SELECTED SOLUTIONS FOR TERENCE TAO'S BOOK "AN INTRODUCTION TO MEASURE THEORY"

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Exercise in the proof of Lemma 1.1.2. We prove that

$$|I| = \lim_{N\to\infty} \frac{1}{N}\#(I \cap \frac{1}{N}\mathbf{Z}).$$

Since $[a,b] \cap \frac{1}{N}\mathbf{Z} \cong [Na, Nb] \cap \mathbf{Z} = \{\lceil Na\rceil, \dots, \lfloor Nb\rfloor\}$, we have

$$\#(I \cap \frac{1}{N}\mathbf{Z}) = \lfloor Nb\rfloor - \lceil Na\rceil + 1.$$

Since $Nb < \lfloor Nb\rfloor + 1 \le Nb + 1$ and $Na \le \lceil Na\rceil < Na + 1$, we have

$$Nb - Na - 1 < \lfloor Nb\rfloor - \lceil Na\rceil + 1 \le Nb - Na + 1,$$

so

$$b - a - \frac{1}{N} < \frac{\lfloor Nb\rfloor - \lceil Na\rceil + 1}{N} \le b - a + \frac{1}{N}.$$

The result follows from the squeeze theorem by sending $N \to \infty$.

Exercise 1.1.3. We first prove the result for $d = 1$. Suppose $m' \colon \mathcal{E}(\mathbf{R}) \to \mathbf{R}^+$ obeys non-negativity, finite additivity and translation invariance. For $n \ge 1$, we have

$$\begin{aligned} c := m'([0,1)) &= m'\left(\bigcup_{i=1}^{n}\Big[\frac{i-1}{n}, \frac{i}{n}\Big)\right) \\ &= \sum_{i=1}^{n} m'\Big(\Big[\frac{i-1}{n}, \frac{i}{n}\Big)\Big) \quad \text{by finite additivity} \\ &= \sum_{i=1}^{n} m'\Big(\Big[0, \frac{1}{n}\Big)\Big) \quad \text{by translation invariance} \\ &= nm'\Big(\Big[0, \frac{1}{n}\Big)\Big), \end{aligned}$$

and so $m'([0,1/n)) = c/n$. Thus $m'([0,k/n)) = ck/n$. Note that non-negativity and finite additivity imply monotonicity, which in turn implies that $m'(\{0\}) < 1/n$ for all n, so that $m'(\{x\}) = 0$ for all $x \in \mathbf{R}$ by translation invariance.

Since elementary sets are finite unions of disjoint boxes, it suffices to show that $m'(B) = cm(B)$ for all boxes B. Since singletons have zero measure as shown above, it suffices by translation invariance to prove the result for $B = [0,a)$ where $a > 0$. By writing $[0,a) = [0,\lfloor a\rfloor) \cup [\lfloor a\rfloor, a)$, we see that it suffices to consider $0 < a < 1$. By considering a sequence in $\mathbf{Q} \cap [0,a)$ converging to a, monotonicity yields the bound $m'([0,a)) \ge ca$, and we may also obtain $m'([0,a)) \le ca$ analogously.

For $\mathbf{R}^d$ we find $m'([0,1/n)^d) = c/n^d$ (recall $\bigcup_i A_i \times \bigcup_j B_j \approx \bigcup_{i,j} A_i \times B_j$). Similar arguments show that $m'(\prod_{1\le i\le d}[0,k_i/n)) = (c/n^d)(\prod_{1\le i\le d} k_i)$, and that degenerate elementary sets (where one of the factor intervals is a singleton) have zero measure under m'. We may finish off with a similar limiting argument:

$$m'\left(\prod_{1\le i\le d}[0,a_i)\right) \ge \sup\left\{m'\left(\prod_{1\le i\le d}[0,q_i)\right) : q_i \in \mathbf{Q}\cap[0,a_i)\right\} = c\prod_{1\le i\le d} a_i.$$

Exercise 1.1.5. To show (1) implies (2), suppose E is Jordan measurable, and let $\epsilon > 0$. Then there exist elementary sets $A \subset E \subset B$ with $m(A) > m(E) - \epsilon/2$ and $m(B) < m(E) + \epsilon/2$, so that $m(B - A) = m(B) - m(A) \le \epsilon$ by finite additivity of elementary measure.

