22 august 2020

ho boon suan

Altman–Kleiman 1

The derivation for (1.8.1) is plainly

$$\begin{aligned}\operatorname{ord}_{(x_\lambda)} F &= \operatorname{ord}_{(0)}(\varphi_{(x_\lambda)} F) \\ &\le \operatorname{ord}_{(0)}(\varphi_{(x_\lambda)} F)_{\mu,0} \\ &= \operatorname{ord}_{(0)} \varphi_{(x_\lambda)} F_{\mu,x_\mu} \\ &= \operatorname{ord}_{(x_\lambda)} F_{\mu,x_\mu}.\end{aligned}$$

1.14. (1) Let $x \in \mathfrak{a}$. We will show that $x \in \mathfrak{a}^{ec} = \varphi^{-1}(\mathfrak{a}^e)$; that is, $\varphi(x) \in \mathfrak{a}^e$. This is plainly true. Now let $x \in \mathfrak{b}^{ce} = (\varphi^{-1}(\mathfrak{b}))^e$. Then $x = \sum_i c_i x_i$ for some $c_i \in R'$ and $x_i \in \varphi(\varphi^{-1}(\mathfrak{b})) = \mathfrak{b}$. Since $\mathfrak{b}$ is an ideal the result follows plainly.

(2) From (1) it suffices to show $\mathfrak{a}^{ece} \subset \mathfrak{a}^e$ and $\mathfrak{b}^{cec} \supset \mathfrak{b}^c$. Let $x \in \mathfrak{a}^{ece}$. Then $x = \sum_i c_i x_i$, where $c_i \in R'$ and $x_i \in \varphi(\mathfrak{a}^{ec}) = \mathfrak{a}^e$; that is, each $x_i = \sum_j c_{ij} x_{ij}$ with $c_{ij} \in R'$ and $x_{ij} \in \varphi(\mathfrak{a})$. Thus $x = \sum_{i,j} c_i c_{ij} x_{ij}$ where $x_{ij} \in \varphi(\mathfrak{a})$, so $x \in \mathfrak{a}^e$ as needed. The proof for $\mathfrak{b}$ is similarly tedious.

Plainly, I am confused.

(3) If $\mathfrak{a}$ be an ideal of R with extension $\mathfrak{b}$ with $x \in \mathfrak{a}$, then $\varphi(x) \in \mathfrak{b}$ so that $x \in \mathfrak{b}^c$. Thus $\mathfrak{a} \subset \mathfrak{b}^c$ so that $\mathfrak{b}^c$ is the largest ideal of R with extension $\mathfrak{b}$.

(4) Let $\mathfrak{a}_1^e$ and $\mathfrak{a}_2^e$ be extensions with the same contraction. We will show that $\mathfrak{a}_1^e = \mathfrak{a}_2^e$. Suppose $x \in \mathfrak{a}_1^e$. Then $x = \sum_i c_i x_i$ with $c_i \in R'$ and $x_i \in \varphi(\mathfrak{a}_1)$. Since $\mathfrak{a}_1 \subset \mathfrak{a}_1^{ec} = \mathfrak{a}_2^{ec}$, it follows that the $x_i \in \varphi(\mathfrak{a}_2^{ec}) = \mathfrak{a}_2^e$ and thus $x \in \mathfrak{a}_2^e$ as needed.

I suspect $\mathfrak{b}'$ will be reserved notation later…

Contraction preserves primality. Let $\varphi\colon R \to R'$ be a ring map, and let $\mathfrak{p}$ be a prime ideal of R'. If $ab \in \mathfrak{p}^c$, then $\varphi(ab) = \varphi(a)\varphi(b) \in \mathfrak{p}$, so that $\varphi(a) \in \mathfrak{p}$ or $\varphi(b) \in \mathfrak{p}$; that is, $a \in \mathfrak{p}^c$ or $b \in \mathfrak{p}^c$. Note that the extension of a prime ideal need not be prime, consider for example $(2) \subset \mathbf{Z}$ extended under the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}[\sqrt{2}]$.

