MA2116 PROBABILITY, 2024/25 S1: WORKED PROBLEMS

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While studying for the MA2116 finals in November 2024, I decided to rewrite the worked problems from L1 to improve my understanding, as well to have it in written form, since much of the context was only spoken and not written down in lectures. I am sharing this in hopes that it will help someone else as well — feel free to share it!

Note that my style can be rather idiosyncratic at times, in part because I am writing these solutions primarily for my own usage, and also because this document was written quite hastily, especially towards the end. You can reach me at `hbs@u.nus.edu` if you have any comments. (*Last updated:* 15 Nov 2024.)

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About notation

In no other branch of mathematics is it so easy
for experts to blunder as in probability theory.

— Martin Gardner (1959)

The *Iverson bracket* $[P]$ is equal to 1 is P is a true proposition and 0 otherwise. I will often express a piecewise function such as

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

simply by writing $f(x) = x^2[x > 0]$.

Sometimes, I write $x := y$ or $y =: x$ to mean that x is *defined* to be equal to y.

The notation $n^{\underline{k}}$ refers to the *falling factorial* $n(n-1)\dots(n-k+1)$, with $n^{\underline{0}} = 1$. Similarly, $n^{\overline{k}} = n(n+1)\dots(n+k-1)$. Notice that $\binom{n}{k} = n^{\underline{k}}/k!$.

I also have sought to give exact decimal values for rational numbers in many places, so fractions such as $\frac{1}{7}$ are expressed with the bar notation $0.\overline{142857}$, meaning that the decimal expansion repeats the part with a bar above it:

$$\frac{1}{7} = 0.142857142857142857142857142857\dots.$$

I write $\mathbf{E}X^2$ for the expectation of X^2, and $(\mathbf{E}X)^2$ for the square of the expectation of X.

The *Stirling numbers of the second kind*, or *Stirling partition numbers*, are denoted by ${n \brace k}$, which counts the number of ways to partition a set of n elements into k parts. For example, ${4 \brace 2} = 7$, since we can partition 1234 into $\{1, 234\}$, $\{2, 134\}$, $\{3, 124\}$, $\{4, 123\}$, $\{12, 34\}$, $\{13, 24\}$, and $\{14, 23\}$. We frequently make use of the identity

$$x^n = \sum_k {n \brace k} x^{\underline{k}},$$

which holds for integers $n \geq 0$.

Table 1. Stirling's triangle for partitions.

n	${n \brace 0}$	${n \brace 1}$	${n \brace 2}$	${n \brace 3}$	${n \brace 4}$	${n \brace 5}$	${n \brace 6}$	${n \brace 7}$	${n \brace 8}$	${n \brace 9}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Once again, I emphasize that **these notes have been written for my own usage**, which means for example that I omit many details when dealing with topics I am comfortable with, or conversely that I sometimes include unnecessary details just because I find them interesting. I hope nonetheless that you may find at least some value in this hastily-prepared document.

01. Basic notions, I

Problem 1. *A dice is biased, with the odd numbers being equally likely to appear, but each even number is three times as likely to appear as any of the odd numbers. (1) Find the probability of getting three. (2) Find the probability of getting one or six.*

Solution. Write p for the probability of getting one. Then p is also the probability of getting three, as well as that of getting five; and $3p$ is the probability for getting two, for getting four, and for getting six. Thus $12p = 1$, or $p = 1/12$. (1) The probability is $1/12 = 0.08\overline{3}$. (2) The probability is $p + 3p = 1/3 = 0.\overline{3}$. □

Problem 2. *(1) Three fair coins are tossed. Find the probability that exactly two heads appear. (2) Four fair coins are tossed. Find the probability that at least two heads appear.*

Solution. (1) $\binom{3}{2}2^{-3} = 3/8 = 0.375$. (2) $1 - 2^{-4}\left(\binom{4}{0} + \binom{4}{1}\right) = 11/16 = 0.6875$. □

Problem 3. *In a large sports club, 47% of members play badminton, 32% play squash, and 13% play both. Find the probability that a randomly selected member plays neither of the two sports mentioned above.*

Solution. The probability that a member plays at least one of those sports is $47\% + 32\% - 13\% = 66\%$, so the desired probability is 34%. □

Problem 4. *In a large language school where students take classes in Chinese, Japanese, Korean, and other languages, 51% of students are enrolled in Chinese, 40% in Japanese, 32% in Korean, 14% in Chinese and Japanese, 17% in Chinese and Korean, 10% is Japanese and Korean, and 3% is Chinese, Japanese, and Korean.*

What is the probability that a random student is enrolled in none of these three language classes?

Solution. By inclusion-exclusion, we have

$$\begin{aligned}
&\Pr(C \cup J \cup K) \\
&\quad = \Pr(C) + \Pr(J) + \Pr(K) - \Pr(CJ) - \Pr(CK) - \Pr(JK) + \Pr(CJK) \\
&\quad = 51\% + 40\% + 32\% - 14\% - 17\% - 10\% + 3\% \\
&\quad = 85\%,
\end{aligned}$$

so the desired probability is 15%. □

Problem 5. *Suppose you assess that there is more than an 80% chance that the weather will be nice tomorrow, and more than a 70% chance tht the wether will be nice the day after tomorrow. (1) Is it valid to infer that there is more than a fair chance that the weather will be nice on both days? (2) Replace the numbers (80,70,50) above with (90,90,90) and answer the question again.*

Solution. (1) The question is asking if $\Pr(A) > 0.8$ and $\Pr(B) > 0.7$ implies $\Pr(AB) > 0.5$. This is valid, since $\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A \cup B) > 1.5 - 1 = 0.5$. (2) No, because the reasoning above gives $0.9 + 0.9 - 1.0 = 0.8 < 0.9$. □

Problem 6. *A four-digit code is to be formed from the digits* $0, 1, \ldots, 9$. *(1) How many codes can be formed? (2) If the digits cannot be repeated, how many codes can be formed?*

Solution. (1) We have ten choices for each of the four digits, which gives us $10^4 = 10000$ choices. (2) The first digit gives us ten choices, and the second digit cannot be the first, giving us nine choices, and so on. Thus we have $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ choices. □

Problem 7. *Five boys and three girls compete in a running race. (Assume there cannot be a tie.) (1) If the boys and girls run together, how many different finishing orders are possible? (2) If the boys and girls run separately, how many different finishing orders are possible?*

Solution. (1) This is the number of ways to arrange eight objects, which is given by $8! = 40320$. (2) There are $3!5! = 720$ ways. □

Problem 8. *I have six textbooks on the shelf to arrange in a row. They are on elementary math, advanced math, elementary physics, advanced physics, elementary chemistry, and advanced chemistry. (1) How many arrangements are possible? (2) If books on the same subject are to be placed together, how many arrangements are possible? (3) If books of the same level (elementary or advanced) are to be placed together, how many arrangements are possible?*

Solution. (1) $6! = 720$. (2) $3!(2!)^3 = 48$. (3) $2!(3!)^2 = 72$. □

02. Basic notions, II

Problem 9. *How many letter arrangements can be formed from the word "statistics"?*

Solution. The counts are $(A, C, I, S, T) = (1, 1, 2, 3, 3)$, so there are

$$\frac{10!}{3!3!2!1!1!} = 50400$$

arrangements. □

Problem 10. *A committee of four people is to be formed from a group of three men and five women, among which there is a married couple. (1) How many committees are possible? (2) If the committee must have two men and two women, how many committees are possible? (3) What if the committee must have two men and two women, and the couple is not allowed to serve together?*

Solution. (1) $\binom{8}{4} = 70$ committees are possible. (2) $\binom{3}{2}\binom{5}{2} = 30$. (3) There are $\binom{3-1}{1}\binom{5-1}{1} = 8$ committees with two men and two women that include the couple, so the answer is $30 - 8 = 22$. □

Problem 11. *A group of nine gamers are playing computer games. (1) The first game consists of three different tasks presented at the same time. The gamers divide themselves into three groups of three to work on the problems simultaneously. How many divisions are possible?*

(2) The second game requires three teams to play simultaneously, each team against the other two. The gamers divide themselves into three groups of three to play this game. How many divisions are possible?

Solution. (1) This is given by

$$\frac{9!}{3!3!3!} = 1680.$$

(2) We have

$$\frac{1}{3!}\binom{9}{3,3,3} = 280.$$

□

Problem 12. *Two fair dice are rolled. (1) What is the probability that the sum of the two values is at least nine? (2) What is the probability that the difference of the two values is at most one?*

Solution. (1) $1/36+2/36+3/36+4/36 = 5/18 = 0.2\overline{7}$. (2) $5/36+6/36+5/36 = 4/9 = 0.\overline{4}$. □

Problem 13. *An urn contains five white balls, four red balls, and three black balls. We take four balls at random from the urn. (1) What is the probability that we get two white and two red balls? (2) What is the probability that we get two white, one red, and one black ball?*

Solution. (1) $\binom{5}{2}\binom{4}{2}/\binom{12}{4} = 4/33 = 0.\overline{12}$. (2) $\binom{5}{2}\binom{4}{1}\binom{3}{1}/\binom{12}{4} = 8/33 = 0.\overline{24}$. □

Problem 14. *A committee is formed by randomly selecting four people from a group of 12 professors consisting of six physicists, four chemists, and two biologists. (1) What is the probability that the committee has two physicists and two chemists? (2) What is the probability that the committee has only physicists, or has only chemists? (3) What is the probability that the committee has no biologists?*

Solution. (1) $\binom{6}{2}\binom{4}{2}/\binom{12}{4} = 2/11 = 0.\overline{18}$. (2) $\binom{6}{4}/\binom{12}{4} + \binom{4}{4}/\binom{12}{4} = 16/495 = 0.0\overline{32}$. (3) $\binom{10}{4}/\binom{12}{4} = 14/33 = 0.\overline{42}$. □

Problem 15. *From a standard deck of 52 cards, we draw four cards at random. (1) What is the probability that the four cards drawn have distinct values? (2) What is the probability that we get three cards with the same value and one card with a different value? (3) What is the probability that we get two cards with the same value, and two cards with a shared value different from the first?*

Express your answers as an integer divided by $\binom{52}{4} = 270725$.

Solution. (1) $4^4\binom{13}{4}/\binom{52}{4} = 183040/\binom{52}{4}$. (2) $13 \cdot 12 \cdot 4 \cdot 4/\binom{52}{4} = 2496/\binom{52}{4}$. (3) $\binom{13}{2}\binom{4}{2}\binom{4}{2}/\binom{52}{4} = 2808/\binom{52}{4}$. □

Problem 16. *A standard deck of 52 cards is dealt out randomly to four players, each getting 13 cards. The picture cards are the J, Q, and K of each suit. What is the probability that each player receives exactly three picture cards?*

Solution. We have

$$\frac{\binom{12}{3,3,3,3}\binom{40}{10,10,10,10}}{\binom{52}{13,13,13,13}} = \frac{257330216}{7937669495} = 0.0324188625\ldots.$$

□

03. Basic notions, III

Problem 17. *(1) Assume that the students in a large class are equally likely to have their birthdays fall on any of the seven days of the week. What is the smallest integer n such that, in a group of n students randomly selected from this class, there is more than a 50% chance for (at least) two of them to have their birthdays fall on the same day of the week?*

(2) Assume that the students in a large class are equally likely to have their birthdays fall on any of the twelve months of the year. What is the smallest integer n such that, in a group of n students randomly selected from this class, there is more than a 50% chance for (at least) two of them to have their birthdays fall on the same month?

Solution. (1) We compute

$$1 - \mathrm{Pr}\left(\substack{\text{no two students have birthdays}\\ \text{on the same day of the week}}\right) = 1 - \frac{7^{\underline{n}}}{7^n},$$

which has values $(0, \frac{1}{7}, \frac{19}{49}, \frac{223}{343}, \frac{2041}{2401}, \frac{16087}{16807}, \frac{116929}{117649}, 1)$ for $n = (1, 2, \ldots, 8)$ respectively. The answer is thus $n = 4$.

(2) The relevant quantity $1 - 12^{\underline{n}}/12^n$ takes values $(0, \frac{1}{12}, \frac{17}{72}, \frac{41}{96}, \frac{89}{144})$ for $n = (1, 2, 3, 4, 5)$, so the answer is 5. □

Problem 18. *From a standard deck of cards, we take the four aces and the two of clubs. We then shuffle the five cards randomly and reveal them one at a time. (1) What is the probability that the two of clubs is the first card revealed? (2) What is the probability that the two of clubs follows the first ace that appears? (3) What is the probability that the ace of spades is the card following the first ace that appears?*

Solution. (1) The probability is 1/5. (2) This happens iff the two of clubs is the second card to appear, so the answer is 1/5. (3) For any arrangement of the four cards that are not the ace of spades, there is only one out of the five possible ways to insert the ace of spades so as to satisfy the desired condition. Thus the answer is 1/5. □

Problem 19. *In an office of workers, there are six men and six women. (1) If the twelve workers are randomly divided into six pairs, what is the probability that exactly four pairs are of mixed gender? (2) If the twelve workers are randomly divided into four teams of three, what is the probability that in every team the three workers are of the same gender?*

Solution. (1) There are $\binom{6}{2}$ ways to pick the men to be paired together and $\binom{6}{2}$ ways to pick the women to be paired together. There are then 4! ways to pair up the remaining men with the remaining women. Thus the probability is

$$\frac{\binom{6}{2}^2 4!6!}{\frac{12!}{(2!)^6}} = \frac{40}{77} = 0.51948051948\ldots,$$

where the 6! in the numerator considers all the ways to order the pairs that have been formed.

(2) There are $\binom{6}{3}$ ways to form the two teams of men and $\binom{6}{3}$ ways to form the two teams of women. Multiply by $\binom{4}{2}$ to choose which two teams will be the teams of men, and divide by $\binom{12}{3,3,3,3}$ to get $5/231 = 0.021645021645\ldots$. □

Problem 20. *Four couples are being seated at a long table. The four women are seated first, along one side of the table. The four men are then assigned seats along the other side, at random. What is the probability that none of the couples end up facing each other?*

Solution. We want the probability that a random permutation of order n is a derangement, and this is given by $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ by a standard inclusion-exclusion argument. When $n = 4$, this is $3/8 = 0.375$. □

Problem 21. *Four couples are seated randomly at a round table. What is the probability that none of the couples end up sitting next to each other?*

Solution. We solve the problem for n couples. Apply inclusion-exclusion with A_i denoting the event that the ith couple is seated together. Then the kth term in the inclusion-exclusion sum is given by

$$\sum_{1\le i_1<\cdots<i_k\le n} \Pr(A_{i_1}\ldots A_{i_k}) = \binom{n}{k}\frac{(2n-k)!}{(2n-k)}2^k \Big/ \frac{(2n)!}{2n} = 2^k\binom{n}{k}\frac{(2n-k-1)!}{(2n-1)!},$$

which is obtained by treating each couple to be seated together as one object, with the 2^k term accounting for the way each individual couple can be ordered. It follows that

$$c_n = \sum_{k=0}^{n}(-1)^k 2^k \binom{n}{k}(2n-k-1)!$$

defines a sequence $(2, 32, 1488, 112512, 12771840, 2036229120, 434469611520)$ for $n = (2,\ldots,8)$ that counts "the number of ways (up to rotations) to seat n married couples at a circular table with no spouses next to each other" (see OEIS A129348). The desired probability is then $p_n = c_n/(2n-1)!$, which goes to $1/e$ as n grows; also $c_n \sim \sqrt{\pi}2^{2n}n^{2n-1/2}/\exp(2n+1)$.

In particular, the answer we seek is $p_4 = 31/105 = 0.295238\ldots$. □

Problem 22. *At the start of today, you assess that there is a 40% chance of rain today, a 50% chance of rain tomorrow, and a 30% chance of rain both today and tomorrow. If it does not rain today, what then would your assessment be of the chance of rain tomorrow?*

Solution. We are seeking a conditional probability given by

$$\frac{\Pr(\text{rain tomorrow and not rain today})}{\Pr(\text{not rain today})} = \frac{50\% - 30\%}{100\% - 40\%} = \frac{1}{3}.$$

□

04. Basic notions, IV

Problem 23. *A fair coin is tossed three times. (1) Given that the first toss is heads, what is the probability that all tosses are heads? (2) Given that at least one toss is a head, what is the probability that all tosses are heads? (3) Given that at least two tosses are heads, what is the probability that all tosses are heads?*

Solution. This is just a standard application of the definition of conditional probability: $\Pr(A \mid B) := \Pr(AB)/\Pr(B)$. (1) This is the probability that the second and third tosses are heads, which is $1/4$. (2) The probability of at least one head is $7/8$, so the answer is $1/7$. (3) The probability of at least two heads is $4/8$, so the answer is $1/4$. □

Problem 24. *A standard deck of 52 cards is dealt out randomly to four players A, B, C, and D, each getting 13 cards. The picture cards are cards with value J, Q, or K. (1) If A has exactly three picture cards, what is the probability that D has exactly three picture cards? (2) If A and B together have exactly six picture cards, what is the probability that D has exactly three picture cards?*

Solution. (1) The probability is

$$\frac{\binom{12}{3,3,6}\binom{40}{10,10,20}\binom{26}{13}}{\binom{12}{3}\binom{40}{10}\binom{39}{13,13,13}} = \frac{115115}{370481} = 0.3107176886\ldots.$$

(2) The probability is

$$\frac{\binom{12}{3,3,6}\binom{40}{10,10,20}\binom{26}{13}}{\binom{12}{6}\binom{40}{20}\binom{26}{13}^2} = \frac{286}{805} = 0.3552795\ldots.$$

□

Problem 25. *At a graduation ceremony with only math and physics majors, 20% of math majors and 30% of physics majors are receiving prizes. There are twice as many math majors as there are physics majors. What is the probability that a randomly selected graduand is a math major receiving a prize?*

Solution. There is probability $2/3$ of being a math major, and 20% of winning a prize as a math major, so multiplying these probabilities gives $2/15 = 0.1\overline{3}$. □

Problem 26. *Five couples are being seated at a long table. The five women are seated first, along one side of the table. The five men are then assigned seats along the other side, at random. What is the probability that exactly two of the couples end up facing each other?*

Solution. By inclusion-exclusion, we have

$$\frac{1}{5!}\binom{5}{2}\left(\binom{3}{0}3! - \binom{3}{1}2! + \binom{3}{2}1! - \binom{3}{3}0!\right) = \frac{1}{6} = 0.1\overline{6}.$$

□

Problem 27. *In a certain city, 30% of the population are smokers, and 60% of smokers eventually die of lung cancer, but only 10% of non-smokers eventually die of lung cancer. (1) What is the probability that a randomly selected person in the city eventually dies of lung cancer? (2) A dead person at autopsy is determined to have died of lung cancer. What is the probability that the person was a smoker?*

Solution. (1) We invoke the law of total probability to compute

$$\begin{aligned}\Pr(C) &= \Pr(C \mid S)\Pr(S) + \Pr(C \mid NS)\Pr(NS)\\ &= (60\%)(30\%) + (10\%)(70\%) = 25\%.\end{aligned}$$

(2) By Bayes's theorem, we have

$$\Pr(S \mid C) = \frac{\Pr(C \mid S)\Pr(S)}{\Pr(C)} = \frac{(60\%)(30\%)}{25\%} = 72\%.$$

□

Problem 28. *A student takes an exam. Each question gives a statement, and the student is required to indicate whether the statement is True or False. The student also has a third alternative of skipping the question — to avoid getting negative marks for indicating a wrong answer. The student knows the correct answer to 75% of the questions. For the remaining 25% of the questions, the student guesses randomly between True, False and Skip. A question is selected at random and the student has answered it correctly. What is the probability that the student actually knew the correct answer to the question?*

Solution. By Bayes's theorem, we have

$$\begin{aligned} \Pr(K \mid C) &= \frac{\Pr(C \mid K)\Pr(K)}{\Pr(C \mid K)\Pr(K) + \Pr(C \mid NK)\Pr(NK)} \\ &= \frac{100\% \cdot 75\%}{100\% \cdot 75\% + \frac{1}{3} \cdot 25\%} = 0.9. \end{aligned}$$

□

Problem 29. *The prevalence of COVID in the population is 1%. An Antigen Rapid Test (ART) for COVID reports a sensitivity of 85% and a specificity of 95%. Not knowing whether I have COVID or not, I took the ART, and the result was positive. What is the probability that I have COVID?*

Solution. Recall that the *sensitivity* is the probability of a true positive, so that $\Pr(+ \mid C) = 85\%$. Similarly, the *specificity* is the probability of a true negative, so that $\Pr(- \mid NC) = 95\%$. Note that this implies $\Pr(+ \mid NC) = 5\%$; also, the question gives $\Pr(C) = 1\%$. By Bayes's theorem, we have

$$\begin{aligned} \Pr(C \mid +) &= \frac{\Pr(+ \mid C)\Pr(C)}{\Pr(+ \mid C)\Pr(C) + \Pr(+ \mid NC)\Pr(NC)} \\ &= \frac{85\% \cdot 1\%}{85\% \cdot 1\% + 5\% \cdot 99\%} \\ &= \frac{17}{116} = 0.1465517241\ldots. \end{aligned}$$

□

05. Basic notions, V

Problem 30. *An experienced policeman in a city claims to have "an eye" for recognizing criminals.*

- *If a criminal is presented to this policeman, he will claim that the person is a criminal 99% of the time.*
- *If a non-criminal is presented to this policeman, he will claim that the person is a non-criminal 95% of the time.*

Historical records show that 3% of the city population are criminals. This policeman catches a person, and claims that the person is a criminal. What is the probability that the person is indeed a criminal?

