Spectral Analysis

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Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

Fourier transform

$$\underbrace{X(f)}_{\substack{\text{Spectrum at} \\ \text{frequency } f}} = \int_{-\infty}^{\infty} x(t) \underbrace{e^{-j2\pi ft}}_{\substack{\text{Basis function} \\ \text{for frequency } f}} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

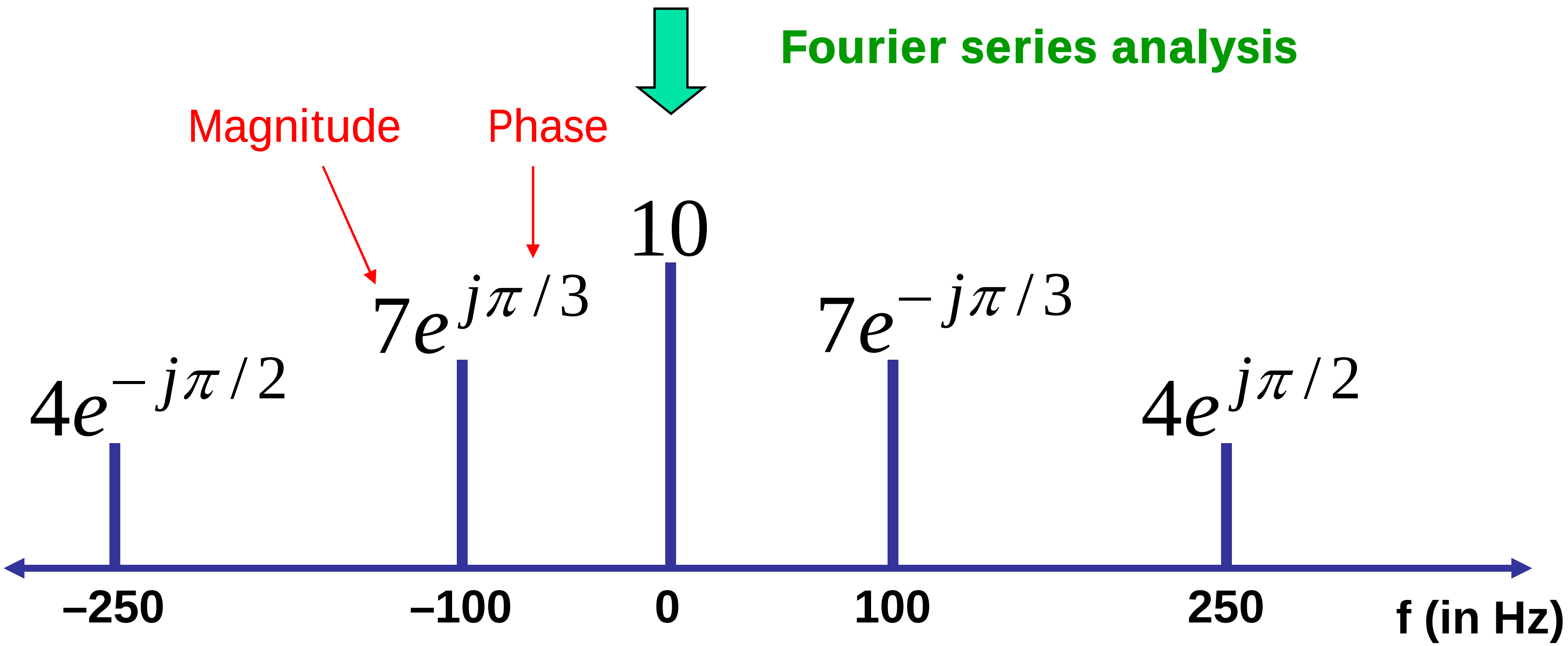
Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Example