Also $\mathbf{Z} \hookrightarrow \mathbf{Q}$ since prime ideals must be proper.

1.15. (1) Any element $P \in \mathfrak{a}(R[\mathcal{X}])$ can be written as $P = \sum_i p_i a_i$ with $p_i \in R[\mathcal{X}]$ and $a_i \in \mathfrak{a}$ (as we're extending by the inclusion $R \hookrightarrow R[\mathcal{X}]$). The a_i absorb the coefficients in the p_i, so $\mathfrak{a}(R[\mathcal{X}]) \subset \mathfrak{a}[\mathcal{X}]$. Conversely, given $P \in \mathfrak{a}[\mathcal{X}]$, since P is the sum of monomials each with coefficient in $\mathfrak{a}$ and the monic part in $R[\mathcal{X}]$; that is, since each monomial in the sum that forms P may be written ax where $a \in \mathfrak{a}$ and $x \in R[\mathcal{X}]$, it follows that $P \in \mathfrak{a}(R[\mathcal{X}])$ as needed.

(2) Immediately follows from (1).

1.16. By exercise 1.15 this amounts to showing that $R[\mathcal{X}]/\mathfrak{a}[\mathcal{X}] \cong (R/\mathfrak{a})[\mathcal{X}]$. This follows from how the mapping $R[\mathcal{X}] \to (R/\mathfrak{a})[\mathcal{X}]$ defined by taking residues of coefficients with the map $\kappa : R \to R/\mathfrak{a}$ and preserving the indeterminates $X \in \mathcal{X}$ has kernel $\mathfrak{a}[\mathcal{X}]$.

I wish I knew how to phrase and write some of this stuff better, but a whole lot of algebra feels trivial to write since the difficulty was in establishing the right definitions precisely to make all these manipulations easy?

Addenda: 23 august 2020. I think I may have messed up some arguments in a minor way involving images of preimages or preimages of images. The proper result is that, for a function $f \colon A \to B$, we have

$$f(f^{-1}(B)) \subset B \quad \text{and} \quad A \subset f^{-1}(f(A)),$$

where the first and second inclusions are equalities only when f is a surjection and injection respectively.

Also, on my earlier remark regarding phrasing and writing — use the universal properties. There exists a unique map having these properties precisely because of the universal properties of quotients, of polynomial rings, etc.

23 august 2020

ho boon suan

> *From [Grothendieck], I have also learned not to take glory in the difficulty of a proof: difficulty means we have not understood. The idea is to be able to paint a landscape in which the proof is obvious.*
>
> — Pierre Deligne, *Théorie des topos et cohomologie étale des schémas* (1963)

Altman–Kleiman 1

17. Parts (1)–(4) generalize Proposition (1.6) to polynomial rings in multiple indeterminates.

(1) Every $F \in P$ may be written $F = \sum b_{(i_1,\dots,i_n)} X_{\lambda_1}^{i_1} \dots X_{\lambda_n}^{i_n}$, so

$$\begin{aligned} F &= \varphi_{(-x_\lambda)} \varphi_{(x_\lambda)} F \\ &= \varphi_{(-x_\lambda)} \sum b_{(i_1,\dots,i_n)} (X_{\lambda_1} + x_{\lambda_1})^{i_1} \dots (X_{\lambda_n} + x_{\lambda_n})^{i_n} \\ &= \varphi_{(-x_\lambda)} \sum a_{(i_1,\dots,i_n)} X_{\lambda_1}^{i_1} \dots X_{\lambda_n}^{i_n} \\ &= \sum a_{(i_1,\dots,i_n)} (X_{\lambda_1} - x_{\lambda_1})^{i_1} \dots (X_{\lambda_n} - x_{\lambda_n})^{i_n}. \end{aligned}$$

Note that the indices are all nonnegative, and $F((x_\lambda)) = a_{(0,\dots,0)}$.