Solution. Let C and R stand for 'criminal' and 'recognize' respectively. We are given the sensitivity $\Pr(R \mid C) = 99\%$, the specificity $\Pr(NR \mid NC) = 95\%$, and the base rate $\Pr(C) = 3\%$. By Bayes's theorem, we calculate

$$\begin{aligned}\Pr(C \mid R) &= \frac{\Pr(R \mid C)\Pr(C)}{\Pr(R \mid C)\Pr(C) + \Pr(R \mid NC)\Pr(NC)} \\ &= \frac{99\% \cdot 3\%}{99\% \cdot 3\% + 5\% \cdot 97\%} \\ &= \frac{297}{782} = 0.3797953964\ldots.\end{aligned}$$

□

Problem 31. *Identical twins are always of the same sex. Fraternal twins are equally likely to be of the same sex as to be of different sex. Data collected in a city shows that 55% of twins are of the same sex. What is the probability that a randomly selected twin in the city is a pair of identical twins?*

Solution. Write $p = \Pr(I)$. We have

$$\begin{aligned}55\% = \Pr(S) &= \Pr(S \mid I)\Pr(I) + \Pr(S \mid F)\Pr(F) \\ &= 100\% \cdot p + 50\% \cdot (1 - p),\end{aligned}$$

which implies $p = 10\%$. □

Problem 32. *An Antigen Rapid Test (ART) for COVID reports a sensitivity of 85% and a specificity of 95%. Due to an outbreak of COVID, the ART was given to all residents of a community, and 10% of those tests came back positive. What is the prevalence of COVID (i.e., the probability of having COVID) in that community?*

Solution. We are given that $\Pr(+ \mid C) = 85\%$ and $\Pr(- \mid NC) = 95\%$. Write $p = \Pr(C)$. Then

$$\begin{aligned}10\% = \Pr(+) &= \Pr(+ \mid C)\Pr(C) + \Pr(+ \mid NC)\Pr(NC) \\ &= 85\% \cdot p + 5\% \cdot (1 - p),\end{aligned}$$

which implies $p = 1/16 = 0.0625$. □

Problem 33. *A man has just been infected with COVID. The man typically spends 60%, 30%, 10% of his time at his home, his workplace, and elsewhere, respectively. It was initially assessed that the probability of the man getting infected at a place is proportional to the length of time he spends at that place. His wife is the only other person living at his home. She was immediately given an Antigen Rapid Test for COVID, and her result was negative. The ART reports a sensitivity of 85% and a specificity of 95%. Given this information, what are the revised assessments of the likely source of infection for this man (1) at home? (2) at the workplace? (3) elsewhere?*

Solution. We are given $\Pr(H) = 60\%$, $\Pr(W) = 30\%$, and $\Pr(E) = 10\%$. Also, $\Pr(+ \mid C) = 85\%$ and $\Pr(- \mid NC) = 95\%$. Denote by N the event of his wife learning that her result was negative. We are seeking $\Pr(H \mid N)$, $\Pr(W \mid N)$, and $\Pr(O \mid N)$. First note that $\Pr(N \mid H) = \Pr(- \mid C) = 15\%$, which says that the probability that his wife tested negative given that he got COVID at home is equal to the probability that the wife tested negative despite having COVID. Also, we have $\Pr(N \mid W) = \Pr(N \mid E) = \Pr(- \mid NC) = 95\%$, since the probability that the wife tests negative given that he got COVID from the workplace is the probability that the wife tested negative given that she did not have COVID.

By Bayes's theorem, we have

$$\begin{aligned}\Pr(H \mid N) &= \frac{\Pr(N \mid H)\Pr(H)}{\Pr(N \mid H)\Pr(H) + \Pr(N \mid W)\Pr(W) + \Pr(N \mid E)\Pr(E)} \\ &= \frac{15\% \cdot 60\%}{15\% \cdot 60\% + 95\% \cdot 30\% + 95\% \cdot 10\%} \\ &= \frac{9}{47} = 0.1914893617\ldots;\end{aligned}$$

similarly $\Pr(H \mid W) = 57/94 = 0.6063829787\ldots$ and $\Pr(H \mid E) = 19/94 = 0.20212765957\ldots.$ □

Problem 34. *We toss two fair dice. Are the events E and F independent?*

(1) E is the event that the sum of the two dice is 8; F is the event that the first dice is 4.
(2) E is the event that the two dice are equal; F is the event that the first dice is 4.
(3) E is the event that the difference of the two dice is 3; F is the event that the first dice is 4.
(4) E is the event that the difference of the two dice is 3; F is the event that the sum of the two dice is 7.

Solution. (1) No. $\Pr(E) = 5/36$, $\Pr(F) = 6/36$, and $\Pr(EF) = 1/36$. (2) Yes. $\Pr(E) = \Pr(F) = 6/36$, and $\Pr(EF) = 1/36$. (3) Yes. $\Pr(E) = \Pr(F) = 6/36$, and $\Pr(EF) = 1/36$. (4) No. $\Pr(E) = \Pr(F) = 6/36$, and $\Pr(EF) = 2/36$. □

Problem 35. *A biased coin has a 40% probability of showing heads and a 60% probability of showing tails. Assume that the outcomes of the tosses are independent. (1) What is the probability of getting at least one head in seven tosses? (2) What is the probability of getting exactly four heads in seven tosses?*

Solution. Write $p = 0.40$ and $q = 1 - p = 0.60$. (1) $1 - \Pr(\text{no heads}) = 1 - q^7 = 75938/78125 = 0.9720064$. (2) $\binom{7}{4} p^4 q^3 = 3024/15625 = 0.193536$. □

06. Discrete distributions, I

Problem 36. *A fair dice is rolled five times independently. (1) What is the probability of 1 occurring at least once? (2) What is the probability of 1, 2 or 3 occurring at least once? (3) Given that 1 has occurred at least once, what is the probability of 2 or 3 occurring at least once?*

Solution. (1) $1 - (\frac{5}{6})^5 = 4651/7776 = 0.5981224\ldots$. (2) $1 - (\frac{3}{6})^5 = 31/32 = 0.96875$. (3) Write X_i for the number of times i is rolled. Then

$$\begin{aligned}
&\Pr(X_2 + X_3 \geq 1 \mid X_1 \geq 1) = \frac{\Pr(X_2 + X_3 \geq 1 \text{ and } X_1 \geq 1)}{\Pr(X_1 \geq 1)} \\
&\quad = \frac{\Pr(X_2 + X_3 \geq 1) + \Pr(X_1 \geq 1) - \Pr(X_2 + X_3 \geq 1 \text{ or } X_1 \geq 1)}{\Pr(X_1 \geq 1)} \\
&\quad = \frac{\Pr(X_2 + X_3 \geq 1) + \Pr(X_1 \geq 1) - \Pr(X_1 + X_2 + X_3 \geq 1)}{\Pr(X_1 \geq 1)} \\
&\quad = \frac{\bigl(1 - (4/6)^5\bigr) + \bigl(1 - (5/6)^5\bigr) - \bigl(1 - (3/6)^5\bigr)}{1 - (5/6)^5} \\
&\quad = \frac{3870}{4651} = 0.83207912\ldots.
\end{aligned}$$

□

Problem 37. *A student applying to a graduate programme asks his professor for a letter of recommendation. He estimates that his chances of getting a strong, average, weak recommendation are 50%, 30%, 20% respectively. He also estimates that his chances of getting accepted by the graduate programme would be 80%, 40%, 10% if the recommendation is strong, average, weak respectively. (1) Based on these estimates, what is his probability of getting accepted by the graduate programme? (2) If he gets accepted by the graduate programme, what is the probability that his letter of recommendation was a strong one? (3) If he gets rejected by the graduate programme, what is the probability that his letter of recommendation was a weak one?*

Solution. We are given $\Pr(S) = 50\%$, $\Pr(A) = 30\%$, $\Pr(W) = 20\%$; we also know that $\Pr(+ \mid S) = 80\%$, $\Pr(+ \mid A) = 40\%$, $\Pr(+ \mid W) = 10\%$. (1) We compute

$$\begin{aligned}\Pr(+) &= \Pr(+ \mid S)\Pr(S) + \Pr(+ \mid A)\Pr(A) + \Pr(+ \mid W)\Pr(W)\\ &= 80\% \cdot 50\% + 40\% \cdot 30\% + 10\% \cdot 20\%\\ &= 54\%.\end{aligned}$$

(2) We have

$$\Pr(S \mid +) = \frac{\Pr(+ \mid S)\Pr(S)}{\Pr(+)} = \frac{80\% \cdot 50\%}{54\%} = \frac{20}{27} = 0.\overline{740}.$$

(3) We have

$$\Pr(W \mid -) = \frac{\Pr(- \mid W)\Pr(W)}{\Pr(-)} = \frac{(100\% - 10\%) \cdot 20\%}{100\% - 54\%} = \frac{9}{23} = 0.3913\ldots.$$

□

Problem 38. *A coin is tossed repeatedly. The outcomes (heads vs tails) of the tosses are assumed to be independent. The coin is biased, with each toss showing head with probability 65%. (1) What is the probability of getting 3 heads before 2 tails?*[1] *(2) What is the probability of getting 2 heads before 3 tails? (3) Solve (1) but with use a fair coin instead. (4) Solve (2) but with use a fair coin instead.*

Solution. This is the *problem of the points*, and we discuss the general solution. The setup is that we have a coin with head probability p, and we are interested in $P_{n,m} = \Pr(n \text{ heads occur before } m \text{ tails})$. The idea is just to use the recurrence

$$P_{n,m} = pP_{n-1,m} + (1-p)P_{n,m-1}, \quad n, m \geq 1;$$

[1] A related result is that the probability of getting a *run* of n successes before a *run* of m failures is given by

$$\frac{p^{n-1}(1-q^m)}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}.$$

with boundary conditions $P_{n,0} = 0$ and $P_{0,m} = 1$.

For this problem, we have $p = 0.65$, and we are interested in

$$P_{3,2} = 4p^3 - 3p^4 \quad \text{and} \quad P_{2,3} = 6p^2 - 8p^3 + 3p^4.$$

(There's no magic here, just tedious expansion.)

We then plug in the values to get (1) $P_{3,2} = 90077/160000 = 0.56298125$; (2) $P_{2,3} = 139763/160000 = 0.87351875$; (3) $P_{3,2} = 5/16 = 0.3125$; and (4) $P_{2,3} = 11/16 = 0.6875$. □

Problem 39. *A quick badminton serve-and-rally match is played following the alternating serve protocol, with player A serving first. The rules are changed so that the match ends when a player wins 2 points (rallies), and that player is declared the winner of the match. (1) If player A has a 60% chance of winning a rally when he serves, but only a 30% chance when player B serves, what is the probability of player A winning the match? (2) If player A has only a 30% chance of winning a rally when he serves, but a 60% chance when player B serves, what is the probability of player A winning the match?*

Solution. In general we have

$$\begin{aligned}
&\Pr(A \text{ wins match}) = \Pr(A \text{ wins} \geq n \text{ points in } 2n-1 \text{ rallies}) \\
&\quad = \sum_{m=n}^{2n-1} \Pr(A \text{ wins exactly } m \text{ points in } 2n-1 \text{ rallies}) \\
&\quad = \sum_{m=n}^{2n-1} \sum_{k=1}^{n} \Pr\left(\substack{A \text{ wins exactly } k \text{ of } n \text{ rallies} \\ \text{in which } A \text{ serves}}\right) \Pr\left(\substack{A \text{ wins exactly } m-k \text{ of } n-1 \text{ rallies} \\ \text{in which } B \text{ serves}}\right) \\
&\quad = \sum_{m=n}^{2n-1} \sum_{k=1}^{n} \binom{n}{k} p_A^k q_A^{n-k} \binom{n-1}{m-k} p_B^{m-k} q_B^{n-m+k-1},
\end{aligned}$$

where p_A and p_B denote respectively the probability that A wins a rally when he serves, and that A wins a rally when B serves; also $q_A = 1 - p_A$ and $q_B = 1 - p_B$. (1) Set $n = 2$, $p_A = 60\%$, and $p_B = 30\%$ to get $63/125 = 0.504$. (2) Set $n = 2$, $p_A = 30\%$, and $p_B = 60\%$ to get $171/500 = 0.342$. □

07. Discrete distributions, II

Problem 40. *An urn contains 5 White balls, 4 Red balls, and 3 Black balls. We take 4 balls at random from the urn. The random variable X denotes the number of Black balls drawn. Let p denote the probability mass function of X. Find $p(i)$ for $0 \le i \le 3$, as well as $\mathbf{E}X$, $\mathbf{E}X^2$, and $\operatorname{Var}(X)$.*

Solution. We have

$$p(i) = \Pr(X = i) = \frac{\binom{3}{i}\binom{9}{4-i}}{\binom{12}{4}},$$

which takes values $(14/55, 28/55, 12/55, 1/55) = (0.25\overline{45}, 0.50\overline{90}, 0.2\overline{18}, 0.0\overline{18})$ for $i = (0, 1, 2, 3)$. Now $\mathbf{E}X = 4 \cdot 3/12 = 1$ by linearity of expectation, since $X = X_1 + X_2 + X_3 + X_4$, where X_j indicates if the jth ball drawn was black. We find $\mathbf{E}X^2 = \sum i^2 p(i)$ by direct calculation, getting $17/11 = 1.\overline{54}$. Finally, we have $\operatorname{Var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 6/11 = 0.\overline{54}$. □

Problem 41. *We toss two fair dice. The random variable X denotes the absolute value of the difference of the two dice values. Find $\mathbf{E}X$, $\mathbf{E}X^2$, and $\operatorname{Var}(X)$.*

Solution. From the table

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

we deduce the probabilities $(6/36, 10/36, 8/36, 6/36, 4/36, 2/36)$ for $i = (0, \dots, 5)$. We can then calculate $\mathbf{E}X = 35/18 = 1.9\overline{4}$, $\mathbf{E}X^2 = 35/6 = 5.8\overline{3}$, and $\operatorname{Var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 665/324 = 2.0524691\dots$. □

Problem 42. *A standard deck of cards is shuffled at random, and the cards are turned up one at a time until a picture card appears. (The picture cards are the J, Q, and K of each suit.) The random variable X denotes the number of cards turned up, including the final picture card. Find $\Pr(X = 1)$ and $\Pr(X > 2)$.*

Solution. We have $\Pr(X = 1) = 12/52 = 0.\overline{230769}$, since there are 12 picture cards. Since $\Pr(X > 2) = 1 - \Pr(X = 1) - \Pr(X = 2)$, we calculate

$$\begin{aligned}
&\Pr(X = 2) \\
&\quad = \Pr(\text{first card not picture})\Pr(\text{second card is picture} \mid \text{first card not picture}) \\
&\quad = \frac{40}{52} \cdot \frac{12}{51} = \frac{40}{221} = 0.180995475\dots;
\end{aligned}$$

thus
$$\Pr(X > 2) = 1 - \frac{12}{52} - \frac{40}{221} = \frac{10}{17} = 0.5882352941\ldots.$$
□

Problem 43. *Every time a coupon is being collected, the coupon is equally likely to be any of four different colours, independent of the coupons previously collected. (1) What is the probability that we collect five or more coupons before seeing all four colours among the coupons we have? (2) What is the probability that we collect exactly five before seeing all four colours among the coupons we have?*

Solution. Let N denote the number of types of coupons, so that $N = 4$ here. Let T denote the number of coupons collected until we obtain a complete set. Then (1) asks for $\Pr(T > 4)$, and (2) asks for $\Pr(T = 5) = \Pr(T > 4) - \Pr(T > 5)$. Now $\Pr(T > n)$ is the probability that we don't have all N types of coupons among the first n coupons. Thus, if we write E_i for the event that the ith color is present among the first n coupons, we have $\Pr(T > n) = \Pr(\bigcup_{i=1}^{N} \bar{E}_i)$, which can be computed by inclusion-exclusion.

In particular, $\Pr(\bar{E}_{i_1} \ldots \bar{E}_{i_k}) = \big((N-k)/N\big)^n$, so that
$$\Pr(T > n) = \sum_{k\geq 1} (-1)^{k+1} \binom{N}{k} \left(\frac{N-k}{N}\right)^n,$$
which takes values
$$\left(1, 1, 1, \frac{29}{32}, \frac{49}{64}, \frac{317}{512}, \frac{499}{1024}, \frac{3089}{8192}, \frac{4729}{16384}, \frac{28757}{131072}\right)$$
for $n = (1, \ldots, 10)$. (1) $\Pr(T > 4) = 29/32 = 0.90625$. (2) $\Pr(T = 5) = 29/32 - 49/64 = 9/64 = 0.140625$. □

Problem 44. *Every time a coupon is being collected, the coupon is equally likely to be any of four different colours, independent of the coupons previously collected. (1) After collecting four coupons, what is the probability that we see exactly two colours among the coupons we have? (2) After collecting five coupons, what is the probability that we see exactly three colours among the coupons we have?*

Solution. In general, given a coupon with N possible types, after collecting r coupons, the probability that we see exactly m types among what we have is

$$N^{-r}\binom{N}{m}\sum_{v=0}^{m}(-1)^v\binom{m}{v}(m-v)^r = N^{-r}m!\binom{N}{m}\left\{{r \atop m}\right\}.$$

That is, there are $\left\{{r \atop m}\right\}$ ways to split up your r coupons into m parts where each part will be assigned one type of coupon, and then there are $m!\binom{N}{m}$ ways to perform this exact assignment.

(1) Set $(N, m, r) = (4, 2, 4)$ to get $21/64 = 0.328125$. (2) Set $(N, m, r) = (4, 3, 5)$ to get $75/128 = 0.5859375$. ☐

Problem 45. *There are four buses, carrying 40, 35, 25, 50 students respectively (not counting the drivers). (1) From the 150 students, one of them is selected at random, and X denotes the number of students on his bus. Find* $\mathbf{E}X$. *(2) From the 4 bus drivers, one of them is selected at random, and Y denotes the number of students on his bus. Find* $\mathbf{E}Y$.

Solution. (1) $\mathbf{E}X = 40\cdot 40/150 + 35\cdot 35/150 + 25\cdot 25/150 + 50\cdot 50/150 = 119/3 = 39.\overline{6}$. (2) $\mathbf{E}Y = (1/4)(40+35+25+50) = 37.5$. ☐

Problem 46. *A fair dice is rolled four times. Assume the outcomes are independent. The random variable X denotes the number of times we see 1 or 2 among the dice rolls. Find* $\Pr(X = i)$ *for* $0 \le i \le 4$.

