國立臺灣大學

Function Generation

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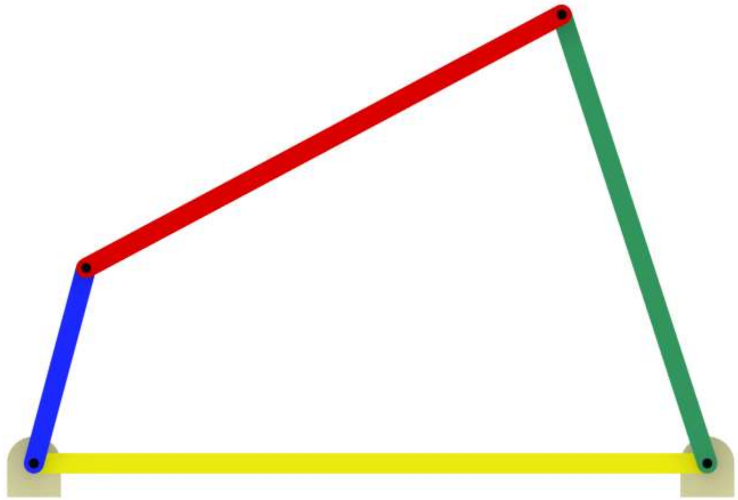
Review: Inversion

- **Inversion** refers to the process of considering the fixed link to be free to move and another link of the mechanism to be the fixed link.



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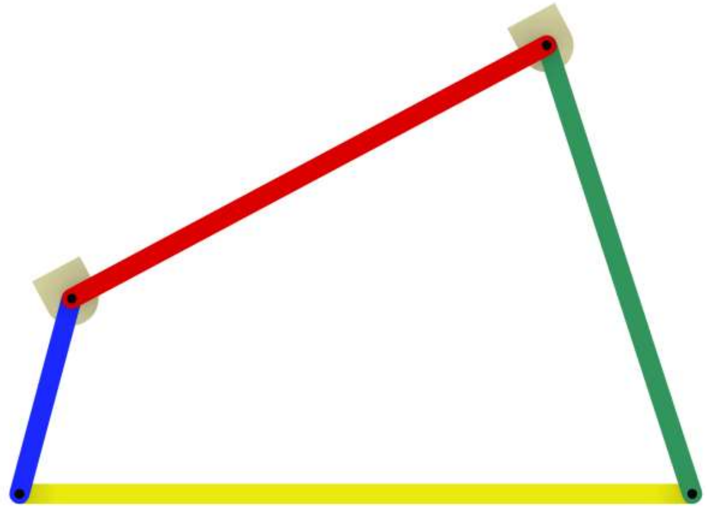
- This is a process which results in a completely different mechanism made up of the same links as the original mechanism.



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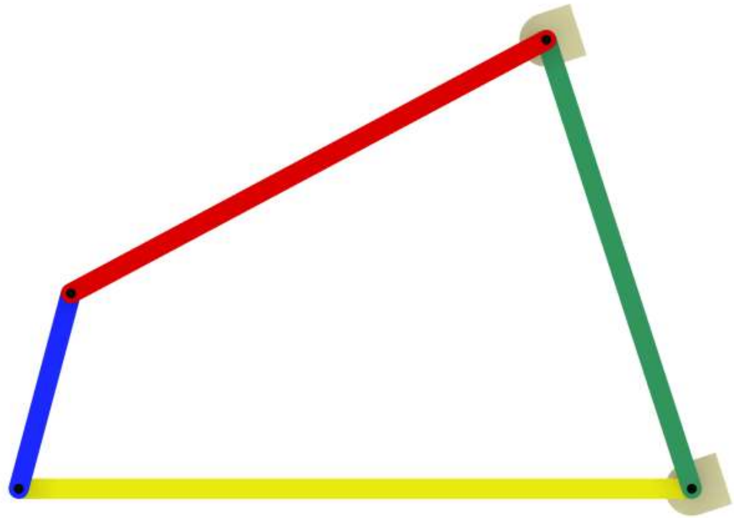
- Since the lengths of the link do not change, the relative rotation between the links is unchanged.



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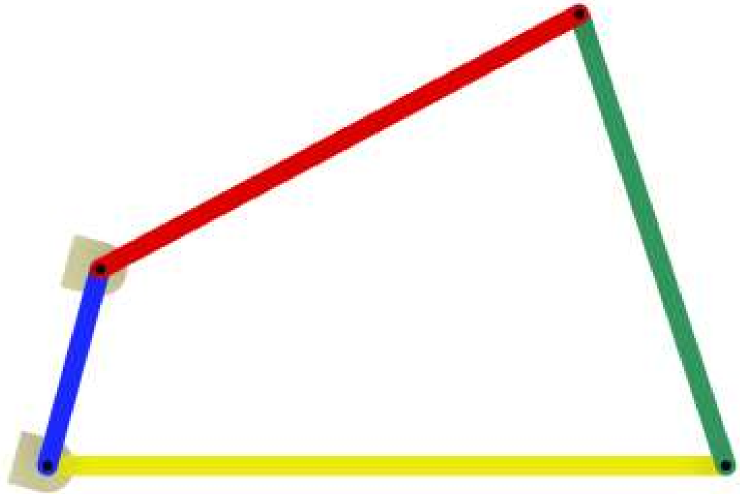

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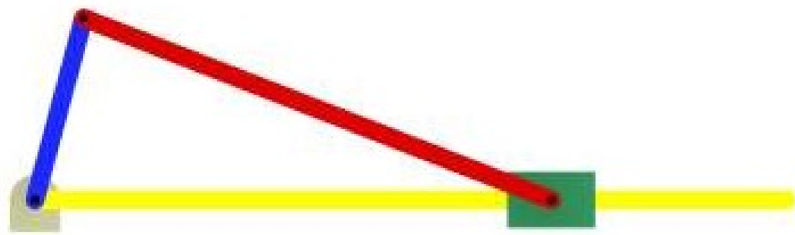
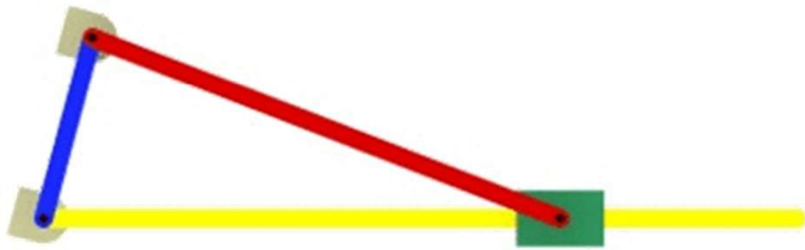
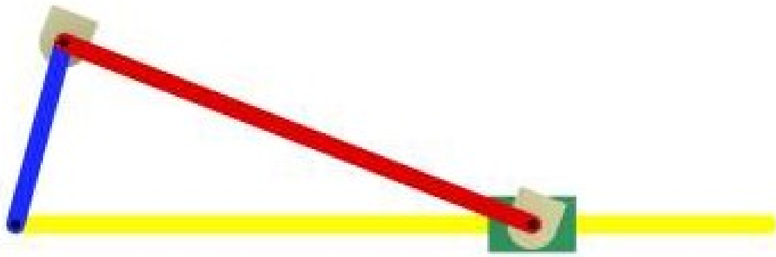
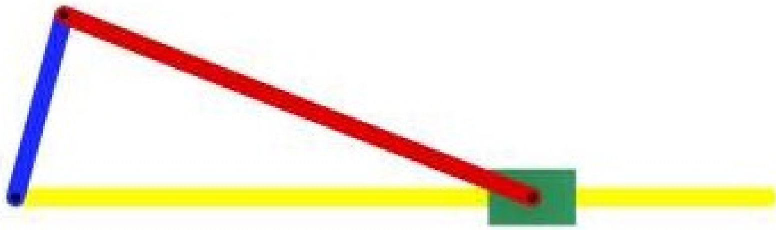
- The significant feature of the inversion technique is that the relative motion between the links is unchanged by the inversion process.









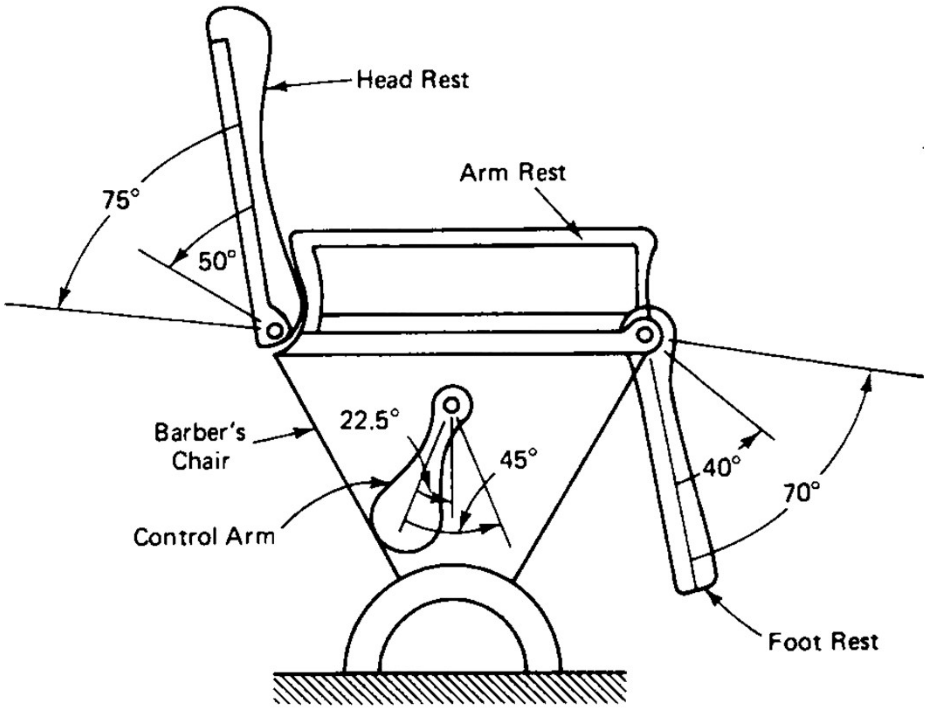
Dimensional Synthesis

Function Generation

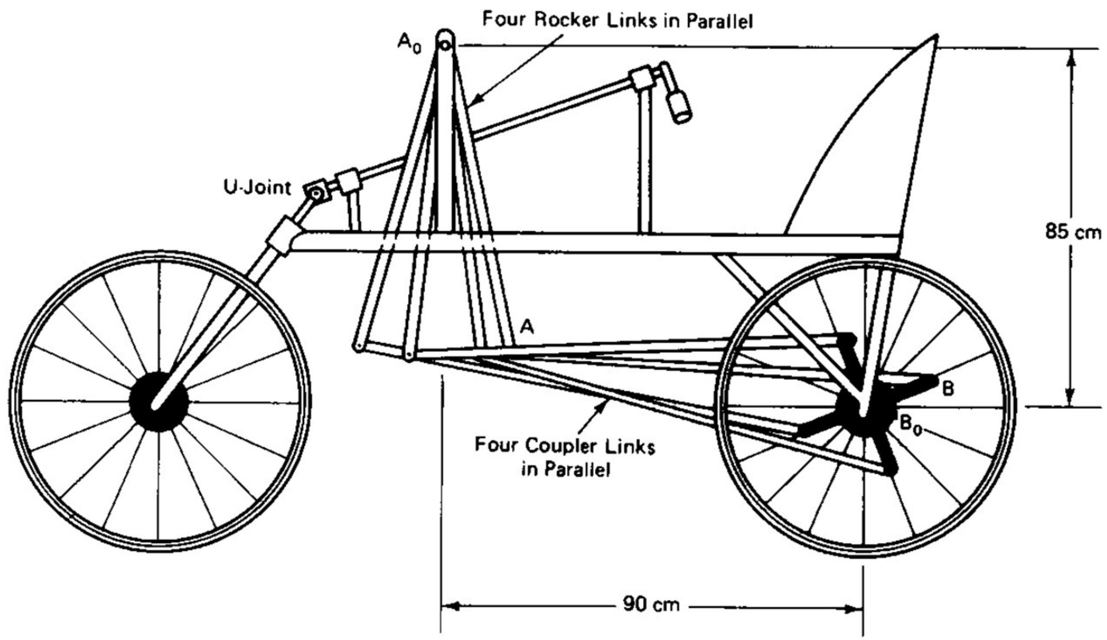
Example: Function Generation

- Several specified sets of coordinated input and output link motion are required to be generated.

Reclining chair mechanism



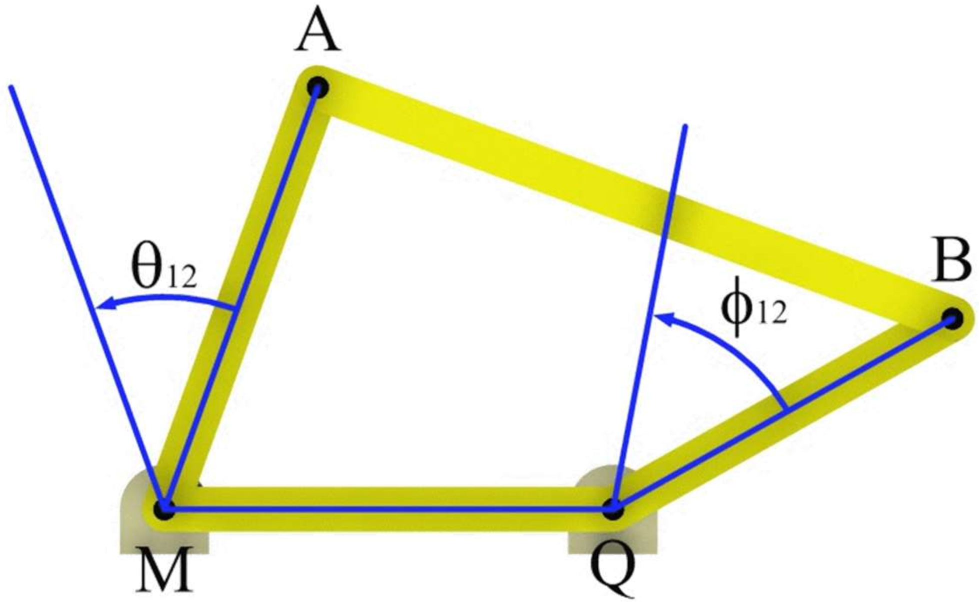
A leg-driven recreational vehicle



Two-Position Synthesis

Two-Position Synthesis

- Design a four-bar linkage for which the output link will rotate through an angle θ_{12} when input link rotates through an angle ϕ_{12}.



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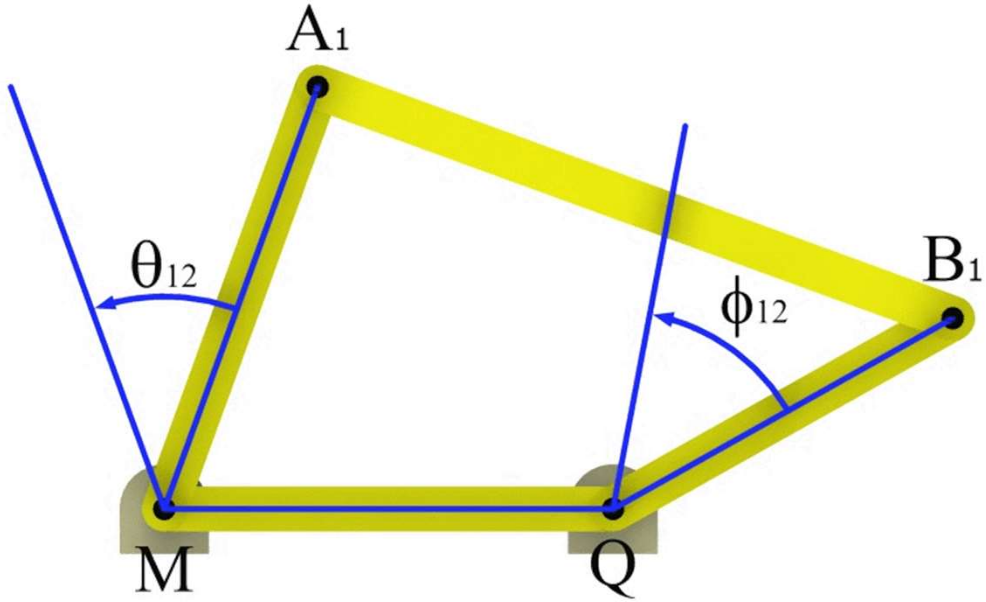
Graphical Solution

Two-Position Synthesis

- Observed from the ground, or link MQ, points A and B appear to move from Position 1 to Position 2 through their respective angles θ_{12} and ϕ_{12}.



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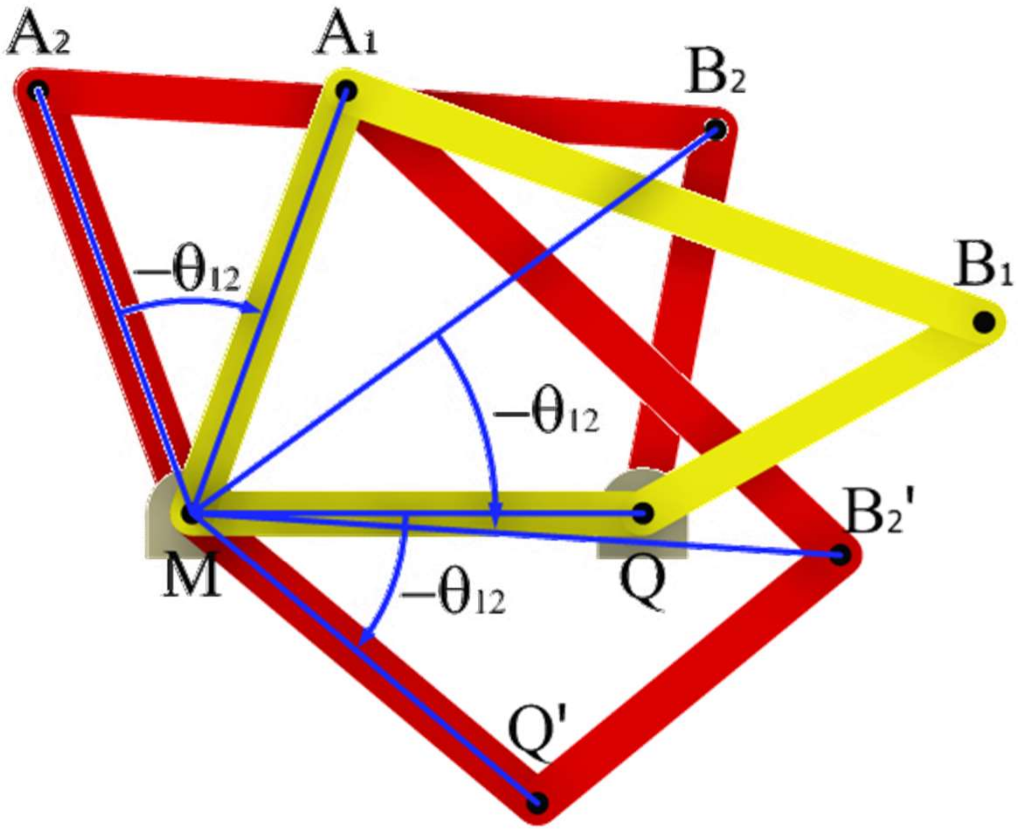
Graphical Solution

Two-Position Synthesis

- If link MA is the reference, then link MA appears to be stationary, and the other links, including the frame, appear to move relative to link MA in the direction of $-\theta_{12}$.



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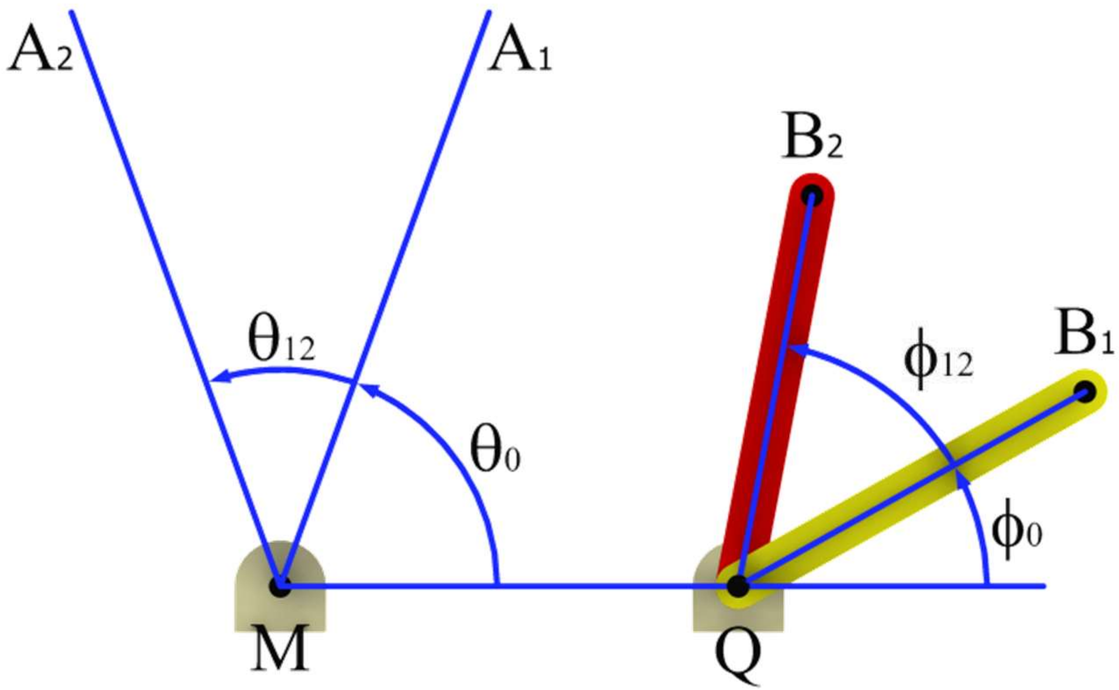
Graphical Solution

Two-Position Synthesis

- Suppose MQ and QB are already given.
- Draw rays corresponding to MA_1 and MA_2.



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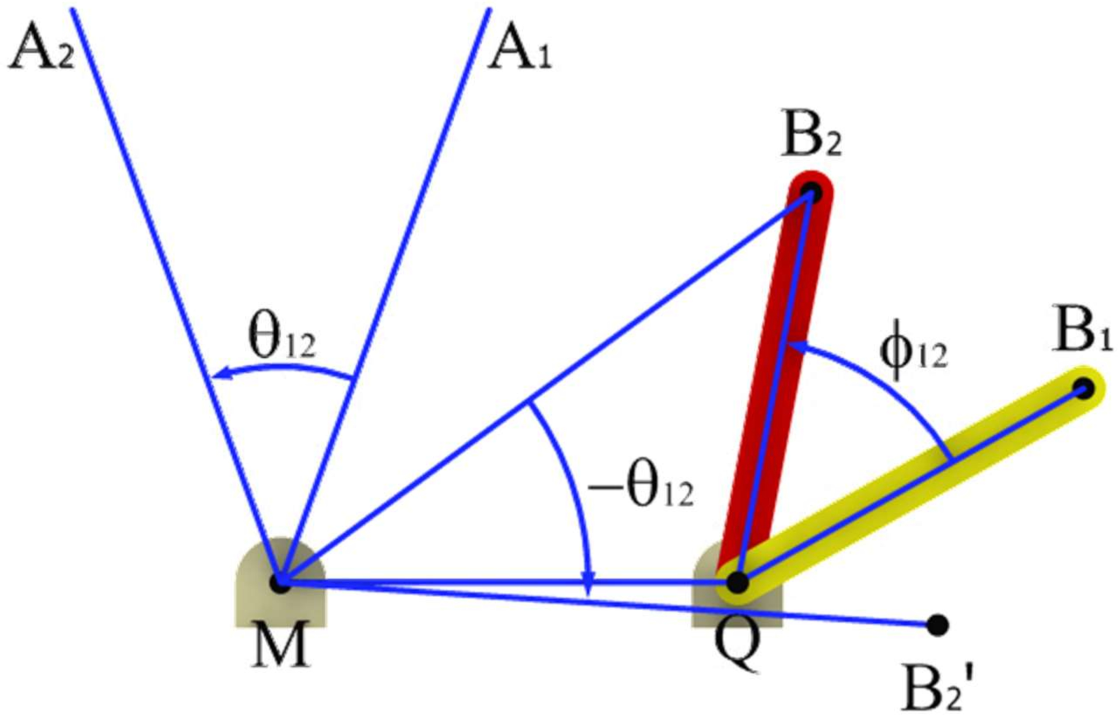
Graphical Solution

Two-Position Synthesis

- Draw a line MB_2 and rotate it by the angle $-\theta_{12}$ about the pivot M. This will locate B_2', which is where B_2 would appear to be if the observer were on link MA.



