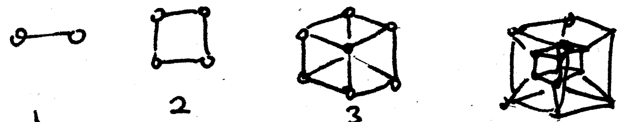
1.1. $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$

1.2. Determine d, $|E|$, diam, g, circumference.



1 2 3 4

The d-cube is d-regular.

To form the $(d+1)$-cube, we take two d-cubes and join the corresponding vertices.

Thus, $|E((d+1)\text{-cube})| = 2|E(d\text{-cube})| + |V(d\text{-cube})|$.

n	1	2	3	4	5	6	7	8	...	k
$\|E\|$	1	4	12	32	80	192	448	1024	...	$k \cdot 2^{k-1}$
$\|V\|$	2	4	8	16	32	64	128	256	...	2^k

An upper bound for the diameter of the d-cube is d, since two d-bit strings necessarily differ in at most d positions. The pair 0000...00 and 1111...11 achieve this bound, and so diam(d-cube) = d.

For $d \geq 2$, there is always a cycle

$$\begin{array}{ccc} 000\cdots00 & - & 000\cdots01 \\ | & & | \\ 000\cdots10 & - & 000\cdots11 \end{array}$$

so $g(d\text{-cube}) \leq 4$. In fact, d-cubes cannot contain triangles since each bit-flip changes the parity of the count of 0's and 1's. (Does the d-cube contain every even cycle?)

Finally, the d-cube circumference (for $d \geq 2$) is $|E| = k \cdot 2^{k-1}$ — that is, the d-cube $(d \geq 2)$ is Hamiltonian. This is proven inductively:

~~00 — 01 — 10 — 11~~ 00 — 01 — 11 — 10 $\rightsquigarrow$

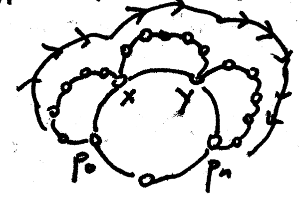
000 — 001 — 011 — 010
100 — 101 — 111 — 110

$\rightsquigarrow$

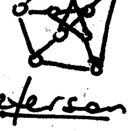
000 — 001 011 — 010
100 — 101 111 — 110

Let us prove this more formally. The 2-cube is Hamiltonian since it is just a 4-cycle:
$$\begin{array}{ccc} 00 & - & 01 \\ | & & | \\ 10 & - & 11 \end{array}$$
Suppose inductively that we have shown the d-cube to be Hamiltonian. Notice that the set of vertices of the $(d+1)$-cube, written $V((d+1)\text{-cube})$, can be expressed as '0' + $V(d\text{-cube}) \cup$ '1' + $V(d\text{-cube})$, where the + denotes string concatenation. This gives us two disjoint cycles in the $(d+1)$-cube; we must now join them to form a Hamiltonian cycle. If the Hamiltonian cycle in the d-cube is written $v_0 \cdots v_n v_0$, then our two cycles are $0v_0 - 0v_1 - \cdots 0v_n - 0v_0$ and $1v_0 - 1v_1 - \cdots - 1v_n - 1v_0$. We claim that $0v_0 - 1v_0 - 1v_n - \cdots - 1v_1 - 0v_1 - \cdots - 0v_n - 0v_0$ is a Hamiltonian cycle for the $(d+1)$-cube. Indeed, it covers all the vertices of the $(d+1)$-cube, and the vertices are indeed all adjacent — for example, $0v_0$ and $1v_0$ differ only in the first digit. This concludes the proof.

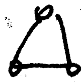
1.3. Suppose C is of length $m < \sqrt{k}$, and suppose we have a path $P = p_0 \cdots p_n$ between vertices p_0 and p_n of C with $n \geq k$. At most m of the p_i's lie on P, and so P must contain a C-path xPy of length at least $\frac{n}{m} \geq \sqrt{k}$, giving us a cycle $xPyCx$ of length at least $\sqrt{k}$.



1.4. Yes. $(g = 2\,\text{diam} + 1)$



Petersen graph
$g = 5$
diam $= 2$



Triangle
$g = 3$
diam $= 1$

(Do such graphs exist for all values diam $\in \mathbb{N}$?)

