

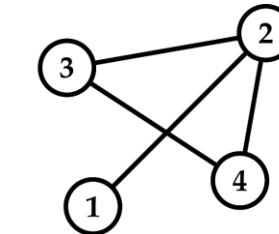
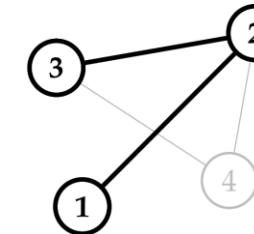
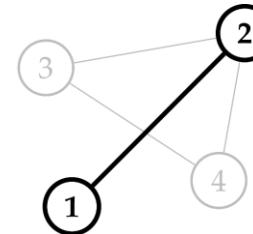
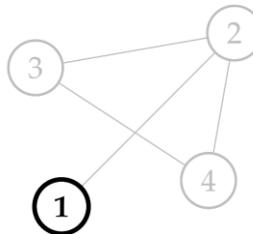
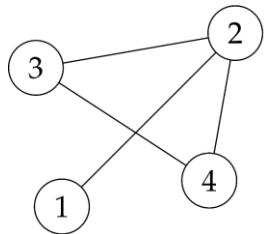
# how many ways are there to traverse a graph?

ho boon suan

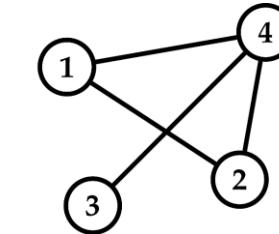
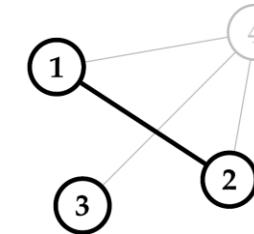
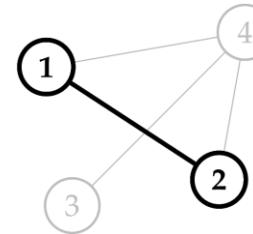
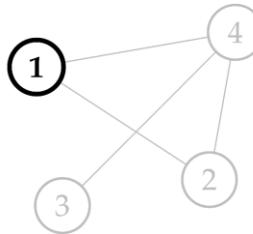
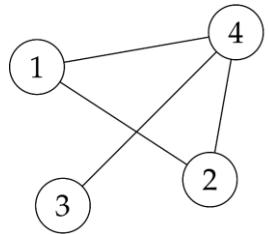
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october 2024

a vertex ordering  $v_1, \dots, v_n$  is *successive* if  
for each  $j > 1$ , there exists  $i < j$  with  $v_i — v_j$

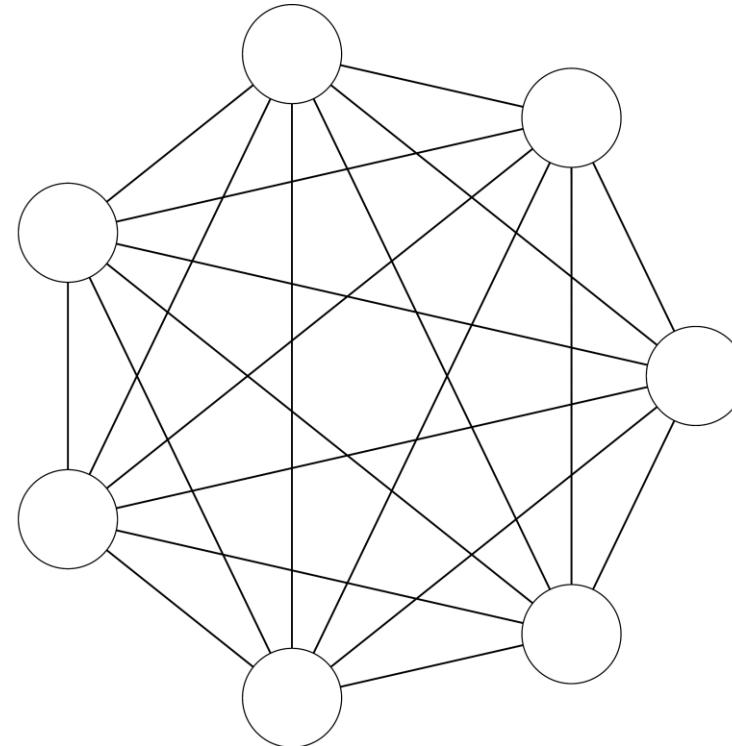


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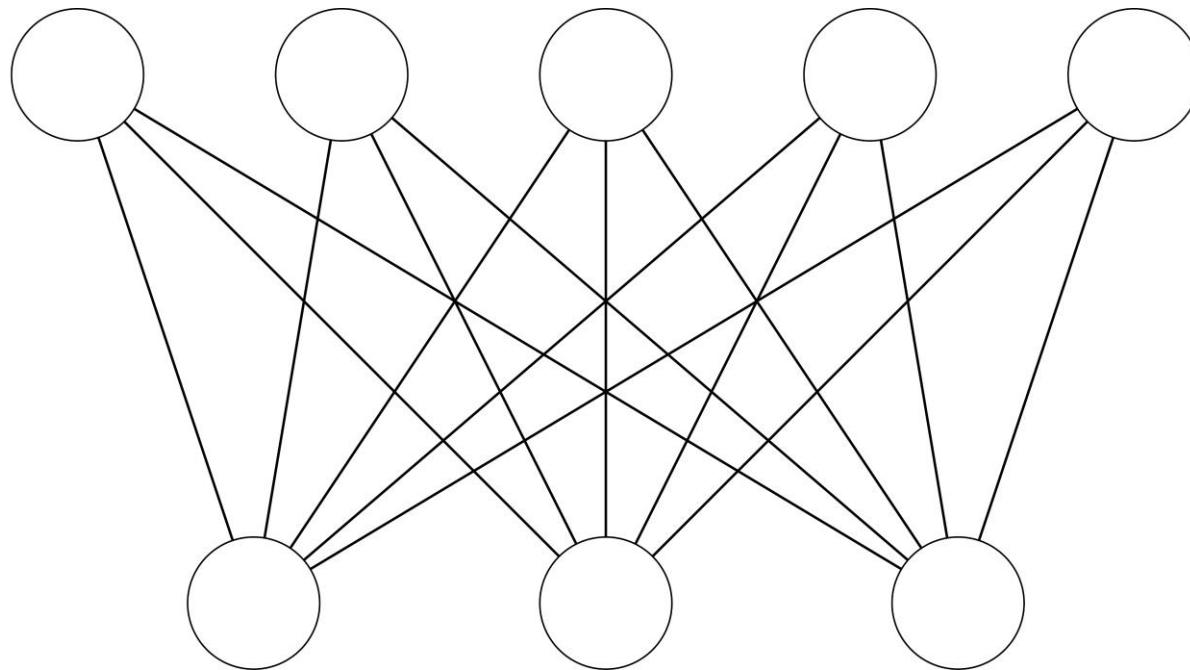
X

given a graph,  
how many successive vertex orderings are there?



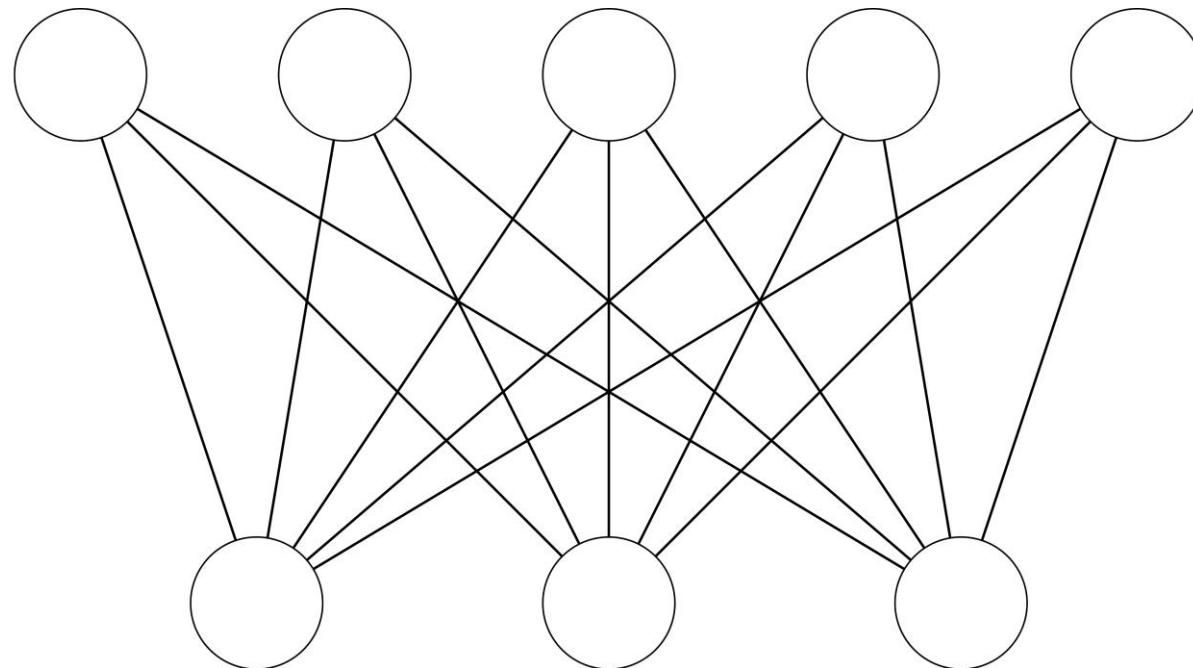
$$\sigma(K_n) = n!$$

given a graph,  
how many successive vertex orderings are there?

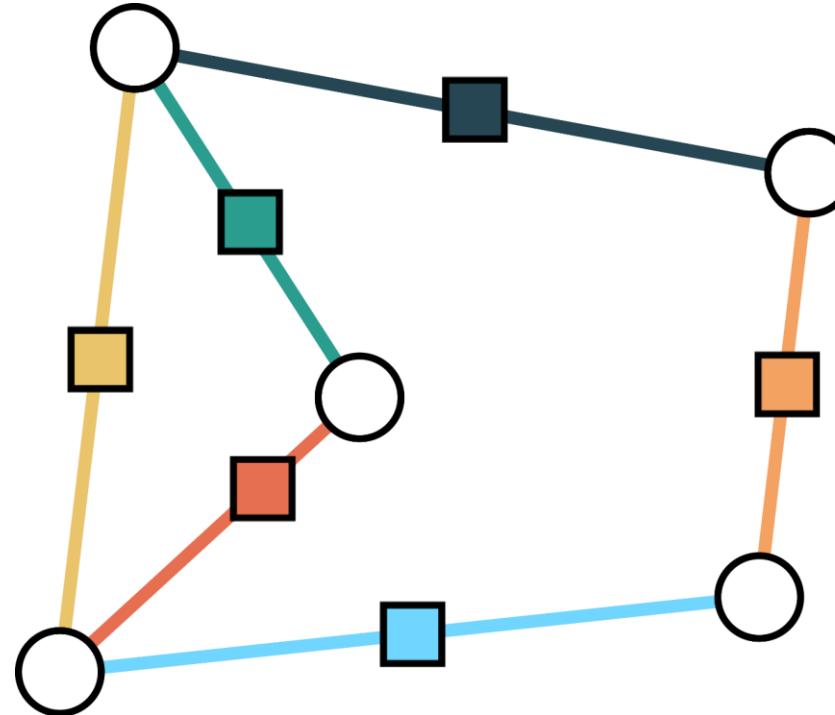


$$\sigma(K_{m,n}) = 2mn(m+n-2)!$$

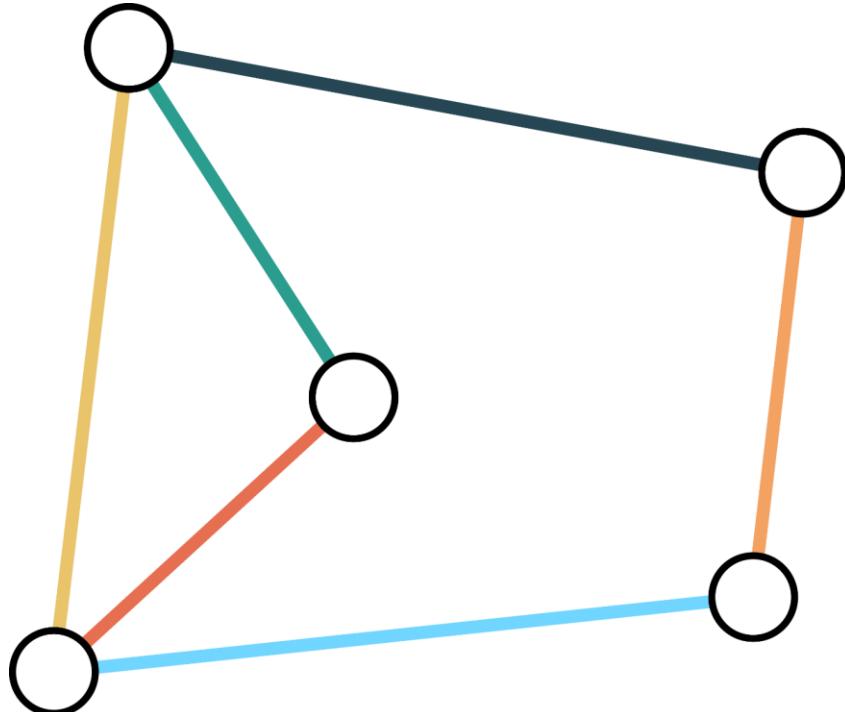
given a graph,  
how many successive *edge* orderings are there?