Solution. We have $X \sim \text{Binomial}(4, 2/6)$, so $\Pr(X = i) = \binom{4}{i}(2/6)^i(4/6)^{4-i}$, which gives values $(16/81, 32/81, 8/27, 8/81, 1/81)$ or, approximating,

$$(0.19753, 0.39506, 0.29630, 0.098765, 0.012346)$$

for $i = (0, \ldots, 4)$. ☐

08. Discrete distributions, III

Problem 47. *The eye colour of a person can be brown (B) or blue (b), with brown being dominant and blue being recessive. Thus, to have blue eyes, a person must have the gene pair bb. (1) A brown-eyed (BB) woman and a blue-eyed man plan to have a child. What is the probability that the child has blue eyes? (2) A brown-eyed (Bb) woman and a blue-eyed man plan to have a child. What is the probability that* *the child has blue eyes? (3) A brown-eyed (Bb) woman and a brown-eyed (Bb) man plan to have a child. What is the probability that the child has blue eyes?*

Solution. (1) 0%. (2) 50%. (3) 25%. □

Problem 48. *A biased coin has a probability of 75% for showing heads. The coin is flipped 10 times independently. (1) Given that a total of 6 heads appeared among the 10 flips, what is the conditional probability that the first 3 flips are head, tail, tail? (2) Given that a total of 6 heads appeared among the 10 flips, what is the conditional probability that exactly 3 heads appear in the first 4 flips?*

Solution. We have $X \sim \text{Binomial}(n = 10, p = 0.75)$. (1) Let $q = 1 - p$. This is

$$\begin{aligned}\Pr(\text{HTT} \mid X = 6) &= \frac{\Pr(X = 6 \mid \text{HTT}) \Pr(\text{HTT})}{\Pr(X = 6)} \\ &= \frac{\binom{7}{5} p^5 q^2 \cdot pq^2}{\binom{10}{6} p^6 q^4} \\ &= \frac{1}{10} = 0.1.\end{aligned}$$

(2) Write E for the event that exactly three heads occur in the first four flips. We compute

$$\begin{aligned}\Pr(E \mid X = 6) = \frac{\Pr(X = 6 \mid E) \Pr(E)}{\Pr(X = 6)} &= \frac{\binom{6}{3} p^3 q^3 \cdot \binom{4}{3} p^3 q}{\binom{10}{6} p^6 q^4} \\ &= \frac{8}{21} = 0.38\overline{095238}.\end{aligned}$$

□

Problem 49. *A fair coin is flipped 100 times independently. What is the probability that exactly 50 heads appear? (Approximate.)*

Solution. Recall that $n! \sim \sqrt{2\pi n}(n/e)^n$. We have

$$\binom{100}{50}\left(\frac{1}{2}\right)^{100} = \frac{100!}{(50)!^2 \cdot 2^{100}} \approx \frac{1}{\sqrt{50\pi}} = 0.0797884560802865\ldots.$$

(Note that the actual value begins $\underline{0.079}58923738717876\ldots$.) □

Problem 50. *We have 9 different coins. The kth coin ($k = 1, \ldots, 9$) has a probability of $k/10$ to show heads when flipped, and X_k denotes the number of heads that appear when that coin is flipped 20 times. Find $\mathbf{E}(X_1 + \cdots + X_9)$.*

Solution. Since $X_k \sim \text{Binomial}(20, k/10)$, we have $\mathbf{E}X_k = (20)(k/10) = 2k$, so the answer is $\sum_{k=1}^{9} 2k = 90$. □

09. Continuous distributions, I

Problem 51. *The random variable X follows a Poisson distribution with parameter π. Find $\Pr(X = k)$ for $0 \le k \le 4$, as well as $\mathbf{E}X^3$ and $\mathbf{E}X^4$.*

Solution. We have $\Pr(X = k) = e^{-\pi}\pi^k/k!$, which gives approximate values $(0.043214, 0.13576, 0.21325, 0.22332, 0.17539)$ for $k = (0, \dots, 4)$. Note that $Y \sim \text{Poisson}(\lambda)$ implies $\mathbf{E}Y^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \lambda^k$ and $\mathbf{E}Y^{\underline{n}} = \lambda^n$. In particular $\mathbf{E}X^3 = \pi + 3\pi^2 + \pi^3 \approx 63.7567$ and $\mathbf{E}X^4 = \pi + 7\pi^2 + 6\pi^3 + \pi^4 \approx 355.6756$. We have $(\mathbf{E}Y, \dots, \mathbf{E}Y^8) = (\lambda, \lambda^2+\lambda, \lambda^3+3\lambda^2+\lambda, \lambda^4+6\lambda^3+7\lambda^2+\lambda, \lambda^5+10\lambda^4+25\lambda^3+15\lambda^2+\lambda, \lambda^6+15\lambda^5+65\lambda^4+90\lambda^3+31\lambda^2+\lambda, \lambda^7+21\lambda^6+140\lambda^5+350\lambda^4+301\lambda^3+63\lambda^2+\lambda, \lambda^8+28\lambda^7+266\lambda^6+1050\lambda^5+1701\lambda^4+966\lambda^3+127\lambda^2+\lambda)$. ☐

Problem 52. *A fair dice is rolled 15 times. Assume that the outcomes are independent. The random variable X denotes the number of times we see 1 among the dice rolls. Let p denote the exact probability of $X \le 2$. Let q denote the probability of $X \le 2$ computed using a Poisson approximation. What is the relative error $|p - q|/p$ expressed as a percentage?*

Solution. We have $X \sim \text{Binomial}(15, 1/6)$, so $p = \Pr(X \le 2) = \frac{250244140625}{470184984576} = 0.53222486645477\dots$.

Recall that we approximate $\text{Binomial}(n, p)$ by $\text{Poisson}(np)$. Thus $\lambda = (15)(1/6)$ with $Y \sim \text{Poisson}(\lambda)$ is such that $q = \Pr(Y \le 2) = 53/(8e^{5/2}) = 0.54381311\dots$.

The relative error is then

$$\frac{3114975522816}{250244140625e^{5/2}} - 1 \approx 2.177322\%.$$

☐

Problem 53. *Assume each person is equally likely to have any of the 365 days of the year as his/her birthday. (We omit 29 Feb for this question.) We gather N people in a group. (1) What is the minimum value of N in order to have >90% chance that at least two people in the group have the same birthday? (2) What is the minimum value of N in order to have >80% chance that at least three people in the group have the same birthday? (3) What is the minimum value of N in order to have >70% chance that at least four people in the group have the same birthday?*

Solution. Let's solve the general problem of having a chance greater than p of having n people share the same birthday. In a group of N there are $\binom{N}{n}$ ways to form a group of n people, and each of these groups has probability $1/365^{n-1}$ of sharing the same birthday, so the number Y of groups of n people with the same birthday follows a distribution $\text{Binomial}\big(\binom{N}{n}, \frac{1}{365^{n-1}}\big)$ approximated by $\text{Poisson}\big(\frac{1}{365^{n-1}}\binom{N}{n}\big)$. To this end we seek to have $\Pr(Y \ge 1) \ge p$, or $\Pr(Y = 0) \le 1 - p$. The Poisson approximation gives $\Pr(Y=0) \approx \exp\big(-\frac{1}{365^{n-1}}\binom{N}{n}\big)$, whence we have (1) $N \geq 42$, (2) $N \geq 110$, and (3) $N \geq 196$ by computer calculation. □

Problem 54. *A biased coin has a probability of 75% for showing head. The coin is flipped 1000 times independently. (1) What is the probability that we see a string of (at least) 20 consecutive heads? (2) What is the probability that we see a string of (at least) 30 consecutive heads?*

Solution. The analysis is difficult; we refer the reader to Example 7d in Chapter 4 of S. Ross's *A First Course in Probability*, 10th edition. We simply state the result obtained: If L_n denotes the largest number of consecutive heads in n flips of a coin with head probability p, we have (as a consequence of the 'Poisson paradigm')

$$\Pr(L_n < k) \approx \exp\Big(-(n-k)p^k(1-p) - p^k\Big).$$

From this it is easy to obtain the following approximations: (1) 0.54164814; (2) 0.042552834. □

Problem 55. *A sample of radioactive substance has been observed to emit on average* 0.1234 *many α-particles per second. (1) What is the probability that at least three α-particles are emitted in a 10 second time interval? (2) What is the shortest time interval we need to wait in order that the probability of the sample emitting any α-particle in that time interval is >90%?*

Solution. We model this as a Poisson process with rate $\lambda = 0.1234$. (1) Let X be the number of α-particles emitted in a ten second interval. Then $X \sim \text{Poisson}(10\lambda)$, so

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2) = 1 - \sum_{k=0}^{2} e^{-10\lambda}\frac{(10\lambda)^k}{k!} \approx 0.127968355.$$

(2) Let t be the length of an interval, and let X_t be the number of α-particles emitted in a length-t interval. Then $X_t \sim \text{Poisson}(\lambda t)$, and we wish to find t such that $\Pr(X_t \geq 1) > 90\%$, or $\Pr(X_t = 0) \leq 0.1$. Since $\Pr(X_t = 0) = e^{-\lambda t}$, we can solve to get $t \geq \log(10)/\lambda \approx 18.65952$. □

Problem 56. *Let X be a random variable with probability density function $f(x) = c(1-x^2)[-1 < x < 1]$. Find the following: (1) c. (2) $\Pr(X < 1/2)$, $\Pr(-1/3 < X < 1/3)$. (3) $\mathbf{E}X$, $\mathrm{Var}(X)$. (4) $\mathbf{E}e^X$.*

Solution. (1) Since $\int_{-1}^{1}(1-x^2)\,dx = 4/3$, we must set $c = 3/4$. (2) We have $\Pr(X < 1/2) = \int_{-1}^{1/2} \frac{3}{4}(1-x^2)\,dx = \frac{27}{32} = 0.84375$ and $\Pr(-1/3 < X < 1/3) = \int_{-1/3}^{1/3} \frac{3}{4}(1-x^2)\,dx = \frac{13}{27} = 0.\overline{481}$. (3) We compute $\mathbf{E}X = \int_{-1}^{1} \frac{3}{4}x(1-x^2)\,dx = 0$ and $\mathbf{E}X^2 = \int_{-1}^{1} \frac{3}{4}x^2(1-x^2)\,dx = \frac{1}{5}$. It follows that $\mathrm{Var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = \frac{1}{5}$. (4) We compute $\mathbf{E}e^X = \int_{-1}^{1} \frac{3}{4}e^x(1-x^2)\,dx = \frac{3}{e} = 1.10363832\ldots$. □

10. Continuous distributions, II

Problem 57. *A point is chosen at random (i.e. in a uniformly distributed way) on a line segment of length L, thus dividing the line segment into two pieces. What is the probability that the longer piece is at least three times as long as the shorter piece?*

Solution. Intuitively it is clear that the length does not matter, and a bit of thought reveals that the cut has to be closer to an end than to the center of the line segment; thus the answer is 1/2.

The rigorous approach goes something like this: We have $X \sim \mathrm{Uniform}([0, L])$, which has density $f(x) = 1/L[0 \le x \le L]$. The two lengths L_{longer} and L_{shorter} then satisfy

$$L_{\text{longer}} = \begin{cases} L - X & \text{if } 0 \le X \le L/2 \\ X & \text{if } L/2 < X \le L \end{cases} \quad \text{and} \quad L_{\text{shorter}} = \begin{cases} X & \text{if } 0 \le X \le L/2 \\ L - X & \text{if } L/2 < X \le L. \end{cases}$$

The desired ratio is then

$$r = \frac{L_{\text{longer}}}{L_{\text{shorter}}} = \begin{cases} \frac{L-X}{X} & \text{if } 0 \le X \le L/2 \\ \frac{X}{L-X} & \text{if } L/2 < X \le L, \end{cases}$$

which implies $\Pr(r \ge 3) = \Pr(X \le L/4) + \Pr(X \ge 3L/4) = 1/2$. □

Problem 58. *The random variable X denotes the lifetime (in hours) of a certain electronic device. Its probability density function is given by $f(x) = 10/x^2[x > 10]$. (1) Find $\Pr(X > 30)$. (2) When ten of these devices are used independently, what is the probability that at least two of them last more than 30 hours?*

Solution. (1) We have

$$\Pr(X > 30) = \int_{30}^{\infty} \frac{10}{x^2}\,dx = \frac{1}{3}.$$

(2) Let N denote the number of devices (out of 10) that last more than 30 hours. Then $N \sim \text{Binomial}(10, 1/3)$, and so we compute

$$\begin{aligned}\Pr(N \geq 2) &= 1 - \Pr(N = 0) - \Pr(N = 1) \\ &= 1 - \binom{10}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10} - \binom{10}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^9 \\ &= \frac{17635}{19683} = 0.89595082\ldots\end{aligned}$$

□

Problem 59. *Let X be a random variable uniformly distributed between $e \approx 2.71828$ and $\pi \approx 3.14159$. Find $\mathbf{E}X^3$ and $\mathbf{E}X^4$.*

Solution. The density function of X is $\frac{1}{\pi - e}1_{[e,\pi]}$. We calculate

$$\mathbf{E}X^n = \int_{-\infty}^{\infty} x^n f(x)\,dx = \frac{1}{\pi - e}\int_e^{\pi} x^n\,dx = \frac{\pi^{n+1} - e^{n+1}}{(n+1)(\pi - e)},$$

which easily yields approximations $\mathbf{E}X^3 \approx 25.283396$ and $\mathbf{E}X^4 \approx 74.463735$. □

Problem 60. *Consider a circle of radius R, together with an inscribed square in the circle. (1) Suppose a chord of the circle is picked randomly in such a way that the distance of the chord from the center of the circle is uniformly distributed between 0 and R. What is the probability that the chord is shorter than the side-length of the square? (2) Suppose now that the chord of the circle is picked randomly in such a way that the angle between the chord and the tangent to the circle at one end is uniformly distributed between 0° and 180°. What is the probability that the chord is shorter than the side-length of the square?*

Solution. (1) Write X for the distance from the chord to the center of the circle, so that $X \sim \text{Uniform}([0, R))$. Some thought reveals that we are seeking $\Pr(X > R/\sqrt{2})$, and this is just $1 - 1/\sqrt{2} \approx 0.2928932$. (Imagine the square rotating to have a side parallel to the chosen chord.) (2) Here the angle A (in degrees) is uniformly distributed between 0 and 180, and if you imagine the tangent line fixed to be perpendicular to a diagonal of the inscribed square, the permissible angles are in the range $(0, 45) \cup (135, 180)$. Thus the desired probability is $1/2$. □

Problem 61. *A biased coin has 30% probability of showing heads. Assume that the outcomes of the tosses are independent. (1) What is the probability that exactly 3 tosses are needed to see the first tail? (2) What is the probability that at least 8 tosses are needed to see the first tail?*

Solution. Write $p = 0.3$ and $q = 0.7$. Then this is a geometric distribution — for example, in the first case we can only begin with HHT. So (1) $p^2q = 0.063$ and (2) $p^7 = 0.0002187$. □

Problem 62. *Let X be a geometric random variable with parameter $1/\pi$. Find* $\mathbf{E}X^3$ *and* $\mathbf{E}X^4$.

Solution. Let $q = 1/\pi$, so that $X \sim \text{Geometric}(q)$. Writing $p = 1 - q$, we have

$$\mathbf{E}X^n = \sum_{k\geq 1} k^n p^{k-1} q = \frac{1}{q^n}\sum_{k=0}^{n-1}(n-k)!\left\{ {n \atop n-k} \right\}(-1)^k q^k.$$

Here are explicit expressions for $(\mathbf{E}X, \ldots, \mathbf{E}X^5)$:

$$\left(\frac{1}{q}, \frac{2-q}{q^2}, \frac{q^2-6q+6}{q^3}, \frac{-q^3+14q^2-36q+24}{q^4}, \frac{q^4-30q^3+150q^2-240q+120}{q^5}\right).$$

We have $\mathbf{E}X^3 \approx 129.9616263$ and $\mathbf{E}X^4 \approx 1356.6250932869$. □

Problem 63. *The random variable X follows a normal distribution with mean π and variance e^2. (1) Find* $\Pr(X = 1)$, $\Pr(-2 < X < 2)$, $\Pr(X \leq 3)$, $\Pr(X > 4)$, *and* $\Pr(|X| > 5)$. *(2) Suppose a, b, c are such that* $\Pr(X < a) = 15\%$, $\Pr(X > b) = 10\%$, *and* $\Pr(|X| > c) = 5\%$. *Find a, b, c.*

Solution. (1) $\Pr(X = 1) = 0$ because the normal distribution is continuous. The remaining questions are exercises in Excel. We have $\Pr(-2 < X < 2) = \Phi(2) - \Phi(-2)$, so we can use

```
=NORM.DIST(2,PI(),EXP(1),TRUE)-NORM.DIST(-2,PI(),EXP(1),TRUE)
```

(notice that we input the standard deviation e instead of the variance e^2). This gives 0.307974511. Similarly $\Pr(X \leq 3) = \Phi(3)$ yields

```
=NORM.DIST(3,PI(),EXP(1),TRUE)
```

or 0.479228878, $\Pr(X > 4) = 1 - \Phi(4)$ yields

```
=1-NORM.DIST(4,PI(),EXP(1),TRUE)
```

or 0.376080797, and $\Pr(|X| > 5) = 1 - (\Phi(5) - \Phi(-5))$ yields

```
=1-(NORM.DIST(5,PI(),EXP(1),TRUE)
    -NORM.DIST(-5,PI(),EXP(1),TRUE))
```

or 0.248463495. (2) We want $a = \Phi^{-1}(0.15)$, so we write

```
=NORM.INV(0.15,PI(),EXP(1))
```

to get 0.324274605. Since $\Pr(X \leq b) = 0.90$, we write

```
=NORM.INV(0.9,PI(),EXP(1))
```

to get 6.625210986. Finally, one can use the Excel Solver to solve for when $1-(\Phi(c)-\Phi(-c))=0.05$, though the quicker method in my view is to manually tweak the value using the usual command

```
=1-(NORM.DIST(A1,PI(),EXP(1),TRUE)
     -NORM.DIST(-A1,PI(),EXP(1),TRUE))
```

(here the value one tweaks is in cell `A1`). □

11. Continuous distributions, III

Problem 64. *The scores of students taking an exam are assumed to follow a normal distribution. It is known that 25% of the students scored less than 45 points, and 25% of the students scored more than 70 points. (1) What is the probability that a randomly chosen student scores less than 50 points? (2) What is the probability that a randomly chosen student scores more than 90 points?*

Solution. The score $X \sim N(\mu, \sigma^2)$ is normal but we do not know the values of μ and σ. We introduce the standard normal $Z = (X-\mu)/\sigma$, so the hypotheses $\Pr(X < 45) = \Pr(X > 70) = 1/4$ become $\Pr(Z < \frac{45-\mu}{\sigma}) = \Pr(Z > \frac{70-\mu}{\sigma}) = 1/4$. Now Excel gives

$$\frac{45-\mu}{\sigma} = -0.67448975 \quad \text{and} \quad \frac{70-\mu}{\sigma} = 0.67448975$$

via

`=NORM.INV(0.25,0,1)` and `=NORM.INV(0.75,0,1)`

respectively, and we may solve the linear system to get $\mu = 57.5$ and $\sigma = 18.53252773132\ldots$. Now that we know μ and σ, it is routine to compute (1) $\Pr(X < 50) = 0.34285128163412\ldots$ and (2) $\Pr(X > 90) = 0.039743247221\ldots$. As an aside, these can be expressed by means of the complementary error function and its inverse: We have

$$\Pr(X < 50) = \frac{1}{2}\operatorname{erfc}\left(\frac{3}{5}\operatorname{erfc}^{-1}\left(\frac{1}{2}\right)\right),$$

recalling that $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$ and $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$. □

Problem 65. *A new test for COVID is designed to work by measuring the concentration of antigens of the virus in a standard nasal sample. The concentration of antigens is assumed to follow a normal distribution:*

- *with mean 10 and variance 2 among COVID infected people, but*
- *with mean 5 and variance 3 among non-COVID (healthy) people.*

The test is calibrated to indicate:

- *positive (for COVID) if the measured concentration is ≥8,*
- *negative (for COVID) if the measured concentration is <8.*

(1) What is the sensitivity of the test? (2) What is the specificity of the test?

