Polynomial Growth

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We say that $g(n)$ is an *asymptotically tight bound* for $f(n)$, and write $f(n) \in \Theta(g(n))$, if there exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$$

whenever $n \ge n_0$.

In this brief note, we prove that polynomials of degree d (with positive leading coefficient) are in $\Theta(n^d)$.

Since $cf(n) \in \Theta(g(n))$ whenever $f(n) \in \Theta(g(n))$ and $c > 0$, as may be seen by scaling the bounding constants c_1 and c_2 by c, it suffices to prove the result for polynomials $\sum_{i=0}^{d} a_i n^i$ with $a_d = 1$. Let $c_1 = 1/2^d$ and $c_2 = (2^{d+1} - 1)/2^d$. The main idea is to have the bound $|a_i|n^i \le 2^{i-d} n^d$ whenever $n \ge n_0$. This can be accomplished by setting $n_0 = 2 \max_{0 \le i < d} |a_i|^{1/(d-i)}$. It follows that $\sum_{i=0}^{d} a_i n^i \le n^d \sum_{i=0}^{d} 2^{i-d} = c_2 n^d$ whenever $n \ge n_0$. To show $\sum_{i=0}^{d} a_i n^i \ge c_1 n^d$, just add the bounds $a_i n^i \ge -2^{i-d} n^d$ for $0 \le i < d$ to the trivial bound $n^d \ge n^d$.