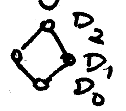
1.5. We have $D_0 = \{v_0\} = \{v \in G : d(v, v_0) = 0\}$ and $D_1 = N_G(v_0) = \{v \in G : d(v, v_0) =$
Now $D_2 = N_G(D_0 \cup D_1) \subseteq N_G(D_1)$ since $N_G(D_0) \subseteq D_1$, and $N_G(D_0 \cup D_1)$
$\subseteq G - D_0 \cup D_1$. In other words: if $v \in N_G(D_0 \cup D_1)$, then $v \in G - D_0 \cup D_1$, and v is a neighbor of D_0 or D_1. But neighbors of D_0 are contained in D_1, which is disjoint from $G - D_0 \cup D_1$. Similarly, to show $N(D_1) \subseteq D_0 \cup D_2$, note that any $v \in N(D_1)$ ~~has~~ is adjacent to a vertex $w \in D_1$, and we may concatenate this edge with an edge ~~from~~ wv_0 to get a path of length 2. Thus $N(D_1) \subseteq \bigcup_{i \leq 2} D_i$, and we get our result since $N(D_1) \cap D_1 = \emptyset$.
Suppose inductively that $D_k = \{v \in G : d(v, v_0) = k\}$ and $D_{k+1} \subseteq N(D_k) \subseteq D_{k-1} \cup D_{k+1}$ for $k < n$. Then, if $v \in D_n = N_G(\bigcup_{i<n} D_i)$, $d(v, v_0) \geq n$ since inductively $\bigcup_{i<n} D_i = \{v \in G : d(v, v_0) < n\}$. On the other hand, v is adjacent to some $w \in \bigcup_{i<n} D_i$, which yields a path $v - \cdots - v_0$ of length at most n. Thus $D_n \subseteq \{v \in G : d(v, v_0) = n\}$. For the other direction, simply write any path $v - \cdots - v_0$ of length n as $v - v' - \cdots - v_0$ and observe that $d(v', v_0) = n - 1$, so $v' \in \bigcup_{i<n} D_i$. Thus $D_n = \{v \in G : d(v, v_0) = n\}$. For the other statement: suppose $v \in D_{n+1}$. Then $v \in N_G(\bigcup_{i<n+1} D_i)$ — that is, v is a neighbor of some w in D_i for some $i < n+1$. Then $v \in N_G(D_i) \subseteq D_{i-1} \cup D_{i+1}$ inductively, which is only possible for $i = n$, since $D_{i-1} \cup D_{i+1} \subseteq \bigcup_{i<n+1} D_i$ otherwise. Thus $v \in N_G(D_n)$ and we have $D_{n+1} \subseteq N(D_n)$. For the second inclusion, given $v \in N(D_n)$ we have the edge $v - w$ for some $w \in D_n$. Joining this to ~~an~~ a path from w to v_0 of length n gives us a path of length at most $n+1$. We can't have $v \in D_n$ by definition, and if $v \in D_{n-2}$, we obtain a path from w to v_0 of length $n-1$. So $v \in D_{n-1} \cup D_{n+1}$. (This proof is admittedly very inelegant and essentially just brute force, but at least it's simple conceptually ...) (Bottom line is that it is (should be) correct.)

P.S. Now that I think about it, I shld prove $D_n = \{v \in G \mid d(v, v_0)$ ... inductively, then ... statement ...

1·6. We have $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ since the value of $\mathrm{rad}(G)$ is the length of some path in G and $\mathrm{diam}(G)$ bounds that. Let u be a central vertex, and let v, w be the ends of a longest path; that is, $d_G(v,w) = \mathrm{diam}(G)$. Thus $d_G(v,w) \leq d_G(v,u) + d_G(u,w) \leq 2\,\mathrm{rad}(G)$ by the triangle inequality.

1·7. Following exercise 5, we fix a vertex v_0 and define sets of vertices $D_k = \{v \in G : d(v,v_0) = k\}$. We have $|D_0| = 1$ and $|D_1| \geq \delta$ as $D_0 = \{v_0\}$ and $D_1 = N(D_0)$. wrt ~~By excercise 5, for 1≤k~~ We consider two cases depending on the parity of the girth g — this is because, for odd girth we have behavior like

, whereas for even girth we have



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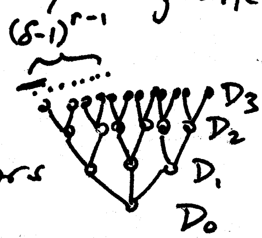
First suppose $g =: 2r+1$. Then $|G| \geq \sum_{i=0}^{r-1} |D_i|$. For $1 \leq i < r$, we know that $D_{i+1} \subseteq N(D_i) \subseteq D_{i-1} \cup D_{i+1}$. Each vertex of D_i has at least δ neighbors — only one lies in D_{i-1}, since otherwise a cycle of length at most $2i < 2r+1 = g$ would be formed. No two vertices of D_i share neighbours in D_{i+1}, and so $|D_{i+1}| \geq (\delta-1)|D_i|$. Thus $|D_{i+1}| \geq (\delta-1)|D_i| \geq \delta(\delta-1)^i$. We compute

