Aufgabe 1

a) $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -1 & 0 & 20'000 \\ 0 & 1 & 0 \\ 1 & 0 & -10'000 \end{pmatrix}$

$\text{cond}(A)_\infty = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 3 \cdot 20'001 = \underline{\underline{60'003}}$

b) $\tilde{b} = \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\|b\|_\infty = 1$

$\|b - \tilde{b}\|_\infty = \left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} 0 \\ 0 \\ -\varepsilon \end{pmatrix} \right\|_\infty = \varepsilon$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq \text{cond}(A)_\infty \cdot \frac{\|b - \tilde{b}\|_\infty}{\|b\|_\infty} \leq 0.01$$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq 60'003 \cdot \frac{\varepsilon}{1} \leq 0.01$$

$$\varepsilon \leq \frac{0.01}{60'003} = \underline{\underline{1.66658 \cdot 10^{-7}}}$$

c) $Ax = b$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x \overset{\text{Python}}{=} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1.66658 \cdot 10^{-7} \end{pmatrix} \Rightarrow \tilde{x} = \begin{pmatrix} -0.99666684 \\ 1.000 \\ 0.9983334 \end{pmatrix}$$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = \underline{\underline{3,3331599999999746 \cdot 10^{-3}}}$$

d) $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix}$ $\tilde{A} = \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix}$

$\|A\|_\infty = 3$

$\|A - \tilde{A}\|_\infty = \left\| \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} - \begin{pmatrix} 1 \pm 10^{-7} & 0 \pm 10^{-7} & 2 \pm 10^{-7} \\ 0 \pm 10^{-7} & 1 \pm 10^{-7} & 0 \pm 10^{-7} \\ 10^{-4} \pm 10^{-7} & 0 \pm 10^{-7} & 10^{-4} \pm 10^{-7} \end{pmatrix} \right\|$

$= \left\| \begin{pmatrix} \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \\ \pm 10^{-7} & \pm 10^{-7} & \pm 10^{-7} \end{pmatrix} \right\| = 3 \cdot 10^{-7}$

Bed. $\text{cond}(A)_\infty \cdot \frac{\|A - \tilde{A}\|_\infty}{\|A\|_\infty} < 1$

Falls ja: relativer Fehler in x ist begrenzt

$60'003 \cdot \frac{3 \cdot 10^{-7}}{3} = 60'003 \cdot 10^{-7} < 1$

dxmax

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq \frac{\text{cond}(A)_\infty}{1 - \text{cond}(A)_\infty \cdot \frac{\|A - \tilde{A}\|_\infty}{\|A\|_\infty}} \cdot \left(\frac{\|A - \tilde{A}\|_\infty}{\|A\|_\infty} + \frac{\|b - \tilde{b}\|_\infty}{\|b\|_\infty} \right)$$

$\Rightarrow$ dx obs

$$= \frac{60'003}{1 - 60'003 \cdot 10^{-7}} \left(\frac{3 \cdot 10^{-7}}{3} + \frac{\varepsilon}{1} \right) \leq 0.01$$

$$\Rightarrow \underline{\underline{\varepsilon \leq 6.56583 \cdot 10^{-8}}}$$