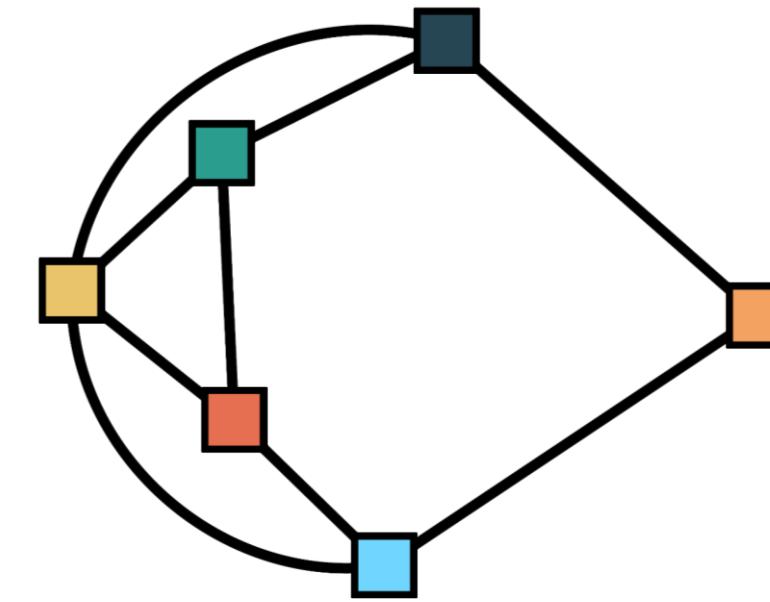
## line graphs



line graphs

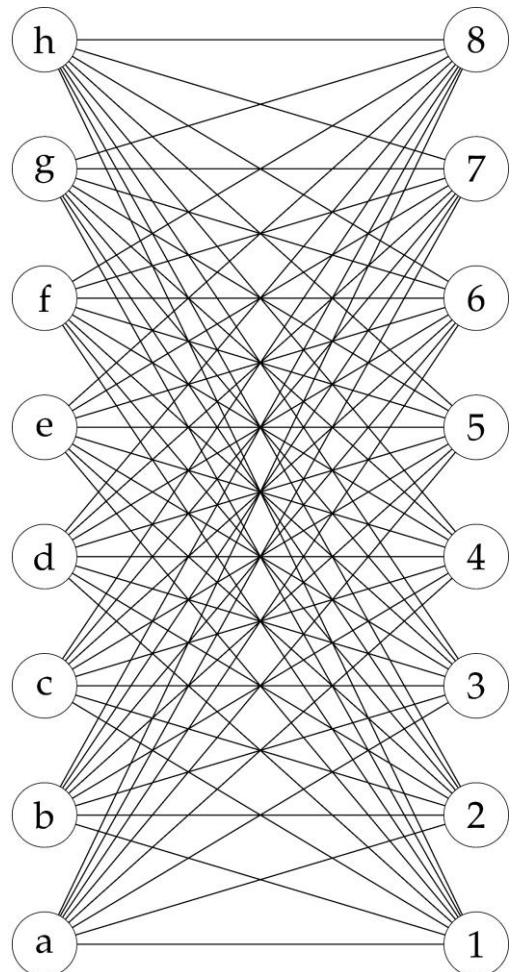


$G$

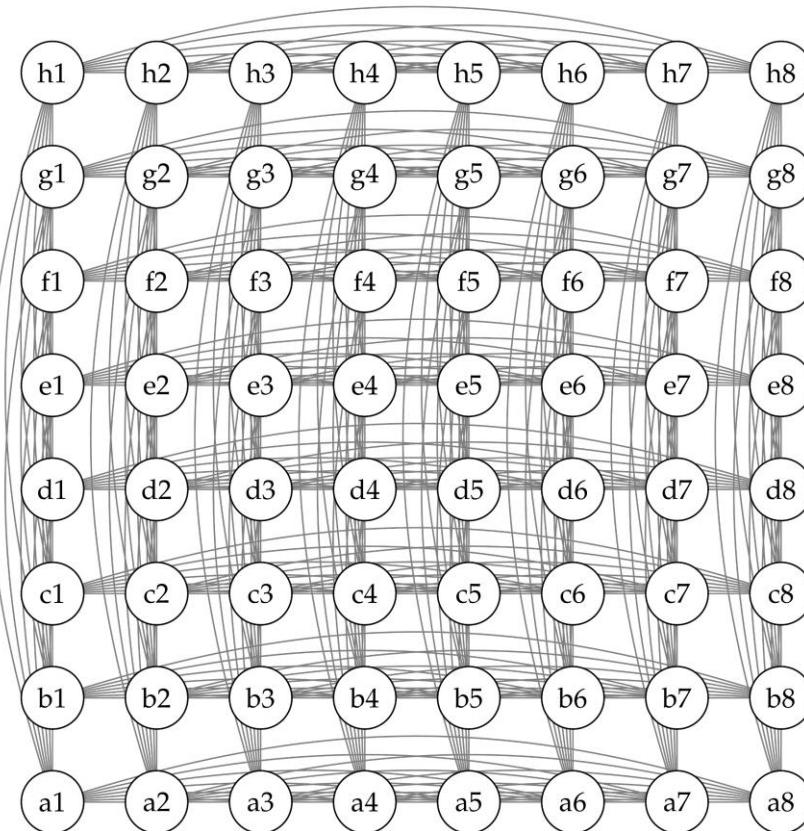


$L(G)$

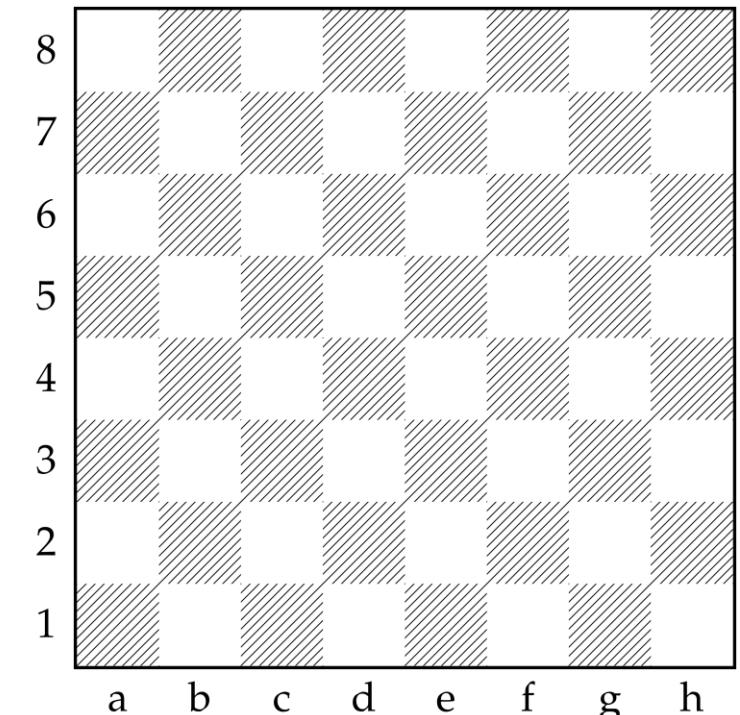
# line graphs



$K_{m,n}$



$L(K_{m,n})$

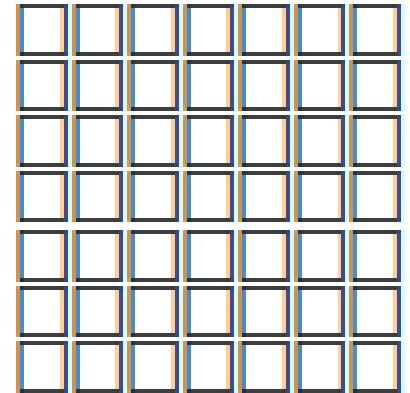


rook graph

successive edge orderings of  $G$   
= successive vertex orderings of  $L(G)$

Gao-Peng, 2021

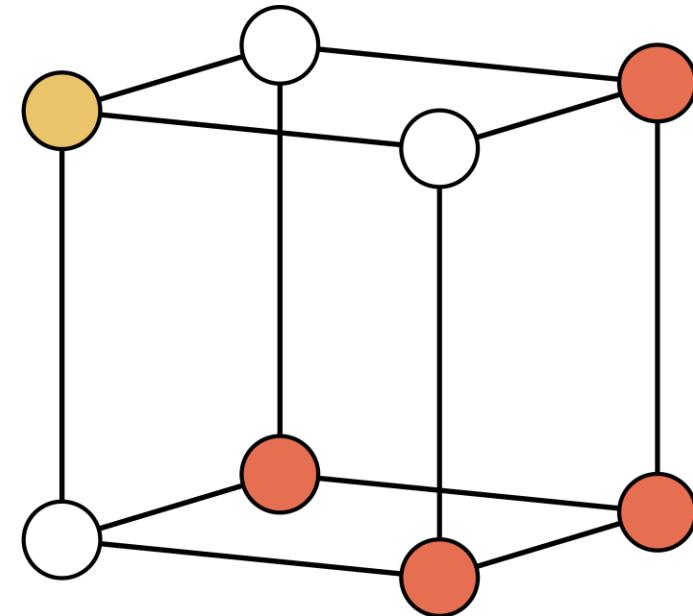
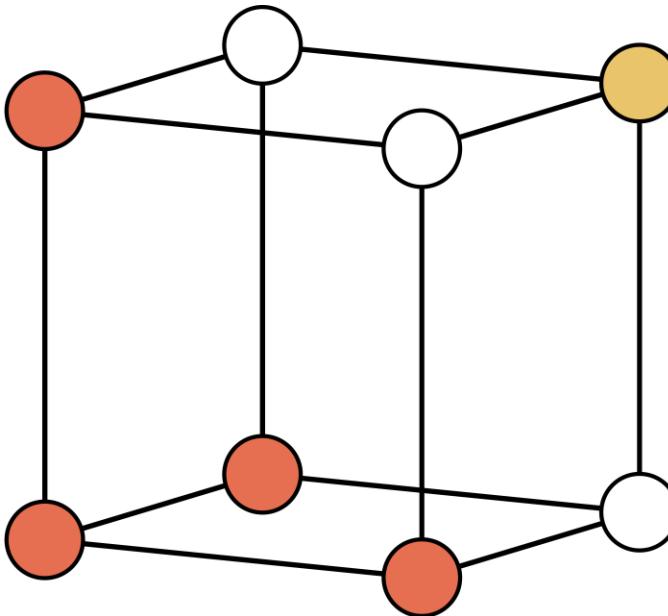
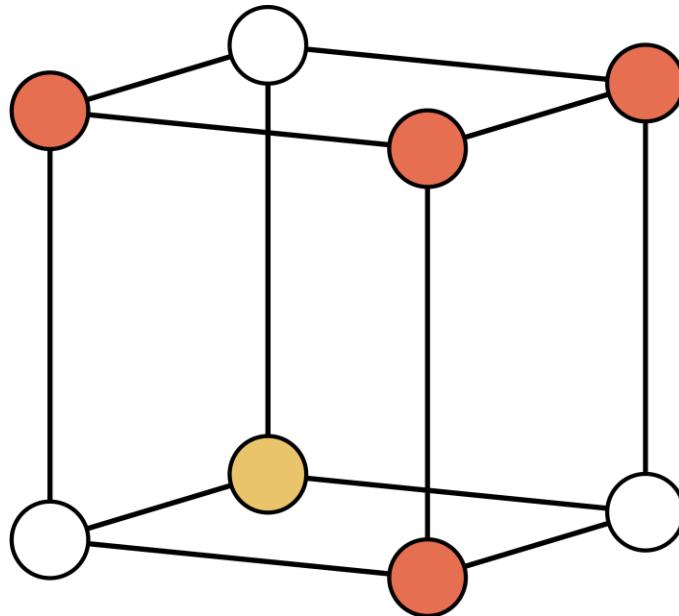
$$\sigma(L(K_{m,n})) = (mn)! \cdot \frac{m+n}{\binom{m+n}{m}}$$