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$



Magnitude and phase

$$X(f) = |X(f)|\, e^{j\phi(f)}$$

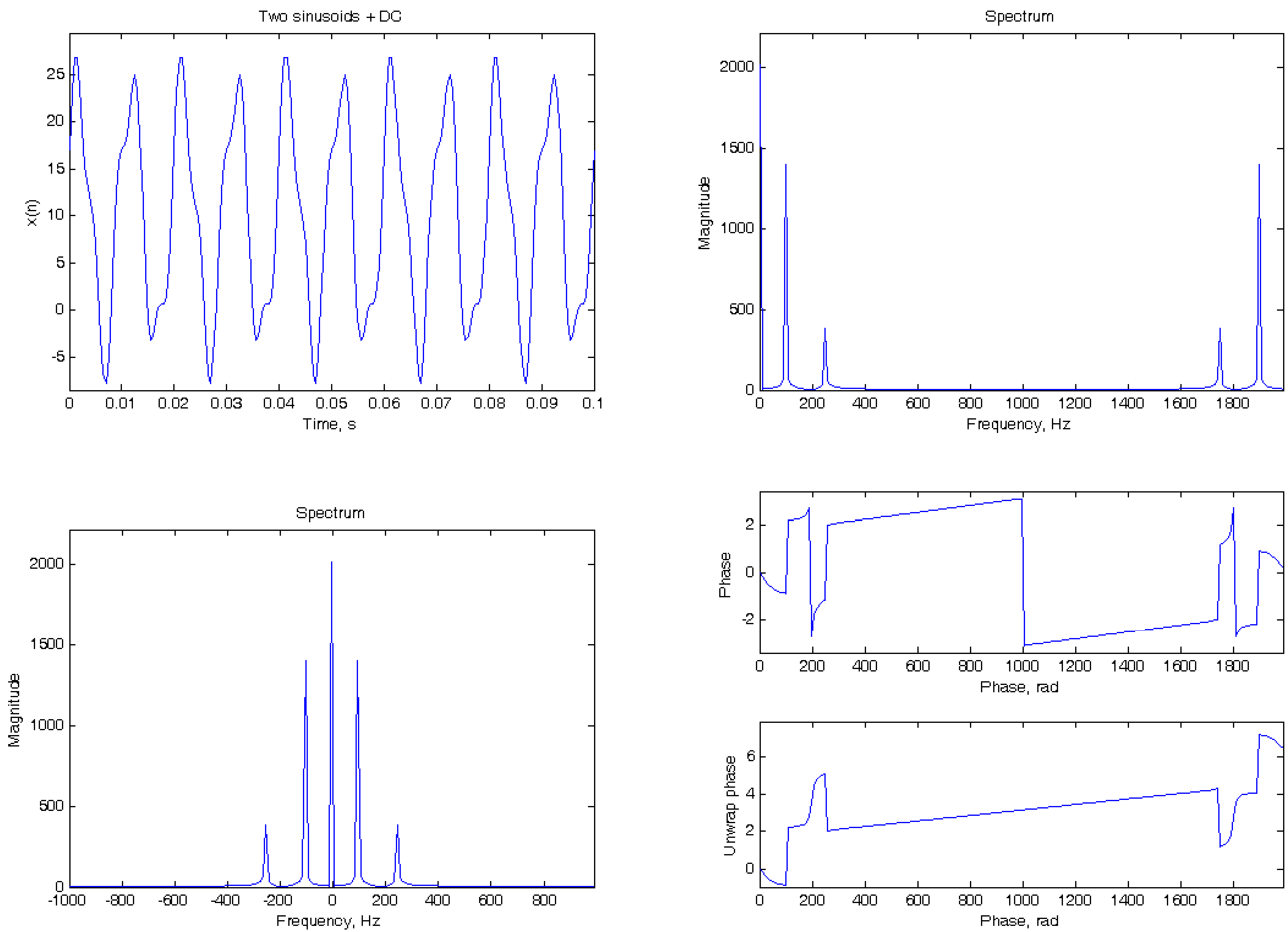
where magnitude and phase spectra

$$|X(f)| = \sqrt{\{\mathrm{Re}[X(f)]\}^2 + \{\mathrm{Im}[X(f)]\}^2}$$

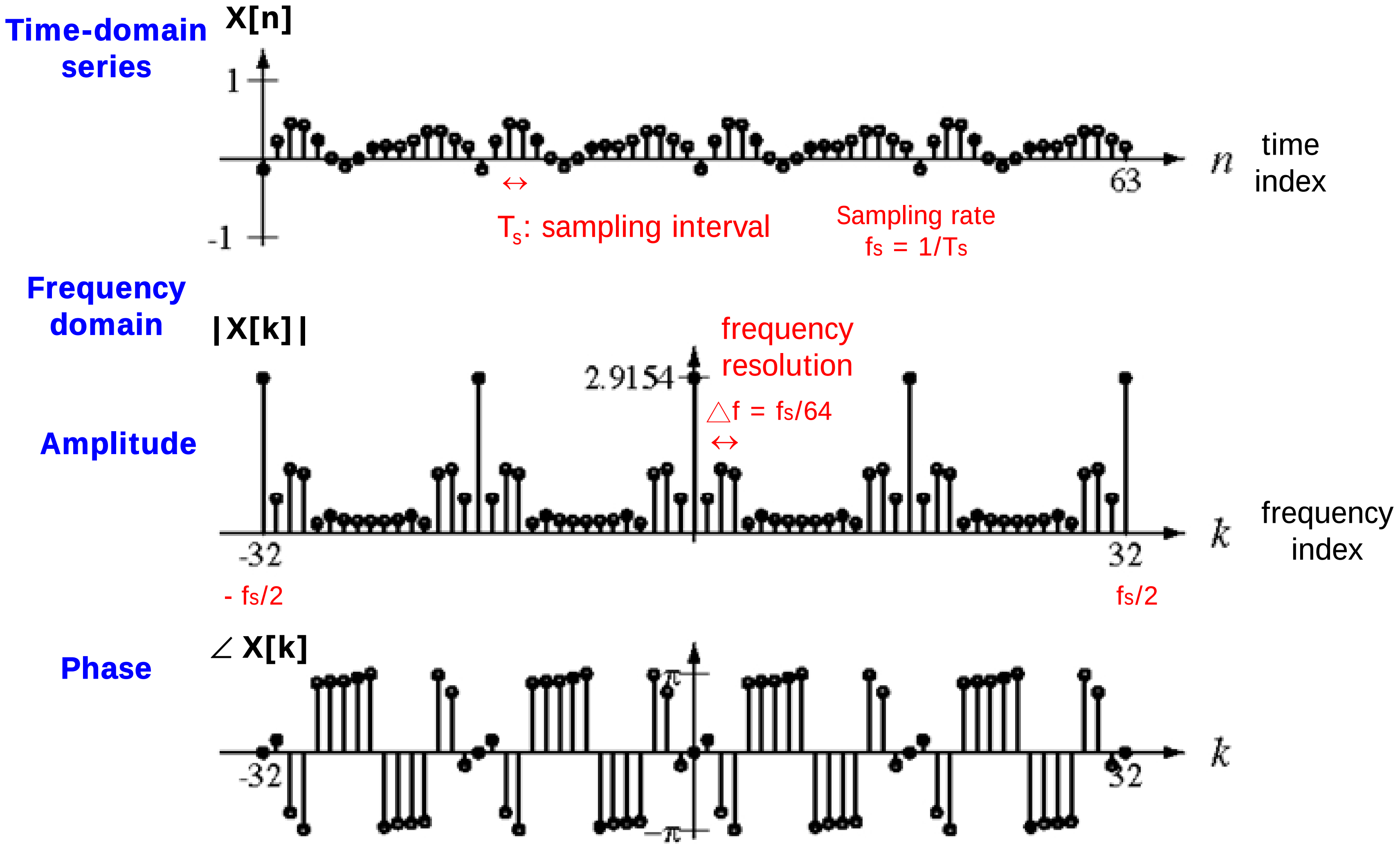
$$\phi(f) = \tan^{-1}\left\{\frac{\mathrm{Im}[X(f)]}{\mathrm{Re}[X(f)]}\right\}$$

```matlab
% Generating two sinusoids and one DC component
fs=2000;   % sampling at 2 kHz
t=0:1/fs:0.1;
x=10 + 14*cos(2*pi*100*t-pi/3) + 4*cos(2*pi*250*t-pi/2);
subplot(2,2,1)
plot(t,x)
ylabel('x(n)')
xlabel('Time, s')
title('Two sinusoids + DC')
axis([min(t) max(t) min(x)*1.1 max(x)*1.1])

% Spectral analysis
Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_mag = abs(Xf);  % magnitude of spectrum
subplot(2,2,2)
plot(f,Xf_mag)
xlabel('Frequency, Hz')
ylabel('Magnitude')
title('Spectrum')
axis([min(f) max(f) 0 max(Xf_mag)*1.1])
```

