Math108A HW 7

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Problem 1

a) Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ be the dual basis of the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Thus,

$$\epsilon_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Consider ϵ_1

$$\begin{aligned}
&\epsilon_1(x, y, z) \\
&= \epsilon_1(xe_1 + ye_2 + ze_3) \\
&= \epsilon_1(xe_1) + \epsilon_1(ye_2) + \epsilon_1(ze_3) \\
&= x\epsilon_1(e_1) \\
&= x
\end{aligned}$$

Likewise, consider ϵ_2

$$\begin{aligned}
&\epsilon_2(x, y, z) \\
&= \epsilon_2(xe_1 + ye_2 + ze_3) \\
&= \epsilon_2(xe_1) + \epsilon_2(ye_2) + \epsilon_2(ze_3) \\
&= y\epsilon_2(e_2) \\
&= y
\end{aligned}$$

Finally, consider ϵ_3

$$\begin{aligned}
&\epsilon_3(x, y, z) \\
&= \epsilon_3(xe_1 + ye_2 + ze_3) \\
&= \epsilon_3(xe_1) + \epsilon_3(ye_2) + \epsilon_3(ze_3) \\
&= z\epsilon_3(e_3) \\
&= z
\end{aligned}$$

Thus, $f(x, y, z) = 3x - 2y + z = 3\epsilon_1(x, y, z) - 2\epsilon_2(x, y, z) + \epsilon_3(x, y, z)$

b) We have some linear function $g \in L(\mathbb{R}^3, F)$. $g(x, y, z) = ax + by + cz$, and we are attempting to solve for a, b, and c. Since we have three equations, we can write this system of equations in the following form:

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & -1 & 1 & -2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & 0 & 2 & -2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

=

Thus, $a = 0, b = 1, c = -1$. Thus, $g(x, y, z) = y - z = \epsilon_2(x, y, z) - \epsilon_3(x, y, z)$

Problem 2

Let V be the vector space given by V $= \{f : [0,1] \to \mathbb{R}$ such that f is a function$\}$, then consider the dual space of $V, V' = L(V, R)$.

Consider $\phi_1 \in V'$ such that $\phi_1(f) = \int_0^1 f(x)dx \forall f \in V$. We will show that ϕ_1 is a linear map. Let $f, g, \in V$, and let $a, b \in \mathbb{R}$. Then $\phi_1(af + bg) = \int_0^1 af(x) + bg(x)dx = \int_0^1 af(x)dx + \int_0^1 bg(x)dx = a\int_0^1 f(x)dx + b\int_0^1 g(x)dx = a\phi_1(f) + b\phi_1(g)$, thus ϕ_1 is a linear map from V to $\mathbb{R}$.

Now, consider $\phi_2 \in V'$ such that $\phi_2(f) = f(0) \forall f \in V$. We will show that ϕ_2 is a linear map. Let $f, g, \in V$, and let $a, b \in \mathbb{R}$. Then $\phi_2(af + bg) = (af + bg)(0) = af(0) + bg(0) = a\phi_2(f) + b\phi_2(g)$. Thus, ϕ_2 is a linear map from V to $\mathbb{R}$.

Finally, consider $\phi_3 \in V'$ such that $\phi_3(f) = f(1) \forall f \in V$. We will show that ϕ_3 is a linear map. Let $f, g, \in V$, and let $a, b \in \mathbb{R}$. Then, $\phi_3(af + bg) = (af + bg)(1) = af(1) + bg(1) = a\phi_3(f) + b\phi_3(g)$. Thus, ϕ_3 is a linear map from V to $\mathbb{R}$.

