SELECTED SOLUTIONS FOR TERENCE TAO'S BOOK "AN INTRODUCTION TO MEASURE THEORY"

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1. Prologue: The Problem of Measure

Exercise in the proof of Lemma 1.1.2. We prove that

$$|I| = \lim_{N\to\infty} \frac{1}{N} \#(I \cap \frac{1}{N}\mathbf{Z}).$$

Since $[a,b] \cap \frac{1}{N}\mathbf{Z} \cong [Na, Nb] \cap \mathbf{Z} = \{\lceil Na \rceil, \ldots, \lfloor Nb \rfloor\}$, we have

$$\#(I \cap \frac{1}{N}\mathbf{Z}) = \lfloor Nb \rfloor - \lceil Na \rceil + 1.$$

Since $Nb < \lfloor Nb \rfloor + 1 \le Nb + 1$ and $Na \le \lceil Na \rceil < Na + 1$, we have

$$Nb - Na - 1 < \lfloor Nb \rfloor - \lceil Na \rceil + 1 \le Nb - Na + 1,$$

so

$$b - a - \frac{1}{N} < \frac{\lfloor Nb \rfloor - \lceil Na \rceil + 1}{N} \le b - a + \frac{1}{N}.$$

The result follows from the squeeze theorem by sending $N \to \infty$.

Exercise 1.1.3. We first prove the result for $d = 1$. Suppose $m' \colon \mathcal{E}(\mathbf{R}) \to \mathbf{R}^+$ obeys non-negativity, finite additivity and translation invariance. For $n \ge 1$, we have

$$\begin{aligned} c := m'([0,1)) &= m'\left(\bigcup_{i=1}^{n} \Big[\frac{i-1}{n}, \frac{i}{n}\Big)\right) \\ &= \sum_{i=1}^{n} m'\Big(\Big[\frac{i-1}{n}, \frac{i}{n}\Big)\Big) \quad \text{by finite additivity} \\ &= \sum_{i=1}^{n} m'\Big(\Big[0, \frac{1}{n}\Big)\Big) \quad \text{by translation invariance} \\ &= nm'\Big(\Big[0, \frac{1}{n}\Big)\Big), \end{aligned}$$

and so $m'([0,1/n)) = c/n$. Thus $m'([0,k/n)) = ck/n$. Note that non-negativity and finite additivity imply monotonicity, which in turn implies that $m'(\{0\}) < 1/n$ for all n, so that $m'(\{x\}) = 0$ for all $x \in \mathbf{R}$ by translation invariance.

Since elementary sets are finite unions of disjoint boxes, it suffices to show that $m'(B) = cm(B)$ for all boxes B. Since singletons have zero measure as shown above, it suffices by translation invariance to prove the result for $B = [0,a)$ where $a > 0$. By writing $[0,a) = [0, \lfloor a \rfloor) \cup [\lfloor a \rfloor, a)$, we see that it suffices to consider $0 < a < 1$. By considering a sequence in $\mathbf{Q} \cap [0,a)$ converging to a, monotonicity yields the bound $m'([0,a)) \ge ca$, and we may also obtain $m'([0,a)) \le ca$ analogously.

For $\mathbf{R}^d$ we find $m'([0,1/n)^d) = c/n^d$ (recall $\bigcup_i A_i \times \bigcup_j B_j \approx \bigcup_{i,j} A_i \times B_j$). Similar arguments show that $m'(\prod_{1\le i\le d}[0,k_i/n)) = (c/n^d)(\prod_{1\le i\le d} k_i)$, and that

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degenerate elementary sets (where one of the factor intervals is a singleton) have zero measure under m'. We may finish off with a similar limiting argument:

$$m'\left(\prod_{1\le i\le d}[0,a_i)\right) \ge \sup\left\{m'\left(\prod_{1\le i\le d}[0,q_i)\right) : q_i \in \mathbf{Q}\cap[0,a_i)\right\} = c\prod_{1\le i\le d} a_i.$$

Exercise 1.1.4. Suppose $E_1 \subset \mathbf{R}^{d_1}$ and $E_2 \subset \mathbf{R}^{d_2}$ are elementary sets. Then $E_1 = \bigcup_i B_i$ and $E_2 = \bigcup_j B_j$, where the B_i and B_j denote boxes, and thus $E_1 \times E_2 = \bigcup_{i,j} B_i \times B_j$. Since the product of boxes is a box, it follows that $E_1 \times E_2$ is elementary. To show $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1)m^{d_2}(E_2)$, we write E_1 and E_2 as unions of disjoint boxes B_i and B_j, so that $E_1 \times E_2 = \bigcup_{i,j} B_i \times B_j$ is a union of disjoint boxes. Then, we have

$$m^{d_1+d_2}(E_1\times E_2) = \sum_{i,j}|B_i||B_j| = \Big(\sum_i |B_i|\Big)\Big(\sum_j |B_j|\Big) = m^{d_1}(E_1)m^{d_2}(E_2)$$

as needed.

Digression: Could we have a result along the lines of this? Let $S \subset \{1,\dots,d\}$ and write $\pi_S(\mathbf{R}^d) \coloneqq \{(x_1,\dots,x_d) \in \mathbf{R}^d : x_i \ne 0 \text{ implies } i \in S\}$. Then $\pi_S(\mathbf{R}^d) \approx \mathbf{R}^{|S|}$ canonically, and so, writing $T = \{1,\dots,d\} - S$, we have $\mathbf{R}^d \approx \pi_S(\mathbf{R}^d) \times \pi_T(\mathbf{R}^d)$ canonically. For example with $\{1,3\} \subset \mathbf{R}^3$, we have $\pi_{\{1,3\}}(\mathbf{R}^3) = \{(x,0,z) \in \mathbf{R}^3\}$ and so there is a natural identification of boxes $[a,b]\times[c,d] \approx [a,b]\times\{0\}\times[c,d]$. Further, together with the complementary identification $\pi_{\{2\}}(\mathbf{R}^3) = \{(0,y,0) \in \mathbf{R}^3\}$ and its associated correspondence of boxes $[e,f] \approx \{0\}\times[e,f]\times\{0\}$, there is a correspondence of products of boxes in a canonical way where products of boxes from both identified subspaces correspond to boxes in $\mathbf{R}^3$. (See the appendix.)

Exercise 1.1.5. To show (1) implies (2), suppose E is Jordan measurable, and let $\epsilon > 0$. Then there exist elementary sets $A \subset E \subset B$ with $m(A) > m(E) - \epsilon/2$ and $m(B) < m(E) + \epsilon/2$, so that $m(B - A) = m(B) - m(A) \le \epsilon$ by finite additivity of elementary measure.

To show (2) implies (3), let $A \subset E \subset B$ be elementary sets with $m(B - A) \le \epsilon$. Then $B \triangle A = B - A \supset B - E$, and so

$$m^{*,(J)}(B\triangle E) = \inf_{\substack{S\supset B-E\\ S\text{ elem.}}} m(S) \le m(B-A) \le \epsilon.$$

To show (3) implies (1), let A be an elementary set with $m^{*,(J)}(A\triangle E) \le \epsilon/4$. Then there exists an elementary set $B \supset A\triangle E$ with $m(B) < \epsilon/2$. This gives us two elementary sets $A - B \subset E \subset A \cup B$. Since

$$m^{*,(J)}(E) \ge m(A-B) \ge m(A) - m(B) > m(A) - \epsilon/2$$

and

$$m_{*,(J)}(E) \le m(A\cup B) \le m(A) + m(B) < m(A) + \epsilon/2,$$

we obtain $m^{*,(J)}(E) - m_{*,(J)}(E) < \epsilon$. It follows that E is Jordan measurable.

Exercise 1.1.6. (1) We begin by proving that $E \cup F$ is Jordan measurable. By exercise 1.1.5(2), there exist elementary sets A, B, A', B' with $A \subset E \subset B$, $A' \subset F \subset B'$, $m(B-A) \le \epsilon/2$, and $m(B'-A') \le \epsilon/2$. Then $A\cup A' \subset E\cup F \subset B\cup B'$. Since $B \cup B' - A \cup A' \subset (B - A) \cup (B' - A')$, it follows from already established properties of elementary measure that

$$\begin{aligned} m(B\cup B' - A\cup A') &\le m((B-A)\cup(B'-A')) \\ &\le m(B-A) + m(B'-A') \\ &\le \epsilon, \end{aligned}$$

and so applying exercise 1.1.5(2) again shows that $E \cup F$ is Jordan measurable. Showing that $E \cap F$ is Jordan measurable is quite similar — one uses the inclusion

$$B \cap B' - A \cap A' = (B \cap B' - A) \cup (B \cap B' - A') \subset (B - A) \cup (B' - A').$$

Showing that $E - F$ is Jordan measurable uses the fact that $A - B' \subset E - F \subset B - A'$ and

$$(B - A') - (A - B') \subset (B - A) \cup (B' - A').$$

Finally, $E \triangle F = E \cup F - E \cap F$ and is thus Jordan measurable.

(2) We have $m(E) \geq m_{*,(J)}(E)$, which is a supremum over elementary measures of elementary sets, which are clearly non-negative by definition.

(3) Let $A \subset E \subset B$, $A' \subset F \subset B'$ be elementary sets with

$$m(B) - \epsilon/2 < m(E) < m(A) + \epsilon/2$$

and

$$m(B') - \epsilon/2 < m(F) < m(A') + \epsilon/2.$$

Then, $E \cup F \supset A \cup A'$, and so

$$m(E \cup F) \geq m(A \cup A') = m(A) + m(A') > m(E) + m(F) - \epsilon.$$

Similarly, $E \cup F \subset B \cup B'$, and we have

$$m(E \cup F) \leq m(B \cup B') \leq m(B) + m(B') < m(E) + m(F) + \epsilon.$$

Since ϵ was arbitrary, this gives $m(E \cup F) = m(E) + m(F)$ as required.

(4) We have $E \uplus (F - E) = F$, where $\uplus$ denotes a disjoint union. By (1), $F - E$ is Jordan measurable, and so $m(E) + m(F - E) = m(F)$ by (3). Since $m(F - E) \geq 0$ by (2), we conclude that $m(E) \leq m(F)$.

(5) Since $E \cup F = E \uplus (F - E)$ and $F - E \subset F$, we have

$$m(E \cup F) = m(E) + m(F - E) \leq m(E) + m(F).$$

(6) This follows immediately from translation invariance of elementary sets — if $A \subset E$ with A elementary, then $A + x \subset E + x$ with $A + x$ elementary and $m(A + x) = m(A)$; similarly for $B \supset E$.

Exercise 1.1.7. (1) Let $f \colon B \to \mathbf{R}$ be a continuous function on a closed box $B \subset \mathbf{R}^d$, and denote by $\Gamma_f := \{(x, f(x)) : x \in B\} \subset \mathbf{R}^{d+1}$ its graph. Since the inner measure is at most the outer measure, the Jordan measurability of Γ_f is immediately established if we find for every $\epsilon > 0$ an elementary set of measure less than ϵ that contains Γ_f. Let $\epsilon > 0$. Since continuous functions on compact sets are uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/m(B)$ whenever $\|x - y\| < \delta$ and $x, y \in B$. Partition B into boxes of diameter less than δ. Each of these boxes $B_\alpha \subset \mathbf{R}^d$ gives rise to a box $B_\alpha \times I_\alpha \subset \mathbf{R}^{d+1}$ containing $\{(x, f(x)) : x \in B'\} \subset \Gamma_f$ with $m(I_\alpha) < \epsilon/m(B)$ by uniform continuity. It follows that $\bigcup_\alpha (B_\alpha \times I_\alpha)$ is an elementary set of measure less than ϵ that contains Γ_f. We conclude that the graph of f is Jordan measurable with Jordan measure zero.

(2) This is essentially the fact that bounded continuous functions are Riemann integrable. Alternatively, letting $U := \{(x, t) : x \in B \text{ and } 0 \leq t \leq f(x)\} \subset \mathbf{R}^{d+1}$, one may consider the sets (as defined in (1))

$$U - \bigcup_\alpha (B_\alpha \times I_\alpha) \subset U \subset U \cup \bigcup_\alpha (B_\alpha \times I_\alpha),$$

which may be shown to be elementary.

Exercise 1.1.8. (1) Suppose AB is horizontal. Then we may translate AB onto the x-axis and use exercise 1.1.7(2) to prove that ABC is Jordan measurable. Note that if the x-coordinate of C does not lie between the x-coordinates of A and B, we may just regard ABC as the difference of two right-angled triangles $AC'C$ and

$BC'C$ where C' is C projected onto the x-axis. We must then add back the line BC, but this has Jordan measure zero by exercise 1.1.7(1).

For the general case translate the triangle so that one point, call it A without loss of generality, lies on the x-axis, and the other two points are above it. Then this can be thought of as the area under a graph again with one or two right triangles removed and lines added appropriately, once again by exercise 1.1.7. It follows that solid triangles are Jordan measurable.

(2) This boils down to finding the area under a line $y = mx$ using the standard Riemann sums arguments.

Exercise 1.1.9. Suppose $P \subset \mathbf{R}^d$ be a compact convex polytope contained in a closed box B. We may write $P = \bigcap_i (B \cap H_i)$, where each $H_i := \{x \in \mathbf{R}^d : x \cdot v_i \leq c_i\}$ is a closed half-space, and so it suffices to prove that sets of the form $B \cap H_i$ are Jordan measurable. We may identify $\mathbf{R}^{d-1} \subset \mathbf{R}^d$ as the subset with $x_i = 0$. Pick an identification where, when the projection of the hyperplane defined by $x \cdot v = c$ onto the identified $\mathbf{R}^{d-1}$ is surjective. Then, projecting the box B down to $\pi(B) \subset \mathbf{R}^{d-1}$, we may use exercise 1.1.7(2) to obtain our result by considering $B \cap H_i$ as the region under an appropriate graph.

