Discretization of the Heston Functional

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Abstract

In this paper, we explore a discretization of the Heston (1993) Stochastic Differential Equations (SDEs) for a mean reverting process with arbitrary correlation between level and volatility. We derive an analytical solution whose computation can be proxied via the trapezoidal properties of an the integrated function.

We begin with the classical Heston (1993) equations for stochastic volatility:

$$dX_t = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t}\left(\rho dW_t + \sqrt{1-\rho^2} dB_t\right) \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t \tag{2}$$

Integrating the process allows for an approximate discretization via the trapezoidal property. Assuming we know the initial variance and spot price, we can simulate the process by integrating (1) between time 0 and t:

$$\int_0^t dX_t = \int_0^t \left(r - \frac{v_s}{2}\right) ds + \int_0^t \sqrt{v_s}\left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right).$$

Distributing terms:

$$X_t - X_0 = \int_0^t r ds - \int_0^t \frac{v_s}{2} ds + \rho \int_0^t \sqrt{v_s} dW_s + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s} dB_s \tag{3}$$

To further simplify the simulation, we can eliminate the W_t Brownian motion by integrating (2) as we did for (1):

$$\int_0^t dv_s = \alpha\beta \int_0^t ds - \alpha \int_0^t v_s ds + \eta \int_0^t \sqrt{v_s} dW_s$$

$$v_t - v_0 = \alpha\beta t - \alpha \int_0^t v_s ds + \eta \int_0^t \sqrt{v_s} dW_s.$$

Rearranging terms:

$$\int_0^t \sqrt{v_s} dW_s = \frac{1}{\eta}\left(v_t - v_0 - \alpha\beta t + \alpha \int_0^t v_s ds\right). \tag{4}$$

Substituting (4) into (3):

$$X_t = X_0 + \int_0^t r ds - \int_0^t \frac{v_s}{2} ds + \frac{\rho}{\eta}\left(v_t - v_0 - \alpha\beta t + \alpha \int_0^t v_s ds\right) + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s} dB_s.$$

Solving to get:

$$X_t = X_0 + rt + \frac{1}{2}\int_0^t v_s ds + \frac{\rho}{\eta}\left(v_t - v_0 - \alpha\beta t\right) + \frac{\rho\alpha}{\eta}\int_0^t v_s ds + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s} dB_s \tag{5}$$

with $\int_0^t v_s ds | \mathcal{F}_{t,t+\delta}$ approximated as $(\varepsilon_1 \hat{v}_0 + \varepsilon_2 \hat{v}_t)\, t$ via the trapezoidal property and $\int_0^t \sqrt{v_s} dB_s$ a scaled *Brownian Motion*.