Solution. (1) The sensitivity (true positive rate) is given by $\Pr\big(N(\mu = 10, \sigma^2 = 2) \geq 8\big) = 0.92135039647\ldots$. (2) The specificity (true negative rate) is given by $\Pr\big(N(\mu = 5, \sigma^2 = 3) < 8\big) = 0.958367741668\ldots$. □

Problem 66. *The random variable Z follows a standard normal distribution. Find $\mathbf{E}Z^3$ and $\mathbf{E}Z^4$.*

Solution. Recall the standard normal density function $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$. We know that $\mathbf{E}Z = 0$ and $\mathbf{E}Z^2 = 1$. We have $\mathbf{E}Z^3 = \int_{-\infty}^{\infty} x^3 f(x)\, dx = 0$, since x^3 is odd and $f(x)$ is even (so their product is odd). Now we integrate by parts to get

$$\mathbf{E}Z^4 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 \exp\Big(-\frac{1}{2}x^2\Big)\, dx = 3 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\Big(-\frac{1}{2}x^2\Big)\, dx = 3.$$

□

Problem 67. *The random variable X follows a normal distribution with mean π and variance e^2. Find $\mathbf{E}X^3$ and $\mathbf{E}X^4$.*

Solution. In general, when $X \sim N(\mu, \sigma^2)$, we have

$$\mathbf{E}X^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!2^k} \mu^{n-2k}\sigma^{2k}$$

and

$$\mathbf{E}(X-\mu)^n = (n-1)!!\sigma^n[n \text{ even}],$$

where $(2k-1)!! = (2k-1)(2k-3)(2k-5)\dots 1$ denotes the double factorial.

n	Non-central moment $\mathbf{E}X^n$	Central moment $\mathbf{E}(X-\mu)^n$
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$
5	$\mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	0
6	$\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$	$15\sigma^6$
7	$\mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$	0
8	$\mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\sigma^8$	$105\sigma^8$

In particular, we have $\mathbf{E}X^3 \approx 100.64649$ and $\mathbf{E}X^4 \approx 698.7659$. □

Problem 68. *A fair dice is rolled 15 times. Assume the outcomes are independent. The random variable X denotes the number of times we see 1 or 2 among the fifteen dice rolls. Let p denote the exact probability of $X = 5$. Let q denote the probability of $X = 5$ computed using a normal approximation. What is the relative error $|(p-q)/p|$ expressed as a percentage?*

Solution. We have $X \sim \text{Binomial}(15, 2/6)$, so

$$p = \Pr(X = 5) = \binom{15}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^{10} = \frac{1025024}{4782969} = 0.214307\dots.$$

Recall that the normal approximation Y to $\text{Binomial}(n, p)$ is given by $N(np, npq)$; in this case we have $Y \sim N(\mu = 5, \sigma^2 = 10/3)$. Then $q = \Pr\big(Y \in (4.5, 5.5)\big) = 0.21580877\dots$, and so

$$\frac{|p-q|}{|p|} \approx 0.700730880464\%.$$

□

Problem 69. *A certain school has a maximum enrolment capacity of 200 students per year. Historical records indicate that 65% of the offers of admission made to applicants of this school were accepted. (1) If the school decides to make 300 offers of admissions this year, what is the probability that more than 200 students accept their offers? (2) The school is only willing to take a risk of 1% chance that more than 200 students accept their offers. What is the maximum number of offers the school can make this year?*

Solution. (1) The number of students that accept is a random variable

$$X \sim \text{Binomial}(300, 0.65),$$

and we have $\Pr(X > 200) = 0.2538995487144\ldots$. You can get pretty close using `=BINOM.DIST.RANGE(300,0.65,201,300)`, though sometimes it's worth using the normal approximation $\Pr(X > 200) \approx \Pr(Y > 200.5)$, where $Y \sim N(300 \cdot 0.65, 300 \cdot 0.65 \cdot 0.35)$. (2) We want the largest n for which $\Pr(X > 200) \leq 0.01$, where $X \sim \text{Binomial}(n, 0.65)$. We can use `=BINOM.DIST.RANGE(A1,0.65,201,A1)` and repeatedly test values in cell `A1` to find that $n = 280$ is the maximum. Alternatively, one can make use of the normal approximation, though you have to be careful about how you handle the values. □

12. Joint distributions, I

Problem 70. *In a country with a large population, a new law is being considered to ban smoking. Suppose 70% of all citizens support this new law. (1) What is the probability that in a random sample of 100 citizens, at least 65% of them support this new law? (2) What is the minimum size of our random sample if we want to be 95% sure that at least 65% of the sampled citizens support this new law?*

Solution. (1) Let X be the number of people who support this new law, so that $X \sim \text{Binomial}(100, 0.7)$. Then we want $\Pr(X \geq 65)$, which is given by

`=BINOM.DIST.RANGE(100,0.7,65,100)`.

This is roughly 0.883921394, and is approximated well by $\Pr(Y > 64.5)$ with $Y \sim N(70, 21)$. (2) We want minimal n satisfying $\Pr(X \geq 0.65n) \geq 0.95$ where $X \sim \text{Binomial}(n, 0.7)$. You can do this by having cell A*j* have value j and having cell B*j* have value

`=BINOM.DIST.RANGE(B`*j*`,0.7,CEILING.MATH(0.65*B`*j*`),B`*j*`)`,

then using conditional formatting ("Highlight Cells Rules" > "Greater Than...") to highlight values that are at least 0.95. It turns out that $n = 220$. □

Problem 71. *The random variable X follows an exponential distribution with parameter $1/\pi$. (1) Find $\Pr(1 < X < 2)$, $\Pr(X > \pi)$, and $\Pr(X < 1/\pi)$. (2) Find $\mathbf{E}X^3$ and $\mathbf{E}X^4$.*

Solution. (1) Recall that $X \sim \text{Exponential}(\lambda)$ has probability density function $f(x) = \lambda e^{-\lambda x}[x > 0]$. We have $\lambda = 1/\pi$, so

$$\Pr(1 < X < 2) = \int_1^2 \frac{1}{\pi} e^{-x/\pi}\,dx \approx 0.198299541,$$
$$\Pr(X > \pi) = \int_\pi^\infty \frac{1}{\pi} e^{-x/\pi}\,dx = \frac{1}{e} \approx 0.36787944117,$$

and

$$\Pr(X < 1/\pi) = \int_0^{1/\pi} \frac{1}{\pi} e^{-x/\pi}\,dx \approx 0.0963572.$$

(2) We have $\mathbf{E}X^n = n!/\lambda^n$ in general, so for this problem we have $\mathbf{E}X^3 = 6\pi^3 \approx 186.03766$ and $\mathbf{E}X^4 = 24\pi^4 \approx 2337.818184816$. □

Problem 72. *When an MRT (subway) line breaks down, the time (in hours) until the resumption of operations is an exponentially distributed random variable with parameter $1/3$. (1) What is the probability that more than three hours is needed to fix a broken down MRT line? (2) One of the MRT lines has broken down six hours ago. What is the probability that it will get fixed within the next three hours?*

Solution. Let X be the time (in hours) until the resumption of operations, so that $X \sim \text{Exponential}(\lambda = 1/3)$. (1) We have $\Pr(X > 3) = \int_3^\infty \frac{1}{3} e^{-x/3}\,dx = \frac{1}{e} \approx 0.36787944117$. (2) We compute (by the memoryless property)

$$\begin{aligned}\Pr(X < 6 + 3 \mid X > 6) &= 1 - \Pr(X > 9 \mid X > 6) = 1 - \Pr(X > 3) \\ &= 1 - \frac{1}{e} \approx 0.63212.\end{aligned}$$

□

Problem 73. *According to a certain model, the lung cancer hazard rate of a t-year old male smoker, for $t \geq 40$, is given by*

$$\lambda(t) = 0.027 + 0.0025(t - 40)^2$$

if $t \geq 40$, and 0 otherwise. A 40-year old male smoker is randomly chosen. Assume that he survives all other hazards. (1) What is the probability that he survives to age 50 without getting lung cancer? (2) What is the probability that he survives to age 55 without getting lung cancer?

Solution. We have $F(t) = \Pr(X < t) = 1 - \exp(-\int_0^t \lambda(s)\,ds)$ by definition, and in this case we get

$$F(t) = 1 - \exp\Big(-0.027(t-40) - \frac{0.0025}{3}(t-40)^3\Big)$$

if $t > 40$ and $F(t) = 0$ otherwise. (1) This is $\Pr(X > 50) = 1 - F(50) = e^{-331/300} \approx 0.33176336$. (2) This is $\Pr(X > 55) = 1 - F(55) = e^{-1287/400} \approx 0.04005507$. $\square$

Problem 74. *The random variable X follows a gamma distribution with parameters $(e, 1/\pi)$. (1) Find $\Pr(1 < X < 2)$. (2) Find $\mathbf{E}X^3$ and $\mathbf{E}X^4$.*

Solution. Recall that $\mathrm{Gamma}(\alpha, \lambda)$ has density $f(x) = \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}/\Gamma(\alpha)[x > 0]$.
(1) We have $\Pr(1 < X < 2) = \int_1^2 f(x)\,dx = 0.035259648\ldots$.
(2) In general $\mathbf{E}X^n = \alpha^{\overline{n}}/\lambda^n$. In particular $\mathbf{E}X^3 = 1478.6666654783\ldots$ and $\mathbf{E}X^4 = 26563.5253272359\ldots$. $\square$

13. Joint distributions, II

Problem 75. *Let X be a random variable uniformly distributed between $-e = -2.71828\ldots$ and $\pi = 3.14159\ldots$. Determine the probability density function of the following random variables: (1) $|X|$; (2) X^3; (3) e^X.*

Solution. The probability density function of X is given by $f_X(a) = \frac{1}{\pi+e}[-e \leq a \leq \pi]$. We will find the cumulative density functions of these functions of X, then we will differentiate to obtain their probability density functions.

(1) When $a < 0$, we have $F_{|X|}(a) = \Pr(|X| \leq a) = 0$; thus $f_{|X|}(a) = 0$ for $a < 0$. When $a \geq 0$, we have

$$F_{|X|}(a) = \Pr(|X| \leq a) = \Pr(-a \leq X \leq a) = \int_{-a}^{a} f_X(x)\,dx,$$

so

$$f_{|X|}(a) = F'_{|X|}(a) = f_X(a) + f_X(-a) = \begin{cases} \frac{2}{\pi+e} & \text{if } 0 \leq a \leq e \\ \frac{1}{\pi+e} & \text{if } e < a \leq \pi \\ 0 & \text{otherwise.} \end{cases}$$

(2) We compute

$$F_{X^3}(a) = \Pr(X^3 \leq a) = \Pr(X \leq a^{1/3}) = \int_{-\infty}^{a^{1/3}} f_X(x)\,dx,$$

which implies

$$f_{X^3}(a) = F'_{X^3}(a) = \frac{1}{3}a^{-2/3} f_X(a^{1/3}).$$

(3) When $a \leq 0$, we have $F_{e^X}(a) = \Pr(e^X \leq a) = 0$, since $e^x > 0$ whenever $x \in \mathbf{R}$. Thus $f_{e^X}(a) = 0$ for $a \leq 0$. Now when $a > 0$, we have

$$F_{e^X}(a) = \Pr(e^X \leq a) = \Pr(X \leq \log a) = \int_{-\infty}^{\log a} f_X(x)\,dx,$$

so

$$f_{e^X}(a) = F'_{e^X}(a) = \frac{1}{a} f_X(\log a).$$

$\square$

Problem 76. *A committee of four persons is to be randomly selected from a group of 12 professors consisting of six physicists, four chemists and two biologists. Let X and Y respectively denote the number of physicists and chemists in the committee.*

(1) Let p denote the joint probability mass function of X and Y. Find $p(1, 2)$.

(2) Let p_X denote the marginal probability mass function of X. Find $p_X(3)$.

(3) Let p_Y denote the marginal probability mass function of Y. Find $p_Y(3)$.

Solution. (1) $p(1, 2) = \Pr(X = 1, Y = 2) = \binom{6}{1}\binom{4}{2}\binom{2}{1}/\binom{12}{4} = 8/55 = 0.1\overline{45}$.

(2) $p_X(3) = \Pr(X = 3) = \binom{6}{3}\binom{4+2}{1}/\binom{12}{4} = 8/33 = 0.\overline{24}$.

(3) $p_Y(3) = \Pr(Y = 3) = \binom{4}{3}\binom{6+2}{1}/\binom{12}{4} = 32/495 = 0.0\overline{64}$. $\square$

Problem 77. *The random variables X and Y are jointly continuous, with a joint probability density function given by*

$$f(x,y) = c\Big(x^2 + \frac{xy}{2}\Big)[0 < x < 1][0 < y < 2],$$

where c is a constant. (1) Find c and $\Pr(X < 1/2$ and $Y > 1/2)$.

(2) Determine the marginal density function f_X of X; find $f_X(0.5)$ and $f_X(1.5)$.

(3) Determine the marginal density function f_Y of Y; find $f_Y(0.5)$ and $f_Y(1.5)$.

(4) Determine the probability density function g of Y/X, assuming $X > 0$. Find $g(1)$ and $g(3)$.

Solution. (1) We need $\int_{\mathbb{R}^2} f(x,y)\,dm = 1$, so we compute

$$\int_0^2 \int_0^1 c\Big(x^2 + \frac{xy}{2}\Big)\,dx\,dy = \int_0^2 c\Big(\frac{1}{3} + \frac{y}{4}\Big)\,dy = c\Big(\frac{2}{3} + \frac{4}{8}\Big),$$

which gives $c = 6/7$. Now we may compute

$$\begin{aligned}\Pr(X < 1/2 \text{ and } Y > 1/2) &= \int_{1/2}^2 \int_0^{1/2} \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)\,dx\,dy \\ &= \frac{69}{448} \approx 0.154.\end{aligned}$$

(2) Recall that $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$. In this case, we have

$$f_X(x) = \int_0^2 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)\,dy[0 \le x \le 1] = \frac{6}{7}(2x^2 + x),$$

so $f_X(0.5) = 6/7 = 0.\overline{857142}$ and $f_X(1.5) = 0$ (because we require $0 \le x \le 1$).

(3) Much like before, we have $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$, and we may compute

$$f_Y(y) = \int_0^1 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)\,dx[0 \le y \le 2] = \frac{1}{14}(3y + 4);$$

this yields $f_Y(0.5) = 11/28 = 0.39\overline{285714}$ and $f_Y(1.5) = 17/28 = 0.60\overline{714285}$.

(4) We compute the CDF (thinking of a as the slope of the line $y = ax$, which coincides with the diagonal of the rectangle $[0,1] \times [0,2]$ when $a = 2$)

$$F_{Y/X}(a) = \Pr(Y/X \le a) = \Pr(Y \le aX)$$

$$= \begin{cases} 0 & \text{if } a < 0 \\ \int_0^1 \int_0^{ax} \frac{6}{7}(x^2 + \frac{xy}{2})\,dy\,dx & \text{if } 0 \le a \le 2 \\ 1 - \int_0^2 \int_0^{y/a} \frac{6}{7}(x^2 + \frac{xy}{2})\,dx\,dy & \text{if } a > 2; \end{cases} = \begin{cases} 0 & \text{if } a < 0 \\ \frac{1}{56}(3a^2 + 12a) & \text{if } 0 \le a \le 2 \\ 1 - \frac{6a+8}{7a^3} & \text{if } a > 2; \end{cases}$$

the PDF is then the derivative

$$g(a) = \begin{cases} 0 & \text{if } a < 0 \\ \frac{1}{56}(6a + 12) & \text{if } 0 \le a \le 2 \\ \frac{12}{7}\frac{a+2}{a^4} & \text{if } a > 2, \end{cases}$$

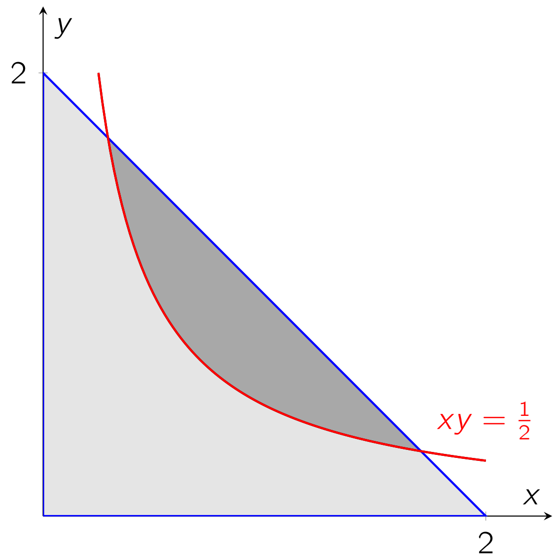
so that $g(1) = 9/28 \approx 0.3214$ and $g(3) = 20/189 \approx 0.10582$. $\square$

14. Joint distributions, III

Problem 78. *Consider the right-angled triangle with vertices at* $(0,0)$, $(2,0)$, *and* $(0,2)$. *A point is chosen randomly within the triangle. (1) The random variable* D *denotes the distance of the chosen point from the origin. Find* $\Pr(D<1)$. *(2) The random variable* A *denotes the area of the rectangle bounded by the* x*- and* y*-axes and the horizontal and vertical lines through the chosen point. Find* $\Pr(A>1/2)$.

Solution. (1) Writing (X,Y) for the randomly chosen point, we have $D=\sqrt{X^2+Y^2}$, so $\Pr(D<1)=\Pr(X^2+Y^2<1)$. Geometrically, this is a quarter circle that lies completely within the triangle we are considering, so this probability is just the area of this radius-1 quarter circle, $\pi/4$, multiplied by the density $1/2$ of the uniform distribution over the triangle (since the triangle has area 2). More formally we are integrating the joint density function $f(x,y)=\frac{1}{2}[x,y\geq 0][x+y\leq 2]$. Therefore $\Pr(D<1)=\pi/8\approx 0.392699$.

(2) We have $A=XY$, so we want $\Pr(XY>1/2)$. This amounts to integrating the joint density function $f(x,y)$ over the region of intersection V depicted below between the triangle and the region $xy>1/2$.



We have

$$\Pr(XY>1/2)=\int_V \frac{1}{2}\,dx\,dy=\int_{1-1/\sqrt{2}}^{1+1/\sqrt{2}}\int_{1/2x}^{2-x}\frac{1}{2}\,dy\,dx$$
$$=\frac{1}{4}\Big(2\sqrt{2}+\log\big(3-2\sqrt{2}\big)\Big)\approx 0.266419987676776.$$

□

Problem 79. *A fair dice is rolled six times.*

(1) What is the probability that three rolls show 1, two rolls show 2 or 3, and one roll shows 4, 5, or 6?

(2) What is the probability that two rolls show 1, two rolls show 2 or 3, and two rolls show 4, 5, or 6?

Solution. (1) We have

$$\binom{6}{3,2,1}\left(\frac{1}{6}\right)^3\left(\frac{2}{6}\right)^2\left(\frac{3}{6}\right)^1 = \frac{5}{324} = 0.01\overline{543209876}.$$

(2) We have

$$\binom{6}{2,2,2}\left(\frac{1}{6}\right)^2\left(\frac{2}{6}\right)^2\left(\frac{3}{6}\right)^2 = \frac{5}{72} = 0.069\overline{4}.$$

□

Problem 80. *A box contains 10 balls, of which some random number R of them are Red and the rest are Green. A sample of 100 balls is drawn at random from the box with replacement. Let X denote the number of Red balls from this sample. Another sample of 100 balls is drawn at random from the box with replacement. Let Y denote the number of Red balls from this sample. Are the random variables X and Y independent? Why?*

Solution. The question is ambiguous. If we interpret R as a number randomly chosen but then fixed for the experiments that follow, then $X, Y \sim \text{Binomial}(100, R/10)$ are independent. If we interpret R as a random variable, it turns out that X and Y are not independent (and in fact highly correlated). □

Problem 81. *(1) Suppose X and Y are independent random variables. Let $g\colon \mathbf{R} \to \mathbf{R}$ and $h\colon \mathbf{R} \to \mathbf{R}$ be (measurable) real-valued functions. Are the random variables $g(X)$ and $h(Y)$ independent?*

(2) Suppose X_1, X_2, Y_1, and Y_2 are independent random variables. Let $g\colon \mathbf{R}^2 \to \mathbf{R}$ and $h\colon \mathbf{R}^2 \to \mathbf{R}$ be (measurable) real-valued functions. Are the random variables $g(X_1, X_2)$ and $h(Y_1, Y_2)$ independent?

