Eventually almost periodic locally bounded functions are sublinear

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Given a function $f\colon [0,\infty) \to \mathbf{R}$ that is *eventually almost periodic* in the sense that $\lim_{x\to\infty}(f(x+1)-f(x)) = 0$ and *locally bounded* on every interval $[n, n+1]$, we prove that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0.$$

Let $\epsilon > 0$, and choose N with $|f(x+1) - f(x)| < \epsilon$ whenever $x \geq N$. Let M_n be a bound for f on the interval $[n, n+1]$ chosen such that $M_n > n\epsilon$. We have $|f(x)| < M_N$ on $[N, N+1]$ by definition, and $|f(x)| < M_N + \epsilon$ on $[N+1, N+2]$ with the triangle inequality. Thus $|f(x)| < M_N/N$ on $[N, N+1]$ and $|f(x)| < \frac{M_N+\epsilon}{N+1}$ on $[N+1, N+2]$. Continuing in this fashion, we see that

$$|f(x)| < M_N + k\epsilon \quad \text{and} \quad \left|\frac{f(x)}{x}\right| < \frac{M_N + k\epsilon}{N+k}$$

on $[N+k, N+k+1]$ with $k \geq 0$. Now

$$\begin{aligned} \lim_{x\to\infty} \left|\frac{f(x)}{x}\right| &\leq \lim_{k\to\infty} \frac{M_N + k\epsilon}{N+k} \\ &= \lim_{k\to\infty} \frac{\frac{M_N}{k} + \epsilon}{\frac{N}{k} + 1} \\ &= \epsilon, \end{aligned}$$

and so the result follows as ϵ was arbitrary.