Exercise 1.1.10. (1) To show that balls are Jordan measurable, it suffices to translate the standard ball $B(x, r)$ by r units in x_d so that it lies in the closed upper half space, then treat it as the difference of two graphs. For example, when $d = 2$, we consider the difference of the regions below the graphs of functions $r \pm \sqrt{r^2 - x^2}$.

Now, if we define the scaling by r of an interval $I = [a, b]$ by $rI := [ra, rb]$ (and similarly for open and half-closed intervals), then $m(rI) = rm(I)$. We may extend this to a box $B = \prod_{1 \leq j \leq d} I_j$ to get $rB := \prod_{1 \leq j \leq d} rI_j$ and $m(rB) = r^d m(B)$, and similarly to elementary sets $A = \bigcup_i B_i$ where $rA := \bigcup_i rB_i$ and $m(rA) = r^d m(A)$.

Denote the open ball of radius r of dimension d centered at 0 by $B_d(r) \subset \mathbf{R}^{d+1}$, and let $c_d := m(B_d(1))$. We will show that $m(B_d(r)) = c_d r^d$. Let $A \subset B_d(1) \subset B$ be elementary sets with

$$c_d - \epsilon / r^d < m(A) \quad \text{and} \quad m(B) < c_d + \epsilon / r^d.$$

Then, $rA \subset B_d(r) \subset rB$ are elementary sets, and so

$$\begin{aligned} c_d r^d - \epsilon < r^d m(A) &= m(rA) \\ &\leq m(B_d(r)) \\ &\leq m(rB) = r^d m(B) < c_d r^d + \epsilon. \end{aligned}$$

Since ϵ was arbitrary we conclude that $m(B_d(r)) = c_d r^d$ as needed.

(2) The bound

$$\left(\frac{2}{\sqrt{d}} \right)^d \leq c_d \leq 2^d$$

is easily established by inscribing and circumscribing cubes in the unit sphere. For the inner cube, note that its diameter is 2, so its side length is $2/\sqrt{d}$ and its volume is $(2/\sqrt{d})^d$. (In fact, $c_d = \frac{1}{d} \frac{2\pi^{d/2}}{\Gamma(d/2)}$.)

Exercise 1.1.11. (1) Recall that exercise 1.1.3 tells us that any map $m' \colon \mathcal{E}(\mathbf{R}^d) \to \mathbf{R}$ satisfying nonnegativity, finite additivity and translation invariance is necessarily a scalar multiple of elementary measure. We prove that $m \circ L$ satisfies these properties. The nonnegativity of $m \circ L$ follows immediately from the nonnegativity of m. If L is invertible, then $(m \circ L)(E \uplus F) = m(L(E) \uplus L(F)) = (m \circ L)(E) + (m \circ L)(F)$. Otherwise, we claim that $m \circ L = 0$. Indeed, $L(E)$ must be a bounded subset of a hyperplane $S \subsetneq \mathbf{R}^d$, and $L(E)$ must be contained in some closed box B, so $m(L(E)) \leq m(B \cap S)$. Choosing an appropriate identification $\mathbf{R}^{d-1} \subset \mathbf{R}^d$ as in exercise 1.1.9, we see that $B \cap S$ is the graph of a linear (and thus continuous) function, and so $m(B \cap S) = 0$ by exercise 1.1.7(1). Finally, translation invariance is immediate from the linearity of L together with the translation invariance of m — we get $m(L(E+x)) = m(L(E) + L(x)) = m(L(E))$. We conclude that $m \circ L = Dm$ for some constant $D \geq 0$.

It is time for a digression. I feel somewhat guilty for the handwavy treatment of the measure zero case both above and in exercise 1.1.9, so I shall make up for it to a small extent by providing some examples. Take $\mathbf{R}^3$ with

$$T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

These are rank 2 matrices; the projection of $\mathrm{im}(T)$ onto the xy-plane is not surjective whereas the projection of $\mathrm{im}(T')$ onto the xy-plane is surjective. If we have an elementary set $E \subset \mathbf{R}^3$, then $\mathrm{im}_{T'}(E) \subset \mathrm{im}(T') \cap B$, where $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbf{R}^3$ is a closed box containing $\mathrm{im}_{T'}(E)$ (which exists, since linear maps are bounded). We may then define the closed box $\pi(B) \coloneqq [a_1, b_1] \times [a_2, b_2] \subset \{(x, y, 0) \in \mathbf{R}^3 : x, y \in \mathbf{R}\} \approx \mathbf{R}^2$ and $f \colon \pi(B) \to \mathbf{R}$ defined by $f(x, y) \coloneqq x + y$. Then $\mathrm{im}_{T'}(E) \subset \Gamma_f \subset \mathrm{im}(T')$, and so we have $m(\mathrm{im}_{T'}(E)) \leq m(\Gamma_f) = 0$ by exercise 1.1.7(1). For $\mathrm{im}(T)$ we may project onto the xz-plane instead, and our mapping will be $(x, z) \mapsto -x$. In general, if we are given a $(d-1)$-dimensional subspace of $\mathbf{R}^d$ represented as the image of a noninvertible linear operator T on $\mathbf{R}^d$, we can always find $d-1$ basis vectors $\{e_i\}_{1 \leq i \leq d; i \neq j}$ such that $\{Te_i\}_{1 \leq i \leq d; i \neq j}$ is independent. We may thus project onto $\{(x_1, \ldots, x_d) \in \mathbf{R}^d : x_j = 0\} =: S \approx \mathbf{R}^{d-1}$, and treat $\mathbf{R}^d \approx S \times \mathbf{R}$ using exercise 1.1.4. (I apologize for how sloppy/handwavy this treatment is. See also the appendix to this section for more thoughts.)

(2) Suppose $E \subset \mathbf{R}^d$ is Jordan measurable. If L is not invertible, then $L(E)$ is Jordan measurable with Jordan measure zero as argued in (1), and $D = 0$, so $m(L(E)) = Dm(E)$. Henceforth we may assume that L is invertible. Let $D > 0$ be such that $m(L(E')) = Dm(E')$, where E' denotes any elementary set. We first prove that $L(E)$ is Jordan measurable. Let $A \subset E \subset B$ be elementary sets with $m(B - A) \leq \epsilon/4D$, or $m(B) \leq m(A) + \epsilon/4D$. We have $L(A) \subset L(E) \subset L(B)$. Since $L(A)$ and $L(B)$ are Jordan measurable, we may choose elementary sets A' and B' such that $A' \subset L(A) \subset L(E) \subset L(B) \subset B'$ with

$$m(A') > m(L(A)) - \epsilon/2 = Dm(A) - \epsilon/2$$

and

$$m(B') < m(L(B)) + \epsilon/4 = Dm(B) + \epsilon/4 \leq Dm(A) + \epsilon/2.$$

It follows that

$$m(B' - A') = m(B') - m(A') < \epsilon.$$

Since ϵ was arbitrary, we conclude that $L(E)$ is Jordan measurable.

Now we prove that $m(L(E)) = Dm(E)$. Let $A \subset E \subset B$ be elementary sets with $m(E) - \epsilon/D < m(A)$ and $m(B) < m(E) + \epsilon/D$. We have $L(A) \subset L(E) \subset L(B)$, and so

$$\begin{aligned} Dm(E) - \epsilon < Dm(A) &= m(L(A)) \\ &\leq m(L(E)) \\ &\leq m(L(B)) = Dm(B) < Dm(E) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we conclude that $m(L(E)) = Dm(E)$.

(3) The case for dimension $d = 1$ is straightforward as all linear maps $\mathbf{R} \to \mathbf{R}$ are scalar multiplication. Henceforth fix $d \geq 2$. We first prove $m(L(E)) = |\det(L)| m(E)$ for Jordan measurable $E \subset \mathbf{R}^d$ and elementary matrices L as in Gaussian elimination. Recall the three classes of elementary matrices:

(A) Row swapping. For $1 \le i < j \le d$, we define $A_{i,j}$ to be the linear operator that swaps entries i and j of an input vector.
(B) Row scaling. For $\alpha \neq 0$ and $1 \le i \le d$, we define B_i^α to be the linear operator that scales row i of an input vector by α.
(C) Row adding. For $\alpha \neq 0$ and $1 \le i, j \le d$, we define $C_{i,j}^\alpha$ to be the linear operator that adds α times row i to row j.

It suffices to prove that $m(L(I^d)) = |\det(L)|$, where $I^d = [0,1]^d$ is the unit d-cube. For row swapping matrices $A_{i,j}$, we know $|\det(A_{i,j})| = |-1| = 1$ and $A_{i,j}(I^d) = I^d$. For row scaling matrices B_i^α, we know $|\det(B_i^\alpha)| = \alpha$ and

$$B_i^\alpha(I^d) = [0,1]^{i-1} \times [0,\alpha] \times [0,1]^{d-i},$$

so $m(B_i^\alpha(I^d)) = \alpha$. For row adding matrices $C_{i,j}^\alpha$, we first consider the 2-dimensional case. The image of a matrix like $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ is a parallelogram that can be realized as a rectangle with two right triangles removed. In particular, the rectangle has vertices $(0,0)$, $(\alpha+1,0)$, $(\alpha+1,1)$ and $(0,1)$. The first triangle has vertices $(1,0)$, $(\alpha+1,0)$ and $(\alpha+1,1)$, and the second triangle has vertices $(0,0)$, $(0,1)$ and $(\alpha,1)$. The boundary lines have zero measure since they are graphs of appropriate functions, so the area of the parallelogram works out to be $(\alpha+1) - \alpha/2 - \alpha/2 = 1$. This is the essential case — in higher dimensions, instead of working with triangles, we end up dealing with the product of triangles with cubes I^k. Let $d \ge 3$. If we define for $S \subset \{1, \dots, d\}$ the identified subspace

$$\pi_S(\mathbf{R}^d) \coloneqq \{(x_1, \dots, x_d) \in \mathbf{R}^d : x_i \neq 0 \text{ implies } i \in S\} \approx \mathbf{R}^{|S|},$$

then we obtain a canonical correspondence $\mathbf{R}^d \approx \pi_{\{i,j\}}(\mathbf{R}^d) \times \mathbf{R}^{d-2}$, where

$$\pi_{\{i,j\}}(C_{i,j}^\alpha) \approx \{(x_j + \alpha x_i, x_i) : x_i, x_j \in I\} =: P$$

is a parallelogram with measure one as argued in the 2-dimensional case. It follows that $C_{i,j}^\alpha \approx \pi_{\{i,j\}}(C_{i,j}^\alpha) \times I^{d-2}$ and thus has measure one as well. Since $\det(C_{i,j}^\alpha) = 1$, the result follows. We conclude that $m(L(E)) = |\det(L)| m(E)$ whenever L is elementary and E is Jordan measurable.

Notice that if $m(L(E)) = Dm(E)$ and $m(L'(E)) = D'm(E)$ for elementary L, L', then $m(L(L'(E))) = Dm(L'(E)) = DD'm(E)$. By Gaussian elimination, we may write any invertible linear map L as the product $L_1 \dots L_k$ of elementary matrices. Since $|\det(AB)| = |\det(A)||\det(B)|$, we compute

$$m(L(E)) = m((L_1 \dots L_k)(E)) = \Big(\prod_{1 \le i \le k} |\det(L_i)|\Big) m(E) = |\det(L)| m(E),$$

and we are done.

Exercise 1.1.12. Suppose F is a Jordan null set and $E \subset F$ is an arbitrary subset. Then, we may find an elementary set $A \supset F$ with $m(A) \le \epsilon$. It follows that $m^{*,(J)}(E) \le \epsilon$. Since ϵ is arbitrary, it follows that $m^{*,(J)}(E) = 0$. But then $0 \le m_{*,(J)}(E) \le m^{*,(J)}(E)$, so the outer and inner measures are identically zero. We conclude that E is a Jordan null set.

Exercise 1.1.13. Recall that we have

$$m(E) = \lim_{N \to \infty} \frac{1}{N^d} \# \Big(E \cap \frac{1}{N} \mathbf{Z}^d\Big)$$

for elementary sets $E \subset \mathbf{R}^d$. We shall prove it for Jordan measurable sets.

Let $E \subset \mathbf{R}^d$ be Jordan measurable, and let $A \subset E \subset B$ be elementary sets with $m(B) \le m(A) + \epsilon/2$. Pick large N with

$$\Big| m(A) - \frac{1}{n^d} \# \Big(A \cap \frac{1}{n} \mathbf{Z}^d\Big) \Big| < \frac{\epsilon}{2} \quad \text{and} \quad \Big| m(B) - \frac{1}{n^d} \# \Big(B \cap \frac{1}{n} \mathbf{Z}^d\Big) \Big| < \frac{\epsilon}{2}$$

whenever $n \geq N$. Then,

$$\begin{aligned} m(E) - \frac{1}{n^d}\#\Big(E \cap \frac{1}{n}\mathbf{Z}^d\Big) &\leq m(B) - \frac{1}{n^d}\#\Big(A \cap \frac{1}{n}\mathbf{Z}^d\Big) \\ &\leq m(A) - \frac{1}{n^d}\#\Big(A \cap \frac{1}{n}\mathbf{Z}^d\Big) + \epsilon/2 \\ &< \epsilon \end{aligned}$$

and

$$\begin{aligned} m(E) - \frac{1}{n^d}\#\Big(E \cap \frac{1}{n}\mathbf{Z}^d\Big) &\geq m(A) - \frac{1}{n^d}\#\Big(B \cap \frac{1}{n}\mathbf{Z}^d\Big) \\ &\geq m(B) - \frac{1}{n^d}\#\Big(B \cap \frac{1}{n}\mathbf{Z}^d\Big) - \epsilon/2 \\ &> -\epsilon \end{aligned}$$

whenever $n \geq N$. Since ϵ is arbitrary, the result follows.