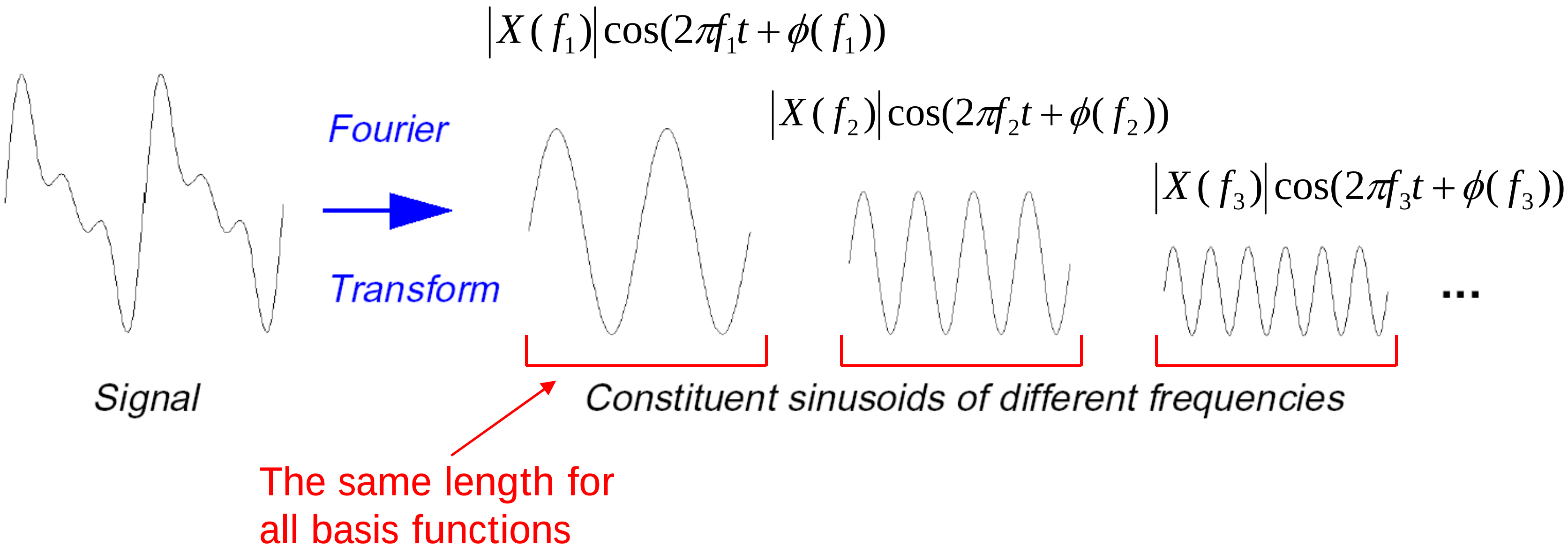
```
Xf_phase = angle(Xf);                 % phase of spectrum
subplot(4,2,6)
plot(f,Xf_phase)
xlabel('Phase, rad');   ylabel('Phase');
axis([min(f) max(f) min(Xf_phase)*1.1 max(Xf_phase)*1.1])

Xf_phase = unwrap(Xf_phase);    % Unwrap phase angle
subplot(4,2,8)
plot(f,Xf_phase)
xlabel('Phase, rad');    ylabel('Unwrap phase');
axis([min(f) max(f) min(Xf_phase)*1.1 max(Xf_phase)*1.1])

% Shift zero-frequency component to center of spectrum
Xf_center=fftshift(Xf);
Xf_mag = abs(Xf_center);         % magnitude of spectrum
Xf_phase = phase(Xf_center);    % phase of spectrum
subplot(2,2,3)
f=f-fs/2;
plot(f,Xf_mag)
xlabel('Frequency, Hz');    ylabel('Magnitude');
title('Spectrum')
axis([min(f) max(f) 0 max(Xf_mag)*1.1])
```


Two sinusoids + DC
x(n)
Time, s
Spectrum
Magnitude
Frequency, Hz
Spectrum
Magnitude
Frequency, Hz
Phase
Phase, rad
Unwrap phase
Phase, rad