Stanley, 2018

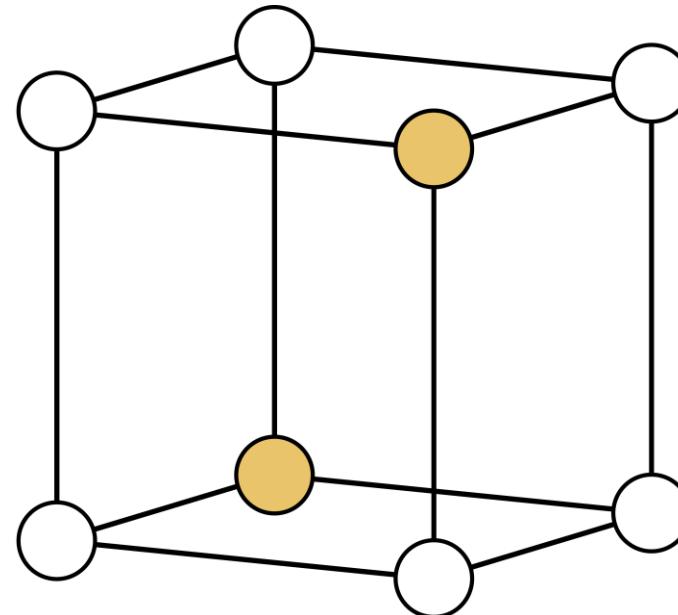
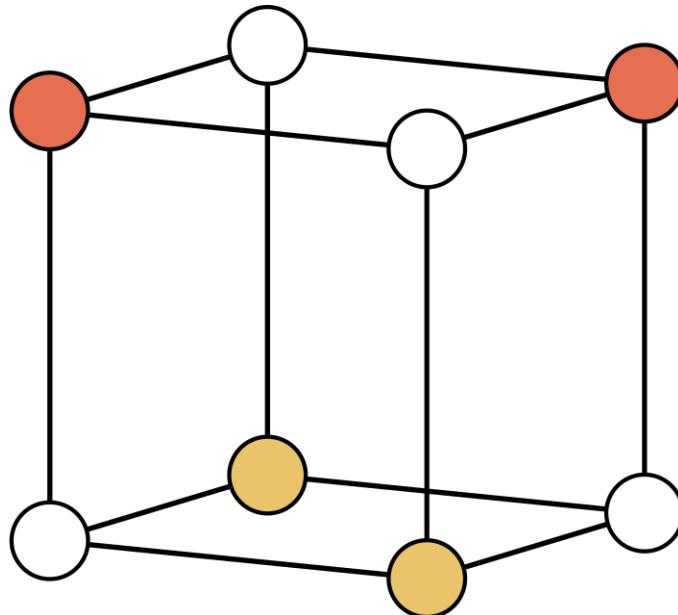
$$\sigma(L(K_n)) = \frac{2^n n! (n-1)! \binom{n}{2}!}{4(2n-2)!}$$

a graph is *regular* if for every  $v_1 \in V$ ,  
the number of vertices in  $V \setminus \{v_1\}$  that are not connected to  $v_1$  is a constant  $a_1$



$$a_1 = 4$$

a graph is *regular* if for every  $v_1 \in V$ ,  
the number of vertices in  $V \setminus \{v_1\}$  that are not connected to  $v_1$  is a constant  $a_1$



a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ ,  
the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

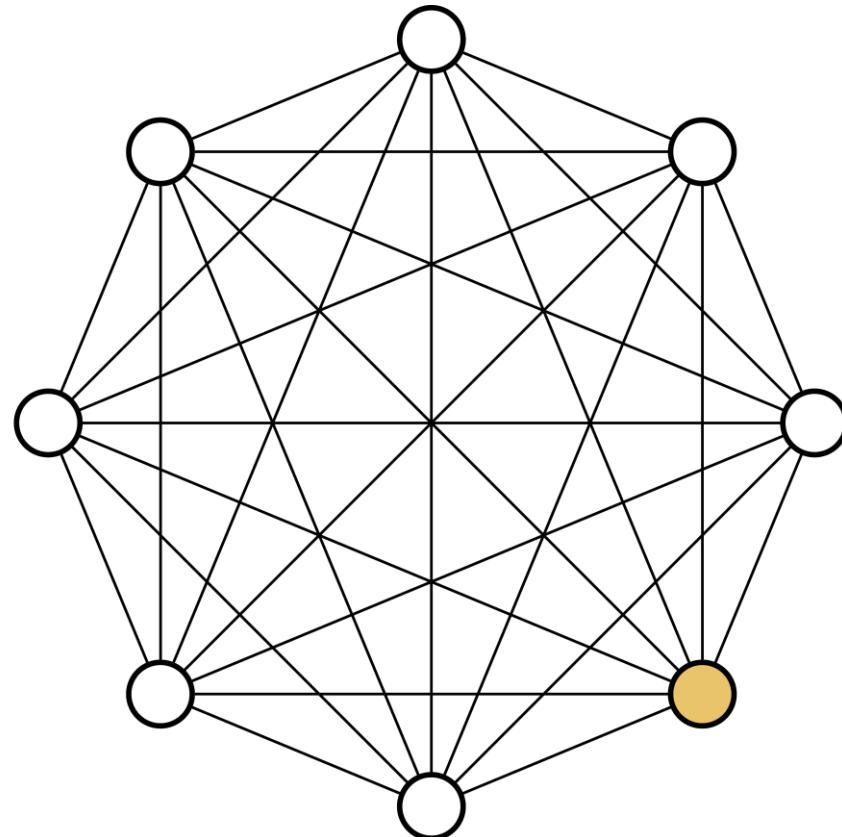
**Theorem** (Fang–Huang–Pach–Tardos–Zuo, 2023). *The number of successive vertex orderings of a fully regular graph  $G$  is given by*

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

*Proof idea.* Use inclusion-exclusion; the fully regular property makes  $\sum_{|I|=m} \Pr(\bigwedge_{v \in I} X_v)$  nice.  $\square$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ ,  
the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

# complete graph $K_n$



$$a_0 = n$$

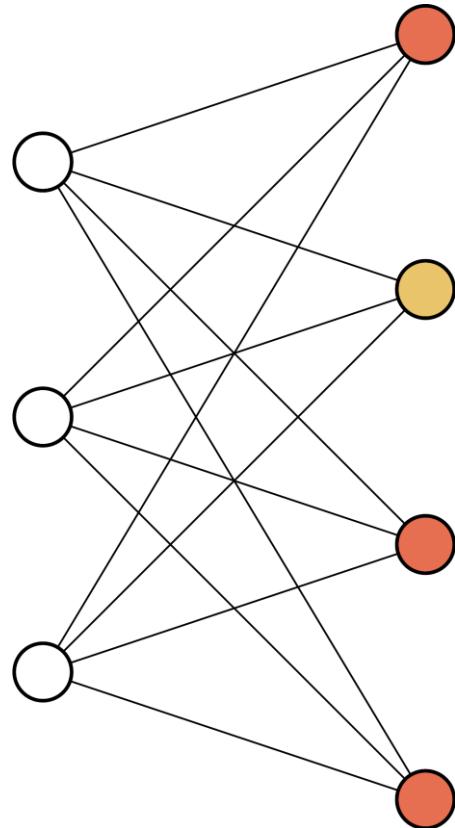
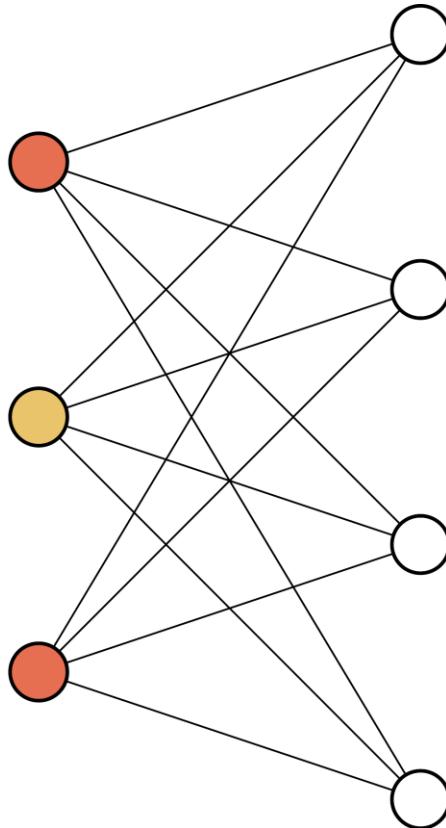
$$a_1 = 0$$

$$\sigma(K_n) = n!$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ ,  
the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

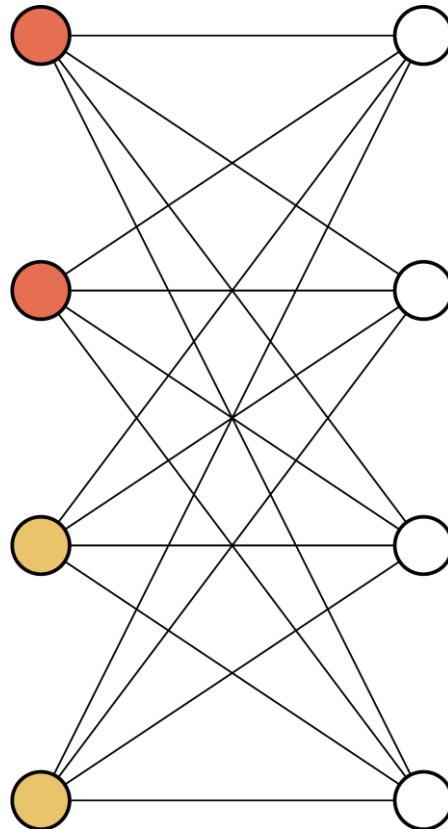
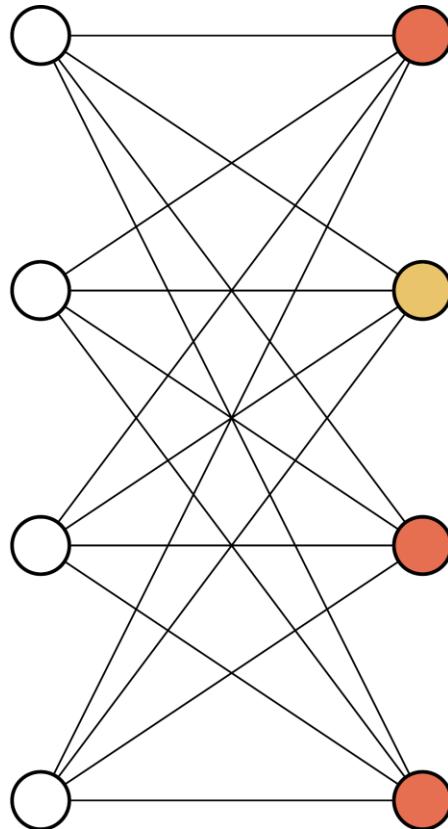
# complete bipartite graph $K_{m,n}$



a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ ,  
the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

