqrt{N})$ Binary search = $O(N)$

→ For CNOT, if control is 1, target is flip

→ For Toffoli, if both control is 1, then target flips.

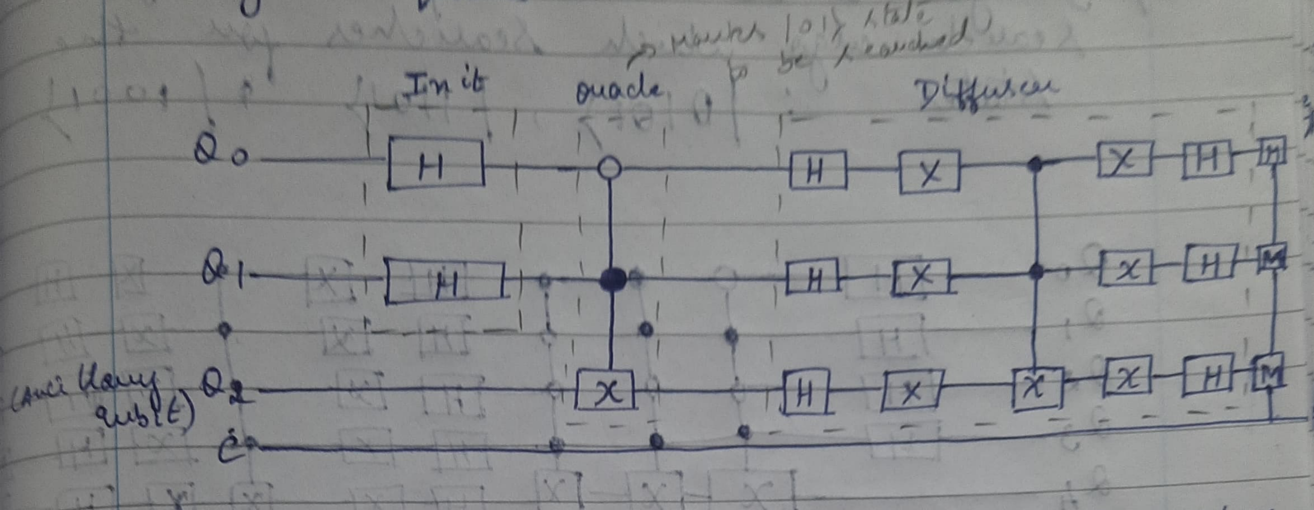
→ MCMT - Multi-Control Multi-target



→ Algorithm has 3 parts :

1. Initialization
2. Oracle : Marks the states searching for
3. Diffuser : Raises the marked state probabi

Using 2 qubits



Input to diff oracle $= \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$

so we apply H gate

Oracle = Takes the state & stores it in Q2 for marking it.

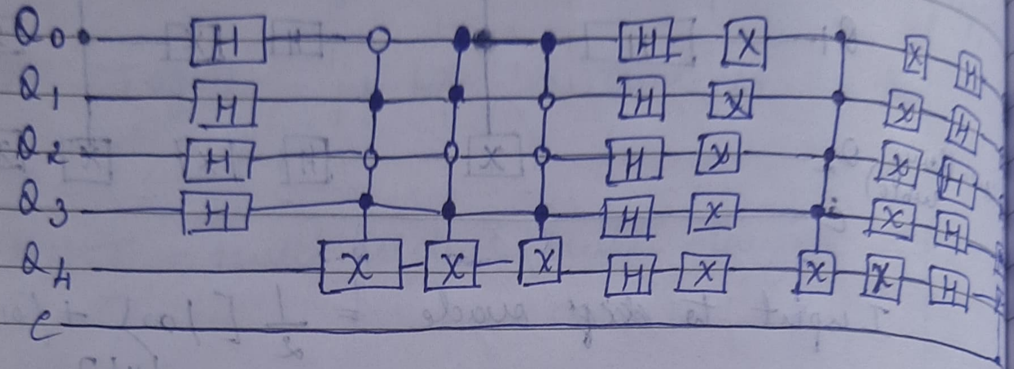
Diffuser = raises probability.

No. of qubits required for grover's search algo (N) = $\lceil \log_2(\#\text{ of states}) \rceil$

n (No. of times oracle & diffuser is applied) $= \lfloor \frac{\pi}{4}\sqrt{N} \rfloor$

Till 32 states i.e. upto 5 qubits, we apply oracle & diffuser only once

Q) Design a QC that implements Grover's search algo which searches for the states : $|0101\rangle$, $|1101\rangle$ & $|1001\rangle$



★ DEUTSCH JOSZA ALGORITHM

→ Consider a function box f, which can function in 2 modes : Const mode & Balanced mode.

Const mode | Balanced mode

a) $\left.\begin{matrix}|0\rangle \\ |1\rangle\end{matrix}\right\} = |0\rangle$

b) $\left.\begin{matrix}|0\rangle \\ |1\rangle\end{matrix}\right\} = |1\rangle$

c) $|0\rangle = |0\rangle$
$|1\rangle = |1\rangle$

d) $|0\rangle = |1\rangle$
$|1\rangle = |0\rangle$

→ If we make a superposition of inputs then for constant mode we get either $|0\rangle$ or $|1\rangle$ and in balanced mode, we get a superposition of $|0\rangle$ & $|1\rangle$.