Discrete Fourier transform



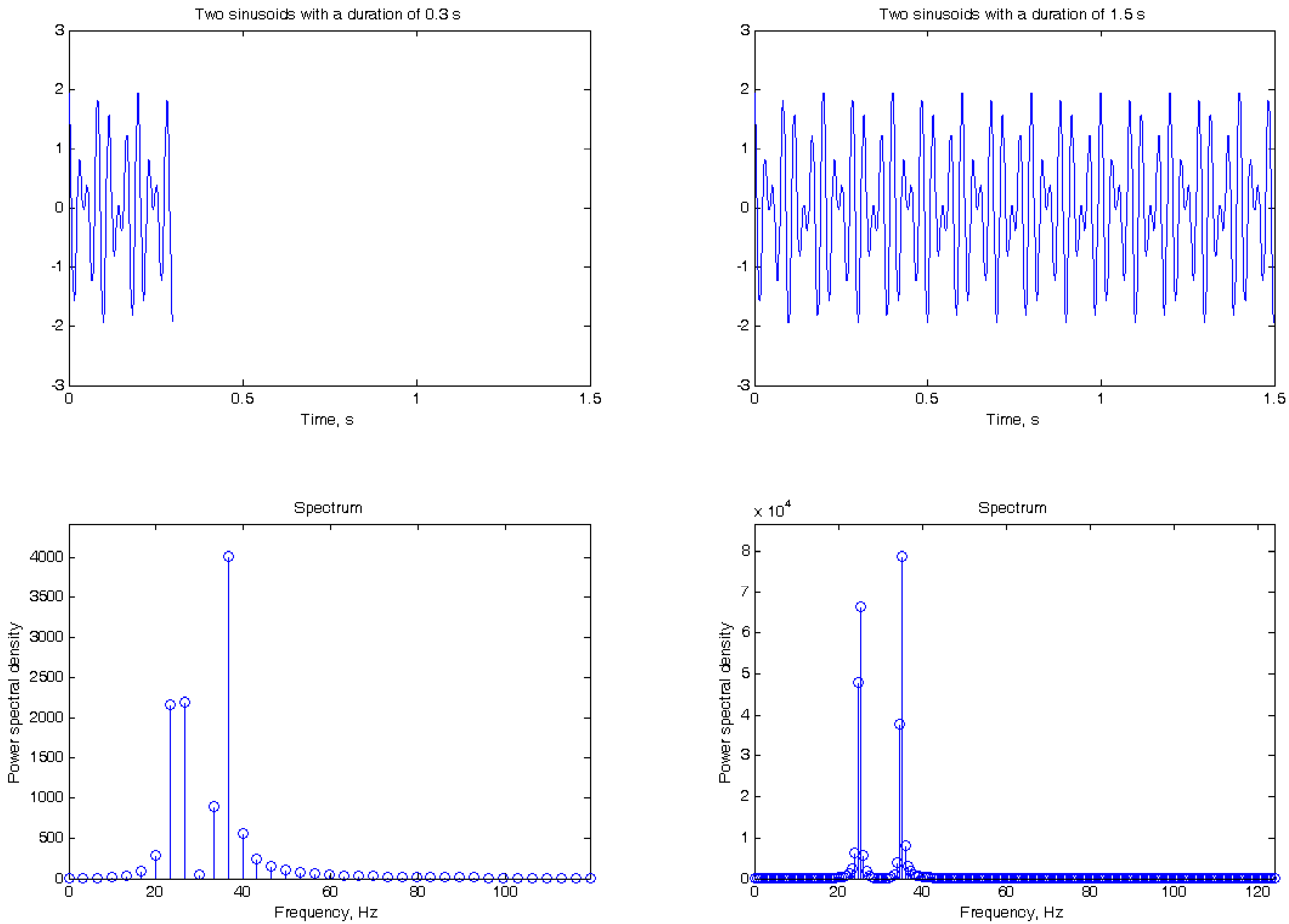
Decomposion of signal



Modified from “Matlab Wavelet Toolbox Manual“

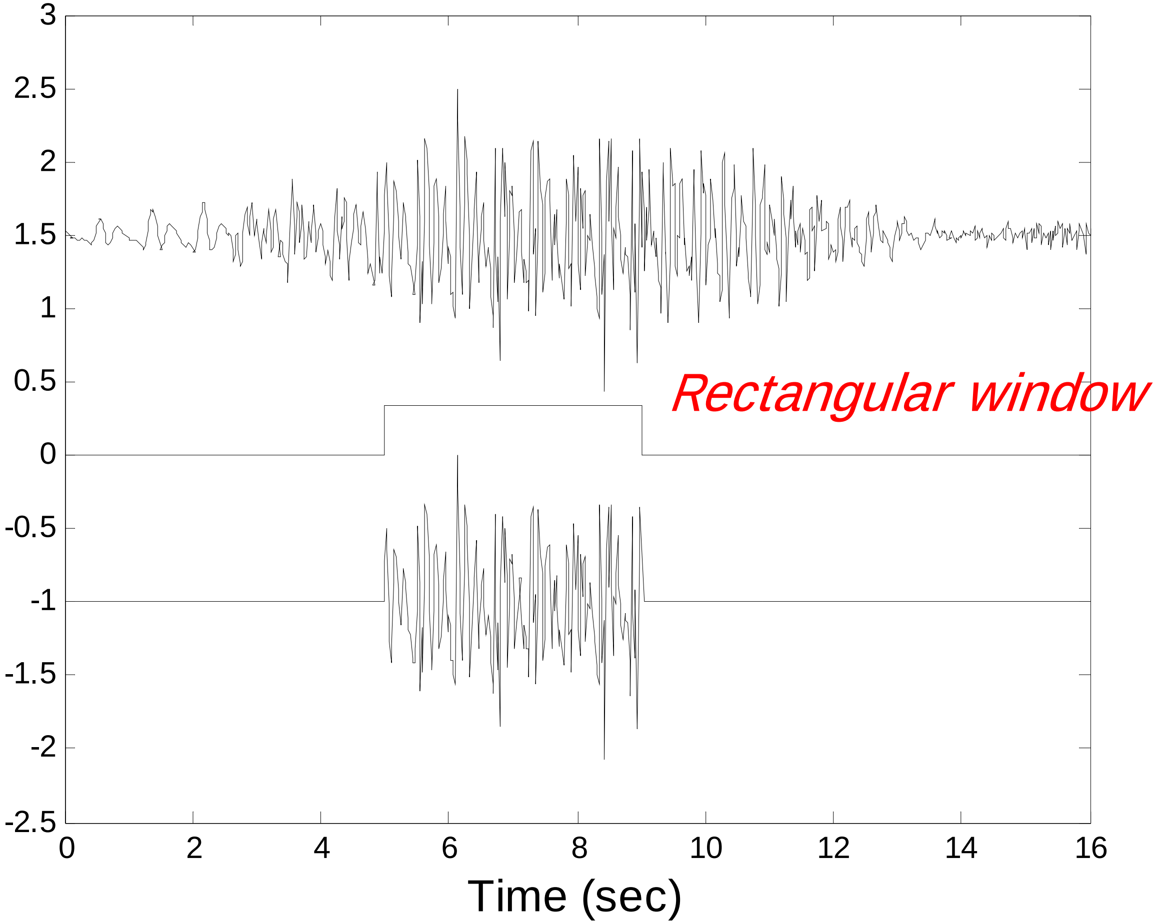
Effect of data points on spectral analysis

```
fs=500;
t=0:1/fs:0.3;
x = cos(2*pi*25*t) + cos(2*pi*35*t+pi/10);

Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);  % power spectral density
index=1:length(Xf)/4;
stem(f(index),Xf_power(index))

% Improve resolution by increase data length
t=0:1/fs:0.3*5;
x = cos(2*pi*25*t) + cos(2*pi*35*t+pi/10);

Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);  % power spectral density
index=1:length(Xf)/4;
stem(f(index),Xf_power(index))
```



Two sinusoids with a duration of 0.3 s
Two sinusoids with a duration of 1.5 s
Time, s
Time, s
Spectrum
Spectrum
Power spectral density
Power spectral density
Frequency, Hz
Frequency, Hz
x 10^4

Data length: truncation



Window functions

Rectangular:

$$w(n) = 1$$

Blackman:

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$$

Hamming:

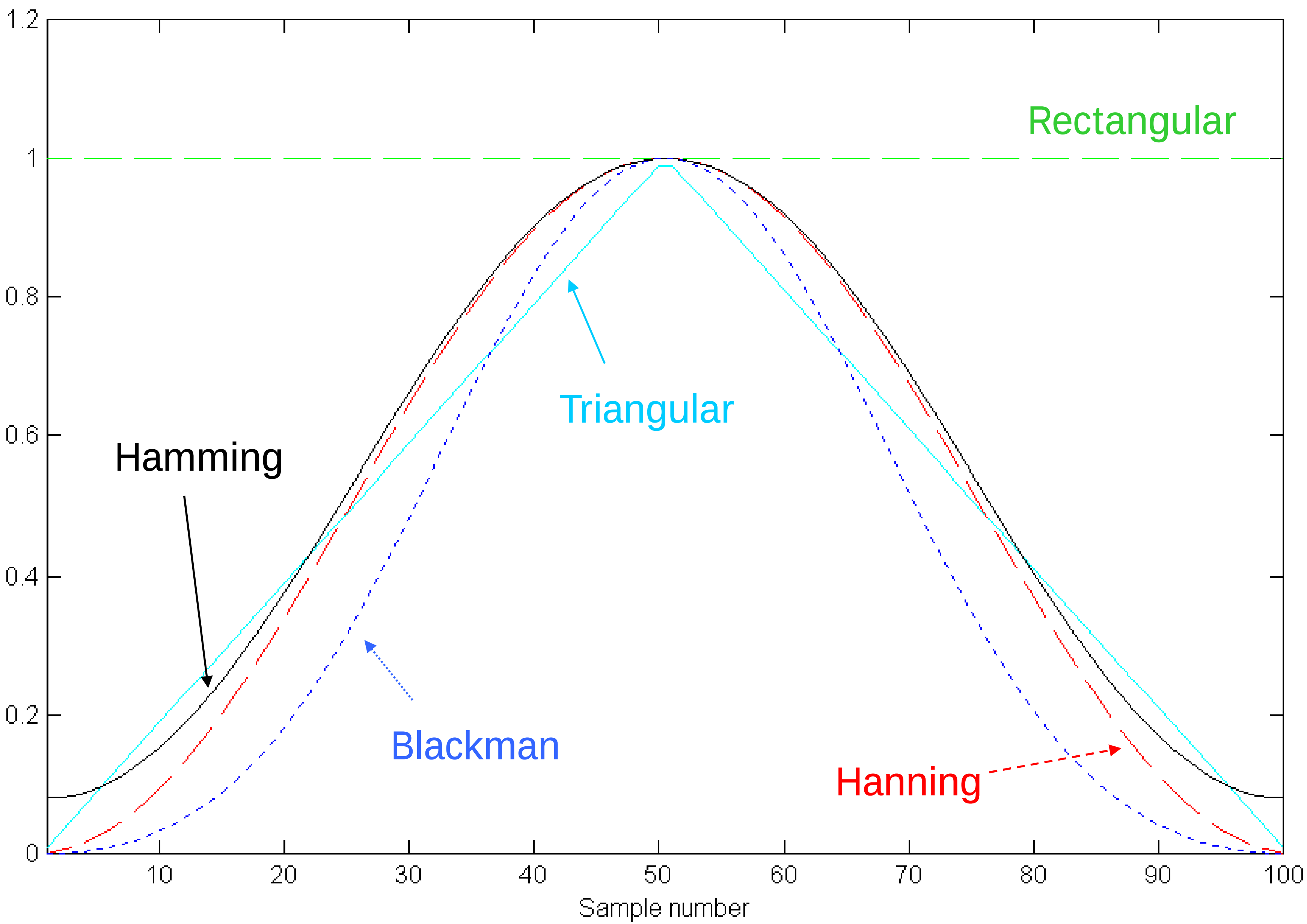
$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$$

Bartlett (Triangular):

$$w(n) = \begin{cases} \dfrac{2n}{N-1}, & 0 \le n \le \dfrac{N-1}{2} \\ 2 - \dfrac{2n}{N-1}, & \dfrac{N-1}{2} \le n \le N-1 \end{cases}$$

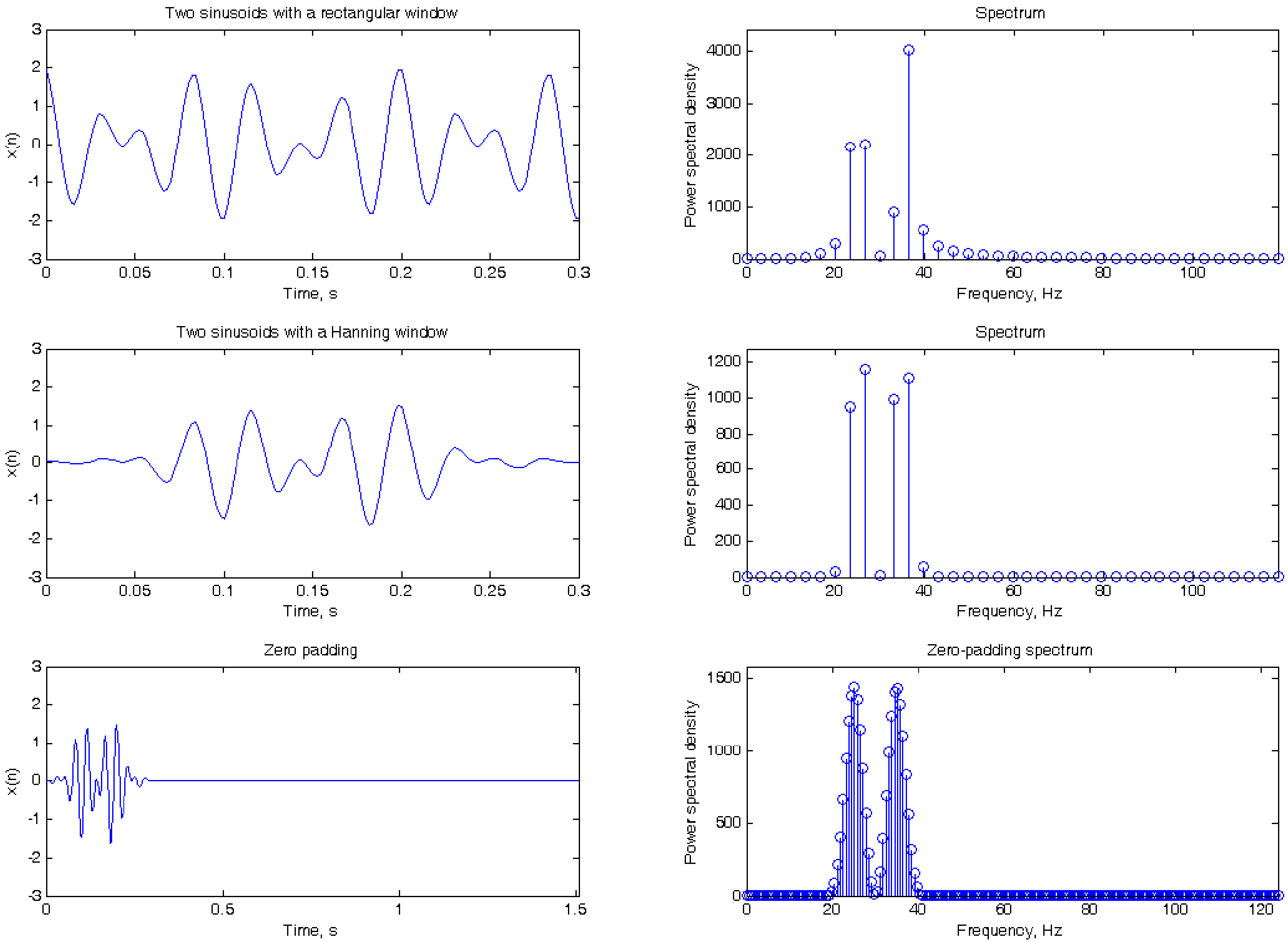
Hanning:

$$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)$$


Rectangular
Triangular
Hamming
Blackman
Hanning
1.2
1
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
90
100
Sample number