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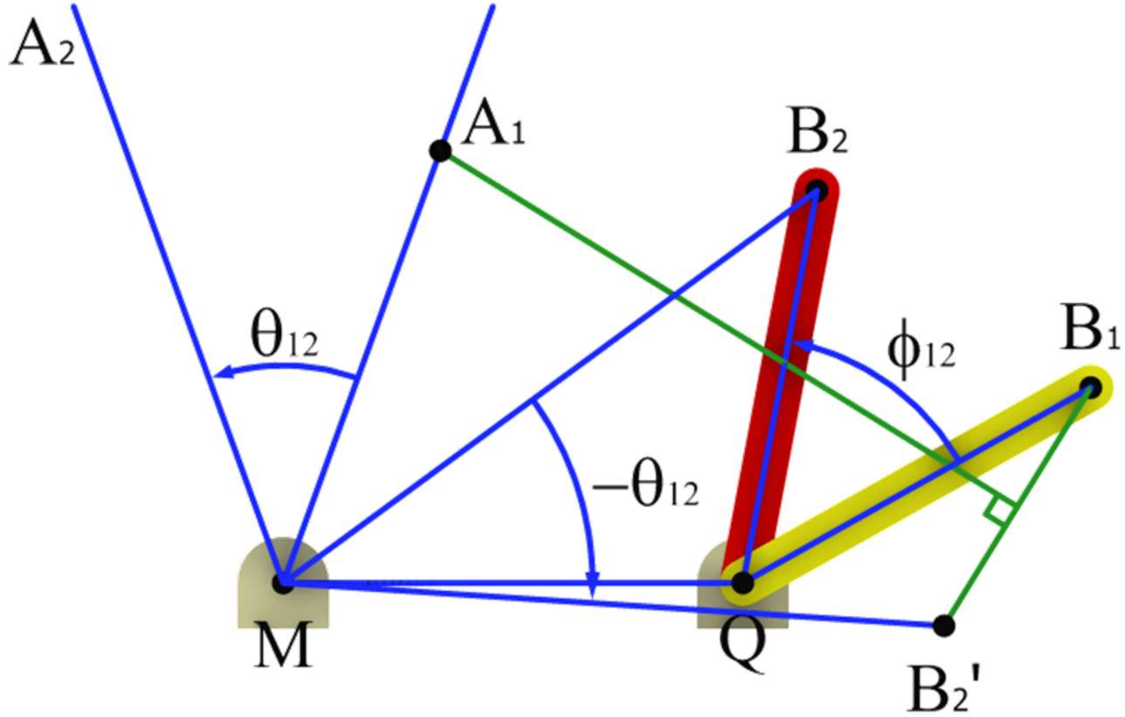
Graphical Solution

Two-Position Synthesis

- Relative to the input link MA in position 1, point B appears to rotate on a circular arc about A_1 as B travels from B_1 to B_2'. Therefore, A_1 must lie on the perpendicular bisector of the line segment B_1B_2'.



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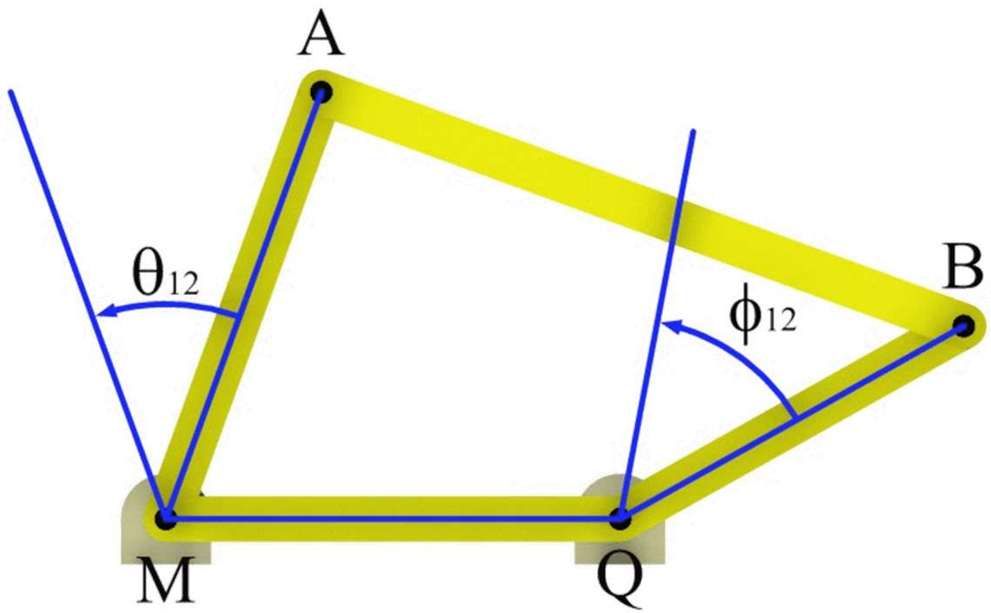
Graphical Solution

Two-Position Synthesis

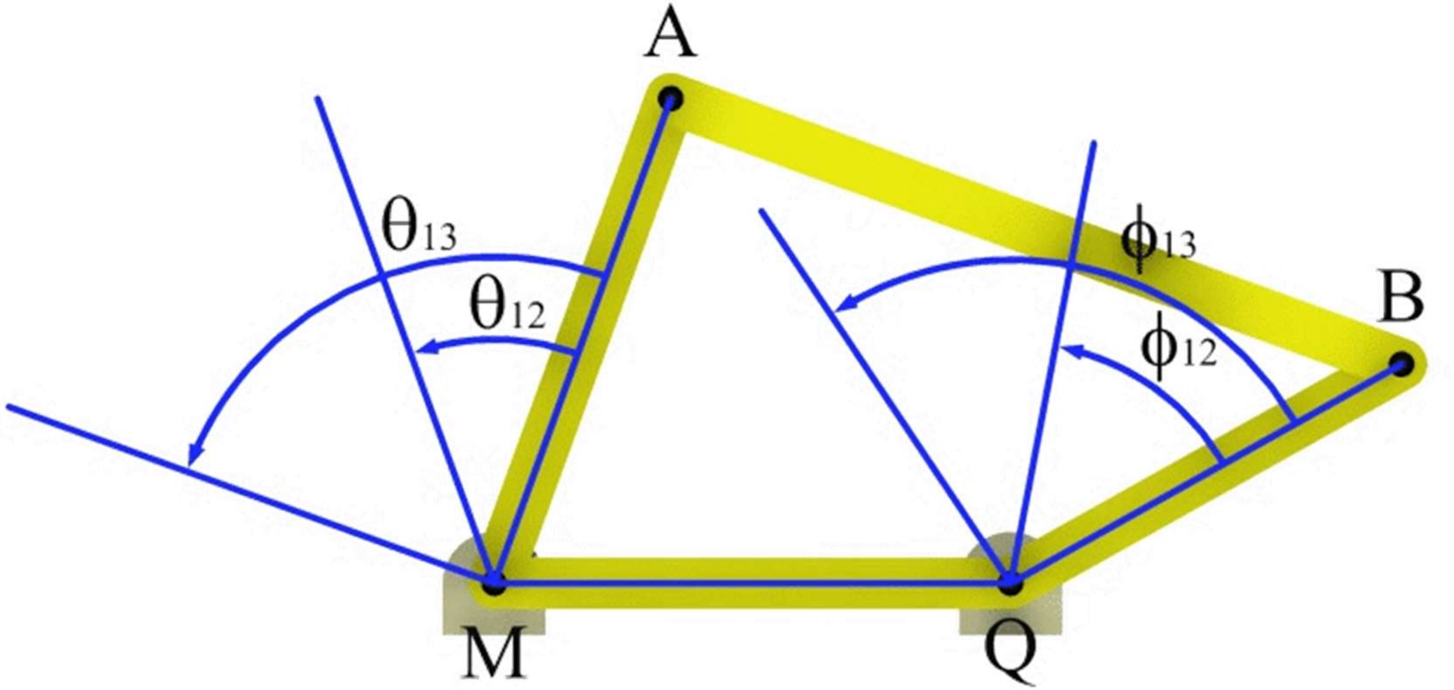
- Once A_1 is determined, the lengths of the input rocker and of the coupler will be known. The input rocker length is MA_1 and the coupler length is A_1B_1. The solution to this problem makes use of **the concept of inversion**.



Three-Position Synthesis

Graphical Solution

Three-Position Synthesis



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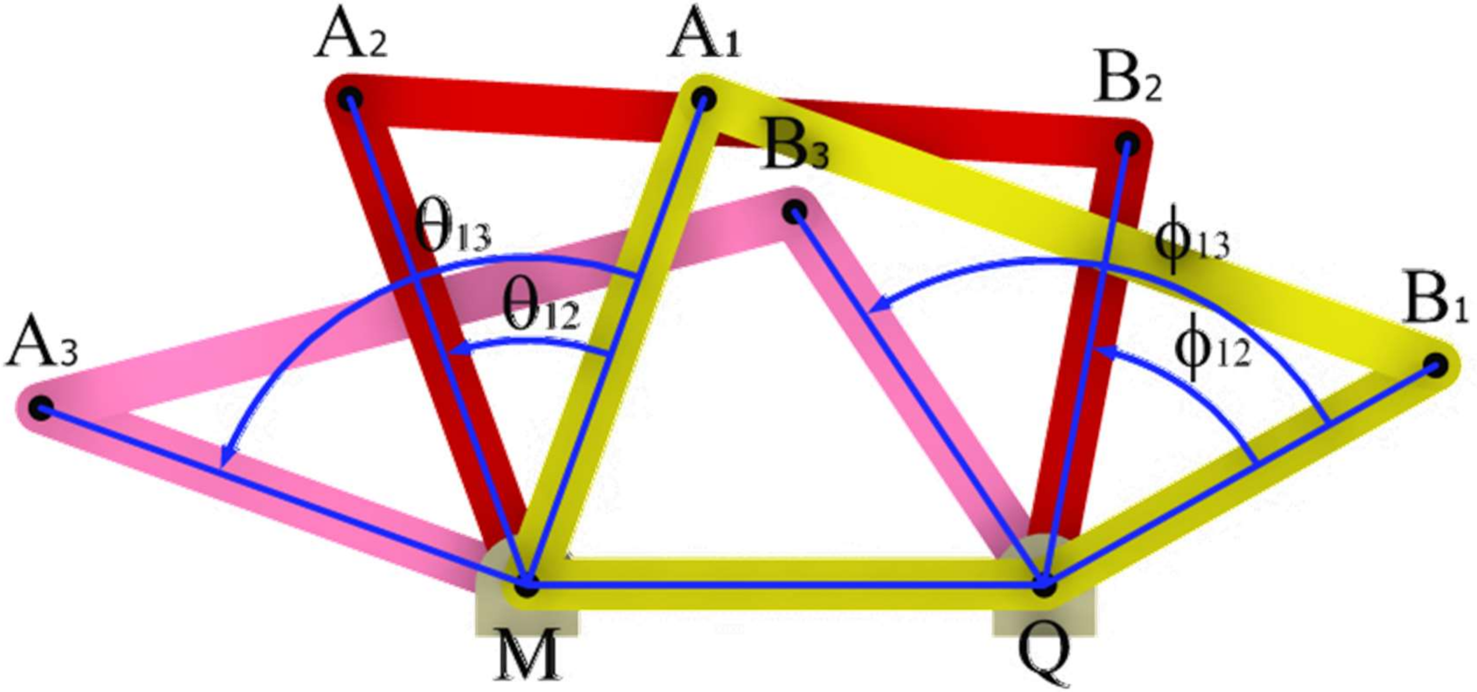
Graphical Solution

Three-Position Synthesis



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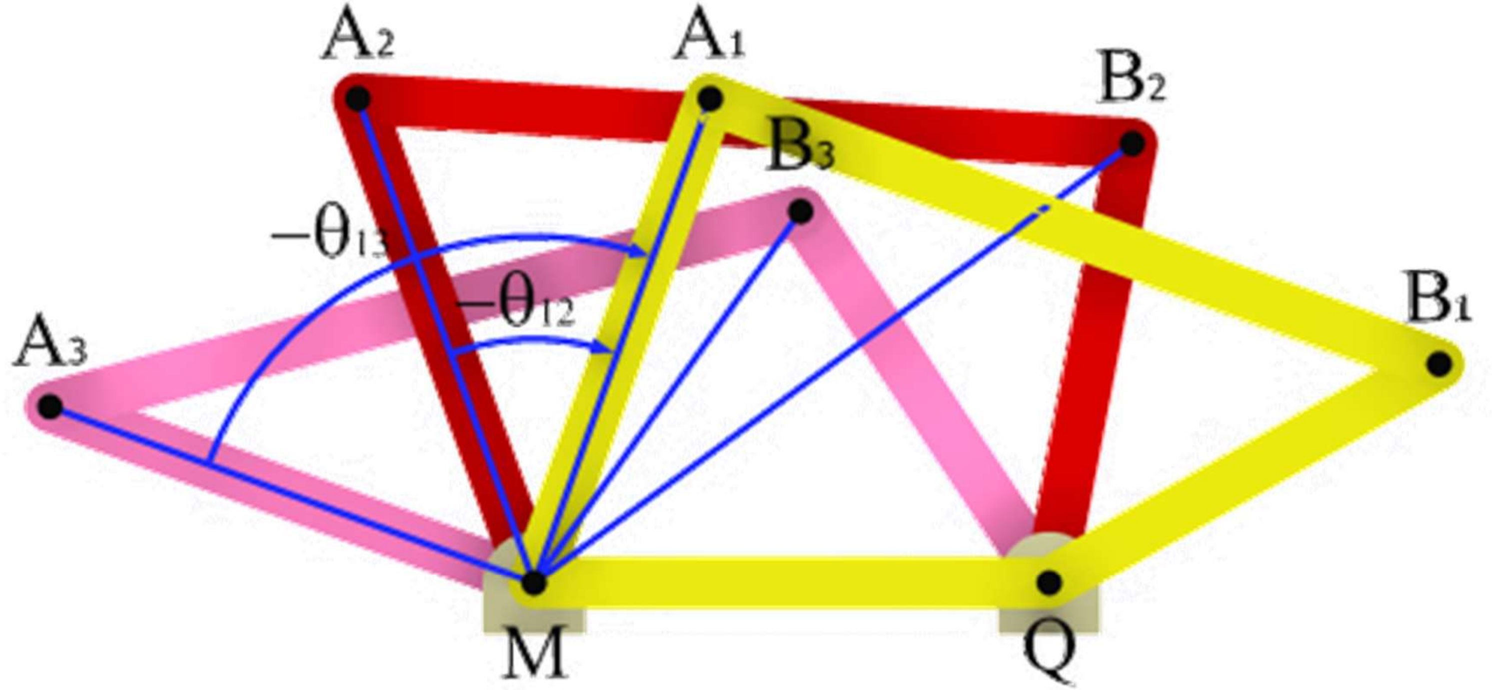
Graphical Solution

Three-Position Synthesis



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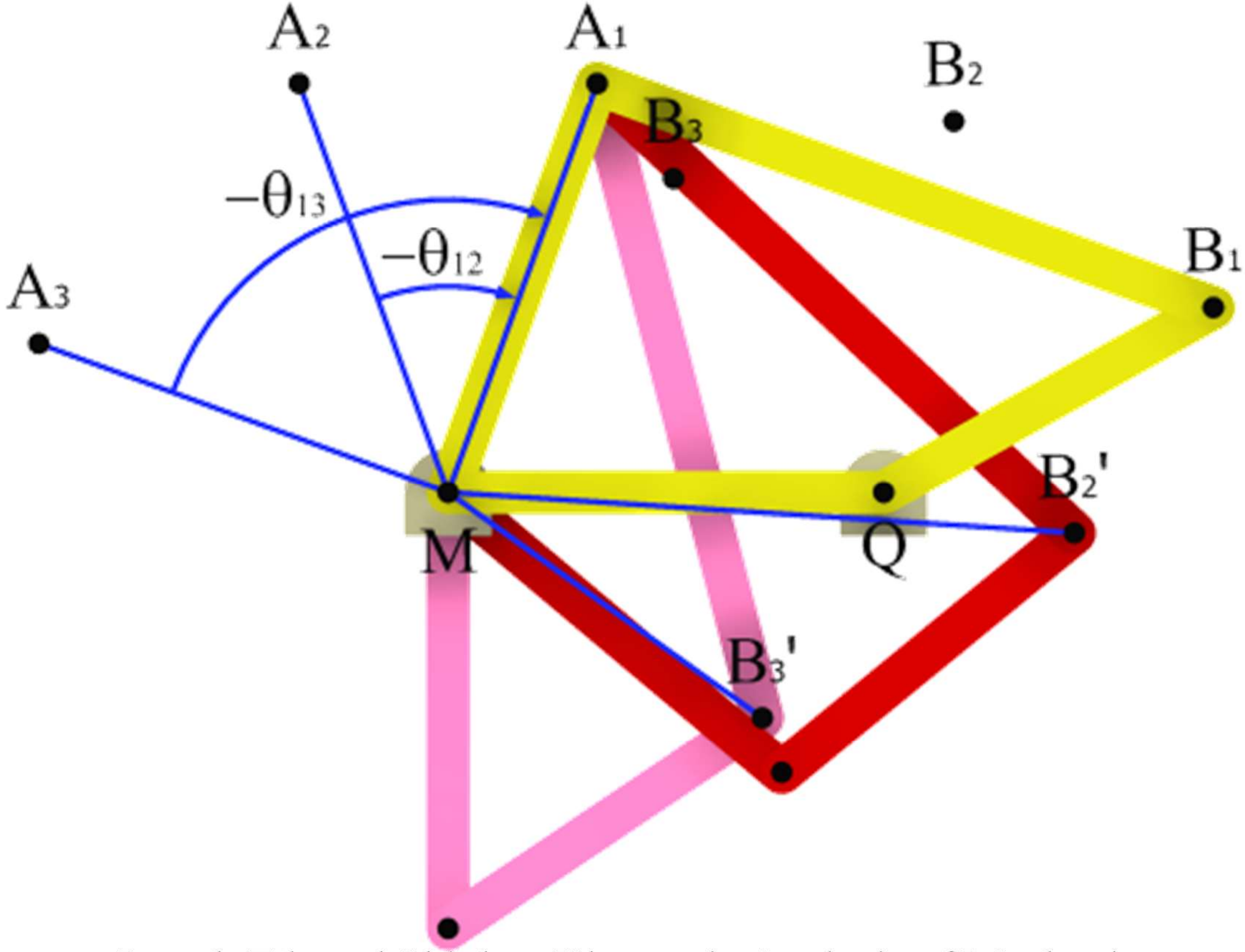
Graphical Solution

Three-Position Synthesis



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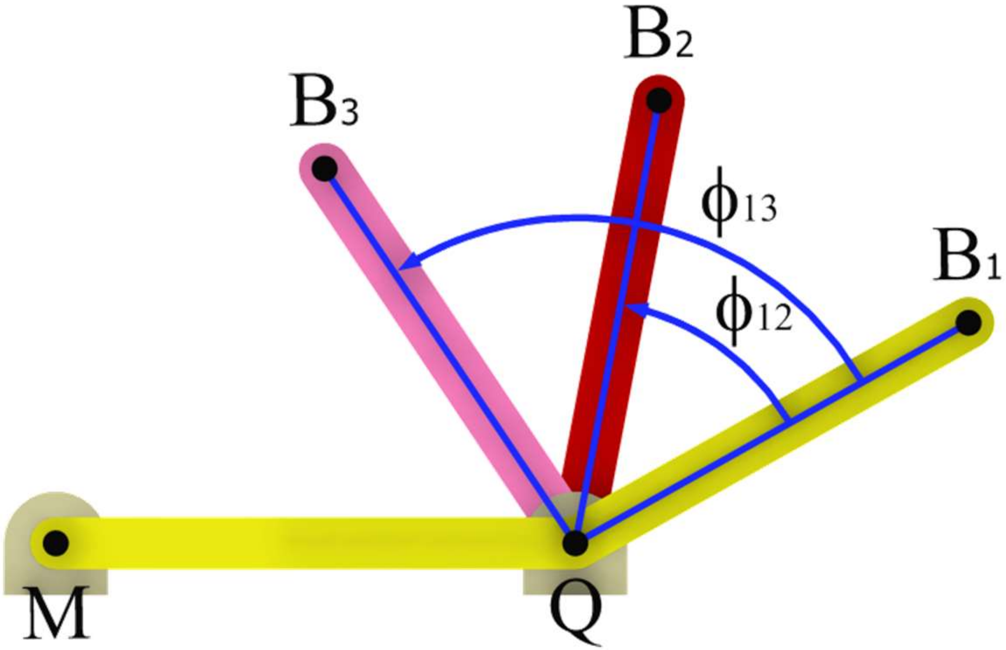
Graphical Solution

Three-Position Synthesis



- This solution is based on inverting the linkage on link MA. Draw the output rocker QB in three positions and locate a desired position for M.

