Yuelin's solution for bounded functions in the circle arc problem (draft 1)

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Suppose we have a function f on the circle and a fixed length α such that, when integrating f over any arc of the circle of length α, we get zero. We are interested in finding conditions for f so that the following properties hold:

- Given an arbitrary length, the integrals of f over arcs of the circle of that length are all equal.
- The integrals of f over arcs of the circle of all lengths are identically zero.
- The function f is zero almost everywhere, or zero everywhere.

In this note, we give a partial solution due to Yuelin for the case where f is everywhere bounded.

For rational $\alpha = m/n$, it is easy to construct nonzero functions with integrals equal to zero over all arcs of length α. Indeed, $\sin(2\pi nx)$ will do. As such, we are only interested in the case where $0 < \alpha < 1$ is irrational. We consider the circle as the quotient $S^1 \approx \mathbf{R}/\mathbf{Z}$, and we write

$$\beta(x, \alpha) := \int_x^{x+\alpha} f(t)\,dt,$$

so the condition that integrating f over any arc of the circle of length α gives zero is equivalent to requiring that $\beta(x, \alpha) = 0$ for $x \in S^1$, which we will assume to hold from now on. We will prove the following result:

Theorem 1. *Let $0 < \alpha < 1$ be irrational, and suppose $f\colon S^1 \to \mathbf{R}$ is bounded, with $\beta(x, \alpha) = 0$ whenever $x \in S^1$. Then $\beta(x + \epsilon', \epsilon) = \beta(x, \epsilon)$ for all $\epsilon, \epsilon' > 0$.*

We begin by establishing the following lemma, which will be useful as $\{k\alpha : k \in \mathbf{Z}\}$ is a dense subset of the circle:

Lemma 1. *Let $k\alpha$ be an integer multiple of α, and suppose $\epsilon > k\alpha$. Then $\beta(x + \epsilon, k\alpha) = \beta(x, k\alpha)$ for all $x \in S^1$.*

Proof. Let $x \in S^1$. Then, given $\epsilon > 0$, we see that $\beta(x, \alpha) = \beta(x + \epsilon, \alpha)$, or

$$\int_x^{x+\alpha} f(t)\,dt = \int_{x+\epsilon}^{x+\epsilon+\alpha} f(t)\,dt.$$

It follows that

$$0 = \int_x^{x+\alpha} f(t)\,dt - \int_{x+\epsilon}^{x+\epsilon+\alpha} f(t)\,dt = \int_x^{x+\epsilon} f(t)\,dt - \int_{x+\alpha}^{x+\epsilon+\alpha} f(t)\,dt,$$

so that $\beta(x + \alpha, \epsilon) = \beta(x, \epsilon)$. Repeated use of this argument then gives $\beta(x + k\alpha, \epsilon) = \beta(x, \epsilon)$. Suppose now that $\epsilon > k\alpha$, and notice that we have $\beta(x, a + b) = \beta(x, a) + \beta(x + a, b)$. We compute

$$\begin{aligned}
\beta(x + k\alpha, \epsilon - k\alpha) + \beta(x + \epsilon, k\alpha) &= \beta(x + k\alpha, \epsilon) \\
&= \beta(x, \epsilon) \\
&= \beta(x, k\alpha) + \beta(x + k\alpha, \epsilon - k\alpha).
\end{aligned}$$

Subtracting $\beta(x + k\alpha, \epsilon - k\alpha)$ from both sides, we conclude that

$$\beta(x, k\alpha) = \beta(x + \epsilon, k\alpha).$$

$\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Suppose for contradiction that

$$|\beta(x, \epsilon) - \beta(x + \epsilon', \epsilon)| =: C > 0$$

for some $\epsilon' > \epsilon > 0$ (requiring $\epsilon' > \epsilon$ is fine because integrals over arcs of length greater than ϵ' can be cut into pieces smaller than ϵ'). Since $\{k\alpha : k \in \mathbf{Z}\}$ is dense in the circle, we may choose k such that $0 < k\alpha < \epsilon < \epsilon'$. By Lemma 1, $\beta(x + \epsilon', k\alpha) = \beta(x, k\alpha)$. Thus we have

$$\begin{aligned}
\beta(x, \epsilon) &= \beta(x, k\alpha) + \beta(x + k\alpha, \epsilon - k\alpha) \\
&= \beta(x + \epsilon', k\alpha) + \beta(x + k\alpha, \epsilon - k\alpha) \\
&= \beta(x + \epsilon', \epsilon) - \beta(x + \epsilon' + k\alpha, \epsilon - k\alpha) + \beta(x + k\alpha, \epsilon - k\alpha),
\end{aligned}$$

so that

$$\beta(x + k\alpha, \epsilon - k\alpha) - \beta(x + \epsilon' + k\alpha, \epsilon - k\alpha) = \beta(x, \epsilon) - \beta(x + \epsilon', \epsilon).$$

It follows that we have

$$|\beta(x + k\alpha, \epsilon - k\alpha)| \geq C/2$$

or

$$|\beta(x + \epsilon' + k\alpha, \epsilon - k\alpha)| \geq C/2.$$

In either case, we obtain an interval I of length $\epsilon - k\alpha$ on which $|\int_I f(t)\, dt| \geq C/2$, so that $|f(x)| \geq C/2(\epsilon - k\alpha)$ for some $x \in I$. The fact that we may chose k with $\epsilon - k\alpha$ arbitrarily small then contradicts the boundedness of f as needed. $\square$

Work to be done. At this point there are still many loose ends to tie up — for example, we should be able to prove that $\beta(x, \epsilon) = 0$ for all ϵ, which would then imply that $f = 0$ almost everywhere (though this isn't as trivial as one may think! One can use the fundamental theorem of calculus. See `https://math.stackexchange.com/q/16244/`.)

I believe the result should also hold for absolutely integrable functions, perhaps by using the fact that continuous functions with compact support (compact support is redundant since the circle is compact) are dense in that space, perhaps taking ideas from `https://math.stackexchange.com/a/3458702/`. Or perhaps one could use the axiom of choice to construct a counterexample, which would be interesting as well.