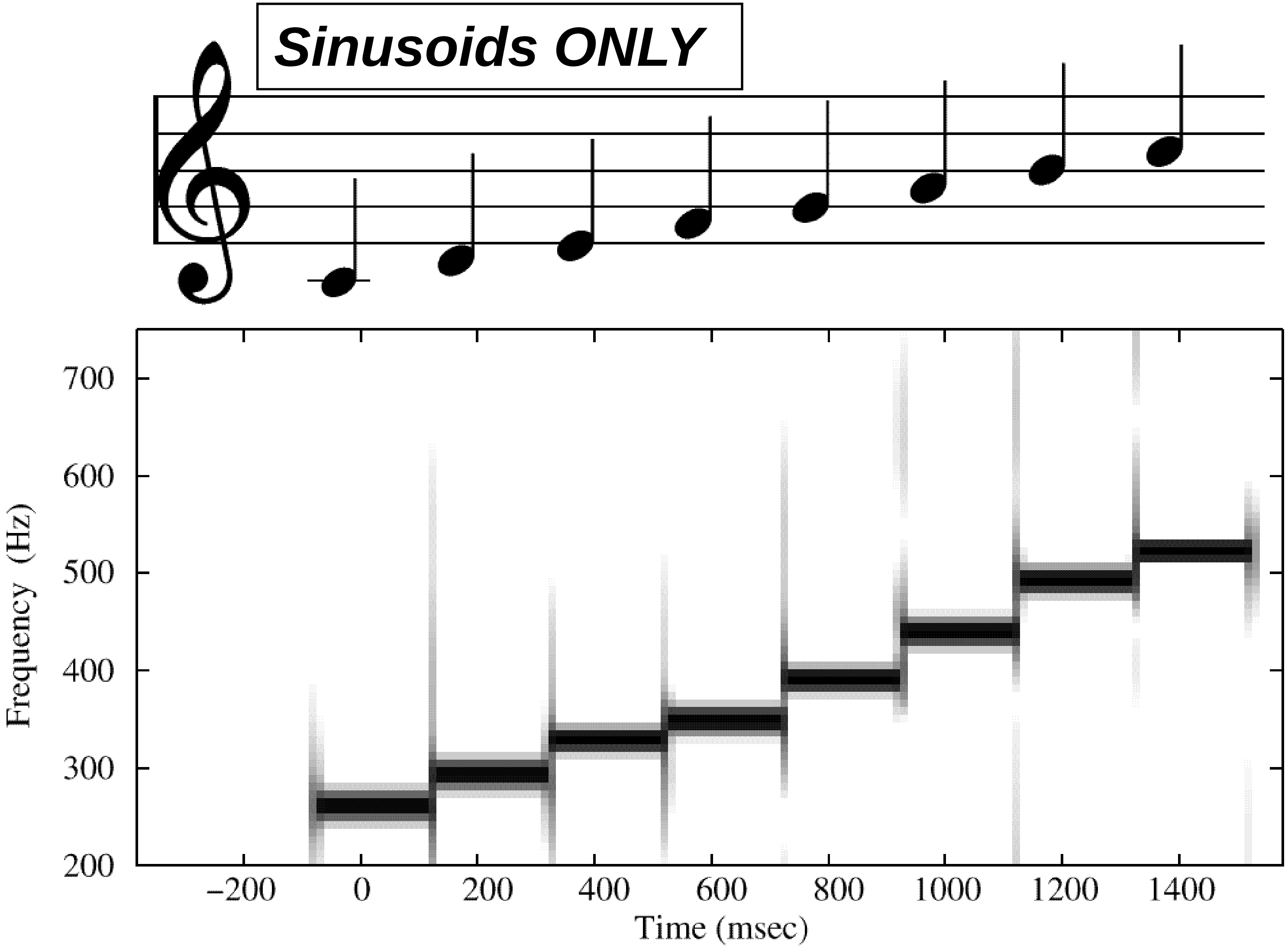
```
fs=500;
t=0:1/fs:0.3;
x = cos(2*pi*25*t) + cos(2*pi*35*t+pi/10);

Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);    % power spectral density
index=1:length(Xf)/4;
stem(f(index),Xf_power(index))

% Using Hanning window
x=x.*hanning(length(x))';
Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);  % power spectral density
index=1:length(Xf)/4;
stem(f(index),Xf_power(index))
```

```
% Zero padding
x_ZP=zeros(1,length(x)*5);
x_ZP(1:length(x))=x;
t_ZP=(0:length(x_ZP)-1)*1/fs;

Xf=fft(x,length(x)*5);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);  % power spectral density
index=1:length(Xf)/4;
stem(f(index),Xf_power(index))
```



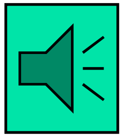
Two sinusoids with a rectangular window
x(n)
Time, s
Spectrum
Power spectral density
Frequency, Hz
Two sinusoids with a Hanning window
x(n)
Time, s
Spectrum
Power spectral density
Frequency, Hz
Zero padding
x(n)
Time, s
Zero-padding spectrum
Power spectral density
Frequency, Hz