(2) Since

$$\begin{aligned} \pi_{(x_\lambda)} F &= \pi_{(x_\lambda)} \left(\sum a_{(i_1,\dots,i_n)} (X_{\lambda_1} - x_{\lambda_1})^{i_1} \dots (X_{\lambda_n} - x_{\lambda_n})^{i_n} \right) \\ &= \sum a_{(i_1,\dots,i_n)} (x_{\lambda_1} - x_{\lambda_1})^{i_1} \dots (x_{\lambda_n} - x_{\lambda_n})^{i_n} \\ &= a_{(0,\dots,0)} \\ &= F((x_\lambda)), \end{aligned}$$

it follows that

$$\ker \pi_{(x_\lambda)} = \{F \in P \mid F((x_\lambda)) = 0\}.$$

That this is equal to $\langle \{X_\lambda - x_\lambda\} \rangle$ follows from (1).

(3) The universal mapping property of residues then induces an isomorphism $P/\langle \{X_\lambda - x_\lambda\} \rangle \xrightarrow{\sim} R$.

(4) Follows from (1) since the only monomial in F without any $(X_\lambda - x_\lambda)$ terms is $a_{(0,\dots,0)} = F((x_\lambda))$.

(5) The P-algebra $P \to R[\mathcal{Y}]$ defined by mapping $p \mapsto \pi_{(x_\lambda)}(p)$ and endowed with the set $\mathcal{Y}$ of distinguished elements induces via the UMP of polynomial rings a morphism $P[\mathcal{Y}] \to R[\mathcal{Y}]$. *(I don't know how to reason nicely that the kernel of this morphism is $\langle \{X_\lambda - x_\lambda\} \rangle \dots$)*

21 (*Chinese Remainder Theorem*)**.** (1a) We have $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b}$ since each term in some element $\sum a_i b_i \in \mathfrak{ab}$ is both in $\mathfrak{a}$ and $\mathfrak{b}$. Conversely, if $x \in \mathfrak{a} \cap \mathfrak{b}$, then $x = ay = bz$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, there exist elements a' and b' with $a' + b' = 1$. Then $x = ay = aya' + ayb' = bza' + ayb' \in \mathfrak{a} + \mathfrak{b}$ as needed.

(1b) There are natural quotients $\pi_{\mathfrak{a}} \colon R \to R/\mathfrak{a}$ and $\pi_{\mathfrak{b}} \colon R \to R/\mathfrak{b}$; we may form the product map $\pi_{\mathfrak{a} \times \mathfrak{b}} \colon R \to R/\mathfrak{a} \times R/\mathfrak{b}$. This map sends r to $(r+\mathfrak{a}, r+\mathfrak{b})$ and thus has kernel $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. It follows that $R/\mathfrak{a}\mathfrak{b} \xrightarrow{\sim} R/\mathfrak{a} \times R/\mathfrak{b}$.

(2) Since $\mathfrak{a}+\mathfrak{b} = \mathfrak{a}+\mathfrak{b}' = R$, there exist elements a, $a' \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $b' \in \mathfrak{b}'$ such that $a+b = a'+b' = 1$. Thus

$$1 = (a+b)(a'+b') = a(a'+b') + a'b + bb' \in \mathfrak{a} + \mathfrak{b}\mathfrak{b}'$$

as needed.

(3) Suppose $\mathfrak{a}^m + \mathfrak{b}^n = R$. Then $a_1 \ldots a_m + b_1 \ldots b_n = 1$. Since ideal powers form a chain $\mathfrak{a} \supset \mathfrak{a}^2 \supset \ldots$, it follows that $\mathfrak{a}+\mathfrak{b} = R$. The converse follows from repeatedly applying (2).

(4) Suppose inductively the result holds up to $n-1$. Then (a) holds since $\mathfrak{a}_1$ is comaximal with $\mathfrak{a}_2 \ldots \mathfrak{a}_{n-1}$ and $\mathfrak{a}_n$. We have (b) as

$$\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cap (\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n) = \mathfrak{a}_1 \cap (\mathfrak{a}_2 \ldots \mathfrak{a}_n) = \mathfrak{a}_1 \ldots \mathfrak{a}_n.$$

Finally, $R/(\mathfrak{a}_1 \ldots \mathfrak{a}_n) \cong R/\mathfrak{a}_1 \times \prod_{2 \le i \le n}(R/\mathfrak{a}_i) \cong \prod(R/\mathfrak{a}_i)$.