# complete bipartite graph $K_{n,n}$



$$\begin{aligned} a_0 &= 2n \\ a_1 &= n - 1 \\ &\vdots \\ a_j &= n - j \end{aligned}$$

$$\sigma(K_{n,n}) = (2n)! \sum_{i=0}^n \prod_{j=1}^i \frac{j-n}{j+n}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

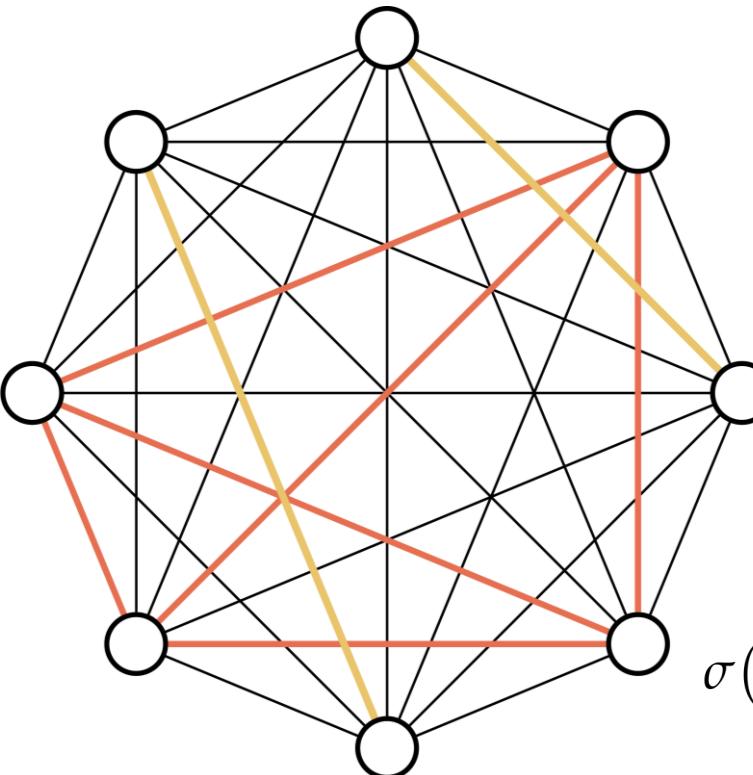
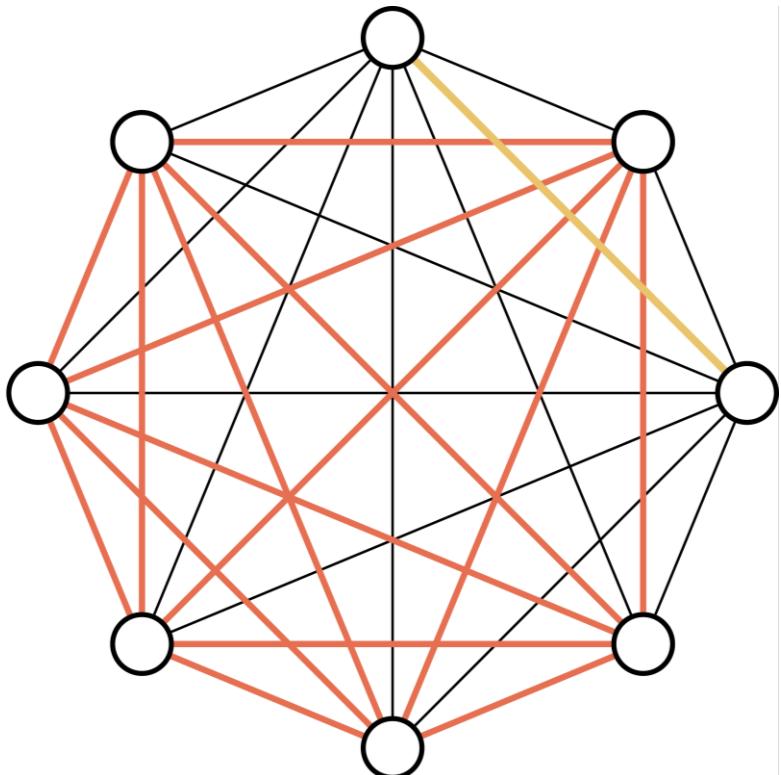
# complete bipartite graph $K_{n,n}$

$$\begin{aligned}
 \sigma(K_{n,n}) &= (2n)! \sum_{i=0}^n \prod_{j=1}^i \frac{j-n}{j+n} \\
 &= (2n)! \left( 1 + \frac{1-n}{1+n} + \frac{1-n}{1+n} \cdot \frac{2-n}{2+n} + \dots \right) \\
 &= (2n)! {}_2F_1 \left( \begin{matrix} 1-n, 1 \\ 1+n \end{matrix} \middle| 1 \right) = (2n)! \frac{n}{2n-1}
 \end{aligned}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ ,  
the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

# line graph $L(K_n)$



$$\begin{aligned} a_0 &= \binom{n}{2} \\ a_1 &= \binom{n-2}{2} \\ \vdots \\ a_j &= \binom{n-2j}{2} \end{aligned}$$

$$\sigma(L(K_n)) = \binom{n}{2}! \sum_{i=0}^{\lfloor n/2 \rfloor} \prod_{j=1}^i \frac{-\binom{n-2j}{2}}{\binom{n}{2} - \binom{n-2j}{2}}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

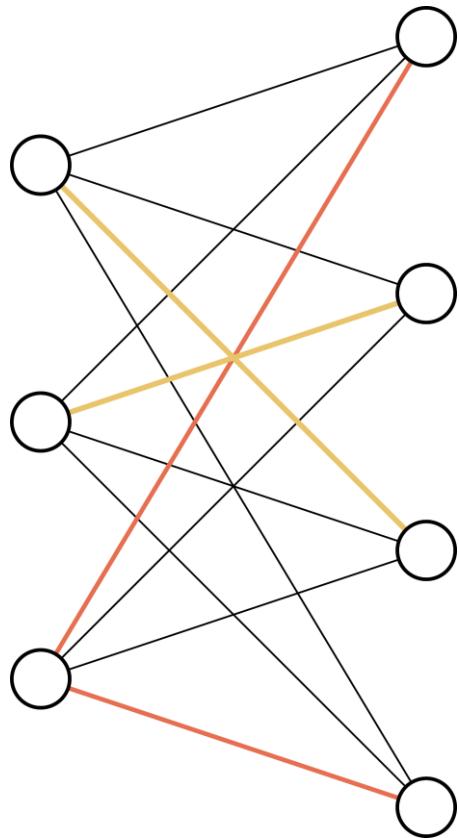
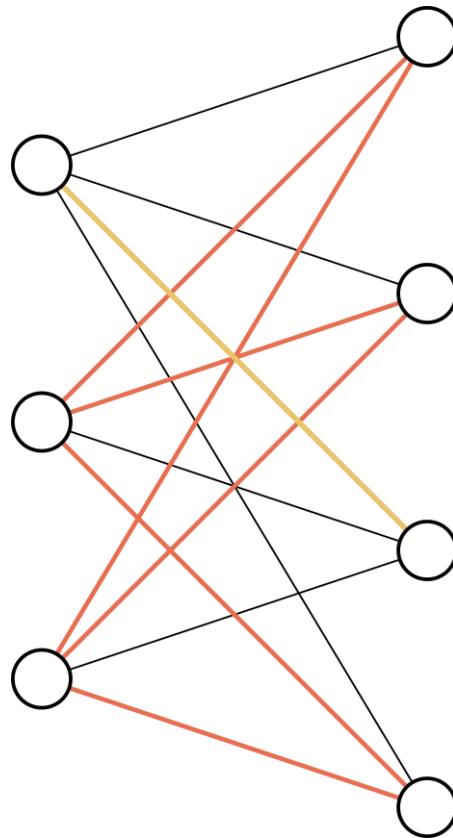
# line graph $L(K_n)$

$$\begin{aligned}
\sigma(L(K_n)) &= \binom{n}{2}! \sum_{i=0}^{\lfloor n/2 \rfloor} \prod_{j=1}^i \frac{-\binom{n-2j}{2}}{\binom{n}{2} - \binom{n-2j}{2}} \\
&= \binom{n}{2}! \left( 1 + \frac{\left(1 - \frac{n}{2}\right) \left(\frac{3}{2} - \frac{n}{2}\right)}{1\left(\frac{3}{2} - n\right)} + \frac{\left(1 - \frac{n}{2}\right) \left(\frac{3}{2} - \frac{n}{2}\right)}{1\left(\frac{3}{2} - n\right)} \cdot \frac{\left(2 - \frac{n}{2}\right) \left(\frac{5}{2} - \frac{n}{2}\right)}{2\left(\frac{5}{2} - n\right)} + \dots \right) \\
&= \binom{n}{2}! {}_2F_1 \left( \begin{matrix} 1 - \frac{n}{2}, \frac{3}{2} - \frac{n}{2} \\ \frac{3}{2} - n \end{matrix} \middle| 1 \right) = \binom{n}{2}! \frac{2^n n! (n-1)!}{4(2n-2)!} = \binom{n}{2}! \frac{2^{n-2}}{C_{n-1}}
\end{aligned}$$

a graph is *fully regular* if for every independent set  $\textcolor{brown}{I} = \{\textcolor{brown}{v}_1, \dots, \textcolor{brown}{v}_j\} \subset V$ ,  
the number of vertices in  $V \setminus \textcolor{brown}{I}$  that are not connected to  $\textcolor{brown}{I}$  is a constant  $\textcolor{red}{a}_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

# line graph $L(K_{m,n})$



$$\begin{aligned} a_0 &= mn \\ a_1 &= (m-1)(n-1) \\ &\vdots \\ a_j &= (m-j)(n-j) \end{aligned}$$

$$\sigma(L(K_{m,n})) = (mn)! \sum_{i=0}^{\min(m,n)} \prod_{j=1}^i \frac{-(m-j)(n-j)}{mn - (m-j)(n-j)}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

# line graph $L(K_{m,n})$

$$\begin{aligned}
\sigma(L(K_{m,n})) &= (mn)! \sum_{i=0}^{\min(m,n)} \prod_{j=1}^i \frac{-(m-j)(n-j)}{mn - (m-j)(n-j)} \\
&= (mn)! \left( 1 + \frac{(1-m)(1-n)}{1(1-m-n)} + \frac{(1-m)(1-n)}{1(1-m-n)} \cdot \frac{(2-m)(2-n)}{2(2-m-n)} + \dots \right) \\
&= (mn)! {}_2F_1 \left( \begin{matrix} 1-m, 1-n \\ 1-m-n \end{matrix} \middle| 1 \right) \\
&= (mn)! \frac{m!}{(n+1)\dots(m+n-1)} = (mn)! \frac{m+n}{\binom{m+n}{m}}
\end{aligned}$$

a graph is *fully regular* if for every independent set  $\textcolor{brown}{I} = \{\textcolor{brown}{v}_1, \dots, \textcolor{brown}{v}_j\} \subset V$ ,  
the number of vertices in  $V \setminus \textcolor{brown}{I}$  that are not connected to  $\textcolor{brown}{I}$  is a constant  $\textcolor{red}{a}_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

a series  $\sum_{k \geq 0} t_k$  is *geometric* if  $\frac{t_{k+1}}{t_k} = c$  is a constant function of  $k$

$$t_k = ct_{k-1} = c^2t_{k-2} = \dots = c^kt_0$$

$$\sum_{k \geq 0} t_k = t_0 \sum_{k \geq 0} c^k$$

a series  $\sum_{k \geq 0} t_k$  is *hypergeometric* if  $\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)}$  is a rational function of  $k$

$$\frac{P(k)}{Q(k)} = \frac{(k + a_1) \dots (k + a_p)}{(k + b_1) \dots (k + b_q)(k + 1)} x$$

$$t_k = \frac{P(k-1)}{Q(k-1)} t_{k-1} = \dots = \frac{P(k-1) \dots P(0)}{Q(k-1) \dots Q(0)} t_0$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} \quad (a)_k = a(a+1) \dots (a+k-1)$$

$$= {}_pF_q \left( \begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \middle| x \right)$$

