245B Real Analysis: Some solutions

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January 2, 2022 to January 15, 2022, 12:53

In this document, I produce some solutions to exercises as I work through Terence Tao's UCLA sequence on graduate real analysis. The lecture notes for the second and third parts of his three part sequence 245ABC are collected in his book *An Epsilon of Room, I: Real Analysis*.

I have attempted most exercises, but for some where I got stuck for too long, I looked up solutions online. I have indicated my references in square brackets at the beginning of such solutions.

Contents

Measure and integration 2

Signed measures 7

L^p spaces 15

(Optional) The Stone and Loomis–Sikorski Representation Theorems 23

Hilbert spaces 24

Duality and the Hahn–Banach theorem 26

Oh! If only someone
would give me time, time,
time to do everything properly,
to read everything at my own tempo,
to take it apart and put it together again.

— Karl Barth (1922)

1.1. A quick review of measure and integration theory

The ultimate measure of a man is not where
he stands in moments of comfort and convenience,
but where he stands at times of challenge and controversy.

— Martin Luther King, Jr., *Strength to Love* (1963)

Exercise 1.1.1. We use a kind of 'structural induction' to prove the claim (see 245A, Remark 1.4.15). We recall the principle here for convenience.

I wonder if it would be sleeker to do this via transfinite induction. I haven't learned the details of this method yet though, so I won't try it for now.

Remark. If $\mathcal{F}$ is a family of sets in X, and $P(E)$ is a property of sets $E \subset X$ which obeys the following axioms:

(i) $P(\varnothing)$ is true.

(ii) $P(E)$ is true for all $E \in \mathcal{F}$.

(iii) If $P(E)$ is true for some $E \subset X$, then $P(X \setminus E)$ is true also.

(iv) If $E_1, E_2, \ldots \subset X$ are such that $P(E_n)$ is true for all n, then $P(\bigcup_{n=1}^\infty E_n)$ is true also.

Then one can conclude that $P(E)$ is true for all $E \in \langle \mathcal{F} \rangle$. Indeed, the set of all E for which $P(E)$ holds is a σ-algebra containing $\mathcal{F}$.

We now prove that a continuous function f between topological spaces X and Y is Borel measurable, by using the remark above with $\mathcal{F}$ being the family of open sets in Y, and $P(E)$ the property that $f^{-1}(E)$ is Borel measurable in X. Claim (i) holds as $f^{-1}(\varnothing) = \varnothing$. Claim (ii) holds by continuity. Claim (iii) follows from the identity $f^{-1}(Y \setminus E) = X \setminus f^{-1}(E)$. Finally, claim (iv) follows from the fact that $f^{-1}(\bigcup_{n=1}^\infty E_n) = \bigcup_{n=1}^\infty f^{-1}(E_n)$.

Remark. The Borel σ-algebra $\mathcal{B}[S]$ of a subspace $S \subset X$ is equal to the Borel σ-algebra of X restricted to S. That is, $\mathcal{B}[S] = \mathcal{B}[X]\downharpoonright_S$. Indeed, they are both generated by sets of the form $U \cap S$, where $U \subset X$ is open. (Be careful not to confuse the notations $\mathcal{B}[X]$ and $\mathcal{B}[\mathcal{F}]$. The first refers to the σ-algebra generated by the open sets of a topological space X, and the second is the smallest σ-algebra containing a family of sets $\mathcal{F} \subset \mathcal{P}(X)$.)

Exercise 1.1.2. We wish to prove that $\mathcal{B}[M]$ is maximal such that

$$\mathcal{B}[M]\downharpoonright_{U_\alpha} = \pi_\alpha^{-1}(\mathcal{B}[\mathbf{R}^n]\downharpoonright_{V_\alpha})$$

for all α. By exercise 1.1.1, we see that a homeomorphism between topological spaces induce a bijection between their σ-algebras. Thus it suffices to prove maximality. Suppose $\mathcal{X}$ is a σ-algebra on M satisfying the above identities, so that

$$\mathcal{X}\downharpoonright_{U_\alpha} = \pi_\alpha^{-1}(\mathcal{B}[\mathbf{R}^n]\downharpoonright_{V_\alpha}) = \mathcal{B}[M]\downharpoonright_{U_\alpha}.$$

Then, it suffices to show that any element of $\mathcal{X}$ is a countable union of sets, each belonging to some $\mathcal{X}\downharpoonright_{U_\alpha}$. By the second countability of

M, we may choose U_{α_i} such that their union covers $X \in \mathcal{X}$. Thus $X = \bigcup_i X \cap U_{\alpha_i}$, so that $X \cap U_{\alpha_i} \in \mathcal{B}[M]\lfloor_{U_{\alpha_i}}$, and we are done.

Exercise 1.1.3. Let $\mathcal{X}$ be a σ-algebra on a finite set X. We define a map $X \to \mathcal{X}$ sending $x \in X$ to the intersection of all sets in $\mathcal{X}$ containing x. We prove that the image of this map is a partition of X, and that $\mathcal{X}$ arises from this partition. Clearly the image covers X. Suppose x and y get sent to sets S_x and S_y with non-empty intersection. Then $x \in S_x \cap (X \setminus S_y) \subsetneq S_x$, contradicting the minimality of S_x. Thus the sets form a partition. Given a set $X \in \mathcal{X}$, we see that $X = \bigcup_{x \in X} S_x$, where the sets in the union are either identical or disjoint. Discarding repeated sets, we obtain the claim.

Exercise 1.1.4. Let $(X_\alpha)_{\alpha \in A}$ be an at most countable family of second countable topological spaces. We prove that

$$\mathcal{B}\Big[\prod_{\alpha \in A} X_\alpha\Big] = \prod_{\alpha \in A} \mathcal{B}[X_\alpha].$$

Let $\prod_{\alpha \in A} B_\alpha \in \prod_{\alpha \in A} \mathcal{B}[X_\alpha]$. Since the projections $\pi_\beta \colon \prod_{\alpha \in A} X_\alpha \to X_\beta$ are Borel measurable (by definition of the product σ-algebra), the sets $\pi_\alpha^{-1}(B_\alpha)$ belong to $\mathcal{B}[\prod_{\alpha \in A} X_\alpha]$, and so

$$\prod_{\alpha \in A} B_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(B_\alpha) \in \mathcal{B}\Big[\prod_{\alpha \in A} X_\alpha\Big]$$

as needed.

For the forward inclusion $\subset$, we see that since the σ-algebra $\mathcal{B}[\prod_{\alpha \in A} X_\alpha]$ is generated by open sets in $\prod_{\alpha \in A} X_\alpha$, it suffices to prove that these open sets belong to $\prod_{\alpha \in A} \mathcal{B}[X_\alpha]$. Expanding the definition of $\prod_{\alpha \in A} \mathcal{B}[X_\alpha]$, we have

$$\prod_{\alpha \in A} \mathcal{B}[X_\alpha] = \bigvee_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}[X_\alpha]) = \mathcal{B}\Big[\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}[X_\alpha])\Big].$$

For each X_α, we let $\mathcal{B}_\alpha$ be a countable base. Then

$$\mathcal{B} := \left\{ \prod_{\alpha \in A} U_\alpha : \begin{array}{c} U_\alpha = X_\alpha \text{ for all but finitely many } \alpha, \\ \text{and if } U_\alpha \neq X_\alpha, \text{ then } U_\alpha \in \mathcal{B}_\alpha. \end{array} \right\}$$

is a countable base for $\prod_{\alpha \in A} X_\alpha$. It remains to show that $\prod_{\alpha \in A} U_\alpha \in \mathcal{B}$ belongs to $\mathcal{B}[\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}[X_\alpha])]$. Writing $U_{\alpha_1}, \ldots, U_{\alpha_n}$ for the finitely many nontrivial sets in the product $\prod_{\alpha \in A} U_\alpha$, we see that such a set is a finite intersection

$$\prod_{\alpha \in A} U_\alpha = \bigcap_{1 \le i \le n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{B}\Big[\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}[X_\alpha])\Big],$$

and thus the result follows.