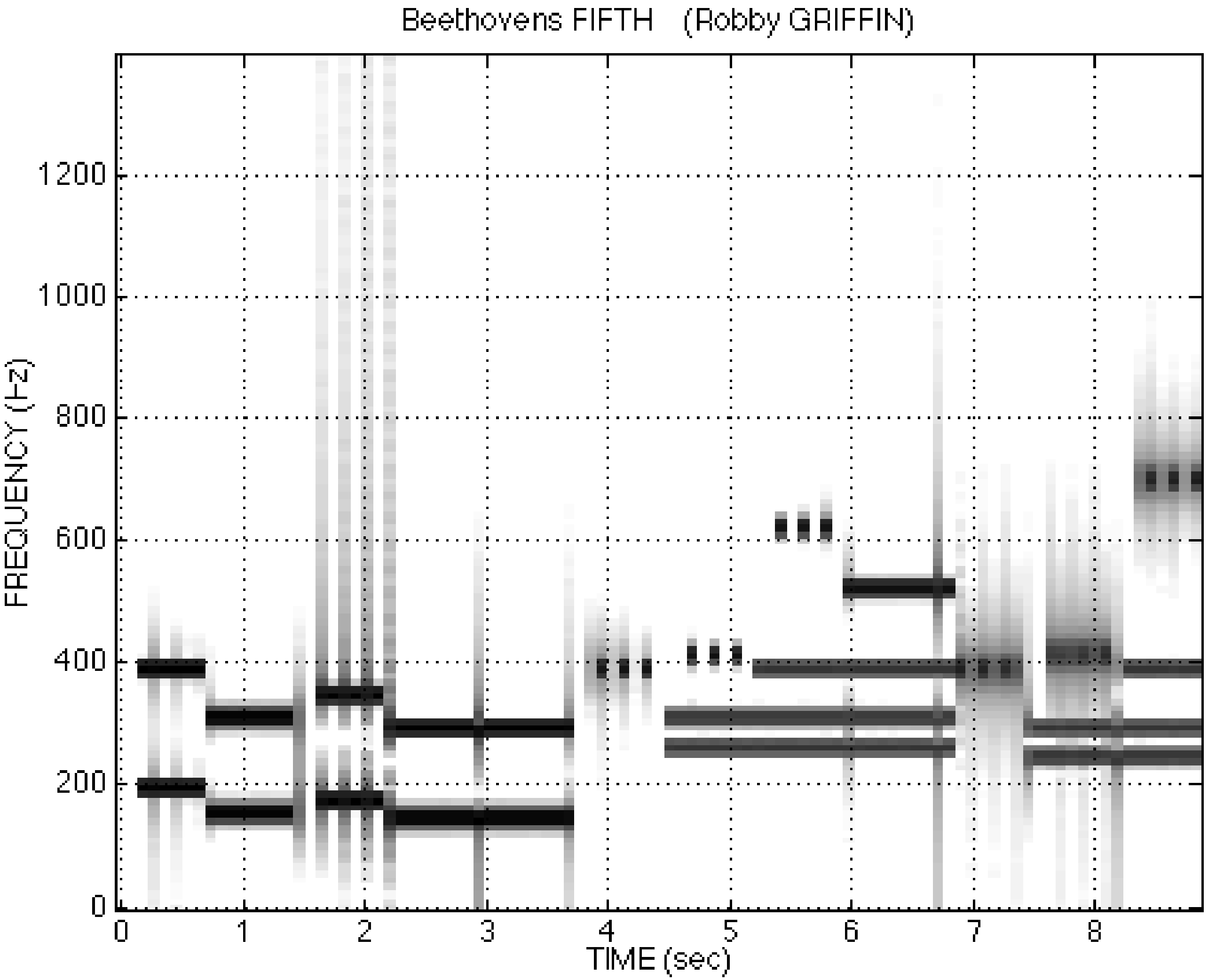
Spectrogram of C-Scale



From J.H. McClellan, R.W. Schafer, Signal Processing First, Prentice-Hall, 2003.

Spectrogram of LAB SONG

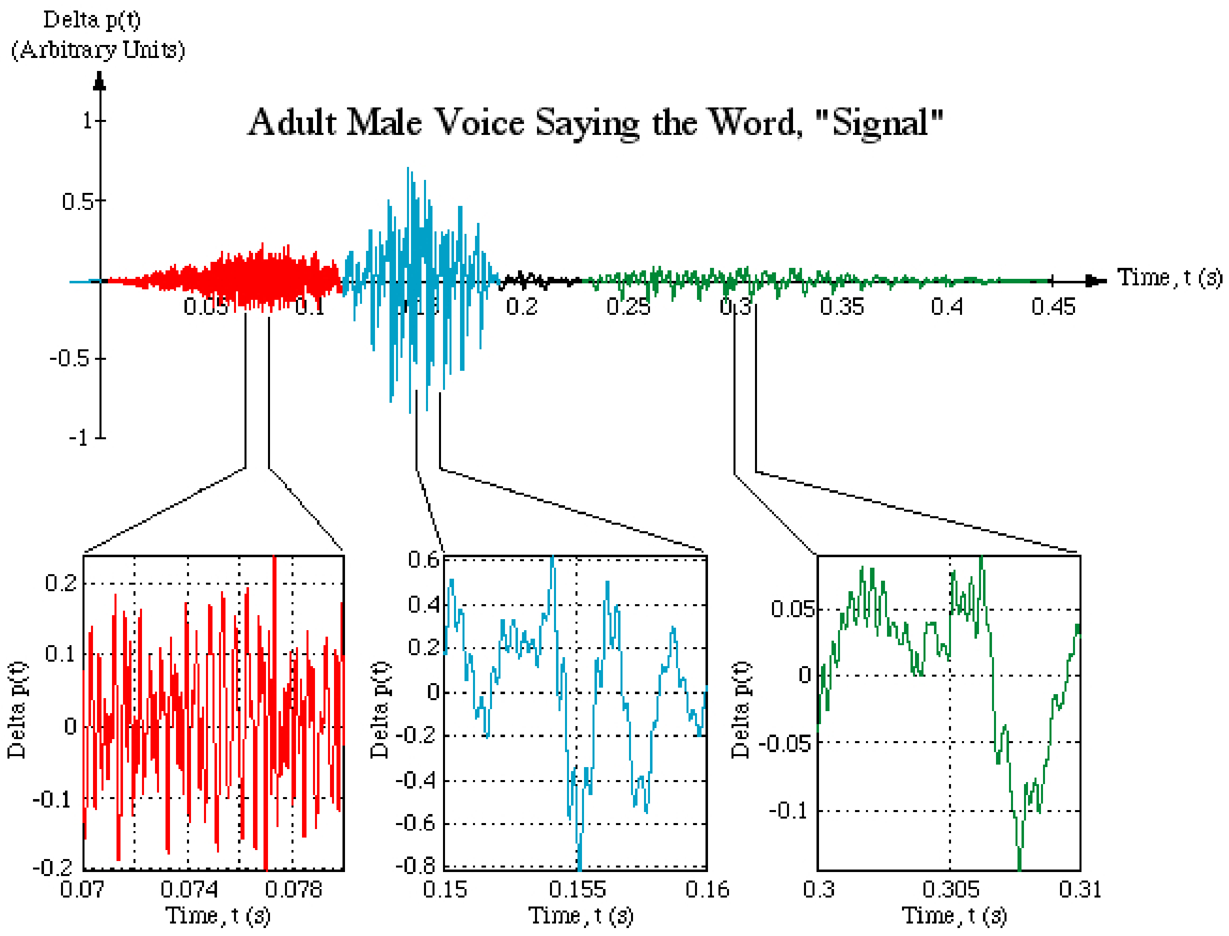




From J.H. McClellan, R.W. Schafer, Signal Processing First, Prentice-Hall, 2003.

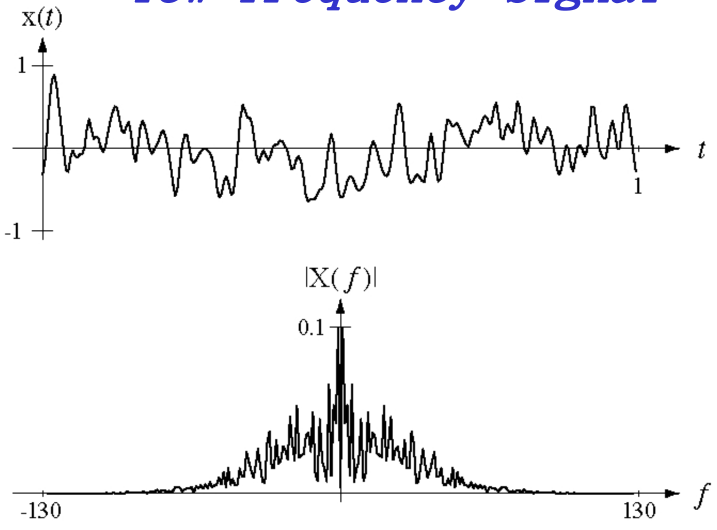
Recorded sound

"s" "i" "gn" "al"