Thus, we have found three distinct elements in the dual space of V. We will show that these elements form a linearly independent set. Let $a, b, c \in \mathbb{R}$ and consider $a\phi_1 + b\phi_2 + c\phi_3 = 0$. Then, in order for this equation to hold, it must hold for all functions, consider $f, g, h \in V$ such that $f(x) = x(x-1), g(x) = \frac{1}{2} - x, h(x) = 1$. Then, $(a\phi_1 + b\phi_2 + c\phi_3)(f) = 0$ implies that $-\frac{1}{6}a = 0$, $(a\phi_1 + b\phi_2 + c\phi_3)(g) = 0$ implies that $.5b - .5c = 0$, and $(a\phi_1 + b\phi_2 + c\phi_3)(h) = 0$ implies that $a + b + c = 0$. Thus, we have a system of equations that can be written in the following form:

$$\begin{bmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$=$

$$\begin{bmatrix} -\frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

=

Thus, a = b = c = 0. Therefore, we can say that the set $\{\phi_1, \phi_2, \phi_3\}$ is linearly independent.

Problem 3

In order to show that $\{1, x-1, (x-1)^2, (x-1)^3\}$ is a basis, we need to show that this set is linearly independent and spans $P_3(\mathbb{R})$.

First, we will show that it is linearly independent, let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, and consider $\lambda_1 + \lambda_2(x-1) + \lambda_3(x-2)^2 + \lambda_4(x-1)^3 = 0$

$$\begin{aligned}
&\lambda_1 + \lambda_2(x-1) + \lambda_3(x-2)^2 + \lambda_4(x-1)^3 = 0 \\
&\lambda_1 + \lambda_2 x - \lambda_2 + \lambda_3(x^2 - 2x + 1) + \lambda_4(x^3 - 3x^2 + 3x - 1) = 0 \\
&\lambda_1 + \lambda_2 x - \lambda_2 + \lambda_3 x^2 - 2\lambda_3 x + \lambda_3 + \lambda_4 x^3 - 3\lambda_4 x^2 + 3\lambda_4 x - \lambda_4 = 0 \\
&\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + x(\lambda_2 - 2\lambda_3 + 3\lambda_4) + x^2(\lambda_3 - 3\lambda_4) + x^3(\lambda_4) = 0
\end{aligned}$$

This implies, clearly, that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Thus, we have shown that the set is linearly independent.

Next, we need to show that the set spans $P_4(\mathbb{R})$. Let $p \in P_4(\mathbb{R})$, then $p = a + bx + cx^2 + dx^3$. Let $\lambda_4 = d, \lambda_3 = 3d + c, \lambda_2 = b + 2c + 3d, \lambda_1 = a + b + c + d$, then consider $\lambda_1 + \lambda_2(x-1) + \lambda_3(x-2)^2 + \lambda_4(x-1)^3$

$$\begin{aligned}
&\lambda_1 + \lambda_2(x-1) + \lambda_3(x-2)^2 + \lambda_4(x-1)^3 = \\
&a + b + c + d + (b + 2c + 3d)(x-1) + (3d + c)(x-2)^2 + d(x-1)^3 = \\
&a + b + c + d + bx + 2cx + 3dx - b - 2c - 3d + (3d + c)(x^2 - 2x + 1) + d(x^3 - 3x^2 + 3x - 1) = \\
&a - c - 2d + bx + 2cx + 3dx + dx^3 - 3dx^2 + 3dx - d + (3d + c)(x^2 - 2x + 1) = \\
&a - c - 3d + bx + 2cx + 6dx + dx^3 - 3dx^2 + 3dx^2 + cx^2 - 6xd + 3d + c - 2xc = \\
&a + bx + dx^3 + cx^2 = p
\end{aligned}$$

Thus, we have shown that the set $\{1, x-1, (x-1)^2, (x-1)^3\}$ spans $P_3(\mathbb{R})$ and is linearly independent, thus, it is a basis.

Next, we need to show that D = $\{\phi_i \in P_3(\mathbb{R})' : i = 0, 1, 2, 3\}$ where $\phi_i(p) = \frac{p^{(i)}(1)}{i!}$ is the dual basis of $\{1, x-1, (x-1)^2, (x-1)^3\}$. Since dim V = dim V' = number of elements in D, and $\phi_i \in P_3(\mathbb{R})'$, it is sufficient to show that

$$\phi_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $e_1 = 1, e_2 = x-1, e_3 = (x-1)^2, e_4 = (x-1)^3$.