Exercise 1.1.5. We proceed via structural induction (see the remark on page 1). Given $x \in X$, we write $E_x := \{y \in Y : (x, y) \in E\}$, and we call it a *slice* of E (we define E^y similarly). Claim (i) is trivial as all slices of the empty set are empty. For claim (ii), we see that the family $\mathcal{X} \times \mathcal{Y}$ of measurable sets has measurable slices — indeed, given $A \times B \in \mathcal{X} \times \mathcal{Y}$ and $x \in A$, any slice $(A \times B)_x \subset Y$ is either $\varnothing$ or B, and is measurable in both cases. Claim (iii) follows from how

$$((X \times Y) \setminus E)_x = \{y \in Y : (x, y) \notin E\} = Y \setminus E_x.$$

Finally, claim (iv) follows from the fact that

$$\Big(\bigcup_{n=1}^{\infty} E_n\Big)_x = \Big\{y \in Y : (x, y) \in \bigcup_{n=1}^{\infty} E_n\Big\} = \bigcup_{n=1}^{\infty} (E_n)_x.$$

Thus the result holds for $x \in X$; the proof is analogous for $y \in Y$.

Exercise 1.1.6. (i) Countable additivity implies finite additivity by setting $E_n := \varnothing$ for $n \geq N$. Therefore, if $E \subset F$, then $\mu(F) = \mu(E) + \mu(F \setminus E)$, and the result follows from nonnegativity of measure.

(ii) Define $E'_n := E_n \setminus \bigcup_{k=1}^{n-1} E_k$ for $n \geq 1$. The sets E'_n are disjoint with $E_n = \bigcup_{k=1}^{n} E'_k$, and consequently $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} E'_n$. Thus, by countable additivity and (i), we have

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} E'_n\Big) = \sum_{n=1}^{\infty} \mu(E'_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

(iii) Since $E_n \subset \bigcup_{k=1}^{\infty} E_k$ for $n \geq 1$, monotonicity implies that $\mu(E_n) \leq \mu(\bigcup_{k=1}^{\infty} E_k)$ for $n \geq 1$, so that $\lim_{n\to\infty} \mu(E_n) \leq \mu(\bigcup_{n=1}^{\infty} E_n)$. Conversely, writing $E'_n := E_n \setminus \bigcup_{k=1}^{n-1} E_k$, we may compute

$$\begin{aligned}
\lim_{n\to\infty} \mu(E_n) &= \lim_{n\to\infty} \mu\Big(\bigcup_{k=1}^{n} E'_k\Big) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E'_k) \\
&= \sum_{n=1}^{\infty} \mu(E'_n) \\
&\geq \mu\Big(\bigcup_{n=1}^{\infty} E'_n\Big) \\
&= \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big).
\end{aligned}$$

(iv) Apply (iii) to the sequence $\varnothing \subset E_1 \setminus E_2 \subset E_1 \setminus E_3 \subset \ldots$ to obtain the identity

$$\begin{aligned}
\mu(E_1) - \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) &= \mu\Big(\bigcup_{n=1}^{\infty} E_1 \setminus E_n\Big) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n) \\
&= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)).
\end{aligned}$$

Note that the claim fails if $\mu(E_1) = +\infty$, consider for example $(1, +\infty) \subset (2, +\infty) \subset (3, +\infty) \subset \ldots$, where each set has infinite measure, but the intersection is empty and thus has zero measure.

Exercise 1.1.7. Given a measure space $(X, \mathcal{X}, \mu)$, we define a new σ-algebra $\overline{\mathcal{X}}$ to be the intersection of all σ-algebras containing $\mathcal{X}$ as well as all subsets of null sets. By definition, this new measurable space $(X, \overline{\mathcal{X}}, \mu)$ is the unique minimal complete refinement of $(X, \mathcal{X}, \mu)$. If a set A is equal a.e. to a set $B \in \mathcal{X}$, then their symmetric difference $A \triangle B$ is a sub-null set, and so $A = (A \triangle B) \triangle B \in \overline{\mathcal{X}}$. Conversely, we may use structural induction. For (i), the empty set belongs to all σ-algebras. For (ii), this is true for all sub-null sets and all elements of $\mathcal{X}$. For (iii), if $E = F$ a.e., then $X \setminus E = X \setminus F$ a.e.. For (iv), if $E_n = F_n$ a.e., then $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$ a.e., since the countable union of sub-null sets is sub-null. Thus the result follows.

Exercise 1.1.8. [Halmos, *Measure Theory*, page 56–57, Theorem D] Suppose $E \subset X$ with $\mu(E) < \infty$. By definition of μ, there exist sets $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$ such that $E \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \mu(E) + \epsilon/2$. By monotone convergence, we have $\lim_{N\to\infty} \mu(\bigcup_{n=1}^{N} A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$. Thus we may choose large N for which $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \mu(\bigcup_{n=1}^{N} A_n) + \epsilon/2$. Since

$$\begin{aligned} \mu\Big(E \setminus \bigcup_{n=1}^{N} A_n\Big) &\leq \mu\Big(\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{n=1}^{N} A_n\Big) \\ &= \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) - \mu\Big(\bigcup_{n=1}^{N} A_n\Big) \\ &\leq \epsilon/2 \end{aligned}$$

and

$$\begin{aligned} \mu\Big(\bigcup_{n=1}^{N} A_n \setminus E\Big) &\leq \mu\Big(\bigcup_{n=1}^{\infty} A_n \setminus E\Big) \\ &= \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) - \mu(E) \\ &\leq \epsilon/2, \end{aligned}$$

we conclude that

$$\mu\Big(E \triangle \bigcup_{n=1}^{N} A_n\Big) \leq \epsilon$$

as desired. [To do: complete the proof for the general σ-finite case.]

Exercise 1.1.9. Define a premeasure on finite unions of boxes $\prod_{i=1}^{n} U_i$ with $U_i \in \mathcal{X}_i$ by $\mu(\prod_{i=1}^{n} U_i) := \prod_{i=1}^{n} \mu_i(U_i)$ and extending to unions by decomposing them into disjoint boxes. We may then apply the Carathéodory extension theorem. (See 245A Proposition 1.7.11.)

Exercise 1.1.10. [I'm skipping this exercise.]

Exercise 1.1.11. The sequence of functions

$$|f| 1_{\{x \in E: |f(x)| > 1\}} \geq |f| 1_{\{x \in E: |f(x)| > 2\}} \geq \cdots$$

converges pointwise a.e. to the zero function, since f is absolutely integrable. Thus we have

$$\lim_{n\to\infty} \int_X |f| 1_{\{x \in E: |f(x)| > n\}} \, d\mu = \int_X \lim_{n\to\infty} |f| 1_{\{x \in E: |f(x)| > n\}} \, d\mu = 0$$

by dominated convergence, and so we may choose large λ for which $\int_{x\in E:|f(x)|>\lambda} |f|\,d\mu \le \epsilon/2$. It follows that

$$\begin{aligned}\int_E |f|\,d\mu &= \int_{x\in E:|f(x)|\le\lambda} |f|\,d\mu + \int_{x\in E:|f(x)|>\lambda} |f|\,d\mu \\ &\le \lambda\mu(E) + \epsilon/2 \\ &\le \epsilon\end{aligned}$$

whenever $\mu(E) \le \epsilon/2\lambda$.

Make use of time, let not advantage slip;
Beauty within itself should not be wasted:
Fair flowers that are not gather'd in their prime
Rot and consume themselves in little time.

— William Shakespeare, *Venus and Adonis* (1593)

1.2. *Signed measures and the Radon–Nikodym–Lebesgue theorem*

Observe due measure, for right timing is in all things the most important factor.

— Hesiod, *Works and Days* (c. 700 B.C.)

Exercise 1.2.1. We first prove that m_f is an unsigned measure. We have

$$m_f(\varnothing) = \int_X 1_\varnothing f\, dm = \int_X 0\, dm = 0.$$

Given disjoint $E_1, E_2, \ldots$, we have

$$\begin{aligned} m_f\Big(\bigcup_{n=1}^\infty E_n\Big) &= \int_X 1_{\bigcup_{n=1}^\infty E_n} f\, dm \\ &= \int_X \sum_{n=1}^\infty 1_{E_n} f\, dm \\ &= \sum_{n=1}^\infty \int_X 1_{E_n} f\, dm \\ &= \sum_{n=1}^\infty m_f(E_n), \end{aligned}$$

where we used monotone convergence for series (Theorem 1.1.21) to swap the sum and integral.

Suppose $g\colon X \to [0,+\infty]$ is a simple unsigned function taking values in $\{a_1,\ldots,a_n\}$. We write $g = \sum_{i=1}^n a_i 1_{g^{-1}(\{a_i\})}$, and compute

$$\begin{aligned} \int_X g\, dm_f &= \sum_{i=1}^n a_i m_f(g^{-1}(\{a_i\})) \\ &= \sum_{i=1}^n a_i \int_X 1_{g^{-1}(\{a_i\})} f\, dm \\ &= \int_X \sum_{i=1}^n a_i 1_{g^{-1}(\{a_i\})} f\, dm \\ &= \int_X gf\, dm. \end{aligned}$$

Since every unsigned measurable function is the pointwise limit of an increasing sequence of unsigned simple functions[1], the result for general g follows from monotone convergence (Theorem 1.1.21).

