1) $Ax = b \qquad A = \begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$

a) A: diagonaldominant

$a_{11} = 8 > 5+2$

$a_{22} = 9 > 5+1$ ✓ => Jacobi konvergiert

$a_{33} = 7 > 4+2$

b) $x^{(k+1)} = -D^{-1}(L+R)\,x^{(k)} + D^{-1}b$

$$\underset{A}{\begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix}} + \underset{D}{\begin{pmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{pmatrix}} + \underset{R}{\begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$D^{-1} = \begin{pmatrix} 1/8 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}$$

$$x^{(k+1)} = -\begin{pmatrix} 1/8 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}\begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 1/8 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}\begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$x^{(k+1)} = \underbrace{\begin{pmatrix} 0 & -0.625 & -0.25 \\ -0.5556 & 0 & -0.1111 \\ -0.5714 & -0.2875 & 0 \end{pmatrix}}_{B} x^{(k)} + \begin{pmatrix} 1/8 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}\begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$x^{(k+1)} = B\,x^{(k)} + C$

$$x^{(1)} = \begin{pmatrix} 0 & -0.625 & -0.25 \\ -0.5556 & 0 & -0.1111 \\ -0.5714 & -0.2875 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1/8 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}\begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 2.2500 \\ -0.3333 \\ 4.5714 \end{pmatrix} \qquad x^{(2)} = \begin{pmatrix} 1.4405 \\ -1.2024 \\ 3.6667 \end{pmatrix} \qquad x^{(3)} = \begin{pmatrix} 2.2098 \\ -0.6521 \\ 4.3775 \end{pmatrix}$$

c) a - posteriori

$$\| x^{(3)} - \bar{x} \|_\infty \leq \frac{\| B \|_\infty}{1 - \| B \|_\infty} \| x^{(3)} - x^{(2)} \|_\infty$$

0.875

$$\| x^{(3)} - \bar{x} \|_\infty \leq \frac{0.875}{0.1250} \left\| \begin{pmatrix} 2.2098 \\ -0.6521 \\ 4.3775 \end{pmatrix} - \begin{pmatrix} 1.4405 \\ -1.2024 \\ 3.6667 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.875}{0.1250} \cdot 0.7693 = 5.3851$$

d) a - priori

$$\| x^{(n)} - \bar{x} \|_\infty \leq \frac{\| B \|_\infty^n}{1 - \| B \|_\infty} \| x^{(1)} - x^{(0)} \|_\infty \overset{!}{\leq} 10^{-4}$$

$$\| x^{(n)} - \bar{x} \|_\infty \leq \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 2.2500 \\ -0.3333 \\ 4.5714 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 1.2500 \\ 0.6777 \\ 1.5714 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.8750^n}{0.1250} \cdot 1.5714 \leq 10^{-4}$$

$$n \geq 88$$

e) $$\|x^{(n)} - \bar{x}\|_\infty \le \frac{\|B\|_\infty^n}{1-\|B\|_\infty} \|x^{(3)} - x^{(2)}\|_\infty \overset{!}{\le} 10^{-4}$$

$$= \frac{0.8750^{n'}}{0.1250} \cdot 0.7693 \le 10^{-4}$$

$n' \geq 82.58$

$x^{(83)} \longrightarrow x^{(85)}$

Iterationsschritt

Aufgabe 2

a) A diagonaldom $\rightarrow$ G-S konvergiert

b)

$$\boxed{x^{(k+1)} = -(D+L)^{-1} R x^{(k)} + (D+L)^{-1} b}$$

$$\begin{pmatrix} 8 & 5 & 2 \\ 5 & 9 & 1 \\ 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = L + D + R$$

$$(D+L)^{-1} = \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix}$$

$$x^{(k+1)} = \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$x^{(k+1)} = \underbrace{\begin{pmatrix} 0 & -0.625 & -0.2500 \\ 0 & 0.3472 & 0.0278 \\ 0 & 0.2579 & 0.1349 \end{pmatrix}}_{B} x^{(k)} + \underbrace{\begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}}_{C}$$

$$x^{(k+1)} = B \quad x^{(k)} + C$$

$$x^{(1)} = \begin{pmatrix} 0 & -0.625 & -0.2500 \\ 0 & 0.3472 & 0.0278 \\ 0 & 0.2579 & 0.1349 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0.125 & 0 & 0 \\ -0.0694 & 0.1111 & 0 \\ -0.0516 & -0.0317 & 0.1429 \end{pmatrix} \begin{pmatrix} 19 \\ 5 \\ 34 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 2.2500 \\ -1.0278 \\ 3.8651 \end{pmatrix} \qquad x^{(2)} = \begin{pmatrix} 2.0511 \\ -1.0134 \\ 3.9746 \end{pmatrix} \qquad x^{(3)} = \begin{pmatrix} 2.0147 \\ -1.0054 \\ 3.9931 \end{pmatrix}$$

c) a - posteriori

$$\| x^{(3)} - \bar{x} \|_\infty \leq \frac{\|B\|_\infty}{1 - \|B\|_\infty} \| x^{(3)} - x^{(2)} \|_\infty$$

$$B = \begin{pmatrix} 0 & -0.625 & -0.2500 \\ 0 & 0.3472 & 0.0278 \\ 0 & 0.2579 & 0.1349 \end{pmatrix} \qquad \|B\|_\infty = 0.8750$$

$$\| x^{(3)} - \bar{x} \|_\infty \leq \frac{0.875}{0.1250} \left\| \begin{pmatrix} 2.0\ldots \\ -1.0054 \\ 3.9\ldots \end{pmatrix} - \begin{pmatrix} 2.0511 \\ -1.0134 \\ 3.9746 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.875}{0.1250} \cdot 0.0364 = \underline{\underline{0.2548}}$$

d) a - priori

$$\| x^{(n)} - \bar{x} \|_\infty \leq \frac{\| B \|_\infty^n}{1 - \| B \|_\infty} \| x^{(1)} - x^{(0)} \|_\infty$$

$$\| x^{(n)} - \bar{x} \|_\infty \leq \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 2.2500 \\ -1.0278 \\ 2.8651 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.8750^n}{0.1250} \left\| \begin{pmatrix} 1.2500 \\ -0.0278 \\ 0.8651 \end{pmatrix} \right\|_\infty$$

$$= \frac{0.8750^n}{0.1250} \cdot 1.25 \leq 10^{-4}$$

$$n \geq \ldots$$

e) $$\| x^{(n)} - \bar{x} \|_\infty \leq \frac{\| B \|_\infty^n}{1 - \| B \|_\infty} \| x^{(3)} - x^{(2)} \|_\infty \overset{!}{\leq} 10^{-4}$$

$$= \frac{0.8750^{n'}}{0.1250} \cdot 0.0369 \leq 10^{-4}$$

$$n' \geq 85$$

$$x^{(85)} \longrightarrow x^{(87)}$$

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