To show (2) implies (3), let $A \subset E \subset B$ be elementary sets with $m(B - A) \le \epsilon$. Then $B \triangle A = B - A \supset B - E$, and so

$$m^{*,(J)}(B\triangle E) = \inf_{\substack{S\supset B-E\\ S \text{ elem.}}} m(S) \le m(B-A) \le \epsilon.$$

To show (3) implies (1), let A be an elementary set with $m^{*,(J)}(A\triangle E) \le \epsilon/4$. Then there exists an elementary set $B \supset A\triangle E$ with $m(B) < \epsilon/2$. This gives us two elementary sets $A - B \subset E \subset A \cup B$. Since

$$m^{*,(J)}(E) \ge m(A-B) \ge m(A) - m(B) > m(A) - \epsilon/2$$

and

$$m_{*,(J)}(E) \le m(A\cup B) \le m(A) + m(B) < m(A) + \epsilon/2,$$

we obtain $m^{*,(J)}(E) - m_{*,(J)}(E) < \epsilon$. It follows that E is Jordan measurable.

Exercise 1.1.6. (1) We begin by proving that $E \cup F$ is Jordan measurable. By exercise 1.1.5(2), there exist elementary sets A, B, A', B' with $A \subset E \subset B$, $A' \subset F \subset B'$, $m(B-A) \le \epsilon/2$, and $m(B'-A') \le \epsilon/2$. Then $A\cup A' \subset E\cup F \subset B\cup B'$. Since $B \cup B' - A \cup A' \subset (B - A) \cup (B' - A')$, it follows from already established properties of elementary measure that

$$\begin{aligned} m(B\cup B' - A\cup A') &\le m((B-A)\cup(B'-A')) \\ &\le m(B-A) + m(B'-A') \\ &\le \epsilon, \end{aligned}$$

and so applying exercise 1.1.5(2) again shows that $E \cup F$ is Jordan measurable. Showing that $E \cap F$ is Jordan measurable is quite similar — one uses the inclusion

$$B\cap B' - A\cap A' = (B\cap B' - A)\cup(B\cap B' - A') \subset (B-A)\cup(B'-A').$$

Showing that $E-F$ is Jordan measurable uses the fact that $A-B' \subset E-F \subset B-A'$ and

$$(B-A') - (A-B') \subset (B-A)\cup(B'-A').$$

Finally, $E\triangle F = E\cup F - E\cap F$ and is thus Jordan measurable.

(2) We have $m(E) \ge m_{*,(J)}(E)$, which is a supremum over elementary measures of elementary sets, which are clearly non-negative by definition.

(3) Let $A \subset E \subset B$, $A' \subset F \subset B'$ be elementary sets with

$$m(B) - \epsilon/2 < m(E) < m(A) + \epsilon/2$$

and

$$m(B') - \epsilon/2 < m(F) < m(A') + \epsilon/2.$$

Then, $E \cup F \supset A \cup A'$, and so

$$m(E\cup F) \ge m(A\cup A') = m(A) + m(A') > m(E) + m(F) - \epsilon.$$

Similarly, $E \cup F \subset B \cup B'$, and we have

$$m(E\cup F) \le m(B\cup B') \le m(B) + m(B') < m(E) + m(F) + \epsilon.$$

Since ϵ was arbitrary, this gives $m(E\cup F) = m(E) + m(F)$ as required.

(4) We have $E \uplus (F - E) = F$, where $\uplus$ denotes a disjoint union. By (1), $F - E$ is Jordan measurable, and so $m(E) + m(F - E) = m(F)$ by (3). Since $m(F - E) \geq 0$ by (2), we conclude that $m(E) \leq m(F)$.

(5) Since $E \cup F = E \uplus (F - E)$ and $F - E \subset F$, we have

$$m(E \cup F) = m(E) + m(F - E) \leq m(E) + m(F).$$

(6) This follows immediately from translation invariance of elementary sets — if $A \subset E$ with A elementary, then $A + x \subset E + x$ with $A + x$ elementary and $m(A + x) = m(A)$; similarly for $B \supset E$.

Exercise 1.1.7. (1) Let $f\colon B \to \mathbf{R}$ be a continuous function on a closed box $B \subset \mathbf{R}^d$, and denote by $\Gamma_f := \{(x, f(x)) : x \in B\} \subset \mathbf{R}^{d+1}$ its graph. Since the inner measure is at most the outer measure, the Jordan measurability of Γ_f is immediately established if we find for every $\epsilon > 0$ an elementary set of measure less than ϵ that contains Γ_f. Let $\epsilon > 0$. Since continuous functions on compact sets are uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/m(B)$ whenever $\|x - y\| < \delta$ and $x, y \in B$. Partition B into boxes of diameter less than δ. Each of these boxes $B_\alpha \subset \mathbf{R}^d$ gives rise to a box $B_\alpha \times I_\alpha \subset \mathbf{R}^{d+1}$ containing $\{(x, f(x)) : x \in B'\} \subset \Gamma_f$ with $m(I_\alpha) < \epsilon/m(B)$ by uniform continuity. It follows that $\bigcup_\alpha (B_\alpha \times I_\alpha)$ is an elementary set of measure less than ϵ that contains Γ_f. We conclude that the graph of f is Jordan measurable with Jordan measure zero.