[1] This result is occasionally called the *Sombrero lemma* due to the construction of the sequence of functions involved. See René L. Schilling, *Measures, Integrals and Martingales* 2e., Theorem 8.8.

Exercise 1.2.2. If $f = g$ m-a.e., then $m_f(E) = \int_E f = \int_E g = m_g(E)$, as the Lebesgue integrals are equal for a.e. equal functions.

Conversely, we prove that $m_f = m_g$ implies that $f = g$ m-a.e.. We first consider the case where $m(X) < \infty$. Suppose contrapositively that $f \neq g$ m-a.e.. Then, without loss of generality, there exists a set E of positive finite measure such that $f > g$ on E. We consider two cases.

Case 1: $\int_E f, \int_E g < \infty$. In this case, f and g must be finite m-a.e., and thus we may safely consider the function $f - g$ on E, which is unsigned measurable as $f > g$ on E by hypothesis. Therefore, we have $\int_E f - g \le \int_E f < \infty$ and

$$\int_E f = \int_E f - g + \int_E g.$$

Since $f - g > 0$ on E, we have $\int_E f - g > 0$, and so $m_f(E) > m_g(E)$ as desired.

Case 2: $\int_E f = \infty$. If $\int_E g < \infty$, there is nothing to prove, so assume that $\int_E f = \infty = \int_E g$. Since $f > g$, we see that g must be finite everywhere. Apply monotone convergence to the sequence $g1_{\{x\in E: g(x)\le 1\}} \le g1_{\{x\in E: g(x)\le 2\}} \le \dots$ to obtain the identity

$$\int_E g = \lim_{N\to\infty} \int_{x\in E: g(x)\le N} g.$$

It follows that there exists N such that $m(\{x \in E : g(x) \le N\}) > 0$. Let $E' := \{x \in E : g(x) \le N\}$. Since $\int_{E'} g \le Nm(E') < \infty$, we are left to consider $\int_{E'} f$. If $\int_{E'} f = \infty$, we are done. Otherwise, we have $\int_{E'} f < \infty$, and we are left with case 1.

This concludes the proof for the finite measure case.

Now suppose that m is σ-finite. Write $X = \bigcup_{n=1}^\infty X_n$, with X_n disjoint and $m(X_n) < \infty$. Then, once again, if $f > g$ on E with $m(E) > 0$ (possibly infinite this time), then we may consider the finite measure sets $E \cap X_n$. At least one of these sets $E \cap X_n$ is non-empty, with $m(E \cap X_n) < \infty$. Thus we may apply the finite measure argument above to obtain a set $E' \subset E \cap X_n$ on which $\int_{E'} f > \int_{E'} g$ as needed.

Finally, we give a counterexample when μ fails to be σ-finite. Consider the measurable space $(\mathbf{N}, 2^{\mathbf{N}})$ equipped with the measure $\mu(E) = +\infty \cdot [E \text{ is non-empty}]$. That is, μ gives all non-empty sets infinite measure. Then, setting $f = 1_{\mathbf{N}}$ and $g = 2 \cdot 1_{\mathbf{N}}$, we see that $\int_X 1_\varnothing f \, d\mu = 0 = \int_X 1_\varnothing g \, d\mu$, and that

$$\int_X 1_E f \, d\mu = +\infty = \int_X 1_E g \, d\mu$$

for all non-empty $E \in 2^{\mathbf{N}}$ (this idea works with a singleton set, but I found $\mathbf{N}$ more comforting).

Exercise 1.2.3. To say that μ has a continuous Radon–Nikodym derivative $d\mu/dm$ is to say that there exists a continuous function $f = d\mu/dm$ such that $\mu = m_f$. We thus compute

$$\mu([0,x]) = m_f([0,x]) = \int_{[0,x]} f \, dm.$$

By the fundamental theorem of calculus, we conclude that

$$\frac{d}{dx}\mu([0,x]) = f(x) = \frac{d\mu}{dm}(x)$$

for all $x \in [0, +\infty)$.

Exercise 1.2.4. Let $\mu\colon X \to [0, +\infty]$ be a measure on X. We would like to write $\mu = \#_f$ for some function $f\colon X \to [0, +\infty]$. Expanding the definitions, we are looking for some f such that

$$\mu(E) = \#_f(E) = \int_E f \, d\# = \sum_{x \in E} f(x).$$

Thus we conclude that the function f, defined by $f(x) := \mu(\{x\})$, is indeed the Radon–Nikodym derivative $d\mu/d\#$ of μ with respect to #.

Remark. If a measure μ on X is differentiable with respect to the Dirac measure δ_x with Radon–Nikodym derivative $d\mu/d\delta_x = f$, then we must have $\mu(E) = (\delta_x)_f(E) = \int_E f \, d\delta_x = f(x)\delta_x(E)$. Since the Radon–Nikodym derivative is defined up to δ_x-a.e. equivalence (which means that $f = g$ δ_x-a.e. iff $f(x) = g(x)$), we see that the only measures differentiable with respect to δ_x are its scalar multiples.

Exercise 1.2.5. Let $\mu = \mu\lfloor_{X_+} - \mu\lfloor_{X_-} = \mu_+ - \mu_-$ be as obtained from the Hahn decomposition theorem, and suppose $\mu = \nu_+ - \nu_-$ is another decomposition such that ν_+ and ν_- are mutually singular unsigned measures. Since ν_+ and ν_- are mutually singular, we may write X as a disjoint union $X = Y_+ \cup Y_-$ such that ν_+ is supported on Y_+ and ν_- is supported on Y_-. Then we may write X as the disjoint union of four sets, namely

$$X = (X_+ \cap Y_+) \cup (X_+ \cap Y_-) \cup (X_- \cap Y_+) \cup (X_- \cap Y_-).$$

On $X_+ \cap Y_+$, we have $\mu_- = \nu_- = 0$, and so

$$\mu_+ = \mu_+ - \mu_- = \nu_+ - \nu_- = \nu_+;$$

consequently $\mu_- = \nu_-$. On $X_+ \cap Y_-$, we have $\mu_- = \nu_+ = 0$, and so

$$\mu_+ = \nu_+ - \nu_- + \mu_+ = -\nu_-.$$

Since μ_+ and ν_- are unsigned, it follows that $\mu_+ = \nu_- = 0$, and so $\mu_+ = \nu_+$ as needed. The remaining cases are handled similarly.

Exercise 1.2.6. We first verify that $|\mu|$ is an unsigned measure. Since μ_+ and μ_- are unsigned, we see that $|\mu| = \mu_+ + \mu_-$ is unsigned as well. Given disjoint $E_1, E_2, \ldots \subset X$, we may compute

$$\begin{aligned} |\mu|\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= \mu_+\Big(\bigcup_{n=1}^{\infty} E_n\Big) + \mu_-\Big(\bigcup_{n=1}^{\infty} E_n\Big) \\ &= \sum_{n=1}^{\infty} \mu_+(E_n) + \sum_{n=1}^{\infty} \mu_-(E_n) \\ &= \sum_{n=1}^{\infty} |\mu|(E_n), \end{aligned}$$

where the last equality is justified by the absolute convergence of both series.

Let ν be an unsigned measure such that $-\nu \leq \mu \leq \nu$, or

$$-\nu_+ + \nu_- \leq \mu_+ - \mu_- \leq \nu_+ - \nu_-.$$

Our goal is to prove that $|\mu| \leq \nu$, or $\mu_+ + \mu_- \leq \nu_+ - \nu_-$. Applying Hahn decomposition to μ, we get $X = X_+ \cup X_-$. Similarly, applying

Hahn decomposition to ν gives $X = Y_+ \cup Y_-$. We may thus write X as a disjoint union

$$X = (X_+ \cap Y_+) \cup (X_+ \cap Y_-) \cup (X_- \cap Y_+) \cup (X_- \cap Y_-).$$

On $X_+ \cap Y_+$, we have $\mu_- = \nu_- = 0$. Thus

$$\mu_+ + \mu_- = \mu_+ - \mu_- \leq \nu_+ - \nu_-$$

as needed. The remaining three cases are handled similarly.