$$|G| \geq \sum_{i=0}^{r-1} |D_i| \geq 1 + \delta \sum_{i=0}^{r-1} (\delta-1)^i$$

as desired. For even g, the analysis is the same up to D_{r-1}, where we get $|D_r| \geq (\delta-1)^{r-1}$, since no two vertices in D_r may originate from the same vertex in D_1, although now in general



we may have vertices of D_r that share neighbors in D_r since now $g =: 2r$. Thus now

$$|G| \geq 1 + \delta \sum_{i=0}^{r-2} (\delta-1)^i + (\delta-1)^{r-1}$$

$$= 1 + \sum_{i=0}^{r-2} (\delta-1)^{i+1} + \sum_{i=0}^{r-2} (\delta-1)^i + (\delta-1)^{r-1}$$

$$= 2 \sum_{i=1}^{r-1} (\delta-1)^i .$$

1.7 (cont.). To establish the bound $|G| \geq n_0(\frac{d}{2}, g)$ for $d(G) \geq d \geq 2$ and $g(G) \geq g \in \mathbb{N}_{\geq 3}$, we use Proposition 1.2.2 to pick a subgraph H of G with $\delta(H) > \varepsilon(H) \geq \varepsilon(G) = \frac{d(G)}{2} \geq \frac{d}{2}$ and $g(H) \geq g(G) \geq g$. Then our above results imply that $|G| \geq |H| \geq n_0(\delta(H), g(H)) \geq n_0(\frac{d}{2}, g)$ as n_0 is increasing in both inputs. (In particular one may compute $n_0(d, g+1) \geq n_0(d,g)$

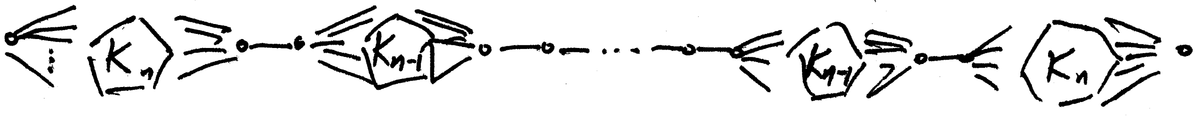
1.8. We use the Moore bound — if $|G| = n$ and $g(G) \geq 5$, then

$$\begin{aligned} n &\geq n_0(\delta(G), g(G)) \\ &\geq n_0(\delta(G), 5) \\ &= 1 + \delta^2, \end{aligned}$$

and so $\delta \leq \sqrt{n-1} \in o(n)$.

1.9. See next sheet.

1.10. Let x_0, x_k be vertices with $d(x_0, x_k) = k$, and let $P = x_0 \dots x_k$ be a path of length k joining them. We may partition G into vertex sets $D_0, \dots, D_k$ where $D_i = \{v \in G : d(v, x_0) = i\}$. Then, given a vertex in some D_i, $0 < i < k$, notice that its neighbors must lie in $D_{i-1} \cup D_i \cup D_{i+1}$. Thus $|D_{i-1} \cup D_i \cup D_{i+1}| \geq d$ and we get a rough bound $|G| \geq \frac{k}{3} \cdot d$. It is easy to construct graphs that near this bound: the graph



has 2 copies of K_n and $m-2$ copies of K_{n-1}. We have $\delta = n$, diam $= 3m-1$ and $|G| \approx \frac{kd+k+d}{3}$.

Minimum path/cycle length (Diestel 1.9)

ho boon suan

August 2021

We prove that every connected graph G of order at least 3 contains a path or cycle of length at least $\min\{2\delta(G), |G|\}$.

We first consider the case $2\delta(G) < |G|$. Suppose for contradiction that $P = x_0 \dots x_m$ is a longest path with $m < 2\delta := 2\delta(G)$. Then the neighbors of x_0 and x_m must belong to P. Let $x_{i_1}, \dots, x_{i_\delta}$ be neighbors of x_m. Then $\{x_{i_1+1}, \dots, x_{i_\delta+1}\}$ is a δ-element subset of $\{x_1, \dots, x_m\}$, so x_0 must have some neighbor $x_{i+1} := x_{i_j+1}$ in that set, since $m < 2\delta$. We may thus form a cycle x_0Px_i — x_mPx_{i+1} — x_0 of length $m+1$ containing all the vertices of P. Since $m < 2\delta < |G|$, the subgraph $G - P$ is nonempty. Connectedness then implies the existence of an edge v — x_k with $v \in G - P$ and $0 < k < m$. Starting from v and following the cycle constructed above, we obtain a path of length $m+1$, contradicting the maximality of P.

Now suppose $2\delta(G) \geq |G|$. Arguing as above we find that the length of a longest path in G must be $|G| - 1$. We may then use the path to construct a cycle of length $|G|$ as needed.