Learning some arithmetic Ramsey theory [draft]

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These are some of my thoughts and solutions to exercises as I work through Terence Tao's expository notes on arithmetic Ramsey theory this holiday season. The notes were taken from `http://www.math.ucla.edu/%7Etao/preprints/Expository/ramsey.dvi`

As it stands, my plan for 2022 will not give me the time to explore combinatorics. Eventually I would like to learn much more about Ramsey theory, additive combinatorics, and many more things. But for now, this short holiday sabbatical will have to suffice.

2. Ramsey's theorem and Schur's theorem

Throughout this document, we use the notation $[n] := \{1, \ldots, n\}$. We also use $[a..b]$ to denote the set $\{a, a+1, \ldots, b\}$.

Theorem 1 (Schur's theorem, infinitary form)**.** *Suppose the positive integers $\mathbf{Z}^+$ are finitely colored and let $k \geq 1$ be arbitrary. Then there exist infinitely many monochromatic sets in $\mathbf{Z}^+$ of the form $\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}$ (the x_i need not be distinct).*

Proof. By Schur's theorem, there exists a positive integer N and a monochromatic set of the desired form in $[1..N]$. Introduce new colors to color the monochromatic set so that each element of the set is the only positive integer with a particular color. Then, applying Schur's theorem again, we obtain a new monochromatic set of the desired form, and by construction it cannot contain any elements of the original monochromatic set. We may repeat this process as many times as we want, and thus there must be infinitely many sets of the form $\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}$. $\square$

Clearly the finitary form implies the infinitary form. Conversely, suppose the infintary form holds, and suppose for contradiction that for every N there exists a coloring $\Phi_N \colon [N] \to [m]$ that avoids monochromatic sets of the form above. Then, since $\Phi_N(k)$ stabilizes along some subsequence for every fixed k, we may apply a diagonalization argument to obtain a subsequence of (Φ_N) that converges pointwise to a coloring $\Phi \colon \mathbf{Z}^+ \to [m]$ avoiding monochromatic sets of the form above.

Let us explore the above argument in more detail. We will construct a subsequence (f_j) of (Φ_N) such that the first j colors are stable from f_j onwards; that is, $f_j|_{[j]} = f_k|_{[j]}$ whenever $k \geq j$. Since $\Phi_N(1)$ is an infinite sequence in $[m]$, the pigeonhole principle tells us that some color must occur infinitely often. Let $1 \leq c \leq m$ be the smallest such color, let Φ_{N_j} be the subsequence of (Φ_N) satisfying $\Phi_{N_j}(1) = c$, and define $f_1 := \Phi_{N_1}$. Now we repeat this argument with the sequence $\Phi_{N_j}(2)$, obtaining a further subsequence on which the first two colors are stable. Repeating in this fashion, we obtain the desired subsequence (f_j), which is easily seen to converge to a coloring

$\Phi\colon \mathbf{Z}^+ \to [m]$. This sequence cannot contain any monochromatic sets of the above form; indeed, since any monochromatic set of the form $\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}$ necessarily lies in some $[N]$, it would then also lie in f_N, which is impossible as f_N belongs to the sequence (Φ_N) which was assumed to avoid monochromatic sets of the above form.

Proposition 1. *Suppose $\mathbf{Z}^+$ is finitely colored. Then there exist infinitely many* distinct *integers x and y such that $\{x, y, x+y\}$ are monochromatic.*