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!} = {}_0F_0 \left( \begin{array}{c} | \\ x \end{array} \right)$$

$\sum_{k \geq 0} t_k$  is *hypergeometric* if  $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$  is a rational function of  $k$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \middle| x \right)$$

$$(a)_k = a(a+1) \dots (a+k-1)$$

$$\begin{aligned}
(1-x)^{-a} &= \sum_{k \geq 0} \binom{-a}{k} (-x)^k = \sum_{k \geq 0} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} (-1)^k x^k \\
&= \sum_{k \geq 0} (a)_k \frac{x^k}{k!} = {}_1F_0 \left( \begin{matrix} a \\ | \\ x \end{matrix} \right)
\end{aligned}$$

$\sum_{k \geq 0} t_k$  is *hypergeometric* if  $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$  is a rational function of  $k$      $\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ | \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$   
 $(a)_k = a(a+1)\dots(a+k-1)$

$$\sin x = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x \sum_{k \geq 0} \frac{(-1)^k x^{2k}}{(2k+1)!} = x {}_0F_1 \left( \begin{matrix} & \\ 3/2 & \end{matrix} \middle| -\frac{x^2}{4} \right)$$

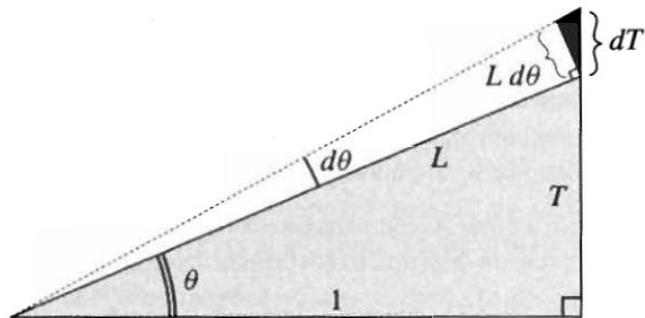
$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} x^{2k+2}/(2k+3)!}{(-1)^k x^{2k}/(2k+1)!} = \frac{1}{(k+3/2)(k+1)} \left( -\frac{x^2}{4} \right)$$

$\sum_{k \geq 0} t_k$  is *hypergeometric* if  $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$  is a rational function of  $k$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$$(a)_k = a(a+1) \dots (a+k-1)$$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k \geq 0} (-1)^k t^{2k} dt = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{2k+1} = x \sum_{k \geq 0} \frac{(-1)^k x^{2k}}{2k+1}$$



$$\frac{dT}{L d\theta} = \frac{L}{1} \quad \Rightarrow \quad \frac{dT}{d\theta} = L^2 = 1 + T^2.$$

$$= x {}_2F_1 \left( \begin{matrix} 1/2, 1 \\ 3/2 \end{matrix} \middle| -x^2 \right)$$

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} x^{2k+2}/(2k+3)}{(-1)^k x^{2k}/(2k+1)} = \frac{k+1/2}{k+3/2}(-x^2) = \frac{(k+1/2)(k+1)}{(k+3/2)(k+1)}(-x^2)$$

$\sum_{k \geq 0} t_k$  is hypergeometric if  $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$  is a rational function of  $k$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$$(a)_k = a(a+1) \dots (a+k-1)$$

Bessel function  $J_\alpha(x) = \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} = \frac{(x/2)^\alpha}{\Gamma(\alpha + 1)} {}_0F_1 \left( \begin{matrix} s \\ \alpha + 1 \end{matrix} \middle| -\frac{x^2}{4} \right)$

incomplete gamma function  $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt = \frac{z^s}{s} {}_1F_1 \left( \begin{matrix} s \\ s+1 \end{matrix} \middle| -x \right)$

Chebyshev polynomial  $T_n(\cos \theta) = \cos n\theta \quad T_n(x) = {}_2F_1 \left( \begin{matrix} -n, n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2} \right)$

dilogarithm  $\text{Li}_2(x) = \sum_{n > 0} \frac{x^n}{n^2} = x {}_3F_2 \left( \begin{matrix} 1, 1, 1 \\ 2, 2 \end{matrix} \middle| x \right)$

Bring radical  $x^5 + x + a = 0 \quad \text{BR}(t) = -a {}_4F_3 \left( \begin{matrix} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ \frac{1}{2}, \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| \frac{5^5}{2^8} t^4 \right)$

$\sum_{k \geq 0} t_k$  is hypergeometric if  $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$  is a rational function of  $k$   $\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$   
 $(a)_k = a(a+1) \dots (a+k-1)$

**Theorem** (Euler). *If  $\Re(c - b) > 0$  and  $x \in \mathbf{C} \setminus [1, \infty)$ , then*

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt.$$

*Proof.* We have

$$\begin{aligned} & \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n \geq 0} \frac{(a)_n}{n!} x^n \int_0^1 t^{n+b-1} (1-t)^{c-b-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n \geq 0} \frac{(a)_n}{n!} x^n \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)} \\ &= \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \end{aligned}$$

□

$$\sum_{k \geq 0} t_k \text{ is hypergeometric if } \frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x \text{ is a rational function of } k \quad \sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

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**Theorem** (Euler). If  $\Re(c - b) > 0$  and  $x \in \mathbf{C} \setminus [1, \infty)$ , then

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt.$$

**Corollary** (Gauss, 1812). If  $\Re(c - a - b) > 0$ , we have

$$\sum_{n \geq 0} \frac{(a)_n (b)_n}{n! (c)_n} = {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

*Proof.* Send  $x \rightarrow 1^-$  to get

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-a-b-1} dt = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

□

$$\sum_{k \geq 0} t_k \text{ is hypergeometric if } \frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x \text{ is a rational function of } k \quad \sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$$(a)_k = a(a+1)\dots(a+k-1)$$

$$\begin{aligned}
{}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-x+xs)^{-a} (1-s)^{b-1} s^{c-b-1} ds \\
&= \frac{(1-x)^{-a}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \left(1 - \frac{xs}{x-1}\right)^{-a} s^{c-b-1} (1-s)^{b-1} ds \\
&= (1-x)^{-a} {}_2F_1 \left( \begin{matrix} a, c-b \\ c \end{matrix} \middle| \frac{x}{x-1} \right) \\
&= (1-x)^{-a} \left(1 - \frac{x}{x-1}\right)^{-c+b} {}_2F_1 \left( \begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x \right) \\
&= (1-x)^{c-a-b} {}_2F_1 \left( \begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x \right)
\end{aligned}$$

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$(a)_k = a(a+1) \dots (a+k-1)$

$${}_2F_1 \left( \begin{matrix} a, & b \\ c & \end{matrix} \middle| x \right) = (1-x)^{c-a-b} {}_2F_1 \left( \begin{matrix} c-a, & c-b \\ c & \end{matrix} \middle| x \right)$$

$$(1-x)^{f-d-e} {}_2F_1 \left( \begin{matrix} f-d, & f-e \\ f & \end{matrix} \middle| x \right) = {}_2F_1 \left( \begin{matrix} d, & e \\ f & \end{matrix} \middle| x \right)$$

$${}_2F_1 \left( \begin{matrix} a, & b \\ c & \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} f-d, & f-e \\ f & \end{matrix} \middle| x \right) = {}_2F_1 \left( \begin{matrix} c-a, & c-b \\ c & \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} d, & e \\ f & \end{matrix} \middle| x \right) \quad c-a-b=f-d-e$$

$$[x^n] = \sum_{k=0}^n \frac{(a)_k(b)_k(f-d)_{n-k}(f-e)_{n-k}}{(c)_k k! (f)_{n-k} (n-k)!} = \frac{(f-d)_n(f-e)_n}{n!(f)_n} {}_4F_3 \left( \begin{matrix} a, & b, & 1-f-n, & -n \\ c, & d-f-n+1, & e-f-n+1 & \end{matrix} \middle| 1 \right)$$

$${}_4F_3 \left( \begin{matrix} -n, & a, & b, & -f-n+1 \\ c, & d-f-n+1, & e-f-n+1 & \end{matrix} \middle| 1 \right) = \frac{(d)_n(e)_n}{(f-d)_n(f-e)_n} {}_4F_3 \left( \begin{matrix} -n, & c-a, & c-b, & 1-f-n \\ c, & 1-d-n, & 1-e-n & \end{matrix} \middle| 1 \right)$$

$${}_2F_1 \left( \begin{matrix} a, & b \\ c & \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \middle| x \right)$$

$(a)_k = a(a+1) \dots (a+k-1)$

$$\begin{aligned}
{}_4F_3 \left( \begin{matrix} -n, a, b, -f-n+1 \\ c, d-f-n+1, e-f-n+1 \end{matrix} \middle| 1 \right) &= \frac{(d)_n(e)_n}{(f-d)_n(f-e)_n} {}_4F_3 \left( \begin{matrix} -n, c-a, c-b, 1-f-n \\ c, 1-d-n, 1-e-n \end{matrix} \middle| 1 \right) \\
{}_4F_3 \left( \begin{matrix} -n, a, b, c \\ d, e, f \end{matrix} \middle| 1 \right) &= \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n} {}_4F_3 \left( \begin{matrix} -n, a, d-b, d-c \\ d, a+1-n-e, a+1-n-f \end{matrix} \middle| 1 \right) \quad \begin{matrix} a+b+c-n+1 \\ = d+e+f \end{matrix} \\
{}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) &= \frac{(e-a)_n}{(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, d-b \\ d, a+1-n-e \end{matrix} \middle| 1 \right) \quad \begin{matrix} f, c \rightarrow \infty \\ f-c \text{ fixed} \end{matrix} \\
&= \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right)
\end{aligned}$$

**Theorem** (Sheppard, 1912). *Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have*

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$(a)_k = a(a+1) \dots (a+k-1)$

## indefinite summation

$$T_n = \sum_{k=0}^n t_k$$

$$\sum_{k=0}^n \binom{N}{k} \quad \text{no simple form}$$

$$\int_{-\infty}^a e^{-x^2} dx \quad \text{no simple form}$$

Gosper's algorithm

## definite summation

$$T_n = \sum_{k=-\infty}^{\infty} t_{n,k}$$

$$\sum_{k=-\infty}^{\infty} \binom{N}{k} = 2^N$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Zeilberger's algorithm

**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$$(a)_k = a(a+1) \dots (a+k-1)$$

## Gosper's algorithm

given hypergeometric  $t_n$ , does there exist hypergeometric  $T_n$  satisfying  $T_{n+1} - T_n = t_n$ ?

(if so, we have  $\sum_{k=0}^n t_k = T_{n+1} - T_0$ )

**Problem.** Find a nice form for

$$\sum_{n=0}^m t_n = \sum_{n=0}^m \frac{1}{(n+1)4^{2n}} \binom{2n}{n}^2.$$

*Solution.* By Gosper's algorithm, we have  $T_{n+1} - T_n = t_n$ , where  $T_n = \frac{n}{4^{2n-1}} \binom{2n}{n}^2$ . Thus

$$\sum_{n=0}^m t_n = T_{m+1} - T_0 = \frac{m+1}{4^{2m+1}} \binom{2m+2}{m+1}^2.$$

□

**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

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## Gosper's algorithm

given hypergeometric  $t_n$ , does there exist hypergeometric  $T_n$  satisfying  $T_{n+1} - T_n = t_n$ ?

