MAT 301 - Problem Set 4

Due Wednesday March 26, 2014

Note: If G and G' are groups, $G \oplus G'$ is the direct product of G and G'.

1. Let $G = D_{12} \oplus U(16)$ and $H = \langle (r^{10}, 5) \rangle$, where $r \in D_{12}$ is an element of order 12.
   a) Prove that H is a normal subgroup of G.
   b) Compute the order of the factor group G/H.
   c) Compute the order of the element $(r^4, 3)H$ in the factor group G/H.
   d) Prove or disprove that G/H is abelian.
   e) Prove or disprove that G/H contains an element of order 8. (*Note*: This can be done without computing the orders of all elements of G/H. Think about relations between orders of elements of G and elements of G/H.)

2. Prove or disprove that D_{12} is isomorphic to $\mathbb{Z}_3 \oplus D_4$.

3. Let p be a prime. Recall that the direct product $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is defined as follows:

$$\mathbb{Z}_p \oplus \mathbb{Z}_p = \{ (i, j) \mid i,\, j \in \mathbb{Z}_p \}$$
$$(i, j) * (k, \ell) = ((i + k)(\bmod p), (j + \ell)(\bmod p)), \quad i,\, j,\, k,\, \ell \in \mathbb{Z}_p.$$

   Note: Do not use material or results on internal direct products here. In particular, do not use Theorem 9.7 of the text. In addition, do not use results from Chapter 11.
   a) Let G be an abelian group of order p^2. Assume that G is not cyclic. Let $a \in G$ be such that $a \neq e$. Let $b \in G$ be such that $b \notin \langle a \rangle$. Prove that an element of G has the form $a^i b^j$ for unique integers i and $j \in \{ 0, 1, 2, \ldots, p-1 \}$. (*Hint*: One way to do this is to first show that $a^i b^j = a^k a^\ell$ if and only if $i = k$ and $k = \ell$, for $i,\, j,\, k,\, \ell \in \{ 0, 1, \ldots, p-1 \}$, and then use $|G| = p^2$.)
   b) Prove that a noncyclic abelian group of order p^2 is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. (*Hint*: Use part a) to define an isomorphism $\phi : G \to \mathbb{Z}_p \oplus \mathbb{Z}_p$.)
   c) Prove that an abelian group of order $2p$ is isomorphic to $\mathbb{Z}_{p^2}$ or to $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

   Remark: At some point in the course, we will prove that a group of order p^2 is abelian.

4. Let k and ℓ be fixed integers. Define $\phi_{k,\ell} : \mathbb{Z} \oplus \mathbb{Z}$ by

$$\phi_{k,\ell}(m, n) = k\,m \,+\, \ell\, n, \qquad m,\, n \in \mathbb{Z}.$$

   a) Prove that $\phi_{k,\ell}$ is a homomorphism.
   b) Let $H = \{ (2\,m, -m) \mid m \in \mathbb{Z} \}$. Prove that H is a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$. Use the First Isomorphism Theorem to prove that the factor group $(\mathbb{Z} \oplus \mathbb{Z})/H$ is isomorphic to $\mathbb{Z}$.

   Remark: You were asked to use the First Isomorphism Theorem to solve part b). There is an alternate way to solve part b), by showing that the factor group $(\mathbb{Z}\oplus\mathbb{Z})/H$ is an infinite cyclic group. An infinite cyclic group is isomorphic to $\mathbb{Z}$.

5. Let $\alpha = (\,2\,3\,7\,)(\,6\,11\,9\,)(\,7\,6\,11\,8\,) \in S_{12}$.
   a) Find $\beta \in S_{12}$ such that $\beta\,\alpha\,\beta^{-1} = \alpha^3$.
   b) Determine which elements of $\langle\alpha\rangle$ are conjugate to α.
   c) Let $T = \{\,\gamma \in S_{12} \mid |\gamma| = |\alpha|\,\}$. Determine the number of distinct conjugacy classes in T.

6. Let $\alpha \in S_n$. Prove that $|\alpha|$ is odd if and only if α and α^2 are conjugate in S_n.

7. Let H be a normal subgroup of a finite group G.
   a) Let n be the number of distinct conjugacy classes in G and let m be the number of distinct conjugacy classes in G/H. Prove that if $H \neq \{e\}$, then $m < n$. (*Hint*: As a first step, show that if a and b are conjugate in G, then aH and bH are conjugate in G/H.)
   b) If $a \in G$, let $C_G(a) = \{\,c \in G \mid cac^{-1} = a\,\}$. Then $C_G(a)$ is a subgroup of G (not necessarily normal in G). We will show in class that the number $|\mathrm{cl}_G(a)|$ of elements in the conjugacy class $\mathrm{cl}_G(a)$ of a in G is equal to $|G|/|C_G(a)|$. Prove that if $a \in H$, then $|\mathrm{cl}_H(a)|$ divides $|\mathrm{cl}_G(a)|$. (*Note*: This part is independent of part a).)