smoothness criteria for covector fields

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Proposition 11.11. *Let M be a smooth manifold with or without boundary, and let $\omega\colon M \to T^*M$ be a rough covector field. The following are equivalent:*

(a) *ω is smooth.*

(b) *In every smooth coordinate chart, the component functions of ω are smooth.*

(c) *Each point of M is contained in some coordinate chart in which ω has smooth component functions.*

(d) *For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.*

(e) *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbf{R}$ is smooth on U.*

Proof. We will prove (a)⇒(b)⇒(c)⇒(a), then (c)⇒(d)⇒(e)⇒(b).

(a)⇒(b). In coordinates we have

$$\hat{\omega}(p) = (x^1(p), \dots, x^n(p), \omega_1(p), \dots, \omega_n(p)).$$

(b)⇒(c). Every point of M lies in a chart.

(c)⇒(a). Smoothness is local, and (c) shows that ω is smooth in a neighborhood of every $p \in M$.

(c)⇒(d). In coordinates, $\omega(X) = \omega_i X^i$. In particular, at a point p we choose a chart with smooth ω_i. Then by hypothesis the X^i are smooth in that chart as well and the result follows.

(d)⇒(e). Suppose $U \subseteq M$ is open and X is a smooth vector field on U. For any $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighborhood of p and supported in U, and define $\tilde{X} := \psi X$, extended to be zero on $M - \operatorname{supp}\psi$. Then $\omega(\tilde{X})$ is smooth by assumption, and is equal to $\omega(X)$ in a neighborhood of p since $\omega(X)(q) = \omega_q(X_q) = \omega_q(\tilde{X}_q) = \omega(\tilde{X})(q)$. This shows that $\omega(X)$ is smooth in a neighborhood of each point of U.

(e)⇒(b). In a fixed chart consider a coordinate vector field ∂x^i; we have $\omega(\partial x^i)$ smooth by hypothesis (e). And so in coordinates we compute

$$\omega(\partial x^i) = \omega_j dx^j(\partial x^i) = \omega_i,$$

which must then be smooth. ∎