(5) We have $(0)(0) = (0) \cap (0) = (0)$. But you probably want something more...

23. Suppose $\mathfrak{a} \subset R' \times R''$ is an ideal. Then $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{a}''$ and we wish to show that both $\mathfrak{a}'$ and $\mathfrak{a}''$ are ideals. If $x, y \in \mathfrak{a}'$, then $(x,0)+(y,0) = (x+y,0) \in \mathfrak{a}$, so $x+y \in \mathfrak{a}'$. Also, $(r',r'')(x,0) = (r'x,0) \in \mathfrak{a}$, so $r'x \in \mathfrak{a}'$. The reasoning for $\mathfrak{a}''$ is the same.

Finally, the map $(R' \times R'')/(\mathfrak{a}' \times \mathfrak{a}'') \cong (R'/\mathfrak{a}') \times (R''/\mathfrak{a}'')$ by the appropriate UMPs. For example, we'll show that $(R' \times R'')/(\mathfrak{a}' \times \mathfrak{a}'')$ satisfies the UMP for products. Let Z be a ring and suppose $f'\colon Z \to R'/\mathfrak{a}'$ and $f''\colon Z \to R''/\mathfrak{a}''$. Also define maps $(R' \times R'')/(\mathfrak{a}' \times \mathfrak{a}'') \to R'/\mathfrak{a}'$ and $R''/\mathfrak{a}''$ by sending $(r',r'') + \mathfrak{a}' \times \mathfrak{a}''$ to $r'+\mathfrak{a}'$ and $r''+\mathfrak{a}''$ respectively. Then a unique map is forced from $Z \to (R' \times R'')/(\mathfrak{a}' \times \mathfrak{a}'')$ defined by sending z to $(z',z'') + \mathfrak{a}' \times \mathfrak{a}''$, where $f'(z) = z'+\mathfrak{a}'$ and $f''(z) = z''+\mathfrak{a}''$. That this is well-defined is left as an exercise to the reader.

this is disgusting

Altman–Kleiman 2

9. This is equivalent to showing that R is a domain iff $R[\mathcal{Y}]$ is a domain. If $ab = 0$ in R, then $ab = 0$ in $R[\mathcal{Y}]$. Conversely, if $pq = 0$ in $R[\mathcal{Y}]$ with both p and q nonzero, then the grlex leading coefficients of p and q are zerodivisors. (There definitely is a nicer way to do this.)

24 august 2020

ho boon suan

Try to finish Altman–Kleiman 2 by tomorrow, latest Wednesday; then start on Ireland–Rosen 4. Also do IR exercise 3.15

Altman–Kleiman 2

23. ($1 \Rightarrow 2$) True as $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}$ or $\mathfrak{b}$, and both are contained in $\mathfrak{p}$.

($2 \Rightarrow 3$) If $x \in \mathfrak{ab}$, then $x = \sum_i a_i b_i$, with each term in both $\mathfrak{a}$ and $\mathfrak{b}$. Thus $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$.

($3 \Rightarrow 1$) We prove the contrapositive. Suppose $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ are such that $a, b \notin \mathfrak{p}$. Then $ab \in \mathfrak{ap} \subset \mathfrak{p}$, so we have $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ by primality of $\mathfrak{p}$.

24. Since $\mathfrak{m}_1 \ldots \mathfrak{m}_n = 0 \subset \mathfrak{p}$, repeated applications of 23(1) yield $\mathfrak{m}_i \subset \mathfrak{p}$ for some i. The result follows from maximality of $\mathfrak{m}_i$ as well as noting that prime ideals are proper by definition.

