Contents

1. <u>Properties of Expectation</u>

1.1. Linearity

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

1.2. Expectation of a Constant

$$\mathbb{E}[c] = c$$

1.3. Additivity

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

1.4. Multiplication by a Constant

$$\mathbb{E}[cX] = c\mathbb{E}[X]$$

1.5. Law of the Unconscious Statistician

$$\mathbb{E}[g(x)] = \sum_x g(x)P(X = x) \quad \textit{for discrete random variables}$$
$$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx \quad \textit{for continuous random variables}$$

1.6. Expectation of a Sum

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

2. <u>Properties of Conditional Expectation</u>

1. $Linear\ Operator: \mathbb{E}(X + Z|g) = \mathbb{E}(X|g) + \mathbb{E}(Z|g)$
2. $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(X|g)) = \mathbb{E}(X)$
3. $if\ g = \{\emptyset, \Omega\}, \mathbb{E}(X|g) = \mathbb{E}(X)$ $(no\ information)$
4. $if\ g = \mathcal{F}, \mathbb{E}(X|g) = \mathbb{E}(X|\mathcal{F}) = X$ $(full\ infomration)$
5. $if\ X \in g, \mathbb{E}(X|g) = X$
6. $if\ Z \in g, \mathbb{E}(ZX|g) = Z\mathbb{E}(X|g)$
7. $Tower\ Property\ (law\ of\ iteratied\ conditional\ expectation)$
   $g_0 \le g \quad \mathbb{E}(\mathbb{E}(X|g)|g_0) = \mathbb{E}(X|g_0)$
   $g_0 \le g \quad \mathbb{E}(\mathbb{E}(X|g_0)|g) = \mathbb{E}(X|g_0)$
8. $if\ X\ is\ independent\ of\ g \quad \mathbb{E}(X|g) = \mathbb{E}(X)$
9. $Jensen\ Inequality$
   $if\ f(\cdot)\ is\ convex, then\ f(\mathbb{E}(X|g)) \le \mathbb{E}(f(x)|g)$

3. <u>Properties of Variance</u>

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$
$$Var(cX) = c^2 Var(X)$$
$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + Cov(X,Y)$$
$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$
$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$$
$$Cov(X,Y) = 0\ if\ X\ and\ Y\ are\ independent$$

4. <u>Probability Density (Mass) Functions</u>

$$PDF_{Gaussian} = f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$PMF_{Poisson} = f_X(x) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$PMF_{Bernoulli} = f_X(x) = \begin{cases} q = 1 - p & if\ k = 0 \\ p & if\ k = 1 \end{cases}$$

$$PDF_{Chi-Squared} = f_X(x) = \frac{1}{\Gamma(x/2)} x^{(k/2)-1} e^{-x/2}$$

5. <u>Taylor Series</u>

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(c)(x-c)^n$$

6. <u>Moment Generating Function</u>

$$M_X(k) = \int_{-\infty}^{\infty} e^{kx} f_X(x)\, dx = \mathbb{E}[e^{kx}]$$

where
$k = k^{th}\ moment, x = random\ variable\ P_x = PDF\ of\ the\ randmom\ variable$

$$M_X(k) = \mathbb{E}[e^{kx}] = \mathbb{E}\left[1 + kx + \frac{1}{2!}k^2x^2 + \frac{1}{3!}k^3x^3 + \cdots\right]$$
$$= 1 + k\mathbb{E}[x] + \frac{1}{2!}k^2\mathbb{E}[x^2] + \frac{1}{3!}k^3\mathbb{E}[x^3] + \cdots \quad for\ \mu = 0$$

6.1. Gaussian Distribution

$$M_x(k) = e^{k\mu + \frac{1}{2}\sigma^2k^2}$$

6.2. Poisson Distribution

$$M_x(k) = e^{\lambda(e^k - 1)}$$

6.3. Bernoulli Distribution

$$M_x(k) = q + pe^k$$

6.4. Chi-Squared Distribution

$$M_x(k) = (1-2t)^{-k/2} \quad for\ 0 < t < \sqrt{e}$$

6.5. First Two Raw Moments of the Gaussian Distribution

$$\frac{\partial M_x(k)}{\partial k} = e^{k\mu + \frac{1}{2}\sigma^2k^2}(\mu + \sigma^2 k) = \mu = \mathbb{E}[X] \quad for\ k = 0$$

$$\frac{\partial^2 M_x(k)}{\partial^2 k} = (\mu + \sigma^2 k)^2 e^{k\mu + \frac{1}{2}\sigma^2k^2} + \sigma^2 e^{k\mu + \frac{1}{2}\sigma^2k^2}$$
$$= \mu^2 + \sigma^2 = \mathbb{E}^2[X] + Var(X) \quad for\ k = 0$$

7. <u>Stochastic Process Definitions</u>

7.1. Martingale

$A\ stochastic\ process\ X = (X_t : t \ge 0)\ is\ a\ martingale\ relative\ to\ (\mathbb{F}, \mathbb{P})\ if:$
1. $\mathbb{E}|X_t| < \infty \quad \forall\, t \ge 0;$
2. $\mathbb{E}(X_t|\mathbb{F}_s) = X_s \quad \forall\, s \le t$

therefore:
$$\mathbb{E}(X_t - X_s|\mathbb{F}_s) = 0 \quad \forall\, s \le t$$

7.2. Brownian Motion

$The\ process\ W := \{W_t : t \ge 0\}\ on\ (\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$
$is\ called\ a\ standard\ brownian\ motion\ if:$
1. $W_0 = 0$
2. $W_t - W_s$
   $is\ independent\ of\ the\ past\ history\ of\ W\ unitl\ time\ s\ \forall\, s \le t$
3. $W_t - W_s \overset{D}{\Leftrightarrow} W_{t-s} \sim N(0, t-s) \quad \forall\, 0 \le s \le t$
4. $W\ has\ continuous\ sample\ paths$
5. $W_t \overset{D}{\Leftrightarrow} Z\sqrt{t} \quad Z \sim N(0,1)$
6. $tW_{\frac{1}{t}} = W_t$