$\phi_1(e_1) = 1/1 = 1$
$\phi_1(e_2) = (1-1)/1 = 0$
$\phi_1(e_3) = (1-1)^2/1 = 0$
$\phi_1(e_4) = (1-1)^3/1 = 0$

$\phi_2(e_1) = (1)'(1)/1 = 0$
$\phi_2(e_2) = (x-1)'(1)/1 = 1/1 = 1$
$\phi_2(e_3) = ((x-1)^2)'(1) = (2x-2)(1)/1 = 0$
$\phi_2(e_4) = ((x-1)^3)'(1) = (3x^2 - 6x + 3)(1)/1 = (3 - 6 + 3)/1 = 0$

$\phi_3(e_1) = (1)''(1)/2 = 0$
$\phi_3(e_2) = (x-1)''(1)/2 = 0/2 = 0$
$\phi_3(e_3) = ((x-1)^2)''(1)/2 = 2/2 = 1$
$\phi_3(e_4) = ((x-1)^3)''(1)/2 = (6x-6)(1)/2 = (6-6)/2 = 0$

$\phi_4(e_1) = (1)'''(1)/6 = 0$
$\phi_4(e_2) = (x-1)''(1)/6 = 0/6 = 0$
$\phi_4(e_3) = ((x-1)^2)''(1)/6 = 0/6 = 0$
$\phi_4(e_4) = ((x-1)^3)'''(1)/6 = 6/6 = 1$

Thus,

$$\phi_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and therefore, D is the dual basis of $\{1, x-1, (x-1)^2, (x-1)^3\}$.

Problem 4

a) We will show that $S^0 = \{\phi \in V' : \phi(s) = 0 \forall s \in S\}$ is a subspace of V'. In order to do this, we will need to show that the set is nonempty, and is closed under addition, as well as scalar multiplication.

Nonempty: Consider the zero map $z \in V'$ such that $z(v) = 0 \forall v \in V$. Since $z(v) = 0 \forall v \in V$, $z(s) = 0 \forall s \in S$, thus, $z \in S^0$, and S^0 is nonempty.

Next, we will show that S^0 is closed under addition, consider $a, b \in S^0$. Then, $(a+b)(s) = a(s) + b(s) = 0$, thus $a + b \in S^0$, and thus S^0 is closed under addition.

Finally, we need to show that S^0 is closed under scalar multiplication. Consider $a \in S^0, \lambda \in F$. Then $(\lambda a)(s) = \lambda a(s) = 0$. Thus, $(\lambda a) \in S^0$, and thus, S^0 is closed under scalar multiplication.

Since S^0 is non-empty, closed under addition and scalar multiplication, it is a subspace of V'.

b) Next, we will show that $S^0 = (Span(S))^0$. Recall that $S^0 = \{\phi \in V' : \phi(s) = 0 \forall s \in S\}$, and that $(\text{Span(S)})^0 = \{\phi \in V' : \phi(s) = 0 \forall s \in \text{Span(S)}\}$.

First, let $\phi \in (\text{Span(S)})^0$, then $\phi(s) = 0 \forall s \in \text{Span(S)}$, and as $S \subset \text{Span(S)}, \phi(s) = 0 \forall s \in S$, and thus $\phi \in S^0$, and therefore $(\text{Span(S)})^0 \subset S^0$.

Next, let $\phi \in S^0$. Consider $v \in \text{Span(S)}$, then $v = \sum_i \lambda_i s_i$ where $s_i \in S$, then $\phi(v) = \phi(\sum_i \lambda_i s_i) = \sum_i \lambda_i \phi(s_i) = \sum_i 0 = 0$, thus $\phi(s) = 0 \forall s \in \text{Span(S)}$, thus $\phi \in (\text{Span(S)})^0$, and therefore, $S^0 \subset (\text{Span(S)})^0$.

Since, $S^0 \subset (\text{Span(S)})^0$ and $(\text{Span(S)})^0 \subset S^0$, $S^0 = (\text{Span(S)})^0$.