26 august 2020

ho boon suan

— Antonín Dvořák, *Symphony No. 9* (1893)

Altman–Kleiman 2

25. (1) We induct on n. The case $n = 2$ is proven in exercise 23. Suppose the result holds for all integers less than n. Then we have $\mathfrak{p} \supset \bigcap_{i=1}^{n} \mathfrak{a}_i = (\bigcap_{i=1}^{n-1} \mathfrak{a}_i) \cap \mathfrak{a}_n$, which is the intersection of two ideals, so either $\mathfrak{p} \supset \mathfrak{a}_n$ and we are done, or $\mathfrak{p} \supset \bigcap_{i=1}^{n-1} \mathfrak{a}_i$ in which case the result follows from the inductive hypothesis.

(2) This follows from (1) by noting that $\mathfrak{p} \subset \bigcap_{i=1}^{n} \mathfrak{a}_i \subset \mathfrak{a}_j$.

26. This is clear for the zero ring, so suppose R is nonzero and let z be a zerodivisor of R. Then $zy = 0$ for some nonzero y. We claim that the principal ideal (z) is proper. Suppose for contradiction that $zx = 1$ for some nonzero $x \in R$. Then $y = yzx = zyx = 0x = 0$, contradicting our assumption that y was nonzero. Thus (z) is proper. Further, every element of (z) is a zerodivisor, as $a \in (z)$ implies $a = rz$ for some $r \in R$ so that $ay = rzy = 0r = 0$.

Finally, $\mathcal{S}$ has maximal elements as a consequence of Zorn's lemma. In more detail, following the proof that every proper ideal of a ring is contained in some maximal ideal, let

$$\mathcal{S}' := \{\mathfrak{b} \mid R \supsetneq \mathfrak{b} \supset (z) \text{ and } \mathfrak{b} \text{ contains only zerodivisors}\}$$

where z is a zerodivisor of R. Then we have $(z) \in \mathcal{S}'$, so $\mathcal{S}'$ is nonempty and partially ordered by inclusion. Now, given a totally ordered subset $\{\mathfrak{b}_\lambda\}$ of $\mathcal{S}'$, set $\mathfrak{b} := \bigcup \mathfrak{b}_\lambda$. Then $\mathfrak{b}$ is a proper ideal consisting purely of zerodivisors of R and is thus an upper bound of $\{\mathfrak{b}_\lambda\}$ in $\mathcal{S}'$. The result now follows from Zorn's lemma.

It follows that $\mathrm{zdiv}(R)$ is a union of prime ideals — indeed, every zerodivisor of R lies in its principal ideal, which in turn lies in a maximal ideal consisting purely of zerodivisors, and maximal ideals are prime.

28. Let $R := R' \times R''$. If both factors are nonzero, then given nonzero elements $r' \in R'$ and $r'' \in R''$, we have $(r', 0) \cdot (0, r'') = (0, 0)$. So if R is a domain, one of the factors must be 0; say R''. Then $R = R' \times 0 \cong R'$, and it is clear that zerodivisors of R' and R correspond by the map $r' \mapsto (r', 0)$. The converse is clear as $R \times 0 \cong R$.

29. Write $\mathfrak{p} = \mathfrak{p}' \times \mathfrak{p}''$. We have

$$\frac{R}{\mathfrak{p}} = \frac{R' \times R''}{\mathfrak{p}' \times \mathfrak{p}''} \cong \frac{R'}{\mathfrak{p}'} \times \frac{R''}{\mathfrak{p}''},$$

which is a domain iff $\mathfrak{p}$ is prime. The result follows from exercise 28. The converse is follows easily using the 'prime iff quotient is domain' characterization.

30. If $x = 0$ then $y = 0$ and the result is clear. Otherwise x is nonzero, so $x = uy$ and $y = vx$ for some u and $v \in R$. It follows that $x = uy = uvx$, so $uv = 1$ as R is a domain. That is, $x = uy$ where u is a unit, as needed.

