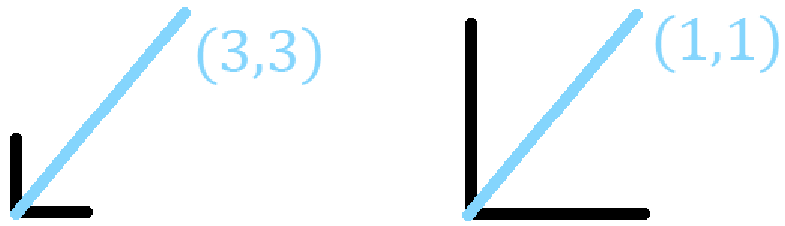
My understanding so far of the intuition behind the terms "covariant" and "contravariant" in differential geometry

Intuitively, some quantities change with or against a change in coordinates. For example, if I scale up the standard coordinate axes, then any vector will have its components decrease, so we say that the components of a vector transform contravariantly with a change of coordinates.



In formulas this goes something like this. Write a vector Z in two coordinate systems (x^i) and (y^i):

$$Z = X^i \frac{\partial}{\partial x^i} = Y^i \frac{\partial}{\partial y^i}.$$

Then we have

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j},$$

and so

$$X^i \frac{\partial}{\partial x^i} = X^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Comparing coefficients, we get our result:

$$Y^j = X^i \frac{\partial y^j}{\partial x^i}.$$

The algebraic intuition is then that, on the right hand side, we are multiplying a component in the old coordinate system (x^i) with the derivative of a coordinate function in the new system (y^j).

For covectors, a dual picture applies:

For concreteness, say our original basis vectors were $\partial/\partial x^i$ for $1 \leq i \leq 3$ as on the left in my beautiful illustration above. Then the right basis vectors are maybe like, $3\partial/\partial x^i$. and so any components of a covector ω must grow, as $\omega_i = \omega(\partial/\partial x^i)$ in the old coordinates and $\omega_i = \omega(3\partial/\partial x^i) = 3\omega(\partial/\partial x^i)$ in the new ones. For this reason we say that the *components* of a covector transform covariantly under change of coordinates. (I'm emphasizing that it's the components since this really tripped me up earlier -- I was trying to figure out why a vector like $\partial/\partial x^i$ was considered to be covariant by some authors, then I realized that covariance was a property of tensor components, and that $\partial/\partial x^i$ behaves as the component function of a covector.)

Once again, we compute and get

$$\omega = \omega_i dx^i = \tilde{\omega}_i d\tilde{x}^i;$$

$$dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$$

$$\omega_i dx^i = \omega_i \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$$

and so

$$\tilde{\omega}_j = \omega_i \frac{\partial x^i}{\partial \tilde{x}^j}.$$

And once again algebraically the idea here is that on the right we're multiplying components from the old coordinates with the derivative of coordinate functions from the old coordinates. The old with the old. Covariance. Something like that.