Exercise 1.1.14. In this exercise, we investigate the epsilon entropy formulation of Jordan measurability. A *dyadic cube* is a half-open box of the form

$$\Big[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\Big) \times \cdots \times \Big[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\Big)$$

for some integers n, i_1, …, i_d. Let $E \subset \mathbf{R}^d$ be a bounded set. For each integer n, let $\mathcal{E}_*(E, 2^{-n})$ denote the number of dyadic cubes of sidelength 2^{-n} that are contained in E, and let $\mathcal{E}^*(E, 2^{-n})$ be the number of dyadic cubes of sidelength 2^{-n} that intersect E. Denote by $\mathcal{S}_*(E, 2^{-n}) \subset E$ the union of dyadic cubes of sidelength 2^{-n} contained in E, and $\mathcal{S}^*(E, 2^{-n}) \supset E$ the union of dyadic cubes of sidelength 2^{-n} intersecting E. These sets are unions of boxes and are thus elementary. Before moving on, we first note the following identities for bounded sets $E, F \subset \mathbf{R}^d$ concerning $\mathcal{S}_*$ (analogous identities hold for $\mathcal{S}^*$):

$$m(\mathcal{S}_*(E, 2^{-n})) = 2^{-dn}\mathcal{E}_*(E, 2^{-n}) \tag{1}$$

$$\mathcal{S}_*(E, 2^{-n}) \subset \mathcal{S}_*(E, 2^{-m}) \quad \text{if } m > n \tag{2}$$

$$\mathcal{S}_*(E, 2^{-n}) \cup \mathcal{S}_*(F, 2^{-n}) \subset \mathcal{S}_*(E \cup F, 2^{-n}) \tag{3}$$

$$m(\mathcal{S}_*(E, 2^{-n})) \leq m_{*,(J)}(E) \tag{4}$$

$$\mathcal{S}_*(E, 2^{-n}) \subset \mathcal{S}_*(F, 2^{-n}) \quad \text{if } E \subset F \tag{5}$$

We now prove that E is Jordan measurable if and only if

$$\lim_{n\to\infty} 2^{-dn}\big(\mathcal{E}^*(E, 2^{-n}) - \mathcal{E}_*(E, 2^{-n})\big) = 0,$$

or, equivalently (by (1)), if

$$\lim_{n\to\infty} \big(m(\mathcal{S}^*(E, 2^{-n})) - m(\mathcal{S}_*(E, 2^{-n}))\big) = 0. \tag{$*$}$$

We start by using dyadic cubes to approximate an interval $I = [a, b] \subset \mathbf{R}$. With n fixed, we are interested in the defect $I - \mathcal{S}_*(I, 2^{-n})$. Since we may choose i with $i/2^n < a \leq (i+1)/2^n$, we see that the first dyadic cube in I is at most distance 2^{-n} from a. We may reason similarly with the endpoint to get

$$m(I - \mathcal{S}_*(I, 2^{-n})) < 2 \cdot 2^{-n} = 2^{-n+1}.$$

Generalizing to a box $B = I_1 \times \cdots \times I_d \subset \mathbf{R}^d$, we obtain

$$B - \mathcal{S}_*(B, 2^{-n}) = \bigcup_{1\leq k\leq d} \Big(I_1 \times \cdots \times I_{k-1} \times (I_k - \mathcal{S}_*(I_k, 2^{-n})) \times I_{k+1} \times \cdots \times I_d\Big)$$

and consequently

$$\begin{aligned} m(B - \mathcal{S}_*(B, 2^{-n})) &\le \sum_{1\le k\le d} |I_1| \dots |I_{k-1}| \cdot m(I_k - \mathcal{S}_*(I_k, 2^{-n})) \cdot |I_{k+1}| \dots |I_d| \\ &\le 2^{-n+1} m(B) \sum_{1\le k\le d} \frac{1}{|I_k|} \to 0 \end{aligned}$$

as $n \to \infty$.

We are now ready to estimate Jordan measurable sets with dyadic cubes. Suppose $E \subset \mathbf{R}^d$ is Jordan measurable, let $\epsilon > 0$, and let $A \subset E$ be an elementary set with $m(A) > m(E) - \epsilon/2$. Write $A = \bigcup_{1\le i\le M} B_i$ as a disjoint union of boxes, where each box is nondegenerate, discarding boxes if necessary. This does not affect $m(A)$, as we are discarding finitely many null sets. Choose large N such that $m(B_i - \mathcal{S}_*(B_i, 2^{-n})) < \epsilon/2M$ whenever $n \ge N$ and $1 \le i \le M$. It follows from (3) that

$$\begin{aligned} m(\mathcal{S}_*(E, 2^{-n})) &\ge m(\mathcal{S}_*(A, 2^{-n})) \\ &\ge \sum_{1\le i\le M} m(\mathcal{S}_*(B_i, 2^{-n})) \\ &> \sum_{1\le i\le M} \Big(m(B_i) - \frac{\epsilon}{2M} \Big) \\ &= m(A) - \epsilon/2 \\ &> m(E) - \epsilon \end{aligned}$$

whenever $n \ge N$. We deduce that $\lim_{n\to\infty} m(\mathcal{S}_*(E, 2^{-n})) = m_{*,(J)}(E) = m(E)$.

One may develop the theory analogously for $\mathcal{E}^*$ and $\mathcal{S}^*$, this time considering $\mathcal{S}^*(B, 2^{-n}) - B$ and estimating E by elementary sets $A \supset E$. It then follows that

$$m(E) = \lim_{n\to\infty} m(\mathcal{S}_*(E, 2^{-n})) = \lim_{n\to\infty} m(\mathcal{S}^*(E, 2^{-n})),$$

and so $(*)$ holds.

Conversely, suppose $(*)$ holds. Then, since

$$m(\mathcal{S}_*(E, 2^{-n})) \le m_{*,(J)}(E) \le m^{*,(J)}(E) \le m(\mathcal{S}^*(E, 2^{-n})) < \infty,$$

it follows from $(*)$ that $m_{*,(J)}(E) = m^{*,(J)}(E)$, and so E is Jordan measurable as needed.

Exercise 1.1.15. Suppose $m' \colon \mathcal{J}(\mathbf{R}^d) \to \mathbf{R}^+$ is a map from the collection $\mathcal{J}(\mathbf{R}^d)$ of Jordan measurable subsets of $\mathbf{R}^d$ to the non-negative reals that obeys non-negativity, finite additivity and translation invariance. By exercise 1.1.3, we have $m'|_{\mathcal{E}(\mathbf{R}^d)} = cm|_{\mathcal{E}(\mathbf{R}^d)}$, where $c = m'([0,1)^d)$. (This is like a density argument.) Now let E be Jordan measurable; we will prove $m'(E) = cm(E)$. If $c = 0$, then, since Jordan measurable sets are bounded by definition, $E \subset B$ for some box B, and $m'(E) \le m'(B) = 0m(B) = 0$ as a consequence. Thus E is a Jordan null set satisfying the identity. Otherwise, suppose $c > 0$ and let $A \subset E$ be elementary with $m(A) > m(E) - \epsilon/c$. Then $m'(E) \ge m'(A) = cm(A) > cm(E) - \epsilon$, and so $m'(E) \ge cm(E)$. Conversely, if $B \supset E$ is elementary with $m(B) - \epsilon/c < m(E)$, then $cm(E) > cm(B) - \epsilon = m'(B) - \epsilon \ge m'(E) - \epsilon$ and so $cm(E) \ge m'(E)$ as needed.

Exercise 1.1.16. Suppose $m(E_1)$ and $m(E_2)$ are non-zero. (The case for $m(E_i) = 0$ is left as an exercise.) Let $A \subset E_1 \subset B$, $A' \subset E_2 \subset B'$ be elementary sets with $\epsilon < 6m(E_1)m(E_2)$ and

$$\begin{cases} m(A) > m(E_1) - \epsilon/2m(E_2) \quad \text{and} \quad m(A') > m(E_2) - \epsilon/2m(E_1); \\ m(B) < m(E_1) + \epsilon/3m(E_2) \quad \text{and} \quad m(B') < m(E_2) + \epsilon/3m(E_1). \end{cases}$$

Then, since $A \times A' \subset E_1 \times E_2 \subset B \times B'$, we find by exercise 1.1.4 that

$$\begin{cases} m(A \times A') = m(A)m(A') > m(E_1)m(E_2) - \epsilon; \\ m(B \times B') = m(B)m(B') < m(E_1)m(E_2) + \epsilon \end{cases}$$

and so $m_{*,(J)}(E_1 \times E_2) \geq m(E_1)m(E_2) \geq m^{*,(J)}(E_1 \times E_2)$.

Exercise 1.1.18. (1) If A is an elementary set, then $m(\overline{A}) = m(A)$. This is because $\overline{\bigcup_i X_i} = \bigcup_i \overline{X}_i$, and $m(\overline{B}) = m(B)$ for boxes B. In particular,

$$m(\overline{A}) = m\Big(\overline{\bigcup_i B_i}\Big) = m\Big(\bigcup_i \overline{B_i}\Big) \leq \sum_i m(\overline{B_i}) = \sum_i m(B_i) = m(A).$$

Now, let $E \subset \mathbf{R}^d$ be bounded, and let $E \subset A$ be elementary with $m(A) < m(E) + \epsilon$. Then $\overline{E} \subset \overline{A}$, and so

$$m^{*,(J)}(\overline{E}) \leq m(\overline{A}) = m(A) < m(E) + \epsilon.$$

Sending $\epsilon \to 0$, we find $m^{*,(J)}(\overline{E}) \leq m(E)$, and so $m^{*,(J)}(E) = m^{*,(J)}(\overline{E})$ as desired.

(2) The proof is essentially dual to that of (1).

(3) Suppose E is Jordan measurable. By (1) and (2) we have

$$m_{*,(J)}(E^\circ) = m(E) = m^{*,(J)}(\overline{E}).$$

Since

$$m^{*,(J)}(E^\circ) \leq m^{*,(J)}(\overline{E}) = m_{*,(J)}(E^\circ),$$

it follows that E° is Jordan measurable. Similarly, since

$$m^{*,(J)}(\overline{E}) = m_{*,(J)}(E^\circ) \leq m_{*,(J)}(\overline{E}),$$

it follows that $\overline{E}$ is Jordan measurable as well, and $m(E^\circ) = m(\overline{E})$. We conclude that ∂E is Jordan measurable, being the difference of Jordan measurable sets, and has measure $m(\partial E) = m(\overline{E}) - m(E^\circ) = 0$.

The converse is trickier — I was unable to figure this out and had to look it up unfortunately. Below I detail the outline given by Silvius Klein in `https://wiki.math.ntnu.no/_media/tma4225/2015h/tma4225-f15-homework.pdf`.

Suppose $m^{*,(J)}(\partial E) = 0$, and let $A \supset \partial E$ be an elementary set with measure $m(A) < \epsilon$. We may assume that A is open — if it isn't, just take its closure $\overline{A}$, which has the same measure as A, and use the fact that every closed box B lies in an open box of measure $(1+\epsilon)m(B)$, which can be obtained by scaling B° and translating appropriately. It follows that $\overline{E} - A$ is closed, and thus compact by the boundedness of E.

Now $\overline{E} - A \subset E^\circ$, and we may consider the cover of $\overline{E} - A$ consisting of all open boxes in E° containing a point of $\overline{E} - A$. We may then use compactness to obtain a finite cover of $\overline{E} - A$ by these boxes, whose union is an elementary set B with $\overline{E} - A \subset B \subset E^\circ$.

It follows that $\overline{E} \subset A \cup B$. Since $A \cup B$ is elementary, we find

$$m^{*,(J)}(E) = m^{*,(J)}(\overline{E}) < m(B) + \epsilon \leq m_{*,(J)}(E^\circ) + \epsilon = m_{*,(J)}(E) + \epsilon.$$

Taking $\epsilon \to 0$, we conclude that E is Jordan measurable.

(4) Denote by $BRS \coloneqq [0,1]^2 - \mathbf{Q}^2$ the bullet-riddled square. Since $\varnothing \subset BRS \subset [0,1]^2$, we see $m_{*,(J)}(BRS) \geq 0$ and $m^{*,(J)}(BRS) \leq 1$. The key fact is that every non-empty open subset of $\mathbf{R}^d$ contains a rational point (of $\mathbf{Q}^d$) and an irrational point (of $\mathbf{R}^d - \mathbf{Q}^d$). So if $m_{*,(J)}(BRS) > 0$, then there exists an elementary set $A \subset BRS$ with $m(A) > 0$. Since A is the finite union of boxes, we must have some non-empty open subset of A, which necessarily contains a rational point. The argument is similar for showing that $m^{*,(J)}(BRS) = 1$, and for analogous results for the set of bullets $[0,1]^2 \cap \mathbf{Q}^2$.