27 august 2020

ho boon suan

I became evil for no reason.
I had no motive for my wickedness except wickedness itself.
It was foul, and I loved it.
I loved the self-destruction, I loved my fall,
not the object for which I had fallen but my fall itself.
My depraved soul leaped down from your firmament to ruin.
I was seeking not to gain anything by shameful means,
but shame for its own sake.

— Augustine of Hippo, *Confessions* (c. 397)

Altman–Kleiman 2

36. If $\mathfrak{p}$ is prime, then $B/\mathfrak{p}$ is a domain. Given nonzero $b + \mathfrak{p} \in B/\mathfrak{p}$, we have $b^2 + \mathfrak{p} = (b + \mathfrak{p})^2 = b + \mathfrak{p}$, and so $b + \mathfrak{p} = 1 + \mathfrak{p}$. Also, since $b + b = (b + b)^2 = b^2 + 2b + b^2 = 4b$, it follows that $b + b = 0$. Thus $B/\mathfrak{p} \cong \mathbf{Z}/(2)$ is a field and $\mathfrak{p}$ is maximal.

37. We reason similarly to the previous exercise. If $x^n = x$ with $n \geq 2$, then $x + \mathfrak{p} = x^n + \mathfrak{p}$ so $(x + \mathfrak{p})(x^{n-2} + \mathfrak{p}) = 1 + \mathfrak{p}$. That is, every nonzero x is invertible, so $R/\mathfrak{p}$ is a field and $\mathfrak{p}$ is maximal.

31 august 2020

ho boon suan

eizouken is the greatest anime of all time

— Ho Boon Suan (2020)

Miranda I

1A. We show that compatibility of charts is a symmetric relation. Since ϕ_1 and ϕ_2 are charts, they are homeomorphisms. It follows that $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ are bijections that are inverses of each other. Finally, given a holomorphic invertible function f, its inverse f^{-1} is holomorphic with derivative

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

My solution here is kinda sketchy.

whenever f' is nonzero on the appropriate domain. In particular, since Lemma 1.7 guarantees that $(\phi_2 \circ \phi_1^{-1})'$ is nonzero, we have that $\phi_1 \circ \phi_2^{-1}$ is also holomorphic.

1G. We verify the compatibility of two charts on the Riemann sphere defined via stereographic projection onto the complex plane. Let $z = a + bi$. Then

$$\begin{aligned}
(\phi_2 \circ \phi_1^{-1})(z) &= \phi_2\left(\frac{2a}{|z|^2+1}, \frac{2b}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \\
&= \left(\frac{2a}{|z|^2+1} - i\frac{2b}{|z|^2+1}\right) \Bigg/ \left(1 + \frac{|z|^2-1}{|z|^2+1}\right) \\
&= \frac{2a-2bi}{2|z|^2} \\
&= \frac{a-bi}{a^2+b^2} \\
&= \frac{1}{z}.
\end{aligned}$$

1H. We verify that the equivalence of atlases is indeed an equivalence relation. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be complex atlases of a topological space X, and denote the equivalence of atlases by $\sim$.

Reflexivity follows directly from the definition of an atlas as all charts of an atlas are required to be pairwise compatible; thus we have $\mathcal{A} \sim \mathcal{A}$ for all atlases $\mathcal{A}$.

Symmetry follows from symmetry of the compatibility relation on charts, see exercise I.1A.

Suppose now that $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$, and let $\phi_{\mathcal{A}}\colon U_{\mathcal{A}} \to V_{\mathcal{A}}$ and $\phi_{\mathcal{C}}\colon U_{\mathcal{C}} \to V_{\mathcal{C}}$ be charts of $\mathcal{A}$ and $\mathcal{C}$ respectively. If $U_{\mathcal{A}}$ and $U_{\mathcal{C}}$ are disjoint, we are done. Otherwise, let $x \in U_{\mathcal{A}} \cap U_{\mathcal{C}}$. Since charts of an atlas cover a space, there exists a chart $\phi_{\mathcal{B}}\colon U_{\mathcal{B}} \to V_{\mathcal{B}}$ with $x \in U_{\mathcal{B}}$.

