Cauchy–Schwarz Master Class

ho boon suan

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Here are some solutions to J. Michael Steele's awesome book, *The Cauchy–Schwarz Master Class*.

Exercise 1.1. Both identities are immediate consequences of the Cauchy–Schwarz inequality (C–S). We have

$$\begin{aligned} a_1 + \cdots + a_n &= 1a_1 + \cdots + 1a_n \\ &\le (1^2 + \cdots + 1^2)^{1/2}(a_1^2 + \cdots + a_n^2)^{1/2} \\ &= \sqrt{n}(a_1^2 + \cdots + a_n^2)^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} a_1 + \cdots + a_n &= a_1^{1/3}a_1^{2/3} + \cdots + a_n^{1/3}a_n^{2/3} \\ &\le (|a_1|^{2/3} + \cdots + |a_n|^{2/3})^{1/2}(|a_1|^{4/3} + \cdots + |a_n|^{4/3})^{1/2} \end{aligned}$$

Exercise 1.2. First note that

$$\frac{a_j}{a_k} + \frac{a_k}{a_j} = \frac{a_j^2 + a_k^2}{a_j a_k} = \frac{(a_j - a_k)^2}{a_j a_k} + \frac{2a_j a_k}{a_j a_k} \ge 2.$$

since $a_j b_j \ge 1$, we have $b_j \ge 1/a_j$, and so

$$\begin{aligned} (\sum_j p_j a_j)(\sum_k p_k b_k) &\ge (\sum_j p_j a_j)(\sum_k p_k/a_k) \\ &= \sum_{j,k} p_j p_k a_j/a_k \\ &= \sum_j p_j^2 + \sum_{j<k} p_j p_k (a_j/a_k + a_k/a_j) \\ &\ge \sum_j p_j^2 + 2\sum_{j<k} p_j p_k \\ &= (\sum_j p_j)^2 \\ &= 1 \end{aligned}$$

as needed.

Exercise 1.3. The first identity follows from applying C–S twice:

$$\begin{aligned} \sum a_k(b_k c_k) &\le (\sum a_k^2)^{1/2}(\sum b_k^2 c_k^2)^{1/2} \\ &\le (\sum a_k^2)^{1/2}(\sum b_k^4)^{1/4}(\sum c_k^4)^{1/4}. \end{aligned}$$

The proof for the second identity starts the same as the proof for the first, but then we use the fact that

$$\sum b_k^2 c_k^2 \le (\sum b_k^2)(\sum c_k^2)$$

to conclude.

Exercise 1.4. (a) Apply C–S with the 1-trick (see exercise 1.1):

$$\begin{aligned}\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} &\le \sqrt{3}\sqrt{(x+y)+(y+z)+(z+x)}\\ &= \sqrt{6}\sqrt{x+y+z}.\end{aligned}$$

Notice that equality holds iff $x=y=z$.

(b) By C–S, we have

$$\begin{aligned}x+y+z &= \frac{x}{\sqrt{y+z}}\sqrt{y+z}+\frac{y}{\sqrt{x+z}}\sqrt{x+z}+\frac{z}{\sqrt{x+y}}\sqrt{x+y}\\ &\le \sqrt{\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}}\sqrt{(y+z)+(x+z)+(x+y)}\\ &= \sqrt{2}\sqrt{\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}}\sqrt{x+y+z};\end{aligned}$$

rearranging and squaring gives the claim. As before, equality holds iff $x=y=z$.

Exercise 1.5. Since

$$\begin{aligned}g(2x) &= \sum p_k\cos(2\beta_k x)\\ &= \sum p_k(2\cos^2(\beta_k x)-1)\\ &= 2\sum p_k\cos^2(\beta_k x)-\sum p_k\\ &= 2\sum p_k\cos^2(\beta_k x)-1,\end{aligned}$$

it suffices to prove that

$$\sum p_k\cos^2(\beta_k x)\ge g^2(x) := (\sum p_k\cos(\beta_k x))^2.$$

This follows from C–S; we have

$$\begin{aligned}(\sum p_k\cos(\beta_k x))^2 &= (\sum \sqrt{p_k}(\sqrt{p_k}\cos(\beta_k x)))^2\\ &\le \sum p_k\sum p_k\cos^2(\beta_k x)\\ &= \sum p_k\cos^2(\beta_k x).\end{aligned}$$

