Module 3b. Hypothesis testing

Data Science & AI

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Contents

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Learning Goals

- Statistical hypothesis testing concepts
- Hypothesis testing procedure
- Apply the *z*- and *t*-test

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Testing procedure

Statistical Hypothesis Testing

Hypothesis Idea that has yet to be proven: statement regarding the numeric value of a population parameter

Hypothesis Test verification of a statement about the values of one or multiple population parameters

Null Hypothesis (H_0) Base hypothesis, we start with assuming it is true

Alternative Hypothesis (H_1, H_a) Conclusion if the null hypothesis is unlikely to be true

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Elements of a testing procedure

Test Statistic The value that is calculated from the sample

Region of Acceptance The region of values supporting the null hypothesis

Critical Region / Region of Rejection The region of values rejecting the null hypothesis

Significance Level The probability of rejecting a true null hypothesis H_0

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Testing procedure

1. Formulate both hypotheses (H_0 and H_1)
2. Determine the significance level (α)
3. Calculate the test statistic
4. Determine the critical region or the probability value
5. Draw conclusions

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Hypotheses about superheroes

A superhero rescues 3.3 persons a day


敵は階段 駅の杉並戦隊

ベビーカーおろすんジャー

Source: `http://www.cracked.com/quick-fixes/4-people-who-saved-day-while-dressed-as-superheroes/`

Assume that, over a period of 30 days, on average 3.483 people were saved each day ($\overline{x} = 3.483$, $n = 30$)

1. Hypothesis: $H_0 : \mu = 3.3$; $H_1 : \mu > 3.3$
2. Significance level: $\alpha = 0.05$
3. Test statistic: $\overline{x} = 3.483$

Standard deviation of the population (assumed to be known): $\sigma = 0.55$

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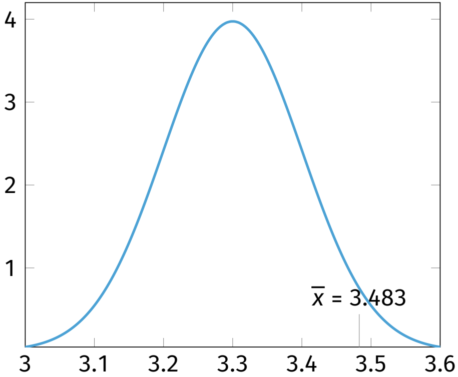
Calculate test statistic

Based on the central limit theorem, we know that:
$M \sim Nor(\mu = 3.3; \sigma = \frac{0.55}{\sqrt{30}} = 0.1)$



Probability Value

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Probability Value

p-value

The p-value is the probability, if the null hypothesis is true, to obtain a value for the test statistic that is at least as extreme as the observed value.

- p-value $< \alpha \Rightarrow$ reject H_0: the discovered value of $\overline{x}$ is too extreme;
- p-value $\geq \alpha \Rightarrow$ do not reject H_0: the discovered value of $\overline{x}$ can still be explained by coincidence.

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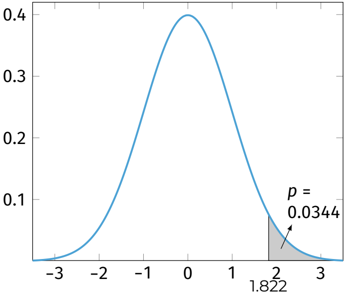
Probability Value



$$P(M > 3.483) = P\left(Z > \frac{3.483 - 3.3}{\frac{\sigma}{\sqrt{n}}}\right) = P(Z > 1.822) = 0.0344$$

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Critical Region

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Critical Region

Critical region

The critical region is the collection of all values of the test statistic for which we can reject the null hypothesis.

We look for a critical value g for which:

$$P(M > g) = \alpha$$

Determine z_α for which:

$$P(Z > z_\alpha) = \alpha \Rightarrow g = \mu + z_\alpha . \frac{\sigma}{\sqrt{n}}$$

- Left of g: region of acceptance (do not reject H_0)
- Right of g: critical region (reject H_0)

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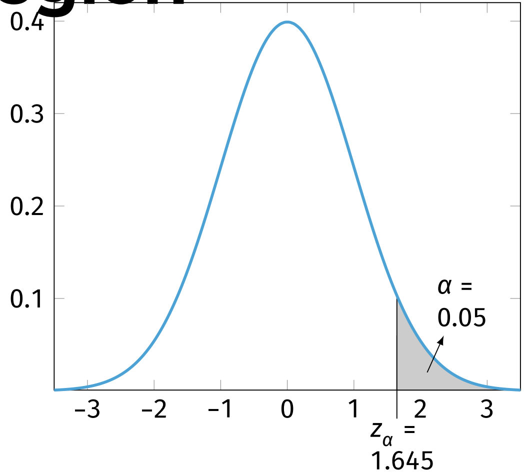
Critical Region



significance level α = 0.05 ⇒ critical value z_α=1.645
(z_α =`stats.norm.isf(1 - 0.95)`)

Left-tailed testing

What would you have to change in the equation in order to calculate the correct critical value?