Exercise 1.1.19. Let $E \subset \mathbf{R}^d$ be bounded and $F \subset \mathbf{R}^d$ be Jordan measurable. We will prove the Carathéodory type identity

$$m^{*,(J)}(E) = m^{*,(J)}(E \cap F) + m^{*,(J)}(E - F).$$

Suppose $A \supset E$ is elementary with $m(A) < m^{*,(J)}(E) + \epsilon$. Then $A \cap F \supset E \cap F$ and $A - F \supset E - F$ are disjoint Jordan measurable sets with $A = (A \cap F) \uplus (A - F)$, so $m(A) = m(A \cap F) + m(A - F)$. It follows that

$$m^{*,(J)}(E \cap F) + m^{*,(J)}(E - F) \leq m(A \cap F) + m(A - F) = m(A) < m^{*,(J)}(E) + \epsilon,$$

and so $m^{*,(J)}(E \cap F) + m^{*,(J)}(E - F) \leq m^{*,(J)}(E)$.

Now suppose $A \supset E \cap F$ and $B \supset E - F$ are elementary sets with

$$m(A) < m^{*,(J)}(E \cap F) + \epsilon/2 \quad \text{and} \quad m(B) < m^{*,(J)}(E - F) + \epsilon/2.$$

Then $A \cup B \supset E$, and so

$$m^{*,(J)}(E) \leq m(A \cup B) \leq m(A) + m(B) < m^{*,(J)}(E \cap F) + m^{*,(J)}(E - F) + \epsilon,$$

and we conclude that $m^{*,(J)}(E) \leq m^{*,(J)}(E \cap F) + m^{*,(J)}(E - F)$. This completes the proof. (In general, we have $m^{*,(J)}(A \uplus B) \leq m^{*,(J)}(A) + m^{*,(J)}(B)$ for bounded sets A and B.)

Exercise 1.1.20. It suffices to prove that refining a partition preserves the quantity $\sum_i c_i |I_i|$, since we may then take the common refinement of two partitions to show that they yield the same value. If an interval I_i of our partition is divided into intervals $I_i \coloneqq I_{i,1} \cup \cdots \cup I_{i,k}$, then f takes the same value c_i on each $I_{i,j}$, and so the contributed value to the summation is $c_i|I_{i,1}| + \cdots + c_i|I_{i,k}| = c_i|I_i|$, and so the quantity is preserved.

Exercise 1.1.21. (1) Suppose f is piecewise constant on $[a, b] = I_1 \cup \cdots \cup I_n$, so that it takes the constant value c_i on I_i. Then, given $c \in \mathbf{R}$, clearly cf is piecewise constant with the same partition, taking the value cc_i on I_i and satisfying $\text{p.c.}\int_a^b cf(x)\,dx = c\ \text{p.c.}\int_a^b f(x)\,dx$. Given piecewise constant $g\colon [a, b] \to \mathbf{R}$, we take the common refinement of the partition associated to f and the partition associated to g, so that $[a, b] = I_1 \cup \cdots \cup I_n$ with $f \equiv c_i$ and $g \equiv d_i$ on I_i. It follows that $f + g \equiv c_i + d_i$ on I_i, so that $\text{p.c.}\int_a^b f(x) + g(x)\,dx = \text{p.c.}\int_a^b f(x)\,dx + \text{p.c.}\int_a^b g(x)\,dx$.

(2) By (1), it suffices to show that the p.c. integral of a p.c. function h is nonnegative whenever h is nonnegative. This is clear, since each c_i is nonnegative.

(3) Write E as the finite union of disjoint intervals contained in $[a, b]$, so that $E = \bigcup_j I_j$. Together with the intervals that form the elementary set $[a, b] - E$, we obtain a partition of $[a, b]$ on which 1_E takes constant values on each interval — namely, 1 on the intervals I_j, and 0 otherwise. It follows that 1_E is piecewise constant on $[a, b]$, and we conclude that $m(E) = \sum_j |I_j| = \text{p.c.}\int_a^b 1_E(x)\,dx$.

Exercise 1.1.22. *The following proof is rough but I believe the ideas are correct.* Suppose f is Riemann integrable. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|\mathcal{R}(f, \mathcal{P}) - \int_a^b f(x)\,dx| < \epsilon$ whenever $\Delta(\mathcal{P}) \leq \delta$. Fix a tagged partition $\mathcal{P}$ with $\Delta(\mathcal{P}) \leq \delta$ — we sometimes call such a partition *δ-fine.* We use $\mathcal{P}$ to define two new tagged partitions $\mathcal{P}_{\text{low}}$ and $\mathcal{P}_{\text{high}}$ with the same points as $\mathcal{P}$ but different tags — in particular, we define $x_i^* \coloneqq \inf_{x \in [x_{i-1}, x_i]} f(x)$ for $\mathcal{P}_{\text{low}}$ and $x_i^* \coloneqq \sup_{x \in [x_{i-1}, x_i]} f(x)$ for $\mathcal{P}_{\text{high}}$. These new partitions satisfy $\Delta \leq \delta$, and so their Riemann sums are within ϵ of $\int_a^b f(x)\,dx$. We see that $\mathcal{P}_{\text{low}}$ corresponds to a p.c. function bounded above by f, whose p.c. integral is precisely the Riemann sum of $\mathcal{P}_{\text{low}}$; similarly for $\mathcal{P}_{\text{high}}$. (The mapping of the endpoints of intervals is inconsequential since both Riemann and Darboux integrability and integral values are unaffected by changes to function values at finitely many inputs, which is easily shown by induction.) It follows that the Darboux integral $\mathcal{D}\int_a^b f(x)\,dx$ exists and is equal to the Riemann integral.

Suppose now that f is Darboux integrable, and let $g \leq f$ and $h \geq f$ be p.c. functions satisfying

$$\text{p.c.}\int_a^b h(x)\,dx - \epsilon < \mathcal{D}\int_a^b f(x)\,dx < \text{p.c.}\int_a^b g(x)\,dx + \epsilon.$$

Since functions differing at finitely many points are identical for the purposes of Riemann and Darboux integration as noted earlier, we may assume that the partitions associated to g and h contain no singletons, and thus correspond to tagged partitions of $[a, b]$. Let $\mathcal{P} = (x_0, \ldots, x_n)$ be their *untagged* common refinement. We will require $\delta < \inf_{1\leq i\leq n} \delta x_i$, so that any subinterval of a δ-fine partition contains at most one point of $\mathcal{P}$. Our goal is to prove that, up to a negligible error, we have $g \leq \phi \leq h$. Fix a δ-fine partition $\mathcal{P}' = ((y_0, \ldots, y_m), (y_1^*, \ldots, y_m^*))$, denote by ϕ the p.c. function that it induces, and write $\{1, \ldots, m\} = I_0 \uplus I_1$, where I_0 consists of all indices for which $[y_{i-1}, y_i]$ contains no points of $\mathcal{P}$, and I_1 consists of all the remaining indices (which necessarily contain exactly one point of $\mathcal{P}$).

Let us first estimate the contributions to the Riemann sum for $\mathcal{P}'$ due to I_0. If $i \in I_0$, then $[y_{i-1}, y_i]$ is completely contained in some $[x_{j-1}, x_j]$, on which g and h are constant. It follows that $g(x) \leq f(y_i^*) \leq h(x)$ on $[y_{i-1}, y_i]$, and so $g \leq \phi \leq h$ on $[y_{i-1}, y_i]$ as needed.

If $i \in I_1$, so that $y_{i-1} < x_j < y_i$, where we are not very careful with endpoints (since there are only finitely many). Write $I_{a(i)} = [y_{i-1}, x_j)$ and $I_{b(i)} = [x_j, y_i]$. Then, either $y_i^* \in I_{a(i)}$, in which case $f(x)$ may not be between $g(x)$ and $h(x)$ for $x \in I_{b(i)}$, or likewise with $a(i)$ and $b(i)$ switched. In either case, the error is bounded above by $2B\delta$, where B is a bound on g and h. Since $|I_1| \leq n$, the total error is bounded by $2nB\delta$, and we may pick δ small enough so that $2nB\delta < \epsilon$. It follows that

$$\begin{aligned}\left|\mathcal{R}(f, \mathcal{P}') - \mathcal{D}\int_a^b f(x)\,dx\right| &\leq \left|\mathcal{R}(f, \mathcal{P}') - \text{p.c.}\int_a^b g(x)\,dx\right| + \epsilon \\ &\leq \left|\sum_{i\in I_0} f(y_i^*)\delta y_i - \sum_{i\in I_0} c_i \delta y_i\right| \\ &\quad + \left|\sum_{i\in I_1} f(y_i^*)\delta y_i - \sum_{i\in I_1} (c_{a(i)}|I_{a(i)}| + c_{b(i)}|I_{b(i)}|)\right| + \epsilon \\ &\leq 2\epsilon + 2nB\delta \\ &< 3\epsilon,\end{aligned}$$

and thus we conclude that f is Riemann integrable.

Exercise 1.1.25. (A sketch.) We prove the result for $f \geq 0$. Suppose f is Riemann integrable. Then it is Darboux integrable. First observe that for p.c. functions g, the set $E_+^g := \{(x, t) : x \in [a, b], 0 \leq t \leq g(x)\}$ is elementary, with measure p.c.$\int_a^b g(x)\,dx$. Darboux integrability gives us p.c. functions $g \leq f \leq h$ within ϵ of $\int_a^b f(x)\,dx$; this yields inclusions $E_+^g \subset E_+ \subset E_+^h$, which show that

$$\text{p.c.}\int_a^b g(x)\,dx \leq m_{*,(J)}(E_+) \quad \text{and} \quad m^{*,(J)}(E_+) \leq \text{p.c.}\int_a^b h(x)\,dx.$$

It follows that E_+ is Jordan measurable, with $m(E_+) = \int_a^b f(x)\,dx$. The converse is similar and also relies on the correspondence between p.c. functions and elementary sets.

Appendix to Section 1: Identifications of subspaces of Euclidean space

This section is a somewhat pedantic treatment of some issues that arise when one identifies a proper subspace of $\mathbf{R}^d$ spanned by unit basis vectors $e_{i_1}, \ldots, e_{i_k}$ with $\mathbf{R}^k$. It is written primarily to assuage some of the author's discomforts concerning certain identifications. Suppose $\{S, T\}$ is a partition of $\{1, \ldots, d\}$, so that $S \cup T = \{1, \ldots, d\}$ and $S \cap T = \varnothing$. Define

$$\begin{aligned} \pi_S \colon \mathbf{R}^d &\to \{(x_1, \ldots, x_d \in \mathbf{R}^d : x_i \neq 0 \text{ implies } i \in S\} \approx \mathbf{R}^{|S|} \\ (x_1, \ldots, x_d) &\mapsto (x_1[1 \in S], \ldots, x_d[d \in S]), \end{aligned}$$

where $[P(x)]$ denotes Iverson's bracket notation — it is equal to 1 if the proposition $P(x)$ is true, and 0 if it is false.

A *box* in $\pi_S(\mathbf{R}^d)$ is defined to be a set of the form

$$\prod_{1 \leq j \leq d} [j \in S] I_j,$$

where $c[a, b] \coloneqq [ca, cb]$. For example, a box in $\pi_{\{1,3\}}(\mathbf{R}^3)$, more commonly known as the xz-plane in 3-dimensional space, is a set of the form $[a_1, b_1] \times \{0\} \times [a_3, b_3] \subset \pi_{\{1,3\}}(\mathbf{R}^3)$. There is a straightforward correspondence between boxes of $\pi_S(\mathbf{R}^d)$ and boxes of $\mathbf{R}^{|S|}$; the forward direction is obtained by removing all the $\{0\}$ factors. (So $[a_1, b_1] \times \{0\} \times [a_3, b_3] \subset \pi_{\{1,3\}}(\mathbf{R}^3)$ corresponds to $[a_1, b_1] \times [a_2, b_2] \in \mathbf{R}^2$.) We thus may define the *elementary measure* of a box in $\pi_S(\mathbf{R}^d)$ as the measure of the box in $\mathbf{R}^{|S|}$ it corresponds to. Given boxes $B \subset \pi_S(\mathbf{R}^d)$ and $B' \subset \pi_T(\mathbf{R}^d)$, we define the *product box* $B \bar{\times} B' \subset \mathbf{R}^d$ by

$$B \bar{\times} B' \coloneqq \prod_{1 \leq j \leq d} I_j,$$

where

$$I_j = \begin{cases} \pi_j(B) & \text{if } j \in S; \\ \pi_j(B') & \text{if } j \in T. \end{cases}$$

Here $\pi_j \colon \mathbf{R}^d \to \mathbf{R}$ is the projection onto the j-th factor defined by $(x_1, \ldots, x_d) \mapsto x_j$. For example, the boxes $B = [a_1, b_1] \times \{0\} \times [a_3, b_3] \subset \pi_{\{1,3\}}(\mathbf{R}^3)$ and $B' = \{0\} \times [a_2, b_2] \times \{0\} \subset \pi_{\{2\}}(\mathbf{R}^3)$ have product $B \bar{\times} B' = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$. We also then define $\pi_S(\mathbf{R}^d) \bar{\times} \pi_T(\mathbf{R}^d) \coloneqq \mathbf{R}^d$. To simplify our language, we will often say things like: "identifying the subspace S spanned by e_1 and e_3 with $\mathbf{R}^2$, we see that $\mathbf{R}^3 \approx S \times \mathbf{R}$. if $[a_1, b_1] \times [a_2, b_2]$ is a box in S and $[a_3, b_3] \in \mathbf{R}$, then the product of those boxes is $[a_1, b_1] \times [a_3, b_3] \times [a_2, b_2]$ under our identifications."