Exercise 1.6. By C–S with the 1-trick, we have

$$\sum\Big(p_k+\frac{1}{p_k}\Big)^2 \ge \frac{1}{n}\Big(\sum\Big(p_k+\frac{1}{p_k}\Big)\Big)^2;$$

thus it suffices to prove that $\sum 1/p_k \geq n^2$. We compute

$$\begin{aligned}
\sum \frac{1}{p_k} &= \sum \frac{p_1 + \cdots + p_n}{p_k} \\
&= \sum_{j,k} \frac{p_j}{p_k} \\
&= n + \sum_{j<k} \Big(\frac{p_j}{p_k} + \frac{p_k}{p_j} \Big) \\
&= n + \sum_{j<k} \frac{p_j^2 + p_k^2}{p_j p_k} \\
&= n + \sum_{j<k} \frac{(p_j - p_k)^2 + 2p_j p_k}{p_j p_k} \\
&\geq n + \frac{n(n-1)}{2} \cdot 2 \\
&= n^2,
\end{aligned}$$

which completes the proof. From the use of C–S at the start, we see that $p_1 + 1/p_1 = \cdots = p_n + 1/p_n$ is a necessary condition for equality to hold. Since $x \mapsto x + 1/x$ is strictly decreasing on $(0, 1]$, it is injective, and so we see that $p_1 = \cdots = p_n = 1/n$. Since $\sum (n + 1/n)^2 = n^3 + 2n + 1/n$, we see that these conditions are sufficient as well.

Exercise 1.7. Define an inner product $\langle \cdot, \cdot \rangle'$ on $\mathbf{R}^2$ by

$$\Big\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle' := 5\alpha x + \alpha y + \beta x + 3\beta y;$$

this is an inner product since it is a linear combination of inner products. The identity in question immediately follows by applying C–S with this inner product.

Exercise 1.8. (a) By C–S,

$$\sum a_k x^k \leq (\sum a_k^2)^{1/2} (\sum x^{2k})^{1/2} = \frac{1}{\sqrt{1 - x^2}} (\sum a_k^2)^{1/2}.$$

(b) By C–S,

$$\sum \frac{a_k}{k} \leq \Big(\sum \frac{1}{k^2} \Big)^{1/2} (\sum a_k^2)^{1/2}.$$

Since $(\sum 1/k^2)^{1/2} \leq (\pi^2/6)^{1/2} < \sqrt{2}$, the result follows.

(c) By C–S,

$$\sum \frac{a_k}{\sqrt{n+k}} \leq \Big(\sum_{k=1}^{n} \frac{1}{n+k} \Big)^{1/2} (\sum a_k^2)^{1/2}.$$

Since $1/(j+1) < 1/x$ for $x \in [j, j+1)$, we may compute

$$\begin{aligned}
\frac{1}{n+1} + \cdots + \frac{1}{2n} &= \int_n^{n+1} \frac{1}{n+1}\, dx + \cdots + \int_{n-1}^{2n} \frac{1}{2n}\, dx \\
&< \int_n^{n+1} \frac{1}{x}\, dx + \cdots + \int_{n-1}^{2n} \frac{1}{x}\, dx \\
&= \int_n^{2n} \frac{1}{x}\, dx \\
&= \log 2,
\end{aligned}$$

from which the result follows.

(d) Apply C–S as before, and use the fact that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n};$$

combinatorially, this is interpreted as the fact that to choose n students from a class of $2n$, you can choose k boys and $n-k$ girls for any $0 \leq k \leq n$. (Alternatively, compare coefficients in the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$.)

Exercise 1.9. Since

$$\begin{aligned} |\sum a_j|^2 + |\sum (-1)^j a_j|^2 &= \sum_{j,k} a_j a_k + \sum_{j,k} (-1)^{j+k} a_j a_k \\ &= \sum_{j+k \text{ even}} 2a_j a_k \\ &\leq 2\sum a_j^2 + 2 \sum_{\substack{j+k \text{ even} \\ j<k}} 2a_j a_k \\ &\leq 2\sum a_j^2 + 2 \sum_{\substack{j+k \text{ even} \\ j<k}} (a_j^2 + a_k^2), \end{aligned}$$

it suffices to prove that

$$\sum_{\substack{j+k \text{ even} \\ j<k}} (a_j^2 + a_k^2) \leq \frac{n}{2} \sum a_i^2.$$

Notice that

$$\sum_{\substack{j+k \text{ even} \\ j<k}} (a_j^2 + a_k^2) = \sum (r_i + c_i) a_i^2$$

with $r_i = \#\{1 \leq j < i : j+i \text{ even}\}$ and $c_i = \#\{i < k \leq n : i+k \text{ even}\}$. Since

$$r_i + c_i = \left\lfloor \frac{i-1}{2} \right\rfloor + \left\lfloor \frac{n-i+1}{2} \right\rfloor \leq \frac{n}{2},$$

the result follows.