some bundle stuff

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Proposition 10.26. *Suppose E and E' are smooth vector bundles over a smooth manifold M with or without boundary, and $F\colon E \to E'$ is a bijective smooth bundle homomorphism over M. Then F is a smooth bundle isomorphism.*

Proof. We first show that F is a bundle isomorphism. Since $\pi' \circ F = \pi$, we have $\pi' = \pi \circ F^{-1}$. It then suffices to show that $F|_{E_p}\colon E_p \to E'_p$ is an isomorphism of vector spaces. Since F is a bijection, its restriction $F|_{E_p}$ is a bijection onto its image $\operatorname{im}(F|_{E_p}) \subset E'_p$. Since $x \notin E_p$ implies that $F(x) \notin E'_p$, it follows that $E'_p \subset \operatorname{im}(F|_{E_p})$. In particular, E and E' are vector bundles of the same rank k.

We now show that F is a local diffeomorphism. Since bijective local diffeomorphisms are diffeomorphisms, this will complete the proof. Let $p \in U \subset M$ for an appropriate neighborhood U of p, and choose local trivializations Φ and Ψ of E and E' as in the diagram

$$\begin{tikzcd}
U \times \mathbf{R}^k \arrow[drr, "\pi_U"'] & \pi^{-1}(U) \arrow[l, "\Phi"'] \arrow[rr, "F"] \arrow[dr, "\pi"] & & \pi'^{-1}(U) \arrow[r, "\Psi"] \arrow[dl, "\pi'"'] & U \times \mathbf{R}^k \arrow[dll, "\pi_U"] \\
& & U & &
\end{tikzcd}.$$

We then have

$$(\Psi \circ F \circ \Phi^{-1})(p, v) = (p, \tau(p)v)$$

for some smooth $\tau\colon U \to \mathrm{GL}(n, \mathbf{R})$. It follows that

$$(\Phi \circ F^{-1} \circ \Psi^{-1})(p, v) = (p, \tau(p)^{-1}v),$$

which implies that F^{-1} is smooth on $\pi'^{-1}(U)$, since inversion is a diffeomorphism in $\mathrm{GL}(n, \mathbf{R})$. This completes the proof. $\square$