Ben Asperheim

Practice Midterm

1) f) False, let $T \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ s.t. $T(x) = -2x$.
Now, consider $T^2 x = \lambda x$
$T \circ T x = \lambda x$
$T(-2x) = \lambda x$
$4x = \lambda x \Rightarrow \lambda = 4$, thus $\lambda = 4$ is an eigenvalue of T^2. Now, assume that $\lambda = 2$ is an eigenvalue of T. Then $Tx = 2x$
$-2x = 2x$
$-x = x$ iff $x = 0$, thus, there is no non-zero vector x where $\lambda = 2$ is an eigenvector, thus, false.

1) g) False, let $V = \mathbb{R}^2$ and let $\beta = (v_1, v_2)$ be a basis of V, then, let $T, S \in \mathcal{L}(V)$ s.t. $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 v_1$ and $S(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_2 v_2$.

Now, let $v \in \ker T$, then $Tv = 0 \Rightarrow \lambda_1 v_1 = 0 \Rightarrow \lambda_1 = 0$, thus $v = \lambda_2 v_2$, and thus $\ker T = \operatorname{span}\{v_2\}$, and thus Nullity $T = 1 \Rightarrow$ Rank $T = 1$

Similarly, it can be shown that Nullity S = Rank $S = 1$,

Thus, we have found 2 elements T, S that exist in $\{T \in \mathcal{L}(V) \mid \text{rank } T < 2\}$.

Now, consider $T+S$, specifically $\ker T+S$. Let $v \in \ker T+S$, then $(T+S)v = 0$
$Tv + Sv = 0$
$\lambda_1 v_1 + \lambda_2 v_2 = 0$, but as v_1 and v_2 form a basis $\Rightarrow \lambda_1 = \lambda_2 = 0$, thus, $v = 0$ and $\ker T+S = \{0\} \Rightarrow$ Rank $T+S = 2$, but then $T+S \notin \{T \in \mathcal{L}(V) \mid \text{rank } T < 2\}$, thus, $\{T \in \mathcal{L}(V) \mid \text{rank } T < 2\}$ is not a subspace, thus false.

1) h) True, let $\beta = (p_0, p_1, p_2, p_3)$ be a basis of $P_3(\mathbb{R})$, and let $p_0 = 1$, $p_1 = x$, $p_2 = x^2$, $p_3 = x^3$. Recall that the associated matrix of T, denoted as A is defined as follows.

$$A = [T]_{\beta,\beta} = \begin{bmatrix} | & | & | & | \\ [Tp_0]_\beta & [Tp_1]_\beta & [Tp_2]_\beta & [Tp_3]_\beta \\ | & | & | & | \end{bmatrix}$$

$(Tp_0)(1) = x(0) + 2(1) = 2$

$(Tp_1)(x) = x(1) + 2(x) = 3x$

$(Tp_2)(x^2) = x(2x) + 2(x^2) = 4x^2$

$(Tp_3)(x^3) = x(3x^2) + 2(x^3) = 5x^3$

$[Tp_0] = (2,0,0,0)$

$[Tp_1] = (0,3,0,0)$

$[Tp_2] = (0,0,4,0)$

$[Tp_3] = (0,0,0,5)$

thus $[T]_{\beta,\beta} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, thus true.

1) i) True, as $T \in \mathcal{L}(P^3)$ and P^3 is finite dimensional, the following are equivalent.

1) T is injective
2) T is surjective
3) T is invertible

So, in order to prove that T is invertible, it suffices to show that T is injective.

Let $v = (x, y, z) \in \ker T$, thus $(x-2y+z, 3y-4z, 5z) = 0$

$\Rightarrow z = 0 \Rightarrow y = 0 \Rightarrow x = 0$

thus, $v = 0$ and $\ker T = \{0\}$, thus, T is invertible.

1) j) True, suppose v is an eigenvector of T, thus, $\exists \lambda \in F$ s.t. $Tv = \lambda v$.

Then, consider

$$(T^2(T-I)^2)(v) = 0$$
$$(T^2(T^2 - 2TI + I^2))v = 0$$
$$(T^2(T^2 - 2T + I))v = 0$$
$$(T^4 - 2T^3 + T^2)v = 0$$
$$T^4(v) - 2T^3(v) + T^2(v) = 0$$
$$\lambda^4 v - 2\lambda^3 v + \lambda^2 v = 0$$
$$\lambda^2(\lambda^2 v - 2\lambda v + v) = 0$$
$$\lambda^2(\lambda^2 - 2\lambda + 1)v = 0$$
$$\lambda^2(\lambda - 1)^2 v = 0$$

$v \neq 0$, thus,

$\lambda = 0, 1$, thus, the only 2 possibilities for λ are 1, and 0, therefore, true.