$$\begin{tikzcd}
\phi_{\mathcal{A}}(U_{\mathcal{A}} \cap U_{\mathcal{B}} \cap U_{\mathcal{C}}) \arrow[d, "\phi_{\mathcal{A}}^{-1}"] \arrow[dd, bend right=60, "\phi_{\mathcal{C}}\circ\phi_{\mathcal{A}}^{-1}"'] & \\
U_{\mathcal{A}} \cap U_{\mathcal{B}} \cap U_{\mathcal{C}} \arrow[d, "\phi_{\mathcal{C}}"] \arrow[r, shift left, "\phi_{\mathcal{B}}"] & \phi_{\mathcal{B}}(U_{\mathcal{A}} \cap U_{\mathcal{B}} \cap U_{\mathcal{C}}) \arrow[l, shift left, "\phi_{\mathcal{B}}^{-1}"] \\
\phi_{\mathcal{C}}(U_{\mathcal{A}} \cap U_{\mathcal{B}} \cap U_{\mathcal{C}}) &
\end{tikzcd}$$

It follows that $\phi_{\mathcal{B}} \circ \phi_{\mathcal{A}}^{-1}$ and $\phi_{\mathcal{C}} \circ \phi_{\mathcal{B}}^{-1}$ are holomorphic, so that

$$\phi_{\mathcal{C}} \circ \phi_{\mathcal{A}}^{-1} = (\phi_{\mathcal{C}} \circ \phi_{\mathcal{B}}^{-1}) \circ (\phi_{\mathcal{B}} \circ \phi_{\mathcal{A}}^{-1})$$

is the composition of holomorphic maps and is thus holomorphic at x. Since x is arbitrary, this establishes the result on the domain $\phi_{\mathcal{A}}(U_{\mathcal{A}} \cap U_{\mathcal{C}})$ as needed, so that equivalence is a transitive relation on complex atlases.

1I. We show that every atlas of a topological space X is equivalent to a unique maximal atlas. Let $\mathcal{A}$ be an atlas of X and consider the set

$$A := \left\{ \mathcal{B} \;\middle|\; \begin{array}{c} \mathcal{B} \text{ is an atlas of } X \text{ containing } \mathcal{A} \\ \text{that is equivalent to } \mathcal{A} \end{array} \right\}.$$

This is a partially ordered set under inclusion. Given a chain $\{\mathcal{B}_\lambda\}$ in A, we let $\mathcal{B} := \bigcup_\lambda \mathcal{B}_\lambda$. Clearly $\mathcal{B}$ is an upper bound of $\{\mathcal{B}_\lambda\}$; we must show that it is in A. Since each $\mathcal{B}_\lambda$ contains $\mathcal{A}$, so does $\mathcal{B}$. Each $\mathcal{B}_\lambda$ is by definition equivalent to $\mathcal{A}$ and are thus pairwise equivalent; so their union forms an atlas. Finally, if $\mathcal{B}_i$ and $\mathcal{B}_j$ are compatible to $\mathcal{A}$, then it is easy to verify directly from the definitions that $\mathcal{B}_i \cup \mathcal{B}_j$ is equivalent to $\mathcal{A}$.

It follows from Zorn's lemma that there exists a maximal element $\mathcal{M} \in A$. We will show that it is unique. Suppose for contradiction that $\mathcal{M} \neq \mathcal{M}'$ where $\mathcal{M}'$ is maximal in A. Then the union $\mathcal{M} \cup \mathcal{M}'$ is an atlas in A properly containing both $\mathcal{M}$ and $\mathcal{M}'$, contradicting their maximality. It follows that every atlas of a topological space X is equivalent to a unique maximal atlas.

2C [incomplete!]. We will show that the map from $\mathbf{P}^1$ to $S^2 \subset \mathbf{R}^3$ defined by

$$[z : w] \mapsto \frac{(2\Re(w\bar{z}), 2\Im(w\bar{z}), |w|^2 - |z|^2)}{|w|^2 + |z|^2}$$

is a homeomorphism.