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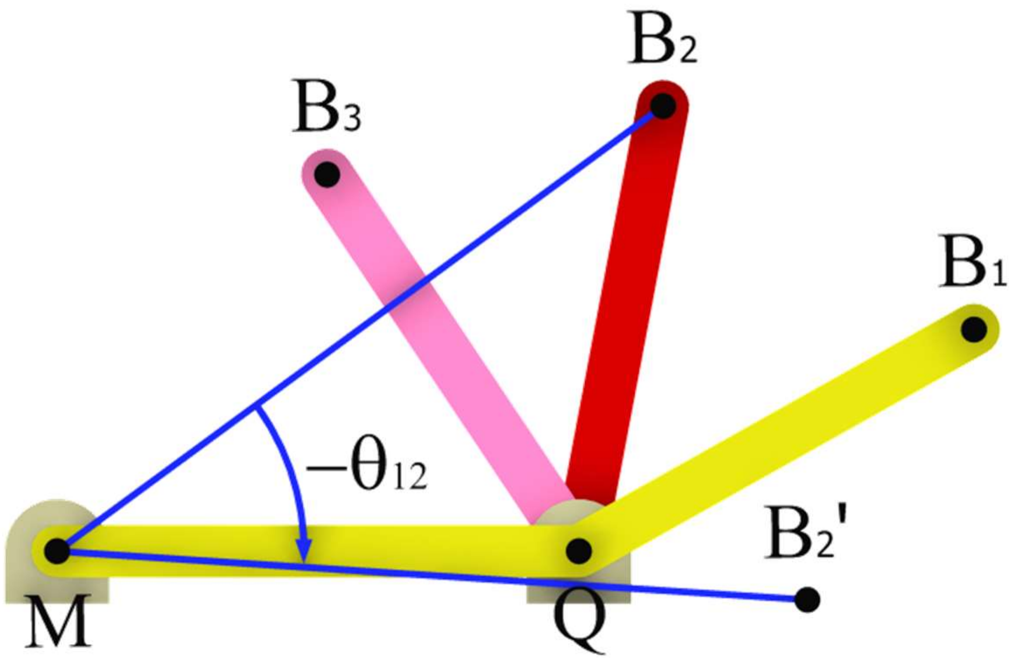
Graphical Solution

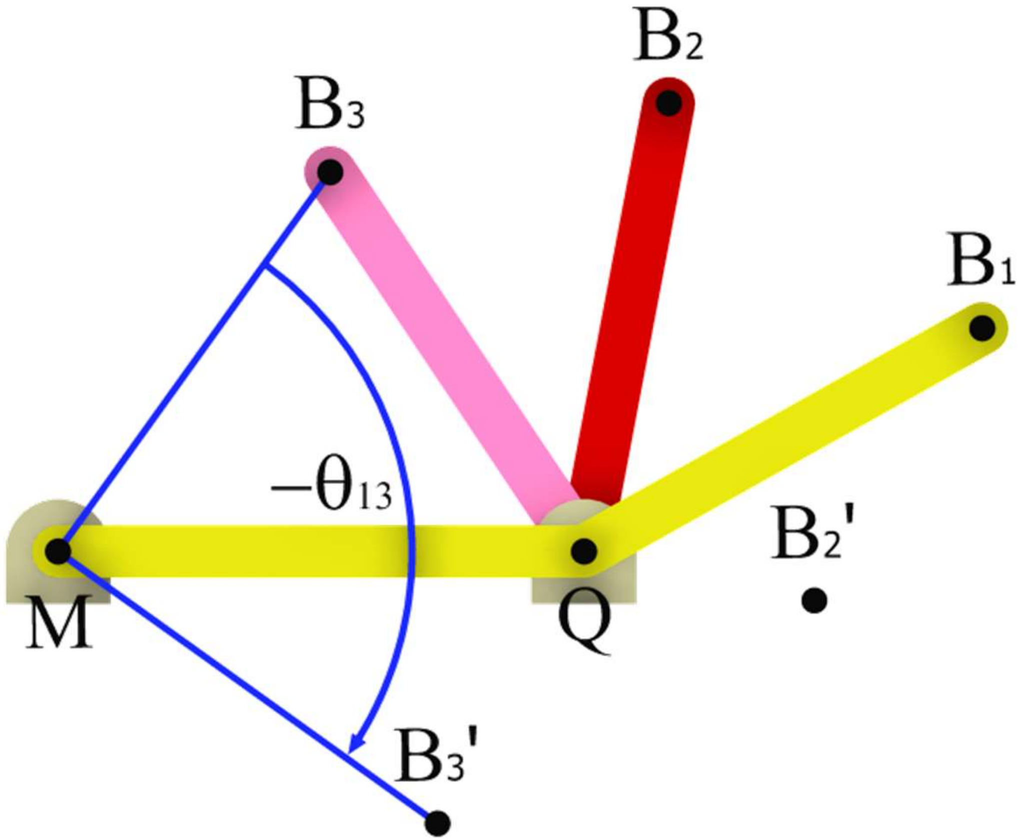
Three-Position Synthesis



- Since we shall invert link MA in the first design position, draw a ray from M to B_2 and rotate it backward through the angle θ_{12} to locate B_2'.

Graphical Solution

Three-Position Synthesis



- Similarly, draw another ray MB_3 and rotate it backward through the angle θ_{13} to locate B_3'.

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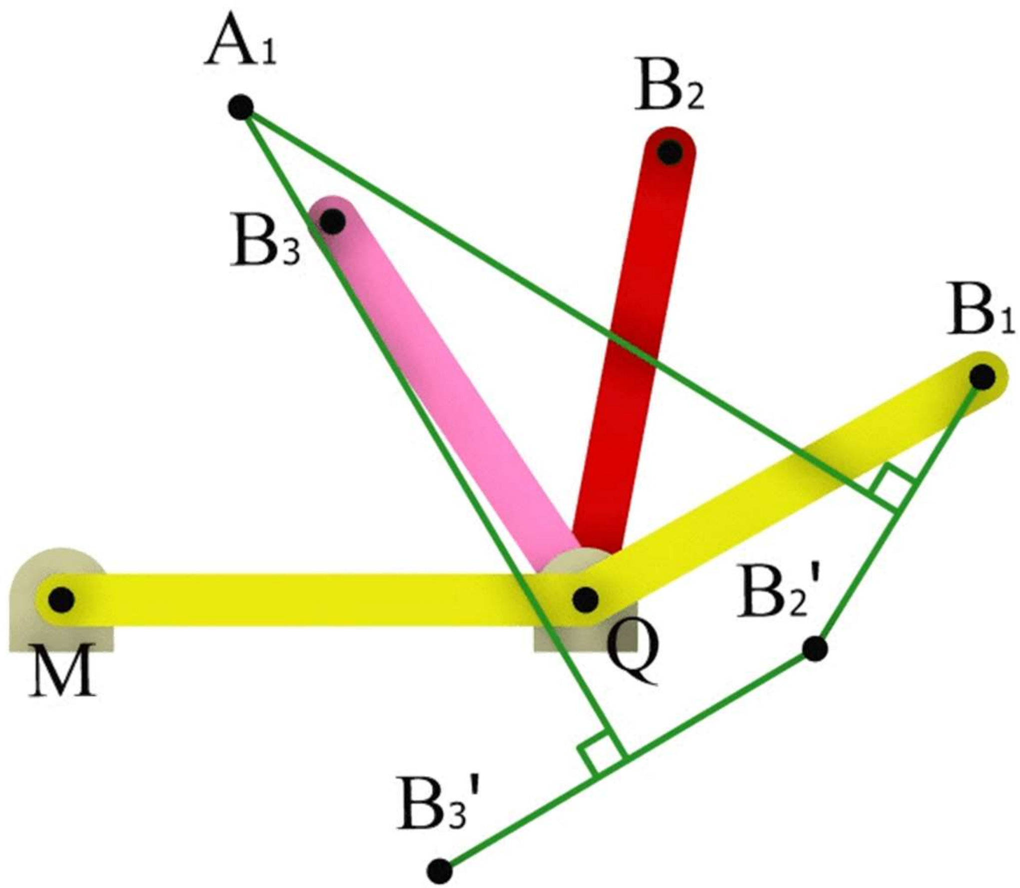
Graphical Solution

Three-Position Synthesis



- Now draw midnormals to the lines B_1B_2' and B_2'B_3. These intersect at A_1 and define the length of the coupler AB and the length and starting position of MA.

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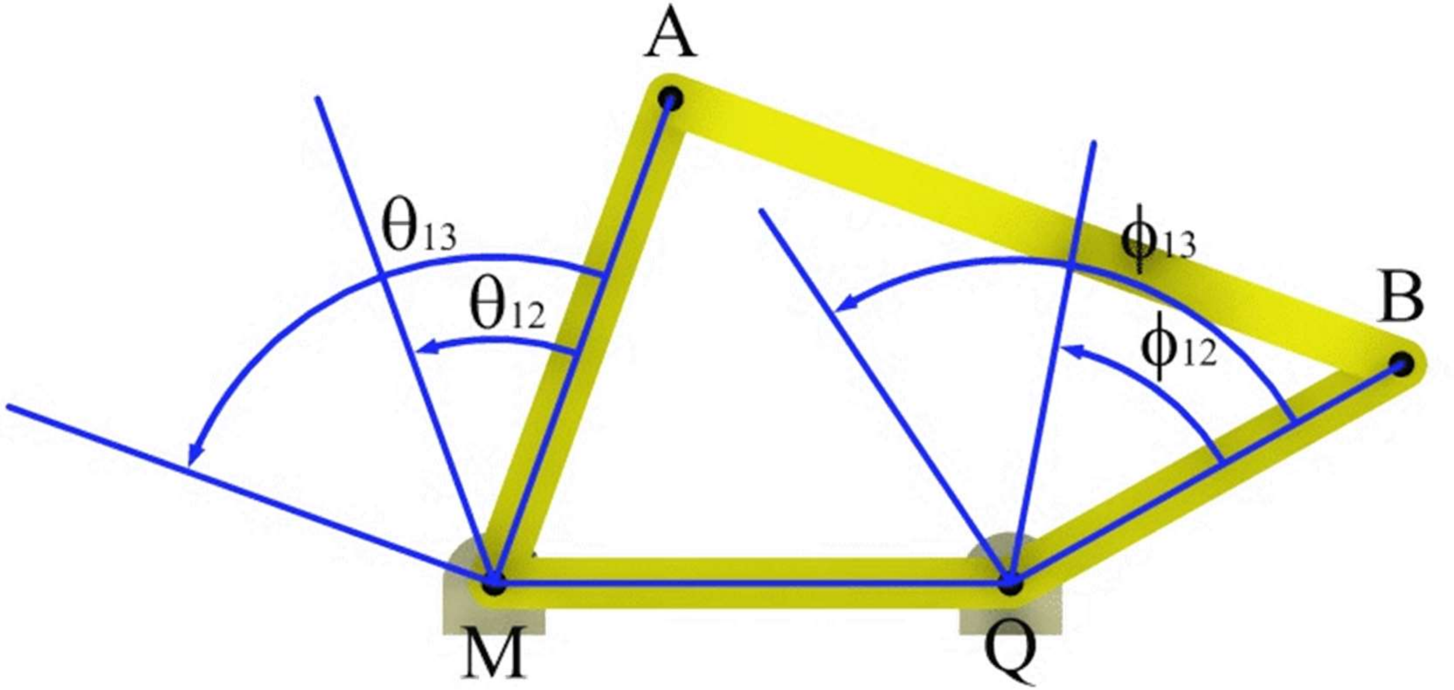
Graphical Solution

Three-Position Synthesis



- Now draw midnormals to the lines B_1B_2' and $B_2'B_3$. These intersect at A1and define the length of the coupler AB and the length and starting position of MA.

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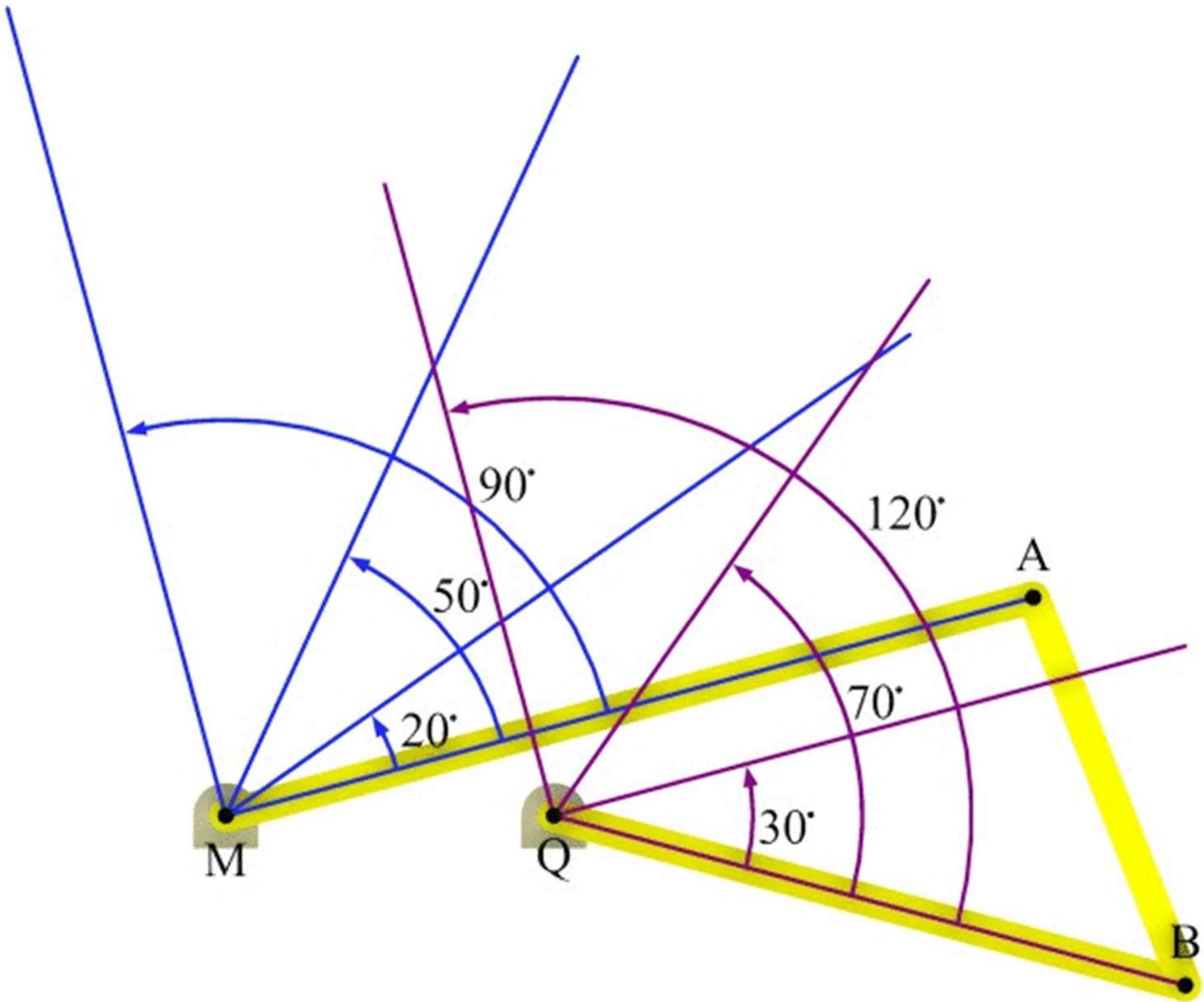
Four-Position Synthesis

Graphical Solution

Four-Position Synthesis



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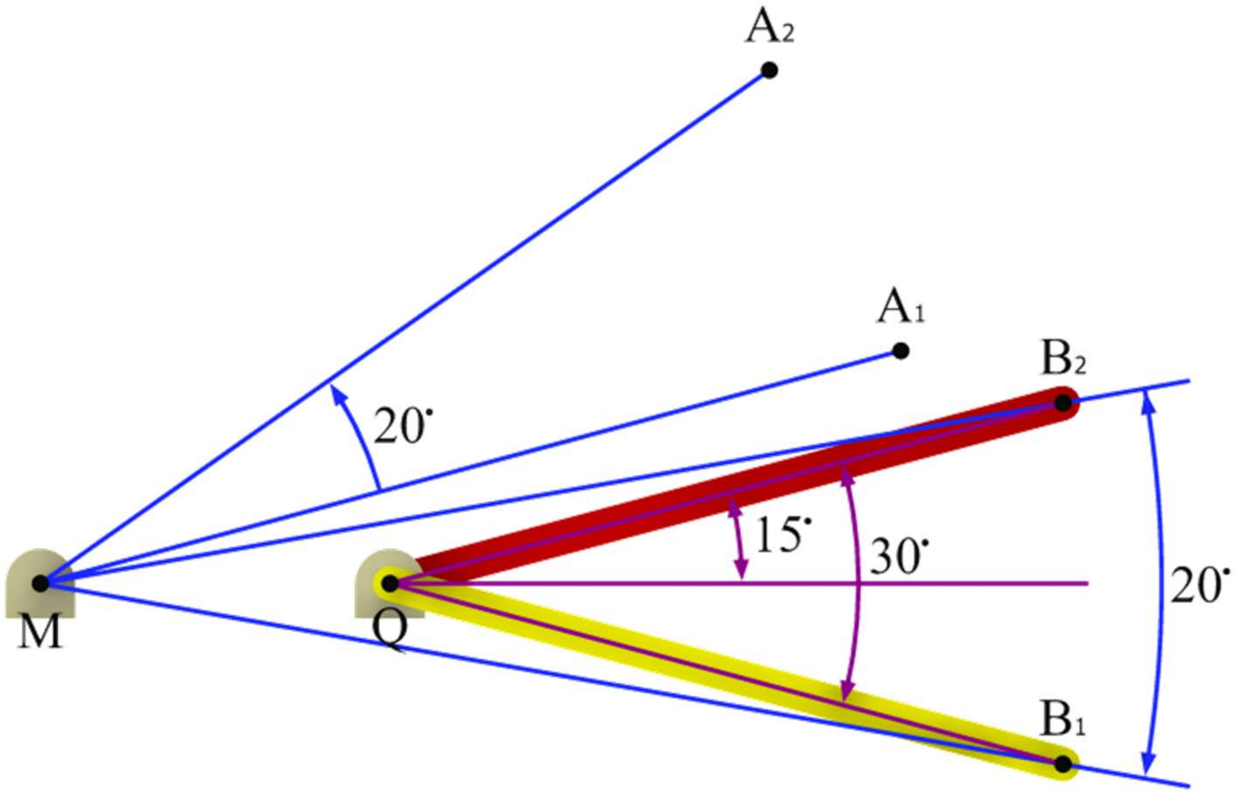
Graphical Solution

Four-Position Synthesis

- **Point-Position Reduction**: In point-position reduction linkage is made symmetrical about the frame centerline so as to cause two of the B' points to be coincident. The effect of this is to produce three equivalent B' points through which a circle can be drawn as in three-position synthesis.



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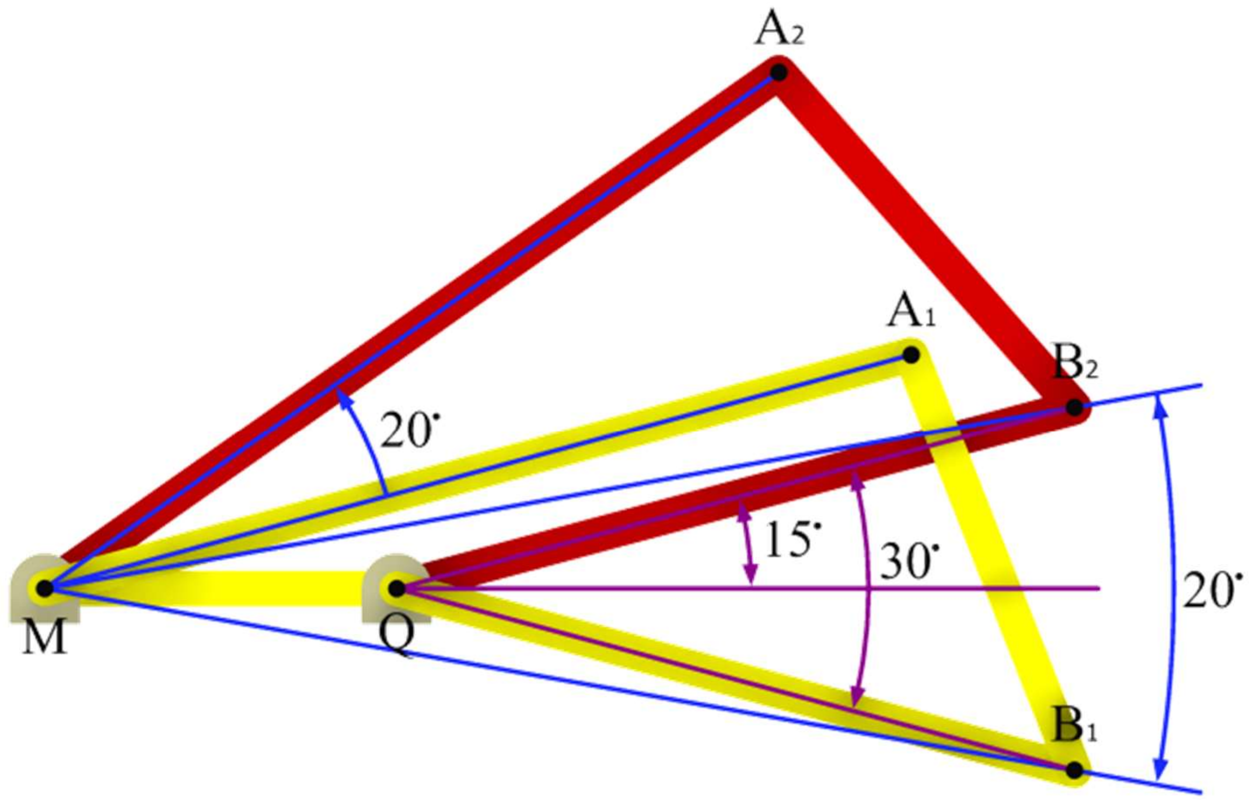
Graphical Solution

Four-Position Synthesis

- The line MQ bisects both θ_{12} (20^o) and ϕ_{12} (30^o); and so, if the output member is turned counterclockwise from the QB_2 position, B_1 and B_2' will be coincident and at B_1.



