Aufgabe 1)

a) $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

↓ Python

$A^{-1} = \begin{pmatrix} -1 & 0 & 20'000 \\ 0 & 1 & 0 \\ 1 & 0 & -10'000 \end{pmatrix}$

$\text{cond}(A)_\infty = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$= 3 \cdot 20'001 = \underline{\underline{60'003}}$

b) $\tilde{b} = \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\|b\|_\infty = 1$

$\|b - \tilde{b}\|_\infty = \left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} 0 \\ 0 \\ -\varepsilon \end{pmatrix} \right\|_\infty = \varepsilon$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq \text{cond}(A)_\infty \cdot \frac{\|b - \tilde{b}\|_\infty}{\|b\|_\infty} < 0.01$$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq 60'003 \cdot \frac{\varepsilon}{1} \leq 0.01$$

$$\varepsilon \leq \frac{0.01}{60'003} = \underline{\underline{1.66658 \cdot 10^{-7}}}$$

c) $Ax = b$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

↖ Python

$A\tilde{x} = \tilde{b}$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1.66658 \cdot 10^{-7} \end{pmatrix}$$

$$\Rightarrow \tilde{x} \begin{pmatrix} -0.99666684 \\ 1.000 \\ 0.9983339 \end{pmatrix}$$

$$\frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} = 3{,}33315 \cdot 10^{-3}$$

d)

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix}$$

$\|A\|_\infty = 3$

$$\|A-\tilde{A}\|_\infty = \left\| \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} - \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix} \right\|_\infty$$

$$= \left\| \begin{pmatrix} \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \end{pmatrix} \right\|_\infty = 3 \cdot 10^{-7}$$

Bedingung:

$$\text{cond}(A)_\infty \cdot \frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty} < 1 \quad ?$$

Falls ja: relativer Fehler in x ist begrenzt

$$60'003 \cdot \frac{3 \cdot 10^{-7}}{3} = 60'003 \cdot 10^{-7} < 1 \quad \checkmark$$

$$\frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} \leq \frac{\text{cond}(A)_\infty}{1 - \text{cond}(A)_\infty \cdot \frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty}} \cdot \left(\frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty} + \frac{\|b-\tilde{b}\|}{\|b\|} \right)$$

$$= \frac{60'003}{1 - 60'003 \cdot 10^{-7}} \cdot \left(\frac{3 \cdot 10^{-7}}{3} + \frac{\varepsilon}{1} \right) \leq 0.01$$

$$\Rightarrow \varepsilon \leq 6.56583 \cdot 10^{-8}$$