Now we prove that

$$|\mu|(E) = \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

where the supremum is taken over all partitions $(E_n)_{n=1}^{\infty}$ of E. Since

$$-\sup \sum_{n=1}^{\infty} |\mu(E_n)| \leq \mu(E) \leq \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

earlier arguments imply that $|\mu|(E) \leq \sup \sum_{n=1}^{\infty} |\mu(E_n)|$. Conversely, since $-|\mu| \leq \mu \leq |\mu|$ means that $|\mu(E)| \leq |\mu|(E)$, we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \sum_{n=1}^{\infty} |\mu|(E_n) = |\mu|\Big(\bigcup_{n=1}^{\infty} E_n\Big) = |\mu|(E)$$

for any partition $(E_n)_{n=1}^{\infty}$ of E, and so we conclude that

$$|\mu|(E) = \sup \sum_{n=1}^{\infty} |\mu(E_n)|$$

as needed.

Exercise 1.2.7. We prove that the following are equivalent:

(i) $\mu(E)$ is finite for every $E \subset X$.

(ii) $|\mu|$ is a finite unsigned measure.

(iii) μ_+ and μ_- are finite unsigned measures.

Claim (i) implies (ii). Indeed, if $\mu(E)$ is finite, then $\mu_+(E) - \mu_-(E)$ is finite. Since the quantities cannot both be infinite, they must both be finite, and so $|\mu|(E) = \mu_+(E) + \mu_-(E)$ is finite as well.

Claim (ii) implies (iii), since

$$\mu_+(E) \leq |\mu|(E) < \infty;$$

similarly for μ_-.

Finally, (iii) implies (i) as

$$\mu(E) = \mu_+(E) - \mu_-(E) < \infty.$$

Remark (Proof of Theorem 1.2.4). [Folland 2e, Lemma 3.7] In the last paragraph of the proof of Theorem 1.2.4, it is shown that $\mu_s \perp m$. Here are some details:

We prove that either $\mu_s \perp m$, or there exist $\epsilon > 0$ and $E \in \mathcal{X}$ such that $m(E) > 0$ and $\mu_s \geq \epsilon m$ on E (that is, E is a totally positive set for $\mu_s - \epsilon m$).

Indeed, let $X = X_+^n \cup X_-^n$ be a Hahn decomposition for $\mu_s - n^{-1}m$, and let $X_+ := \bigcup_{n=1}^\infty X_+^n$ and $X_- := \bigcap_{n=1}^\infty X_-^n = X \setminus X^+$. Then X_- is a totally negative set for $\mu_s - n^{-1}m$ for all n; i.e., $0 \leq \mu_s(X_-) \leq n^{-1}m(X_-)$ for all n, so $\mu_s(X_-) = 0$. If $m(X_+) = 0$, then $\mu_s \perp m$. Otherwise, if $m(X_+) > 0$, then $m(X_+^n) > 0$ for some n, and so $X_+^n =: E$ is a totally positive set for $\mu_s - n^{-1}m$.

Exercise 1.2.8. [Folland 2e, Theorem 3.8; Math.SE answer 3713882] (I'm still a bit sketchy on this solution.) Suppose μ, m are σ-finite. Then we may write $X = \bigcup_{n=1}^\infty X_n$ with $\mu(X_n) < \infty$, $m(X_n) < \infty$, and X_n disjoint. Defining $\mu_n(E) := \mu(E \cap X_n)$ and $m_n(E) := m(E \cap X_n)$, we may apply the result for the finite measure case to obtain decompositions

$$\mu_n = (m_n)_{f_n} + (\mu_n)_s$$

with $(\mu_n)_s \perp m_n$. Let $f := \sum_n f_n$ and $\mu_s := \sum_n (\mu_n)_s$. We may assume that $f_n = 0$ on $X \setminus X_n$, so that

$$\begin{aligned}\sum_n (m_n)_{f_n}(E) &= \sum_n \int_E f_n \, dm_n \\ &= \sum_n \int_E f_n \, dm \\ &= \int_E \sum_n f_n \, dm \\ &= \int_E f \, dm \\ &= m_f(E).\end{aligned}$$

Thus, we have

$$\mu = \sum_n \mu_n = \sum_n (m_n)_{f_n} + \sum_n (\mu_n)_s = m_f + \mu_s.$$

Finally, we prove that $\mu_s \perp m$. Since $(\mu_n)_s \perp m_n$, we may write $X = A_n \cup B_n$ with A_n, B_n disjoint such that $(\mu_n)_s$ is null outside A_n and m_n is null outside B_n. Then, setting $\tilde{A}_n := A_n \cap X_n$ and $\tilde{B}_n := B_n \cap X_n$, we may define $A := \bigcup_n \tilde{A}_n$ and $B := \bigcup_n \tilde{B}_n$, so that $X = A \cup B$ with A, B disjoint. Since $\mu_s = \sum_n (\mu_n)_s$ is null outside A and $m = \sum_n m_n$ is null outside B, we conclude that $\mu_s \perp m$ as needed.

Exercise 3.9 from Folland 2e. *Suppose (ν_n) is a sequence of unsigned measures. If $\nu_n \perp \mu$ for all n, then $\sum_n \nu_n \perp \mu$; and if $\nu_n \ll \mu$ for all n, then $\sum_n \nu_n \ll \mu$.*

Say ν_n is supported on X_n, so that μ is supported on $X \setminus X_n$. Then $\sum_n \nu_n$ is supported on $\bigcup_n X_n$, and μ is supported on $\bigcap_n (X \setminus X_n) = X \setminus \bigcup_n X_n$, so that $\sum_n \nu_n \perp \mu$. Suppose $\nu_n(E) = 0$ whenever $\mu(E) = 0$. Then $\sum_n \nu_n(E) = 0$ whenever $\mu(E) = 0$, and so $\sum_n \nu_n \ll \mu$.

Exercise 1.2.9. Let m be an unsigned σ-finite measure. As before, by Hahn decomposition, we may assume that μ is an *unsigned* σ-finite measure. Suppose every point is measurable, and that $m(\{x\}) = 0$ for all $x \in X$. (We say that m is *continuous*.) By the Lebesgue decomposition theorem, we may write $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$ uniquely, with $\mu_{\mathrm{ac}} \ll m$ and $\mu_{\mathrm{s}} \perp m$. We will further decompose

$$\mu_{\mathrm{s}} = \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}},$$

where μ_{pp} is supported on an at most countable set, and where μ_{sc} is continuous with $\mu_{\mathrm{sc}} \perp m$. The natural idea is to define the set

$$E := \{x \in X : \mu_{\mathrm{s}}(\{x\}) > 0\}.$$

Let $\mu_{\mathrm{sc}} := \mu_{\mathrm{s}}\lfloor_{X \setminus E}$ and $\mu_{\mathrm{pp}} := \mu_{\mathrm{s}}\lfloor_E$. Then we must show:

(1) E is at most countable.

(2) $\mu_{\mathrm{sc}}(\{x\}) = 0$ for all $x \in X$.

(3) $\mu_{\mathrm{sc}} \perp m$.

We first prove (1). Suppose for contradiction that E is uncountable. Since $\mu_{\mathrm{s}} \leq \mu$ and μ is σ-finite, it follows that μ_{s} is σ-finite as well. Thus we may write $X = \bigcup_{n=1}^\infty X_n$ such that $\mu_{\mathrm{s}}(X_n) < \infty$, with X_n disjoint. Then, $E = \bigcup_{n=1}^\infty (E \cap X_n)$, with $\mu_{\mathrm{s}}(E \cap X_n) < \infty$ and $E \cap X_n$ disjoint. Since the countable union of countable sets is countable, there exists n such that $E \cap X_n$ is uncountable. Define

$$E_{n,k} := \left\{ x \in E \cap X_n : \frac{1}{k} \leq \mu_s(\{x\}) < \frac{1}{k-1} \right\}$$

for $k \geq 2$, with $E_{n,1} := \{x \in E \cap X_n : \mu_s(\{x\}) \geq 1\}$. Then $E \cap X_n = \bigcup_{k=1}^\infty E_{n,k}$ with $E_{n,k}$ disjoint, and so $E_{n,k}$ is uncountable for some k. Taking a countable subset $S \subset E_{n,k}$, we see that

$$\mu(E \cap X_n) \geq \mu(E_{n,k}) \geq \sum_{j=1}^\infty \frac{1}{k} = +\infty,$$

contradicting the finiteness of $\mu(E \cap X_n)$.

Claim (2) holds as μ_{sc} is supported on $X \setminus E$, and all positive measure singletons are in E by definition.

Claim (3) follows from the fact that $\mu_{\mathrm{s}} \perp m$. Indeed, $\mu_{\mathrm{sc}} \leq \mu_{\mathrm{s}}$, which implies that the support of μ_{sc} is a subset of the support of μ_{s}. This completes the proof.

