Derivations

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In a MathOverflow answer, Terence Tao once wrote[1]: *... whenever one has an n-dimensional smooth manifold M, and (locally) one has n smooth coordinate functions $x^1, \dots, x^n \colon M \to \mathbf{R}$ on this manifold, whose differentials $dx^1, \dots, dx^n$ form a basis of the cotangent space at every point p of the manifold M, then (locally at least) there is a unique "dual basis" of derivations $\partial_1, \dots, \partial_n$ on $C^\infty(M)$ with the property $\partial_i x^j = \delta_i^j$ for $i, j = 1, \dots, n$. (By the way, proving this claim is an excellent exercise for someone who really wants to understand the modern foundations of differential geometry.)*

Well, I really do want to understand! So here I have sketched my understanding as it currently stands, which I guess serves as a decent distillation of what I've understood about manifolds thus far, learning from John M. Lee's *Introduction to Smooth Manifolds*. (Needless to say, I still have a long way to go.)

[1] https://mathoverflow.net/a/308784/

Let M be an n-dimensional smooth manifold, and let (U, φ) be a chart of M. That is, $\varphi \colon U \to \mathbf{R}^n$ with U an open set, and so we may define coordinate functions $x^i \colon U \to \mathbf{R}$ for $1 \le i \le n$ by $x^i := \pi^i \circ \varphi$, where the π^i denote the usual projection maps. As such we may write $\hat{p} := \varphi(p) = (x^1(p), \dots, x^n(p))$. Similarly, we may represent smooth maps $f \in C^\infty(M)$ in local coordinates by maps $\hat{f} := f \circ \varphi^{-1}$. Here $f \colon M \to \mathbf{R}$ and $\hat{f} \colon \mathbf{R}^n \to \mathbf{R}$, so we can handle $\hat{f}$ using our familiar methods in Euclidean space.

We define the *differential* df of a smooth map $f \in C^\infty(M)$ as a covector field specified at a point $p \in M$ by

$$\begin{aligned} df_p \colon T_pM &\to \mathbf{R} \\ v &\mapsto v(f). \end{aligned}$$

By taking differentials of our coordinate functions with the usual local understandings (e.g., identification of $T_pU \hookrightarrow T_pM$), we obtain local covector fields $dx^1, \dots, dx^n$ that yield bases on the cotangent spaces at each point $p \in U$. Now we may define our desired derivations ∂_i. Since derivations are vector fields, we define $\partial_i|_p \colon C^\infty(M) \to \mathbf{R}$ by sending f to $\frac{\partial \hat{f}}{\partial x^i}(\hat{p})$. We must verify that $\partial_i|_p(x^j) = \delta_i^j$. To do this we simply compute

$$\begin{aligned} \partial_i|_p(x^j) &= \frac{\partial \widehat{x^j}}{\partial x^i}(\hat{p}) \\ &= \lim_{h \to 0} \frac{x^j \circ \varphi^{-1}(x^1(p), \dots, x^i(p) + h, \dots, x^n(p)) - x^j \circ \varphi^{-1}(\hat{p})}{h} \\ &= \lim_{h \to 0} \frac{\pi^j(x^1(p), \dots, x^i(p) + h, \dots, x^n(p)) - x^j(p)}{h} \\ &= \lim_{h \to 0} \frac{h\delta_i^j}{h} \\ &= \delta_i^j. \end{aligned}$$

As for uniqueness, here I am rather sketchy for really I'm not sure myself. But an idea is that, at each point p the vectors $\partial_i|_p$ must arise as dual vectors to $dx^j|_p$, in the sense that we may take the dual basis of the $dx^j|_p$ to get a basis for the double dual $(T_pM)^{**}$, and make it into a basis for T_pM via the canonical isomorphism. This gives us $\partial_i|_p(x^j) = dx^j|_p(\partial_i|_p) = \delta_i^j$, where the first equality is a consequence of the definition of the differential. But I don't know, this feels too cheap, like I've forgotten to account for something...