SELECTED SOLUTIONS FOR TERENCE TAO'S BOOK "AN INTRODUCTION TO MEASURE THEORY"

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Exercise in the proof of Lemma 1.1.2. We prove that

$$|I| = \lim_{N\to\infty} \frac{1}{N}\#(I \cap \frac{1}{N}\mathbf{Z}).$$

Since $[a,b] \cap \frac{1}{N}\mathbf{Z} \cong [Na, Nb] \cap \mathbf{Z} = \{\lceil Na\rceil, \dots, \lfloor Nb\rfloor\}$, we have

$$\#(I \cap \frac{1}{N}\mathbf{Z}) = \lfloor Nb\rfloor - \lceil Na\rceil + 1.$$

Since $Nb < \lfloor Nb\rfloor + 1 \le Nb + 1$ and $Na \le \lceil Na\rceil < Na + 1$, we have

$$Nb - Na - 1 < \lfloor Nb\rfloor - \lceil Na\rceil + 1 \le Nb - Na + 1,$$

so

$$b - a - \frac{1}{N} < \frac{\lfloor Nb\rfloor - \lceil Na\rceil + 1}{N} \le b - a + \frac{1}{N}.$$

The result follows from the squeeze theorem by sending $N \to \infty$.

Exercise 1.1.3. We first prove the result for $d = 1$. Suppose $m' \colon \mathcal{E}(\mathbf{R}) \to \mathbf{R}^+$ obeys non-negativity, finite additivity and translation invariance. For $n \ge 1$, we have

$$\begin{aligned} c := m'([0,1)) &= m'\left(\bigcup_{i=1}^{n}\Big[\frac{i-1}{n}, \frac{i}{n}\Big)\right) \\ &= \sum_{i=1}^{n} m'\Big(\Big[\frac{i-1}{n}, \frac{i}{n}\Big)\Big) \quad \text{by finite additivity} \\ &= \sum_{i=1}^{n} m'\Big(\Big[0, \frac{1}{n}\Big)\Big) \quad \text{by translation invariance} \\ &= n m'\Big(\Big[0, \frac{1}{n}\Big)\Big), \end{aligned}$$

and so $m'([0,1/n)) = c/n$. Thus $m'([0,k/n)) = ck/n$. Note that non-negativity and finite additivity imply monotonicity, which in turn implies that $m'(\{0\}) < 1/n$ for all n, so that $m'(\{x\}) = 0$ for all $x \in \mathbf{R}$ by translation invariance.

Since elementary sets are finite unions of disjoint boxes, it suffices to show that $m'(B) = cm(B)$ for all boxes B. Since singletons have zero measure as shown above, it suffices by translation invariance to prove the result for $B = [0,a)$ where $a > 0$. By writing $[0,a) = [0,\lfloor a\rfloor) \cup [\lfloor a\rfloor, a)$, we see that it suffices to consider $0 < a < 1$. By considering a sequence in $\mathbf{Q} \cap [0,a)$ converging to a, monotonicity yields the bound $m'([0,a)) \ge ca$, and we may also obtain $m'([0,a)) \le ca$ analogously.

For $\mathbf{R}^d$ we find $m'([0,1/n)^d) = c/n^d$ (recall $\bigcup_i A_i \times \bigcup_j B_j \approx \bigcup_{i,j} A_i \times B_j$). Similar arguments show that $m'(\prod_{1\le i\le d}[0,k_i/n)) = (c/n^d)(\prod_{1\le i\le d} k_i)$, and that degenerate elementary sets (where one of the factor intervals is a singleton) have zero measure under m'. We may finish off with a similar limiting argument:

$$m'\left(\prod_{1\le i\le d}[0,a_i)\right)\ge\sup\left\{m'\left(\prod_{1\le i\le d}[0,q_i)\right):q_i\in\mathbf{Q}\cap[0,a_i)\right\}=c\prod_{1\le i\le d}a_i.$$

Exercise 1.1.5. To show (1) implies (2), suppose E is Jordan measurable, and let $\epsilon>0$. Then there exist elementary sets $A\subset E\subset B$ with $m(A)>m(E)-\epsilon/2$ and $m(B)<m(E)+\epsilon/2$, so that $m(B-A)=m(B)-m(A)\le\epsilon$ by finite additivity of elementary measure.

To show (2) implies (3), let $A\subset E\subset B$ be elementary sets with $m(B-A)\le\epsilon$. Then $B\triangle A=B-A\supset B-E$, and so

$$m^{*,(J)}(B\triangle E)=\inf_{\substack{S\supset B-E\\ S\text{ elem.}}}m(S)\le m(B-A)\le\epsilon.$$

To show (3) implies (1), let A be an elementary set with $m^{*,(J)}(A\triangle E)\le\epsilon/4$. Then there exists an elementary set $B\supset A\triangle E$ with $m(B)<\epsilon/2$. This gives us two elementary sets $A-B\subset E\subset A\cup B$. Since

$$m^{*,(J)}(E)\ge m(A-B)\ge m(A)-m(B)>m(A)-\epsilon/2$$

and

$$m_{*,(J)}(E)\le m(A\cup B)\le m(A)+m(B)<m(A)+\epsilon/2,$$

we obtain $m^{*,(J)}(E)-m_{*,(J)}(E)<\epsilon$. It follows that E is Jordan measurable.