Proof. Let $c\colon \mathbf{Z}^+ \to [m]$ be a finite coloring of the positive integers. Define a new coloring $c'\colon \mathbf{Z}^+ \to [2m]$ as follows: write $k = 2^r \cdot s$ with s odd. If r is odd, then define $c'(k) := 2c(k)$; otherwise, define $c'(k) := c(k)$. We say that the integers k with odd r are colored with tinted colors; given a color $j \in [m]$, we think of the color $j + m \in [2m]$ as its tinted version. Then no monochromatic sets of the form $\{x, 2x\}$ arise in c', as all such pairs consist of one untinted and one tinted number. Thus, by Schur's theorem, there exist infinitely many monochromatic sets of the form $\{x, y, x+y\}$ with x, y distinct in this new coloring. To complete the proof, we must transport infinitely many of these sets back to the original coloring. Either a set is untinted, in which case it also arises in c as its coloring has not changed, or it is tinted, in which case the set arises in c in its untinted form. Thus we have transported all the monochromatic sets of our desired form back to the original coloring, and we are done. $\square$

This result holds for sets of the form $\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}$, though the proof is much more involved. I have not worked out the details, but my thoughts for the $k = 3$ case are that we have to somehow deal with obstructions of the form $x + 2y = z$ and $3x = y$ in a compatible way, perhaps using multiple levels of tinting.

Proposition 2 (Schur's theorem, multiplicative variant)**.** *Suppose the positive integers $\mathbf{Z}^+$ are finitely colored, and let $k \geq 1$. Then there exist infinitely many monochromatic sets of the form $\{x_1, \ldots, x_k, x_1 \ldots x_k\}$.*

Proof. It suffices to prove the finitary form, as we may then argue as in Theorem 1 to obtain the result. The idea of the proof is to modify the proof of Schur's theorem by using exponentials. Let $N := R(k+1, \ldots, k+1; m) - 1$, and suppose $\mathbf{c}\colon [2^N] \to [m]$ is a finite coloring of $[2^N]$. Let G be the complete graph on the set $[N+1]$, and define a coloring $\tilde{\mathbf{c}}\colon E(G) \to [m]$ by $\tilde{\mathbf{c}}(\{i, j\}) := \mathbf{c}(2^{i-j})$ with $i > j$. Then, Ramsey's theorem implies the existence of a monochromatic complete subgraph G' on $k+1$ vertices in G. Writing the vertices of G' in order as $v_0 < \cdots < v_k$, we see that the quantities $\mathbf{c}(2^{v_i - v_j})$ for $i > j$ are all equal to each other. The result then follows by setting $x_j := 2^{v_j - v_{j-1}}$. $\square$

Proposition 3. *The quantity $N(m, k)$ in Schur's theorem can be taken to be C^{k^m} for some absolute constant $C > 1$.*

Proof. We show that we can take $C = 4$. When $m = 2$, we have

$$N(2, k) = R(k+1, k+1; 2) \leq \binom{2k}{k} \leq 4^k.$$

For $m \geq 3$, we first establish the bound

$$\binom{4^{k^l}+k-1}{k} \leq 4^{k^{l+1}}$$

whenever $k, l \geq 1$. Indeed, the identity reduces to $\binom{4}{1} \leq 4$ if $k = 1$, which is clearly true. If $k > 1$, then we compute

$$\begin{aligned}\binom{4^{k^l}+k-1}{k} &= \frac{(4^{k^l}+k-1)!}{k!(4^{k^l}-1)!} \\ &= \frac{(4^{k^l}+k-1)(4^{k^l}+k-2)\dots(4^{k^l})}{k!} \\ &= \frac{4^{k^l}+k-1}{k} \cdot \frac{4^{k^l}+k-2}{k-1} \cdots \frac{4^{k^l}}{1} \\ &\leq 4^{k^l} \cdot 4^{k^l} \dots 4^{k^l} \\ &= 4^{k^{l+1}}.\end{aligned}$$