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Graphical Solution

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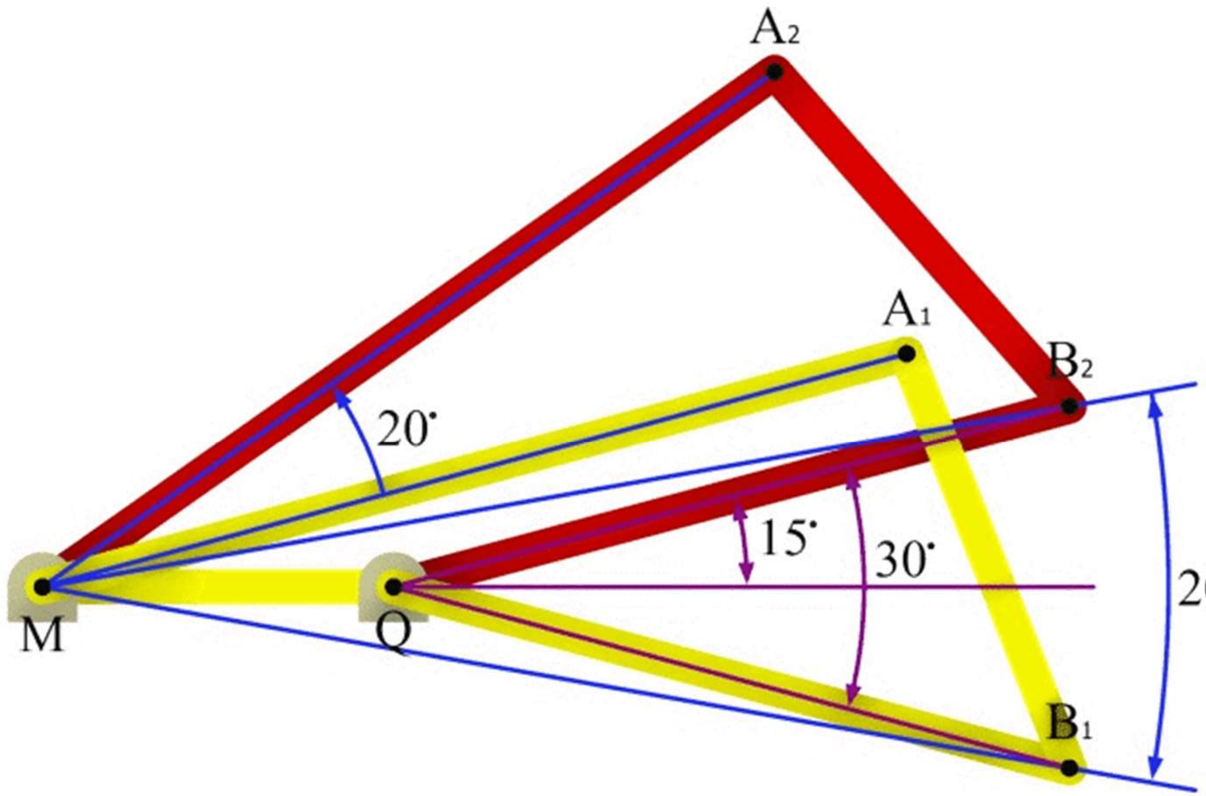

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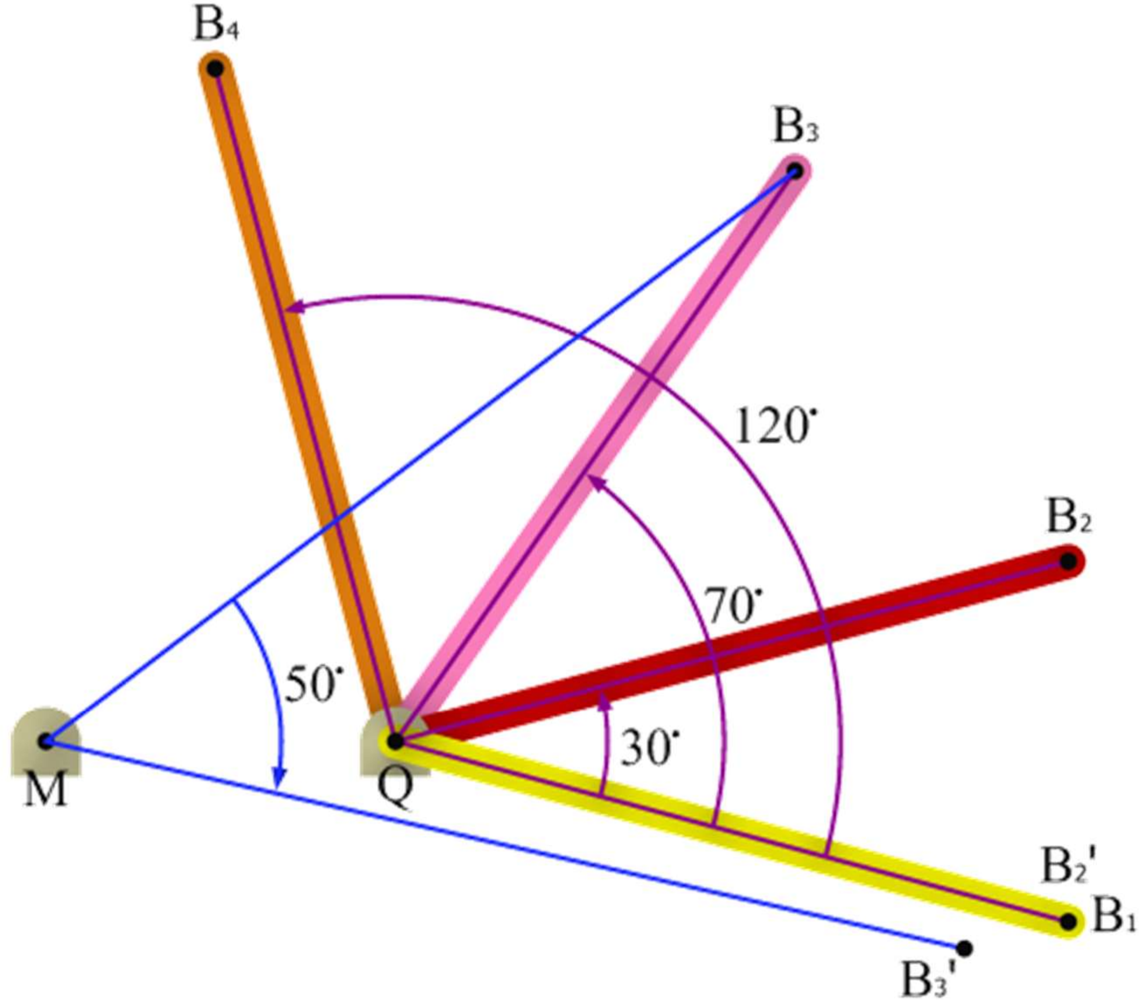
Graphical Solution

Four-Position Synthesis

- The procedure is exactly the same as that for three precision points. Point A_1 is obtained at the intersection of the midnormals to B_2'B_3' and B_3B_4'.



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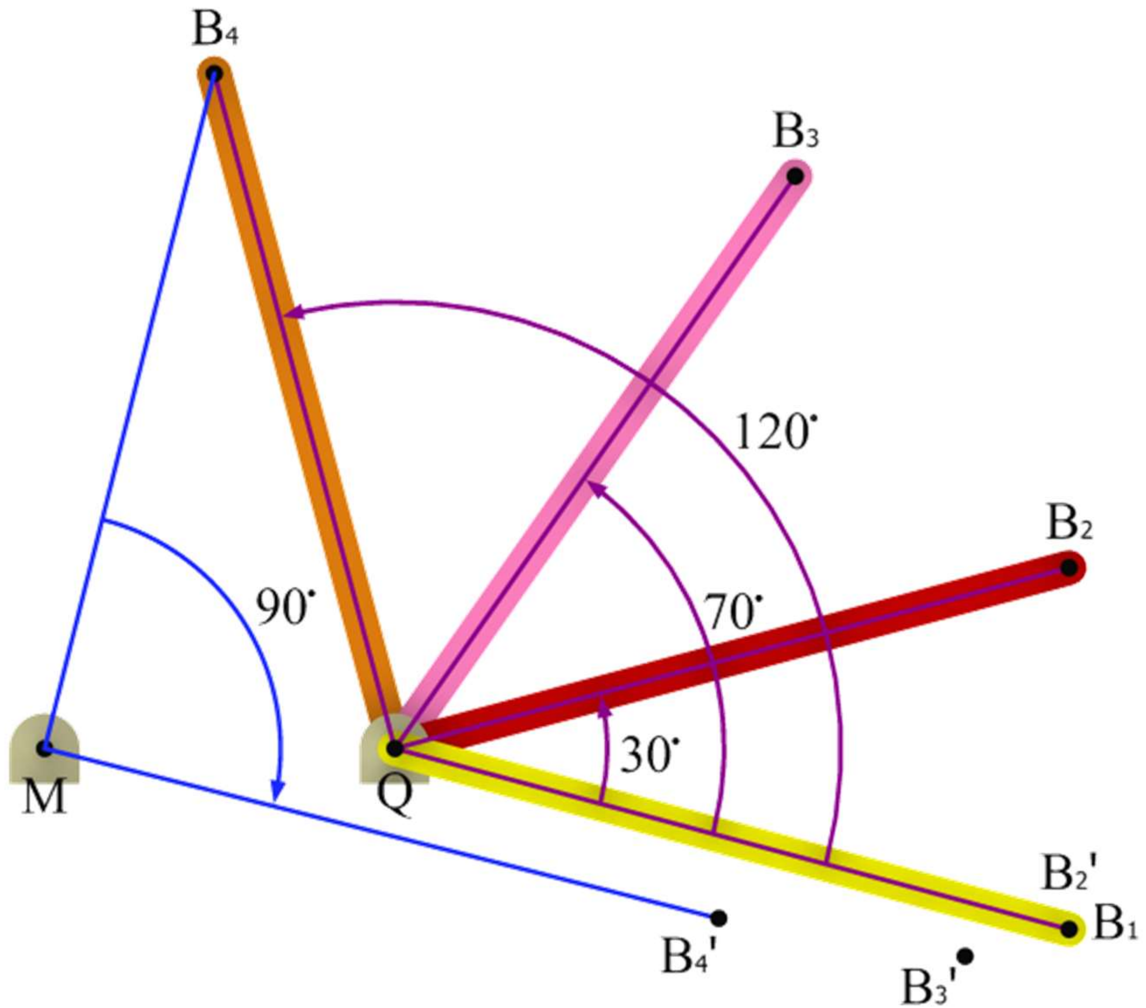
Graphical Solution

Four-Position Synthesis

- The procedure is exactly the same as that for three precision points. Point A_1 is obtained at the intersection of the midnormals to $B_2'B_3'$ and B_3B_4'.



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Graphical Solution

Four-Position Synthesis

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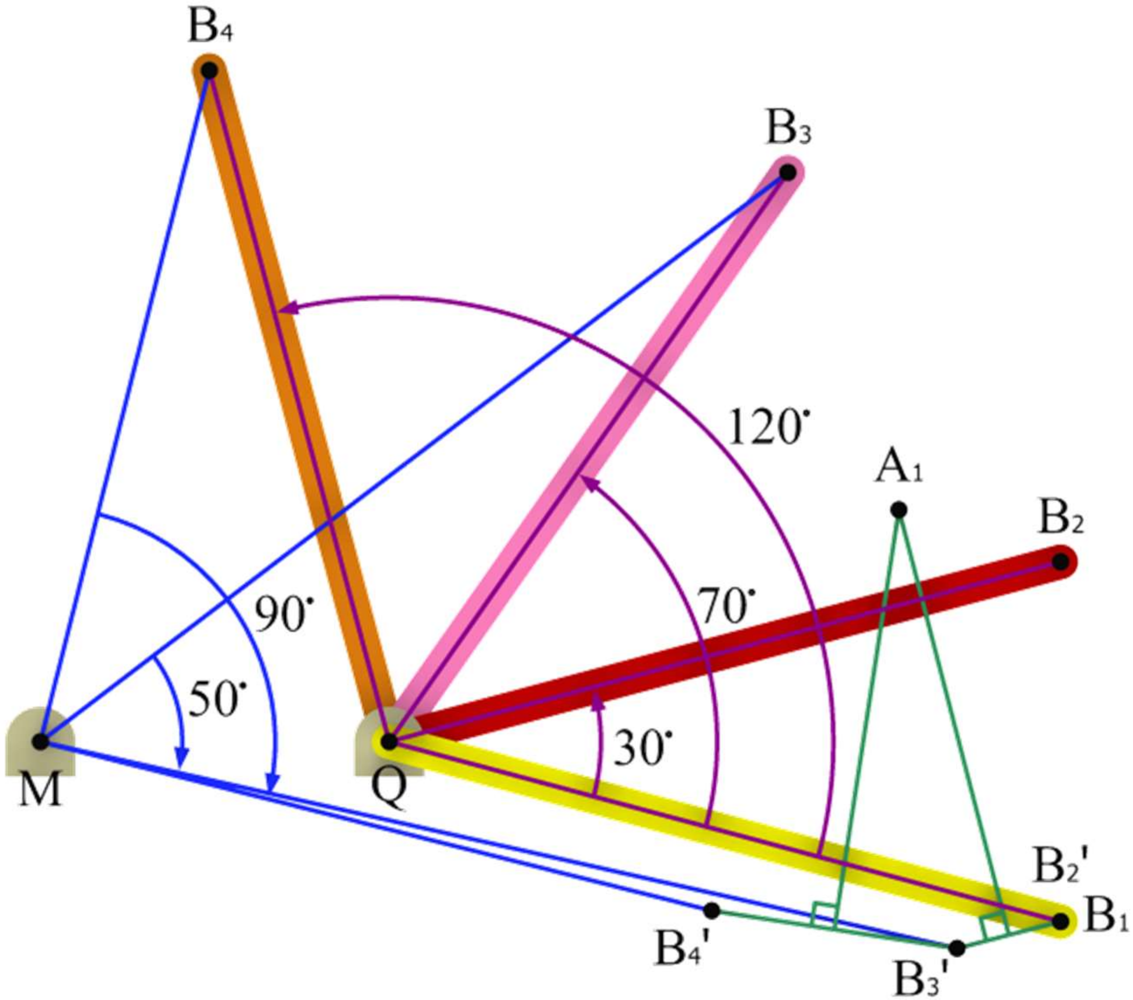

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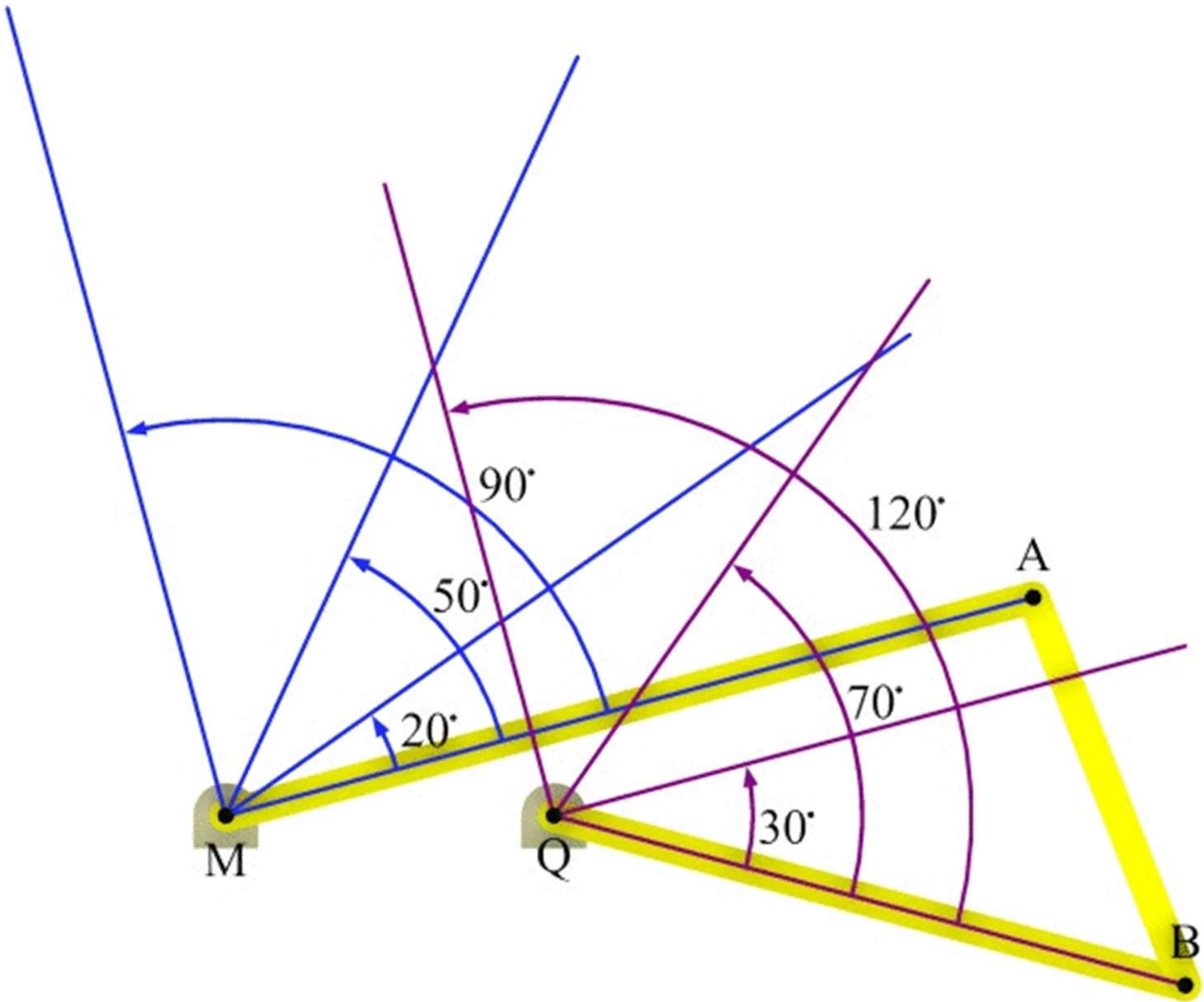
Graphical Solution

Four-Position Synthesis



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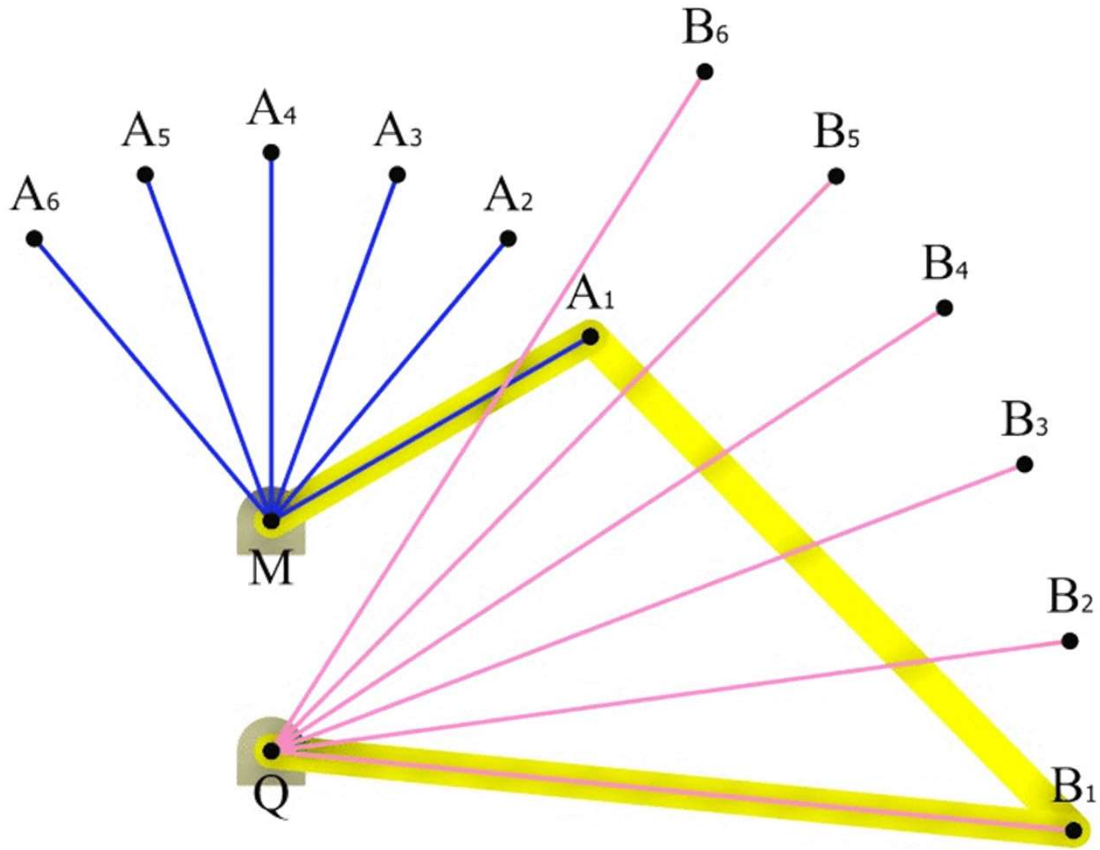
Multiple Positions Synthesis

Graphical Solution

Multiple Positions Synthesis

- Synthesis of a function generator, say, using **the overlap method**, is the easiest and quickest of all methods to use. It is not always possible to obtain a solution, and sometimes the accuracy is rather poor. Theoretically, however, one can employ as many points as are desired in the process.



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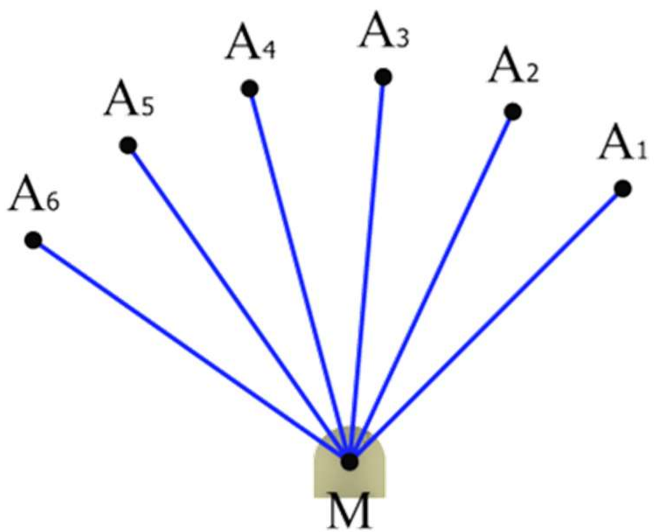
Graphical Solution

Multiple Positions Synthesis

- Use a sheet of tracing paper and construct the input rocker in all its position. This requires a choice for the length of MA.



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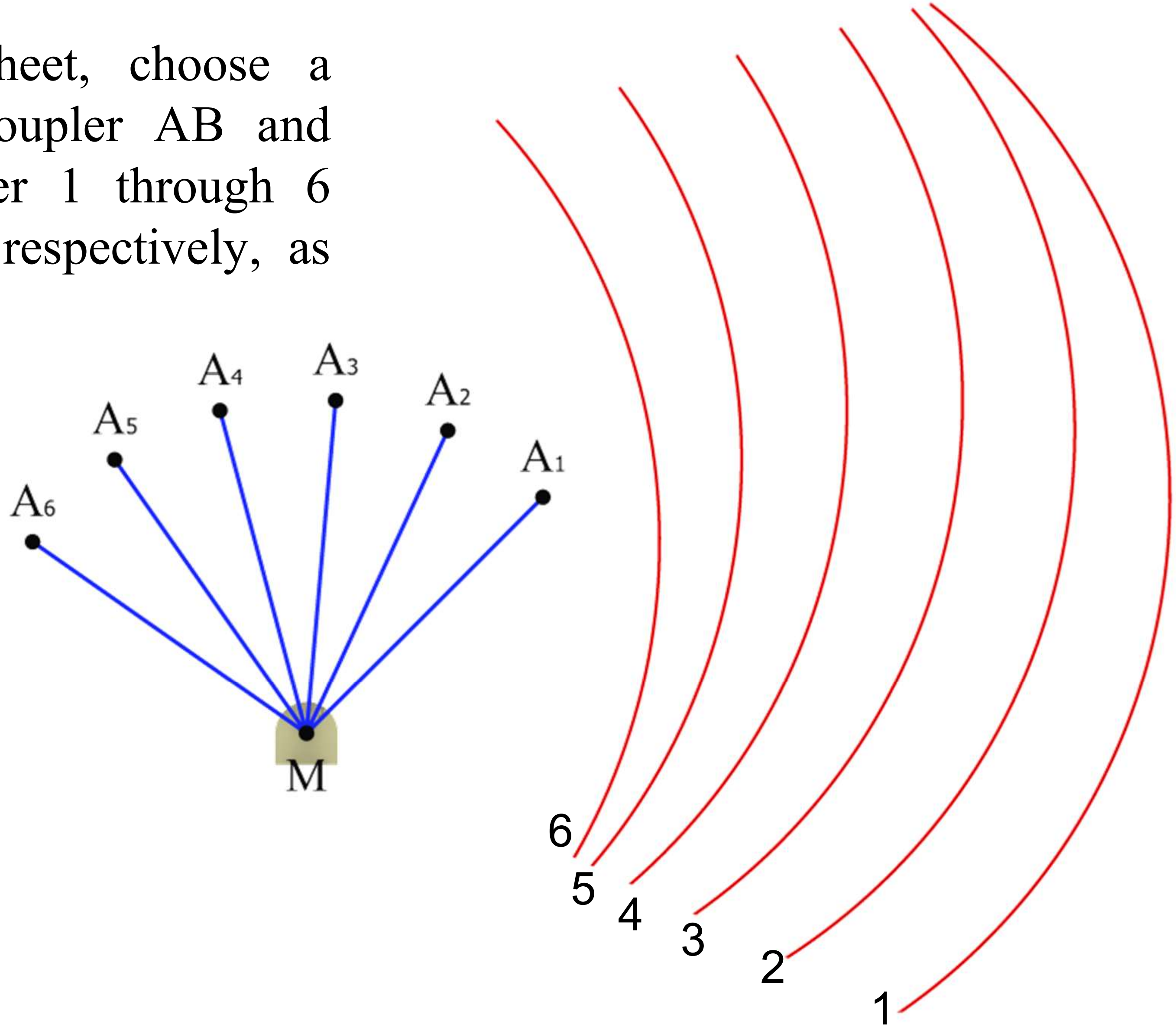
Graphical Solution

Multiple Positions Synthesis

- Also, on this sheet, choose a length for the coupler AB and draw arcs number 1 through 6 using A_1 to A_6. respectively, as centers.