Remark (Absolute continuity). [C. Heil, *Introduction to Real Analysis*, Problem 6.1.9] Using the definition in the text, we can prove that a function $f\colon I \to \mathbf{R}$ is absolutely continuous if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^\infty |f(y_i) - f(x_i)| \leq \epsilon$ whenever $[x_1, y_1], \ldots$ is a family of *countably many* disjoint intervals in I of total length at most δ.

Indeed, given $\epsilon > 0$ and a countably infinite family $[x_1, y_1], \ldots,$ we choose $\delta > 0$ as in the finite case. If $\sum_{i=1}^\infty (y_i - x_i) < \delta$, then $\sum_{i=1}^n (y_i - x_i) < \delta$ for all n, and therefore $\sum_{i=1}^n |f(y_i) - f(x_i)| \leq \epsilon$

for all n. Thus we conclude that $\sum_{i=1}^{\infty} |f(y_i) - f(x_i)| \leq \epsilon$ as needed. The converse follows from setting all sufficiently large intervals to be empty.

Exercise 1.2.10. (i) Suppose μ is continuous. We first prove that $x \mapsto \mu([0,x])$ is right-continuous. This does not require the continuity hypothesis for μ (it is a general property of *cumulative distribution functions*). Indeed,

$$\lim_{h\downarrow 0} \mu([0,x+h]) - \mu([0,x]) = \lim_{h\downarrow 0} \mu((x,x+h]) \leq \mu((x,x+1/n))$$

for all $n \geq 1$, and so

$$\begin{aligned}
\lim_{h\downarrow 0} \mu((x,x+h]) &\leq \lim_{n\to\infty} \mu((x,x+1/n)) \\
&= \mu\Big(\bigcap_{n=1}^{\infty} (x,x+1/n)\Big) \\
&= \mu(\varnothing) \\
&= 0
\end{aligned}$$

by dominated convergence for sets.

Now we prove left-continuity. We must prove that

$$\lim_{h\downarrow 0} \mu([0,x-h]) - \mu([0,x]) = 0,$$

or equivalently, that

$$\lim_{h\downarrow 0} \mu((x-h,x]) = 0.$$

Arguing as before, we see that

$$\begin{aligned}
\lim_{h\downarrow 0} \mu((x-h,x]) &\leq \lim_{n\to\infty} \mu((x-1/n,x]) \\
&= \mu\Big(\bigcap_{n=1}^{\infty} (x-1/n,x]\Big) \\
&= \mu(\{x\}) \\
&= 0
\end{aligned}$$

as needed.

Conversely, suppose $x \mapsto \mu([0,x])$ is continuous. Fix $x \in [0,+\infty]$. We prove that $\mu(\{x\}) \leq \epsilon$ for all $\epsilon > 0$. By continuity,

$$\lim_{h\downarrow 0} \mu([0,x-h]) = \mu([0,x]),$$

so that

$$\lim_{h\downarrow 0} \mu((x-h,x]) = 0.$$

Thus, for small h we have

$$\mu(\{x\}) \leq \mu((x-h,x]) \leq \epsilon$$

as needed.

(ii) Let $\epsilon > 0$. If $\mu \ll m$, then the Radon–Nikodym theorem tells us that there exists $\delta > 0$ such that $|\mu(E)| < \epsilon$ whenever $m(E) \le \delta$. Suppose $[x_1, y_1], \ldots, [x_n, y_n]$ are disjoint intervals in $[0, +\infty]$ of total length at most δ, so that $m(\bigcup_{i=1}^n (x_i, y_i]) \le \delta$. Then

$$\sum_{i=1}^n |\mu([0, y_i]) - \mu([0, x_i])| = \sum_{i=1}^n \mu((x_i, y_i]) = \mu\Big(\bigcup_{i=1}^n (x_i, y_i]\Big) < \epsilon,$$

proving that $x \mapsto \mu([0, x])$ is absolutely continuous. Conversely, suppose $x \mapsto \mu([0, x])$ is absolutely continuous. By the remarks above, there exists $\delta > 0$ such that $\mu(\bigcup_{i=1}^\infty (x_i, y_i]) < \epsilon$ whenever $[x_1, y_1], \ldots$ is a countable family of disjoint intervals in $[0, +\infty]$ of total length at most δ. Suppose $m(E) = 0$. Then outer regularity of Lebesgue measure implies that we may cover E with an open set U of m-measure at most δ. Open sets in $\mathbf{R}$ can be written as countable unions of open intervals; thus we may write $U = \bigcup_{i=1}^\infty (a_i, b_i)$ so as to conclude that $\mu(E) \le \mu(U) < \epsilon$. Since ϵ is arbitrary, we conclude that $\mu(E) = 0$ as needed.

Exercise 1.2.11. [I'm skipping this exercise.]

I saw myself sitting in the crotch of this fig-tree, starving to death,
just because I couldn't make up my mind which of the figs I would choose.
I wanted each and every one of them,
but choosing one meant losing all the rest, and,
as I sat there, unable to decide, the figs began to wrinkle and go black, and,
one by one, they plopped to the ground at my feet.

— Sylvia Plath, *The Bell Jar* (1963)

1.3. L^p spaces

...if one were to refuse to have direct, geometric, intuitive insights,
if one were reduced to pure logic, which does not permit a choice among
every thing that is exact, one would hardly think of many questions,
and certain notions ... would escape us completely.

— Henri Lebesgue, *Sur le développement de la notion d'intégrale* (1926)

Exercise 1.3.1. We are given the space $L^p(X, \mathcal{X}, \mu)$ together with its completion $L^p(X, \overline{\mathcal{X}}, \overline{\mu})$. Every function $\overline{f}\colon X \to \mathbf{C}$ that is measurable with respect to $(X, \overline{\mathcal{X}}, \overline{\mu})$ can be associated with a function $f\colon X \to \mathbf{C}$ that is measurable with respect to $(X, \mathcal{X}, \mu)$, such that

$$\overline{\mu}(\{x \in X : f(x) \neq \overline{f}(x)\}) = 0.$$

It suffices to prove this for simple functions, as a measurable function is the supremum of a sequence of simple functions. Suppose $\overline{E}_i$ is $\overline{\mathcal{X}}$-measurable. Then, by definition of the completion, $\overline{E}_i$ must differ from an $\mathcal{X}$-measurable set E_i by a sub-null set, so that $\overline{\mu}(\overline{E}_i) = \mu(E_i)$. Thus

$$\int_X \sum_{i=1}^n c_i 1_{\overline{E}_i}\, d\overline{\mu} = \sum_{i=1}^n c_i \overline{\mu}(\overline{E}_i) = \sum_{i=1}^n c_i \mu(E_i) = \int_X \sum_{i=1}^n c_i 1_{E_i}\, d\mu.$$

Exercise 1.3.2. (i) We would like to argue that

$$\begin{aligned} \|f+g\|_{L^p}^p &= \int_X |f(x)+g(x)|^p\, d\mu \\ &\overset{?}{\leq} \int_X |f(x)|^p + |g(x)|^p\, d\mu \\ &= \|f\|_{L^p}^p + \|g\|_{L^p}^p. \end{aligned}$$

Thus, given $x \in X$, it suffices to prove that

$$|f(x)+g(x)|^p \leq |f(x)|^p + |g(x)|^p, \qquad 0 < p < 1 \tag{$*$}$$

whenever $f(x)$ and $g(x)$ are both non-zero. This in turn follows from the real inequality

$$(1+t)^p \leq 1 + t^p, \qquad t \geq 0,\ \ 0 < p < 1,$$

as then, for $\alpha \in \mathbf{C}$, the complex triangle inequality implies that

$$|1+\alpha|^p \leq (1+|\alpha|)^p \leq 1 + |\alpha|^p;$$

the inequality $(*)$ then follows by setting $\alpha = f(x)/g(x)$. Since the function $h(t) := 1 + t^p - (1+t)^p$ for $t \geq 0$ is such that $h(0) = 0$ and $h'(t) = pt^{p-1} - p(1+t)^{p-1} = p(1/t^{1-p} - 1/(1+t)^{1-p}) \geq 0$, it must be a non-decreasing function, and thus the result follows.

