Serie 10:

Aufg. 1:

$$Ax = b \,, \quad A = \begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

a) diagonal dominant

$$a_{11} = 8 > 5+2$$

$$a_{22} = 9 > 5+1 \qquad \checkmark \implies \text{Jacobi konvergiert}$$

$$a_{33} = 7 > 4+2$$

b) $x^{(k+1)} = -D^{-1}(L+R)\,x^{(k)} + D^{-1} b$

$$\underset{A}{\begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix}} + \underset{D}{\begin{pmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{pmatrix}} + \underset{R}{\begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$x^{(k+1)} = \begin{pmatrix} -\frac{1}{8} & 0 & 0 \\ 0 & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} \cdot x^{(k)} + \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & -5/8 & -1/4 \\ -5/9 & 0 & -1/9 \\ -4/7 & -2/7 & 0 \end{pmatrix}}_{B} x^{(k)} + \underbrace{\begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}}_{C}$$

$$x^{(1)} = \begin{pmatrix} 0 & -5/8 & -1/4 \\ -5/9 & 0 & -1/9 \\ -4/7 & -2/7 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix} = \begin{pmatrix} -1/8 \\ -8/9 \\ -2/7 \end{pmatrix} + \begin{pmatrix} 19/8 \\ 5/9 \\ 34/7 \end{pmatrix} = \begin{pmatrix} 2.25 \\ -1/3 \\ 32/7 \end{pmatrix} \approx \underline{\underline{\begin{pmatrix} 2.25 \\ -0.3333 \\ 4.5714 \end{pmatrix}}}$$

$$x^{(2)} = \begin{pmatrix} 0 & -5/8 & -1/4 \\ -5/9 & 0 & -1/9 \\ -4/7 & -2/7 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2.25 \\ -1/3 \\ 32/7 \end{pmatrix} + \begin{pmatrix} 19/8 \\ 5/9 \\ 34/7 \end{pmatrix} = \begin{pmatrix} -157/168 \\ -443/252 \\ -25/21 \end{pmatrix} + \begin{pmatrix} 19/8 \\ 5/9 \\ 34/7 \end{pmatrix} = \begin{pmatrix} 121/84 \\ -101/84 \\ 11/3 \end{pmatrix} \approx \underline{\underline{\begin{pmatrix} 1.4405 \\ -1.2024 \\ 3.6667 \end{pmatrix}}}$$

$$x^{(3)} = \begin{pmatrix} 0 & -5/8 & -1/4 \\ -5/9 & 0 & -1/9 \\ -4/7 & -2/7 & 0 \end{pmatrix} \cdot \begin{pmatrix} 121/84 \\ -101/84 \\ 11/3 \end{pmatrix} + \begin{pmatrix} 19/8 \\ 5/9 \\ 34/7 \end{pmatrix} = \begin{pmatrix} -37/224 \\ -913/756 \\ -47/98 \end{pmatrix} + \begin{pmatrix} 19/8 \\ 5/9 \\ 34/7 \end{pmatrix} = \begin{pmatrix} 495/224 \\ -493/756 \\ 429/98 \end{pmatrix} \approx \underline{\underline{\begin{pmatrix} 2.2098 \\ -0.6521 \\ 4.3776 \end{pmatrix}}}$$

c) a-posteriori

$$\|x^{(3)}-\bar{x}\|_\infty \leq \frac{\overset{0.875}{\|B\|_\infty}}{1-\|B\|_\infty}\|x^{(3)}-x^{(2)}\|_\infty$$

$$\Rightarrow \|x^{(3)}-\bar{x}\|_\infty \leq \frac{0.875}{0.1250}\left\|\begin{pmatrix}2.2098\\-0.6521\\4.3776\end{pmatrix}-\begin{pmatrix}1.4405\\-1.2024\\3.6667\end{pmatrix}\right\|_\infty = \frac{0.875}{0.1250}\left\|\begin{pmatrix}0.7693\\0.5503\\0.7109\end{pmatrix}\right\|_\infty$$

$$\Rightarrow \|x^{(3)}-\bar{x}\|_\infty \leq \frac{0.875}{0.1250}\cdot 0.7693 = \underline{\underline{5.3851}}$$

d) a-priori:

$$\|x^{(n)}-\bar{x}\|_\infty \leq \frac{\|B\|_\infty^n}{1-\|B\|_\infty}\|x^{(1)}-x^{(0)}\|_\infty \overset{!}{\leq} 10^{-4}$$

$$\|x^{(n)}-\bar{x}\|_\infty \leq \frac{0.8750^n}{0.1250}\left\|\begin{pmatrix}2.2500\\-0.3333\\4.5714\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\|_\infty = \frac{0.8750^n}{0.1250}\left\|\begin{pmatrix}1.25\\0.6666\\1.5714\end{pmatrix}\right\|_\infty = \frac{0.8750^n}{0.1250}\cdot 1.5714$$

$$\Rightarrow \frac{0.8750^n}{0.1250}\cdot 1.5714 \leq 10^{-4}$$

$$\Rightarrow 0.8750^n \leq 7.955\cdot 10^{-6} \quad\Rightarrow\quad n\cdot\log(0.8750) \leq \log(7.955\cdot 10^{-6}) \quad\Rightarrow\quad n \geq \frac{\log(7.955\cdot 10^{-6})}{\log(0.8750)}$$

$$\Rightarrow \quad n \geq 87.9325 \quad\Rightarrow\quad \underline{\underline{n=88}}$$

e)

$$\|x^{(n')}-\bar{x}\|_\infty \leq \frac{\|B\|_\infty^{n'}}{1-\|B\|_\infty}\|x^{(3)}-x^{(2)}\|_\infty \overset{!}{\leq} 10^{-4}$$

$$\|x^{(n')}-\bar{x}\|_\infty \leq \frac{0.8750^{n'}}{0.1250}\left\|\begin{pmatrix}2.2098\\-0.6521\\4.3776\end{pmatrix}-\begin{pmatrix}1.4405\\-1.2024\\3.6667\end{pmatrix}\right\|_\infty = \frac{0.8750^{n'}}{0.1250}\left\|\begin{pmatrix}0.7693\\0.5503\\0.7109\end{pmatrix}\right\|_\infty = \frac{0.8750^{n'}}{0.1250}\, 0.7693$$

$$\Rightarrow \frac{0.8750^{n'}}{0.1250}\cdot 0.7693 \leq 10^{-4}$$

$$\Rightarrow 0.8750^{n'} \leq 1.625\cdot 10^{-5} \quad\Rightarrow\quad n'\cdot\log(0.8750) \leq \log(1.625\cdot 10^{-5}) \quad\Rightarrow\quad n' \geq \frac{\log(1.625\cdot 10^{-5})}{\log(0.8750)}$$

$$\Rightarrow \quad n' \geq 82.5841 \quad\Rightarrow\quad \underline{\underline{n'=83}} \qquad n = n'+2$$

$$\Rightarrow \underline{\underline{n = 85 \text{ Iterationsschritte}}}$$