(3) Suppose X and Y are discrete random variables whose joint probability mass function is given by $p(1,1) = 1/8$, $p(1,2) = 1/4$, $p(2,1) = 1/8$, and $p(2,2) = 1/2$. Are X and Y independent?

Solution. The answer to both (1) and (2) is yes, because a statement like $g(X) \in A$ is equivalent to $X \in g^{-1}(A)$, so independence is inherited in this way. (3) No, since $p_Y(1) = p(1,1) + p(2,1) = 1/4$ and $p_X(1) = p(1,1) + p(1,2) = 3/8$, but $p(1,1) = 1/8 \neq p_X(1)p_Y(1)$. □

Problem 82. *Suppose X and Y are independent random variables, each of them taking the values $0, 1, \ldots, 10$ with equal probability. Find $\Pr(X \le Y)$ and $\Pr(X = Y)$.*

Solution. The joint density function satisfies $p(i, j) = 1/11^2$ for all $0 \le i, j \le 10$. We have $\Pr(X = Y) = 1/11 = 0.\overline{09}$. By symmetry $\Pr(X \le Y) = \Pr(X \ge Y)$; since $\Pr(X \le Y) + \Pr(X \ge Y) - \Pr(X = Y) = 1$, we have $\Pr(X \le Y) = 6/11 = 0.\overline{54}$. □

Problem 83. *Suppose X and Y are independent exponential random variables with (the same) parameter $\pi = 3.14159\ldots$. Find $\Pr(X \le Y)$ and $\Pr(X = Y)$.*

Solution. We have $\Pr(X = Y) = 0$ since it integrates over a null set. Then $\Pr(X \le Y) = \Pr(X \ge Y)$ by symmetry and so $\Pr(X \le Y) + \Pr(X \ge Y) = 1 - \Pr(X = Y) = 1$ implies $\Pr(X \le Y) = 1/2$. Interestingly, this gives a proof of the integral identity

$$\int_0^\infty \int_x^\infty \lambda^2 e^{-\lambda(x+y)}\, dy\, dx = \frac{1}{2}$$

for $\lambda > 0$. □

15. Conditioning, I

Problem 84. *Suppose X and Y are jointly continuous random variables whose joint density function is given by $f(x, y)$. Are X and Y independent?*

(1) $f(x, y) = xe^{-(x+y)}[x > 0][y > 0]$
(2) $f(x, y) = 2[0 < x < y < 1]$
(3) $f(x, y) = (x + y)[0 < x < 1][0 < y < 1]$

Solution. (1) Yes. A natural guess would be $f_X(x) = cxe^{-x}[x > 0]$ and $f_Y(y) = c^{-1}e^{-y}[y > 0]$; integration shows we can take $c = 1$ and have both functions be valid density functions.

(2) No. Suppose for contradiction that $f(x, y) = f_X(x)f_Y(y)$. Given $0 < a < 1$ there exists b and c satisfying $0 < b < a < c < 1$; then $f(a, c) = 2$ implies $f_X(a) > 0$ and $f(b, a) = 2$ implies $f_Y(a) > 0$. Thus f_X and f_Y are nonzero over $(0, 1)$, which implies that $f(x, y)$ is nonzero over $(0, 1) \times (0, 1)$, contrary to the definition of f.

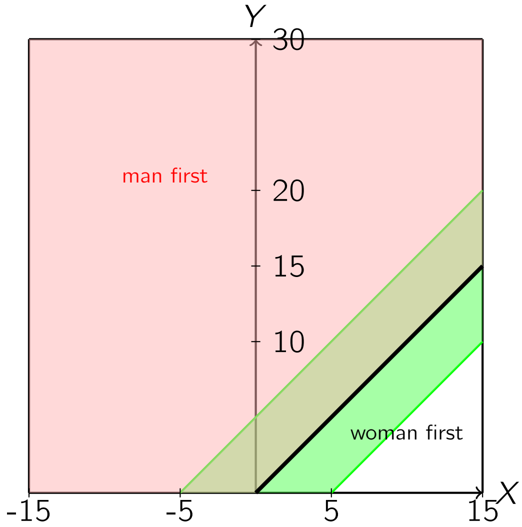
(3) No. Compute $f_X(x) = \int_{\mathbb{R}} f(x, y)\, dy = (x + 1/2)[0 < x < 1]$; similarly $f_Y(y) = (y + 1/2)[0 < y < 1]$. But clearly $f(x, y) \ne f_X(x)f_Y(y)$. □

Problem 85. *A man and a woman agree to meet at a location at 12 p.m. The man arrives at the location at a time uniformly distributed between 11:45 a.m. and 12:15 p.m. The woman arrives at the location at a time uniformly distributed between 12:00 p.m. and 12:30 p.m.*

(1) What is the probability that the first person to arrive waits less than five minutes for the second person?

(2) What is the probability that the man arrives first?

Solution. Let X and Y be the times that the man and woman arrive at, relative to 12 p.m., so that $X \sim \text{Uniform}([-15, 15])$ and $Y \sim \text{Uniform}([0, 30])$. We assume X and Y are independent. The situation is best understood geometrically:



The answer to (1), $\frac{1}{6} = 0.1\overline{6}$, is the area of the small shaded region beginning from the bottom between $X = -5$ and $X = 5$, divided by the area of the square; whereas the answer to (2), $\frac{7}{8} = 0.875$, is the area of the region of the square above the thick diagonal $Y = X$, divided by the area of the square. □

Problem 86. *A table is ruled with equidistant parallel lines at distance* $\sqrt{3}$ *cm apart. A needle of length 2 cm is randomly thrown on the table. What is the probability that the needle will intersect (at least) one of the lines?*

Solution. Imagine the lines drawn horizontally. Let X be the vertical distance between the midpoint of the needle and the closest line on the table, and let θ be the acute angle between the needle and the closest horizontal line. Then $X \sim \text{Uniform}([0, \sqrt{3}/2])$ and $\theta \sim \text{Uniform}([0, \pi/2])$, and we assume they are independent. The point is that an intersection occurs if and only if $\frac{L}{2}\sin\theta > X$. In this case $L = 2$, so this just becomes $\sin\theta > X$. The joint density function is $f(x, \theta) = \frac{4}{\pi\sqrt{3}}[0 \le x \le \frac{\sqrt{3}}{2}][0 \le \theta \le \frac{\pi}{2}]$, and we can integrate this over the region satisfying $X < \sin\theta$ to get the desired probability:

$$\frac{4}{\pi\sqrt{3}}\int_0^{\pi/3} \sin\theta \, d\theta + \frac{4}{\pi\sqrt{3}}\int_{\pi/3}^{\pi/2} \frac{\sqrt{3}}{2}\, d\theta = \frac{1}{3} + \frac{2}{\sqrt{3}\pi} = 0.70088593\ldots.$$

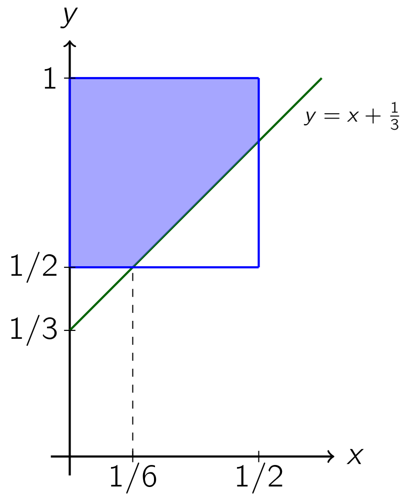
Write l for the needle length and t for the distance between parallel lines. We note that in general, the probability of the needle crossing a line is $\frac{2l}{t\pi}$ when $l \le t$, and $\frac{2}{\pi}\arccos\frac{t}{l} + \frac{2l}{\pi t}\left(1 - \sqrt{1 - (t/l)^2}\right)$ when $l > t$. (In Excel, use `=ACOS()`.) We finally note that a needle intersects two lines iff $\frac{L}{2}\cos\theta > X$ and $\frac{L}{2}\cos\theta > D - X$, where D is the distance between parallel lines. □

Problem 87. *One point is randomly selected on the interval* $[0, 1/2]$. *Another point is randomly selected on the interval* $[1/2, 1]$. *What is the probability that the distance between the two points is* $>1/3$*?*

Solution. We have independent random variables $X \sim \text{Uniform}([0, 1/2])$ and $Y \sim \text{Uniform}([1/2, 1])$, and we are interested in $\Pr(Y - X > 1/3)$. The joint density is $f(x, y) = 4[0 \le x \le 1/2][1/2 \le y \le 1]$, and we can integrate to get

$$\Pr(Y - X > 1/3) = \int_{Y > X + \frac{1}{3}} f(x, y)\, dx\, dy = \int_V 4\, dx\, dy = \frac{7}{9} = 0.\overline{7},$$

where V is the shaded region below.



□

Problem 88. *In each trial, two fair dice are tossed together, and the sum of their values is observed. Let N denote the number of trials needed until the sum observed is a multiple of 3. Let X denote the sum observed for the final trial. Find* $\Pr(X = 9)$, $\Pr(N = 4)$, $\Pr(N = 4$ *and* $X = 9)$, *and* $\Pr(N = 4 \mid X = 9)$.

Solution. Write $p(n)$ for the probability of getting a sum with value n. Then $(p(3), p(6), p(9), p(12)) = (\frac{2}{36}, \frac{5}{36}, \frac{4}{36}, \frac{1}{36})$, so the probability of stopping at a round is $p(3) + p(6) + p(9) + p(12) = 1/3$. We have

$$\Pr(X = 9) = \frac{p(9)}{p(3) + p(6) + p(9) + p(12)} = \frac{1}{3}.$$

We also have $\Pr(N = 4) = (1 - \frac{1}{3})^3(\frac{1}{3}) = 8/81 = 0.\overline{098765432}$, a geometric distribution; and $\Pr(N = 4 \text{ and } X = 9) = (1 - \frac{1}{3})^3 p(9) = \frac{8}{243} = 0.03292181\ldots$. Finally, from the previous results we can calculate

$$\Pr(N = 4 \mid X = 9) = \frac{\Pr(N = 4 \text{ and } X = 9)}{\Pr(X = 9)} = \frac{8}{81} = 0.\overline{098765432}.$$

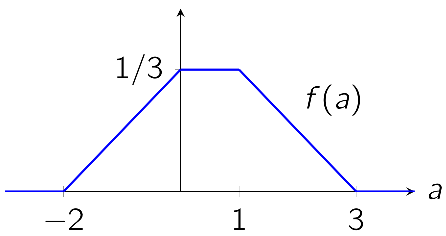
□

16. Expectations and moments, I

Problem 89. *Let X and Y be independent random variables uniformly distributed on the unit interval $[0,1]$. Determine the probability density function f of $3X-2Y$. Use this to find $f(-1.5)$, $f(0.5)$, and $f(1.5)$.*

Solution. We have $f_X = f_Y = 1_{[0,1]}$, so $f_{3X} = \frac{1}{3}1_{[0,3]}$ and $f_{-2Y} = \frac{1}{2}1_{[-2,0]}$. It follows that

$$\begin{aligned} f(a) = (f_{3X} * f_{-2Y})(a) &= \int_{-\infty}^{\infty} f_{3X}(a-y)f_{-2Y}(y)\,dy \\ &= \frac{1}{6}\int_{-\infty}^{\infty} [0 \le a-y \le 3][-2 \le y \le 0]\,dy \\ &= \frac{1}{6}\int_{a-3}^{a} [-2 \le y \le 0]\,dy \\ &= \frac{1}{6}\cdot \begin{cases} a+2 & \text{if } -2 \le a \le 0; \\ 2 & \text{if } 0 \le a \le 1; \\ 3-a & \text{if } 1 \le a \le 3; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



It follows that $f(-1.5) = 1/12$, $f(0.5) = 1/3$, and $f(1.5) = 1/4$. □

Problem 90. *Let X and Y be independent random variables following gamma distributions with parameters $(e, 4/\pi)$ and $(2e, 2/\pi)$ respectively.*

(1) Determine the probability density function f_{2X} of $2X$. Use this to find $f_{2X}(1)$ and $f_{2X}(\pi)$.

(2) Determine the probability density function f of $2X+Y$. Use this to find $f(8)$ and $f(10)$.

Solution. (1) Recall that the gamma distribution with parameters (α, λ) where $\alpha, \lambda > 0$ is given by the density function[2]

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} 1_{x\geq 0}. \tag{1}$$

Since $f_{2X}(a) = F'_{2X}(a)$ and

$$F_{2X}(a) = \Pr(2X \leq a) = \Pr(X \leq a/2) = \int_{-\infty}^{a/2} f_X(x)\,dx = F_X(a/2),$$

the chain rule gives

$$f_{2X}(a) = \frac{1}{2} f_X(a/2) = \frac{(2/\pi)e^{-2a/\pi}(2a/\pi)^{e-1}}{\Gamma(e)} 1_{a\geq 0}.$$

Thus

$$f_{2X}(1) = \frac{(2/\pi)e^{-2/\pi}(2/\pi)^{e-1}}{\Gamma(e)} = 0.09890350748\ldots$$

and

$$f_{2X}(\pi) = \frac{(2/\pi)e^{-2}2^{e-1}}{\Gamma(e)} = 0.18086176979\ldots.$$

(2) In general, $X \sim \text{Gamma}(\alpha, \lambda)$ implies $cX \sim \text{Gamma}(\alpha, \lambda/c)$ for $c > 0$. Thus $2X \sim \text{Gamma}(e, 2/\pi)$ and $Y \sim \text{Gamma}(2e, 2/\pi)$. Since their rate parameters are equal, they can be added to give $2X + Y \sim \text{Gamma}(2e + e, 2/\pi)$. Plugging these parameters into (1) gives

$$f(x) = \frac{(2/\pi)e^{-2x/\pi}(2x/\pi)^{3e-1}}{\Gamma(3e)} 1_{x\geq 0},$$

so we conclude that

$$f(8) = \frac{(2/\pi)e^{-16/\pi}(16/\pi)^{3e-1}}{\Gamma(3e)} = 0.06480826988\ldots$$

and

$$f(10) = \frac{(2/\pi)e^{-20/\pi}(20/\pi)^{3e-1}}{\Gamma(3e)} = 0.089545896402\ldots.$$

□

[2] The parameters α and λ are called the *shape* and *rate* respectively. Confusingly, it is also common to parameterize the gamma distribution with the *scale parameter* $\theta = 1/\lambda$ in place of λ.

Problem 91. *A man and a woman agree to meet at a location at 12 p.m. The man arrives at the location at a time normally distributed with mean 12:00 p.m. and standard deviation 5 minutes. The woman arrives at the location at a time normally distributed with mean 12:02 p.m. and standard deviation 3 minutes.*

(1) What is the probability that the first person to arrive waits less than 3 minutes for the second person?

(2) What is the probability that the man arrives first?

Solution. We consider independent random variables $X \sim N(0, 5^2)$ and $Y \sim N(2, 3^2)$. Here X and Y correspond respectively to the time taken for the man and woman to arrive, counting from 12:00 p.m.

(1) We use the fact that $A \sim N(\mu_A, \sigma_A^2)$ and $B \sim N(\mu_B, \sigma_B^2)$ with the two variables independent implies $A - B \sim N(\mu_A - \mu_B, \sigma_A^2 + \sigma_B^2)$. We are thus interested in $\Pr(-3 < Y - X < 3) = \Phi(3) - \Phi(-3)$ for the distribution $Y - X \sim N(2, 5^2 + 3^2)$. This is given by

```
=NORM.DIST(3,2,SQRT(5^2+3^2),TRUE)
 -NORM.DIST(-3,2,SQRT(5^2+3^2),TRUE);
```

which yields 0.372497867.

(2) This is $\Pr(Y - X > 0) = 1 - \Phi(0)$, which we can find using

```
=1-NORM.DIST(0,2,SQRT(5^2+3^2),TRUE);
```

this gives 0.634199706. □

Problem 92. *A sample of radioactive substance is observed to emit* 0.1234 *α-particles per second, on average. A sample of another substance is observed to emit* 0.2345 *α-particles per second, on average. The two samples are now combined.*

(1) What is the probability that at least three α-particles are emitted from the combined sample in a ten second interval?

(2) What is the shortest time interval we need to wait so that the probability of the combined sample emitting any α-particle in that interval is $>$90%*?*

Solution. We model this problem using *Poisson processes*. Write $N(t)$ and $M(t)$ for the number of α-particles observed in t seconds for the first and second samples respectively, so that $N(t) \sim \text{Poisson}(0.1234t)$ and $M(t) \sim \text{Poisson}(0.2345t)$. Then the number $N(t) + M(t)$ of α-particles observed in t seconds from the combined sample is a Poisson process whose rate is given by the sum of individual weights; that is, $N(t) + M(t) \sim \text{Poisson}(0.3579t)$.

(1) This probability is given by $\Pr(N(10) + M(10) \geq 3) = 1 - F(3)$, or

```
=1-POISSON.DIST(2,0.3579*10,TRUE),
```

which gives 0.69351758. Note that we set `x` to 2, since the Poisson distribution is discrete.

(2) We are interested in finding minimal t for which $\Pr(N(t) + M(t) \geq 1) > 0.9$. Now this is equivalent to $e^{-0.3579t} = \Pr(N(t) + M(t) = 0) < 0.1$, so $t > \ln(10)/0.3579 = 6.4335990\ldots$ works. □

Problem 93. *Suppose X and Y are discrete random variables whose joint probability mass function is given by $p(1,1) = 1/8$, $p(1,2) = 1/4$, $p(2,1) = 1/8$, and $p(2,2) = 1/2$. Find the conditional probability mass function $p_{X|Y}(x|y)$ of X given that $Y = y$; that is, evaluate $p_{X|Y}(1 \mid 1)$, $p_{X|Y}(1 \mid 2)$, $p_{X|Y}(2 \mid 1)$, and $p_{X|Y}(2 \mid 2)$.*

Solution. The marginal probabilities are $p_Y(1) = p(1,1) + p(2,1) = 1/4$, $p_Y(2) = p(1,2) + p(2,2) = 3/4$, $p_X(1) = p(1,1) + p(1,2) = 3/8$, and $p_X(2) = p(2,1) + p(2,2) = 5/8$. Thus $p_{X|Y}(1 \mid 1) = p(1,1)/p_Y(1) = 1/2$, $p_{X|Y}(1 \mid 2) = p(1,2)/p_Y(2) = 1/3$, $p_{X|Y}(2 \mid 1) = p(2,1)/p_Y(1) = 1/2$, and $p_{X|Y}(2 \mid 2) = p(2,2)/p_Y(2) = 2/3$. □

Problem 94. *A fair dice is rolled. Let N denote the number shown on the dice. A fair coin is then tossed N times. Let X denote the number of heads seen. Find $\Pr(X = 4 \mid N = 5)$, $\Pr(N = 5 \mid X = 4)$, and $\Pr(X = 4)$.*

Solution. Notice that $N \sim \text{Uniform}(\{1, \ldots, 6\})$ and $X \sim \text{Binomial}(N, 1/2)$. The probability $\Pr(X = 4 \mid N = 5)$ is just the usual binomial probability $\binom{5}{4}(1/2)^5 = 5/32 = 0.15625$. We compute

$$\begin{aligned}
\Pr(X = 4) &= \sum_{n=1}^{6} \Pr(X = 4 \mid N = n) \Pr(N = n) \\
&= \frac{1}{6} \sum_{n=4}^{6} \binom{n}{4} \left(\frac{1}{2}\right)^n \\
&= \frac{29}{384} = 0.07552083\overline{3}.
\end{aligned}$$

Finally, Bayes's rule gives

$$\begin{aligned}
\Pr(N = 5 \mid X = 4) &= \frac{\Pr(X = 4 \mid N = 5) \Pr(N = 5)}{\Pr(X = 4)} \\
&= \frac{(5/32)(1/6)}{(29/384)} = \frac{10}{29} = 0.3448275862\ldots.
\end{aligned}$$

□

17. Expectations and moments, II

Problem 95. *The random variables X and Y are jointly continuous with joint density function given by*

$$f(x,y) = \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)[0 < x < 1][0 < y < 2].$$

(1) Determine the conditional density function $f_{X|Y}(x \mid y)$ of X given $Y = y$. Use this to find $f_{X|Y}(1/3 \mid 1)$, $f_{X|Y}(2/3 \mid 1)$, and $\Pr(1/3 < X < 2/3 \mid Y = 1)$.