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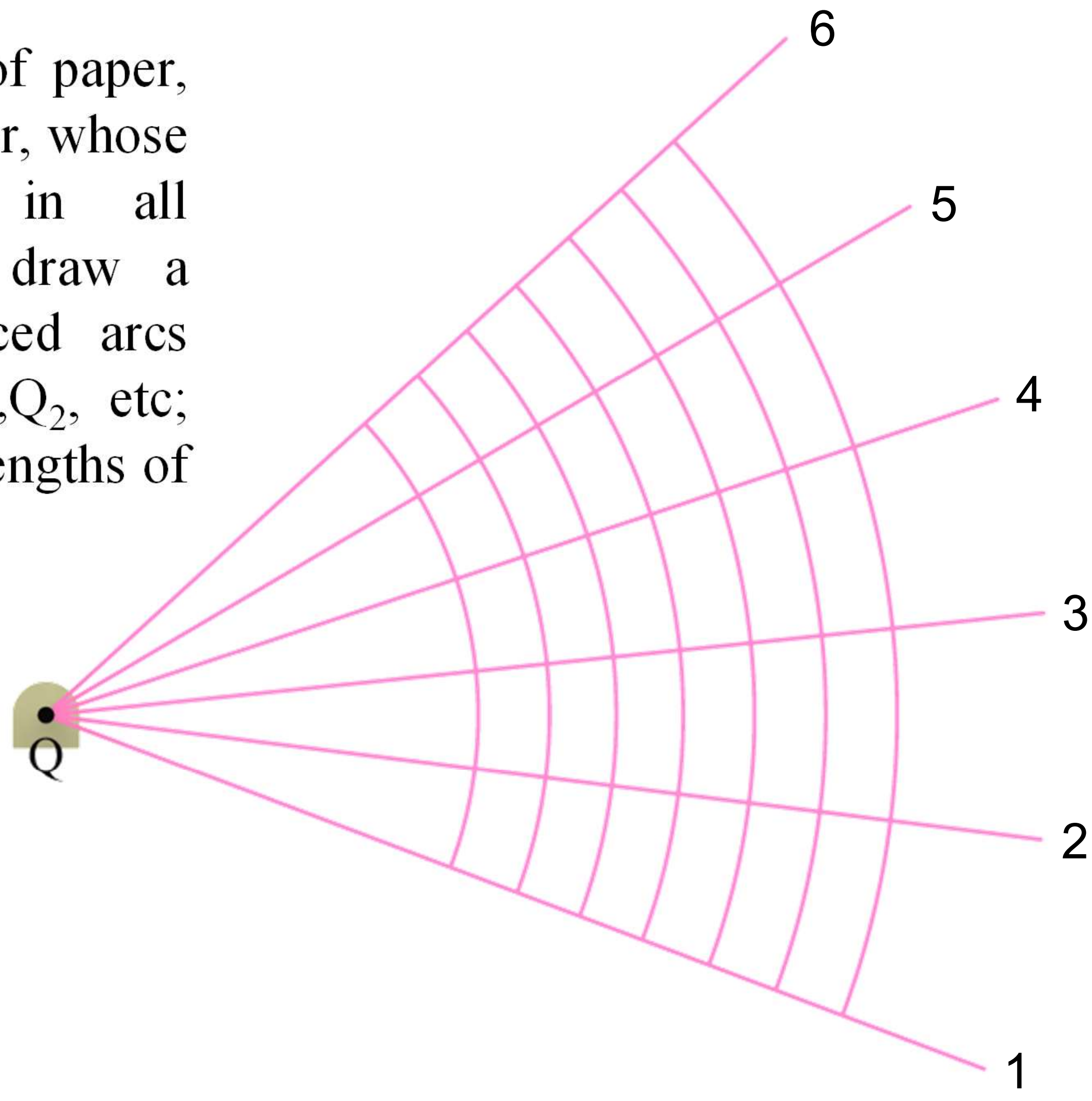
Graphical Solution

Multiple Positions Synthesis

- Now, on another sheet of paper, construct the output rocker, whose length is unknown, in all positions. Through Q draw a number of equally spaced arcs intersecting the lines Q_1,Q_2, etc; these represent possible lengths of the output rocker.



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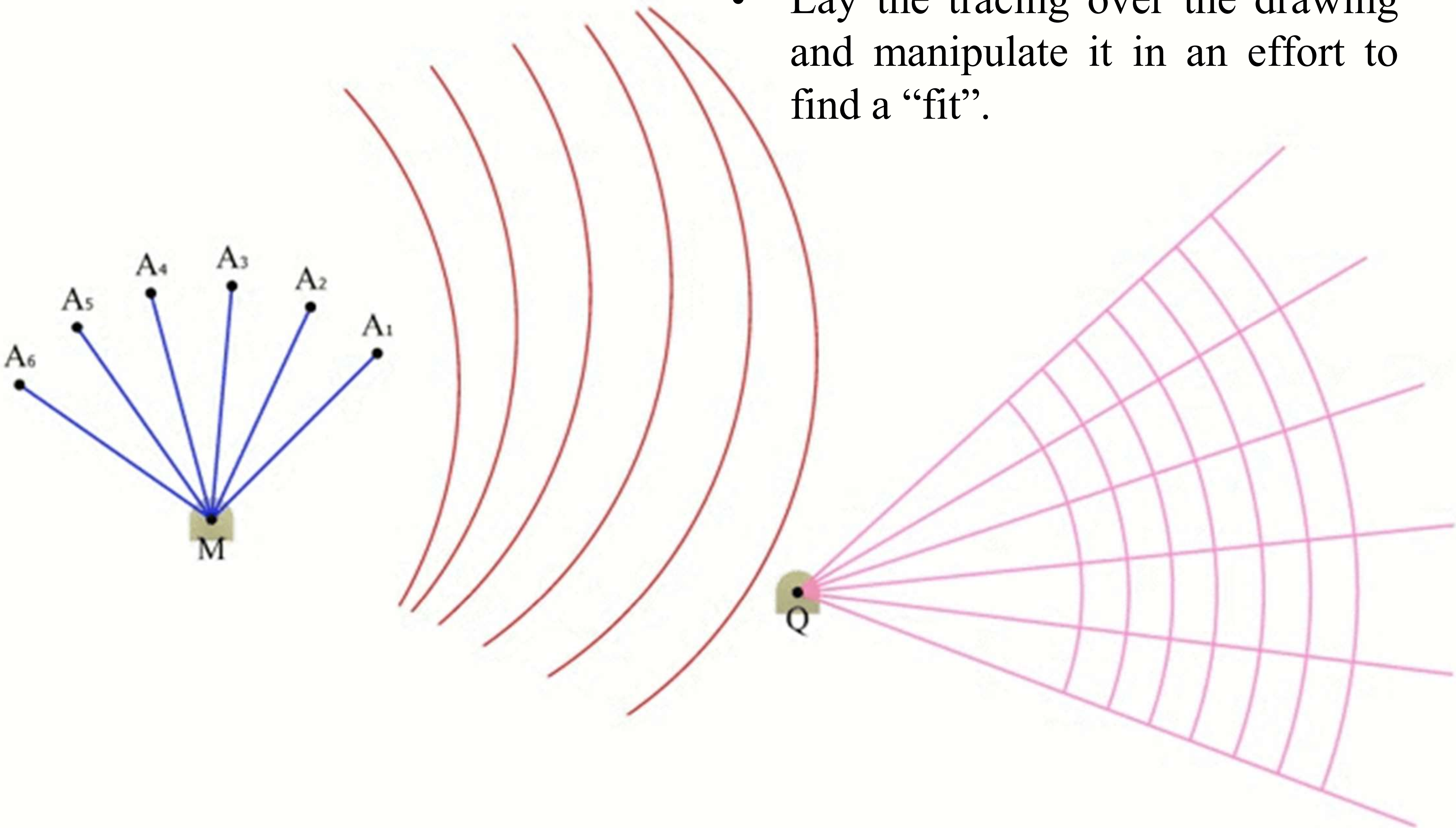
Graphical Solution

Multiple Positions Synthesis

- Lay the tracing over the drawing and manipulate it in an effort to find a “fit”.



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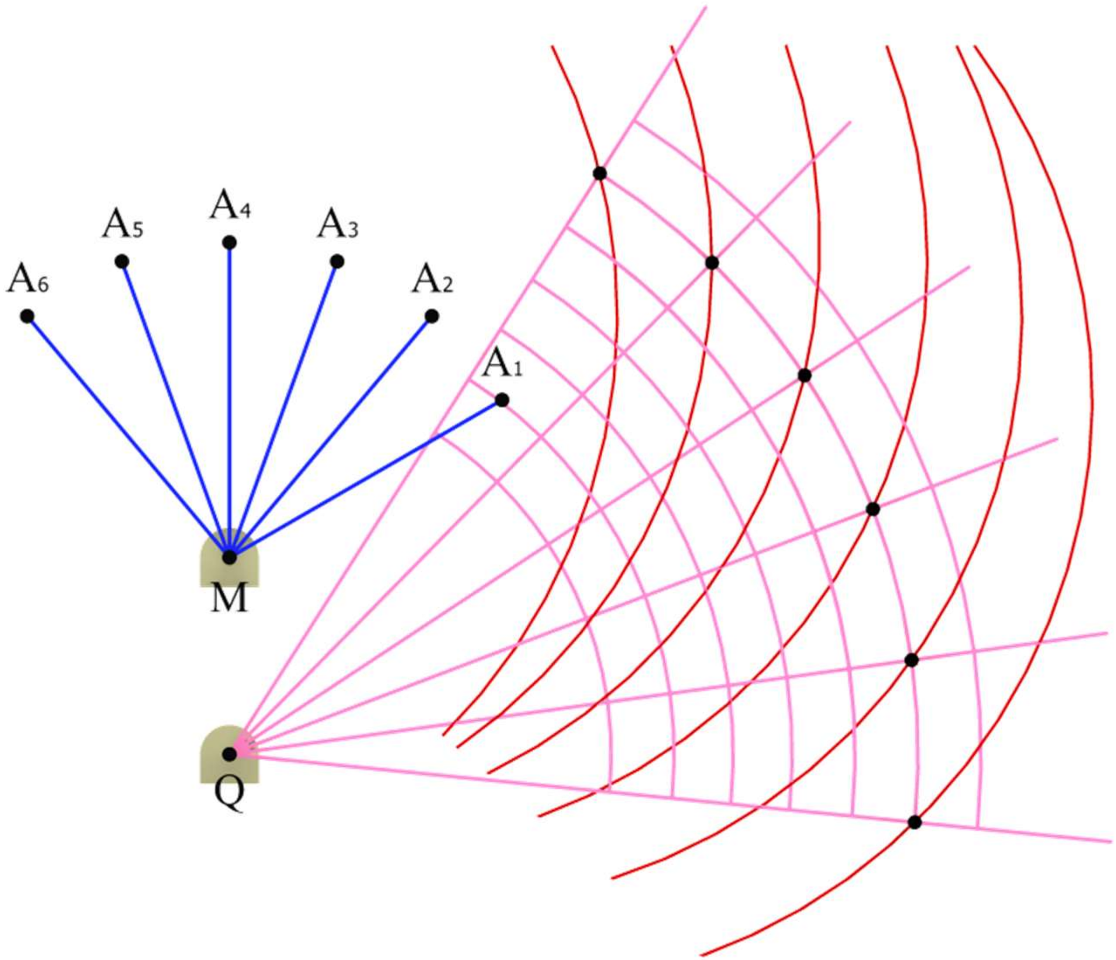
Graphical Solution

Multiple Positions Synthesis

- In this case a fit was found, and the result is shown below.



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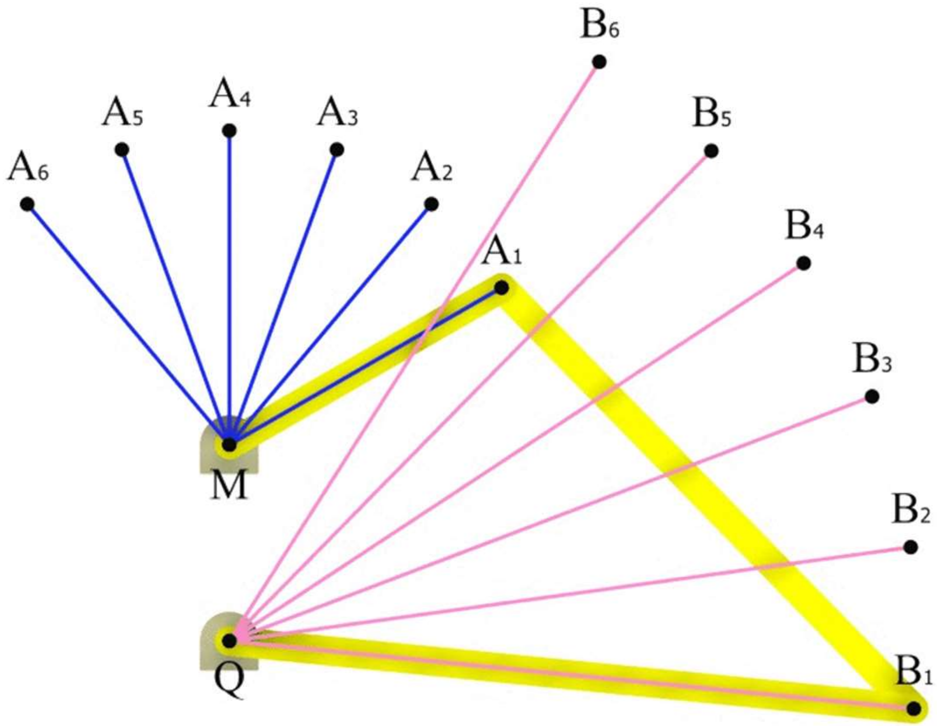
Graphical Solution

Multiple Positions Synthesis



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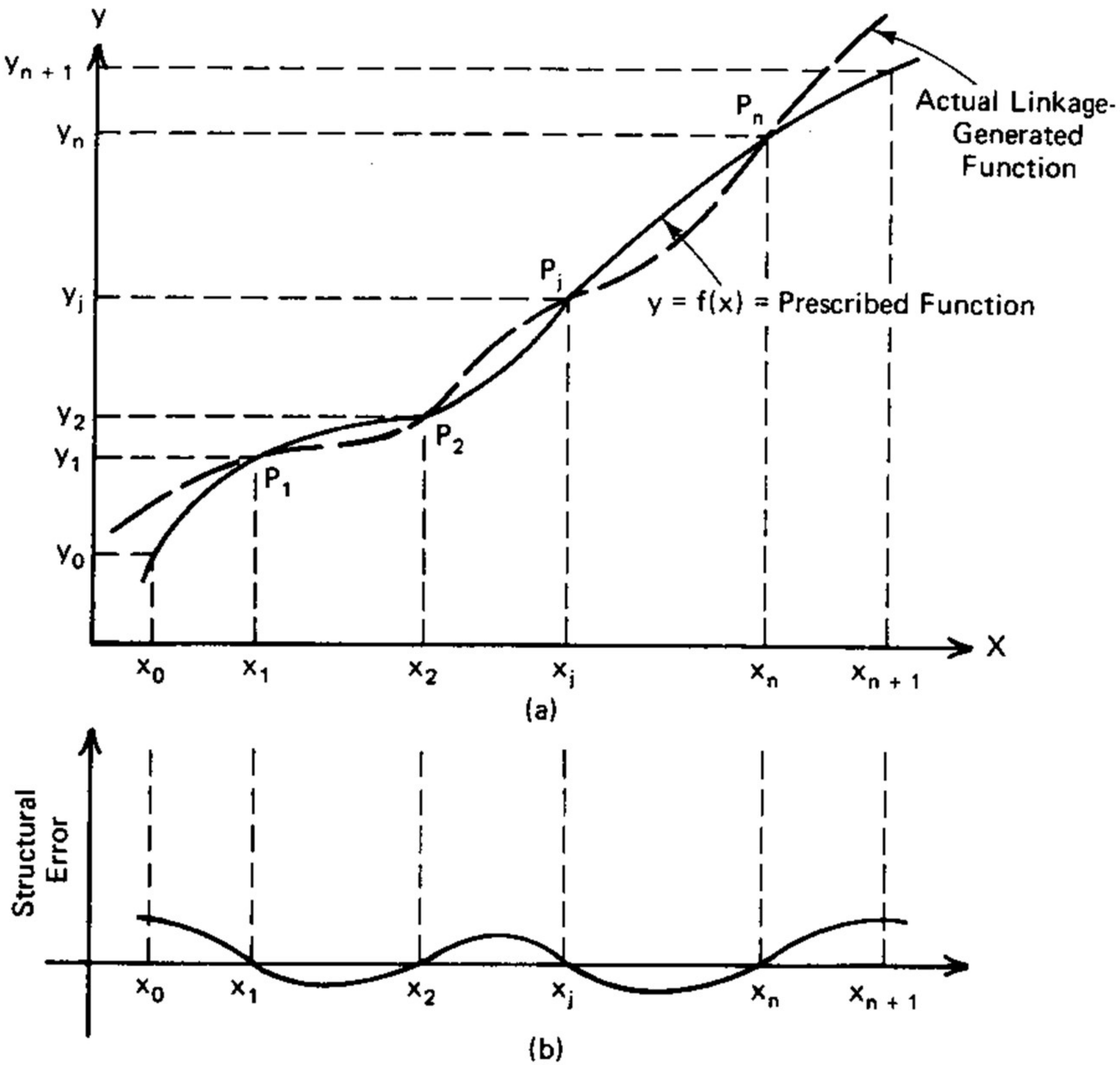
Function Generation

- The four-bar linkage is not capable of error-free generation of an arbitrary function and can match the function at only a limited number of precision points It is, however, widely used in industry in applications where high precision at many points is not required because the four-bar is simple to construct and maintain.



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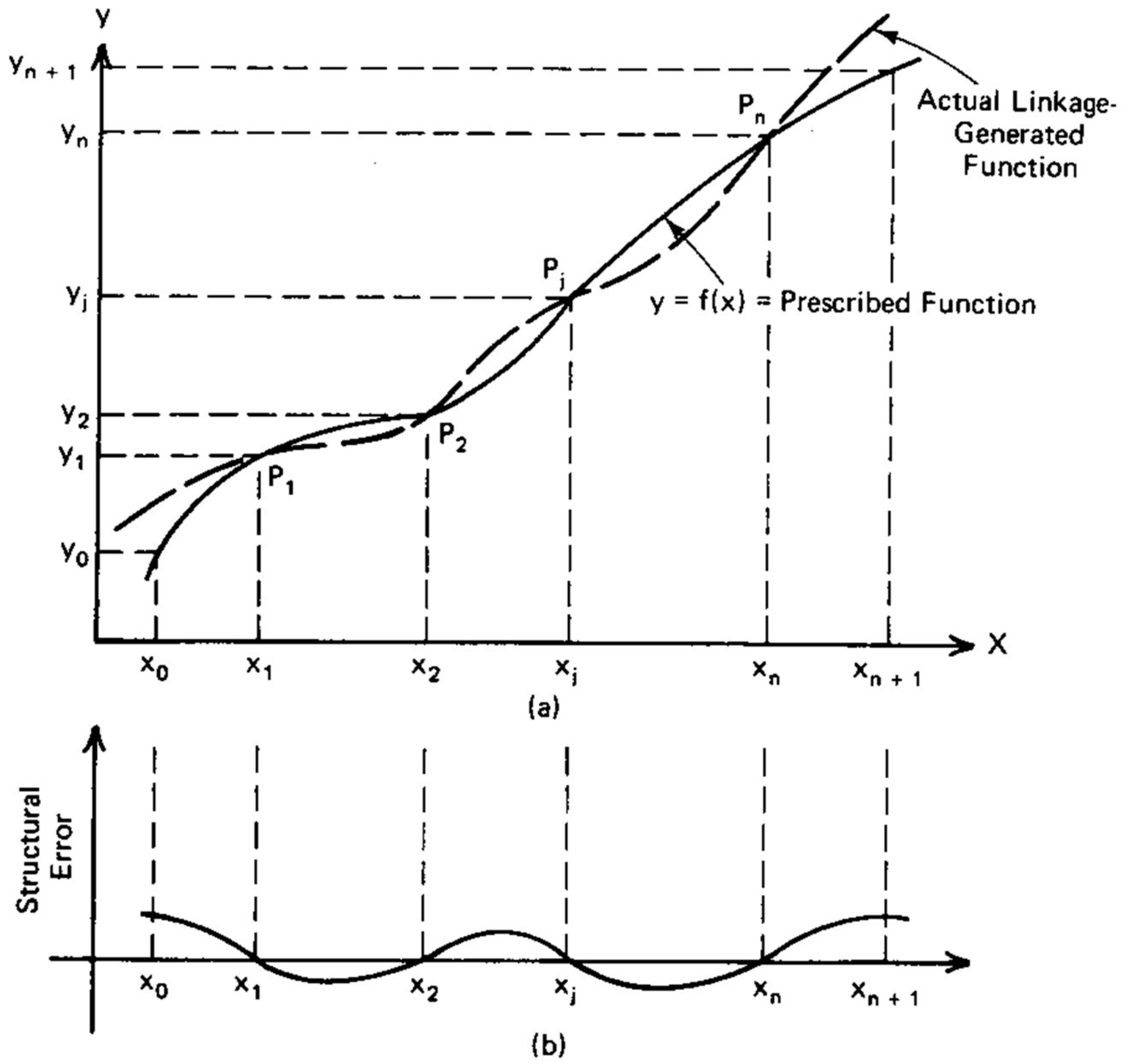
Function Generation

- The number of precision points that are used in the dimensional synthesis of the four-bar linkage varies in general between two and five.

- It is often desirable to space the precision points over the range of the function in such a way as to minimize the structural error of the linkage.