Left-tailed testing

What would you have to change in the equation in order to calculate the correct critical value? Answer:

$$g = \mu - z \times \frac{\sigma}{\sqrt{n}}$$

because

$$P(M < g) = P\left(Z < \frac{g - \mu}{\frac{\sigma}{\sqrt{n}}}\right) = 0.05$$

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Left-tailed testing

Because of symmetry:

$$P\left(Z > -\left(\frac{g-\mu}{\frac{\sigma}{\sqrt{n}}}\right)\right) = 0.05$$

The corresponding *z*-value is 1.645, and therefore:

$$z = \frac{-g+\mu}{\frac{\sigma}{\sqrt{n}}} \Leftrightarrow -g = \frac{\sigma}{\sqrt{n}} z_\alpha - \mu$$

$$\Leftrightarrow g = \mu - z_\alpha \frac{\sigma}{\sqrt{n}}$$

Two-tailed testing

Sometimes it can be necessary to perform a two-tailed test. In this case, two critical values need to be calculated, namely the left and right critical value.

$$g = \mu \pm z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}} \tag{1}$$

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Summary

Goal	Test regarding the value of the population mean μ using a sample of n independent values		
Prerequisite	De population has a random distribution, n is sufficiently large		
Test Type	Two-tailed	Left-tailed	Right-tailed
H_0	$\mu = \mu_0$	$\mu = \mu_0$	$\mu = \mu_0$
H_1	$\mu \neq \mu_0$	$\mu < \mu_0$	$\mu > \mu_0$
Critical Region	$\lvert\overline{x}\rvert > g$	$\overline{x} < -g$	$\overline{x} > g$
Test statistic	$z = \frac{\overline{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$		

Table: Summary of Testing Procedures

Requirements for z-test

- The sample needs to be random
- The sample size needs to be sufficiently large ($n \geq 30$)
- The test statistic needs to have a normal distribution
- The standard deviation of the population, σ, is known

Sometimes these assumptions will not hold and in this case we can *not* use the Z-test!

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Examples

Example 1

When drawing a random sample consisting of 50 observations from a population with variance $\sigma^2 = 55$ we obtain as sample mean $\overline{x} = 25$.
We now want to find out if there is a reason to assume that the population mean is smaller than 27.

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Example 1

1. Formulate both hypotheses
   $H_0 : \mu = 27$ en $H_1 : \mu < 27$.
2. Determine significance level α and sample size n
   $\alpha = 0.05$ en $n = 50$
3. Test statistic & value: sample mean $\overline{x} = 25$

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Example 1 (cont.)

4a Probability Value

According to the central limit theorem:

$$M \sim Nor(\mu = 27, \frac{\sigma^2}{\sqrt{n}})$$

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{25 - 27}{\sqrt{\frac{55}{50}}} \approx -1.91$$

We therefore have a probability value of **0.0281**.

Using a significance level of 0.05, we can reject H_0.

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Example 1 (cont.)

4b Calculate and plot the critical region

$$g = \mu - z \times \frac{\sigma}{\sqrt{n}}$$
$$= 27 - 1.645 \times \sqrt{\frac{\sigma}{n}}$$
$$= 25.27470944$$

We have that $\overline{x} < g$ and therefore we can reject H_0.

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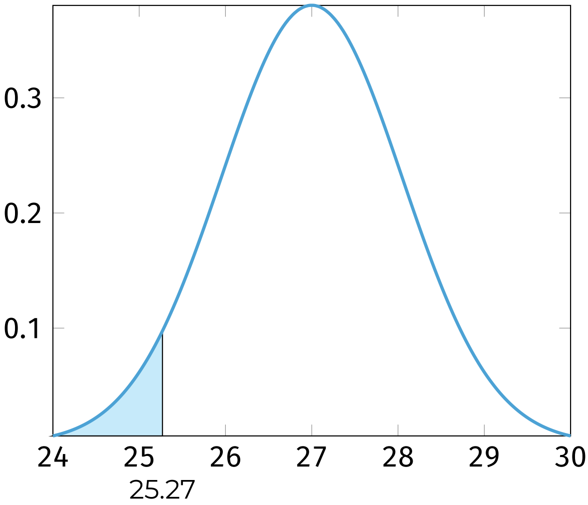
Example 1 (cont.)



Example 1 (cont.)

5 Conclusion
We can conclude, based on the sample, that $\mu < 27$ for a significance level of 0.05

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Example 2

In a reseach about the amount of change in the pockets of our superheroes, researchers state that on average a superhero carries €25 of cash. We assume that the standard deviation of the population $\sigma = 7$. For a random sample of size $n = 64$, the average amount of money a superhero carries is $\overline{x} = €23$. For the significance level, $\alpha = 0.05$ is selected.

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Example 2

1. Formulate both hypotheses
   $H_0 : \mu = 25$ en $H_1 : \mu \neq 25$
2. Determine significance level α and sample size n
   $\alpha = 0.05$ en $n = 64$.
3. Test statistic & value: $\overline{x} = 23$

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Example 2 (cont.)

4b Calculate the critical region

$$g_1 = \mu - z \times \frac{\sigma}{\sqrt{n}} = 23.28$$

$$g_2 = \mu + z \times \frac{\sigma}{\sqrt{n}} = 26.72$$

We have that $\overline{x}$ is inside the critical region (because 23 < 23.28) so we can reject H_0.

5 Based on this sample we can conclude that superheroes carry on average *less* than 25 €, using a significance level of 5%

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Student’s t-test

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Student's t-test

What if the requirements for a z-test are not met? E.g.

- Sample size too small
- Population stdev (σ) unknown

If the variable is normally distributed, you can use the t-test

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The *t*-test

Determine critical value:

$$g = \mu \pm t \times \frac{s}{\sqrt{n}}$$

- t-value is derived from the Student's t-distribution, based on the number of *degrees of freedom*, $n - 1$
- Look for value using the function `t.isf()` in Python
- Apart from this, the procedure is identical to the procedure of the z-test

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Errors in Hypothesis Tests

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Errors in Hypothesis Tests

Conclusion	Reality H_0 **True**	 H_1 **True**
H_0 **not rejected**	Correct inference	Type II error (false negative)
H_0 **rejected**	Type I error (false positive)	Correct inference

P(type I error) = α (= significance level)
P(type II error) = β
Calculating β is ***not*** trivial, but if α ↘ then β ↗

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