We may give an upper bound for the Ramsey numbers. The base case is given by the bound $R(k+1, k+1; 2) \leq 4^{k^{2-1}}$, which was proven earlier. Suppose inductively that $R(k+1, \dots, k+1; m-1) \leq 4^{k^{m-2}}$. Then, we have

$$\begin{aligned}R(k+1, \dots, k+1; m) &:= R(R(k+1, \dots, k+1; m-1), k+1; 2) \\ &\leq R(4^{k^{m-2}}, k+1; 2) \\ &\leq \binom{4^{k^{m-2}}+k-1}{k} \\ &\leq 4^{k^{m-1}}\end{aligned}$$

as needed. □

Proposition 4 (Erdős 1947). *Suppose $n \geq 2$ and $N < 2^{n/2}$. Then there exists a two-coloring of the complete graph K_N on N vertices which does not contain a monochromatic complete subgraph of n vertices. That is, $R(n, n) \geq 2^{n/2}$.*

Proof. It is easily to verify manually that the result holds for $n = 2$ and $n = 3$ as $R(2, 2) = 2$ and $R(3, 3) = 6$. Assume that $n \geq 4$. Then[1]

$$\begin{aligned}&\Pr\begin{pmatrix}\text{some coloring of the}\\ \text{edges of } K_N \text{ contains}\\ \text{a monochromatic } K_n\end{pmatrix} \\ &\leq \#\{K_n\text{'s in } K_N\} \cdot \Pr\begin{pmatrix}\text{a randomly colored}\\ K_n \text{ is monochromatic}\end{pmatrix} \\ &= \binom{N}{n} \cdot 2^{-\binom{n}{2}+1}.\end{aligned}$$

Now, using the fact[2] that $\binom{N}{n} \leq \frac{N^n}{2^{n-1}}$ for $n \geq 2$, and that $N^n < 2^{n^2/2}$, it follows that

$$\binom{N}{n} 2^{-\binom{n}{2}+1} \leq \frac{N^n}{2^{n-1}} 2^{-\binom{n}{2}+1} < 2^{n^2/2-\binom{n}{2}+2-n} = 2^{2-n/2} \leq 1$$

whenever $n \geq 4$. Therefore the probability that some coloring of the edges of K_N contains a monochromatic K_n is less than one under the hypotheses of the theorem, and so there must exist some edge-colored K_N that does not contain any monochromatic K_n as a subgraph. □

[1] I'm still unfamiliar with probability so I didn't phrase things well with events and stuff like that. My understanding is that the first inequality here is the union bound $\Pr(\bigcup_i A_i) \leq \sum_i \Pr(A_i)$, which captures the idea that the events involved aren't independent.

[2] Proof:

$$\begin{aligned}\binom{N}{n} &= \frac{N \cdot (N-1) \cdots (N-n+1)}{n \cdot (n-1) \cdots 2 \cdot 1} \\ &\leq \frac{N \cdot N \cdots N}{2 \cdot 2 \cdots 2 \cdot 1} \\ &= \frac{N^n}{2^{n-1}}\end{aligned}$$

3. *Van der Waerden's theorem*

We begin by verifying a claim that Tao makes in his proof of Van der Waerden's theorem.

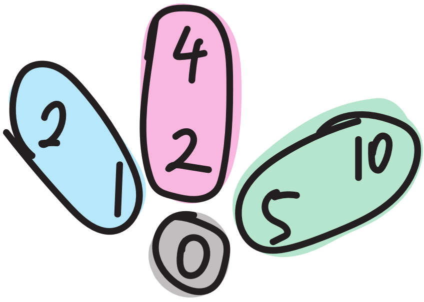

Figure 1: A strongly polychromatic fan of radius 3, degree 3, and base point 0.

Proposition 5. *If $F = (a + [0, k) \cdot r_1, \ldots, a + [0, k) \cdot r_d)$ is a fan of radius k and degree d, and $a_0, r \in Z$ are such that the $k-1$ fans $a_0 + jr + F$, $1 \leq j \leq k-1$ and the origin $\{a_0 + a\}$ all lie in P and are disjoint from each other, and furthermore the fans $a_0 + jr + F$ are all strongly polychromatic with the same colors $\mathbf{c}(a_0 + jr + F) = c$, then the fan $\tilde{F} := (a_0 + a + [0, k) \cdot r, a_0 + a + [0, k) \cdot (r + r_1), \ldots, a_0 + a + [0, k) \cdot (r + r_d))$ also lies in P and is a weakly polychromatic fan of radius k and degree $d + 1$. In other words, arithmetic progressions of strongly polychromatic fans contain a weakly polychromatic fan of one higher degree.*