(if so, we have  $\sum_{k=0}^n t_k = T_{n+1} - T_0$ )

**Problem.** Find a nice form for

$$\sum_{n=0}^{m-1} t_n = \sum_{n=0}^{m-1} \frac{2^n}{n+1}.$$

*Solution.* Gosper's algorithm halts without finding a solution. This constitutes a rigorous proof that  $T_n = \sum_{k=0}^{n-1} t_k$  is not a hypergeometric term.  $\square$

**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

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## Gosper's algorithm

given hypergeometric  $t_n$ , does there exist hypergeometric  $T_n$  satisfying  $T_{n+1} - T_n = t_n$ ?

(if so, we have  $\sum_{k=0}^n t_k = T_{n+1} - T_0$ )

$$\sum_{k=0}^m t_k = \sum_{k=0}^m (-1)^k \binom{n}{k} \quad \text{integer } n > 0$$

1. Write the term ratio as  $\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$  for polynomials  $p(k), q(k), r(k)$  such that if  $(k + \alpha) \mid q(k)$  and  $(k + \beta) \mid r(k)$ , then  $\alpha - \beta$  is not a positive integer. (This is always possible and is not hard.)

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} n(n-1)\dots(n-k)/(k+1)!}{(-1)^k n(n-1)\dots(n-k+1)/k!} = \frac{k-n}{k+1}$$

$$p(k) = 1, \quad q(k) = k - n, \quad r(k) = k$$

2. Write  $T_k = \frac{r(k)s(k)}{p(k)} t_k$ . Then the condition  $T_{n+1} - T_n = t_n$  implies a recurrence  $p(k) = q(k)s(k+1) - r(k)s(k)$  for  $s(k)$ . (The condition from step 1 and the requirement that  $T_n$  is a hypergeometric term force  $s(k)$  to be polynomial.) Solve for  $s(k)$  to get  $T_k$ ; if  $s(k)$  does not exist, it means that no hypergeometric  $T_k$  exists.

$$1 = (k-n)s(k+1) - ks(k)$$

$$s(k) = -1/n$$

$$T_k = \frac{k(-1/n)}{1} (-1)^k \binom{n}{k} = (-1)^{k-1} \binom{n-1}{k-1}$$

**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$$(a)_k = a(a+1)\dots(a+k-1)$$

## Gosper's algorithm

given hypergeometric  $t_n$ , does there exist hypergeometric  $T_n$  satisfying  $T_{n+1} - T_n = t_n$ ?

(if so, we have  $\sum_{k=0}^n t_k = T_{n+1} - T_0$ )

$$\sum_{k=0}^m t_k = \sum_{k=0}^m \binom{n}{k} \quad \text{integer } n > 0$$

1. Write the term ratio as  $\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$  for polynomials  $p(k), q(k), r(k)$  such that if  $(k + \alpha) \mid q(k)$  and  $(k + \beta) \mid r(k)$ , then  $\alpha - \beta$  is not a positive integer. (This is always possible and is not hard.)

$$\frac{t_{k+1}}{t_k} = \frac{n(n-1)\dots(n-k)/(k+1)!}{n(n-1)\dots(n-k+1)/k!} = \frac{n-k}{k+1}$$

$$p(k) = 1, \quad q(k) = n - k, \quad r(k) = k$$

2. Write  $T_k = \frac{r(k)s(k)}{p(k)} t_k$ . Then the condition  $T_{n+1} - T_n = t_n$  implies a recurrence  $p(k) = q(k)s(k+1) - r(k)s(k)$  for  $s(k)$ . (The condition from step 1 and the requirement that  $T_n$  is a hypergeometric term force  $s(k)$  to be polynomial.) Solve for  $s(k)$  to get  $T_k$ ; if  $s(k)$  does not exist, it means that no hypergeometric  $T_k$  exists.

$$\begin{aligned} 1 &= (n - k)s(k + 1) - ks(k) \\ 2 &= (n - 2k)(s(k + 1) + s(k)) + n(s(k + 1) - s(k)) \\ &\quad \deg(s(k)) = d \\ d &= -1 \end{aligned}$$

no hypergeometric  $T_n$  exists

**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a+b-n-d-e+1 \\ a-n-d+1, a-n-e+1 \end{matrix} \middle| 1 \right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$(a)_k = a(a+1)\dots(a+k-1)$

## Zeilberger's algorithm

**Proposition.** For integer  $n \geq 0$ , we have

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}.$$

*Proof.* Let  $F(n, k) = \binom{n}{k}^2 / \binom{2n}{n}$ , and set  $G(n, k) := R(n, k)F(n, k)$ , where

$$R(n, k) = -\frac{k^2(3n - 2k + 3)}{2(2n + 1)(n - k + 1)^2}.$$

Then it is straightforward to verify that

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k);$$

summing both sides over  $k$  then gives

$$\sum_k F(n + 1, k) = \sum_k F(n, k)$$

for  $n \geq 0$ . Since  $\sum_k F(0, k) = 1$ , we are done. □

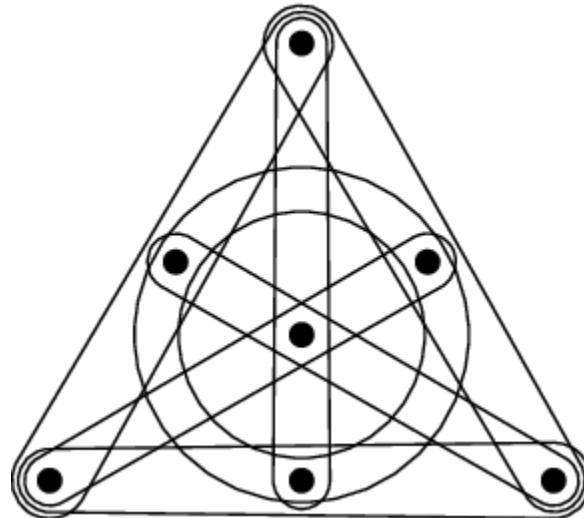
**Theorem** (Sheppard, 1912). Given  $a, b, d, e \in \mathbf{C}$  and integer  $n \geq 0$ , we have

$${}_3F_2 \left( \begin{matrix} -n, a, b \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d - a)_n (e - a)_n}{(d)_n (e)_n} {}_3F_2 \left( \begin{matrix} -n, a, a + b - n - d - e + 1 \\ a - n - d + 1, a - n - e + 1 \end{matrix} \middle| 1 \right).$$

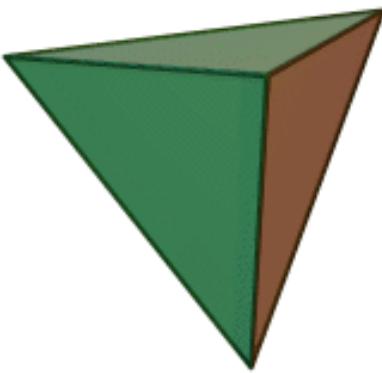
$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right)$$

$(a)_k = a(a + 1) \dots (a + k - 1)$

An *r-uniform hypergraph* or *r-graph* consists of a finite vertex set  $V$  along with an edge set  $E \subseteq \binom{V}{r}$ . Each edge of the *r*-graph is an *r*-element subset of vertices.



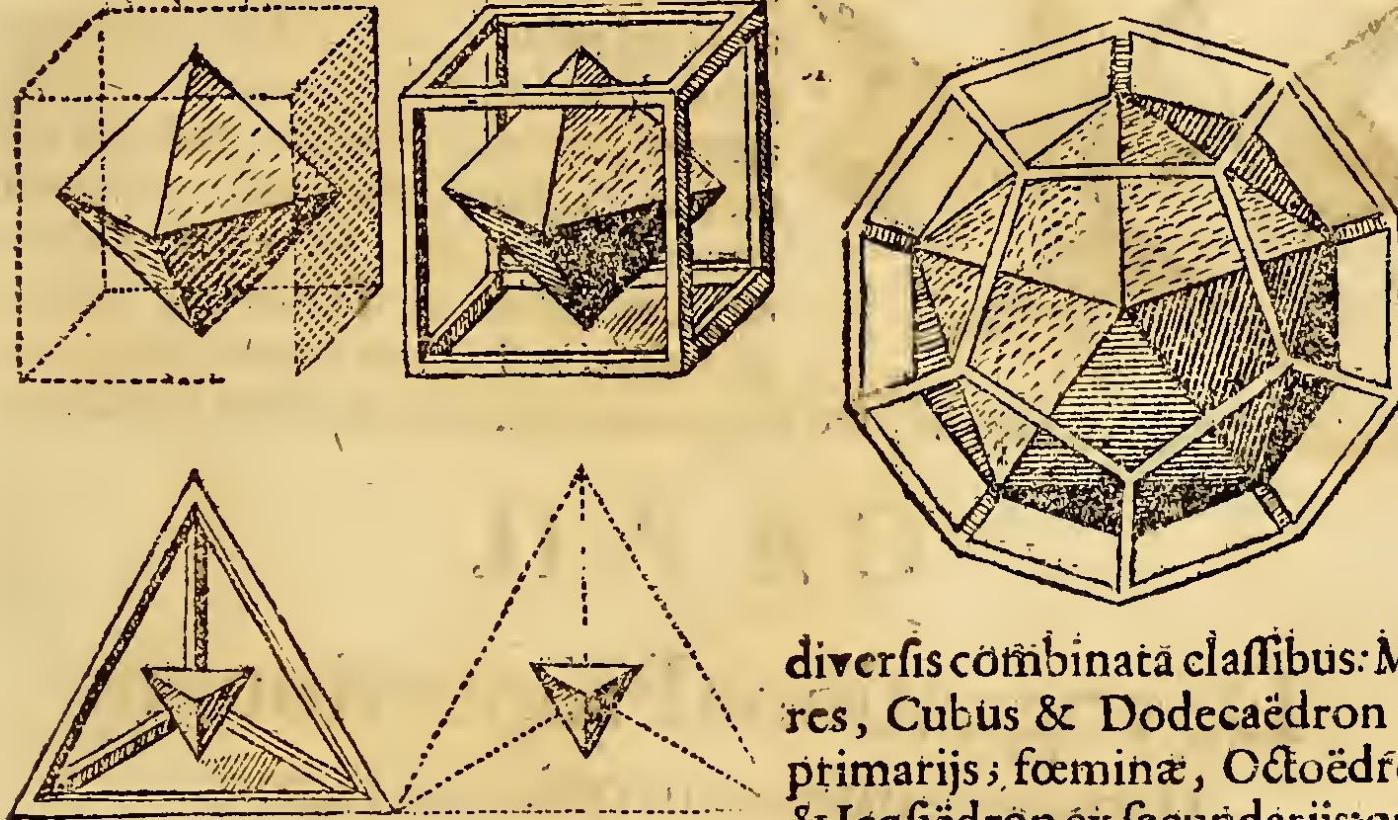
$K_4^{(3)}$  is the complete 3-uniform hypergraph on four vertices