Firstly, the mapping is well-defined as replacing z and w by λz and λw causes factors of λ^2 to show up on both the numerator and denominator which promptly cancel each other out.

Now we verify that $[z : w]$ is mapped into the 2-sphere. Indeed, we have

$$\begin{aligned}
&(2\Re(w\bar{z}))^2 + (2\Im(w\bar{z}))^2 + (|w|^2 - |z|^2)^2 \\
&\quad = 4\Re(w\bar{z})^2 + 4\Im(w\bar{z})^2 + |w|^4 - 2|w|^2|z|^2 + |z|^4 \\
&\quad = 4|w\bar{z}|^2 + |w|^4 - 2|w|^2|z|^2 + |z|^4 \\
&\quad = |w|^4 + 2|w|^2|z|^2 + |z|^4 \\
&\quad = (|w|^2 + |z|^2)^2.
\end{aligned}$$

Next, we show surjectivity. Let $(a, b, c) \in S^2 \subset \mathbf{R}^3$. Then ...

Finally, we show injectivity. Suppose

$$\begin{aligned}
&\frac{(2\Re(w\bar{z}), 2\Im(w\bar{z}), |w|^2 - |z|^2)}{|w|^2 + |z|^2} \\
&\quad = \frac{(2\Re(w'\bar{z}'), 2\Im(w'\bar{z}'), |w'|^2 - |z'|^2)}{|w'|^2 + |z'|^2}.
\end{aligned}$$

Then ...

2D. Given any $x \in L$, the open balls of radius $\epsilon = \frac{1}{2}\min\{|\omega_1|, |\omega_2|\}$ centered at x intersect no other points of L, so each singleton in L is closed and open; that is, L is discrete.

Addendum, 1 september 2020.

I'm guessing that the proper approach to problem I.2C of *Miranda* is probably to construct an inverse rather than show that the map is injective and surjective. I'm not going to pursue it because I've decided that I'm not going to work on *Miranda* for the time being.

I have a bad habit of changing plans too often and leaving books or projects incomplete. For now, the plan is to work on *Ireland–Rosen* and *Altman–Kleiman* side-to-side; then *LeeSM* and *Vakil*, and then *tom Dieck*. The years have taught me that, well, this will take years. But I am better now, and faster; hopefully wiser — so I will try to work harder. That said, time, the world, and most importantly I, will judge me by my actions, by what I ultimately produce. So I hope for the best, and will be off to do some more work now.

1 september 2020

ho boon suan

Like a bird on the wire
Like a drunk in a midnight choir
I have tried in my way to be free

— Leonard Cohen, *Bird on The Wire* (1979)

In an ideal University, as I conceive it, a man should be able to obtain instruction in all forms of knowledge, and discipline in the use of all the methods by which knowledge is obtained. In such a University, the force of living example should fire the student with a noble ambition to emulate the learning of learned men, and to follow in the footsteps of the explorers of new fields of knowledge. And the very air he breathes should be charged with that enthusiasm for truth, that fanaticism of veracity, which is a greater possession than much learning; a nobler gift than the power of increasing knowledge; by so much greater and nobler than these, as the moral nature of man is greater than the intellectual; for veracity is the heart of morality.

— Thomas Henry Huxley, *Universities: Actual and Ideal* (1874)

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10. (1) Suppose $\mathfrak{q}$ is an ideal of R' with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. We will show that $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$. Given $x \in \varphi^{-1}(\mathfrak{p}R')$, we have $\varphi(x) = \sum_i r_i' \varphi(p_i)$ with $r_i' \in R'$ and $p_i \in \mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Thus $\varphi(p_i) \in \mathfrak{q}$ and $\varphi(x) \in \mathfrak{q}$; that is, $x \in \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. For the reverse inclusion, let $x \in \mathfrak{p}$. Then $\varphi(x) \in \mathfrak{p}R'$, so that $x \in \varphi^{-1}(\mathfrak{p}R')$. It follows that $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$.

The converse is straightforward as $\mathfrak{p}R'$ is an ideal of R'.