Most importantly for our purposes, we may prove a useful generalization of exercises 1.1.4 and 1.1.7. Define elementary sets in $\pi_S(\mathbf{R}^d)$ as finite unions of boxes (where boxes in $\pi_S(\mathbf{R}^d)$ are defined above). We may then define the elementary measure of elementary sets in $\pi_S(\mathbf{R}^d)$. We then have the following results, which are proven in the same ways as their normal counterparts, just with clunkier notation:

Proposition. *If $E_1 \subset \pi_S(\mathbf{R}^d)$ and $E_2 \subset \pi_T(\mathbf{R}^d)$ are elementary sets, then $E_1 \bar{\times} E_2 \subset \pi_S(\mathbf{R}^d) \bar{\times} \pi_T(\mathbf{R}^d) = \mathbf{R}^d$ is elementary, and $m(E_1 \bar{\times} E_2) = m(E_1) m(E_2)$.*

Proposition. *Suppose $1 \leq i \leq d+1$, and define $S = \{i\}$, $T = \{1, \ldots, d+1\} - \{i\}$. Let B be a closed box in $\pi_T(\mathbf{R}^{d+1})$, and let $f \colon \pi_T(\mathbf{R}^{d+1}) \to \mathbf{R}$ be a continuous function.*

(1) The graph $\{(x, f(x)) \in \pi_T(\mathbf{R}^{d+1}) \times \pi_S(\mathbf{R}^{d+1}) = \mathbf{R}^{d+1} : x \in B\}$ is Jordan measurable in $\mathbf{R}^{d+1}$ with Jordan measure zero.

(2) The set $\{(x, t) : x \in B; 0 \leq t \leq f(x)\}$ is Jordan measurable.

In conclusion, this whole business is rather pedantic and reminds me of how working mathematicians casually abuse identifications such as $\mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$ (which appears to be very much justifiable); set theory also leads to things like $2 \in 3$ as 'technical artefacts' of sorts. I guess this is motivation for type theory?

2. Lebesgue Measure

Exercise 1.2.1. Enumerate $\mathbf{Q}^2 \cap [0,1]^2$ by $q_1, q_2, \dots$. Then, although the sets $\{q_i\}$ are Jordan measurable, the union $\bigcup_{i\geq 1}\{q_i\} = \mathbf{Q}^2 \cap [0,1]^2$ is not measurable, as we saw earlier. Similarly, the sets $[0,1]^2 - \{q_i\}$ are Jordan measurable, being the difference of Jordan measurable sets. But their intersection is the bullet-riddled square, which we showed was not Jordan measurable earlier. This demonstrates the failure of Jordan measure to behave nicely with countable sets of objects, which feature prominently in analysis, for example whenever sequences arise. We will remedy this by introducing the *Lebesgue measure* shortly.

Exercise 1.2.2. Another flaw of the Jordan and Riemann theories is that we may have sequences of Riemann integrable functions that converge pointwise to a non-integrable function. For example, enumerating $\mathbf{Q} \cap [0,1] = \{q_1, q_2, \dots\}$, we may define a sequence of functions $f_i \colon [0,1] \to \mathbf{R}$ by sending $q_1, \dots, q_i$ to 0 and all other inputs to 0. Then f_i converges to the function

$$f(x) \coloneqq \begin{cases} 0 & \text{if } x \in \mathbf{Q}, \\ 1 & \text{if } x \in \mathbf{R} - \mathbf{Q}. \end{cases}$$

Since f has uncountably many discontinuities, it is not Riemann integrable.

While this problem is resolved if one requires uniform continuity (the standard proof involving the '$\epsilon/3$ trick'), we will find it fruitful to investigate the problem of 'completing' the gaps in the space of Riemann integrable functions, seeking a theory that behaves well generally with respect to limits, much as one completes the rationals to form the reals.

Exercise 1.2.3. (i) The empty set is contained in a singleton, which can be thought of as the union of countably many copies the same degenerate box, which has measure zero.

(ii) Any cover of F by boxes also covers E.

(iii) We show that Lebesgue outer measure m^* satifies countable subadditivity; that is, if $E_1, E_2, \dots \subset \mathbf{R}^d$ is a sequence of sets, then $m^*(\bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty m^*(E_n)$. By the axiom of countable choice, we may choose for each $n \geq 1$ a sequence of boxes $B_{n,1}, B_{n,2}, \dots$ whose union contains E_n such that $\sum_{i=1}^\infty |B_{n,i}| < m^*(E_n) + \epsilon/2^n$. Then $(B_{n,i})_{n,i\geq 1}$ is a countable set of boxes whose union contains $\bigcup_{n=1}^\infty E_n$. It then follows from Tonelli's theorem for series that

$$\begin{aligned} m^*\Big(\bigcup_{n=1}^\infty E_n\Big) &\leq \sum_{n,i\geq 1} |B_{n,i}| \\ &= \sum_{n=1}^\infty \sum_{i=1}^\infty |B_{n,i}| \\ &< \sum_{n=1}^\infty \Big(m^*(E_n) + \frac{\epsilon}{2^n}\Big) \\ &= \sum_{n=1}^\infty m^*(E_n) + \epsilon. \end{aligned}$$

Sending $\epsilon \to 0$, we conclude that $m^*(\bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty m^*(E_n)$.

Exercise 1.2.4. Let $E, F \subset \mathbf{R}^d$ be disjoint. Suppose E is compact and F is closed. We will prove $\operatorname{dist}(E,F) > 0$. Suppose contrapositively that $\operatorname{dist}(E,F) = 0$. Since $\operatorname{dist}(E,F) = 0$, there exist sequences $(x_n)_{n\geq 1} \subset E$ and $(y_n)_{n\geq 1} \subset F$ with $|x_n - y_n| < 1/n$ for $n \geq 1$. Since E is compact, the sequence $(x_n)_{n\geq 1}$ contains a convergent subsequence $(x_{n_i})_{i\geq 1}$ that converges to a limit x. This limit x is in E as E is closed. We will show that $x \in F$ as well, so that $E \cap F$ is nonempty. Let $\epsilon > 0$, and choose N such that $|x_{n_i} - x| < \epsilon/2$ whenever $i \geq N$. Then

$$|y_{n_i} - x| \leq |y_{n_i} - x_{n_i}| + |x_{n_i} - x| < \frac{1}{n_i} + \epsilon/2$$

whenever $i \geq N$. Since $1/n \to 0$, this quantity can be made smaller than ϵ for large i, and so we conclude that $y_{n_i} \to x$. But F is closed, and thus we have $x \in F$.

The compactness assumption is necessary — consider the graphs of the functions $x \mapsto 1/x$ and $x \mapsto -1/x$ in $\mathbf{R}^2$.

Exercise 1.2.5. If E is unbounded, we have $m^*(E) \geq m_{*,(J)}(E) = \infty$, and so the claim holds trivially. Thus we may assume that E is bounded. Suppose $E = \bigcup_{n=1}^\infty B_n$, where the boxes B_n are almost disjoint. To prove $m^*(E) = m_{*,(J)}(E)$, it suffices by (1.2) to prove $m^*(E) \leq m_{*,(J)}(E)$. By Lemma 1.2.9, we have $m^*(E) = \sum_{n=1}^\infty |B_n|$. We prove $\sum_{n=1}^N |B_n| \leq m_{*,(J)}(E)$ for every finite N. Since $\bigcup_{n=1}^N B_n$ is elementary, it follows from the monotonicity of Jordan inner measure that

$$\sum_{n=1}^N |B_n| = m\Big(\bigcup_{n=1}^N B_n\Big) = m_{*,(J)}\Big(\bigcup_{n=1}^N B_n\Big) \leq m_{*,(J)}(E).$$

We conclude that $m^*(E) = m_{*,(J)}(E)$.

Exercise 1.2.6. Let $E = [0,1] - \mathbf{Q}$. Then $m^*(E) \geq m^*([0,1]) - m^*([0,1] \cap \mathbf{Q})$ by subadditivity. By Lemma 1.2.6, $m^*([0,1]) = 1$. Since $[0,1] \cap \mathbf{Q}$ is countable, it has Lebesgue outer measure zero. It follows that $m^*(E) \geq 1$. (In fact, since $E \subset [0,1]$, monotonicity implies $m^*(E) = 1$.) But since $\mathbf{Q}$ is dense in $\mathbf{R}$, the set E cannot contain any non-empty open sets, and so we have $\sup_{U \subset E, U \text{ open}} m^*(U) = 0$.

Exercise 1.2.7. Claim (i) is equivalent to (ii); this is our definition of Lebesgue measurability. To see that (ii) implies (iii), notice that since $E \subset U$, we have $U \triangle E = U \cup E - U \cap E = U - E$. Similarly, (iv) implies (v). Claim (iii) implies (vi) by Lemma 1.2.13(i), and similarly (v) implies (vi) by Lemma 1.2.13(ii).

Now we show that (ii) implies (iv). Suppose E is Lebesgue measurable. Then $\mathbf{R}^d - E$ is Lebesgue measurable by Lemma 1.2.13(v), and so there exists open $U \supset \mathbf{R}^d - E$ with $m^*(U - (\mathbf{R}^d - E)) \leq \epsilon$. Since $U - (\mathbf{R}^d - E) = E - (\mathbf{R}^d - U)$, $\mathbf{R}^d - U$ is closed, and $\mathbf{R}^d - U \subset E$, the result follows.

Finally, we prove (vi) implies (ii). Let $\epsilon > 0$, and let $E_{\epsilon/4}$ be a Lebesgue measurable set with $m^*(E_{\epsilon/4} \triangle E) \leq \epsilon/4$. Then we may find boxes $B_1, B_2, \ldots$ whose union covers $E_{\epsilon/4} \triangle E$, such that $\sum_{n=1}^\infty |B_n| \leq \epsilon/2$. For each n, let B_n' be an open box containing B_n with $|B_n'| \leq |B_n| + \epsilon/2^{n+1}$. Then $E_{\epsilon/4} \triangle E \subset \bigcup_{n=1}^\infty B_n'$ with $\sum_{n=1}^\infty |B_n'| \leq \epsilon$. Now, $\bigcup_{n=1}^\infty B_n'$ is the countable union of measurable sets, and is thus measurable by Lemma 1.2.13(vi). It follows that $E_\epsilon' := E_{\epsilon/4} \cup \bigcup_{n=1}^\infty B_n'$ is a measurable set containing E. Since $E_\epsilon' - E \subset \bigcup_{n=1}^\infty B_n'$, we deduce that

$$m^*(E_\epsilon' \triangle E) = m^*(E_\epsilon' - E) \leq m^*\Big(\bigcup_{n=1}^\infty B_n'\Big) \leq \sum_{n=1}^\infty |B_n'| \leq \epsilon.$$

We have thus constructed for every $\epsilon > 0$ a measurable set E_ϵ' containing E with $m^*(E_\epsilon' - E) \leq \epsilon$. Let $E' := \bigcup_{n=1}^\infty E_{1/n}'$. This set is measurable by Lemma 1.2.13(vii). Since $m^*(E' - E) \leq m^*(E_{1/n}' - E) \leq 1/n$ for all $n \geq 1$, it follows that $m^*(E' - E) = 0$. Thus E differs from a measurable set by a null set, and is thus measurable. (In more detail, if $E' \subset U$ where U is open and $m^*(U - E') \leq \epsilon$, then by finite subadditivity we have $m^*(U - E) \leq m^*(U - E') + m^*(E' - E) \leq \epsilon$.)

Exercise 1.2.8. Suppose $E \subset \mathbf{R}^d$ is Jordan measurable. By exercise 1.1.5(3), there exists an elementary set A such that $m^{*,(J)}(A \triangle E) \leq \epsilon$. Since E is Jordan measurable, it follows that $A \triangle E$ is Jordan measurable as well. By (1.2), we find that $m^{*,(J)}(A \triangle E) = m^*(A \triangle E)$, and so the result follows from exercise 1.2.7(vi).

Exercise 1.2.9. Since each I_n is the finite union of closed intervals, they are closed sets. Since C is the countable intersection of closed sets I_n, it follows that C is closed. Since $C \subset [0,1]$, we conclude that C is compact.

There is an injection from the set of countable sequences $(a_i)_{i=1}^{\infty}$ with each $a_i \in \{0,2\}$. This set is isomorphic to the powerset of $\mathbf{N}$, which is uncountable. Thus C is uncountable.

Each I_n has measure $(2/3)^n$, so $m^*(C) \leq (2/3)^n$ for all n by monotonicity. Thus $m^*(C) = 0$ and we conclude that C is a null set.

Exercise 1.2.11. (1) Let $A_n := E_n - \bigcup_{k=1}^{n-1} E_k$. Then the sets A_n are disjoint, with $E_n = \bigcup_{k=1}^{n} A_k$ and $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k$. By countable additivity, we have

$$m(E_n) = m\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} m(A_k)$$

and

$$m\Big(\bigcup_{k=1}^{\infty} E_k\Big) = m\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} m(A_k).$$

Since $\lim_{n\to\infty} \sum_{k=1}^{n} m(A_k) = \sum_{k=1}^{\infty} m(A_k)$, the result follows.

(2) Without loss of generality suppose $m(E_1) < \infty$. Then, applying (1) to the sequence $E_1 - E_2 \subset E_1 - E_3 \subset \ldots$, we get $\lim_{n\to\infty} m(E_1 - E_n) = m(\bigcup_{n=1}^{\infty}(E_1 - E_n))$. It follows that

$$\lim_{n\to\infty} m(E_n) = m(E_1) - \lim_{n\to\infty} m(E_1 - E_n) = m(E_1) - m\Big(\bigcup_{n=1}^{\infty}(E_1 - E_n)\Big) = m\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

(3) Consider the sets $[0,\infty) \subset [1,\infty) \subset [2,\infty) \subset \ldots$; each set has infinite measure and yet their intersection is empty.