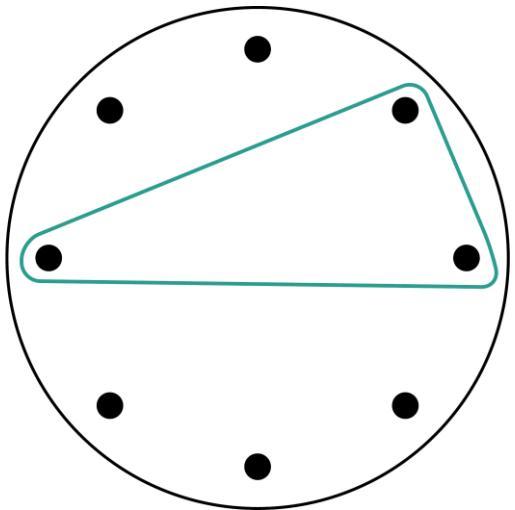
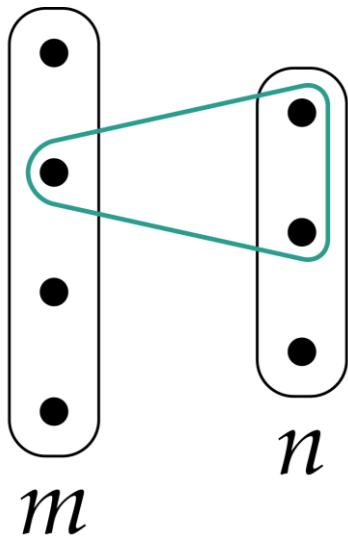
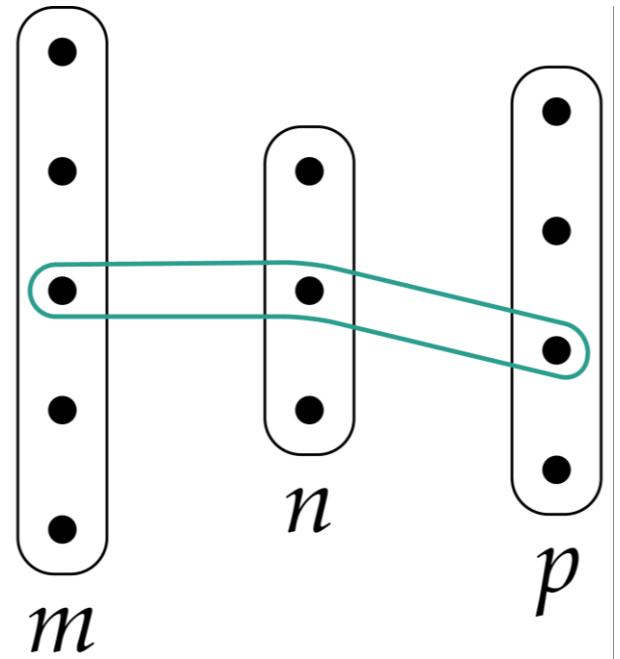
also known as a tetrahedron

the *line graph*  $L(H)$  of a hypergraph  $H$  has each edge of  $H$  as a vertex, with two  $H$ -edges adjacent if they intersect

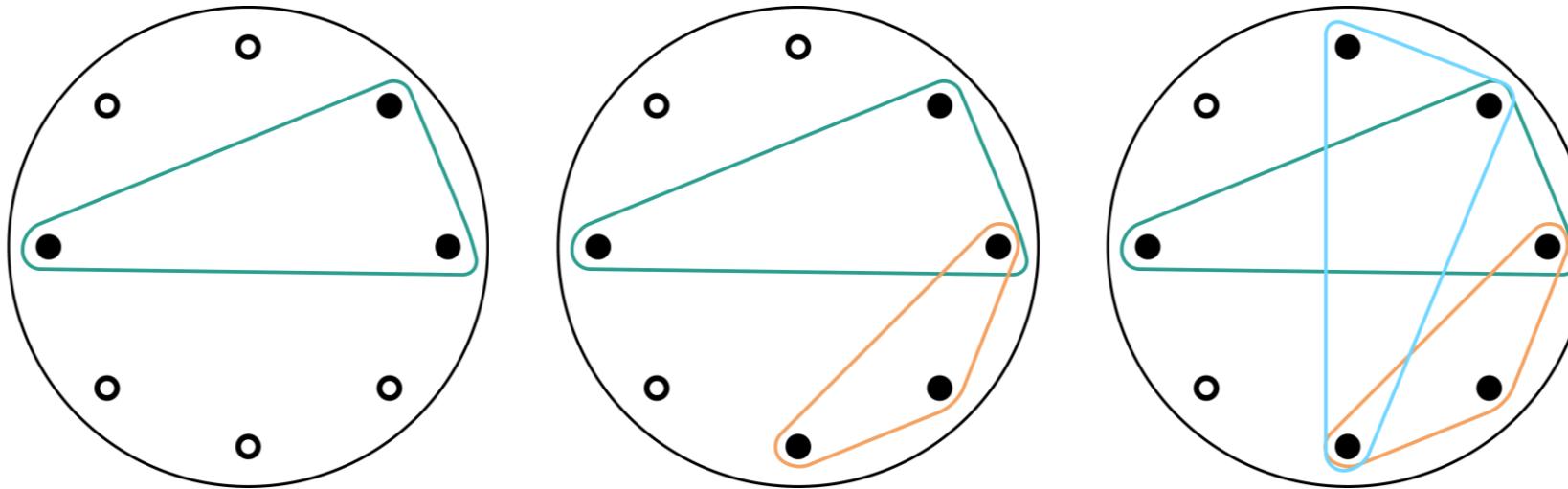
Sunt autem notabilia duō veluti conjugia harum figurarum, ex



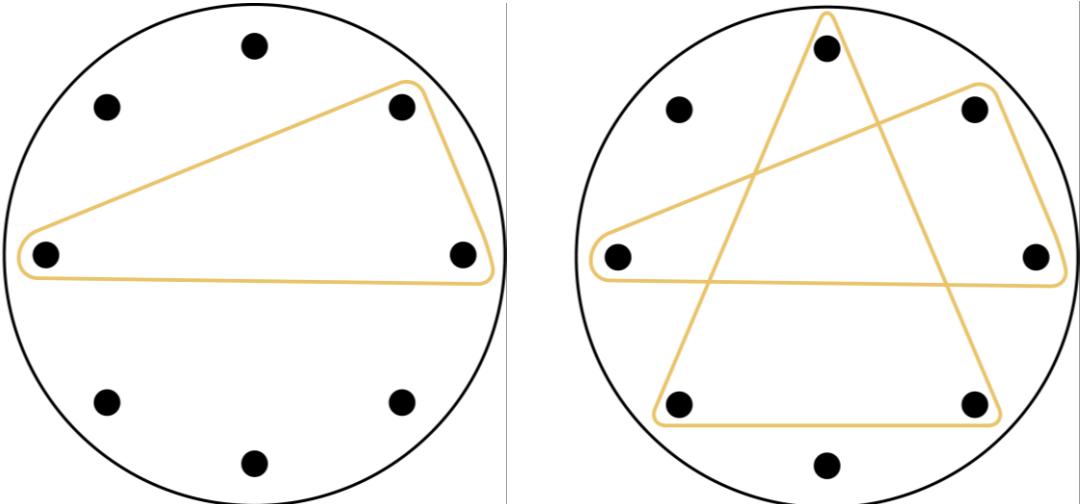
diversis combinata classibus: Ma-  
res, Cubus & Dodecaëdron ex  
primarijs; fœminæ, Octoëdron  
& Icosiëdron ex secundarijs; qui-  
bus accedit una veluti cœlebs aut Androgynos, Tetraëdron; quia sibi  
ipsi inscribitur, ut illæ fœmellæ maribus inscriventur & veluti subji-  
ciuntur, & signia sexus fœminina masculinis opposita habent, angulos  
scilicet planiciebus.

$K_n^{(3)}$  $\binom{n}{3}$  edges $K_{m,n}^{(1,2)}$  $m \binom{n}{2}$  edges $K_{m,n,p}^{(1,1,1)}$  $mnp$  edges

a successive vertex ordering of  $L(K_n^{(3)})$  is a way to order the 3-element subsets of  $\{1, 2, \dots, n\}$  so that each set after the first one contains some element of a previously chosen set



# line graph $L(K_n^{(3)})$



$$a_0 = \binom{n}{3}$$

$$a_1 = \binom{n-3}{3}$$

$$\vdots$$

$$a_j = \binom{n-3j}{3}$$

$$\sigma(L(K_n^{(3)})) = \binom{n}{3}! \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

# line graph $L(K_n^{(3)})$

$$\sigma(L(K_n^{(3)})) = \binom{n}{3}! \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}}$$

```
In[52]:= Table[ Sum[ Floor[n/3] i Product[ -Binomial[n - 3 j, 3], {j, 1, i}], Binomial[n, 3] - Binomial[n - 3 j, 3]], {n, 3, 20}]
```

```
Out[52]= {1, 1, 1, 18/19, 27/31, 18/23, 459/664, 297/493, 1755/3379, 162/365, 368631/979151, 3726/11741, 4363065/16398253, 21141/95215, 100602/545179, 855961182/5601577967, 898724539971/7125324321331, 308367/2970431}
```

```
In[53]:= %52 // FactorInteger
```

```
Out[53]= {{ {1, 1}}, {{1, 1}}, {{1, 1}}, {{2, 1}}, {3, 2}, {19, -1}}, {{3, 3}}, {31, -1}}, {{2, 1}}, {3, 2}, {23, -1}}, {{2, -3}}, {3, 3}, {17, 1}, {83, -1}}, {{3, 3}}, {11, 1}, {17, -1}, {29, -1}}, {{3, 3}}, {5, 1}, {13, 1}, {31, -1}, {109, -1}}, {{2, 1}}, {3, 4}, {5, -1}, {73, -1}}, {{3, 5}}, {37, 1}, {41, 1}, {47, -1}, {83, -1}, {251, -1}}, {{2, 1}}, {3, 4}, {23, 1}, {59, -1}, {199, -1}}, {{3, 8}}, {5, 1}, {7, 1}, {19, 1}, {29, -1}, {47, -1}, {53, -1}, {227, -1}}, {{3, 6}}, {5, -1}, {29, 1}, {137, -1}, {139, -1}}, {{2, 1}}, {3, 7}, {23, 1}, {67, -1}, {79, -1}, {103, -1}}, {{2, 1}}, {3, 8}, {19, -1}, {37, 1}, {41, 1}, {43, 1}, {61, -1}, {149, -1}, {163, -1}, {199, -1}}, {{3, 10}}, {41, 1}, {43, 1}, {89, 1}, {97, 1}, {193, -1}, {283, -1}, {409, -1}, {467, -1}, {683, -1}}, {{3, 8}}, {47, 1}, {97, -1}, {113, -1}, {271, -1}}}
```

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

**Conjecture** (Fang–Huang–Pach–Tardos–Zuo, 2023). *We have, for  $n \geq 3$ ,*

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = \left\lfloor \frac{n}{3} \right\rfloor \prod_{\substack{j=n+1 \\ 3 \nmid j}}^{3n/2-2} c_j \Bigg/ \prod_{\substack{j=3 \\ 3 \mid j}}^{n-3} c_j,$$

where  $c_j = \frac{6}{j} \left( \binom{n}{3} - \binom{n-j}{3} \right) = j^2 + (3-3n)j + 3n^2 - 6n + 2$ .

**Theorem** (H., 2023). We have, for  $n \geq 3$ ,

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = \left\lfloor \frac{n}{3} \right\rfloor \prod_{\substack{j=n+1 \\ 3 \nmid j}}^{3n/2-2} c_j \Bigg/ \prod_{\substack{j=3 \\ 3 \mid j}}^{n-3} c_j,$$

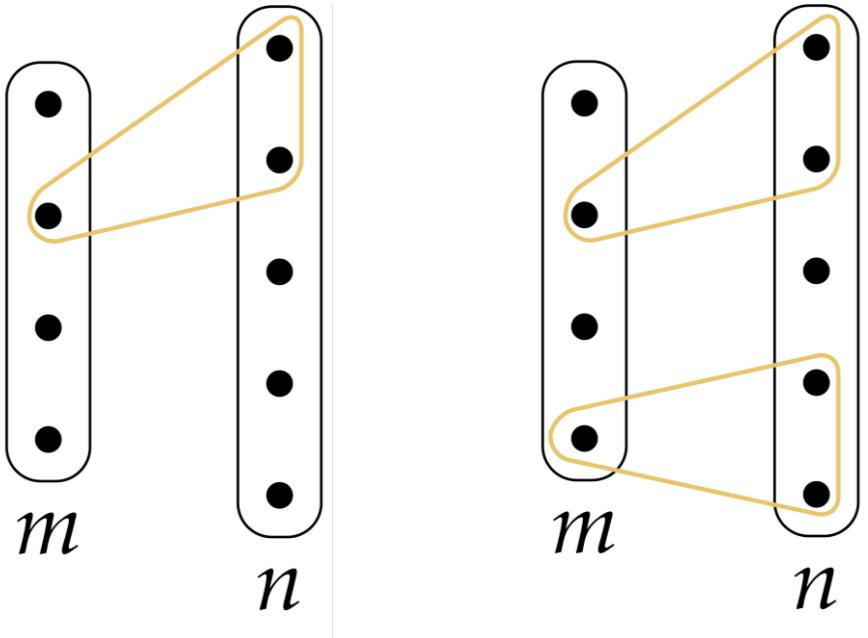
where  $c_j = \frac{6}{j} \left( \binom{n}{3} - \binom{n-j}{3} \right) = j^2 + (3-3n)j + 3n^2 - 6n + 2$ .

