MAT301 Problem Set 3

Daniel Kats
997492468

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1 Question 1

1.(a)

Suppose that $x = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \in G$. Then we have

$$xH = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab + c \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

Also

$$Hx = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + c \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

Note that (1.2) is not equal to (1.1) when $a \neq 1$ and $b \neq 0$. Therefore H is not a normal subgroup of G.

1.(b)

Consider $x = (1, 2, 3) \in S_4$. Then $x^{-1} = (3, 2, 1)$. Using the normal subgroup test from Theorem 9.1 in the book, we look at xHx^{-1} and see if this is a subset of H. We know that any element in H has the form $(1, 3, 4)^n$ where n is a natural number. Consider the case where $n = 2$

$$(1, 2, 3)(1, 3, 4)^2(3, 2, 1) = (1, 2, 3)(1, 4, 3)(3, 2, 1) = (1, 2, 4) \notin H \tag{1.3}$$

Therefore $xHx^{-1} \not\subset H$, so H is not a normal subgroup of G.

2 Question 2

2.(a)

By Lagrange's Theorem, the number of distinct cosets is precisely $|G|/|H|$. Moreover, we can see that $H = \langle 69 \rangle = \langle 3 \rangle$. Therefore the order of the factor group G/H is simply 3.

2.(b)

The element $46 + \langle 69 \rangle$ is the same as the element $46 + \langle 3 \rangle$, and is actually the same as the element $1 + \langle 3 \rangle$. Therefore, the order of this element is 3.

3 Question 3

Proof. Suppose that H is a normal subgroup of S_4 such that $|H| = 8$. I will show that every element of order 2 in S_4 must belong to H.

Since H is normal, the factor group S_4/H exists and has order 3 (by Lagrange's Theorem, and because $|S_n| = n!$). Since the order of each element divides the order of the group, for all $\alpha \in S_4$, it is the case that $(\alpha H)^3 = \alpha^3 H = H$ (because each coset of the form αH is actually an element of the factor group). This means that $\alpha^3 \in H$ for all $\alpha \in S_4$.

Any α of order 2 has the property that $\alpha^3 = \alpha$. So H contains all elements of order 2. But there are at least 9 of these: 6 of the form (a, b) and at least 3 of the form $(a, b)(c, d)$. Contradiction, because cardinality of H should be 8 ¡ 9.

□

4 Question 4

4.(a)

I will show H' is a subgroup of G.

Proof. Consider $e \in G$. Clearly $e \in H$, and $e^2 = e$, so $e \in H'$, so H' is non-empty. Not consider any two elements $a, b \in H'$. Clearly $a, b \in G$ and we have $a^2, b^2 \in H$ by definition. Notice that because G is a group, $ab^{-1} \in G$. All that remains to be proven is that $ab^{-1}ab^{-1} \in H$.

Now, suppose that G/H is Abelian. Then for every $a, b \in G$, we have:

$$(aH)(bH) = (bH)(aH) \tag{4.1}$$

$$abH = baH \tag{4.2}$$

$$abH = baH \tag{4.3}$$

This means that for every $x \in H$, there exists $y \in H$ such that

$$abx = bay \tag{4.4}$$

$$(ba)^{-1}abx = y \tag{4.5}$$

Therefore it must be the case that $(ba)^{-1}abx \in H$. Since $x \in H$ and H is a group, $(ba)^{-1}ab \in H$. And by closure of H, we can multiply this quantity on the left by a^2 and on the right by b^{-2}, and we get $ab^{-1}ab^{-1} \in H$. So we have shown that H' is a subgroup of G.

$\square$

I will show H' is normal in G.

Proof. I will show that for every $g \in G$ and $a \in H'$, it follows that $gag^{-1} \in H'$. First, it is trivial that $gag^{-1} \in G$, since G is a group. Now I must show $(gag^{-1})^2 \in H$. $(gag^{-1})^2 = ga^2g^{-1}$. But this is trivial when $a^2 \in H$, since we know H is normal, which is precisely this statement. $\square$

5 Question 5

5.(a)

Suppose $x = a^i \in G$ and $y = a^j \in G$ where i and j are non-negative integers.

$$\phi(x)\phi(y) = \phi(a^i)\phi(a^j) = b^{15i}b^{15j} = b^{15(i+j)} \tag{5.1}$$

$$\phi(xy) = \phi(a^ia^j) = \phi(a^{i+j}) = b^{15(i+j)} \tag{5.2}$$

Therefore ϕ is a homomorphism by definition.

Since the order of G' is 18, we have $b^{18k} = e$ where k is a non-negative integer. So we are looking for all non-negative integers j such that $15j = 18k$. $j = \frac{6k}{5}$. Since j must be an integer, k must be divisible by 5, and so j may be any multiple of 6. However, note that the order of G is 12, so there are actually only 2 elements in G that fit this criteria: $a^0 = e$ and a^6.

$$Ker(\phi) = \{e, a^6\} \tag{5.3}$$

5.(b)

Let $x, y \in \mathbb{R}$.

$$\phi(x)\phi(y) = \begin{pmatrix} cos(x) & -sin(x) \\ sin(x) & cos(x) \end{pmatrix} \begin{pmatrix} cos(y) & -sin(y) \\ sin(y) & cos(y) \end{pmatrix}$$

$$= \begin{pmatrix} cos(x)cos(y) - sin(x)sin(y) & -cos(x)sin(y) - sin(x)cos(y) \\ sin(x)cos(y) + cos(x)sin(y) & -sin(x)sin(y) + cos(x)cos(y) \end{pmatrix}$$

$$= \begin{pmatrix} cos(x+y) & -sin(x+y) \\ sin(x+y) & cos(x+y) \end{pmatrix} \tag{5.4}$$

$$\phi(x+y) = \begin{pmatrix} cos(x+y) & -sin(x+y) \\ sin(x+y) & cos(x+y) \end{pmatrix} \tag{5.5}$$

Therefore ϕ is a homomorphism by definition. Note that this is under the assumption that the group operation is addition in the reals.

