MAT 301 - Problem Set 3

Due Wednesday, March 12, 2014

1. In each case below, prove or disprove that the subgroup H is normal in the group G. If H is normal in G, determine whether the factor group G/H is abelian. (*Note*: You do not need to show that H is a subgroup of G.)
   a) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a,\, b \in \mathbb{R},\, a \neq 0 \right\}$ (under matrix multiplication).
      Let $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R}, \right\}$.
   b) Let $G = S_4$ and let $H = \langle (1\,3\,4) \rangle$.
2. Let $G = \mathbb{Z}_{84}$ and $H = \langle 69 \rangle$.
   a) Compute the order $|G/H|$ of the factor group G/H.
   b) Determine the order of the element $46 + H$ in the factor group G/H.
3. Prove that S_4 does not have a normal subgroup of order 8. (*Hint*: Suppose that H is a normal subgroup of S_4 such that $|H| = 8$. Prove that every element of order 2 in S_4 must belong to H.)
4. Let H be a normal subgroup of a group G. Let $H' = \{\, a \in G \mid a^2 \in H \,\}$.
   a) Suppose that G/H is abelian. Prove that H' is a subgroup of G. Prove that H' is normal in G.
   b) *Do not hand in this part.* Find an example where $H \neq \{e\}$, G/H is not abelian and H' is not a subgroup of G. (*Hint*: Try dihedral groups with nontrivial centre.)
5. In each case below, determine whether the function $\phi : G \to G'$ is a homomorphism. If ϕ is a homomorphism, find all elements in the kernel $\operatorname{Ker}\phi$ of ϕ.
   a) Let $G = \langle a \rangle$ be a cyclic group of order 12 and let $G' = \langle b \rangle$ be a cyclic group of order 18. Define $\phi(a^j) = b^{15j}$, $0 \leq j \leq 11$.
   b) Let $G = \mathbb{R}$ and $G' = GL(2, \mathbb{R})$. Define
   $$\phi(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \qquad x \in \mathbb{R}.$$
6. Let $r \in D_{20}$ be an element of order n and let $s \in D_{20}$ be a reflection. Suppose that $\phi : D_{20} \to D_{20}$ is a homomorphism such that $\phi(r) = r^{12}$.
   a) Prove that $\phi(s)$ is a reflection.
   b) Find all elements in the kernel $\operatorname{Ker}\phi$ of ϕ.
   c) Prove that the factor group $D_{20}/\operatorname{Ker}\phi$ is nonabelian.
   d) Prove that $D_{20}/\ker\phi$ is isomorphic to a dihedral group D_m. (What is m?)
7. Let G be a finite group. Suppose that $\phi : G \to S_4$ is a homomorphism and ϕ is onto.
   a) Prove that G is not abelian.
   b) Prove that G contains an element of order 4.
   c) Let $H = \{\, a \in G \mid \phi(a) \in A_4 \,\}$. Prove that H is a subgroup of G.
   d) Determine whether the subgroup H defined in part c) is a normal subgroup of G.