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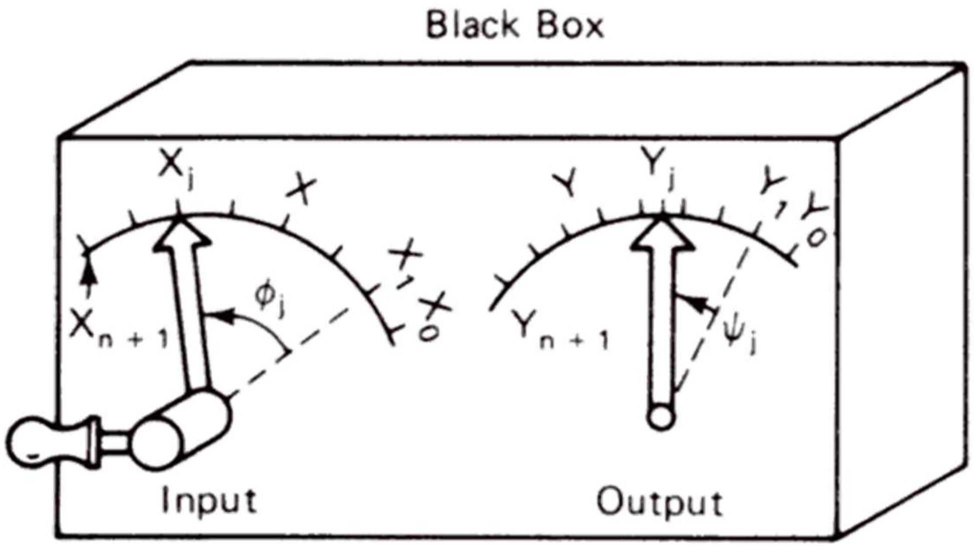
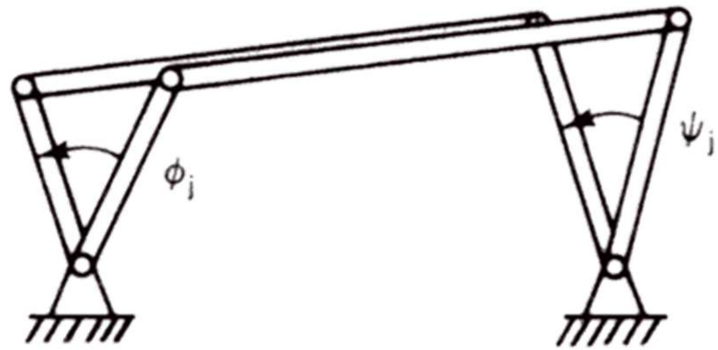
- To correlate input and output such that as the input moves by x, the output moves by $y = f(x)$ for the range $x_0 \le x \le x_{n+1}$.
- In the case of rotary input and output, the angles of rotation θ and ϕ are linear analogs of x and y.

$$\Delta x = |x_{n+1} - x_0|$$

$$\Delta \phi = |\phi_{n+1} - \phi_0|$$

$$\frac{\phi_j - \phi_1}{x_j - x_1} = \frac{\Delta\phi}{\Delta x}, \quad \frac{\psi_j - \psi_1}{y_j - y_1} = \frac{\Delta\psi}{\Delta y}$$





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Chebyshev Spacing

Chebyshev Spacing

- Chebyshev spacing of precision points is employed to minimize the structural error. This technique is often used as a "first guess".

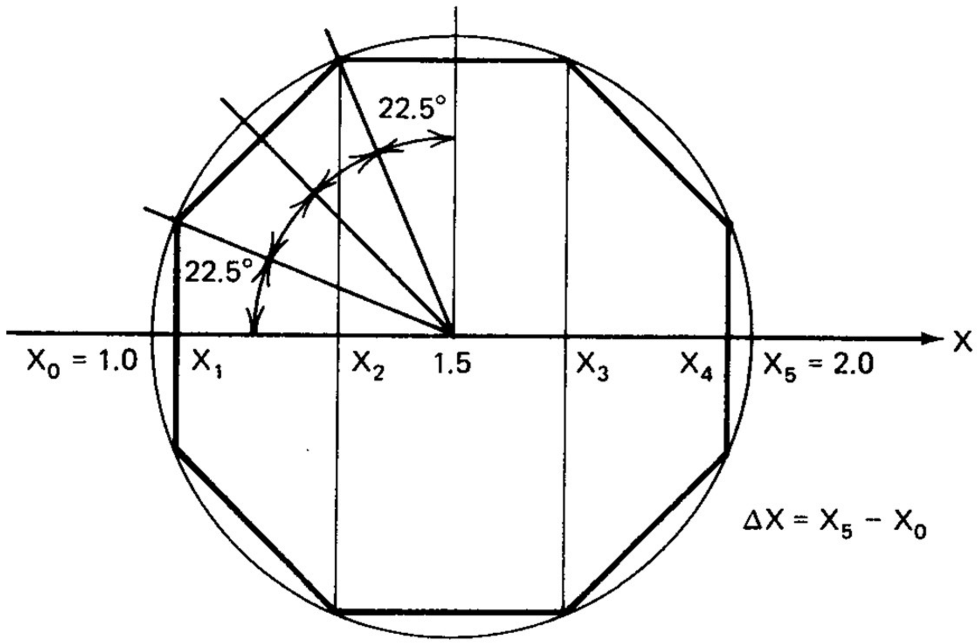
- Precision points may be located graphically:

1. A circle is drawn whose diameter is proportional to the range of the independent parameter (Δx).
2. A regular equilateral polygon having 2n sides (where n = the number of prescribed precision points) is then inscribed in the circle such that two sides of the polygon are vertical.
3. Lines drawn perpendicular to the horizontal diameter through each corner of the polygon intersect the diameter at points spaced at distances proportional to Chebyshev spacing of precision points.

Chebyshev Spacing

- **Example 1**: Determine the Chebyshev spacing for a four-bar linkage generating the function $y = 2x^2 - 1$, in the range $1 \leq x \leq 2$, where four precision points are to be prescribed ($n = 4$).

1. Draw a circle with diameter $\Delta x = x_{n+1} - x_0 = 2.0 - 1.0 = 1.0$.
2. Construct a polygon of $2n = 8$ sides, with two sides vertical.
3. The corners of the polygon projected vertically onto the horizontal axis are the prescribed precision points.



$x_0 = 1.00,\ x_3 = 4.71$

$x_1 = 1.04,\ x_4 = 1.96$

$x_2 = 1.31,\ x_5 = 2.00$

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Chebyshev Spacing

- **Example 2**: Given the Chebyshev precision points derived in **Example 1** and the range in the input and output link rotations $\Delta\phi = 60$, $\Delta\psi = 90$, find ϕ_2, ϕ_3, ϕ_4, ψ_2, ψ_3, and ψ_4.

$$y = 2x^2 - 1$$

$$\frac{\phi_j - \phi_1}{x_j - x_1} = \frac{\Delta\phi}{\Delta x}, \quad \frac{\psi_j - \psi_1}{y_j - y_1} = \frac{\Delta\psi}{\Delta y}$$

$x_0 = 1.00$, $x_3 = 1.69$
$x_1 = 1.04$, $x_4 = 1.96$
$x_2 = 1.31$, $x_5 = 2.00$

$y_0 = 1.00$, $y_3 = 4.71$
$y_1 = 1.16$, $y_4 = 6.68$
$y_2 = 2.43$, $y_5 = 7.00$

$\phi_2 = 16.2^o$
$\phi_3 = 39.0^o$
$\phi_4 = 55.2^o$

$\psi_2 = 19.1^o$
$\psi_3 = 53.3^o$
$\psi_4 = 82.8^o$

G. Erdman, G. N. Sandor, S. Kota, Mechanism Design: Analysis and Synthesis

Algebraic Methods of Synthesis using Displacement Equation

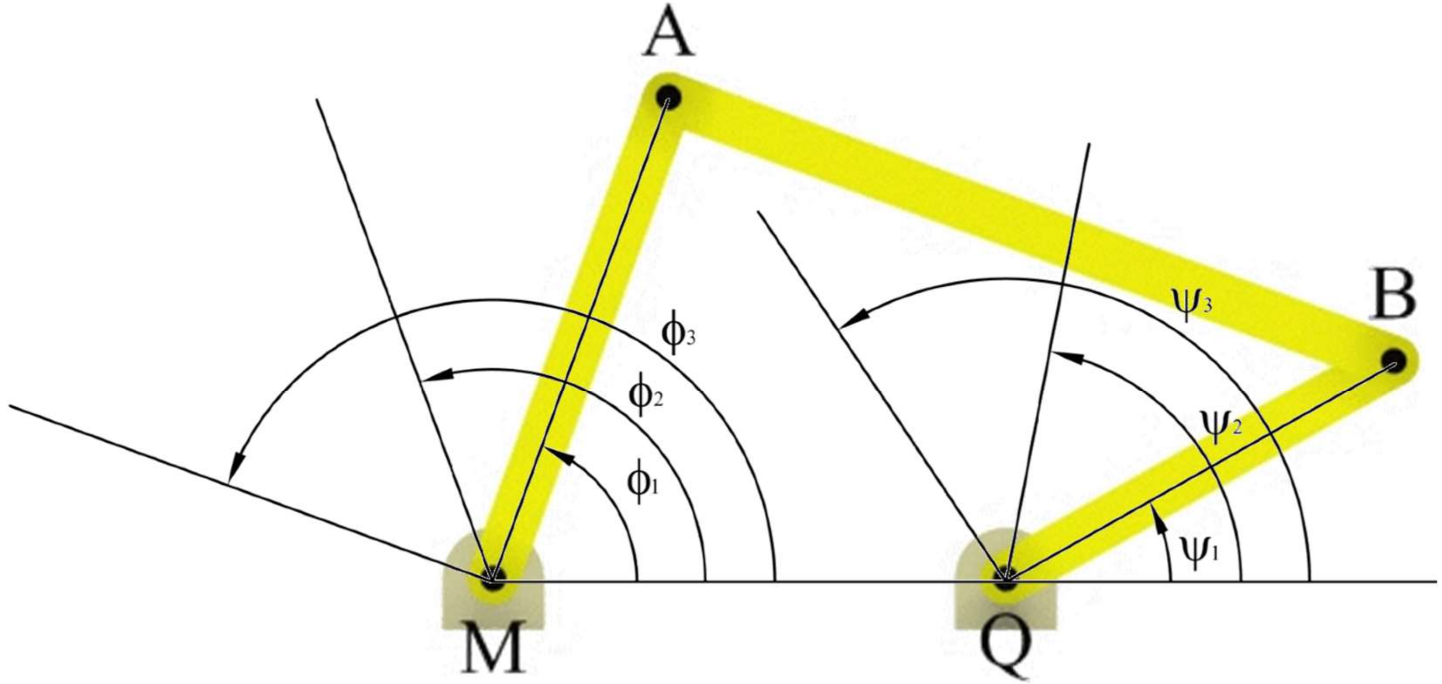
R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Crank and Follower Synthesis: Three Accuracy Points

Three given positions of the crank : ϕ_1, ϕ_2, ϕ_3

Corresponding positions of the follower : ψ_1, ψ_2, ψ_3

Find the proper values of a_1, a_2, a_3, and a_4 for three related pairs (ϕ_1, ψ_1), (ϕ_2, ψ_2), and (ϕ_3, ψ_3)



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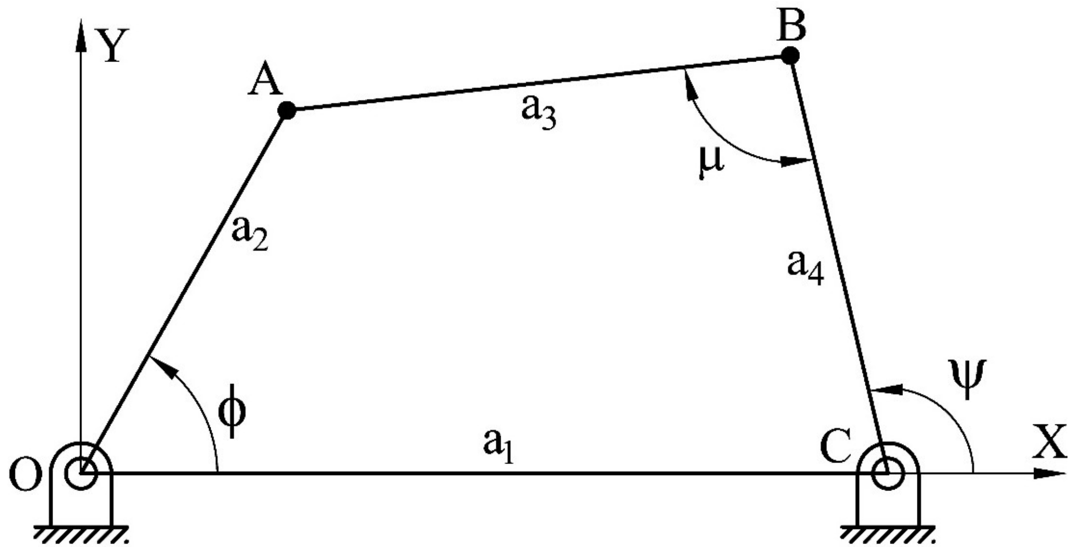
Displacement Equation

$A(a_2\cos\phi, a_2\sin\phi)$

$B(a_1 + a_4\cos\psi, a_4\sin\psi)$

$\overline{AB} = a_3$



$$(a_1 + a_4\cos\psi - a_2\cos\phi)^2 + (a_4\sin\psi - a_2\sin\phi)^2 = a_3{}^2$$

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$$(a_1 + a_4\cos\psi - a_2\cos\phi)^2 + (a_4\sin\psi - a_2\sin\phi)^2 = a_3{}^2$$

$$K_1\cos\phi - K_2\cos\psi + K_3 = \cos(\phi - \psi)$$

Freudenstein Euqation

where $K_1 = -\frac{a_1}{a_4}$ $K_2 = -\frac{a_1}{a_2}$ $K_3 = \frac{a_1{}^2 + a_2{}^2 - a_3{}^2 + a_4{}^2}{2a_2a_4}$

When written for three pairs of values (ϕ_1, ψ_1), (ϕ_2, ψ_2), (ϕ_3, ψ_3) this equation yields a system of three equations linear with respect to K_1, K_2, K_3

$$K_1\cos\phi_1 - K_2\cos\psi_1 + K_3 = \cos(\phi_1 - \psi_1)$$
$$K_1\cos\phi_2 - K_2\cos\psi_2 + K_3 = \cos(\phi_2 - \psi_2)$$
$$K_1\cos\phi_3 - K_2\cos\psi_3 + K_3 = \cos(\phi_3 - \psi_3)$$

Freudenstein, F.: Approximate synthesis of four-bar linkages. Transactions of ASME

Solving the resulting system yields K_1, K_2, K_3 as

$$K_1 = \frac{w_2 w_6 - w_3 w_5}{w_2 w_4 - w_1 w_5} \qquad K_2 = \frac{w_1 w_6 - w_3 w_4}{w_2 w_4 - w_1 w_5}$$

where $w_1 = \cos\phi_1 - \cos\phi_2$

$$w_2 = \cos\psi_1 - \cos\psi_2$$

$$w_3 = \cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2)$$

$$w_4 = \cos\phi_1 - \cos\phi_3$$

$$w_5 = \cos\psi_1 - \cos\psi_3$$

$$w_6 = \cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3)$$

$$K_3 = \cos(\phi_i - \psi_i) - K_1\cos\phi_i + K_2\cos\psi_i \qquad i = 1, 2, 3$$

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With the values of K_1, K_2, K_3 known, the parameters of the linkage may be found from the relations

$$a_2 = -\frac{a_1}{K_2}$$

$$a_4 = -\frac{a_1}{K_1}$$

$$a_3 = \sqrt{{a_2}^2 + {a_4}^2 + {a_1}^2 - 2a_2a_4K_3}$$

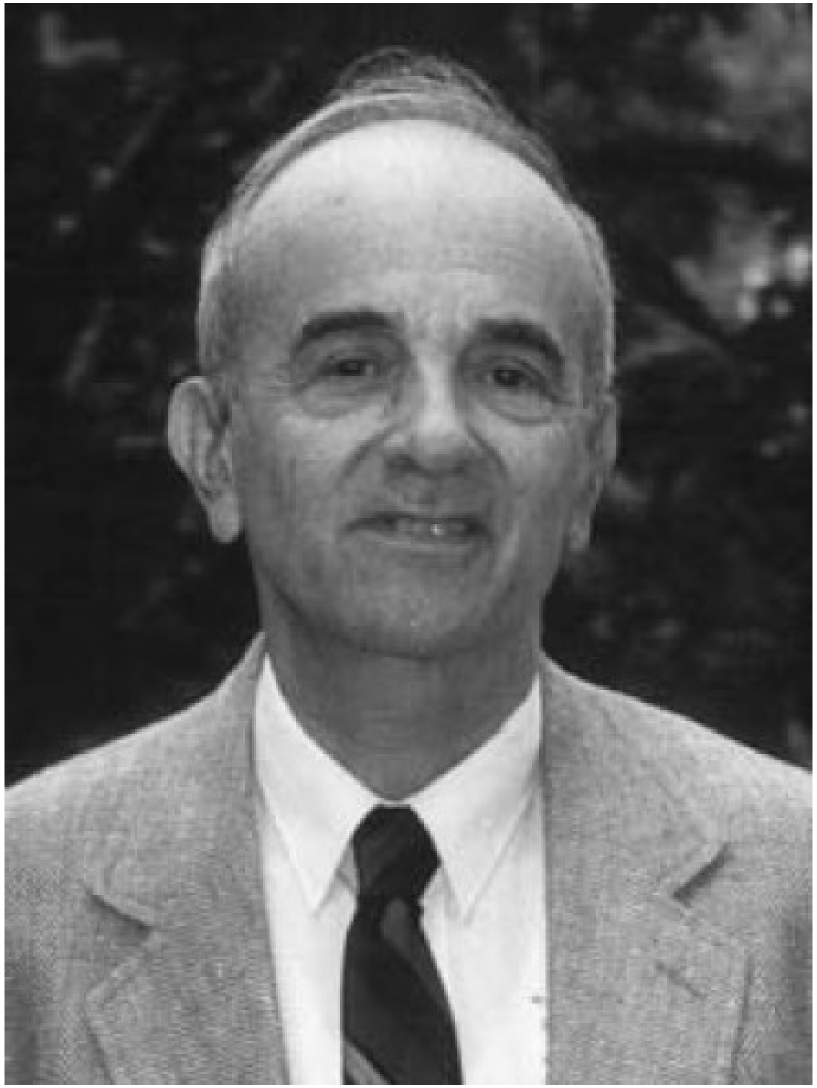
The parameter a_1 may be given a positive but arbitrary value. This parameter merely determines the size of the linkage and has no effect on the angular relationships.

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

```
clc
clear

phi_1 = 0 / 180 * pi; phi_2 = 26 / 180 * pi; phi_3 = 52 / 180 * pi;
psi_1 = 0 / 180 * pi; psi_2 = 29.4 / 180 * pi; psi_3 = 51.4 / 180 * pi;

A = [ cos(phi_1)  -cos(psi_1)  1;
      cos(phi_2)  -cos(psi_2)  1;
      cos(phi_3)  -cos(psi_3)  1];

B = [ cos(phi_1 - psi_1);
      cos(phi_2 - psi_2);
      cos(phi_3 - psi_3);];

X = A \ B;

K1 = X(1,1); K2 = X(2,1); K3 = X(3,1);
a1 = 1; a2 = -a1 / K2; a4 = -a1 /K1; a3 = sqrt(a2^2+a4^2+a1^2-2*a2*a4*K3);
```

Ferdinand Freudenstein

Ferdinand Freudenstein (12 May 1926 – 30 March 2006) was an American physicist and engineer who is considered to be the "**Father of Modern Kinematics**." Freudenstein made revolutionary contribution applying digital computation to the kinematic synthesis of mechanisms. In his Ph.D dissertation, Freudenstein developed what is to become the Freudenstein Equation, which uses a simple algebraic method to synthesize planar four-bar function generators.

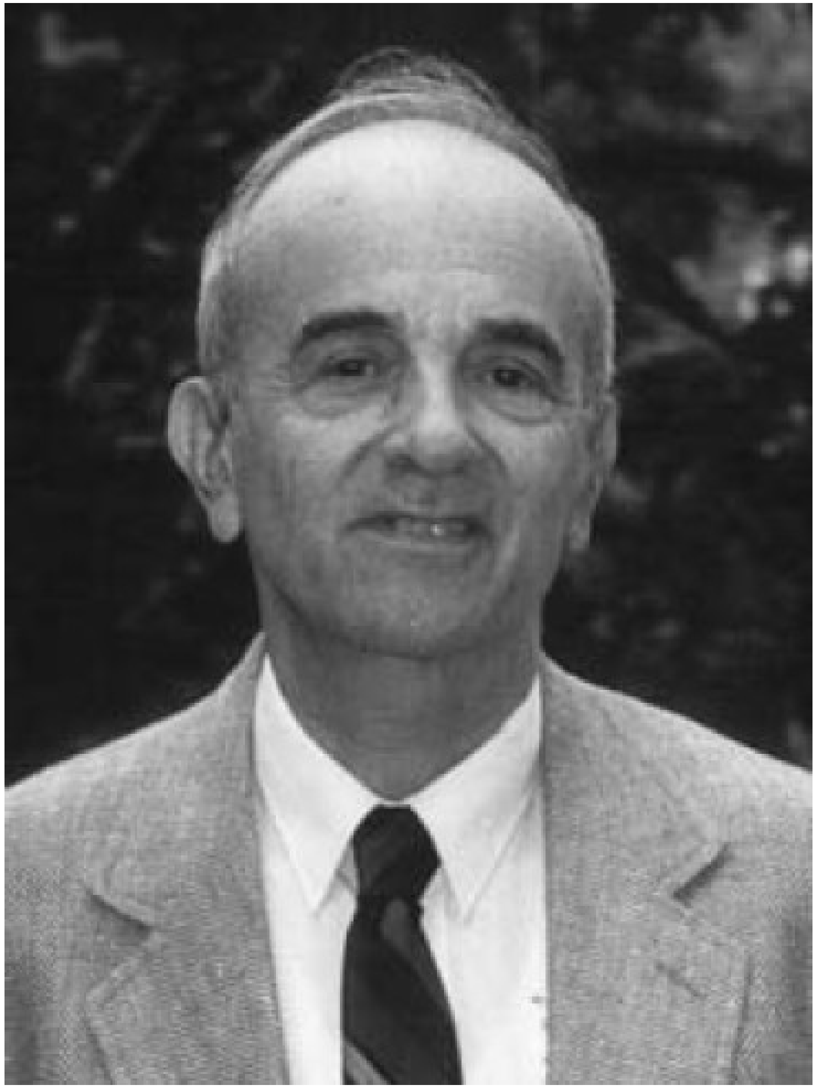


https://en.wikipedia.org/wiki/Ferdinand_Freudenstein

Ferdinand Freudenstein

Freudenstein spent his entire career working and teaching in **Columbia University** as the Higgins Professor of Mechanical Engineering. Over his life-time, Freudenstein mentored over 500 Ph.D. students in hundreds of universities across the world.