(ii) We emulate the proof of Lemma 1.3.3(iii), except this time the function $x \mapsto |x|^p$ for $x > 0$ is *concave* as we have $0 < p < 1$. As before, by non-degeneracy we may take both $\|f\|_{L^p}$ and $\|g\|_{L^p}$ to be non-zero. By homogeneity we normalize $\|f\|_{L^p} + \|g\|_{L^p} = 1$, and by homogeneity again we write $f = (1-\theta)F$ and $g = \theta G$ for some $0 < \theta < 1$ and $F, G \in L^p$ with $\|F\|_{L^p} = \|G\|_{L^p} = 1$. Our task is then to show that

$$\int_X \big((1-\theta)F(x) + \theta G(x)\big)^p \, d\mu \geq 1. \qquad (*)$$

Since the function $x \mapsto x^p$ is concave for $x > 0$ and $0 < p < 1$, we have

$$\big((1-\theta)F(x) + \theta G(x)\big)^p \geq (1-\theta)F(x)^p + \theta G(x)^p.$$

Together with the normalizations of $\|F\|_{L^p}$ and $\|G\|_{L^p}$, this implies $(*)$ as desired.

(iii) By (i), we have $\|f+g\|_{L^p} \leq (\|f\|_{L^p}^p + \|g\|_{L^p}^p)^{1/p}$. Since $x \mapsto x^{1/p}$ is convex for $x > 0$ and $0 < p < 1$, we have

$$\Big(\frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p\Big)^{1/p} \leq \frac{1}{2}\|f\|_{L^p} + \frac{1}{2}\|g\|_{L^p}.$$

It follows that

$$\|f+g\|_{L^p} \leq 2^{1/p-1}(\|f\|_{L^p} + \|g\|_{L^p}).$$

This constant is in fact best possible, since we may take, say, $f = 1_{[0,1]}$ and $g = 1_{[1,2]}$ to get

$$\|f+g\|_{L^p} = 2^{1/p} = 2^{1/p-1}(1+1) = 2^{1/p-1}(\|f\|_{L^p} + \|g\|_{L^p}).$$

(iv) First suppose $0 < p < 1$. Since $x \mapsto x^p$ is non-linear, the only way equality can occur in Jensen's inequality

$$\big((1-\theta)F(x) + \theta G(x)\big)^p \geq (1-\theta)F(x)^p + \theta G(x)^p$$

is when $F(x) = G(x)$. This implies that $f = cg$ for some $c > 0$. The case for $p > 1$ is analogous.

When $p = 1$, the identity becomes

$$\int_X |f(x) + g(x)| \, d\mu = \int_X |f(x)| \, d\mu + \int_X |g(x)| \, d\mu,$$

which holds for all non-negative measurable functions f and g by linearity of the integral.

Exercise 1.3.3. Let $\|\cdot\|$ be a norm, and let $v, w \in \{x \in V : \|x\| \leq 1\}$. Then, given $0 < t < 1$, homogeneity and the triangle equality imply that

$$\|tv + (1-t)w\| \leq |t|\|v\| + |1-t|\|w\| \leq t + (1-t) = 1,$$

so that the line joining v and w is contained in the closed unit ball. Conversely, suppose that the closed unit ball is convex. Then, given $v, w \in V$, we must prove the triangle inequality. By non-degeneracy, we may assume both vectors are non-zero. By homogeneity, we may assume that $\|v\| + \|w\| = 1/2$. By homogeneity again, we can write $v = (1-\theta)v'$ and $w = \theta w'$ for some $0 < \theta < 1$ and $v', w' \in V$ with $\|v'\| = \|w'\| = 1/2$. Convexity then implies that

$$\|v + w\| = \|(1-\theta)v' + \theta w'\| \leq (1-\theta)\|v'\| + \theta\|w'\| = \|v\| + \|w\|,$$

as desired. The proofs for the open unit ball are analogous.

Exercise 1.3.4. Note that $\operatorname{supp} f = \operatorname{supp} |f|^p$. Markov's inequality implies that

$$\mu\Big(\Big\{x \in X : |f(x)|^p \geq \frac{1}{n}\Big\}\Big) \leq n \int_X |f(x)|^p \, d\mu < \infty.$$

Thus

$$\operatorname{supp} f = \bigcup_{n=1}^{\infty} \Big\{x \in X : |f(x)|^p \geq \frac{1}{n}\Big\}$$

is σ-finite.

Exercise 1.3.5.

(i) [I could not solve this. This solution is an elaboration of `https://math.stackexchange.com/a/242792/` for my own understanding.]

Let $0 < \delta < \|f\|_{L^\infty}$, and let $S_\delta := \{x \in X : |f(x)| \geq \|f\|_{L^\infty} - \delta\}$. By definition of $\|\cdot\|_{L^\infty}$, we have $\mu(S_\delta) > 0$. We compute

$$\|f\|_{L^p} \geq \Big(\int_{S_\delta} (\|f\|_{L^\infty} - \delta)^p \, d\mu\Big)^{1/p} = (\|f\|_{L^\infty} - \delta)\mu(S_\delta)^{1/p} \qquad (*)$$

for $0 < p < \infty$. Setting $p = p_0$, we see that

$$(\|f\|_{L^\infty} - \delta)\mu(S_\delta)^{1/p_0} \leq \|f\|_{L^{p_0}} < \infty,$$

so that $\mu(S_\delta) < \infty$. Taking the limit inferior as $p \to \infty$ of $(*)$, we thus have

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}.$$

Conversely, since $|f(x)| \leq \|f\|_{L^\infty}$ for almost every x, we have

$$\begin{aligned}
\|f\|_{L^p} &= \Big(\int_X |f(x)|^{p-p_0} |f(x)|^{p_0} \, d\mu\Big)^{1/p} \\
&\leq \Big(\int_X \|f\|_{L^\infty}^{p-p_0} |f(x)|^{p_0} \, d\mu\Big)^{1/p} \\
&= \|f\|_{L^\infty}^{(p-p_0)/p} \Big(\int_X |f(x)|^{p_0} \, d\mu\Big)^{1/p} \\
&= \|f\|_{L^\infty}^{(p-p_0)/p} \|f\|_{L^{p_0}}^{p_0/p}
\end{aligned}$$

whenever $p > p_0$. Taking the limit superior as $p \to \infty$, we conclude that

$$\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}.$$

Therefore, the limit $\lim_{p\to\infty} \|f\|_{L^p}$ exists and is equal to $\|f\|_{L^\infty}$.

(ii) The argument is a modification of (i), except this time we use sets of the form $S_N := \{x \in X : |f(x)| \geq N\}$. We also handle the case $\mu(S_N) = +\infty$ directly, and we do not need the limit superior case.

Exercise 1.3.6. These are routine verifications. We first verify that the function d is a metric:

- (Non-degeneracy) By non-degeneracy of $\|\cdot\|$, we have $d(f, g) = 0$ iff $\|f - g\| = 0$ iff $f - g = 0$ iff $f = g$.

- (Symmetry) By homogeneity of $\|\cdot\|$, we have

$$d(f,g) = \|f-g\| = |-1|\|g-f\| = d(g,f).$$

- (Triangle inequality) By the triangle inequality for $\|\cdot\|$, we have

$$d(f,h) = \|f-h\| \leq \|f-g\| + \|g-h\| = d(f,g) + d(g,h).$$

This metric d satisfies:

- (Translation-invariance) We have

$$d(f+h, g+h) = \|(f+h)-(g+h)\| = \|f-g\| = d(f,g).$$

- (Homogeneity) By homogeneity of $\|\cdot\|$, we have

$$d(cf, cg) = \|cf - cg\| = \|c(f-g)\| = |c|\|f-g\| = |c|d(f,g).$$

Conversely, given a translation-invariant homogeneous metric d, we may define a function $\|\cdot\|\colon V \to [0,+\infty)$ by $\|f\| := d(0,f)$. We verify that this function $\|\cdot\|$ is a norm:

- (Non-degeneracy) By the non-degeneracy of d, we have $\|f\| = d(0,f) = 0$ iff $f = 0$.
- (Homogeneity) By homogeneity of d, we have

$$\|cf\| = d(0,cf) = |c|d(0,f) = |c|\|f\|.$$

- (Triangle inequality) By the triangle inequality for d, and by the translation-invariance of d, we have

$$\begin{aligned}\|f+g\| &= d(0,f+g)\\ &\leq d(0,f) + d(f,f+g)\\ &= d(0,f) + d(0,g)\\ &= \|f\| + \|g\|.\end{aligned}$$

We may establish analogous claims relating quasi-norms and quasi-metrics, as well as seminorms and semimetrics.