*Proof sketch.* Let  $X_n = \sqrt{1 + 6n - 3n^2}$ . Observe that the sum is hypergeometric:

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = {}_3F_2 \left( \begin{matrix} 1 - \frac{n}{3}, \frac{4}{3} - \frac{n}{3}, \frac{5}{3} - \frac{n}{3} \\ \frac{1}{6}(9 - 3n + X_n), \frac{1}{6}(9 - 3n - X_n) \end{matrix} \middle| 1 \right).$$

Apply Sheppard's theorem and Gosper's algorithm, then simplify.  $\square$

# line graph $L(K_{m,n}^{(1,2)})$



$$a_0 = m \binom{n}{2}$$

$$a_1 = (m-1) \binom{n-2}{2}$$

⋮

$$a_j = (m-j) \binom{n-2j}{2}$$

$$\sigma(L(K_{m,n}^{(1,2)})) = \left(m \binom{n}{2}\right)! \sum_{i=0}^{\min(m, \lfloor n/2 \rfloor)} \prod_{j=1}^i \frac{-(m-j)(\binom{n-2j}{2})}{m \binom{n}{2} - (m-j)(\binom{n-2j}{2})}$$

a graph is *fully regular* if for every independent set  $I = \{v_1, \dots, v_j\} \subset V$ , the number of vertices in  $V \setminus I$  that are not connected to  $I$  is a constant  $a_j$

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^i \frac{-a_j}{a_0 - a_j}$$

**Conjecture** (Fang–Huang–Pach–Tardos–Zuo, 2023). *We have, for  $m \geq 1$  and  $n \geq 2$ ,*

$$\sum_{i=0}^{\min\{m, \lfloor n/2 \rfloor\}} \prod_{j=1}^i \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2} - (m-j)\binom{n-2j}{2}} = m \prod_{j=1}^{m-1} \frac{mn - \binom{m+1}{2} + \binom{j}{2}}{d_j},$$

where  $d_j = \frac{1}{j}(m\binom{n}{2} - (m-j)\binom{n-2j}{2})$ , and where any numerators or denominators on the right that are equal to zero are ignored.

**Theorem** (H., 2023). We have, for  $m \geq 1$  and  $n \geq 2$ ,

$$\sum_{i=0}^{\min\{m, \lfloor n/2 \rfloor\}} \prod_{j=1}^i \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2} - (m-j)\binom{n-2j}{2}} = m \prod_{j=1}^{m-1} \frac{mn - \binom{m+1}{2} + \binom{j}{2}}{d_j},$$

where  $d_j = \frac{1}{j}(m\binom{n}{2} - (m-j)\binom{n-2j}{2})$ , and where any numerators or denominators on the right that are equal to zero are ignored.

*Proof sketch.* Let  $Y_{m,n} = \sqrt{(2m+1)^2 - 8mn}$ . Observe that the sum is hypergeometric:

$$\sum_{i=0}^{\min\{m, \lfloor n/2 \rfloor\}} \prod_{j=1}^i \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2} - (m-j)\binom{n-2j}{2}} = {}_3F_2 \left( \begin{matrix} 1 - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}, 1-m \\ \frac{1}{4}(5-2m-2n+Y_{m,n}), \frac{1}{4}(5-2m-2n-Y_{m,n}) \end{matrix} \middle| 1 \right)$$

Apply Sheppard's theorem and Gosper's algorithm, then simplify carefully. □

```
In[64]:= Table[Sum[Product[(-Binomial[n - 4 j, 4])/(Binomial[n, 4] - Binomial[n - 4 j, 4]), {j, 1, Floor[n/4]}], {i, 0, Floor[n/4]}], {n, 40, 44}] // FactorInteger
```

```
Out[64]= {{{{2, 21}, {13, -1}, {23, -1}, {41, -1}, {53, -2}, {61, -1}, {73, -1}, {89, -1}, {241, -1}, {331, -1}, {743, -1}, {761, -1}, {911, -1}, {1091, -1}, {19154778839631970056937, 1}}, {{2, 18}, {5, -2}, {13, -1}, {17, -1}, {43, -1}, {47, -1}, {59, -1}, {67, -1}, {71, -1}, {89, -1}, {157, -1}, {211, -1}, {269, -1}, {557, 1}, {1409, -1}, {1415797460315131, 1}}, {{2, 18}, {3, -1}, {5, -1}, {7, -4}, {11, -2}, {17, 1}, {23, -1}, {53, -1}, {61, -1}, {73, -1}, {191, -1}, {331, 1}, {373, -1}, {449, -1}, {829, -1}, {108971, 1}, {207004691, 1}}, {{2, 20}, {5, -2}, {7, -1}, {29, -1}, {47, -1}, {59, -1}, {67, -1}, {71, -1}, {79, -2}, {131, 1}, {181, -1}, {311, 1}, {317, -1}, {521, -1}, {1013, -1}, {1091, -1}, {15393787603028743, 1}}, {{2, 22}, {3, -3}, {7, -1}, {23, -1}, {29, -1}, {31, -1}, {53, -3}, {61, -1}, {71, -1}, {73, -1}, {179, -1}, {181, -1}, {449, 1}, {499, -1}, {1367, -1}, {265423, 1}, {347851378367, 1}}}}
```

```
In[65]:= Sum[Product[(-Binomial[n - 4 j, 4])/(Binomial[n, 4] - Binomial[n - 4 j, 4]), {j, 1, i}], {i, 0, Floor[n/4]}], Method → HypergeometricTermPFQ]
```

```
Out[65]= Sum[Gamma(i + 1) (1 - n/4)_i (5/4 - n/4)_i (3/2 - n/4)_i (7/4 - n/4)_i / ((1/4 - n/2)_i (1/8 (-2 n - Sqrt[-4 n^2 + 12 n + 1] + 11))_i (1/8 (-2 n + Sqrt[-4 n^2 + 12 n + 1] + 11))_i), {i, 0, Floor[n/4]}], Method → HypergeometricTermPFQ]
```

terminating 0-balanced  ${}_4F_3(1)$

```
In[68]:= FullSimplify[Sum[Product[ $\frac{i}{j}$ , {j, 1, i}], {i, 0, Floor[n/5]}], Method -> HypergeometricTermPFQ]
```

## future work

counting successive vertex orderings for nice graphs (e.g., hypercube  $Q_n$ ), and nice classes of graphs (e.g., strongly regular / distance-transitive graphs)

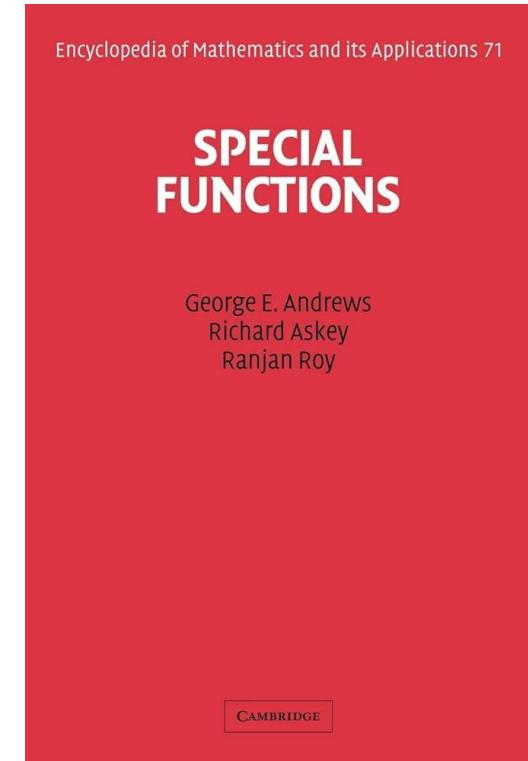
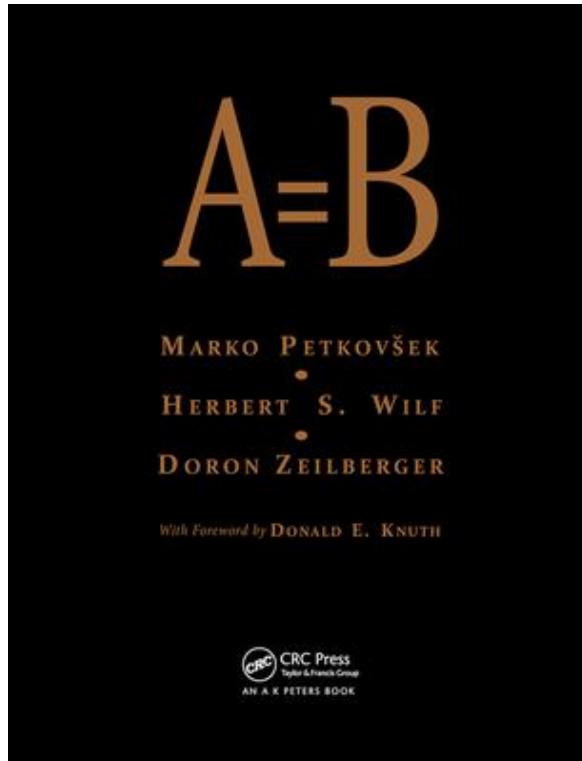
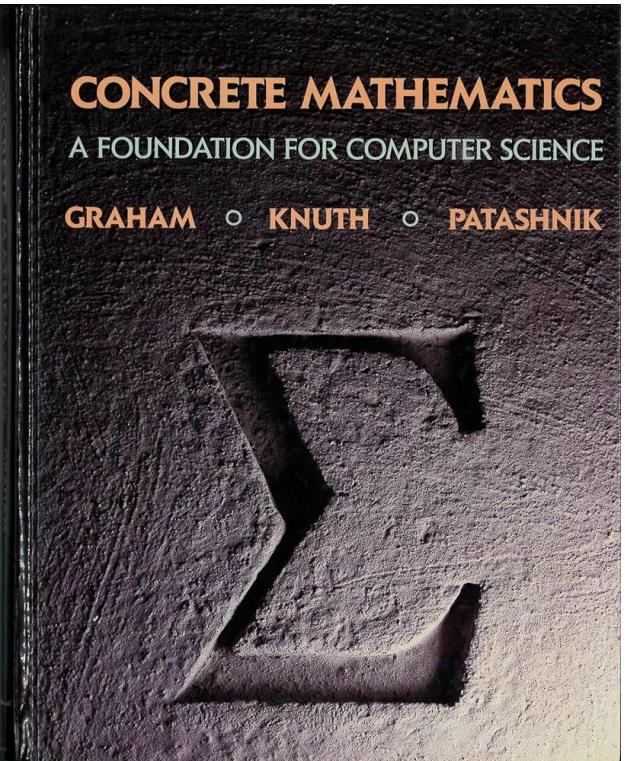
recently, T. Amdberhan proved that  $\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n-3j}{3}}{\binom{n}{3}-\binom{n-3j}{3}} = \frac{n}{3} \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^i \frac{-\binom{n+3-1-3j}{3}}{\binom{n}{3}-\binom{n-3j}{3}}$  (in fact, this generalizes from 3 to  $t$ ); together with the Pfaff–Saalschütz identity, this gives an alternate proof of the  $L(K_n^{(3)})$  theorem avoiding Sheppard's theorem and Gosper's algorithm. Does a similar method apply for proving the  $L(K_{m,n}^{(1,2)})$  theorem?

How do we apply hypergeometric techniques to the  $K_{m,n,p}^{(1,1,1)}$  case?

Are there combinatorial arguments for the product formulas like Stanley's for  $L(K_n)$ ?

Algorithms for computing  $\sigma(G)$  for symmetric graphs / general graphs

# further reading



Thank you for listening!