some bundle stuff

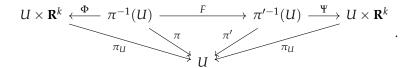
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Proposition 10.26. Suppose E and E' are smooth vector bundles over a smooth manifold M with or without boundary, and $F: E \to E'$ is a bijective smooth bundle homomorphism over M. Then F is a smooth bundle isomorphism.

Proof. We first show that F is a bundle isomorphism. Since $\pi' \circ F = \pi$, we have $\pi' = \pi \circ F^{-1}$. It then suffices to show that $F|_{E_p} \colon E_p \to E'_p$ is an isomorphism of vector spaces. Since F is a bijection, its restriction $F|_{E_p}$ is a bijection onto its image $\operatorname{im}(F|_{E_p}) \subset E'_p$. Since $x \notin E_p$ implies that $F(x) \notin E'_p$, it follows that $E'_p \subset \operatorname{im}(F|_{E_p})$. In particular, E and E' are vector bundles of the same rank E.

We now show that F is a local diffeomorphism. Since bijective local diffeomorphisms are diffeomorphisms, this will complete the proof. Let $p \in U \subset M$ for an appropriate neighborhood U of p, and choose local trivializations Φ and Ψ of E and E' as in the diagram



We then have

$$(\Psi \circ F \circ \Phi^{-1})(p,v) = (p,\tau(p)v)$$

for some smooth $\tau: U \to GL(n, \mathbf{R})$. It follows that

$$(\Phi \circ F^{-1} \circ \Psi^{-1})(p,v) = (p,\tau(p)^{-1}v),$$

which implies that F^{-1} is smooth on $\pi'^{-1}(U)$, since inversion is a diffeomorphism in $GL(n, \mathbf{R})$. This completes the proof.