Problem 5

Clearly, the set S is linearly independent, thus, the dim(Span(S)) = 2; also, recall that the span of any set is a subspace, and therefore, U = Span(S) is a subspace of V. Recall the formula dim U + dim U^0 = dim V. As dim U = 2, dim V = 2, dim $U^0 = 2$. Also, as seen in problem 4a, $S^0 = (\text{Span(S)})^0$, so therefore, $U^0 = S^0$, and therefore, dim $S^0 = 2$. So, we need to find two linearly independent elements in S^0, and then we will have a basis for S^0.

Consider, $\phi_1, \phi_2 \in P_3(\mathbb{R})'$ such that $\phi_1(p) = p^{(3)}(1), \phi_2(p) = p(-1) - p^{(2)}(1)$. Now, consider the following:

$\phi_1(1+x) = (1+x)'''(1) = 0$
$\phi_1(1+x^2) = (1+x^2)'''(1) = 0$

$\phi_2(1+x) = (1+(-1)) + (1+x)''(1) = 0 + 0 = 0$
$\phi_2(1+x^2) = (1+(-1)^2) - (1+x^2)''(1) = (1+1) - (2) = 2 - 2 = 0$

Thus, we have found two elements in S^0. In order for these elements to form a basis, they need to be linearly independent.

Now, letting $a, b \in \mathbb{R}$, consider $a\phi_1 + b\phi_2 = 0$, then, consider $1, x^3 \in P_3(\mathbb{R})$. Then,

$$\begin{aligned} &0 = (a\phi_1 + b\phi_2)(1) = \\ &(a\phi_1(1) + b\phi_2(1)) = \\ &a * 0 + b(1 - 0) = b \end{aligned}$$

Thus, b = 0

$$0 = (a\phi_1 + b\phi_2)(x^3) =$$
$$a\phi_1(x^3) + b\phi_2(x^3) =$$
$$6a + b(-1-6) = 6a$$

Thus, a = 0. Since $a\phi_1 + b\phi_2 = 0$ implies that a = b = 0, we have that $\phi_1$andϕ_2 are linearly independent. Thus, we have found a basis for S^0, $\{\phi_1, \phi_2\}$.

Problem 6

Let $\phi \in (U+W)^0$, then $\phi(s) = 0 \forall s \in U+W$. Letting W = 0, $\phi(s) = 0 \forall s \in U+0 = U$. Therefore, $\phi(s) = 0 \forall s \in U$, therefore, $\phi \in U^0$. Additionally, when letting U = 0, $\phi(s) = 0 \forall s \in 0+W = W$. Therefore, $\phi(s) = 0 \forall s \in W$, therefore, $\phi \in W^0$. As $\phi \in U^0 \text{and} \phi \in W^0, \phi \in U^0 \cap W^0$. Thus, $(U+W)^0 \subset U^0 \cap W^0$.

Now, let $\phi \in U^0 \cap W^0$, then $\phi \in U^0 \text{and} \phi \in W^0$. Therefore we have that $\phi(u) = 0 \forall u \in U$ and $\phi(w) = 0 \forall w \in W$. As ϕ is a linear map, $\phi(x) + \phi(y) = \phi(x+y)$, therefore, $\phi(u) + \phi(w) = 0 \forall u \in U, \forall w \in W$ implies that $\phi(u+w) = 0 \forall v \in V, \forall w \in W$, therefore, $\phi \in (U+W)^0$, and thus, $U^0 \cap W^0 \subset (U+W)^0$.

As, $(U+W)^0 \subset U^0 \cap W^0$ and $U^0 \cap W^0 \subset (U+W)^0$, $(U+W)^0 = U^0 \cap W^0$.

b) Let $\phi_1 \in U^0$ and let $\phi_2 \in W^0$, and thus $\phi_1 + \phi_2 \in U^0 + W^0$. Then, for any $z \in U \cap W, \phi_1(z) = \phi_2(z) = 0$, and as ϕ_1, ϕ_2 are linear maps, $(\phi_1 + \phi_2)(z) = 0$. This implies that $\phi_1 + \phi_2 = 0 \forall z \in U \cap W$, and therefore, $\phi_1 + \phi_2 \in (U \cap W)^0$. Thus $U^0 + W^0 \subset (U \cap W)^0$.