(2) Determine the conditional density function $f_{Y|X}(y|x)$ of Y given $X = x$. Use this to find $f_{Y|X}(1 \mid 1/3)$, $f_{Y|X}(1 \mid 2/3)$, and $\Pr(Y = 1 \mid 1/3 < X < 2/3)$.

Solution. (1) We compute the marginal distribution (for $0 \le y \le 2$):

$$f_Y(y) = \int_0^1 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)\,dx = \frac{1}{14}(4 + 3y).$$

Consequently

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{6x(2x+y)}{4+3y},$$

and so $f_{X|Y}(1/3 \mid 1) = 10/21 = 0.\overline{476190}$, $f_{X|Y}(2/3 \mid 1) = 4/3 = 1.\overline{3}$, and

$$\Pr\Big(\frac{1}{3} < X < \frac{2}{3} \,\Big|\, Y = 1\Big) = \int_{1/3}^{2/3} f_{X|Y}(x \mid 1)\,dx$$
$$= \frac{6}{7}\int_{1/3}^{2/3}(2x^2 + x)\,dx = \frac{55}{189} = 0.\overline{291005}.$$

(2) As before, we compute the marginal

$$f_X(x) = \int_0^2 \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big)\,dy = \frac{6}{7}x(1+2x)$$

for $0 \le x \le 1$, so that

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2x+y}{4x+2}.$$

Consequently $f_{Y|X}(1 \mid 1/3) = f_{Y|X}(1 \mid 2/3) = 1/2$. Finally, $\Pr(Y = 1 \mid 1/3 < X < 2/3) = 0$, since we are conditioning on an event of positive measure. □

Problem 96. *Let X, Y, and Z be independent standard normal random variables. Let*

$$T = \frac{Z}{\sqrt{(X^2+Y^2)/2}}.$$

Find $\Pr(-1 < T < 1)$ and $\Pr(T > 3)$.

Solution. All we do is realize that this is a t-distribution with two degrees of freedom. Then $\Pr(-1 < T < 1) = F(1) - F(-1)$, or

```
=T.DIST(1,2,TRUE)-T.DIST(-1,2,TRUE),
```

which gives 0.577350269; similarly $\Pr(T > 3) = 1 - F(3)$, or

```
=1-T.DIST(3,2,TRUE)
```

with value 0.047732983. □

Problem 97. *Let X and Y be independent standard normal random variables. Find* $\Pr(|X| < |Y|)$ *and* $\Pr(X > 3|Y|)$*.*

Solution. Observe that $T = X/|Y|$ is a t-distribution with one degree of freedom. We then have $\Pr(|X| < |Y|) = \Pr(-1 < T < 1) = F(1) - F(-1)$ and

```
=T.DIST(1,1,TRUE)-T.DIST(-1,1,TRUE)
```

(which has value exactly 0.5) and $\Pr(X > 3|Y|) = \Pr(T > 3)$, which can be found using

```
=1-T.DIST(3,1,TRUE),
```

giving 0.102416382. □

Problem 98. *Suppose the random variables X and Y have a bivariate normal distribution, in which:*

- *the marginal distribution of X is a standard normal, and*
- *the marginal distribution of Y has mean 2 and variance 3, and*
- *the correlation parameter ρ is $1/2$.*

(1) The condiitional distribution of X given that $Y = 2.5$ is distributed following $N(\mu_1, \sigma_1^2)$. Find the mean μ_1 and the variance σ_1^2.

(2) The conditional distribution of Y given that $X = 0.5$ is distributed following $N(\mu_2, \sigma_2^2)$. Find the mean μ_2 and the variance σ_2^2.

Solution. (1) Recall that

$$X|(Y = y) \sim N\Big(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y), \sigma_X^2(1 - \rho^2)\Big),$$

so that

$$\mu_1 = 0 + \frac{1}{2}\frac{1}{\sqrt{3}}(2.5 - 2) = \frac{1}{4\sqrt{3}} = 0.1443375672974\ldots$$

and

$$\sigma_1^2 = 1^2\big(1 - (1/2)^2\big) = \frac{3}{4} = 0.75.$$

(2) Much like before, we have

$$Y|(X = x) \sim N\Big(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X), \sigma_Y^2(1 - \rho^2)\Big),$$

so that

$$\mu_2 = 2 + \frac{1}{2}\frac{\sqrt{3}}{1}(0.5 - 0) = 2 + \frac{\sqrt{3}}{4} = 2.4330127\ldots$$

and

$$\sigma_2^2 = 3\big(1 - (1/2)^2\big) = \frac{9}{4} = 2.25.$$

□

Problem 99. *We are given a coin which can be tossed any number of times to generate independent outcomes of heads or tails. Its probability p of showing heads is assumed to be uniformly distributed on the unit interval $(0, 1)$.*

(1) Before tossing the coin at all, what is the probability that when we toss the coin 10 times, we see exactly 7 heads among them?

(2) We now toss the coin 10 times and see exactly 7 heads. What is the probability that p is between 0.65 and 0.75?

Solution. (1) We have $p \sim \text{Uniform}([0, 1])$. Let X be the number of heads observed from ten tosses of the p-biased coin. We know that $X|(p = \alpha) \sim \text{Binomial}(10, \alpha)$

for any $0 \leq \alpha \leq 1$. We also know that the conditional density function of $p|(X = 7)$ is given by

$$f_{p|X}(\alpha \mid X = 7) = \frac{\Pr(X = 7 \mid p = \alpha) f_p(\alpha)}{\Pr(X = 7)} = \frac{\Pr(X = 7 \mid p = \alpha)}{\Pr(X = 7)}[0 \leq \alpha \leq 1].$$

Putting this together with the fact that

$$\int_0^1 f_{p|X}(\alpha \mid X = 7)\, d\alpha = 1,$$

it follows that

$$\Pr(X = 7) = \int_0^1 \binom{10}{7} \alpha^7 (1 - \alpha)^3\, d\alpha = \frac{1}{11} = 0.\overline{09}.$$

(I think this is fairly intuitive — the probability of getting the desired outcome averages over the individual probabilities of that outcome over all of the possible initial configurations in a uniform way.)

(2) We compute

$$\begin{aligned}\Pr(0.65 \leq p \leq 0.75 \mid X = 7) &= \frac{1}{\Pr(X = 7)} \int_{0.65}^{0.75} \binom{10}{7} \alpha^7 (1 - \alpha)^3\, d\alpha \\ &= 1320 \int_{0.65}^{0.75} \alpha^7 (1 - \alpha)^3\, d\alpha \\ &= \frac{736651192417}{2560000000000} \\ &= 0.287754372\ldots.\end{aligned}$$

$\square$

Problem 100. *A point P is picked at random from the part of the unit disk lying in the right half plane, i.e., from the set*

$$\Big\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1 \text{ and } x > 0\Big\}.$$

(1) What is the probability that the line joining P to the origin makes an angle of less than 5° with the x-axis?

(2) What is the probability that the point P is closer to the boundary unit circle than to the origin?

Solution. We perform a change of coordinates $(r, \theta) \mapsto (r\cos\theta, r\sin\theta) = (x, y)$, with Jacobian derivative r, to obtain the joint density function $\phi(r, \theta) = 2r/\pi$ inside the right half disk. From this we get the marginal density functions

$$\phi_R(r) = \int_{-\pi/2}^{\pi/2} \phi(r, \theta)\, d\theta = 2r[0 \leq r \leq 1]$$

and

$$\phi_\Theta(\theta) = \int_0^1 \phi(r, \theta)\, dr = \frac{1}{\pi}[-\pi/2 \leq \theta \leq \pi/2].$$

(1) We compute

$$\Pr\Big(-\frac{5\pi}{180} \leq \Theta \leq \frac{5\pi}{180}\Big) = \int_{-5\pi/180}^{5\pi/180} \phi_\Theta(\theta)\, d\theta = \frac{1}{18} = 0.0\overline{5}.$$

(Notice that this corresponds to the obvious intuition that there are 180 degrees to choose from, and we are seeking the probability of our point lying in a 10° slice.)

(2) Compute

$$\Pr\Big(\frac{1}{2} \leq R \leq 1\Big) = \int_{1/2}^1 \phi_R(r)\, dr = \frac{3}{4} = 0.75.$$

(This also corresponds to the obvious intuition — the half-disk of radius $1/2$ has a quarter of the area of the half-disk of radius 1.) □

18. Conditioning, II

Problem 101. *Suppose the random variables X and Y have a joint density function given by*

$$f(x, y) = \frac{1}{x^2y^2}[x \geq 1][y \geq 1].$$

Determine the joint density function $g(u, v)$ of $U = XY$ and $V = X/Y$. Use this to find $g(1/2, 2)$, $g(2, 1/2)$, and $g(2, 2)$.

Solution. We consider the change of variables $\Phi(x, y) = (xy, x/y) =: (u, v)$. Its Jacobian is given by

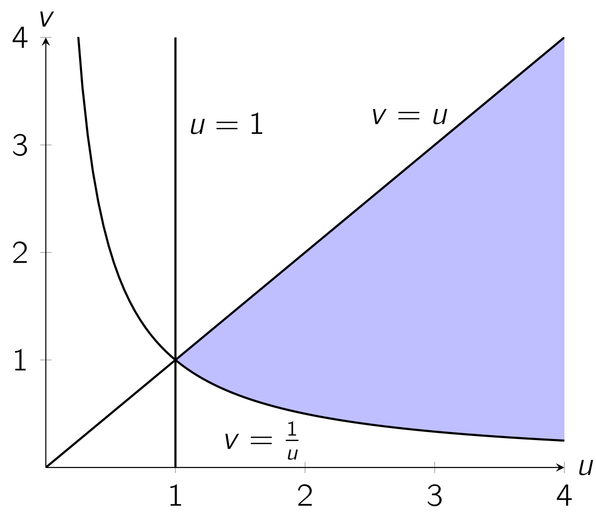
$$J = \det \begin{pmatrix} y & x \\ 1/y & -x/y^2 \end{pmatrix} = -\frac{2x}{y} = -\frac{2\sqrt{uv}}{\sqrt{u/v}},$$

so the joint density function is

$$\begin{aligned} g(u, v) = f(x, y) &= f(\Phi^{-1}(u, v))|J|^{-1} \\ &= \frac{1}{(\sqrt{uv})^2(\sqrt{u/v})^2} \frac{2\sqrt{u/v}}{\sqrt{uv}} = \frac{1}{2u^2v}, \end{aligned}$$

over the region

$$\left\{(u, v) \;\middle|\; \sqrt{uv} \geq 1 \text{ and } \sqrt{u/v} \geq 1\right\} = \left\{(u, v) \;\middle|\; u \geq 1 \text{ and } 1/u \leq v \leq u\right\}.$$



Thus $g(1/2, 2) = 0$ since $1/2 < 1$. Finally, we have

$$g(2, 1/2) = \frac{1}{2 \cdot 2^2 \cdot 1/2} = 0.25 \quad \text{and} \quad g(2, 2) = \frac{1}{2 \cdot 2^2 \cdot 2} = \frac{1}{16} = 0.0625.$$

$\square$

Problem 102. *Let X and Y be independent random variables uniformly distributed on the unit interval $[0,1]$. Find $\mathbf{E}(X/(X+Y))$, $\mathbf{E}(e^X \sin(\pi Y))$, $\mathbf{E}(\max(X,Y))$, and $\mathbf{E}(\min(X,Y))$.*

Solution. In general, if the (real-valued) random variables X and Y have joint distribution function $f(x,y)$, we have $\mathbf{E}(g(X,Y)) = \int_{\mathrm{R}^2} g(x,y)f(x,y)\,dm$. We have

$$\mathbf{E}\Big(\frac{X}{X+Y}\Big) = \int_{[0,1]^2} \frac{x}{x+y}\,dm = \frac{1}{2}$$

by symmetry. We have

$$\begin{aligned}\mathbf{E}(e^X \sin(\pi Y)) &= \int_{[0,1]^2} e^x \sin(\pi y)\,dm \\ &= \int_0^1 e^x\,dx \int_0^1 \sin(\pi y)\,dy \\ &= (e-1)\cdot\frac{2}{\pi} = 1.093892\ldots.\end{aligned}$$

We have

$$\begin{aligned}\mathbf{E}(\max(X,Y)) &= \int_0^1 \Pr(\max(X,Y) \geq t)\,dt \\ &= \int_0^1 1 - \Pr(\max(X,Y) < t)\,dt \\ &= \int_0^1 1 - \Pr(X < t)\Pr(Y < t)\,dt \\ &= \int_0^1 (1-t^2)\,dt = \frac{2}{3} = 0.\overline{6}.\end{aligned}$$

Finally, $\mathbf{E}(\min(X,Y)) = 1/3$ since $\min(X,Y) + \max(X,Y) = X + Y$ has expectation $1/2 + 1/2 = 1$. $\square$

Problem 103. *Suppose X and Y are jointly continuous random variables whose joint density function is given by*

$$f(x,y) = \frac{1}{y}[0 < y < 1][0 < x < y].$$

Find $\mathbf{E}(XY)$ and $\mathbf{E}(X+Y)$.

Solution. We compute

$$\begin{aligned}\mathbf{E}(XY) &= \int_{\mathbb{R}^2} xy\frac{1}{y}[0<y<1][0<x<y]\,dm \\ &= \int_0^1\int_0^y x\,dx\,dy \\ &= \int_0^1 \frac{y^2}{2}\,dy = \frac{1}{6} = 0.1\overline{6}\end{aligned}$$

and

$$\mathbf{E}(X+Y) = \int_0^1\int_0^y \frac{x+y}{y}\,dx\,dy = \int_0^1 \frac{y}{2}+y\,dy = \frac{3}{4} = 0.75.$$

□

Problem 104. *A biased coin has a probability of* 60% *for showing heads.*

(1) If we flip the coin 666 *times, what is the expected number of heads obtained?*

(2) If we flip the coin 666 *times, what is the expected number of changeovers? (A* changeover *occurs whenever an outcome differs from the one preceding it. For instance, if HHTHT is the outcome of five flips, there are three changeovers.)*

(3) What is the expected number of flips needed until we obtain 100 *tails?*

Solution. (1) This is the expectation of a random variable following the distribution Binomial$(666, 0.6)$, which is just $666 \times 0.6 = 399.6$.

(2) Let I_k be the indicator random variable for the event that a changeover occurs at the kth coin. Writing $C_k \in \{\mathrm{H}, \mathrm{T}\}$ for the outcome of the kth flip, linearity of expectation gives

$$\begin{aligned}\mathbf{E}\Big(\sum_{k=2}^{666} I_k\Big) &= \sum_{k=2}^{666}\mathbf{E}I_k \\ &= \sum_{k=2}^{666}\Pr(C_{k-1}\neq C_k).\end{aligned}$$

Now $C_{k-1}\neq C_k$ occurs when $C_{k-1}C_k \in \{\mathrm{HT}, \mathrm{TH}\}$, and thus the probability in the sum above is $2p(1-p)$, where $p = 0.6$. It follows that the desired expectation has value $665\cdot 2p(1-p) = 319.2$.

(3) The idea is that this is equivalent to 100 repeats of the event where we flip heads repeatedly until obtaining tails; i.e., a geometric distribution. Formally, we can define the random variable N_k to be the number of flips following the $(k-1)$st tail until the kth tail, so that the desired expectation becomes $\mathbf{E}(\sum N_k) = \sum \mathbf{E}(N_k)$. Then $\mathbf{E}(N_k) = 1/(1-p)$ (since our 'success' here is a tail which has probability $1-p$), so the desired expectation is $100/(1-p) = 250$. □

Problem 105. *Suppose that in this auditorium, there are:*

- *31 students in MA2116,*
- *72 students in ST2131, and*
- *10 students who are auditing (guests).*

We randomly select 15 students to form a group.

(1) What is the expected number of MA2116 students in the group?

(2) What is the expected number of auditing students in the group?

Solution. There are $n = 113$ students in total, so given any student, the expected number of students in the group who are him is $15/113$. Consequently the answer to (1) is $31 \cdot 15/113 = 465/113 = 4.1150442\ldots$ and the answer to (2) is $10 \cdot 15/113 = 1.327433628\ldots$. $\square$

19. Conditioning, III

Problem 106. *Every time a coupon is being collected, the coupon is equally likely to be any of the different colours available, independent of the coupons previously collected.*

(1) If there are 4 different colours available, what is the expected number of coupons collected until a complete set of all colours is obtained?

(2) If there are 5 different colours available, what is the expected number of coupons collected until a complete set of all colours is obtained?

Solution. In general, if T is the number of draws needed to collect all n colors, we have $T = T_1 + \cdots + T_n$, where T_k is the number of draws needed to get k colors *after collecting $k-1$ colors*. Now $\mathbf{E}T_k = (n-k+1)/n$, so $\mathbf{E}T = nH_n$ by linearity. For $1 \leq n \leq 13$, the values of nH_n are

$$\left(1, 3, \frac{11}{2}, \frac{25}{3}, \frac{137}{12}, \frac{147}{10}, \frac{363}{20}, \frac{761}{35}, \frac{7129}{280}, \frac{7381}{252}, \frac{83711}{2520}, \frac{86021}{2310}, \frac{1145993}{27720}\right).$$

$\square$

Problem 107. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with probability mass function given by*

$$\Pr(X_n = 0) = \Pr(X_n = 2) = \frac{1}{2}$$

for $n \geq 1$. The random variable $X = \sum_{n=1}^{\infty} X_n/3^n$ is said to have the Cantor distribution. *Find $\mathbf{E}(X)$.*

Solution. We have $\mathbf{E}X = \sum_{n=1}^{\infty} \mathbf{E}(X_n/3^n) = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$. $\square$

Problem 108. *The random variable X follows a binomial distribution with parameters 7 and $1/\pi$.*

(1) Find $\mathbf{E}(X(X-1)(X-2))$.

(2) Find $\mathbf{E}(X^3)$.

Solution. We have $\mathbf{E}X^{\underline{k}} = n^{\underline{k}}p^k$ and

$$\mathbf{E}X^k = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} n^{\underline{j}} p^j;$$

in particular

$$\mathbf{E}X^3 = np\Big(1 + 3(n-1)p + (n-2)(n-1)p^2\Big).$$

□

Problem 109. *Ten couples are being seated at a long table. The ten women are seated first along one side of the table, and then the ten men are assigned seats on the other side at random. The random variable X denotes the number of couples facing each other.*

(1) Find $\mathbf{E}(X^3)$.

(2) Find $\mathbf{E}(X^4)$.

Solution. Let $n = 10$. Write $X = \sum_{k=1}^{n} I_k$ where I_k is the indicator random variable for if the kth couple is seated facing one another. Notice that $\mathbf{E}I_k = 1/n$, so $\mathbf{E}X = 1$. Similarly $\mathbf{E}(I_j I_k) = \frac{1}{n}\frac{1}{n-1}$, so

$$\mathbf{E}\binom{X}{2} = \sum_{j<k} \mathbf{E}(I_j I_k) = \binom{n}{2}\frac{1}{n}\frac{1}{n-1} = \frac{1}{2!};$$

in general $\mathbf{E}\binom{X}{k} = \frac{1}{k!}$, and $\mathbf{E}X^{\underline{k}} = 1$, so $\mathbf{E}X^k = \varpi_k$ (for $k < n$), the kth Bell number. □

Problem 110. *A fair dice is rolled 20 times. Let X denote the number of 1's that occurred. Let Y denote the number of 5's or 6's that occurred. Find* $\operatorname{cov}(X, Y)$.

