A brief, terse combinatorial excursion,
also a small experiment in pedagogy and notation

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We fill in some details in the derivation of the combinatorial identity

$$\sum_{k\ge 0}\binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1+m} = (-1)^n m^{\underline{n}} m^{\underline{-n-1}}, \quad \text{integers } m, n \ge 0,$$

as it appears in Graham, Knuth and Patashnik's *Concrete Mathematics*, pp. 184–185.

Since

$$\begin{aligned}\binom{j+k}{j} &= \binom{j+k}{k}\\ &= \frac{k+1}{j+k+1}\binom{j+k+1}{k+1}\\ &= \frac{k+1}{j+k+1}\binom{j+k+1}{j}\\ &= \frac{k+1}{j+k+1}(-1)^j\binom{-k-2}{j},\end{aligned}$$

we have

$$\frac{1}{k+1}\binom{-k-2}{j}^{-1} = \frac{(-1)^j}{j+k+1}\binom{j+k}{j}^{-1} = (-1)^j\frac{j!k!}{(j+k+1)!}. \qquad (1)$$

It follows that

$$\begin{aligned}&\sum_{k\ge 0}\binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1}\sum_{j\ge 0}\binom{m}{j}\binom{-k-2}{j}^{-1}\\ &= \sum_{j\ge 0}\sum_{k}\binom{n}{k}\binom{m}{j}(-1)^j\frac{j!k!}{(j+k+1)!}(-1)^k\binom{n+k}{k} && \text{by (1)}\\ &= \sum_{j\ge 0}\sum_{k}\frac{n!}{\cancel{k!}(n-k)!}\,\frac{m!}{\cancel{j!}(m-j)!}(-1)^j\frac{\cancel{j!}\cancel{k!}}{(j+k+1)!}\binom{-n-1}{k} && \text{negate upper index and expand binomial coefficients}\\ &= m!n!\sum_{j\ge 0}(-1)^j\frac{1}{(m-j)!}\sum_{k}\frac{1}{(n-k)!(j+k+1)!}\binom{-n-1}{k}\\ &= \frac{m!n!}{(m+n+1)!}\sum_{j\ge 0}(-1)^j\frac{(m+n+1)!}{(n+1+j)!(m-j)!}\sum_{k}\frac{(n+1+j)!}{(n-k)!(j+k+1)!}\binom{-n-1}{k} && \text{multiply by appropriate terms to create new binomial coefficients}\\ &= \frac{m!n!}{(m+n+1)!}\sum_{j\ge 0}(-1)^j\binom{m+n+1}{n+1+j}\sum_{k}\binom{n+1+j}{k+j+1}\binom{-n-1}{k}.\end{aligned}$$