Exercise 1.3.7. Suppose the series $\sum_{j=1}^{\infty} f_j$ converges absolutely, so that $\sum_{j=1}^{\infty} \|f_j\| < \infty$. We claim that $(\sum_{j=1}^{n} f_j)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$. Then there exists N such that $\sum_{j=N}^{\infty} \|f_j\| < \epsilon$. Thus, given $m, n \geq N$, we have by repeated applications of the triangle inequality

$$\begin{aligned}\Big\|\sum_{j=m}^{n} f_j\Big\| &\leq \sum_{j=m}^{n} \|f_j\|\\ &\leq \sum_{j=N}^{\infty} \|f_j\|\\ &< \epsilon.\end{aligned}$$

Therefore $(\sum_{j=1}^n f_j)_{n=1}^\infty$ is Cauchy and converges to a limit f, which must be equal to $\sum_{j=1}^\infty f_j$ by definition of summation. Thus $\sum_{j=1}^\infty f_j$ is conditionally convergent as needed.

Conversely, suppose that absolute convergence implies conditional convergence, and let $(f_j)_{j=1}^\infty$ be a Cauchy sequence. Choose a sequence of integers $N_1 < N_2 < N_3 < \dots$ such that $\|f_m - f_n\| \le \epsilon/2^k$ whenever $m, n \ge N_k$. Then $\sum_{j=k}^\infty \|f_{N_j} - f_{N_{j-1}}\| \le \epsilon/2^{k-1}$, where $k \ge 2$. Therefore the series $\sum_{j=2}^\infty \|f_{N_j} - f_{N_{j-1}}\|$ converges, and by hypothesis the series $\sum_{j=2}^\infty (f_{N_j} - f_{N_{j-1}})$ converges as well. Since

$$\lim_{k\to\infty} \sum_{j=2}^k (f_{N_j} - f_{N_{j-1}}) = \lim_{k\to\infty} (f_{N_k} - f_{N_1}) = \lim_{k\to\infty} f_{N_k} - f_{N_1},$$

we see that the limit $\lim_{k\to\infty} f_{N_k}$ exists. Thus we have a convergent subsequence of a Cauchy sequence, which implies that the original sequence converges as desired.

Remark. Here is an equivalent formulation of convergence in L^p norm that is useful for understanding the last sentence of the proof of Proposition 1.3.7. Let $1 \le p < \infty$. Given a sequence $(f_n)_{n=1}^\infty$ of L^p functions together with an L^p function f, we have

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0 \quad \text{iff} \quad \lim_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p}.$$

For the forward implication, the reverse triangle inequality gives

$$\big| \|f_n\|_{L^p} - \|f\|_{L^p} \big| \le \|f_n - f\|_{L^p} \to 0$$

as $n \to \infty$.

Conversely, since $|f_n - f|^p \le 2^{p-1}(|f_n|^p + |f|^p)$ by convexity[2] of $x \mapsto |x|^p$, we may apply the reverse Fatou lemma to get

[2] Or, by (1.16),
$$|f_n - f|^p \le 2^p(|f_n|^p + |f|^p).$$

$$\limsup_{n\to\infty} \int_X |f_n - f|^p \, d\mu \le \int_X \limsup_{n\to\infty} |f_n - f|^p \, d\mu = 0,$$

so that $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$ as needed.

Exercise 1.3.8. The argument is similar to the proof of Proposition 1.3.8, except that we exclude the step where horizontal truncation is used to limit our consideration to bounded L^∞ functions of finite measure support. This is because in L^∞, functions do not necessarily 'decay at infinity.' Such decay allows us to use Markov's inequality to write the support of any L^p function with $0 < p < \infty$ as a countable union of finite measure sets, which lets us use horizontal truncation for measure spaces (245A exercise 1.4.36(x)). Consider the measure space with a singleton set $\{*\}$, where $\mu(\{*\}) = \infty$. Then $1_{\{*\}} \in L^\infty$, and $\int_{\{*\}} 1_{\{*\}} \, d\mu = \infty$, but the only function with finite measure support is the zero function, which has integral equal to zero.

Exercise 1.3.9. [I learned the following answer from `https://math.stackexchange.com/a/538087/`. The key idea I missed was that one could use a generating set to approximate a σ-algebra.] Suppose $\mathcal{X} = \langle \mathcal{A} \rangle$ for some countable set $\mathcal{A}$. By σ-finiteness, we may partition

$X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$. We prove that $L^p(X_n, \mathcal{X}\downharpoonright_{X_n}, \mu\downharpoonright_{X_n})$ is separable. Let $\epsilon > 0$ and $f \in L^p(X_n)$. Choose a simple function $f' = \sum_{j=1}^m c_j 1_{E_j}$ with rational coefficients c_j such that $\|f - f'\|_{L^p(X_n)} \leq \epsilon/2$. We define $\mathcal{A}_n := \{A \cap X_n : A \in \mathcal{A}\}$. Then $\mathcal{A}_n$ is countable, with $\mathcal{X}\downharpoonright_{X_n} = \langle \mathcal{A}_n \rangle$. Thus, by 245A exercise 1.4.28, we may approximate E_j by some set $E_j' \in \mathcal{A}_n$, so that $\mu(E_j \triangle E_j') \leq (\epsilon/2m|c_j|)^p$. Defining $f'' := \sum_{j=1}^m c_j 1_{E_j'}$, we see that

$$\begin{aligned}
\|f - f''\|_{L^p(X_n)} &\leq \|f - f'\|_{L^p(X_n)} + \|f' - f''\|_{L^p(X_n)} \\
&\leq \epsilon/2 + \sum_{j=1}^m |c_j| \|1_{E_j} - 1_{E_j'}\|_{L^p(X_n)} \\
&\leq \epsilon/2 + \sum_{j=1}^m |c_j| \|1_{E_j \triangle E_j'}\|_{L^p(X_n)} \\
&\leq \epsilon/2 + \sum_{j=1}^m |c_j| (\epsilon/2m|c_j|) \\
&= \epsilon.
\end{aligned}$$

Since the set D_n of rational linear combinations $\sum_{j=1}^m c_j 1_{E_j}$ of indicator functions 1_{E_j} with sets $E_j \in \mathcal{A}_n$ is countable, it follows that $L^p(X_n)$ is separable.

The general case then follows from letting D be the set of finite sums of simple functions with at most one taken from each D_n; that is, we let $D = \bigcup_{N=1}^{\infty} \{\sum_{n=1}^N f_n : f_n \in D_n\}$. Then D is countable, and given $f \in L^p(X)$, we may choose an approximation of $f\downharpoonright_{X_n}$ by some function $f_n \in D_n$ such that $\|f\downharpoonright_{X_n} - f_n\|_{L^p(X_n)} \leq \epsilon/2^{n+1}$, so that $\sum_{n=1}^{\infty} \|f\downharpoonright_{X_n} - f_n\|_{L^p(X_n)} \leq \epsilon/2$. By the completeness of L^p, we then have

$$\sum_{n=1}^{\infty} (f\downharpoonright_{X_n} - f_n) = f - \sum_{n=1}^{\infty} f_n \in L^p(X),$$

and so $\sum_{n=1}^{\infty} f_n \in L^p(X)$ as well. Thus we may choose sufficiently large N for which $\sum_{n=1}^N f_n \in D$ is a good approximation for f, so that $\|f - \sum_{n=1}^N f_n\|_{L^p(X)} \leq \epsilon$ as needed.

We note that L^∞ need not be separable. Consider $(\mathbf{N}, 2^{\mathbf{N}}, \#)$ for example, where we have $\|f\|_{L^\infty} = \sup_{n \in \mathbf{N}} |f(n)|$. If we look at the uncountably many maps of the form $f\colon \mathbf{N} \to \{0, 1\} \subset \mathbf{C}$, we see that they all belong to L^∞. Given any two distinct maps f and g of this form, we see that $\|f - g\|_{L^\infty} = 1$. Thus we may take small open balls in L^∞ around each function of this form, to obtain uncountably many disjoint open sets. It follows that $L^\infty(\mathbf{N}, 2^{\mathbf{N}}, \#)$ is not separable.

Exercise 1.3.10. Let us first note that we are dealing with Young's inequality: if $a, b \geq 0$ are nonnegative real numbers and $p, q > 1$ are dual (so that $1/p + 1/q = 1$), then $ab \leq a^p/p + b^q/q$.

Consider the use of convexity in the proof of Hölder's inequaliy. In particular, we used the fact that

$$e^{(1-t)\alpha + t\beta} \leq (1-t)e^\alpha + te^\beta.$$

When $0 < t < 1$, equality holds iff $\alpha = \beta$. Since we used this with $\alpha = p \log |f(x)|$ and $\beta = q \log |g(x)|$, it follows that $|f(x)|^p = |g(x)|^q$.