Solution. Recall that the covaraiance is a bilinear symmetric positive semidefinite ($\operatorname{cov}(X, X) \geq 0$) operator satisfying $\operatorname{cov}(X, X) = \operatorname{Var}(X)$ defined by

$$\operatorname{cov}(X, Y) = \mathbf{E}\Big((X - \mu_X)(Y - \mu_Y)\Big) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y).$$

Write $X = \sum X_i$ and $Y = \sum Y_j$, where $X_i =$ [the ith roll has value 1] and similarly for Y_j. Then we may compute (using independence of X_i and Y_j when $i \neq j$)

$$\operatorname{cov}(X, Y) = \sum_{i,j} \operatorname{cov}(X_i, Y_j) = \sum_i \operatorname{cov}(X_i, Y_i) = 20 \cdot -\frac{1}{18} = -\frac{10}{9},$$

since

$$\mathbf{E}(X_i Y_i) - \mathbf{E}(X_i)\mathbf{E}(Y_i) = 0 - \frac{1}{6} \cdot \frac{2}{6}.$$

□

20. Expectations and moments, III

Problem 111. *Let Z be a standard normal random variable. Let $W = 1 + eZ + \pi Z^2$, $X = 1 + \pi Z^2$, and $Y = -1 + eZ^2$. Find* $\operatorname{cov}(X, Y)$ *and* $\operatorname{cov}(W, Z)$.

Solution. Compute

$$\begin{aligned}
&\operatorname{cov}(1 + \pi Z^2, -1 + eZ^2) \\
&\quad = -\operatorname{cov}(1, 1) + e \operatorname{cov}(1, Z^2) - \pi \operatorname{cov}(Z^2, 1) + \pi e \operatorname{cov}(Z^2, Z^2) \\
&\quad = \pi e \operatorname{Var}(Z^2) \\
&\quad = \pi e\big(\mathbf{E}Z^4 - (\mathbf{E}Z^2)^2\big) \\
&\quad = \pi e(3 - 1^2) = 2\pi e \approx 17.0795
\end{aligned}$$

(see Problem 67 on page 29). Similar ideas express $\operatorname{cov}(W, Z)$ in terms of odd moments of the standard normal which all vanish, so the answer then is 0. □

Problem 112. *Suppose X and Y are jointly continuous random variables whose joint density function is given by*

$$f(x, y) = 2e^{-2x}/x[x \geq 0][0 < y < x].$$

Find $\operatorname{cov}(X, Y)$.

Solution. Since $\operatorname{cov}(X, Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y)$, we compute these terms first. To do this we need the marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^x 2e^{-2x}/x\, dy = 2e^{-2x}[x > 0],$$

and (for $y > 0$)

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{y}^{\infty} 2e^{-2x}/x\, dx.$$

Now we have

$$\mathbf{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{0}^{\infty} x \cdot 2e^{-2x}\, dx = \frac{1}{2}$$

and

$$\mathbf{E}Y = \int_{0}^{\infty} y \int_{y}^{\infty} 2e^{-2x}/x\, dx\, dy = \frac{1}{4}$$

(change the order of integration!); together with

$$\mathbf{E}(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)\, dy\, dx = \frac{1}{4},$$

we see that $\operatorname{cov}(X, Y) = 1/8$. □

Problem 113. *Let X_1, X_2, X_3, X_4 be a sequence of i.i.d. random variables with mean 0 and variance 1. Determine the correlation coefficients $\rho(X_1+X_2+X_3, X_2+X_3+X_4)$ and $\rho(X_1 + 2X_2, X_2 + 2X_3 + 3X_4)$.*

Solution. Recall that the *correlation coefficient* $\rho(X, Y)$ of two random variables is defined by

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{Var}(X) \operatorname{Var}(Y)}}.$$

Independence kills many of the terms so we get

$$\begin{aligned}
\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4) &= \frac{\operatorname{cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)}{\sqrt{\operatorname{Var}(X_1 + X_2 + X_3) \operatorname{Var}(X_2 + X_3 + X_4)}} \\
&= \frac{\operatorname{cov}(X_2, X_2) + \operatorname{cov}(X_3, X_3)}{\sqrt{3 \cdot 3}} \\
&= \frac{\operatorname{Var}(X_2) + \operatorname{Var}(X_3)}{3} = \frac{2}{3}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\rho(X_1 + 2X_2, X_2 + 2X_3 + 3X_4) &= \frac{\operatorname{cov}(X_1 + 2X_2, X_2 + 2X_3 + 3X_4)}{\sqrt{\operatorname{Var}(X_1 + 2X_2) \operatorname{Var}(x_2 + 2X_3 + 3X_4)}} \\
&= \frac{2 \operatorname{cov}(X_2, X_2)}{\sqrt{(1 + 2^2)(1 + 2^2 + 3^2)}} \\
&= \frac{2}{\sqrt{70}} \approx 0.239.
\end{aligned}$$

□

Problem 114. *The random variables X and Y are jointly continuous, with a joint density function given by*

$$f(x, y) = e^{-y}/y[0 < x < y].$$

Find $\mathbf{E}(X^2 \mid Y = 2)$ *and* $\mathbf{E}(X^3 \mid Y = 3)$.

Solution. We have $\mathbf{E}(X^2 \mid Y = 2) = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x \mid 2)\, dx$, where $f_{X|Y}(x \mid y) = f(x, y)/f_Y(y)$ is the conditional density function, and where $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$ is the marginal density function.

To this end we compute

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^y \frac{e^{-y}}{y}\, dx = e^{-y}[y > 0],$$

so that

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{y}[0 < x < y].$$

Then

$$\mathbf{E}(X^2 \mid Y = 2) = \int_0^2 \frac{x^2}{2}\, dx = \frac{4}{3}$$

and

$$\mathbf{E}(X^3 \mid Y = 3) = \int_0^3 \frac{x^3}{3}\, dx = \frac{27}{4}.$$

□

Problem 115. *A biased coin A with probability p of showing heads is flipped five times. (1) Given that exactly two of the first three flips are heads, what is the conditional expected number of heads obtained in the five flips of coin A? (2) Type A and B coins have head probabilities p and p' respectively. Three coins of type A and two coins of type B are put in a box. One coin is chosen from the box at random and it is flipped five times. Given that exactly two of the first three flips are heads, what is the conditional expected number of heads obtained in the five flips of the chosen coin?*

Solution. (1) Let X be the number of heads in five flips, and let Y be the number of heads in the first three flips. Then $\mathbf{E}(X \mid Y = 2)$ depends only on the outcome of the last two flips, and thus has value

$$\begin{aligned} &2\,\Pr(\mathrm{TT}) + 3\,\Pr(\mathrm{TH\ or\ HT}) + 4\,\Pr(\mathrm{HH}) \\ &\quad = 2(1-p)^2 + 3 \cdot 2p(1-p) + 4p^2 = 2 + 2p. \end{aligned}$$

(2) This is tricky! We have

$$
\begin{aligned}
&\mathbf{E}(X \mid Y=2)\\
&\quad = \mathbf{E}\big((X \mid Y=2) \mid A\big)\Pr(A \mid Y=2) + \mathbf{E}\big((X \mid Y=2) \mid B\big)\Pr(B \mid Y=2)\\
&\quad = (2+2p)\Pr(A \mid Y=2) + (2+2p')\Pr(B \mid Y=2),
\end{aligned}
$$

$$
\Pr(A \mid Y=2) = \frac{\Pr(Y=2 \mid A)\Pr(A)}{\Pr(Y=2 \mid A)\Pr(A) + \Pr(Y=2 \mid B)\Pr(B)} = \frac{\frac{3}{5}p^2 q}{\frac{3}{5}p^2 q + \frac{2}{5}p'^2 q'},
$$

and similarly

$$
\Pr(B \mid Y=2) = \frac{\frac{2}{5}p'^2 q'}{\frac{3}{5}p^2 q + \frac{2}{5}p'^2 q'}.
$$

□

21. Expectations and moments, IV

Problem 116. *A lab mouse is trapped in a cell containing three doors.*

- *Door A leads it to a tunnel that returns to the cell after two meters of travel.*
- *Door B leads it to a tunnel that returns to the cell after four meters of travel.*
- *Door C leads it to the exit after six meters of travel.*

The mouse has no memory of its past selections, and each time it selects doors A, B, C with probabilities 50%, 30%, 20% respectively. What is the expected distance that the mouse travels until it reaches the exit?

Solution. We have

$$
\begin{aligned}
\mathbf{E}D &= \Pr(A)\mathbf{E}(D \mid A) + \Pr(B)\mathbf{E}(D \mid B) + \Pr(C) + \mathbf{E}(D \mid C)\\
&= 50\% \cdot (2+\mathbf{E}D) + 30\% \cdot (4+\mathbf{E}D) + 20\% \cdot 6,
\end{aligned}
$$

which can be solved to give $\mathbf{E}D = 17$. □

Problem 117. *A group of four hunters are waiting for ducks to fly by. When a flock of ducks fly by, the number of ducks in the flock follows a Poisson distribution with mean 5. When the hunters see the flock, they fire at the same time, but each hunter chooses his target at random, independently of the others. Each hunter independently hits his chosen target with probability 60%. What is the expected number of ducks hit in a flock?*

Solution. Write $N \sim \text{Poisson}(\lambda = 5)$ and $p = 0.6$, and let X be the number of ducks hit in a flock. We are seeking $\mathbf{E}X$, but it is easier to first consider $Y := N - X$, the number of ducks not hit. We can write $N - X = \sum_{j=1}^{N} Y_j$, where Y_j indicates if the jth duck is not hit. Then

$$\mathbf{E}Y_j = \prod_{1 \le k \le 4} \Pr(j\text{th duck not hit by } k\text{th hunter}),$$

and each factor is equal to

$$\begin{aligned} &\Pr(j\text{th duck not selected by } k\text{th hunter}) + \Pr\left(\substack{j\text{th duck selected by } k\text{th hunter} \\ \text{and yet does not get hit}}\right) \\ &= \Big(1 - \frac{1}{N}\Big) + \frac{1}{N}(1 - p) = 1 - \frac{p}{N}. \end{aligned}$$

It follows that $\mathbf{E}Y_j = (1 - p/N)^4$. But here $N \ge 0$ is a discrete random variable, and so we are really conditioning on $N = n$ for some $n \ge 0$. Thus $\mathbf{E}(Y \mid N = n) = n(1 - p/n)^4$, and so

$$\mathbf{E}Y = \sum_{n=0}^{\infty} \mathbf{E}(Y \mid N = n) \Pr(N = n) = \sum_{n=1}^{\infty} n\Big(1 - \frac{p}{n}\Big)^4 e^{-\lambda} \frac{\lambda^n}{n!}.$$

This has no nice form. Numerically, we have $\mathbf{E}X = 5 - \mathbf{E}Y = 1.906883476\ldots$. □

Problem 118. *The building S17 in NUS has eight floors, including the ground floor. The number of people who enter an elevator on the ground floor follows a Poisson distribution with mean 3. Each person who enters the elevator on the ground floor is equally likely to get off the elevator at any of the seven floors above, independently of where the others get off. What is the expected number of stops that the elevator will make before all its passengers get off?*

Solution. Write $N \sim \text{Poisson}(\lambda = 3)$ and $p = 1/7$. As usual we exploit linearity of expectation. Write Y for the number of stops the elevator makes, so that $Y = \sum_{j=2}^{8} Y_j$, where Y_j indicates if the elevator stopped at floor j. Then

$$\begin{aligned} \mathbf{E}(Y_j \mid N = n) &= 1 - \Pr(\text{elevator does not stop at floor } j \mid n \text{ people entered}) \\ &= 1 - (1 - p)^n, \end{aligned}$$

so $\mathbf{E}(Y \mid N = n) = 7\big(1 - (1-p)^n\big)$ and consequently

$$\begin{aligned}\mathbf{E}Y &= \sum_{n\geq 1} \mathbf{E}(Y \mid N = n) \Pr(N = n)\\ &= \sum_{n\geq 1} 7\Big(1 - (1-p)^n\Big) e^{-\lambda}\frac{\lambda^n}{n!}\\ &= 7 - 7e^{-3/7} \approx 2.43992659728261.\end{aligned}$$

□

Problem 119. *A dice is biased in such a way that it shows:*

- *1, 2, 3 with probability 10% each,*
- *4, 5 with probability 20% each,*
- *6 with probability 30%.*

The dice is rolled ten times. For $i \in 1, 2, 3, 4, 5, 6$, we let N_i be the number of times the value i occurs in these ten rolls. Find $\mathbf{E}(N_6 \mid N_4 > 0)$ and $\mathbf{E}(N_6 \mid N_4 > 1)$.

Solution. We have $N_6 \sim \text{Binomial}(10, 0.3)$, so that $\mathbf{E}N_6 = 10(0.3) = 3$. Now

$$3 = \mathbf{E}N_6 = \mathbf{E}(N_6 \mid N_4 = 0) \Pr(N_4 = 0) + \mathbf{E}(N_6 \mid N_4 > 0) \Pr(N_4 > 0),$$

$\Pr(N_4 = 0) = (0.8)^{10}$, and $\Pr(N_4 > 0) = 1 - (0.8)^{10}$, so we are left to find two conditional expectations. The key is to realize that $N_6 | (N_4 = 0) \sim \text{Binomial}(10, 3/8)$, because $3/8$ is the probability of a roll yielding six given that it does not yield four. Then $\mathbf{E}(N_6 \mid N_4 = 0) = 10(3/8) = 15/4$, and we can solve to get $\mathbf{E}(N_6 \mid N_4 > 0) \approx 2.90978231$. Finding $\mathbf{E}(N_6 \mid N_4 > 1)$ is the same idea, though slightly more tedious. The main point is that $N_6 | (N_4 = 1) \sim \text{Binomial}(9, 3/8)$. □

Problem 120. *We are given a coin. It can be tossed any number of times to generate independent outcomes (heads vs tails). Its probability X of showing heads is assumed to be uniformly distributed on $(0,1)$. The random variable T denotes the number of tosses needed until a head occurs. Find $\Pr(T=k)$ for $k=2,3,4$.*

Solution. We are given $T|(X=p) \sim \text{Geometric}(p)$, where p is the success probability. Consequently we have

$$\begin{aligned}\Pr(T=k) &= \int_{-\infty}^{\infty} \Pr(T=k \mid X=p) f_X(p)\,dp \\ &= \int_0^1 \Pr(T=k \mid X=p)\,dp \\ &= \int_0^1 (1-p)^{k-1} p\,dp \\ &= \frac{1}{k(k+1)}.\end{aligned}$$

□

22. Limit theorems, I

Problem 121. *We are given a coin. It can be tossed any number of times to generate independent outcomes (heads vs tails). Its probability X of showing head is assumed to have a continuous distribution with density function given by*

$$f(x) = cx^3(1-x)^2[0<x<1].$$

The random variable N denotes the number of heads occuring when we toss the coin 10 times. (1) Find $\Pr(N=k)$ for $k=3,5,7$. (2) Find $\mathbf{E}N$. (3) Find $\operatorname{Var}(N)$.

Solution. (1) The constraint $\int_{-\infty}^{\infty} f(x)\,dx = 1$ quickly implies $c=60$. Now $N \sim \text{Binomial}(10, X)$, so $N|(X=x) \sim \text{Binomial}(10,x)$ and $\Pr(N=k \mid X=x) = \binom{10}{k} x^k (1-x)^{10-k}$. It follows that

$$\begin{aligned}\Pr(N=k) &= \int_0^1 \Pr(N=k \mid X=x) f(x)\,dx \\ &= \int_0^1 \binom{10}{k} x^k (1-x)^{10-k} \cdot 60x^3(1-x)^2\,dx \\ &= 60\binom{10}{k} \int_0^1 x^{k+3}(1-x)^{12-k}\,dx.\end{aligned}$$

It is convenient now to recall the *beta function*

$$\mathrm{B}(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

as the integral above is $\mathrm{B}(k+4, 13-k)$.

(2) The law of total expectation gives

$$\mathbf{E}N = \int_{-\infty}^{\infty} \mathbf{E}(N \mid X = x) f(x)\, dx = \int_0^1 10x \cdot 60x^3(1-x)^2\, dx$$
$$= 600\,\mathrm{B}(5,3) = \frac{40}{7} \approx 5.71.$$

(3) It suffices to find $\mathbf{E}N^2$. This is easy from observing $\mathbf{E}(N^2 \mid X = x) = 10x(1-x) + (10x)^2$ (using the fact that $\mathrm{Var}(N \mid X = x) = 10x(1-x)$).

Remarks. The random variable $\mathrm{Var}(N \mid X)$ is equal to $10X(1-X)$, and the random variable $\mathbf{E}(N \mid X)$ is equal to $10X$. It turns out that we have

$$\mathrm{Var}(N) = \mathbf{E}\big(\mathrm{Var}(N \mid X)\big) + \mathrm{Var}\big(\mathbf{E}(N \mid X)\big),$$

an identity known as the *law of total variance*. $\square$

Problem 122. *Let $U_1, U_2, \ldots$ be a sequence of independent random variables uniformly distributed on the unit interval $[0,1]$. For any $t \in [0,2]$, let the random variable N_t denote the number of U_i's that need to be summed up to exceed the value t:*

$$N_t := \min\Big\{ n \geq 0 \;\Big|\; U_1 + \cdots + U_n > t \Big\}.$$

Find $\mathbf{E}N_1, \mathbf{E}N_{1.5}$, and $\mathbf{E}N_2$.

Solution. Conditioning on $U_1 = s$, we see that for $t < 1$ we have

$$\mathbf{E}N_t = 1 + \int_0^t \mathbf{E}N_{t-s}\,ds = 1 + \int_0^t \mathbf{E}N_s\,ds,$$

whereas for $t \geq 1$ we have

$$\mathbf{E}N_t = 1 + \int_0^1 \mathbf{E}N_{t-s}\,ds = 1 + \int_{t-1}^t \mathbf{E}N_s\,ds.$$

Writing $n(t) = \mathbf{E}N_t$, we see that $n(t)$ solves $n'(t) = n(t)$ on $(0,1)$, and $n(0) = 1$, implying $n(t) = e^t$ on $(0,1)$. For $t \geq 1$ we have $n'(t) = n(t) - n(t-1)$. In general, we have

$$\mathbf{E}N_t = \sum_{k=0}^{\lfloor t \rfloor} \frac{(-1)^k}{k!}(t-k)^k e^{t-k},$$

which has as special cases the following:

$$\begin{aligned}
0 \leq t < 1 : &\quad e^t \\
1 \leq t < 2 : &\quad e^{t-1}(-t + e + 1) \\
2 \leq t < 3 : &\quad \frac{1}{2}e^{t-2}\left((t-2)^2 - 2e(t-1) + 2e^2\right) \\
3 \leq t < 4 : &\quad \frac{1}{6}e^{t-3}\left(-(t-3)^3 + 3e(t-2)^2 - 6e^2(t-1) + 6e^3\right)
\end{aligned}$$

□

23. Limit theorems, II

Problem 123. *We are given a coin. It can be tossed any number of times to generate independent outcomes (heads or tails). Its probability X of showing heads is assumed to be uniformly distributed on $(0,1)$.*

Before any toss of the coin is made: (1) What would be our best prediction on the number of heads occuring when the coin is tossed 10 times? (2) What is the mean squared error of this prediction?

We just tossed the coin five times, and we saw exactly three heads: (3) What would now be our best prediction of the number of heads occuring when the coin is tossed another 10 times? (4) What is the mean squared error of this prediction?