Proof. All of $\tilde{F}$'s elements are part of the k fans which lie in P by hypothesis. (If we think of each of the k fans as lying in parallel planes stacked on top of one another, then $\tilde{F}$ lies in a plane that starts at the origin and slices diagonally through the k planes, taking the j-th element of every spoke on the j-th plane.) In more detail, we have $a_0 + a + j(r + r_n) \in a_0 + jr + F$, since $a + jr_n \in F$. Since the first spoke $a_0 + a + [0, k) \cdot r$ of $\tilde{F}$ consists of the base points of each fan, we see that it causes $\tilde{F}$ to be weakly monochromatic, of one degree higher than F. This completes the proof. □

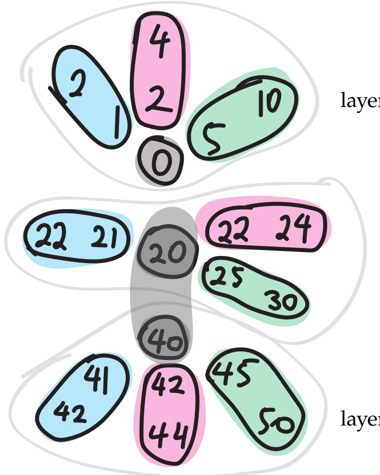

Figure 2: The base points of the strongly polychromatic fans form one spoke of $\tilde{F}$ with the same color as the original base point, giving us a weakly polychromatic fan of one higher degree. The other spokes of $\tilde{F}$ cut through the levels — for example, the pink spoke $\{22, 42\}$ contains an element from each fan below the original fan.

Note that in Lemma 3.4, the case of $d = 1$ follows from the outer inductive hypothesis — at that point we know that for every m, there exists $N(k-1, m)$ such that any m-coloring of a proper AP of length at least $N(k-1, m)$ contains a proper monochromatic AP of length $k-1$. We may take $\tilde{N}(k-1, m, 1) := N(k-1, m)$. Then, a proper monochromatic AP of length $k-1$, written as $a + [0, k-1) \cdot r$, is either such that $a - r$ is colored the same color, in which case we obtain a k-AP, or it is a different color, in which case we get a strongly polychromatic fan of radius k and degree 1.

The number of fans of radius k and degree $d-1$ in P_0 is at most N_1^d. We have N_1 choices for the base point. The $d-1$ spokes are determined by their first element, and so we have another N_1^{d-1} choices, giving the claim.

The possible colors $c(bN_1 v + F(b))$ of a strongly polychromatic fan is at most m^d. There are $d-1$ spokes and 1 base point, and they all have m possible color choices (but no more, since a spoke has only one color).

From what I understand, the moral of the color focusing proof of van der Waerden is that color focusing is the key tool for getting from $(k-1)$-AP's to k-AP's in the induction step. If we have a bunch of color focused progressions of length $k-1$, say, $\{2,5\}$, $\{4,6\}$, and $\{6,7\}$, which are all focused on 8, then they form d spokes of a polychromatic fan, and if d exhausts all the colors, then one of the spokes must have the same color as the focus, which gives us an AP of length k. See Imre Leader's Michaelmas 2000 notes on Ramsey theory at `https://www.dpmms.cam.ac.uk/~par31/notes/ramsey.pdf` (I rewrote his Proposition 5 below for my learning), and the illuminating illustrations in Blondal and Jungić's article *Proof of Van der Waerden's theorem in nine figures* (`https://doi.org/10.1007/s40598-018-0090-5`).

