

Aufgabe 1)

$$a) \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Python

$$A^{-1} = \begin{pmatrix} -1 & 0 & 20'000 \\ 0 & 1 & 0 \\ 1 & 0 & -10'000 \end{pmatrix}$$

$$\text{cond}(A)_{\infty} = \|A\|_{\infty} \cdot \|A^{-1}\|_{\infty} \\ = 3 \cdot 20'001 = \underline{\underline{60'003}}$$

$$b) \quad \tilde{b} = \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\|b\|_{\infty} = 1 \\ \|b - \tilde{b}\|_{\infty} = \left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \right\|_{\infty} = \left\| \begin{pmatrix} 0 \\ 0 \\ -\varepsilon \end{pmatrix} \right\|_{\infty} = \varepsilon$$

$$\frac{\|x - \tilde{x}\|_{\infty}}{\|x\|_{\infty}} \leq \text{cond}(A)_{\infty} \cdot \frac{\|b - \tilde{b}\|_{\infty}}{\|b\|_{\infty}} < 0.01$$

$$\frac{\|x - \tilde{x}\|_{\infty}}{\|x\|_{\infty}} \leq 60'003 \cdot \frac{\varepsilon}{1} \leq 0.01$$

$$\varepsilon \leq \frac{0.01}{60'003} = \underline{\underline{1.66658 \cdot 10^{-7}}}$$

$$c) \quad Ax = b$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Python

$$A\tilde{x} = \tilde{b}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1.66658 \cdot 10^{-7} \end{pmatrix}$$

$$\Rightarrow \tilde{x} = \begin{pmatrix} -0.99666684 \\ 1.000 \\ 0.9983339 \end{pmatrix}$$

$$\frac{\|x - \tilde{x}\|_{\infty}}{\|x\|_{\infty}} = \underline{\underline{3,33315 \cdot 10^{-3}}}$$

$$d) \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \quad \tilde{A} = \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix}$$

$$\|A\|_{\infty} = 3$$

$$\begin{aligned} \|A - \tilde{A}\|_{\infty} &= \left\| \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} - \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix} \right\|_{\infty} \\ &= \left\| \begin{pmatrix} \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \end{pmatrix} \right\|_{\infty} = 3 \cdot 10^{-7} \end{aligned}$$

Bedingung:

$$\text{cond}(A)_{\infty} = \frac{\|A - \tilde{A}\|_{\infty}}{\|A\|_{\infty}} < 1 \quad ?$$

Falls ja: relativer Fehler in x ist begrenzt

$$60'003 \cdot \frac{3 \cdot 10^{-7}}{3} = 60'003 \cdot 10^{-7} < 1 \quad \checkmark$$

$$\frac{\|x - \tilde{x}\|_{\infty}}{\|x\|_{\infty}} \leq \frac{\text{cond}(A)_{\infty}}{1 - \text{cond}(A)_{\infty} \cdot \frac{\|A - \tilde{A}\|_{\infty}}{\|A\|_{\infty}}} \cdot \left(\frac{\|A - \tilde{A}\|_{\infty}}{\|A\|_{\infty}} + \frac{\|b - \tilde{b}\|}{\|b\|} \right)$$

$$= \frac{60'003}{1 - 60'003 \cdot 10^{-7}} \cdot \left(\frac{3 \cdot 10^{-7}}{3} + \frac{\varepsilon}{1} \right) \leq 0.01$$

$$\Rightarrow \underline{\underline{\varepsilon \leq 6.56583 \cdot 10^{-8}}}$$