Exercise 1.2.12. Suppose m' is a map from the space of Lebesgue measurable sets to elements of $[0,+\infty]$ that obeys countable additivity and satisfies $m'(\varnothing) = 0$. Then, if $A \subset B$ are measurable sets, the set $B - A$ is measurable with $B = A \uplus (B - A)$, and so countable additivity together with $m'(\varnothing) = 0$ implies $m'(A) \leq m'(B)$, since $m'(B - A) \geq 0$ by hypothesis.

Given a sequence of measurable sets $A_1, A_2, \ldots$, we may define a corresponding sequence of disjoint sets by $A'_n := A_n - \bigcup_{i=1}^{n-1} A_i \subset A_n$. Then, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$, and so we have

$$m'\Big(\bigcup_{i=1}^{\infty} A_i\Big) = m'\Big(\bigcup_{i=1}^{\infty} A'_i\Big) = \sum_{i=1}^{\infty} m'(A'_i) \leq \sum_{i=1}^{\infty} m'(A_i)$$

by countable additivity and monotonicity.

Exercise 1.2.13. (i) Since $1_E(x) = \liminf_{n\to\infty} 1_{E_n}(x)$, we see that $x \in E$ if and only if there exists n such that $x \in E_k$ whenever $k \geq n$. Thus $E = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$. Similarly, since $1_E(x) = \limsup_{n\to\infty} 1_{E_n}(x)$, we have $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. Since countable unions and countable intersections of measurable sets are measurable by Lemma 1.2.13, it follows that E is Lebesgue measurable.

(ii) Since $\bigcap_{k=1}^{\infty} E_k \subset \bigcap_{k=2}^{\infty} E_k \subset \ldots$, upward monotone convergence implies $m(E) = \lim_{n\to\infty} m(\bigcap_{k=n}^{\infty} E_k)$. By monotonicity, we get $m(E) \leq \lim_{n\to\infty} m(E_n)$. Similarly, since $\bigcup_{k=1}^{\infty} E_k \supset \bigcup_{k=2}^{\infty} E_k \supset \ldots$, and since $m(\bigcup_{k=1}^{\infty} E_k) \leq m(F) < \infty$, we may apply downward monotone convergence to obtain $m(E) = \lim_{n\to\infty} m(\bigcup_{k=n}^{\infty} E_k)$, and so monotonicity implies $m(E) \geq \lim_{n\to\infty} m(E_n)$. We conclude that $m(E) = \lim_{n\to\infty} m(E_n)$.

(iii) The functions $1_{[n,n+1]}$ 'escape to infinity,' converging to the zero function, although each function is non-zero on a set of measure one, namely $[n, n+1]$.

Exercise 1.2.14. Given $\epsilon > 0$, let $(B_n^{\epsilon})_{n=1}^{\infty}$ be a sequence of boxes with $\bigcup_{n=1}^{\infty} B_n^{\epsilon} \supset E$ and $\sum_{n=1}^{\infty} |B_n^{\epsilon}| \leq m^*(E) + \epsilon$. Then $A := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} B_n^{1/k}$ is a

Lebesgue measurable set containing E. By monotonicity, $m(A) \leq m^*(E) + 1/k$ for any $k \geq 1$, and so we conclude that $m(A) = m^*(E)$.

Exercise 1.2.15. By monotonicity, we have $m(E) \geq \sup_{K \subset E,\, K \text{ cpt.}} m(K)$. Therefore we must prove that $m(E) \leq \sup_{K \subset E,\, K \text{ cpt.}} m(K)$. Since E is measurable, we may spend an epsilon to approximate E by a closed subset F, so that $m(F) \geq m(E) - \epsilon$. Denoting by B_n the closed ball of radius n about the origin, we may then take intersections $F \cap B_1 \subset F \cap B_2 \subset \dots$ to obtain a monotone increasing sequence of compact sets, and so upward monotone convergence implies $m(F) = \lim_{n \to \infty} m(F \cap B_n)$. Thus for some N we have $m(F \cap B_N) \geq m(F) - \epsilon$, and so

$$m(E) - 2\epsilon \leq m(F) - \epsilon \leq m(F \cap B_N) \leq \sup_{K \subset E,\, K \text{ cpt.}} m(K).$$

Since ϵ was arbitrary, the result follows.

Exercise 1.2.16. Claim (i) implies (ii), since if E is Lebesgue measurable, then $m(E) = m^*(E) < \infty$, and one may contain E in an open set U with $m^*(U - E) \leq \epsilon$, so that $m(U) \leq m(E) + \epsilon < \infty$ as well.

We show that (ii) implies (iii). Let $E \subset U$ be open with $m^*(U - E) \leq \epsilon/2$. Now $m^*(E \triangle U) \leq \epsilon/2$, but U need not be bounded. Form an increasing sequence of bounded sets $U \cap B_1^\circ \subset U \cap B_2^\circ \subset \dots$, and apply upward monotone convergence to get $\lim_{n \to \infty} m(U \cap B_n^\circ) = m(U)$. Thus for some N we have $m(U \cap B_N^\circ) \geq m(U) - \epsilon/2$, or $m(U - B_N^\circ) \leq \epsilon/2$. Since

$$E \triangle (U \cap B_N^\circ) = (E - B_N^\circ) \cup (U \cap B_N^\circ - E) \subset (U - B_N^\circ) \cup (U - E),$$

we obtain $m^*(E \triangle (U \cap B_N^\circ)) \leq \epsilon$ by subadditivity as desired.

Claim (i) implies (iv) by inner regularity. Claim (iv) implies (v) trivially. Claim (iii) implies (vi) as open sets are measurable. Claim (v) implies (vi), since compact sets are bounded and measurable. Claim (vi) implies (vii) as bounded measurable sets have finite measure by monotonicity.

We show (vii) implies (viii). We first note that given sets A, B, C, we have $m^*(A \triangle C) \leq m^*(A \triangle B) + m^*(B \triangle C)$. This follows from the fact that

$$A \triangle C = (A \triangle B) \triangle (B \triangle C) \subset (A \triangle B) \cup (B \triangle C).$$

(Here it is useful to think of the symmetric difference as addition modulo 2.) Now suppose that E differs from a measurable set A by a set of outer measure at most ϵ, so that $m^*(A \triangle E) \leq \epsilon$. Now, by definition of measurability, $m(A) = m^*(A)$, and so there exists a sequence of boxes $B_1, B_2, \dots$ with $\sum_{n=1}^\infty |B_n| \leq m(A) + \epsilon < \infty$. Since this series converges to a finite value, there exists N with $m^*(\bigcup_{n=N+1}^\infty B_n) = \sum_{n=N+1}^\infty B_n < \epsilon$, and so we may take $\bigcup_{n=1}^N B_n$ to be our elementary set. It follows that

$$\begin{aligned} m^*\Big(E \triangle \bigcup_{n=1}^N B_n\Big) &\leq m^*\Big(A \triangle \bigcup_{n=1}^N B_n\Big) + m^*(A \triangle E) \\ &\leq m^*\Big(A \triangle \bigcup_{n=1}^\infty B_n\Big) + m^*\Big(\bigcup_{n=N+1}^\infty B_n\Big) + m^*(A \triangle E) \\ &\leq 3\epsilon. \end{aligned}$$

That claim (viii) implies (ix) follows from our solution to exercise 1.1.14, where we proved that dyadic cubes of fixed sidelength 2^{-n} approximate elementary sets arbitrarily well.

Finally, (ix) implies (i), since the finite union of closed dyadic cubes is measurable with finite measure $m(F)$. Thus E is almost a measurable set, and so it is measurable by exercise 1.2.7(vi) with measure at most $m(F) + \epsilon < \infty$.

Exercise 1.2.17. We first prove (i) implies (ii). Suppose E is measurable and A is elementary (and thus measurable). Then $A \cap E$ and $A - E$ are measurable and disjoint, with $m(A) = m(A \cap E) + m(A - E)$ as needed.

Claim (ii) implies (iii) trivially, since boxes are elementary sets.

Finally, we prove (iii) implies (i). Since $E \subset \mathbf{R}^d$ may have infinite outer measure, we prove the result for $A \cap E$, where A is a box. Since $\mathbf{R}^d$ is the countable union of disjoint boxes, and since countable unions of measurable sets are measurable, this will suffice to prove the claim. By hypothesis, we have $|A| = m^*(A \cap E) + m^*(A - E)$. Cover $A \cap E$ with boxes $B_1, B_2, \dots$, with $A \cap E \subset \bigcup_{n=1}^\infty B_n$ and $\sum_{n=1}^\infty |B_n| \leq m^*(A \cap E) + \epsilon$. (We may replace B_i with $A \cap B_i$, so we may assume $B_i \subset A$.) Then

$$\begin{aligned} m^*(A \cap E) + m^*\Big(\bigcup_{n=1}^\infty B_n - A \cap E\Big) &\leq \sum_{n=1}^\infty m^*(B_n \cap E) + \sum_{n=1}^\infty m^*(B_n - E) \\ &= \sum_{n=1}^\infty |B_n| \\ &\leq m^*(A \cap E) + \epsilon. \end{aligned}$$

Thus $m^*(\bigcup_{n=1}^\infty B_n - A \cap E) \leq \epsilon$, and so $A \cap E$ differs from a measurable set by measure at most ϵ. We conclude that $A \cap E$ is Lebesgue measurable.

Exercise 1.2.18. (i) It suffices to prove that, if $E \subset A \subset B$ with A, B elementary, then $m(A) - m^*(A - E) = m(B) - m^*(B - E)$, or $m^*(B - E) = m^*(A - E) + m(B - A)$. The general result then follows, since the intersection of elementary sets containing E is once again an elementary set containing E. We prove something slightly more general:

Lemma. *Let $E \subset \mathbf{R}^d$ be bounded, and suppose A and B are elementary sets with $A \cap E \neq \varnothing$ and $A \cup E \subset B$. Then $m^*(B - E) = m^*(A - E) + m(B - A)$.*

Proof. By subadditivity, we have $m^*(B - E) \leq m^*(A - E) + m(B - A)$. To show $m^*(B - E) \geq m^*(A - E) + m(B - A)$, we let $B_1, B_2, \dots$ be a sequence of boxes with $\bigcup_{n=1}^\infty B_n \supset B - E$ and $\sum_{n=1}^\infty |B_n| \leq m^*(B - E) + \epsilon$. Then, $\bigcup_{n=1}^\infty B_n - (B - A)$ is the countable union of sets, each set being a box with an elementary set removed, which is itself an elementary set and thus the finite union of boxes. It follows that $\bigcup_{n=1}^\infty B_n - (B - A)$ is itself the countable union of boxes $B_1', B_2', \dots$. Since $\sum_{n=1}^\infty |B_n'| = \sum_{n=1}^\infty |B_n| - m(B - A)$ and $A - E \subset \bigcup_{n=1}^\infty B_n'$, it follows that

$$m^*(B - E) - m(B - A) \leq \sum_{n=1}^\infty |B_n'| \leq m^*(B - E) - m(B - A) + \epsilon.$$

Since ϵ was arbitrary, we conclude that $m^*(B - E) \geq m^*(A - E) + m(B - A)$. $\square$

(ii) To show $m_*(E) \leq m^*(E)$, we must show $m(A) - m^*(A - E) \leq m^*(E)$ for any elementary $A \supset E$. But this is just subadditivity. We now show that $m_*(E) = m^*(E)$ iff E is Lebesgue measurable. Suppose $m_*(E) = m^*(E)$. We will use the Carathéodory criterion to prove that E is measurable — in particular, we will show that $m(A) = m^*(A \cap E) + m^*(A - E)$ for all elementary sets A. We know

$$m(A) = m^*(E) + m^*(A - E)$$

for all elementary sets $A \supset E$. If $A \cap E = \varnothing$, the criterion is satisfied trivially. Finally, if $A \cap E \neq \varnothing$, the criterion is an easy consequence of the lemma above — indeed, since $m(B) = m^*(E) + m^*(B - E)$ by hypothesis, it follows that

$$m(A) - m^*(A - E) = m^*(E)$$

as needed. Thus E is Lebesgue measurable.

The converse follows from the finite additivity of Lebesgue measure.

Exercise 1.2.19. That (ii) implies (i) and (iii) implies (i) are easy consequences of Lemma 1.2.13. Showing that (i) implies (ii) amounts to constructing a sequence of open sets $U_n \supset E$ with $m^*(U_n - E) \leq 1/n$ and taking their intersection; (i) implies (iii) follows dually by inner approximations by closed sets $F_n \subset E$ with $m^*(E - F_n) \leq 1/n$ and taking their union.

Exercise 1.2.25. Since continuously differentiable curves are Lipschitz, there exists K with $\|f(x) - f(y)\| \leq K|x - y|$ for all $x, y \in [a, b]$. Partition $[a, b]$ into subintervals of length at most ϵ. Then the image of such a subinterval is contained in a cube of sidelength $2K\epsilon$, and so we may cover the curve with an elementary set of measure bounded by

$$\frac{b-a}{\epsilon}(2K\epsilon)^d.$$

This can be made arbirarily small for $d \geq 2$, and so we conclude that such curves have measure zero.