As a professor, Freudenstein had over 500 Ph.D. students including current Stanford professor of mechanical engineering **Bernard Roth**, engineer **George Sandor**, and Texas A&M professor Norris Stubbs.



https://en.wikipedia.org/wiki/Ferdinand_Freudenstein

Freudenstein Academic Tree

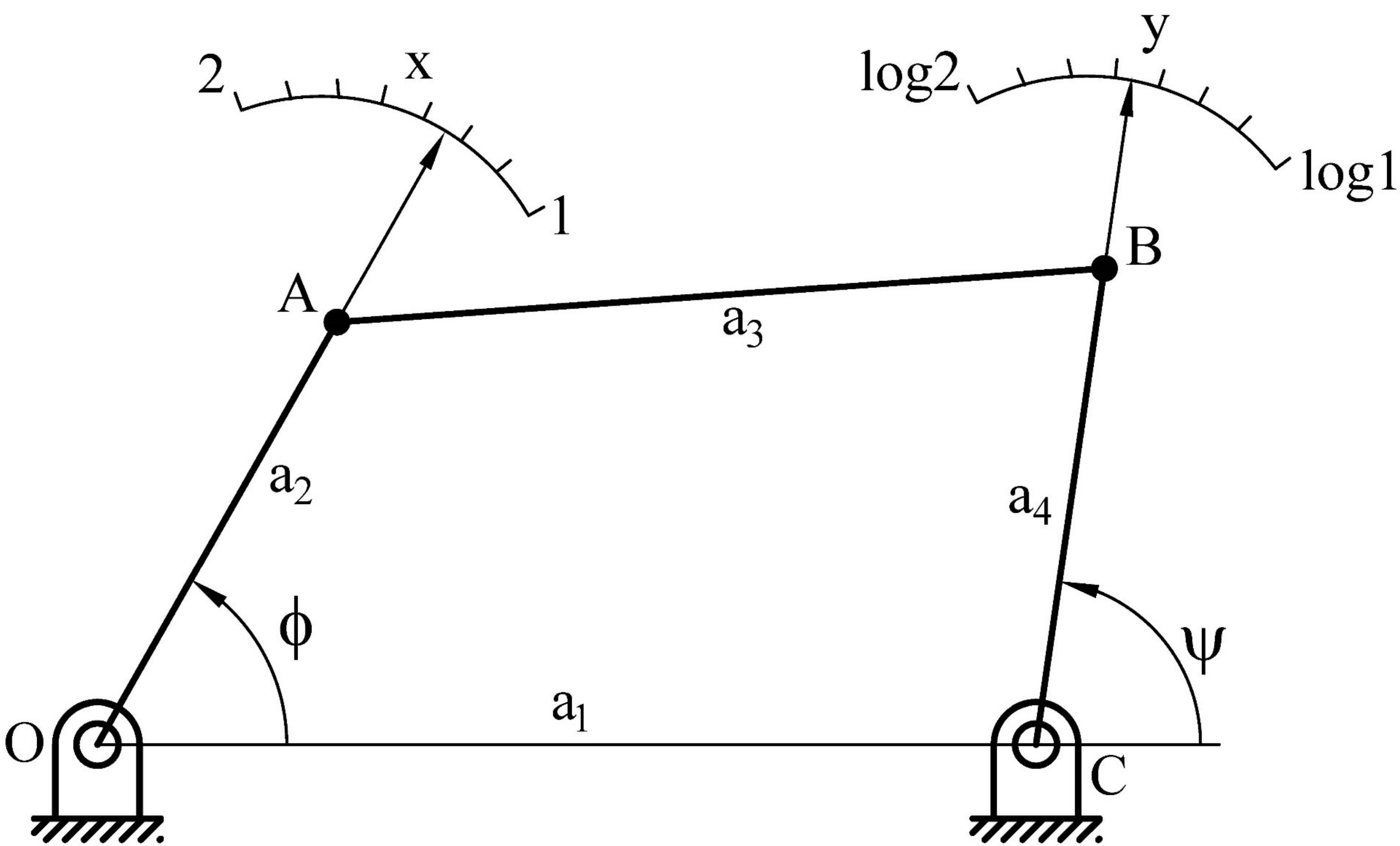
Freudenstein, Ferdinand

- Sandor, George (Columbia 1959) Univ. of Florida (deceased)
    - Kaufman, Roger (RPI 1969) George Washington Univ.
    - Erdman, Art (RPI 1971) Univ. of Minnesota
        - Midha, Ashok (Univ. of Minnesota 1977) Univ. of Missouri-Rolla
            - Turcic, David (Penn State 1982) Portland State
            - Her, Innchyn (Purdue 1986) National Sun Yat-Sen Univ
            - Farhang, Kambiz (Purdue 1989) Southern Illinois Univ.
            - Nahvi, Hassan (Purdue 1991) Isfahan Univ. (Iran)
            - Howell, Larry (Purdue 1993) BYU
                - Lyon, Scott (BYU 2003) L3
                - Lusk, Craig (BYU 2005) Univ. of South Florida
                - Wittwer, Jonathan (BYU 2005) Sandia Nat. Labs
            - Mahyuddin, Andi (Purdue 1993) Inst. Teknologi Bandung (Indonesia)

Freudenstein Academic Tree

- Pisano, Albert (Columbia 1981) U.C. Berkeley
  - Chan, C.-Y. (U.C. Berkeley 1988)
  - Lin, Yuyi (U.C. Berkeley 1989) Univ. of Missouri Columbia
    - Wen, G. (Univ. of Missouri Columbia 1998)
    - Cai, X. (Univ. of Missouri Columbia 2005)
    - Harby, D. (Univ. of Missouri Columbia 2006)
  - Cho, Y.-H. (U.C. Berkeley 1990)
  - Hodges, P. (U.C. Berkeley 1991)
  - Kim, C.-J. (U.C. Berkeley 1991) U.C. Los Angeles
    - Simon, Jonathan (U.C. Los Angeles 1997) Dust Networks
    - Sherman, Faiz (U.C. Los Angeles 1998) Proctor & Gamble
    - Tseng, Fan-Gang (Kevin) (U.C. Los Angeles 1998) Nat. Tsing Hua Univ.
    - Huang, Long-Sun (U.C. Los Angeles 1999) Nat. Taiwan Univ.
    - Lee, Jung-Hoon (U.C. Los Angeles 2000) Seoul Nat. Univ.
    - Yi, Taechung (U.C. Los Angeles 2000) FormFactor Inc.
    - Yao, Da-Jeng (Jeffery) (U.C. Los Angeles 2001) Nat. Tsing Hua Univ.
    - Kim, Joonwon (U.C. Los Angeles 2003) Pohang Univ. of Sci. & tech.
    - Fan, Shih-Kang (Scott) (U.C. Los Angeles 2003) Nat. Chiao Tung Univ.

Example 1: The function $y = \log x$ is to be generated in the interval $1 \leq x \leq 2$ by means of a four-bar linkage. The ranges of variation of ϕ and ψ must be selected. They are chosen as $\Delta\phi = \Delta\psi = 60°$. Three accuracy points are taken in the interval $1 \leq x \leq 2$.



Example 1: The function y = logx is to be generated in the interval $1 \le x \le 2$ by means of a four-bar linkage. The ranges of variation of ϕ and ψ must be selected. They are chosen as $\Delta\phi = \Delta\psi = 60°$. Three accuracy points are taken in the interval $1 \le x \le 2$.

$$x_1 = 1.067;\ y_1 = 0.0282;$$
$$x_2 = 1.5;\quad y_2 = 0.1761;$$
$$x_3 = 1.933;\ y_3 = 0.2862;$$

$$\frac{\phi_2 - \phi_1}{\Delta\phi} = \frac{x_2 - x_1}{\Delta x} \qquad \phi_2 - \phi_1 = 26° \qquad (\Delta x = 2 - 1 = 1)$$

$$\frac{\phi_3 - \phi_1}{\Delta\phi} = \frac{x_3 - x_1}{\Delta x} \qquad \phi_3 - \phi_1 = 52°$$

$$\frac{\psi_2 - \psi_1}{\Delta\psi} = \frac{y_2 - y_1}{\Delta y} \qquad \psi_2 - \psi_1 = 29.4° \qquad (\Delta y = \log 2 - \log 1)$$

$$\frac{\psi_3 - \psi_1}{\Delta\psi} = \frac{y_3 - y_1}{\Delta y} \qquad \psi_3 - \psi_1 = 51.4°$$

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Example 1: The function $y = \log x$ is to be generated in the interval $1 \leq x \leq 2$ by means of a four-bar linkage. The ranges of variation of ϕ and ψ must be selected. They are chosen as $\Delta\phi = \Delta\psi = 60°$. Three accuracy points are taken in the interval $1 \leq x \leq 2$.

Choosing $\phi_1 = 0°$ and $\psi_1 = 0°$ yields

$$\phi_2 = 26° \quad \phi_3 = 52° \quad \psi_2 = 29.4° \quad \psi_3 = 51.4°$$

From which $\quad w_1 = 0.1011 \quad w_2 = 0.1285 \quad w_3 = 0.0017$

$$w_4 = 0.3839 \quad w_5 = 0.3766 \quad w_6 = 0$$

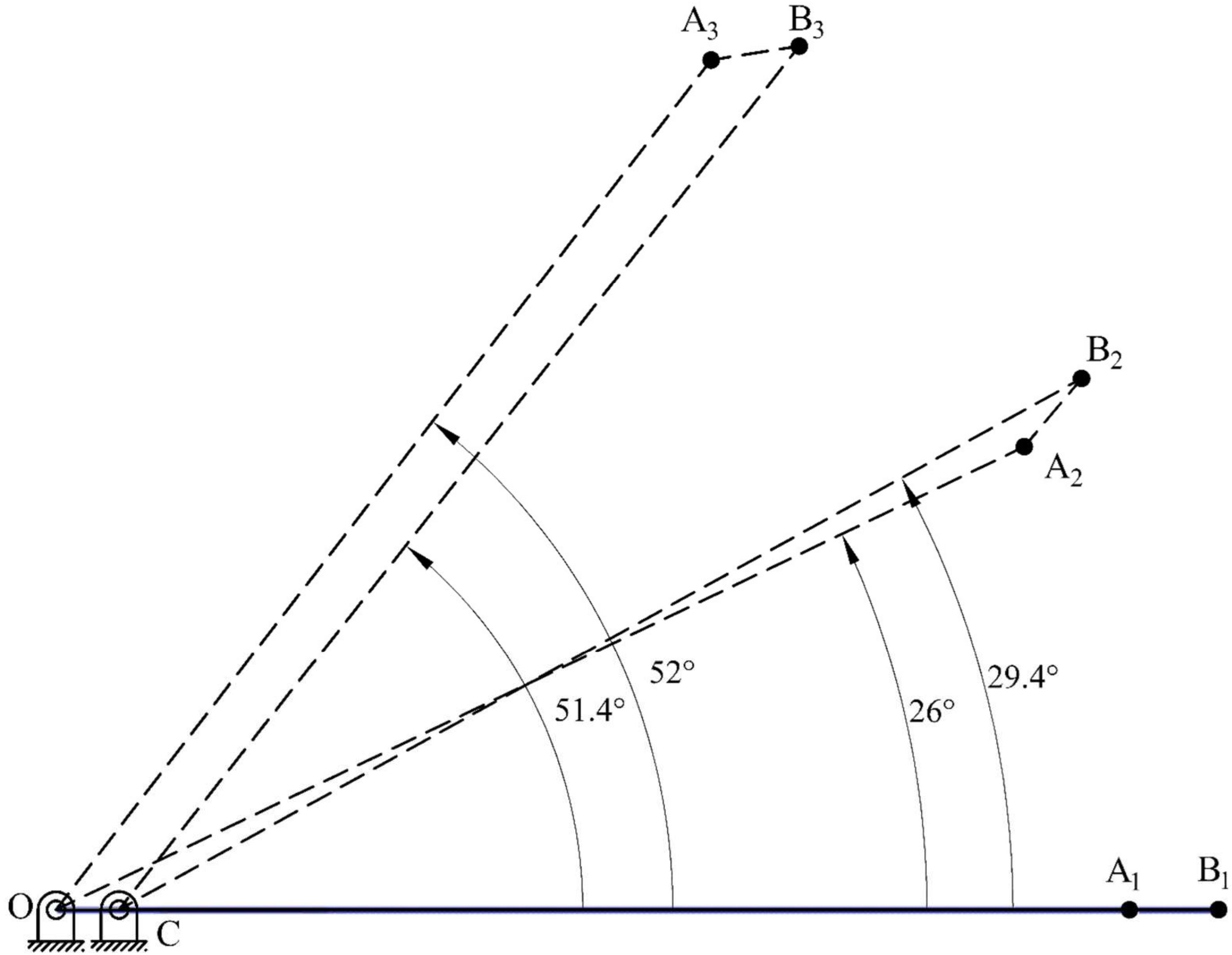
Giving $K_1 = -0.05777 \quad K_2 = -0.059 \quad K_3 = 0.99877$

With the frame $a_1 = 1$ unit of length, we find

$$a_2 = 17.04 \quad a_3 = 1.42 \quad a_4 = 17.45$$

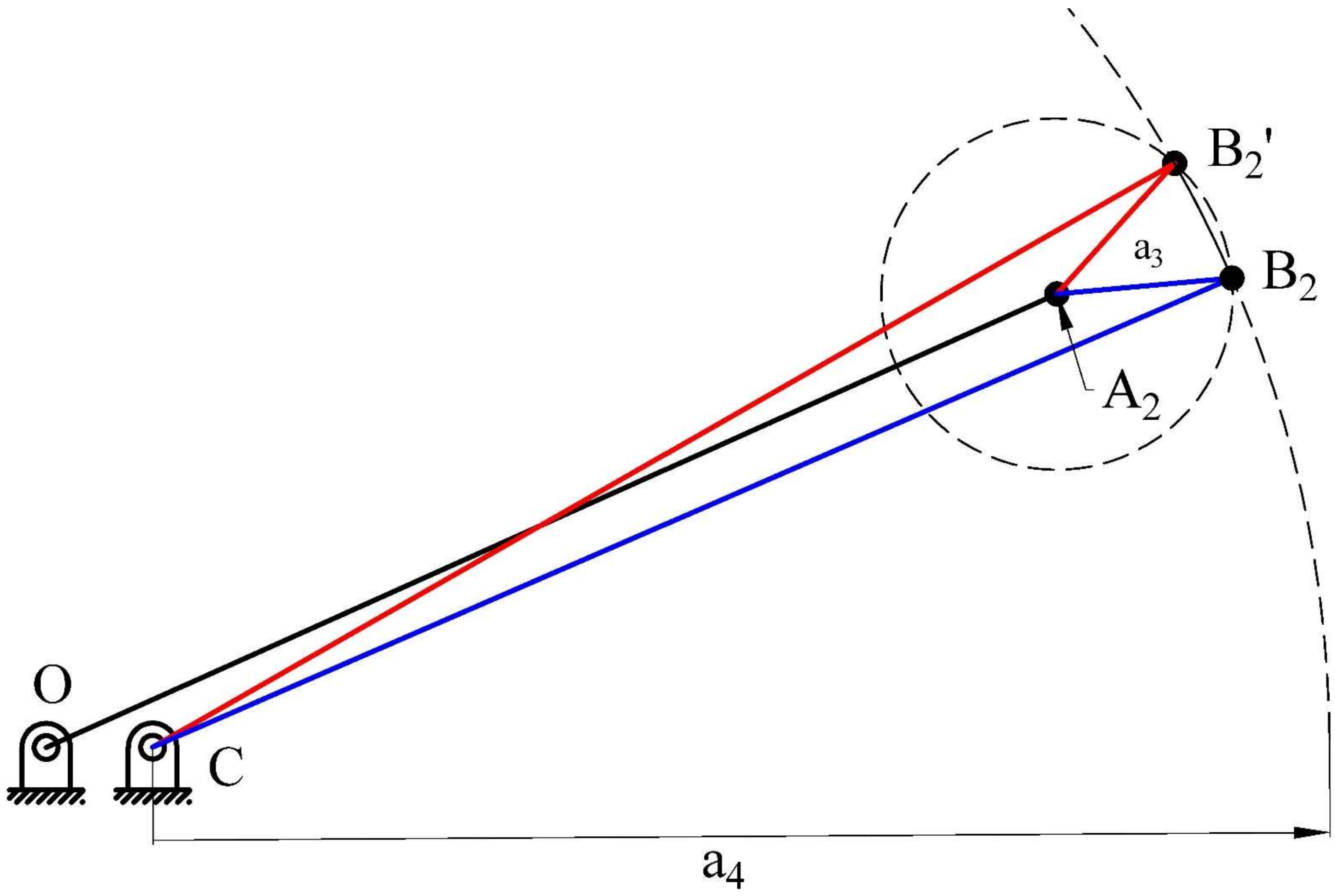
R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

This linkage, with two long links ($a_2 = 17.038, a_4 = 17.454$) and two short links ($a_1 = 1, a_3 = 1.416$), has poor force-transmission qualities and is not an acceptable solution. On second thought, both crank and follower had starting angles of 0°. The unsatisfactory solution was compounded from unhappy choices of arbitrary values-starting angles and ranges of motion.

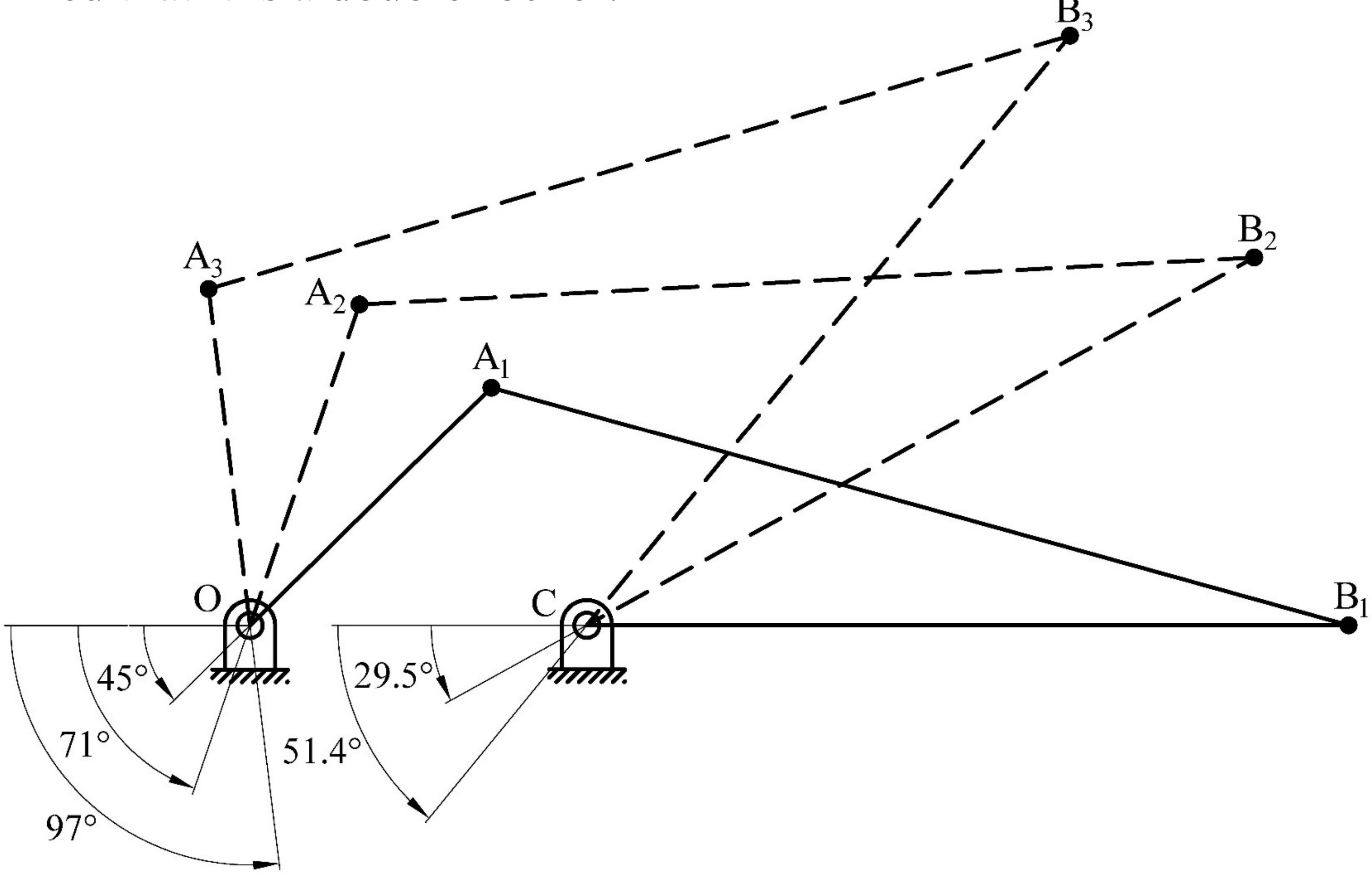


R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

For any synthesized linkage using the analytical method, its motion must be simulated through graphical method so as to check the its **branch defect**. For example, when input link occupies the second precision point CA_2, output link occupies position OB_2. However, output link is supposed to arrive at position OB_2', which is the specified design positions. B_2 and B_2' are output-link positions (branches) which belong to the same input.



The 60° ranges seems desirable so this feature is to be retained. A second attempt, in which $\phi_1 = 45°\ (\phi_2 = 71°, \phi_3 = 97°)$ and $\psi_1 = 0°\ (\psi_2 = 29.4°, \psi_3 = 51.4°)$ were assumed, with $a_1 = 1$, yielded $a_2 = 1.005, a_3 = 2.646, a_4 = 2.259$. These linkage proportions are favorable to force-transmission, and the design may be considered as acceptable, if it is recognized that it is a double rocker.



R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

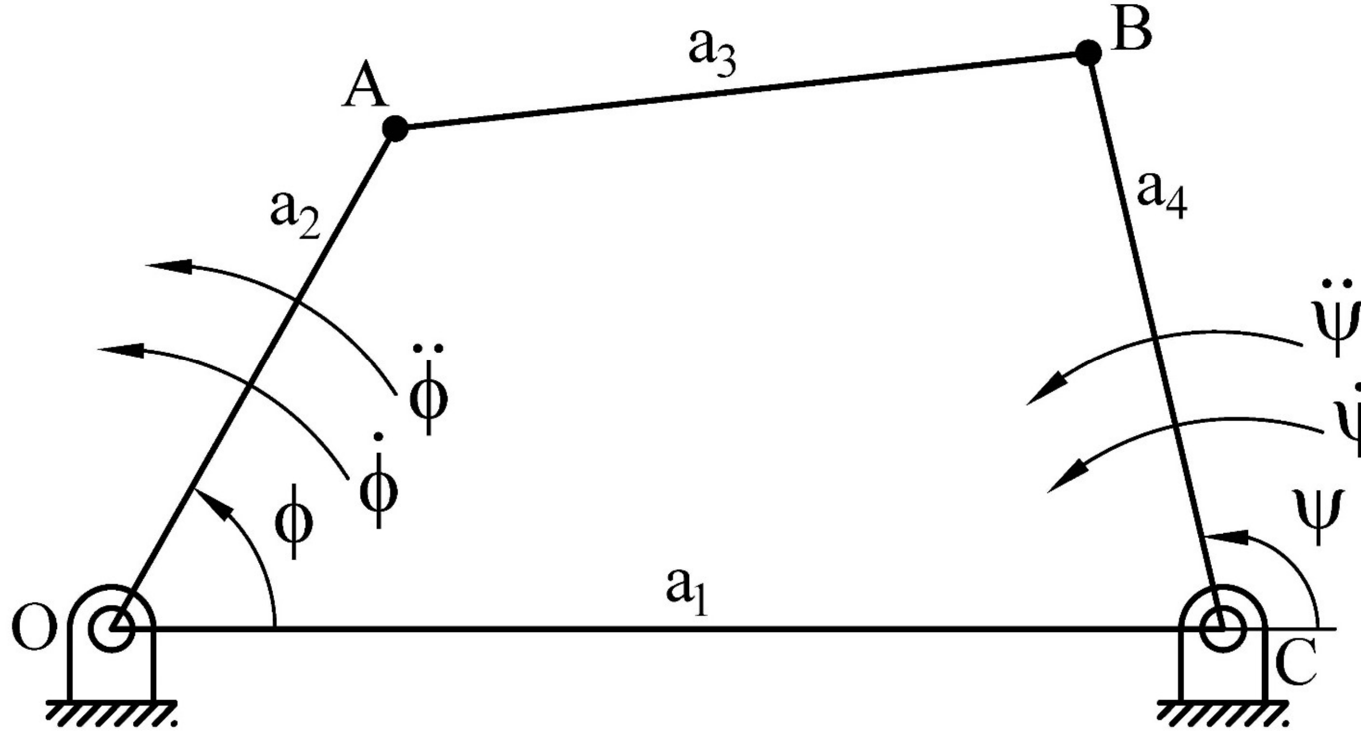
The difference between the values of y_{mech} given by the linkage and the corresponding values of logx, is shown in the last column. As expected, this structural error vanishes at the accuracy points.

x	ϕ, DEG	ψ, DEG	$\log x$	y_{mech}	$y_{\text{mech}} - \log x$
1.0	41.0	−6.1	0	−0.025	−0.025
1.1	47.0	2.8	0.041	0.042	0.001
1.2	53.0	10.4	0.079	0.080	0.001
1.3	59.0	17.3	0.114	0.115	0.001
1.4	65.0	23.5	0.146	0.146	0
1.5	71.0	29.4	0.176	0.176	0
1.6	77.0	34.9	0.204	0.203	−0.001
1.7	83.0	40.1	0.230	0.229	−0.001
1.8	89.0	45.1	0.255	0.254	−0.001
1.9	95.0	49.9	0.279	0.278	−0.001
2.0	101.0	54.5	0.301	0.302	0.001

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Crank and Follower Synthesis Angular Velocities and Accelerations

A planar four-bar linkage may be designed such that, when the crank has a specified position (ϕ), angular velocity ($\dot{\phi}$) and acceleration ($\ddot{\phi}$), the follower will also have a specified position (ψ), angular velocity ($\dot{\psi}$) and acceleration ($\ddot{\psi}$). In other words, it is necessary to determine the parameters a_1, a_2, a_3, a_4 such that a given set of values ϕ, $\dot{\phi}$, $\ddot{\phi}$ will give rise to desired values of ψ, $\dot{\psi}$, $\ddot{\psi}$.



