Stable Television Schedules

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In Chapter 1 of Jon Kleinberg and Éva Tardos's influential book *Algorithm Design*, there is the following exercise:

> Suppose we have two television networks, whom we'll call $\mathcal{A}$ and $\mathcal{B}$. There are n prime-time programming slots, and each network has n TV shows. Each network wants to devise a *schedule* — an assignment of each show to a distinct slot — so as to attract as much market share as possible. Here is the way we determine how well the two networks perform relative to each other, given their schedules. Each show has a fixed *rating*, which is based on the number of people who watched it last year; we'll assume that no two shows have exactly the same rating. A network *wins* a given time slot if the show that it schedules for the time slot has a larger rating than the show the other network schedules for that time slot. The goal of each network is to win as many time slots as possible.
>
> Suppose in the opening week of the fall season, Network $\mathcal{A}$ reveals a schedule S and Network $\mathcal{B}$ reveals a schedule T. On the basis of this pair of schedules, each network wins certain time slots, according to the rule above. We'll say that the pair of schedules (S, T) is *stable* if neither network can unilaterally change its own schedule and win more time slots. That is, there is no schedule S' such that Network $\mathcal{A}$ wins more slots with the pair (S', T) than it did with the pair (S, T); and symmetrically, there is no schedule T' such that Network $\mathcal{B}$ wins more slots with the pair (S, T') than it did with the pair (S, T).
>
> For every set of TV shows and ratings, is there always a stable pair of schedules?

It turns out that it is usually not possible to obtain a stable pair from a set of shows and ratings. That said, there is a simple condition for the existence of a stable pair of schedules. We say a set of shows and ratings 'has *contiguous ratings*,' or just 'is *contiguous*,' if one of the Networks is such that any show of the other Network is either rated higher or lower than all of the first Network's shows. It is straightforward to show that *any* scheduling that arises from a contiguous set of shows and ratings is stable.

If we arrange the relative ratings of shows from a pair of schedules in the form of a matrix, then contiguous sets give rise to matrices where there is a row of consecutive numbers. An example:

$$\begin{array}{c} \begin{array}{ccc} \text{Slot 1} & \text{Slot 2} & \text{Slot 3} \end{array} \\ \begin{pmatrix} 2 & 1 & \mathbf{6} \\ \mathbf{3} & \mathbf{4} & 5 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Network } \mathcal{A} \\ \text{Network } \mathcal{B} \end{array}$$

Notice that we have typeset the numbers corresponding to shows that won a slot in **bold**.

We now show that a stable pair must arise from a contiguous set of shows and ratings. Suppose we have a pair of schedules (S, T), and Network $\mathcal{A}$ changes its schedule S to a new schedule S' so that it wins more slots with the pair (S', T). Then, Network $\mathcal{B}$ may effectively undo the changes of Network $\mathcal{A}$ by rearranging its schedule in the same way. (For example, if Network $\mathcal{A}$ moved the show that it originally put in slot 1 to slot 3, Network $\mathcal{B}$ would just do the same.) Therefore, any pair obtained by a Network changing its schedule to win more slots cannot be stable.

Given a stable pair, a Network cannot reschedule so as to lose slots. (If it could, it would be able to undo its rescheduling so as to obtain a stable pair while winning back the slots it lost initially.) Similarly, given a stable pair, a Network rescheduling so as to gain slots would be equivalent to the other Network rescheduling so as to lose slots, so that is not possible as well. It follows that any rescheduling in a stable pair cannot change the number of slots each Network already has.

Finally, suppose we have a set of shows and ratings that are not contiguous, and consider a fixed scheduling corresponding to this set. For convenience, we write $a_1, \ldots, a_n$ for the ratings of the shows of Network $\mathcal{A}$ in order of scheduling, and similarly write $b_1, \ldots, b_n$ for Network $\mathcal{B}$. By our hypotheses, we may find a sequence $a_i < b_j < a_k < b_l$ (or $b_i < a_j < b_k < a_l$). We may then switch the slots of the shows corresponding to the ratings a_i and a_k, and the reader may verify that this increases the number of slots belonging to Network $\mathcal{A}$ (or $\mathcal{B}$), as illustrated in the following example:

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ \mathbf{4} & \mathbf{5} & \mathbf{7} & \mathbf{8} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{6} & 2 & 3 & 1 \\ 4 & \mathbf{5} & \mathbf{7} & \mathbf{8} \end{pmatrix}, \quad \text{with } 1 < 4 < 6 < 7$$

It follows that a stable pair must arise from a contiguous set of shows and ratings, completing the proof.

In summary:

- Suppose a Network could improve its score, and suppose it results in a stable pairing. But then the other Network could improve its score by undoing the rescheduling. So the result could not have been a stable pairing.
- Thus a stable pairing cannot be obtained via a Network increasing its number of slots.
- Thus a network cannot decrease its number of slots in a stable pairing, because then it could undo its rescheduling from that state to obtain the original stable pairing while increasing its number of slots, violating the point above.
- Therefore every rescheduling of a stable pairing by either Network cannot change the number of slots each Network already has.
- The result follows from the fact that the only sets of shows and ratings for which no rescheduling can change the number of slots each Network has are precisely the contiguous sets.

References & Acknowledgements

This write-up is a fleshed out version of a StackOverflow answer of Craig Gidney, available at `https://stackoverflow.com/a/1328844/`.

The exercise from Kleinberg and Tardos's book is exercise 3, which is on page 23.