Exercise 1.2.26. Since sets with zero outer measure are Lebesgue measurable, the Vitali set E must have positive outer measure. Pick N with $Nm(E) > 1$. Then, taking $N - 1$ disjoint cyclic translates of E by rationals in $[0, 1]$, we see that the outer measure of their union is at most one by monotonicity, whereas the sum of their outer measures is greater than one by construction.

Exercise 1.2.27. The Vitali set E is nonmeasurable, but $E \times \{0\} \subset \mathbf{R}^2$ is a null set and is thus measurable. So projections of measurable sets need not be measurable.

3. The Lebesgue Integral

Exercise 1.3.1. (i) Choose representations of f and g that are compatible, in the sense that we may write $f = \sum_{1\le i\le k} c_i 1_{E_i}$ and $g = \sum_{1\le i\le k} c_i' 1_{E_i}$ for measurable sets $E_1, \dots, E_k$. This is possible since if we are given any representations of f and g, where f involves k measurable sets and g involves k' measurable sets, then the $k+k'$ measureable sets involved partition $\mathbf{R}^d$ into $2^{k+k'}$ disjoint sets, each of which is an intersection of the original sets or their complement, which we may take as our E_i. (One may think of each set as described by a binary string of $k+k'$ digits — the i-th bit is set to 1 iff it is involved in the intersection.) Then $f+g = \sum_{1\le i\le k}(c_i + c_i')1_{E_i}$, and so

$$\begin{aligned}\operatorname{Simp}\int_{\mathbf{R}^d} f(x)+g(x)\,dx &= \sum_{1\le i\le k}(c_i+c_i')m(E_i)\\ &= \sum_{1\le i\le k} c_i m(E_i) + \sum_{1\le i\le k} c_i' m(E_i)\\ &= \operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx + \operatorname{Simp}\int_{\mathbf{R}^d} g(x)\,dx.\end{aligned}$$

Showing scalar multiplication follows from the observation that

$$c\sum_{1\le i\le k} c_i 1_{E_i} = \sum_{1\le i\le k} c c_i 1_{E_i}.$$

(ii) Suppose $\operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx < \infty$, and write $f = \sum_{1\le i\le k} c_i 1_{E_i}$. Then by definition we have $\sum_{1\le i\le k} c_i m(E_i) < \infty$, and so each term $c_i m(E_i)$ is finite. In particular, we may only have $c_i = \infty$ if $m(E_i) = 0$, and so f can only take infinite values on a null set. Thus f is finite almost everywhere. Similarly, we may only have $m(E_i) = \infty$ if $c_i = 0$, and so the support of f has finite measure. The converse follows from the same ideas.

(iii) Write $f = \sum_{1\le i\le k} c_i 1_{E_i}$. If $\operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx = 0$, then each term $c_i m(E_i)$ is zero, which tells us that the sets E_i on which f is nonzero necessarily have measure zero; that is, f is zero almost everywhere. The converse is similar.

(iv) This follows from (i) and (iii), since $f-g$ is zero almost everywhere.

(v) It suffices to prove that the simple integral is nonnegative whenever $f \ge 0$. . . .

(vi) This is clear from the definition of the simple integral.

Finally, we prove that the simple integral is the unique map from the space of unsigned simple functions to $[0,+\infty]$ obeying the above properties. Indeed, suppose $f \mapsto \mathrm{S}\int_{\mathbf{R}^d} f(x)\,dx$ is an operator $\operatorname{Simp}^+(\mathbf{R}^d) \to [0,+\infty]$ obeying the above properties. Then, writing $f = \sum_{1\le i\le k} c_i 1_{E_i}$, we find that

$$\mathrm{S}\int_{\mathbf{R}^d} f(x)\,dx = \sum_{1\le i\le k} c_i \cdot \mathrm{S}\int_{\mathbf{R}^d} 1_{E_i}(x)\,dx = \sum_{1\le i\le k} c_i m(E_i) = \operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx.$$

Exercise 1.3.2. (i) This part is just a lot of computations. We first prove additivity for real-valued (absolutely integrable) simple functions. Indeed, since

$$f+g = (f+g)_+ - (f+g)_- = (f_+ - f_-) + (g_+ - g_-),$$

we have

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+,$$

and so

$$\begin{aligned}&\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)_+(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}f_-(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}g_-(x)\,dx\\&\quad=\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)_-(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}f_+(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}g_+(x)\,dx.\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}&\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)_+(x)\,dx-\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)_-(x)\,dx\\&\quad=\Big(\mathrm{Simp}\int_{\mathbf{R}^d}f_+(x)\,dx-\mathrm{Simp}\int_{\mathbf{R}^d}f_-(x)\,dx\Big)\\&\qquad+\Big(\mathrm{Simp}\int_{\mathbf{R}^d}g_+(x)\,dx-\mathrm{Simp}\int_{\mathbf{R}^d}g_-(x)\,dx\Big),\end{aligned}$$

and so

$$\mathrm{Simp}\int_{\mathbf{R}^d}f(x)+g(x)\,dx=\mathrm{Simp}\int_{\mathbf{R}^d}f(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}g(x)\,dx$$

as needed.

Now we may prove additivity for complex-valued functions. We compute

$$\begin{aligned}&\mathrm{Simp}\int_{\mathbf{R}^d}f(x)+g(x)\,dx\\&=\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Re}(f(x)+g(x))+i\,\mathrm{Im}(f(x)+g(x))\,dx\\&:=\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Re}(f(x)+g(x))\,dx+i\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Im}(f(x)+g(x))\,dx\\&=\Big(\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Re}\,f(x)\,dx+i\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Im}\,f(x)\,dx\Big)\\&\quad+\Big(\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Re}\,g(x)\,dx+i\mathrm{Simp}\int_{\mathbf{R}^d}\mathrm{Im}\,g(x)\,dx\Big)\\&=\mathrm{Simp}\int_{\mathbf{R}^d}f(x)\,dx+\mathrm{Simp}\int_{\mathbf{R}^d}g(x)\,dx.\end{aligned}$$

To prove the scalar multiplication property for real-valued functions and real constants c, it suffices to prove it for the three cases $c=-1$, $c=0$ and $c>0$. For $c=-1$, the result follows from the fact that $(-f)_+=f_-$, $(-f)_-=f_+$, and $-f=f_--f_+$. The case $c=0$ follows from the fact that the simple integral is zero iff the function is zero almost everywhere. Finally, if $c>0$, we may use the scalar multiplicativity of the unsigned simple integral (as established in exercise 1.3.1(i)) to compute

$$\begin{aligned}\mathrm{Simp}\int_{\mathbf{R}^d}cf(x)\,dx&=\mathrm{Simp}\int_{\mathbf{R}^d}cf_+(x)\,dx-\mathrm{Simp}\int_{\mathbf{R}^d}cf_-(x)\,dx\\&=c\Big(\mathrm{Simp}\int_{\mathbf{R}^d}f_+(x)\,dx-\mathrm{Simp}\int_{\mathbf{R}^d}f_-(x)\,dx\Big)\\&=c\mathrm{Simp}\int_{\mathbf{R}^d}f(x)\,dx.\end{aligned}$$

We may now prove scalar multiplicativity for $c=a+bi\in\mathbf{C}$ and $f\colon\mathbf{R}^d\to\mathbf{C}$. Since

$$cf(x)=(a+bi)(\mathrm{Re}\,f(x)+i\,\mathrm{Im}\,f(x))=(a\,\mathrm{Re}\,f(x)-b\,\mathrm{Im}\,f(x))+i(b\,\mathrm{Re}\,f(x)+a\,\mathrm{Im}\,f(x)),$$

we have

$$\begin{aligned}
&\operatorname{Simp}\int_{\mathbf{R}^d} cf(x)\,dx\\
&= \operatorname{Simp}\int_{\mathbf{R}^d} a\operatorname{Re} f(x) - b\operatorname{Im} f(x)\,dx + i\operatorname{Simp}\int_{\mathbf{R}^d} b\operatorname{Re} f(x) + a\operatorname{Im} f(x)\,dx\\
&= a\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx - b\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx\\
&\quad + bi\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx + ai\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx\\
&= c\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx + ci\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx\\
&= c\operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx
\end{aligned}$$

as needed.

Finally, we prove the $*$-linearity of our integral. We have

$$\begin{aligned}
\operatorname{Simp}\int_{\mathbf{R}^d} \overline{f}(x)\,dx &= \operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x) - i\operatorname{Im} f(x)\,dx\\
&= \operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx - i\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx\\
&= \overline{\operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx}.
\end{aligned}$$

(ii) If real-valued functions f and g are equal almost everywhere (a.e.), then f_+ and g_+ are equal a.e. and so are f_- and g_-. Similarly, if complex-valued functions f and g are equal a.e., then their real and imaginary parts are equal a.e. as well. Thus the result follows from the corresponding result concerning unsigned simple integrals.

(iii) This is immediate from the corresponding result for unsigned simple integrals.

To establish uniqueness, suppose $f \mapsto \mathrm{S}\int_{\mathbf{R}^d} f(x)\,dx$ is an operator satisfying the above axioms. If f is simple unsigned, then $\mathrm{S}\int_{\mathbf{R}^d} \equiv \operatorname{Simp}\int_{\mathbf{R}^d}$ by exercise 1.3.1. If f is real-valued, then by (i) we have

$$\begin{aligned}
\mathrm{S}\int_{\mathbf{R}^d} f(x)\,dx &= \mathrm{S}\int_{\mathbf{R}^d} f_+(x) - f_-(x)\,dx\\
&= \mathrm{S}\int_{\mathbf{R}^d} f_+(x)\,dx - \mathrm{S}\int_{\mathbf{R}^d} f_-(x)\,dx\\
&= \operatorname{Simp}\int_{\mathbf{R}^d} f_+(x)\,dx - \operatorname{Simp}\int_{\mathbf{R}^d} f_-(x)\,dx\\
&= \operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx.
\end{aligned}$$

Finally, if f is complex-valued, then

$$\mathrm{S}\int_{\mathbf{R}^d} f(x)\,dx = \mathrm{S}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx + i\mathrm{S}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx$$

by (i), which is in turn equal to

$$\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Re} f(x)\,dx + i\operatorname{Simp}\int_{\mathbf{R}^d} \operatorname{Im} f(x)\,dx = \operatorname{Simp}\int_{\mathbf{R}^d} f(x)\,dx$$

by our above result for real-valued functions.

Exercise 1.3.3. (i) The preimage of an open set is open, and open sets are measurable, so continuous functions are measurable by Lemma 1.3.9(x).

(ii) Every unsigned simple function f is the limit of the constant sequence $(f)_{i\in\mathbf{N}}$, and is thus measurable by Lemma 1.3.9(ii).

(iii) Suppose $\{f_n : n \in \mathbf{N}\}$ is a countable set of unsigned measurable functions. Then

$$\left\{x \in \mathbf{R}^d : \sup_{n\in\mathbf{N}} f_n(x) > \lambda\right\} = \bigcup_{n\in\mathbf{N}} \{x \in \mathbf{R}^d : f_n(x) > \lambda\}$$

is the countable union of measurable sets and is thus measurable. The proof for infimums is similar, and the proof for limit superiors and limit inferiors follows from the fact that we may express them as supremums of infimums and vice versa.

(iv) Suppose f is an unsigned function that is equal a.e. to an unsigned measurable function g. Then, since g is the pointwise limit of a sequence of unsigned simple functions g_n, it follows that f is the pointwise limit *a.e.* of the sequence g_n, and so f is measurable by Lemma 1.3.9(iii).

(v) We have $\lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$, which is measurable by (iii).

(vi) The preimage of an open set under $\phi \circ f$ is the preimage of an open set under f (since ϕ is continuous), which is measurable.

(vii) If (f_n) and (g_n) are sequences of simple functions that pointwise approach f and g, then $(f_n + g_n)$ and $(f_n g_n)$ are sequences of simple functions that pointwise approach $f + g$ and fg.

Exercise 1.3.4. Suppose $f\colon \mathbf{R}^d \to [0, +\infty]$ is bounded unsigned measurable. Then, if $A \leq f \leq B$ for constants A and B, we may define $f_n(x)$ to be the largest integer multiple of 2^{-n} that is at most equal to $f(x)$, along the lines of the construction given in the proof of Lemma 1.3.9. Each function f_n takes on finitely many values (at most $2^n(B - A)$), and the convergence is uniform as f_n always lies in the tube of radius 2^{-n} of f.

Exercise 1.3.5. Suppose $f\colon \mathbf{R}^d \to [0, +\infty]$ is unsigned measurable and takes on at most finitely many values. Call these values $c_1, \ldots, c_n$. Then each set $E_i := f^{-1}(\{c_i\})$ is measurable by hypothesis, and we have $f = \sum_{1\leq i\leq n} c_i 1_{E_i}$.

Exercise 1.3.6. Since f is measurable, there exists an increasing sequence of unsigned simple functions f_n converging to f. They induce measurable sets

$$\{(x, t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f_n(x)\}$$

which form a monotone increasing sequence — their (countable) union is the set

$$\{(x, t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f(x)\}$$

which is thus measurable.

Exercise 1.3.7. (i) and (ii) are equivalent by definition. (iv) and (v) are equivalent as may be seen by taking complements.

To see that (ii) implies (iii), note that since $f_n \to f$, we have $\operatorname{Re} f_n \to \operatorname{Re} f$, which implies $(\operatorname{Re} f_n)_+ \to (\operatorname{Re} f)_+$, which in turn implies that $|(\operatorname{Re} f_n)_+| \to |(\operatorname{Re} f)_+|$. Since $|(\operatorname{Re} f_n)_+|$ is unsigned simple, we conclude that $|(\operatorname{Re} f)_+|$ is measurable, and the other results follow similarly.