This problem may be solved by taking the first and second time derivatives of the equation mentioned before.

$$K_1\cos\phi - K_2\cos\psi + K_3 = \cos(\phi - \psi)$$

$$K_1\dot{\phi}\sin\phi - K_2\dot{\psi}\sin\psi = (\dot{\phi} - \dot{\psi})\sin(\phi - \psi)$$

$$K_1\left(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi\right) - K_2\left(\ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi\right)$$
$$= (\ddot{\phi} - \ddot{\psi})\sin(\phi - \psi) + (\dot{\phi} - \dot{\psi})^2\cos(\phi - \psi)$$

Solving the two equations for K_1, K_2.

$$K_1 = \frac{w_2 w_6 - w_3 w_5}{w_2 w_4 - w_1 w_5} \qquad K_2 = \frac{w_1 w_6 - w_3 w_4}{w_2 w_4 - w_1 w_5}$$

With $w_1 = \dot{\phi}\sin\phi \quad w_2 = \dot{\psi}\sin\psi \quad w_3 = (\dot{\phi} - \dot{\psi})\sin(\phi - \psi)$

$w_4 = \ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi \quad w_5 = \ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi$

$w_6 = (\ddot{\phi} - \ddot{\psi})\sin(\phi - \psi) + (\dot{\phi} - \dot{\psi})^2\cos(\phi - \psi)$

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

$$K_1 = \frac{w_2 w_6 - w_3 w_5}{w_2 w_4 - w_1 w_5} \qquad K_2 = \frac{w_1 w_6 - w_3 w_4}{w_2 w_4 - w_1 w_5}$$

where $w_1 = \dot{\phi}\sin\phi \quad w_2 = \dot{\psi}\sin\psi \quad w_3 = (\dot{\phi} - \dot{\psi})\sin(\phi - \psi)$

$$w_4 = \ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi \quad w_5 = \ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi$$

$$w_6 = (\ddot{\phi} - \ddot{\psi})\sin(\phi - \psi) + (\dot{\phi} - \dot{\psi})^2\cos(\phi - \psi)$$

$$K_3 = \cos(\phi - \psi) - K_1\cos\phi + K_2\cos\psi$$

The parameters of the linkage are given by

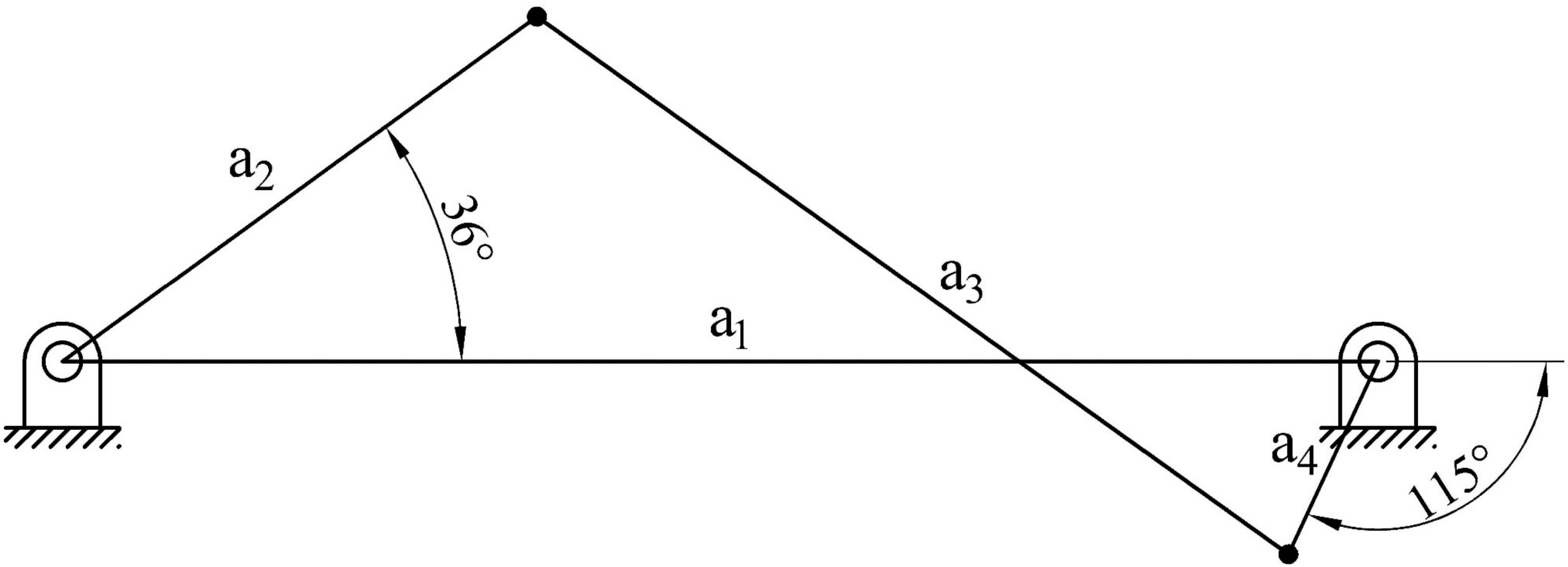
$$a_2 = -\frac{a_1}{K_2} \qquad a_4 = -\frac{a_1}{K_1} \qquad a_3 = \sqrt{{a_2}^2 + {a_4}^2 + {a_1}^2 - 2a_2 a_4 K_3}$$

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Example 2: Design a four-bar linkage meeting the following specifications.

$$\phi = 36° \quad \dot{\phi} = -3 \text{ rad/sec} \quad \ddot{\phi} = 0$$
$$\psi = 65° \quad \dot{\psi} = 8 \text{ rad/sec} \quad \ddot{\psi} = 0$$

Solution: Taking $a_1 = 3.76$, the remaining parameters of the linkage are found as $a_2 = 1.6759 \quad a_3 = 2.6398 \quad a_4 = -0.6064$.



R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Analysis of Mechanical Errors in Linkages

- Consider a linkage with n constant parameters q_1, q_2, …, q_n transforming a motion, defined by an input variable ϕ, into another motion, defined by an output variable ψ. This linkage has been designed to generate a given function such that, if the linkage was built to perfection, the maximum deviation between the desired function and the function generated by the linkage would not exceed ε_s, the structural or mathematical error.
- An additional error, mechanical error (ε_m), due to manufacturing tolerances will now be evaluated in terms of dimensional variations.
- The displacement equation of the linkage may be written in general form as

$$\psi = f(q_1, q_2, \dots, q_n, \phi)$$

The errors are assumed to be Δq_1, Δq_2, …, Δq_n, $\Delta\phi$. The output variable will be $\psi + \Delta\psi$, and we may write

$$\psi + \Delta\psi = f(q_1 + \Delta q_1, q_2 + \Delta q_2, \dots, q_n + \Delta q_n, \phi + \Delta\phi)$$

The mechanical error $\varepsilon_m = \Delta\psi$

The displacement equation of the linkage may be written in general form as

$$\psi = f(q_1, q_2, \dots, q_n, \phi)$$

Total differential of the equation may be written as

$$d\psi = \frac{\partial f}{\partial q_1} \times dq_1 + \frac{\partial f}{\partial q_2} \times dq_2 + \cdots + \frac{\partial f}{\partial q_n} \times dq_n + \frac{\partial f}{\partial \phi} \times d\phi$$

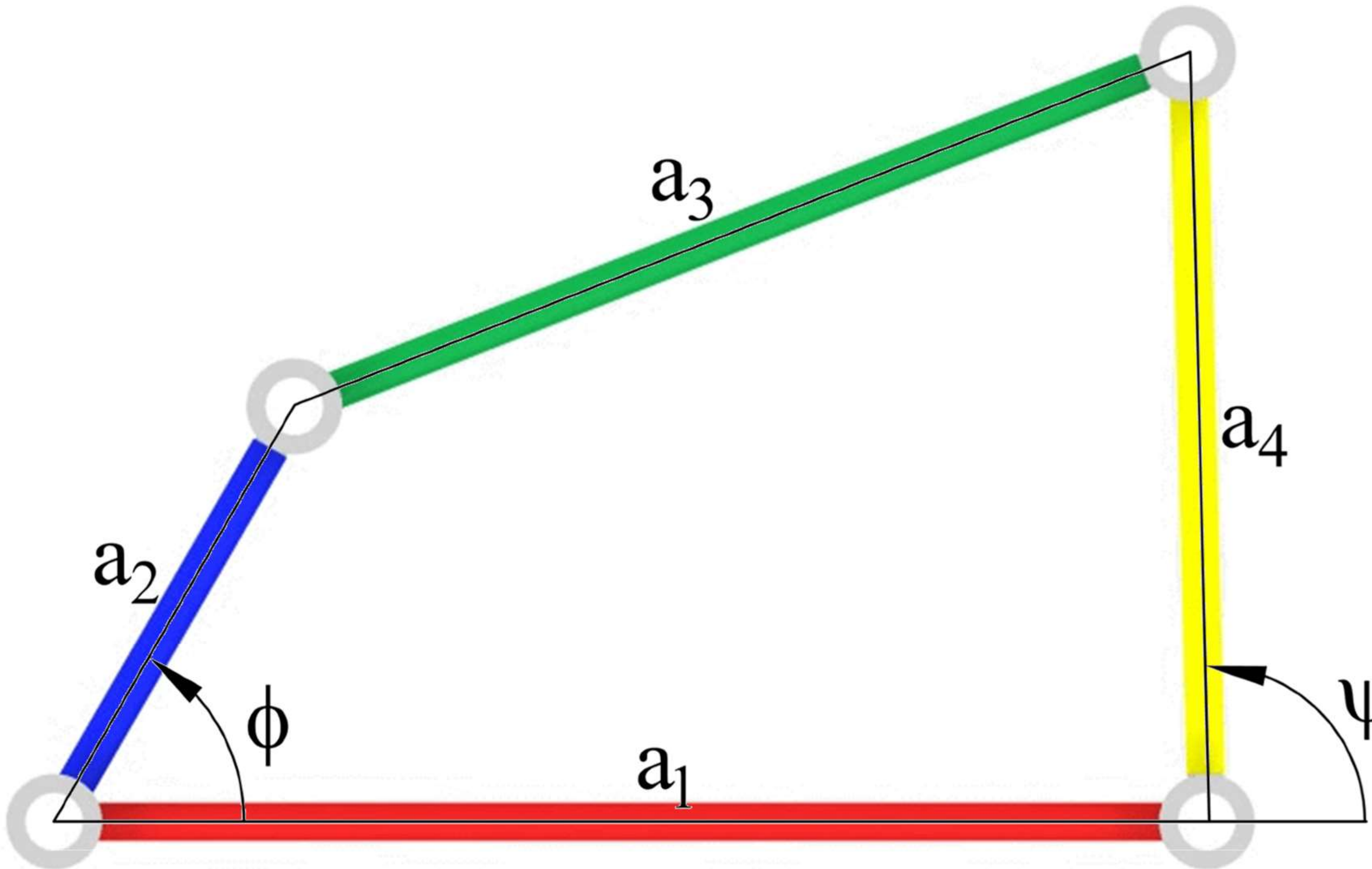
For small values of the errors Δq_1, Δq_2, …, Δq_n, $\Delta\phi$, $\Delta\psi$.

$$\Delta\psi = \varepsilon_m = \frac{\partial f}{\partial q_1} \times \Delta q_1 + \frac{\partial f}{\partial q_2} \times \Delta q_2 + \cdots + \frac{\partial f}{\partial q_n} \times \Delta q_n + \frac{\partial f}{\partial \phi} \times \Delta\phi$$

The total mechanical error ε_m in the linkage is therefore the sum of the individual errors due to each of the parameters considered separately.

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Mechanical Errors in Four-Bar Linkages

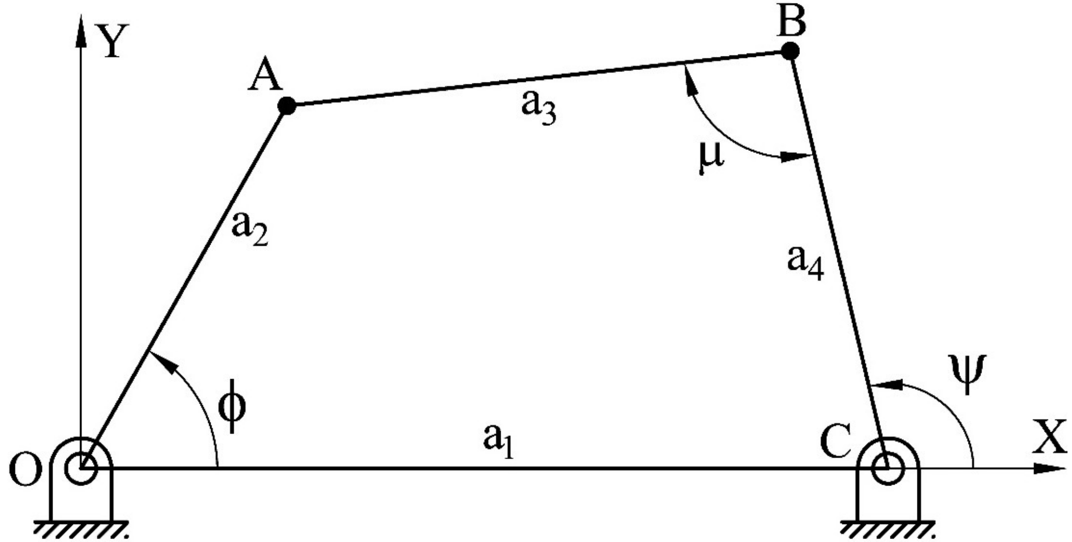


Displacement Equation

$A(a_2\cos\phi, a_2\sin\phi)$

$B(a_1 + a_4\cos\psi, a_4\sin\psi)$

$\overline{AB} = a_3$



$$(a_1 + a_4\cos\psi - a_2\cos\phi)^2 + (a_4\sin\psi - a_2\sin\phi)^2 = {a_3}^2$$

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$$(a_1 + a_4\cos\psi - a_2\cos\phi)^2 + (a_4\sin\psi - a_2\sin\phi)^2 = {a_3}^2$$

$$D\sin\psi + E\cos\psi = F$$

where

$$D = 2a_2a_4\sin\phi$$

$$E = 2a_2a_4\cos\phi - 2a_1a_4$$

$$F = {a_1}^2 + {a_2}^2 - {a_3}^2 + {a_4}^2 - 2a_1a_2\cos\phi$$

We could express $\sin\psi$ and $\cos\psi$ in terms of $\tan(\psi/2)$

$$\sin\psi = \frac{2\tan(\psi/2)}{1 + \tan^2(\psi/2)} \qquad \cos\psi = \frac{1 - \tan^2(\psi/2)}{1 + \tan^2(\psi/2)}$$

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$$D\sin\psi + E\cos\psi = F$$

$$\sin\psi = \frac{2\tan(\psi/2)}{1+\tan^2(\psi/2)} \qquad \cos\psi = \frac{1-\tan^2(\psi/2)}{1+\tan^2(\psi/2)}$$

$$2D\tan\frac{\psi}{2} + E\left(1-\tan^2\frac{\psi}{2}\right) = F\left(1+\tan^2\frac{\psi}{2}\right)$$

$$(E+F)\tan^2\frac{\psi}{2} - 2D\tan\frac{\psi}{2} - E + F = 0$$

$$\tan\frac{\psi}{2} = \frac{D \mp \sqrt{D^2+E^2-F^2}}{E+F}$$

$$\therefore\ \psi = 2\arctan\frac{D \mp \sqrt{D^2+E^2-F^2}}{E+F}$$

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

Mechanical Errors in Four-Bar Linkages

The displacement equation of the four-bar linkage is written as

$$D\sin\psi + E\cos\psi = F$$

in which

$D = 2a_2a_4\sin\phi$

$E = 2a_2a_4\cos\phi - 2a_1a_4$

$F = {a_1}^2 + {a_2}^2 - {a_3}^2 + {a_4}^2 - 2a_1a_2\cos\phi$

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Errors in the link lengths Δa_1, Δa_2, Δa_3, Δa_4 and $\Delta\phi$ will modify the coefficients D, E, F by amounts ΔD, ΔE, ΔF and will produce an error $\Delta\psi$ in the output.

In the presence of errors the displacement equation may be written as

$$(D + \Delta D)\sin(\psi + \Delta\psi) + (E + \Delta E)\cos(\psi + \Delta\psi) = (F + \Delta F)$$

After expansion of this equation ($\Delta\psi \approx 0$, $\sin\Delta\psi = \Delta\psi$, $\cos\Delta\psi = 1$, and neglect of higher-order terms), we find

$$\Delta\psi = \frac{-\Delta D\sin\psi - \Delta E\cos\psi + \Delta F}{D\cos\psi - E\sin\psi} = \varepsilon_m$$ This is the link-error equation

R. S. Hartenberg and J. Denavit, Kinematic Synthesis of Linkages

$$\Delta\psi = \frac{-\Delta D\sin\psi - \Delta E\cos\psi + \Delta F}{D\cos\psi - E\sin\psi} = \varepsilon_m$$

where $D = 2a_2a_4\sin\phi$

$E = 2a_2a_4\cos\phi - 2a_1a_4$

$F = {a_1}^2 + {a_2}^2 - {a_3}^2 + {a_4}^2 - 2a_1a_2\cos\phi$

Error due only to Δa_1 :

$\Delta D = 0$

$\Delta E = (E + \Delta E) - E = -2(a_1 + \Delta a_1)a_4 - (-2a_1a_4) = -2\Delta a_1a_4$

$\Delta F = 2a_1\Delta a_1 - 2\Delta a_1a_2\cos\phi$

$$\Delta\psi_1 = \varepsilon_1 = 2\Delta a_1 \frac{a_4\cos\psi + a_1 - a_2\cos\phi}{D\cos\psi - E\sin\psi}$$

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$$\text{Error due only to } \Delta a_1 : \varepsilon_1 = 2\Delta a_1 \frac{a_4\cos\psi + a_1 - a_2\cos\phi}{D\cos\psi - E\sin\psi}$$

$$\text{Error due only to } \Delta a_2 : \varepsilon_2 = -2\Delta a_2 \frac{a_4\cos(\phi-\psi) - a_2 + a_1\cos\phi}{D\cos\psi - E\sin\psi}$$

$$\text{Error due only to } \Delta a_3 : \varepsilon_3 = -2\Delta a_3 \frac{a_3}{D\cos\psi - E\sin\psi}$$

$$\text{Error due only to } \Delta a_4 : \varepsilon_4 = 2\Delta a_4 \frac{a_4 + a_1\cos\psi - a_2\cos(\phi-\psi)}{D\cos\psi - E\sin\psi}$$

$$\text{Error due only to } \Delta\phi : \varepsilon_\phi = 2\Delta\phi \frac{a_2(a_1\sin\phi + a_4\sin(\phi-\psi))}{D\cos\psi - E\sin\psi}$$

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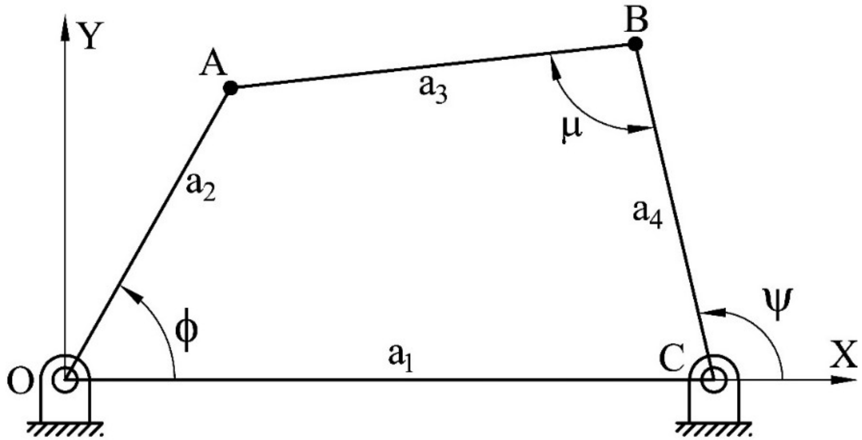
Example 3: Consider a four-bar linkage, the dimensions of this linkage are $a_1 = 1$ in, $a_2 = 3.23$, $a_3 = 0.84$ in, and $a_4 = 3.48$ in. The tolerance of the link lengths are $|\Delta a_1| = |\Delta a_2| = |\Delta a_3| = |\Delta a_4| = 0.001$ in. The mechanical error will now be evaluated at the a accuracy point at $\phi = 53°(\psi = 59.16°)$



$\sin\psi = -0.86 \quad \cos\psi = -0.51$
$D = 17.95 \quad E = 6.57 \quad F = 18.95$
$D\cos\psi - E\sin\psi = 3.56$

$$\varepsilon_1 = 2\Delta a_1 \frac{a_4\cos\psi + a_1 - a_2\cos\phi}{D\cos\psi - E\sin\psi} = 0.471\Delta a_1 \text{ rad} = 0.027°$$

$$\varepsilon_2 = -0.466\Delta a_2 \text{ rad} = -0.0267°$$

$$\varepsilon_3 = -0.471\Delta a_3 \text{ rad} = -0.027°$$

$$\varepsilon_4 = 0.438\Delta a_4 \text{ rad} = 0.0251°$$

The worst-case error : $\varepsilon_{m(max)} = |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4| = 0.1059°$

The maximum expected error: $\varepsilon_{m(rms)} = \sqrt{\varepsilon_1{}^2 + \varepsilon_2{}^2 + \varepsilon_3{}^2 + \varepsilon_4{}^2} = 0.0529°$