Thus the claim follows from the normalizations of $|f|$ and $|g|$ as in the proof.

Alternatively, one could see this geometrically by proving a more general form of Young's inequality: given a real-valued continuous strictly increasing function $f\colon [0,a] \to [0,+\infty)$ with $f(0) = 0$, we have

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx,$$

where $b \in \operatorname{im} f$. Indeed, the areas given by the two integrals cover the rectangle $[0,a] \times [0,b]$, which gives the result geometrically. Equality then holds iff $b = f(a)$, and we may recover the equality case for Hölder's inequality by setting $f(x) = x^{p/q}$.

If p is infinite, then we get

$$\|fg\|_{L^q} \leq \|f\|_{L^\infty} \|g\|_{L^q},$$

or

$$\Big(\int_X |f(x)g(x)|^q\,d\mu\Big)^{1/q} \leq \|f\|_{L^\infty} \Big(\int_X |g(x)|^q\,d\mu\Big)^{1/q}.$$

Thus equality holds iff $|f| = \|f\|_{L^\infty}$; that is, if $|f|$ is constant a.e..

Exercise 1.3.11. For $q = p$ the result is clear, so suppose $0 < q < p$. By Hölder's inequality, we have

$$\|f\|_{L^q} \leq \|1_E\|_{L^{pq/(p-q)}} \|f\|_{L^p} = \mu(E)^{1/q-1/p} \|f\|_{L^p}.$$

Equality holds iff $|f| = 1_E$.

Exercise 1.3.12. [This problem is hard! The solution below is from `https://math.stackexchange.com/a/669971/`.]

The idea is to use level sets. Let $E_\lambda := \{x \in X : |f(x)| \geq \lambda\}$, and suppose $0 < p < q < \infty$. Then

$$\|f\|_{L^p}^p = \int_X |f(x)|^p\,d\mu \geq \int_{E_\lambda} |f(x)|^p\,d\mu \geq \lambda^p \mu(E_\lambda).$$

In particular, if $\lambda > m^{-1/p}\|f\|_{L^p}$, then

$$\|f\|_{L^p}^p > m^{-1}\|f\|_{L^p}^p \mu(E_\lambda),$$

so that $\mu(E_\lambda) < m$. By definition of m, we must have $\mu(E_\lambda) = 0$. Thus

$$|f| \leq m^{-1/p}\|f\|_{L^p} \quad \text{a.e..}$$

It follows that

$$\begin{aligned}\int_X |f(x)|^q\,d\mu &\leq \||f|^{q-p}\|_{L^\infty} \int_X |f(x)|^p\,d\mu \\ &\leq (m^{-1/p}\|f\|_{L^p})^{q-p} \int_X |f(x)|^p\,d\mu,\end{aligned}$$

so that

$$\begin{aligned}\|f\|_{L^q} &\leq (m^{-1/p}\|f\|_{L^p})^{1-p/q} \Big(\int_X |f(x)|^p\,d\mu\Big)^{1/p+(1/q-1/p)} \\ &= m^{1/q-1/p}\|f\|_{L^p}\end{aligned}$$

as needed. Equality holds iff $|f|$ is constant. The case for $q = \infty$ then follows from taking the limit $q \to \infty$, noting that $\|f\|_{L^q} \le C < \infty$ for some constant C and sufficiently large q.

Exercise 1.3.13. By Hölder's inequality, we have

$$\begin{aligned} \|f\|_{L^p} &= \big\| |f|^{1-\theta} |f|^{\theta} \big\|_{L^p} \\ &\le \big\| |f|^{1-\theta} \big\|_{L^{p_0/(1-\theta)}} \big\| |f|^{\theta} \big\|_{L^{p_1/\theta}} \\ &= \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}. \end{aligned}$$

Equality holds when $|f|^{p_0} = |f|^{p_1}$; that is, when $|f|^{p_1 - p_0} = 1$, or when $|f| = 1_X$.

Exercise 1.3.14. By exercise 1.3.11, $\|f\|_{L^p} \le \mu(E)^{1/p - 1/p_0} \|f\|_{L^{p_0}}$, so that

$$\|f\|_{L^p}^p \le \mu(E)^{1 - p/p_0} \|f\|_{L^{p_0}}^p.$$

Thus

$$\limsup_{p \to 0} \|f\|_{L^p}^p \le \mu(E).$$

By Fatou's lemma,

$$\liminf_{n \to \infty} \int_X |f(x)|^{1/n}\, d\mu \ge \int_X \liminf_{n \to \infty} |f(x)|^{1/n}\, d\mu = \mu(E),$$

and so $\liminf_{p \to 0} \|f\|_{L^p}^p = \liminf_{n \to \infty} \|f\|_{L^{1/n}}^{1/n} \ge \mu(E)$ by continuity, which gives the result.

Time is a waste of money.

— OSCAR WILDE, *Phrases and Philosophies for the Use of the Young* (1894)

2.3. *The Stone and Loomis–Sikorski Representation Theorems*

When DEK taught Concrete Mathematics at Stanford for the first time, he explained the somewhat strange title by saying that it was his attempt to teach a math course that was hard instead of soft. He announced that, contrary to the expectations of some of his colleagues, he was not *going to teach the Theory of Aggregates, nor Stone's Embedding Theorem, nor even the Stone–Čech compactification. (Several students from the civil engineering department got up and quietly left the room.)*

— Ronald L. Graham, Donald E. Knuth, & Oren Patashnik, *Concrete Mathematics* (1988)

Exercise 2.3.1.

Exercise 2.3.2.

Exercise 2.3.3.

Unfortunately, it appears that there is now in your world
a race of vampires, called referees, who clamp down mercilessly
upon mathematicians unless they know the right passwords.
I shall do my best to modernize my language and notations,
but I am well aware of my shortcomings in that respect;
I can assure you, at any rate, that my intentions are honourable
and my results invariant, probably canonical, perhaps even functorial.
But please allow me to assume that the characteristic is not 2.

— André Weil, in *Annals of Mathematics* **69** (1959)

1.4. Hilbert spaces

Dr. von Neumann,
I would very much like to know,
what after all is a Hilbert space?

— David Hilbert, apocryphally (1929)

Exercise 1.4.1.

Exercise 1.4.2.

Exercise 1.4.3.

Exercise 1.4.4.

Exercise 1.4.5.

Exercise 1.4.6.

Exercise 1.4.7.

Exercise 1.4.8.

Exercise 1.4.9.

Exercise 1.4.10.

Exercise 1.4.11.

Exercise 1.4.12.

Exercise 1.4.13.

Exercise 1.4.14.

Exercise 1.4.15.

Exercise 1.4.16.

Exercise 1.4.17.

Exercise 1.4.18.

Exercise 1.4.19.

Exercise 1.4.20.

Exercise 1.4.21.

Exercise 1.4.22.

Exercise 1.4.23.

Exercise 1.4.24.

Exercise 1.4.25.

One moral of the above story is, of course, that we must be very careful when we give advice to younger people; sometimes they follow it!

— Edsger W. Dijkstra, *The Humble Programmer* (1972)

1.5. Duality and the Hahn–Banach theorem

迷生寂亂 *Rest and unrest derive from illusion;*
悟無好惡 *with enlightenment there is no liking and disliking.*
一切二邊 *All dualities come from*
妄自斟酌 *ignorant inference.*
夢幻虛華 *They are like dreams of flowers in the air:*
何勞把捉 *foolish to try to grasp them.*
得失是非 *Gain and loss, right and wrong:*
一時放卻 *such thoughts must finally be abolished at once.*

— 鑑智僧璨,《信心銘》 (c. 600)

Exercise 1.5.1.

Exercise 1.5.2.

Exercise 1.5.3.

Exercise 1.5.4.

Exercise 1.5.5.

Exercise 1.5.6.

Exercise 1.5.7.

Exercise 1.5.8.

Exercise 1.5.9.

Exercise 1.5.10.

Exercise 1.5.11.

Exercise 1.5.12.

Exercise 1.5.13.

Exercise 1.5.14.

Exercise 1.5.15.

Exercise 1.5.16.

Exercise 1.5.17.

Exercise 1.5.18.

Exercise 1.5.19.

Exercise 1.5.20.

Ask whatever questions you please, but do not ask me for reasons.
A young woman may be forgiven for not being able to give reasons,
since they say she lives in her feelings. Not so with me.

I generally have so many reasons,
and most often such mutually contradictory reasons,
that for this reason it is impossible for me to give reasons.

— Søren Kierkegaard, *Either/Or* **I** (1843)