(iii) implies (ii) as we may use the measurability of the functions $|\operatorname{Re}(f)_+|$ and $|\operatorname{Re}(f)_-|$ to construct sequences of unsigned simple functions converging to them, which may be combined to form a sequence of signed simple functions converging to $\operatorname{Re}(f)$. The same may be done with $\operatorname{Im}(f)$, and the two may then be combined to give a sequence of simple functions converging to f. We now know that (i)–(iii) are equivalent.

On the equivalence of (i)–(iii) and (iv)–(v): at the moment I do not know the solution. An idea though: we know that

$$f^{-1}(U) = \{x \in \mathbf{R}^d : f(x) \in U\},$$

and $f(x) \in U$ implies $\mathrm{Re}(f(x)) \in \mathrm{Re}(U)$ and $\mathrm{Im}(f(x)) \in \mathrm{Im}(U)$, with $\mathrm{Re}(U), \mathrm{Im}(U)$ open. So $\{x \in \mathbf{R}^d : \mathrm{Re}(f(x)) \in \mathrm{Re}(U)\} \dots$

Exercise 1.3.9. Riemann integrable functions are continuous a.e., and thus measurable a.e..

Exercise 1.3.10. (i) If f is simple, then

$$\underline{\int_{\mathbf{R}^d}} f(x)\,dx = \sup_{0 \le g \le f;\, g \text{ simple}} \mathrm{Simp}\int_{\mathbf{R}^d} g(x)\,dx = \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\,dx$$

by monotonicity of the simple integral; similarly for the upper integral.

(ii) We wish to show

$$\sup_{0 \le h \le f;\, h \text{ simple}} \mathrm{Simp}\int_{\mathbf{R}^d} h(x)\,dx \le \sup_{0 \le h \le g;\, h \text{ simple}} \mathrm{Simp}\int_{\mathbf{R}^d} h(x)\,dx.$$

Any simple function $h \le f$ is bounded above by g almost everywhere; we may modify h on a null set so that it still has the same simple integral and the result follows.

(iii) The case for $c = 0$ is simple. If $c \in (0, +\infty)$, then the result follows from the fact that, if $g \le f$ is a simple function, then $cg \le cf$ is simple with $\mathrm{Simp}\int_{\mathbf{R}^d} cg(x)\,dx = c\,\mathrm{Simp}\int_{\mathbf{R}^d} g(x)\,dx$ by earlier results.

(iv) If $f = g$ a.e., let S be the null set of points where they disagree. Then, given a simple function $h \le f$, we may define a new simple function h' which is equal to h outside S and is zero on S. This proves that

$$\underline{\int_{\mathbf{R}^d}} f(x)\,dx \le \underline{\int_{\mathbf{R}^d}} g(x)\,dx,$$

and equality follows by symmetry.

(v) If $h \le f$ and $h' \le g$ are simple, then $h + h' \le f + g$ is simple, with

$$\mathrm{Simp}\int_{\mathbf{R}^d} f(x) + g(x)\,dx = \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\,dx + \mathrm{Simp}\int_{\mathbf{R}^d} g(x)\,dx.$$

(vi) Similar to proof of (v).

(vii) Since $f(x) = f(x)1_E(x) + f(x)1_{\mathbf{R}^d - E}(x)$, superadditivity settles one direction for us. To show

$$\underline{\int_{\mathbf{R}^d}} f(x)\,dx \le \underline{\int_{\mathbf{R}^d}} f(x)1_E(x)\,dx + \underline{\int_{\mathbf{R}^d}} f(x)1_{\mathbf{R}^d - E}(x)\,dx,$$

let $h \le f$ be simple. Then we may write h as the sum of simple functions $h = h1_E + h1_{\mathbf{R}^d - E}$. They are simple because $1_A 1_B = 1_{A \cap B}$. Since $h1_E \le f1_E$ and similarly for $h1_{\mathbf{R}^d - E}$, the result follows.

(viii) By monotonicity, $\underline{\int_{\mathbf{R}^d}} \min(f(x), n)\,dx$ is an increasing sequence bounded above by $\underline{\int_{\mathbf{R}^d}} f(x)\,dx$. Thus we must prove

$$(*) \qquad \sup_{n \in \mathbf{N}} \underline{\int_{\mathbf{R}^d}} \min(f(x), n)\,dx = \underline{\int_{\mathbf{R}^d}} f(x)\,dx.$$

Let us first settle the case where $m(f^{-1}(\infty)) > 0$. In this case, we construct simple functions f_n which are equal to n on the set $f^{-1}(\infty)$. Such functions satisfy

$$\mathrm{Simp}\int_{\mathbf{R}^d} f_n(x)\,dx \ge n \cdot m(f^{-1}(\infty)) \to \infty$$

as $n \to \infty$, and so both sides of $(*)$ are infinite. Now suppose $m(f^{-1}(\infty)) = 0$, so that f is finite a.e.. Let $g \le f$ be simple with

$$\mathrm{Simp}\int_{\mathbf{R}^d} g(x)\,dx \ge \underline{\int_{\mathbf{R}^d}} f(x)\,dx - \epsilon.$$

Then, $g(x)$ is finite a.e., and we may modify it on a null set so that it is finite everywhere without affecting the value of its simple integral. Since $g(x)$ is simple, it attains finitely many values, and thus it is bounded by some natural number n. It follows that $g(x) \le \min(f(x), n)$, so

$$\underline{\int_{\mathbf{R}^d}} \min(f(x), n)\, dx \ge \underline{\int_{\mathbf{R}^d}} f(x)\, dx - \epsilon,$$

and we are done.

(ix) Let $h \le f$ be simple (as discussed earlier, we may take the inequality to hold almost everywhere) with

$$\mathrm{Simp} \int_{\mathbf{R}^d} h(x)\, dx \ge \underline{\int_{\mathbf{R}^d}} f(x)\, dx - \epsilon.$$

If we write $h = \sum_k c_k 1_{E_k}$, then $h(x)1_{|x|\le n}(x) = \sum_k c_k 1_{E_k \cap \{|x|\le n\}}(x)$ is simple for all n, and we have

$$\mathrm{Simp} \int_{\mathbf{R}^d} h(x)1_{|x|\le n}(x)\, dx = \sum_k c_k m(E_k \cap \{|x| \le n\}).$$

By upward monotone convergence, we have $m(E_k) = \lim_{n\to\infty} m(E_k \cap \{|x| \le n\})$. It follows that

$$\lim_{n\to\infty} \mathrm{Simp} \int_{\mathbf{R}^d} h(x)1_{|x|\le n}(x)\, dx \le \mathrm{Simp} \int_{\mathbf{R}^d} h(x)\, dx,$$

and so we may pick N such that

$$\mathrm{Simp} \int_{\mathbf{R}^d} h(x)1_{|x|\le N}(x)\, dx \ge \mathrm{Simp} \int_{\mathbf{R}^d} h(x)\, dx - \epsilon.$$

Thus

$$\underline{\int_{\mathbf{R}^d}} f(x)1_{|x|\le N}\, dx \ge \underline{\int_{\mathbf{R}^d}} f(x)\, dx - 2\epsilon,$$

and the result follows since ϵ was arbitrary.

(x) Since $f + g$ is simple, we may write $f + g = \sum_k c_k 1_{E_k}$. Let $\sum_k g_k 1_{E''_k} \ge g$ be simple with

$$\overline{\int_{\mathbf{R}^d}} g(x)\, dx \ge \sum_k g_k m(E''_k) - \epsilon.$$

Since

$$\sum_k c_k 1_{E_k} - f = g \le \sum_k g_k 1_{E''_k},$$

we have

$$\sum_k c_k 1_{E_k} - \sum_k g_k 1_{E''_k} \le f.$$

Thus

$$\sum_k c_k m(E_k) - \sum_k g_k m(E''_k) \le \underline{\int_{\mathbf{R}^d}} f(x)\, dx.$$

We may then compute

$$\begin{aligned} \mathrm{Simp} \int_{\mathbf{R}^d} f(x) + g(x)\, dx = \sum_k c_k m(E_k) &\le \underline{\int_{\mathbf{R}^d}} f(x)\, dx + \sum_k g_k m(E''_k) \\ &\le \underline{\int_{\mathbf{R}^d}} f(x)\, dx + \overline{\int_{\mathbf{R}^d}} g(x)\, dx + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we conclude that

$$\mathrm{Simp} \int_{\mathbf{R}^d} f(x) + g(x)\, dx \le \underline{\int_{\mathbf{R}^d}} f(x)\, dx + \overline{\int_{\mathbf{R}^d}} g(x)\, dx,$$

and the reverse inequality follows from the analogous argument with $\sum_k c_k 1_{E_k} - g = f \geq \sum_k f_k 1_{E'_k}$ instead.

4. Abstract measure spaces

Exercise 1.4.4. Let $\mathcal{B}$ be a finite Boolean algebra on a set X, and define a map $f\colon X \to \mathcal{B}$ sending $x \in X$ to the intersection of $\mathcal{B}$-measurable sets containing x, so that $f(x)$ is the smallest $\mathcal{B}$-measurable set containing x. This intersection is finite since $\mathcal{B}$ is finite, and so $f(x) \in \mathcal{B}$. The image $\mathrm{im}(f) =: (A_\alpha)_{\alpha \in I}$ is a subset of $\mathcal{B}$; we claim that it is in fact a partition of X into atoms $(A_\alpha)_{\alpha\in I}$ with $\mathcal{B} = \mathcal{A}((A_\alpha)_{\alpha\in I})$. The sets A_α cover X. If distinct sets $f(x) = A_\alpha$ and $f(y) = A_\beta$ had a non-empty intersection, then $A_\alpha \setminus A_\beta$ or $A_\alpha \cap A_\beta$ would be a smaller $\mathcal{B}$-measurable set containing x. It follows that $\mathrm{im}(f)$ partitions X. Clearly $\mathcal{A}(\mathrm{im}(f)) \subset \mathcal{B}$. Conversely, suppose $A \in \mathcal{B}$. Arguing as above, we see that $\mathrm{im}_f(A)$ is a partition of A into atoms, and so $A \in \mathcal{A}(\mathrm{im}(f))$ as needed.

Exercise 1.4.5. Suppose these algebras were atomic. Then, since they contain every singleton, they must have these singletons as atoms, and so they would contain every subset of Euclidean space, which would be absurd.

Exercise 1.4.8. Let n be a natural number, and suppose $\mathcal{F} = \{X_1, \dots, X_n\}$ is a finite collection of n sets. Then $\mathcal{F}$ partitions $X := \bigcup_{i=1}^n X_i$ into at most 2^n disjoint sets, which yields an atomic algebra with at most 2^{2^n} elements containing $\langle \mathcal{F}\rangle_{\mathrm{bool}}$. Thus $|\langle \mathcal{F}\rangle_{\mathrm{bool}}| \leq 2^{2^n}$.

This bound is in fact best possible. Indeed, let $X = \{0,1\}^n$, and consider the family $\mathcal{F} = \{X_1, \dots, X_n\}$ where X_i contains the 2^{n-1} elements of X with the i-th coordinate equal to 0. We will show that all 2^n singleton subsets of X are contained in $\langle \mathcal{F}\rangle_{\mathrm{bool}}$; this will imply that $\langle \mathcal{F}\rangle_{\mathrm{bool}} = 2^X$ as needed. Write an element $x \in X$ as a string of n binary digits. Then, since Boolean algebras are closed under complements and intersections, we may form the set $\{x\}$ as an intersection of n sets which are either X_i or $X \setminus X_i$ — if the i-th digit of x is 0, we include X_i in the intersection; otherwise we include $X \setminus X_i$. For example, if $n = 3$, then $X_1 = \{000, 001, 010, 011\}$, $X_2 = \{000, 001, 100, 101\}$, $X_3 = \{000, 010, 100, 110\}$, and $\{110\} = (X \setminus X_1) \cap (X \setminus X_2) \cap X_3$.

Exercise 1.4.9. We first show that the collection $\bigcup_{n=0}^\infty \mathcal{F}_n$ is a Boolean algebra. Notice that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$. By (ii), $\bigcup_{n=0}^\infty \mathcal{F}_n$ contains the empty set as an empty union. Suppose $A, B \in \bigcup_{n=0}^\infty \mathcal{F}_n$. Then $A \in \mathcal{F}_n$ for some n, and so its complement is contained in $\mathcal{F}_{n+1}$ by (ii). Since the $\mathcal{F}_n$ are nested, we may find n for which $A, B \in \mathcal{F}_n$. It then follows from (ii) that $A \cup B$ is in $\mathcal{F}_{n+1}$. Since $\bigcup_{n=0}^\infty \mathcal{F}_n$ is a Boolean algebra containing $\mathcal{F}$, we conclude that $\langle \mathcal{F}\rangle_{\mathrm{bool}} \subset \bigcup_{n=0}^\infty \mathcal{F}_n$. For the reverse inclusion, notice that $\mathcal{F}_0 \subset \langle \mathcal{F}\rangle_{\mathrm{bool}}$ trivially, and if $\mathcal{F}_n \subset \langle \mathcal{F}\rangle_{\mathrm{bool}}$, then the axioms of a Boolean algebra guarantee that $\mathcal{F}_{n+1} \subset \langle \mathcal{F}\rangle_{\mathrm{bool}}$. The inclusion $\bigcup_{n=0}^\infty \mathcal{F}_n \subset \langle \mathcal{F}\rangle_{\mathrm{bool}}$ then follows from induction.

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