4) a) Let $d = (d_1, d_2, d_3)$ be the standard basis of F^3 and let $e = (e_1, e_2)$ be the standard basis of F^2.

Recall that $A = [T]_{d,e} = \begin{bmatrix} | & | & | \\ [Td_1] & [Td_2] & [Td_3] \\ | & | & | \end{bmatrix}$

$Td_1 = (1 - 2(0) + 0, 2(1) + 3(0) - 5(0)) = (1, 2)$
$Td_2 = (0 - 2(1) + 0, 2(0) + 3(1) - 5(0)) = (-2, 3)$
$Td_3 = (0 - 2(0) + 1, 2(0) + 3(0) - 5(1)) = (1, -5)$

$[Td_1] = (1, 2)$
$[Td_2] = (-2, 3)$
$[Td_3] = (1, -5)$

Thus, $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -5 \end{bmatrix}$

b) Recall that $B = [S]_{e,d} = \begin{bmatrix} | & | \\ [Se_1] & [Se_2] \\ | & | \end{bmatrix}$

$Se_1 = (2(1) + 3(0), 1 - 0, 1 + 3(0)) = (2, 1, 1)$
$Se_2 = (2(0) + 3(1), 0 - 1, 0 + 3(1)) = (3, -1, 3)$

Thus, $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$

c) Recall that $[ST]_{d,d} = BA$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & -13 \\ -1 & -5 & 6 \\ 7 & 7 & -14 \end{bmatrix}$$

d) Recall that $[TS]_{e,e} = AB$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 2 & -12 \end{bmatrix}$$

5) a) $(T'(\varphi))(p)$

$= (\varphi \circ T)(p)$

$= \varphi \circ T(p)$

$= \varphi \circ (x p'(x) + p''(x))$

$= x p''(x) + p'(x) + p''(x) \big|_{x=1}$

$= 1p'''(1) + p'(1) + p''(1)$

$= p'(1) + p''(1) + p'''(1)$

b) $(T'(\psi))(q)$

$= (\psi \circ T)(q)$

$= \psi \circ T(q)$

$= \psi \circ (x(2x+1) + 2)$

$= \psi \circ (2x^2 + x + 2)$

$= \int_{-1}^{1} 2x^2 + x + 2 \, dx$

$= \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x \Big]_{-1}^{1}$

$= \frac{2}{3} + \frac{1}{2} + 2 - (-\frac{2}{3} + \frac{1}{2} - 2) = 16/3$

6) Suppose $v = \alpha v_1 + \beta v_2$ is an eigenvector of T.

Thus, $Tv = \lambda v$

$$T(\alpha v_1 + \beta v_2) = \lambda \alpha v_1 + \lambda \beta v_2$$

$$\alpha T(v_1) + \beta T(v_2) = \lambda(\alpha v_1 + \beta v_2)$$

$$\alpha(4v_1 + v_2) + \beta(-5v_1 - 2v_2) = \lambda(\alpha v_1 + \beta v_2)$$

$$\alpha 4 v_1 - 5\beta v_1 + \alpha v_2 - \beta 2 v_2 = \lambda(\alpha v_1 + \beta v_2)$$

$$(4\alpha - 5\beta)v_1 + (\alpha - 2\beta)v_2 = \lambda \alpha v_1 + \lambda \beta v_2$$

$$4\alpha - 5\beta = \lambda\alpha \qquad \alpha - 2\beta = \beta\lambda$$

$$4(\beta\lambda + 2\beta) - 5\beta = \lambda(\lambda\beta + 2\beta) \qquad \alpha = \beta\lambda + 2\beta$$

$$4\beta\lambda + 8\beta - 5\beta = \lambda^2\beta + \lambda 2\beta$$

$$2\beta\lambda + 3\beta = \lambda^2\beta$$

$$\beta(\lambda^2 - 2\lambda - 3) = 0$$

$$\cancel{\lambda s} \; \beta(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3, -1$$