(2) This is essentially the fact that bounded continuous functions are Riemann integrable. Alternatively, letting $U := \{(x, t) : x \in B \text{ and } 0 \leq t \leq f(x)\} \subset \mathbf{R}^{d+1}$, one may consider the sets (as defined in (1))

$$U - \bigcup_\alpha (B_\alpha \times I_\alpha) \subset U \subset U \cup \bigcup_\alpha (B_\alpha \times I_\alpha),$$

which may be shown to be elementary.

Exercise 1.1.8. (1) Suppose AB is horizontal. Then we may translate AB onto the x-axis and use exercise 1.1.7(2) to prove that ABC is Jordan measurable. Note that if the x-coordinate of C does not lie between the x-coordinates of A and B, we may just regard ABC as the difference of two right-angled triangles $AC'C$ and $BC'C$ where C' is C projected onto the x-axis. We must then add back the line BC, but this has Jordan measure zero by exercise 1.1.7(1).

For the general case translate the triangle so that one point, call it A without loss of generality, lies on the x-axis, and the other two points are above it. Then this can be thought of as the area under a graph again with one or two right triangles removed and lines added appropriately, once again by exercise 1.1.7. It follows that solid triangles are Jordan measurable.

(2) This boils down to finding the area under a line $y = mx$ using the standard Riemann sums arguments.

Exercise 1.1.9. Suppose $P \subset \mathbf{R}^d$ be a compact convex polytope contained in a closed box B. We may write $P = \bigcap_i (B \cap H_i)$, where each $H_i := \{x \in \mathbf{R}^d : x \cdot v_i \leq c_i\}$ is a closed half-space, and so it suffices to prove that sets of the form $B \cap H_i$ are Jordan measurable. We may identify $\mathbf{R}^{d-1} \subset \mathbf{R}^d$ as the subset with $x_i = 0$. Pick an identification where the hyperplane defined by $x \cdot v = c$ is not orthogonal to the identified $\mathbf{R}^{d-1}$. Then, projecting the box B down to $\pi(B) \subset \mathbf{R}^{d-1}$, we may use exercise 1.1.7(2) to obtain our result by considering $B \cap H_i$ as the region under an appropriate graph.

Exercise 1.1.10. (1) To show that balls are Jordan measurable, it suffices to translate the standard ball $B(x, r)$ by r units in x_d so that it lies in the closed upper half space, then treat it as the difference of two graphs. For example, when $d = 2$, we consider the difference of the regions below the graphs of functions $r \pm \sqrt{r^2 - x^2}$.

Now, if we define the scaling by r of an interval $I = [a, b]$ by $rI := [ra, rb]$ (and similarly for open and half-closed intervals), then $m(rI) = rm(I)$. We may extend this to a box $B = \prod_{1 \le j \le d} I_j$ to get $rB := \prod_{1 \le j \le d} rI_j$ and $m(rB) = r^d m(B)$, and similarly to elementary sets $A = \bigcup_i B_i$ where $rA := \bigcup_i rB_i$ and $m(rA) = r^d m(A)$.

Denote the open ball of radius r of dimension d centered at 0 by $B_d(r) \subset \mathbf{R}^{d+1}$, and let $c_d := m(B_d(1))$. We will show that $m(B_d(r)) = c_d r^d$. Let $A \subset B_d(1) \subset B$ be elementary sets with

$$c_d - \epsilon/r^d < m(A) \quad \text{and} \quad m(B) < c_d + \epsilon/r^d.$$

Then, $rA \subset B_d(r) \subset rB$ are elementary sets, and so

$$\begin{aligned} c_d r^d - \epsilon < r^d m(A) &= m(rA) \\ &\le m(B_d(r)) \\ &\le m(rB) = r^d m(B) < c_d r^d + \epsilon. \end{aligned}$$

Since ϵ was arbitrary we conclude that $m(B_d(r)) = c_d r^d$ as needed.

(2) The bound

$$\left(\frac{2}{\sqrt{d}}\right)^d \le c_d \le 2^d$$

is easily established by inscribing and circumscribing cubes in the unit sphere. For the inner cube, note that its diameter is 2, so its side length is $2/\sqrt{d}$ and its volume is $(2/\sqrt{d})^d$. (In fact, $c_d = \frac{1}{d}\frac{2\pi^{d/2}}{\Gamma(d/2)}$.)

Exercise 1.1.11.