The identity in G' is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $cos(x) = 1$ and $sin(x) = 0$. Therefore the kernel consists of all x that are an integer multiple of 2π.

$$Ker(\phi) = \{2\pi k \mid k \in \mathbb{Z}\} \tag{5.6}$$

6 Question 6

6.(a)

Prove that $\phi(s)$ is a reflection.

Proof. Consider the property of dihedral groups where r is any rotation and s is any reflection: $rs = sr^{-1}$. In particular, note that $\phi(rs) = \phi(sr^{-1})$

$$\phi(rs) = \phi(r)\phi(s) = r^{12}\phi(s) \tag{6.1}$$

$$\phi(sr^{-1}) = \phi(s)[\phi(r)]^{-1} = \phi(s)(r^{12})^{-1} = \phi(s)r^8 \tag{6.2}$$

Note that these two quantities must equal each other.

$$r^{12}\phi(s) = \phi(s)r^8 \tag{6.3}$$

Note that $\phi(s) \neq e$, since $r^{12} \neq r^8$ (which we know because $|r| = 20$).

Suppose that $\phi(s)$ is a rotation. Note that rotations commute, so we have $\phi(s)r^{12} = \phi(s)r^8$. Now we are able to left-multiply by $\phi(s)^{-1}$ (which must exist because D_{20} is a group) to obtain $r^{12} = r^8$, which is a contradiction. Therefore $\phi(s)$ cannot be a rotation either. It follows that $\phi(s)$ must be a reflection.

□

6.(b)

Because of part (a), we know that no reflection is in the kernel. For any rotation x, note that $x = r^k$ where k is a non-negative integer. We have

$$\phi(x) = \phi(r^k) = [\phi(r)]^k = r^{12k} \tag{6.4}$$

If we have $r^{12k} = e$, then $12k$ must be divisible by 20. It follows that $3k$ must be divisible by 5, so k must be a multiple of 5.

$$Ker(\phi) = \{e, r^5, r^{10}, r^{15}\} \tag{6.5}$$

These are the only elements in the kernel, because D_{20} contains only reflections and rotations (and identity maps to identity under all homomorphisms).

6.(c)

Proof. Let s be any reflection from D_{20}, and let r be the rotation of degree 20 from D_{20}.

On the one hand, we have:

$$sr\{e, r^5, r^{10}, r^{10}\} = \{sr, sr^6, sr^{11}, sr^{16}\} \tag{6.6}$$

On the other hand, we have:

$$rs\{e, r^5, r^{10}, r^{10}\} = sr^{-1}\{e, r^5, r^{10}, r^{10}\} = \{sr^{19}, sr^4, sr^9, sr^{14}\} \tag{6.7}$$

Clearly these two sets are not equal, so the factor group is not Abelian. □

6.(d)

Consider the Theorem 10.3 from the textbook. Since ϕ is a homomorphism from D_{20} to D_{20}, we have that there exists an isomorphism between $D_{20}/Ker\phi$ and D_{20} given by $gKer\phi \to \phi(g)$. This works because ϕ is onto. m in this case is 20.

7 Question 7

7.(a)

I will show that G is not Abelian.

Proof. Suppose that G is Abelian. Consider any two elements $x, y \in S_4$. Because ϕ is onto, there exist elements $a, b \in G$ such that $\phi(a) = x$ and $\phi(b) = y$. Therefore we have:

$$xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx \tag{7.1}$$

And we have shown that S_4 is Abelian, which we know is not the case. Contradiction. □

7.(b)

I will show that G contains an element of order 4.

Proof. We know that S_4 has an element of order 4. Call it α. Because ϕ is onto, there exists $x \in G$ such that $\phi(x) = \alpha$. Moreover, by properties of isomorphisms, we have $|x|$ is divisible by 4. □

7.(c)

I will prove that H is a subgroup of G.

Proof. First, notice that $e \in A_4$ and ϕ maps identity to identity. Therefore $e \in H$ (and H is non-empty). Now, consider $a, b \in H$. I will show that $ab^{-1} \in H$ as well.

This is evident simply by properties of homomorphisms.

$$\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)[\phi(b)]^{-1} \tag{7.2}$$

We already know $\phi(a) \in A_4$, and because A_4 is a group and $\phi(b) \in A_4$, it must be the case that $[\phi(b)]^{-1} \in A_4$ as well. And again because A_4 is a group, the whole thing must be in A_4. So by definition $ab^{-1} \in H$ and we are done.

□

7.(d)

It is normal. Note that if $x \in G$ is part of H, then it is trivial. And if $x \notin H$, then clearly $x^{-1} \notin H$. It must be the case that both are mapped to something not in A_4, otherwise they would be in A_4. And we know that the product of two odd permutations and an even permutation is an even permutation. So we are done.