Since we have shown that $U^0 + W^0 \subset (U \cap W)^0$, in order to prove that $U^0 + W^0 = (U \cap W)^0$, it is sufficient to show that dim $U^0 + W^0 =$ dim $(U \cap W)^0$.

Since V is a finite dimensional vector space, let dim V = n. We know that dim $(U \cap W)^0$ = n - dim $U \cap W$. This will help us with the following:

$$\dim U^0 + W^0 = \dim W^0 + \dim U^0 - \dim U^0 \cap W^0$$
$$\dim U^0 + W^0 = 2n - \dim U - \dim W - \dim (U+W)^0$$
$$\dim U^0 + W^0 = 2n - \dim U - \dim W - (n - \dim (U+W)$$
$$\dim U^0 + W^0 = n - \dim U - \dim W + \dim (U+W)$$
$$\dim U^0 + W^0 = n - \dim (U \cap W)$$

Since $\dim(U \cap W)^0 = n - \dim(U \cap W) = \dim(U^0 + W^0), \dim(U \cap W)^0 = \dim(U^0 + W^0)$. Thus as $U^0 + W^0 \subset (U \cap W)^0$ and $\dim(U^0 + W^0) = \dim(U \cap W)^0, (U \cap W)^0 = U^0 + W^0$.

Problem 7

Problem 8

a) Since T: $P(\mathbb{R}) \to P(\mathbb{R})$, T': $P(\mathbb{R})' \to P(\mathbb{R})'$. By definition, $(T'(\phi))(p) = \phi \circ T \circ p$

$$\begin{aligned}
&(T'(\phi))(p) = \\
&\phi \circ T \circ p = \\
&\phi \circ (xp(x) + p'(x)) = \\
&p(2) + 2p'(2) + p''(2)
\end{aligned}$$

b) Since T: $P(\mathbb{R}) \to \mathbb{R}, T' : P(\mathbb{R})' \to P(\mathbb{R})'$. Additionally, by definition, $(T'(\phi))(p) = \phi(T(p))$.

$$\begin{aligned}
&(T'(\phi))(x^2) = \\
&\phi(T(x^2)) = \\
&\phi(x(x^2) + 2x) = \\
&\phi(x^3 + 2x) = \\
&[\frac{1}{4}x^4 + x^2]\Big|_{-1}^{1} = \\
&\frac{1}{4} + 1 - (\frac{1}{4} + 1) = \\
&0
\end{aligned}$$

Problem 9

$\rightarrow$ First, we will show that if $\phi_1, ..., \phi_m$ spans V', then Γ is injective. Suppose, v $\in$ null Γ, thus $\Gamma(v) = (0, ...0)$. Therefore, $\phi_1(v) = 0, ..., \phi_m(v) = 0$. Now, let ϕ be some arbitrary element of V', since $\phi_1, ..., \phi_m$ spans V', $\phi = \lambda_1\phi_1 + ... + \lambda_m\phi_m$ for some $\lambda_1, ..., \lambda_m \in F$. Now, consider $\phi(v)$.

$$\begin{aligned}
&\phi(v) = \\
&(\lambda_1\phi_1 + ... + \lambda_m\phi_m)(v) = \\
&\lambda_1\phi_1(v) + ... + \lambda_m\phi_m(v) = \\
&0 + ... + 0 = \qquad 0
\end{aligned}$$

Since $\phi(v) = 0 \forall \phi \in V'$, v = 0. Since null $\Gamma = 0$, then Γ is injective.

$\leftarrow$ Now, we will show that if Γ is injective, then $\phi_1, ..., \phi_m$ is spans V'. Suppose, looking for a contradiction, that $\phi_1, ..., \phi_m$ does not span V'. Now, let U = span($\phi_1, ..., \phi_m$. Now let $x \in V' - U$, and let $w \in F^m$. Since U is a proper subspace of V', there exists some $\phi \in V'$ such that $\phi(U) = 0$ and $\phi(w)$