Exercise 1.1.6. (1) We begin by proving that $E\cup F$ is Jordan measurable. By exercise 1.1.5(2), there exist elementary sets A,B,A',B' with $A\subset E\subset B$, $A'\subset F\subset B'$, $m(B-A)\le\epsilon/2$, and $m(B'-A')\le\epsilon/2$. Then $A\cup A'\subset E\cup F\subset B\cup B'$. Since $B\cup B'-A\cup A'\subset(B-A)\cup(B'-A')$, it follows from already established properties of elementary measure that

$$\begin{aligned}m(B\cup B'-A\cup A')&\le m((B-A)\cup(B'-A'))\\&\le m(B-A)+m(B'-A')\\&\le\epsilon,\end{aligned}$$

and so applying exercise 1.1.5(2) again shows that $E\cup F$ is Jordan measurable. Showing that $E\cap F$ is Jordan measurable is quite similar — one uses the inclusion

$$B\cap B'-A\cap A'=(B\cap B'-A)\cup(B\cap B'-A')\subset(B-A)\cup(B'-A').$$

Showing that $E-F$ is Jordan measurable uses the fact that $A-B'\subset E-F\subset B-A'$ and

$$(B-A')-(A-B')\subset(B-A)\cup(B'-A').$$

Finally, $E\triangle F=E\cup F-E\cap F$ and is thus Jordan measurable.

(2) We have $m(E)\ge m_{*,(J)}(E)$, which is a supremum over elementary measures of elementary sets, which are clearly non-negative by definition.

(3) Let $A\subset E\subset B$, $A'\subset F\subset B'$ be elementary sets with

$$m(B)-\epsilon/2<m(E)<m(A)+\epsilon/2$$

and

$$m(B')-\epsilon/2<m(F)<m(A')+\epsilon/2.$$

Then, $E\cup F\supset A\cup A'$, and so

$$m(E\cup F)\ge m(A\cup A')=m(A)+m(A')>m(E)+m(F)-\epsilon.$$

Similarly, $E\cup F\subset B\cup B'$, and we have

$$m(E\cup F)\le m(B\cup B')\le m(B)+m(B')<m(E)+m(F)+\epsilon.$$

Since ϵ was arbitrary, this gives $m(E\cup F)=m(E)+m(F)$ as required.

(4) We have $E \uplus (F - E) = F$, where $\uplus$ denotes a disjoint union. By (1), $F - E$ is Jordan measurable, and so $m(E) + m(F - E) = m(F)$ by (3). Since $m(F - E) \geq 0$ by (2), we conclude that $m(E) \leq m(F)$.

(5) Since $E \cup F = E \uplus (F - E)$ and $F - E \subset F$, we have

$$m(E \cup F) = m(E) + m(F - E) \leq m(E) + m(F).$$

(6) This follows immediately from translation invariance of elementary sets — if $A \subset E$ with A elementary, then $A + x \subset E + x$ with $A + x$ elementary and $m(A + x) = m(A)$; similarly for $B \supset E$.

Exercise 1.1.7. (1) Let $f \colon B \to \mathbf{R}$ be a continuous function on a closed box $B \subset \mathbf{R}^d$, and denote by $\Gamma_f := \{(x, f(x)) : x \in B\} \subset \mathbf{R}^{d+1}$ its graph. Since the inner measure is at most the outer measure, the Jordan measurability of Γ_f is immediately established if we find for every $\epsilon > 0$ an elementary set of measure less than ϵ that contains Γ_f. Let $\epsilon > 0$. Since continuous functions on compact sets are uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/m(B)$ whenever $\|x - y\| < \delta$ and $x, y \in B$. Partition B into boxes of diameter less than δ. Each of these boxes $B_\alpha \subset \mathbf{R}^d$ gives rise to a box $B_\alpha \times I_\alpha \subset \mathbf{R}^{d+1}$ containing $\{(x, f(x)) : x \in B'\} \subset \Gamma_f$ with $m(I_\alpha) < \epsilon/m(B)$ by uniform continuity. It follows that $\bigcup_\alpha (B_\alpha \times I_\alpha)$ is an elementary set of measure less than ϵ that contains Γ_f. We conclude that the graph of f is Jordan measurable with Jordan measure zero.

(2) This is essentially the fact that continuous functions are Riemann integrable. Alternatively, letting $U := \{(x, t) : x \in B \text{ and } 0 \leq t \leq f(x)\} \subset \mathbf{R}^{d+1}$, one may consider the sets (as defined in (1))

$$U - \bigcup_\alpha (B_\alpha \times I_\alpha) \subset U \subset U \cup \bigcup_\alpha (B_\alpha \times I_\alpha),$$

which may be shown to be elementary.

Exercise 1.1.8.