Solution. Write N for the number of heads occuring when the coin is tossed 10 times as in (1). By 'best prediction' we mean the expectation $\mathbf{E}N$, and by 'mean squared error' we mean the expectation of the square of the error, that is, $\mathbf{E}((N-\mu)^2) = \mathrm{Var}(N)$. So these are just the mean and variance, at least for our purposes. We work through the routine computation: Since $N|(X = x) \sim \mathrm{Binomial}(10, x)$, we have

$$\begin{aligned}
\Pr(N = k) &= \int_{-\infty}^{\infty} \Pr(N = k \mid X = x) f_X(x)\, dx \\
&= \int_0^1 \binom{10}{k} x^k (1-x)^{10-k}\, dx \\
&= \binom{10}{k} B(k+1, 11-k) \\
&= \frac{1}{11}.
\end{aligned}$$

In other words, $N \sim \mathrm{Uniform}(\{0, 1, \dots, 10\})$. Consequently $\mathbf{E}N = \sum_{k=0}^{10} k \cdot \frac{1}{11} = 5$, and $\mathrm{Var}(N) = \sum_{k=0}^{10}(k-5)^2 \frac{1}{11} = 10$.

Write N' for the new number of heads occuring when the coin is tossed another 10 times, as in (3). Write X' for our current knowledge of the proability of heads; that is, X conditioned on the event where we saw three heads after tossing the coin five times. We have $N'|(X' = x) \sim \mathrm{Binomial}(10, x)$ and $\mathbf{E}(N' \mid X' = x) = 10x$, and $f_{X'}(x) = 60x^3(1-x)^2[0 \le x \le 1]$ (see Example 5e in Chapter 6 of Ross). Then the rest is exactly as in Problem 121 on page 62. □

Problem 124. *A sender sends a signal whose value S is normally distributed with mean π and variance e^2. It is known that if the actual signal value sent is $S = s$, the receiver receives a value R that is normally distributed with mean s and variance* 1.

(1) If the receiver receives a signal value of $R = 2$, what is the best estimate of the signal value S that was sent?

(2) What is the mean squared error of this prediction?

(3) Find $\mathrm{cov}(S, R)$.

Solution. (1) is asking for $\mathbf{E}(S \mid R = 2)$ and (2) is asking for $\mathrm{Var}(S \mid R = 2)$. I cannot beat the treatment in Example 6b in Chapter 7 of Ross, so I refer the reader there. (3) We have $R \sim N(\mu, \sigma^2 + 1)$ and $S \sim N(\mu, \sigma^2)$, so $\mathrm{cov}(R, S) = \rho\sqrt{\mathrm{Var}(R)\,\mathrm{Var}(S)} = \sigma^2$, where $\rho = \sigma/\sqrt{\sigma^2+1}$ is the correlation coefficient. Once again the reader is referred to Chapter 7 of Ross for details; see in particular the discussion on bivariate normals. □

Problem 125. *The random variable X is uniformly distributed on $(0,1)$. The random variable Y is the optimal discretizer / quantizer (with respect to the mean squared error) of X over the subintervals $(0,1/3)$ and $[1/3,1)$.*
(1) What is the value of Y when $X \in (0,1/3)$?
(2) What is the value of Y when $X \in [1/3,1)$?
(3) Find $\mathbf{E}(Y-X)^2$.

Solution. Recall that the *optimal discretizer Y* is a discrete random variable that approximates X over a specified subinterval in a way that minimizes the mean squared error. (1) This is asking for $\mathbf{E}(X \mid X \in (0,1/3))$, which is given by the integral

$$\int_0^{1/3} x f_{X\mid(X\in(0,1/3))}(x)\,dx = \int_0^{1/3} x \cdot 3\,dx = \frac{1}{6}.$$

(2) We have

$$\mathbf{E}\big(X \mid X \in [1/3,1)\big) = \int_{1/3}^{1} x \cdot \frac{1}{1-1/3}\,dx = \frac{2}{3}.$$

(3) Use the law of total probability to write $\mathbf{E}(Y-X)^2$ as

$$\begin{aligned}
&\mathbf{E}((Y-X)^2 \mid X \in (0,1/3))\Pr(X \in (0,1/3)) \\
&\quad + \mathbf{E}((Y-X)^2 \mid X \in [1/3,1))\Pr(X \in [1/3,1)) \\
&= \frac{1}{3}\operatorname{Var}\big(\text{Uniform}((0,1/3))\big) + \frac{2}{3}\operatorname{Var}\big(\text{Uniform}([1/3,1))\big).
\end{aligned}$$

The rest is easy to find, noting that the variance of Uniform$((a,b))$ is given by $\frac{1}{12}(b-a)^2$. □

Problem 126. *Suppose X and Y are independent random variables, whose moment generating functions are respectively*

$$M_X(s) = \exp(e^{s+1} - e) \quad \textit{and} \quad M_Y(t) = (e^{t-1} + 1 - 1/e)^4.$$

Find $\Pr(X+Y=2)$ and $\Pr(XY=0)$.

Solution. Recall that the *moment generating function M_X* of a random variable X is the function $s \mapsto \mathbf{E}(e^{sX})$ (in an interval containing $s=0$). Its name is justified by the fact that it is equal to $\sum_k t^k m_k/k!$, where $m_k = \mathbf{E}X^k$ is the kth moment of X. Thus $m_k = M_X^{(k)}(0)$, where $f^{(k)}$ denotes the kth derivative of a function f.

In this case we simply lookup our table of moment generating functions to find that $X \sim \text{Poisson}(e)$ and $Y \sim \text{Binomial}(4, 1/e)$. We have

$$\begin{aligned}
&\Pr(X+Y=2)\\
&= \Pr(X=0)\Pr(Y=2) + \Pr(X=1)\Pr(Y=1) + \Pr(X=2)\Pr(Y=0)\\
&= e^{-e}\binom{4}{2}\left(\frac{1}{e}\right)^2\left(1-\frac{1}{e}\right)^2 + e^{-e}e\binom{4}{1}\left(\frac{1}{e}\right)\left(1-\frac{1}{e}\right)^3 + e^{-e}\frac{e^2}{2}\binom{4}{0}\left(1-\frac{1}{e}\right)^4\\
&\approx 0.1270042490.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Pr(XY=0) &= \Pr(X=0) + \Pr(Y=0) - \Pr(X=0)\Pr(Y=0)\\
&= \left(1-\frac{1}{e}\right)^4 + e^{-e} - \left(1-\frac{1}{e}\right)^4 e^{-e}\\
&\approx 0.2151136.
\end{aligned}$$

□

24. Review, I

Problem 127. *The random variable S has the log-normal distribution with parameters $\mu = \pi$ and $\sigma^2 = e^2$. Find $\mathbf{E}S$ and $\operatorname{Var}(S)$.*

Solution. We are given that $X = \log S \sim N(\pi, e^2)$. We know that the moment generating function of a normal $Y \sim N(\mu, \sigma^2)$ is given by $M_Y(t) = \exp(\mu t + \sigma^2 t^2/2)$, so $M_X(t) = \exp(\pi t + e^2 t^2/2)$.

We have $\mathbf{E}S = \mathbf{E}e^X = M_X(1) = \exp(e^2/2 + \pi) \approx 930.871$. Similarly to find the variance we compute $\mathbf{E}S^2 = \mathbf{E}e^{2X} = M_X(2) = \exp(2e^2 + 2\pi)$, which gives $\operatorname{Var}(S) = \mathbf{E}S^2 - (\mathbf{E}S)^2 \approx 1401318386.4$. □

Problem 128. *The random variable X has moment generating function given by*

$$M_X(t) = \frac{1}{10}e^t + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}.$$

Find $\mathbf{E}X$ and $\mathbf{E}X^2$.

Solution. We have $\mathbf{E}X = M_X'(0) = (1+4+9+16)/10 = 3$ and $\mathbf{E}X^2 = M_X''(0) = (1+8+27+64)/10 = 10$.

Note in general that a random variable taking values in a discrete set $S \subset \mathbf{R}$ with probability mass function $p_X(x) = \Pr(X = x)$ satisfies $M_X(t) = \sum_{s \in S} p_X(s)e^{st}$. □

Problem 129. *The random variable X has moment generating function given by $M_X(t) = (2 - e^t)^{-3}$. Find $\mathbf{E}X$ and $\mathbf{E}X^2$.*

Solution. Tedious calculations reveal that

$$M_X'(t) = \frac{3e^t}{(e^t - 2)^4} \quad \text{and} \quad M_X''(t) = -\frac{3e^t\,(3e^t + 2)}{(e^t - 2)^5},$$

which imply that $\mathbf{E}X = M_X'(0) = 3$ and $\mathbf{E}X^2 = M_X''(0) = 15$. □

Problem 130. *Suppose X and Y are independent random variables, whose moment generating functions are respectively*

$$M_X(s) = \exp(e^{s+1} - e) \quad \textit{and} \quad M_Y(t) = (e^{t-1} + 1 - 1/e)^4.$$

Find $\mathbf{E}(XY)$, $\mathbf{E}(X^2Y)$, and $\mathbf{E}(XY^2)$.

Solution. Looking up a table, we find $X \sim \text{Poisson}(e)$ and $Y \sim \text{Binomial}(4, 1/e)$. Now one could compute these quantities by exploiting the independence of X^m and Y^n, but we note the following method: By defining the joint mgf $M_{X,Y}(s,t) := M_X(s)M_Y(t)$, we can use the identity

$$\frac{\partial^{k+l}}{\partial s^k \partial t^l} M_{X,Y}(s,t)\Big|_{s=0,t=0} = \mathbf{E}(X^kY^l).$$

The actual computation is left as an exercise; the final answers are 4, $4e+4$, and $12/e+4$. □

Problem 131. *The random variables X and Y have joint moment generating function given by*

$$M_{X,Y}(s,t) = \frac{4}{10}e^{s+t} + \frac{3}{10}e^{s+2t} + \frac{2}{10}e^{2s+3t} + \frac{1}{10}e^{3s+2t}.$$

Find $\mathbf{E}X$, $\text{Var}(X)$*, $\mathbf{E}Y$,* $\text{Var}(Y)$*, and* $\text{cov}(X,Y)$*.*

Solution. We invoke the definition of the joint moment generating function to compute

$$\begin{aligned} M_{X,y}(s,t) = \mathbf{E}(e^{sX+tY}) &= \sum_{(x,y)} \Pr(X = x, Y = y)e^{sx+ty} \\ &= \frac{4}{10}e^{s+t} + \frac{3}{10}e^{s+2t} + \frac{2}{10}e^{2s+3t} + \frac{1}{10}e^{3s+2t}. \end{aligned}$$

We then have

$$\mathbf{E}X = \frac{\partial}{\partial s} M_{X,Y}(s,t)\Big|_{s=t=0} = \frac{1}{10}(4+3+4+3) = \frac{7}{5}$$

and

$$\mathbf{E}X^2 = \frac{\partial^2}{\partial s^2} M_{X,Y}(s,t)\Big|_{s=t=0} = \frac{12}{5},$$

from which we have $\mathrm{Var}(X) = 12/5 - (7/5)^2$. Similarly we can find $\mathbf{E}Y$ and $\mathrm{Var}(Y)$. And $\mathrm{cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y)$ can be found using the fact that

$$\mathbf{E}(XY) = \frac{\partial^2}{\partial s \partial t} M_{X,Y}(s,t)\Big|_{s=t=0}.$$

□

Problem 132. *Let $X_1, \dots, X_{10}$ be i.i.d. standard normal random variables. Write $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$ for their sample mean, and $S^2 = \frac{1}{9}\sum_{i=1}^{10}(X_i - \bar{X})^2$ for their (unbiased) sample variance.*

(1) Find $\Pr(\bar{X} > 0.1$ and $S^2 < 1)$.

(2) Find $\Pr(|\bar{X}| < 0.05$ and $S^2 > 1.5)$.

Solution. We know that $\bar{X}$ and S^2 are independent, and we have

$$\bar{X} \sim N\Big(\mu = 0, \sigma_{\bar{X}}^2 = \frac{1}{10}\sigma^2 = \frac{1}{10}\Big)$$

and

$$\frac{9S^2}{\sigma^2} = 9S^2 \sim \chi^2(\text{degrees of freedom} = n - 1 = 9).$$

We can then compute $\Pr(\bar{X} > 0.1)$ by

```
=1-NORM.DIST(0.1,0,SQRT(1/10),TRUE)
```

and $\Pr(S^2 < 1)$ by `=CHISQ.DIST(9.0,9,TRUE)` to solve (1), which is the product of these two values. Similarly for (2), we compute $\Pr(-0.05 < \bar{X} < 0.05)$ by

```
=NORM.DIST(0.05,0,SQRT(1/10),TRUE)
-NORM.DIST(-0.05,0,SQRT(1/10),TRUE)
```

and $\Pr(S^2 > 1.5) = \Pr(9S^2 > 9 \cdot 1.5)$ by

```
=1-CHISQ.DIST(9*1.5,9,TRUE)
```

to get the answer to (2) by multiplying both probabilities. □

25. Review, II

Problem 133. *(Lesson 13) Three points X, Y and Z are selected independently at random from the unit interval $[0,1]$. What is the probability that Y lies between X and Z?*

Solution. By symmetry, each possible ordering of X, Y, Z has equal probability. There are six possible orderings, of which two have Y between X and Z; thus the desired probability is $2/6 = 1/3$.

Cryptic comment: Rotate a cube so that it looks like a hexagon, then divide that hexagon into six equilateral triangles. □

Problem 134. *(Lesson 17) Suppose that in this auditorium, there are:*

- *31 students in MA2116,*
- *72 students in ST2131, and*
- *10 students who are auditing (guests).*

We randomly select 15 students to form a group. The random variable X denotes the number of MA2116 students in this group. Find $\mathbf{E}(X(X-1)(X-2))$ *and* $\mathbf{E}X^3$*.*

Solution. Write $n = 15$, $m = 31$ and $N = 113$. We found in Problem 105 on page 54 that $\mathbf{E}X = n \cdot m/N$, by writing $X = \sum_{i=1}^{n} X_i$ with X_i indicating if the ith selected student is an MA2116 student. Now $\mathbf{E}\binom{X}{2}$ is the expected number of pairs of MA2116 students among the n randomly selected students, so we may write $\binom{X}{2} = \sum_{1 \le i < j \le n} Y_{ij}$, where Y_{ij} indicates if both the ith and jth student selected are taking MA2116. Since $\mathbf{E}Y_{ij} = \frac{m(m-1)}{N(N-1)}$, we deduce that $\mathbf{E}\binom{X}{2} = \binom{n}{2}\frac{m(m-1)}{N(N-1)}$, from which $\mathbf{E}X^2$ can be found. In this way for $r < 15$ we have the formulas

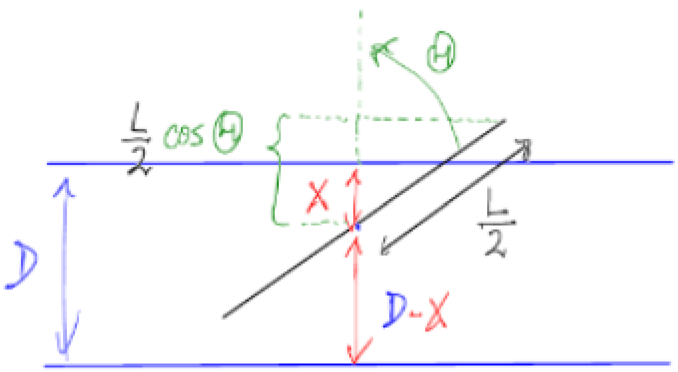
$$\mathbf{E}\binom{X}{r} = \binom{n}{r}\frac{m^{\underline{r}}}{N^{\underline{r}}}, \quad \mathbf{E}X^{\underline{r}} = \frac{n^{\underline{r}}m^{\underline{r}}}{N^{\underline{r}}}, \quad \text{and} \quad \mathbf{E}X^r = \sum_{k=1}^{r} \left\{ {r \atop k} \right\} \frac{n^{\underline{k}}m^{\underline{k}}}{N^{\underline{k}}}.$$

□

Extra problems

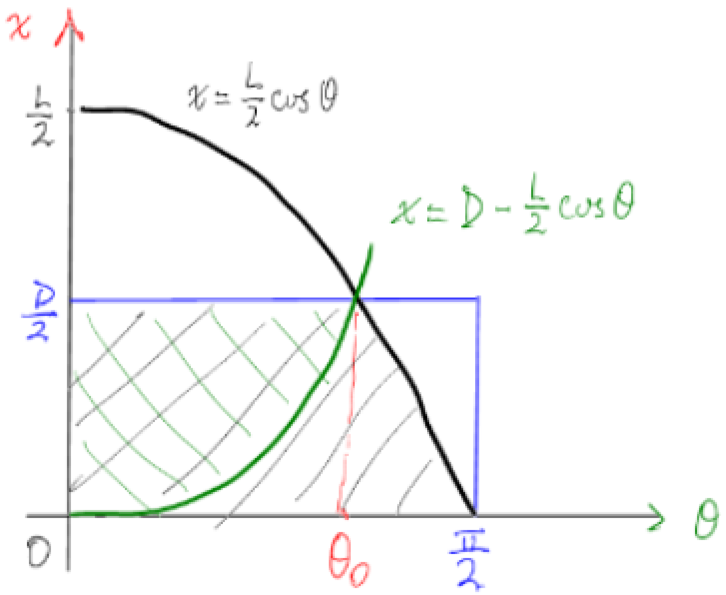
Problem 135. *A table is ruled with equidistance parallel lines at distance 1 cm apart. A needle of length 2 cm is randomly thrown on the table. What is the probability that the needle intersects two lines on the table simultaneously?*

Solution. Set $D = 1$ and $L = 2$. We have $X \sim \text{Uniform}([0, D/2])$ and $\Theta \sim \text{Uniform}([0, \pi/2])$, and the needle intersects two lines iff $\frac{L}{2}\cos\Theta > X$ and $\frac{L}{2}\cos\Theta > D - X$.



Now the density function $f(\theta, x)$ is $4/\pi D$ in the rectangle and 0 everywhere else. We can then calculate

$$\Pr\Big(\frac{L}{2}\cos\Theta > X \text{ and } \frac{L}{2}\cos\Theta > D - X\Big) = \int_0^{\theta_0}\int_{D-\frac{L}{2}\cos\theta}^{D/2} \frac{4}{\pi D}\,dx\,d\theta$$
$$= \frac{2L}{\pi D}\sin\theta_0 - \frac{2\theta_0}{\pi} \approx 43.60\%.$$



∎

Problem 136. *The total rainfall in a certain region follows a normal distribution with mean 25 mm and variance 25 mm^2. Assume that the total rainfall in different years are independent. What is the probability that more than ten consecutive years pass without encountering a year with greater than 30 mm rainfall?*

Solution. Write X for the rainfall in a year, so that $X \sim N(\mu = 25, \sigma^2 = 25)$. Then $\Pr(X > 30) \approx 0.158655254 =: p$ via `=1-NORM.DIST(30,25,5,TRUE)`, so the desired event has probability $(1-p)^{10} \approx 0.177721459$ using the function `=POWER()`. ∎

Problem 137. *Trains heading to destination A arrive at the station at 15 minute intervals starting at 7 a.m. Trains heading to destination B arrive at 25 minute intervals starting at 7:05 a.m.*

A man arrives at the station at a time uniformly distributed between 7 a.m. and 8 a.m., and takes the first train that arrives. What is the probability that he goes to destination A?

Solution. Let us consider the time interval from 0700 to 0800. A-trains arrive at 0700, 0715, 0730, 0745, and 0800. B-trains arrive at 0705, 0730, and 0755. Putting the two together on a timeline gives

$$\begin{array}{ccccccc} A & B & A & AB & A & B & A \\ 0700 & 0705 & 0715 & 0730 & 0745 & 0755 & 0800. \end{array}$$

The man definitely goes to A if he arrives in the time intervals (0705..0715), (0730..0745), or (0755..0800), which constitutes $(10 + 15 + 5)/60 = \frac{1}{2}$ of the entire hour. We interpret the interval (0715..0730) as giving the man probability 1/2 of going to A, though strictly speaking this is an ambiguity in the question. This contributes $\frac{15}{60} \cdot \frac{1}{2} = \frac{1}{8}$ to the probability of him going to A. Thus the desired probability is $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$. □

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