Note: The nxet two results and their proofs are almost entirely copied from the notes of Imre Leader cited above.

Proposition 6. *Let $m \in \mathbf{Z}^+$. Then there exists $N = N(m) \in \mathbf{Z}^+$ such that, given a m-coloring of $[N]$, there exists a monochromatic AP of length three.*

Proof. We first introduce the notion of *color focusing*. A d-tuple $(a_1 + [0,k) \cdot r_1, \ldots, a_d + [0,k) \cdot r_d)$ of k-AP's is said to be *color focused at focus f* if $a_1 + kr_1 = \cdots = a_d + kr_d = f$. Notice how this concept is 'isomorphic' to Tao's notion of a polychromatic fan!

It suffices to prove this claim: *for all $r \leq m$, there exists N such that whenever $[N]$ is m-colored, then either $[N]$ contains a monochromatic 3-AP, or $[N]$ contains r color focused monochromatic 2-AP's.* Then the result follows from taking $r = m$, since then the focus must share a color with one of the 2-AP's, giving us a 3-AP as needed.

We induct on r. When $r = 1$, the claim follows from setting $N = k+1$ by the Pigeonhole principle. Suppose inductively that $N = N(r-1)$ satisfies the claim for $r-1$; we prove that $N(r) := (k^{2N} + 1)2N$ works. Given a k-coloring of $[(k^{2N}+1)2N]$ not containing any monochromatic 3-AP's, we break up $[(k^{2N}+1)2N]$ into blocks of length $2N$; namely $B_j := [2Nj + 1..2Nj + 2N]$ for $0 \leq j \leq k^{2N}$. Then, our choice of N guarantees that each block B_j contains $r-1$ monochromatic color focused 2-AP's in the first half of B_j (that is, $[2Nj + 1, 2Bj + N]$), together with their focus (since if the 2-AP is in the first half of B_j then its focus must lie in the second half of B_j). Since there are k^{2N} ways to color a block, it follows that two blocks B_j and B_{j+t} are colored in the exact same way. Say B_j contains $\{a_i, a_i + d_i\}$ for $1 \leq i \leq r-1$, all color focused at f. Then B_{j+t} contains $\{a_i + 2Nt, a_i + 2Nt + d_i\}$ for $1 \leq i \leq r-1$, all color focused at $f + 2Nt$ with the same colors as of B_j, and so the $r-1$ progressions $\{a_i, a_i + 2Nt + d_i\}$ are color focused at $a_i + 4Nt + 2d_i = f + 4Nt$. Together with $\{f, f+2Nt\}$, we obtain r color focused progressions, which completes the induction. $\square$

The proof of the general case is similar, except now we need a double induction, with outer induction on the length k of an AP, and inner induction on the number $r \leq k$ of color focused AP's of length $k-1$.

Proposition 7. *Let $m, k \in \mathbf{Z}^+$. Then there exists $N = N(k, m) \in \mathbf{Z}^+$ such that whenever $[n]$ is m-colored, there exists a monochromatic k-AP.*