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5.2 Itô processes and stochastic calculus

Definition 37 (Itô processes) *Let W be a one dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)$. A one dimensional Itô process is a stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}_t)$ of the form*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \tag{15}$$

where X_0 is a $\mathbb{F}_0$-measurable random variable, b_t is a $\mathbb{F}_t$-adapted process such that

$$\int_0^t |b_s|\, ds < \infty\ a.s.\ \forall t \ge 0,$$

σ_t is a $\mathbb{F}_t$-adapted process such that

$$\int_0^t |\sigma_s|^2\, ds < \infty\ a.s.\ \forall t \ge 0.$$

The shorthand notation for the above is

$$dX_t = b_t dt + \sigma_t dW_t,$$

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

$$W_t = \rho X_t + \sqrt{1 - \rho^2} B_t$$

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The shorthand notation for the above is

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8. Pricing, Risk, & Hedging

8.1. Put-Call Parity

$$C_0 + ke^{-r_{\Delta t}} = P_0 + S_0$$

8.2. Binomial Option Pricing

8.2.1. Cox-Ross-Rubinstein (1979)

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = 1/u$$

8.2.2. Log-Normal Tree (Jarrow and Rudd, 1983)

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}}$$
$$d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}$$

8.2.3. Risk Neutral Probability

$$\hat{p}_{UP} = \frac{e^{(rT)} - d}{u - d}$$
$$\hat{p}_{DOWN} = \frac{u - e^{(rT)}}{u - d} = 1 - \hat{p}_{UP}$$

8.2.4. Binomial Option Pricing Payoffs

$$C_N^i = max\left(S_0 u^{N-i} d^i - K, 0\right)$$
$where\ N = periods\ to\ maturity, i = state\ 0 \leq i \leq N,$
$u = up\ movement, d = down\ movement$

$$P_N^i = max\left(k - S_0 u^{N-i} d^i, 0\right)$$
$where\ N = periods\ to\ maturity, i = state\ \forall\ 0 \leq i \leq N,$
$u = up\ movement, d = down\ movement$

8.2.5. European Options Risk Neutral Valuation (2 Period)

$$C_0 = e^{-r_{\Delta t}N}\left(\hat{p}^2 C_N^0 + 2\hat{p}(1-\hat{p})C_N^1 + (1-\hat{p})^2 C_N^2\right)$$

8.2.6. European Options Risk Neutral Valuation (No Dividends)

$$C_0, P_0 = e^{-r_{\Delta t}N} \sum_{i=0}^{N} \left(\left(C_N^i, P_N^i\right) \binom{N}{i} \hat{p}^{N-i} (1-\hat{p})^i\right)$$
$where\ \binom{N}{i} = \frac{N!}{i!\,(N-i)!},\ \ N = periods\ to\ maturity,$
$i = state\ 0 \leq i \leq N,\ \ k = strike\ price$

8.3. Hedging with Derivatives

8.3.1. Hedging a Long Position with Futures

$$S_0 \cong S_1 + (F_0 - F_1) = F_0 + (S_1 - F_1) = F_0 - basis \quad for\ F_t \cong S_t$$

8.3.2. Hedging a Short Position with Futures

$$S_0 \cong S_1 - (F_1 - F_0) = F_0 + (S_1 - F_1) = F_0 - basis \quad for\ F_t \cong S_t$$

8.3.3. Minimum Variance Hedge Ratio

$$N_F = -\frac{V_p}{V_F}\beta_p \quad where\ \beta_p = \frac{\sigma_{R_p,R_m}}{\sigma^2_{R_m}}$$

8.3.4. Leveraging Beta

$$N_F = \frac{V_p}{V_F}\left(\beta_d - \beta_p\right) \quad where\ \beta_p = \frac{\sigma_{R_p,R_m}}{\sigma^2_{R_m}},\ \beta_d = desired\ \beta$$

8.3.5. Single Index Model (SIM)

$$R_p = \alpha + \beta_p R_m + \varepsilon_p$$

8.3.6. Capital Asset Pricing Model (CAPM)

$$R_p - r = \alpha + \beta_p(R_m - r) + \varepsilon_p$$

8.3.7. Risk

$$\beta_p R_m, \beta_p(R_m - r) = Market\ (Systematic)\ Risk$$
$$\varepsilon_p = Specific\ (Idiosyncratic)\ Risk$$

9. Linear Regression Analysis

9.1. Assumption 1

$$E(\varepsilon) = 0$$

$$E(\varepsilon\varepsilon') = Var(\varepsilon) = \sigma^2 I = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

9.2. Assumption 2

$$E(X'\varepsilon) = 0$$

9.3. Assumption 3

$$rank(X'X) = rank(x) = k$$

9.4. Ordinary Least Squares Estimator

$$\hat{\beta} = (X'X)^{-1}X'Y$$
$$(X'X)^{-1}X' = P_x : R^T \to R^K$$
$$\hat{\beta} = P_x Y$$