If $\lambda = 3 \Rightarrow 4\alpha - 5\beta = 3\alpha$
$\alpha = 5\beta$

If $\lambda = -1 \Rightarrow 4\alpha - 5\beta = -\alpha$
$-5\beta = -5\alpha$
$\beta = \alpha$

When $\lambda = 3$, $v = \alpha v_1 + \frac{\alpha}{5} v_2 = \alpha(v_1 + \frac{1}{5} v_2)$ are the eigenvectors, $\alpha \in F$

When $\lambda = -1$, $v = \alpha v_1 + \alpha v_2 = \alpha(v_1 + v_2)$ are the eigenvectors, $\alpha \in F$

7) As $Tv_1 = 3v_1$, v_1 is an eigenvector of T with eigenvalue v_1. Now, consider $u_2 = \alpha v_1 + \beta v_2$, we want to find α, β s.t. u_2 is an eigenvector of T.

$$Tu_2 = \lambda(\alpha v_1 + \beta v_2) \quad {}^* \alpha\beta \neq 0$$
$$T(\alpha v_1 + \beta v_2) = \lambda(\alpha v_1 + \beta v_2)$$
$$\alpha(3v_1) + \beta(2v_1 + 2v_2) = \lambda(\alpha v_1 + \beta v_2)$$
$$3\alpha v_1 + 2\beta v_1 + 2\beta v_2 = \lambda\alpha v_1 + \beta\lambda v_2$$
$$(3\alpha + 2\beta)v_1 + (2\beta)v_2 = \lambda\alpha(v_1) + \beta\lambda(v_2)$$

$$3\alpha + 2\beta = \lambda\alpha \qquad 2\beta = \beta\lambda$$
$$3\alpha + 2\beta = 2\alpha \qquad 2 = \lambda$$
$$2\beta = -\alpha$$
$$\alpha = 2\beta$$

thus, $2v_1 - v_2$ is an eigenvector of T with eigenvalue $\lambda = 2$.

Next let $u_3 = \alpha v_1 + \beta v_2 + cv_3$, we want to find ~~ofp~~ α, β, c s.t. u_3 is an eigenvector of T.

$$Tu_3 = \lambda u_3$$
$$T(\alpha v_1 + \beta v_2 + cv_3) = \lambda(\alpha v_1 + \beta v_2 + cv_3) \quad {}^* \alpha, \beta, c \neq 0$$
$$\alpha(3v_1) + \beta(2v_1 + 2v_2) + c(v_1 + v_2 + 4v_3) = \lambda(\alpha v_1 + \beta v_2 + cv_3)$$
$$3\alpha v_1 + 2\beta v_1 + cv_1 + 2\beta v_2 + cv_2 + 4cv_3 = \lambda\alpha v_1 + \lambda\beta v_2 + \lambda c v_3$$
$$(3\alpha + 2\beta + c)v_1 + (2\beta + c)v_2 + (4c)v_3 = (\lambda\alpha)v_1 + (\lambda\beta)v_2 + (\lambda c)v_3$$

$$3\alpha + 2\beta + c = \lambda\alpha \qquad 2\beta + c = \lambda\beta \qquad 4c = \lambda c$$
$$3\alpha + 4\beta = 4\alpha \qquad 2\beta + c = 4\beta \qquad 4 = \lambda$$
$$\alpha \; 4\beta \qquad c = 2\beta$$

thus, $u_3 = 4v_1 + v_2 + 2v_3$ is an eigenvector with eigenvalue $\lambda = 4$

Thus, we consider the set $\{v_1, 2v_1 - v_2, 4v_1 + v_2 + 2v_3\}$, we want to show that this is a basis of V. Since there are 3 elements in the set, it is sufficient to show that the elements are linearly independent.

Consider $\alpha v_1 + \beta(2v_1 - v_2) + c(4v_1 + v_2 + 2v_3) = 0$

$$\alpha v_1 + 2\beta v_1 - \beta v_2 + 4cv_1 + cv_2 + 2cv_3 = 0$$

$$(\alpha + 2\beta + 4c)v_1 + (-\beta + c)v_2 + (2c)v_3 = 0$$

$\Rightarrow c = 0 \Rightarrow \beta = 0 \Rightarrow \alpha = 0$, thus, the elements are linearly independent, and thus, form a basis of V. Thus, as we have found a basis of V s.t. all elements are eigenvectors of T, the matrix w.r.t. this basis is diagonalizable.