Proof. We induct on k. The case $k = 1$ is trivial. Now suppose inductively that we are given $k \geq 3$ such that $N(k-1, m)$ exists for all m. It suffices to prove this claim: *for all $r \leq k$, there exists N such that whenever $[N]$ is m-colored, there exists either a monochromatic k-AP, or r color focused $(k-1)$-AP's.* The result will follow by setting $r = k$, as then the focus must share its color with one of the $(m-1)$-AP's, giving us a monochromatic m-AP. We prove the claim by induction on r. For $r = 1$, we may take $N = N(k-1, m)$. Suppose inductively that $r \geq 2$ and we have $N = N(r-1)$ suitable for $r-1$. We prove that $N(r) := N(k-1, m^{2N})2N$ works. Given an m-coloring of $[N(k-1, m^{2N})2N]$ with no monochromatic k-AP's, we can break up $[N(k-1, m^{2N})2N]$ into $N(k-1, m^{2N})$ many blocks of length $2N$, so that $B_j := [2Nj + 1..2Nj + 2N]$ for $0 \leq j \leq N(k-1, m^{2N}) - 1$. We may think of the blocks $B_1, \ldots, B_{N(k-1, m^{2N})-1}$ as a sequence, and think of the mapping $B_i \mapsto (c(2Nj+1), \ldots, c(2Nj+2N))$ as a m^{2N}-coloring of this sequence. It follows that there exists a monochromatic $(k-1)$-AP of B_i, so that we have $k-1$ identically colored blocks $B_s, B_{s+t}, \ldots, B_{s+(k-2)t}$. Now, B_s contains $r-1$ color focused $(k-1)$-AP's (in the first half of B_s), as well as their focus; say $a_i + jd_i \in B_s$ are focused at f, where $1 \leq i \leq r-1$ and $0 \leq j \leq k-2$. Then we have the $r-1$ 'diagonal' $(k-1)$-AP's $a_i + j(d_i + 2Nt)$ for $1 \leq i \leq r-1$ and $0 \leq j \leq k-2$, which are all color focused at $f + (k-1)2Nt$. Finally, we have the $(k-1)$-AP formed by the foci, given by $f + j \cdot 2Nt$ for $0 \leq j \leq k-2$. Thus we have r color focused $(k-1)$-AP's, which closes the induction and completes the proof. $\square$

Traditionally, we write $W(m, k)$ for the smallest integer for which any m-coloring of $[W(m, k)]$ admits a monochromatic k-AP. These integers are known as the Van der Waerden numbers. The best upper bound in general is due to Timothy Gowers, who proved in 2001 that

$$W(m, k) \leq 2^{2^{m^{2^{2^{k+9}}}}}.$$

So we see the concepts of a polychromatic fan, of a set of color focused AP's, and the idea of a n-dimensional arithmetic progrsesion, which all get at the same essential ideas, though each framing comes with its own emphases.

Proposition 8. *To prove Van der Waerden's theorem, it suffices to prove it for the case where there are $m = 2$ colors.*

Proof. We induct on m. Suppose we are given m such that $N(k, m-1)$ exists for all k. Then we show that $N = N(N(k, m-1), 2)$ works. Indeed, given an m-coloring $\mathbf{c}$ of $[N]$, we define a 2-coloring $\mathbf{c}/\sim$ by identifying the first $m-1$ colors. By Van der Waerden's theorem for two colors, either we have an arithmetic progression of length

$N(k, m-1) \geq k$ and we are done, or there is an arithmetic progression P of length $N(k, m-1)$ that only takes colors in the first $m-1$ colors. By the inductive hypothesis, we may thus find a monochromatic subprogression P' of P of length k. $\square$

Note that the above proposition only holds because of the generality of the version of Van der Waerden's theorem Tao uses in the notes. If we only had the version which considered only subprogressions of $[N]$, we would then need to prove the result for all m.

Proposition 9 (Infinitary form of Van der Waerden's theorem)**.** *Suppose the positive integers* $\mathbf{Z}^+$ *are finitely colored. Then, there exists a color* c *such that the set* $\{n \in \mathbf{Z}^+ : \mathbf{c}(n) = c\}$ *contains arbitrarily long proper AP's.*

Proof. Suppose not. Then ... $\square$

The above result implies the finitary form of Van der Waerden's theorem. Indeed, ...

Proposition 10. *If* $N = N(k, m)$ *is sufficiently large, and* $\mathbf{c} \colon [N] \to [m]$ *is an* m*-coloring of* $[N]$*, then* $[N]$ *will contain at least* $c(k, m)N^2$ **(not sure what $c(k, m)$ means, but hopefully i'll find out)** *monochromatic* k*-AP's.*

4. Rado's theorem

To be written...

5. The Hales–Jewett theorem

To be written...

6. Polynomial analogues

To be written...