Aufg. 2:

a) A diagonal dominant $\Longrightarrow$ Gauss-Seidel konv.

b) $x^{(k+1)} = -(D+L)^{-1} R x^{(k)} + (D+L)^{-1} b$

$$\underbrace{\begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix}}_{L} + \underbrace{\begin{pmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{R}$$

$$(D+L)^{-1} = \begin{pmatrix} 8 & 0 & 0 \\ 5 & 9 & 0 \\ 4 & 2 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix}$$

$$x^{(k+1)} = -\begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & -0.625 & -0.25 \\ 0 & 0.3472 & 0.0278 \\ 0 & 0.2579 & 0.1349 \end{pmatrix}}_{B} x^{(k)} + \underbrace{\begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}}_{C}$$

$x^{(k+1)} = B\, x^{(k)} + C$

$$x^{(1)} = \begin{pmatrix} 0 & -0.625 & -0.25 \\ 0 & 0.3472 & 0.0278 \\ 0 & 0.2579 & 0.1349 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix} = \begin{pmatrix} 2.25 \\ -1.0278 \\ 3.8651 \end{pmatrix}$$

mit Python: $x^{(2)} = \begin{pmatrix} 2.0511 \\ -1.0134 \\ 3.9746 \end{pmatrix}$ $\qquad x^{(3)} = \begin{pmatrix} 2.0147 \\ -1.0054 \\ 3.9931 \end{pmatrix}$

c) $\|x^{(3)} - \bar{x}\|_\infty \leq \dfrac{\|B\|_\infty}{1-\|B\|_\infty} \|x^{(3)} - x^{(2)}\|_\infty$

$$\Longrightarrow \|x^{(3)} - \bar{x}\|_\infty \leq \frac{0.875}{0.1250} \left\| \begin{pmatrix} 2.0\ldots \\ -1.0054 \\ 3.9\ldots \end{pmatrix} - \begin{pmatrix} 1.4405 \\ -1.2024 \\ 3.6667 \end{pmatrix} \right\|_\infty$$

$$\Longrightarrow \|x^{(3)} - \bar{x}\|_\infty \leq \frac{0.875}{0.1250} \cdot 0.0364 = \underline{\underline{0.2548}}$$

d) a-priori:

$$\|x^{(n)} - \bar{x}\|_\infty \leq \frac{\|B\|_\infty^n}{1 - \|B\|_\infty} \|x^{(1)} - x^{(0)}\|_\infty \overset{!}{\leq} 10^{-4}$$

$$\|x^{(n)} - \bar{x}\|_\infty \leq \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 2.25 \\ -1.0278 \\ 3.8651 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\|_\infty = \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 1.25 \\ 2.0278 \\ 0.8651 \end{pmatrix} \right\|_\infty = \frac{0.8750^n}{0.1250} \cdot 2.0278$$

$$\Rightarrow \frac{0.8750^n}{0.1250} \cdot 2.0278 \leq 10^{-4} \Rightarrow n \cdot \log(0.8750) \leq \log(6.164 \cdot 10^{-6})$$

$$\Rightarrow n \geq 89.842 \Rightarrow n = 90$$

e) $$\|x^{(n')} - \bar{x}\|_\infty \leq \frac{\|B\|_\infty^{n'}}{1 - \|B\|_\infty} \|x^{(3)} - x^{(2)}\|_\infty \overset{!}{\leq} 10^{-4}$$

$$\|x^{(n')} - \bar{x}\|_\infty \leq \frac{0.8750^{n'}}{0.1250} \left\| \begin{pmatrix} 2.0147 \\ -1.0054 \\ 3.9931 \end{pmatrix} - \begin{pmatrix} 2.0511 \\ -1.0134 \\ 3.9746 \end{pmatrix} \right\|_\infty = \frac{0.8750^{n'}}{0.1250} \cdot 0.0364$$

$$\Rightarrow \frac{0.8750^{n'}}{0.1250} \cdot 0.0364 \leq 10^{-4} \Rightarrow n' \cdot \log(0.875) \leq \log(3.434 \cdot 10^{-4})$$

$$\Rightarrow n' \geq 59.736 \Rightarrow n' = 60 \Rightarrow n = n' + 2 = \underline{\underline{62 \text{ Iterationsschritte}}}$$