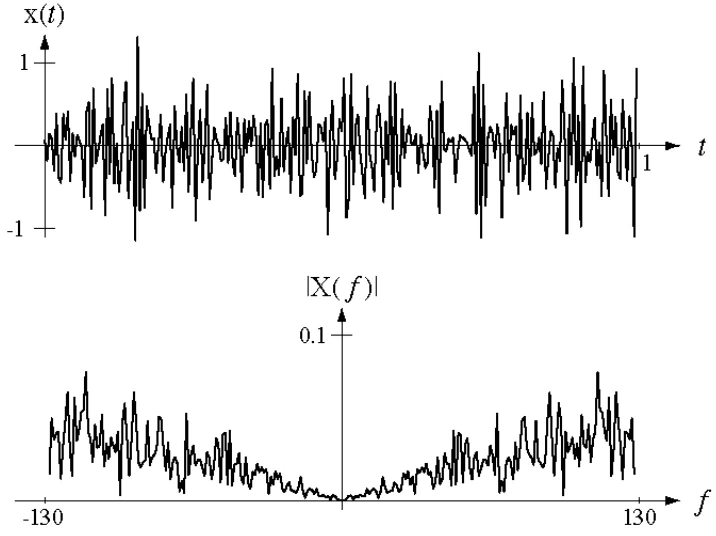


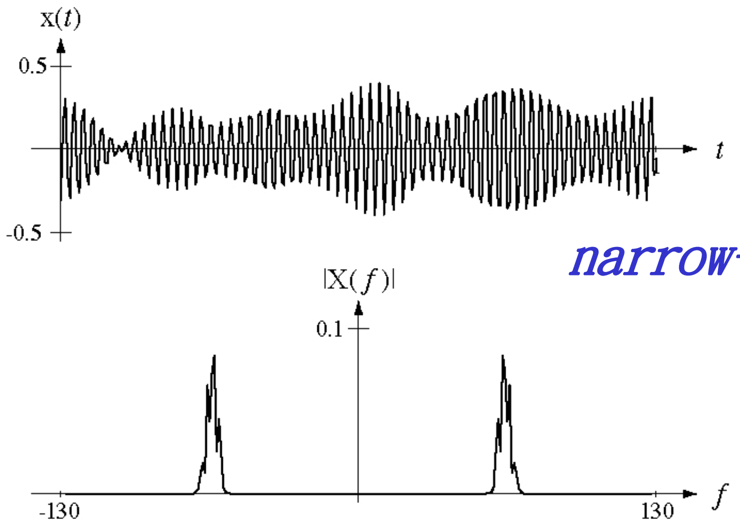
Fourier transform of signals

low-frequency signal



high-frequency signal





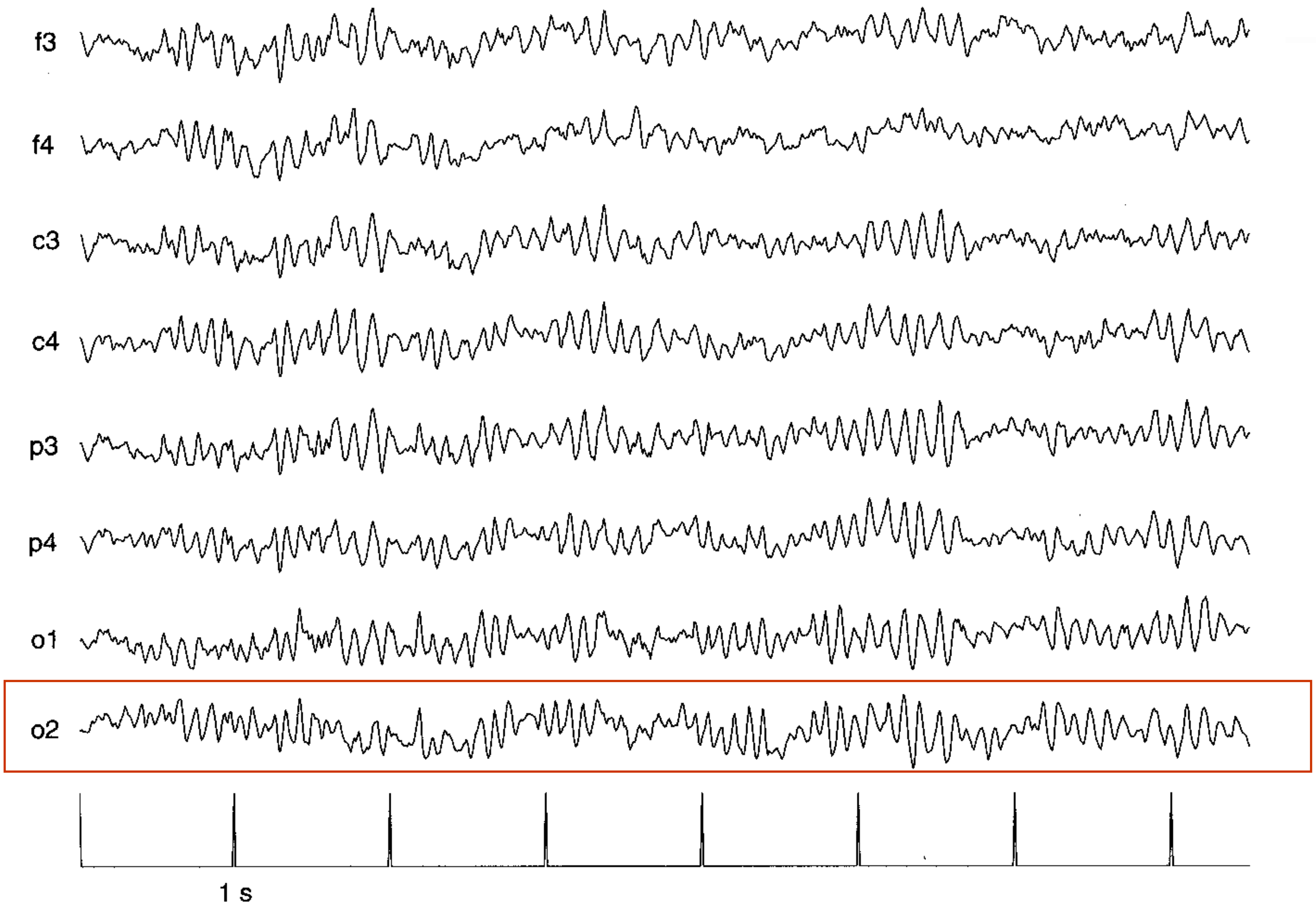
narrow-band signal

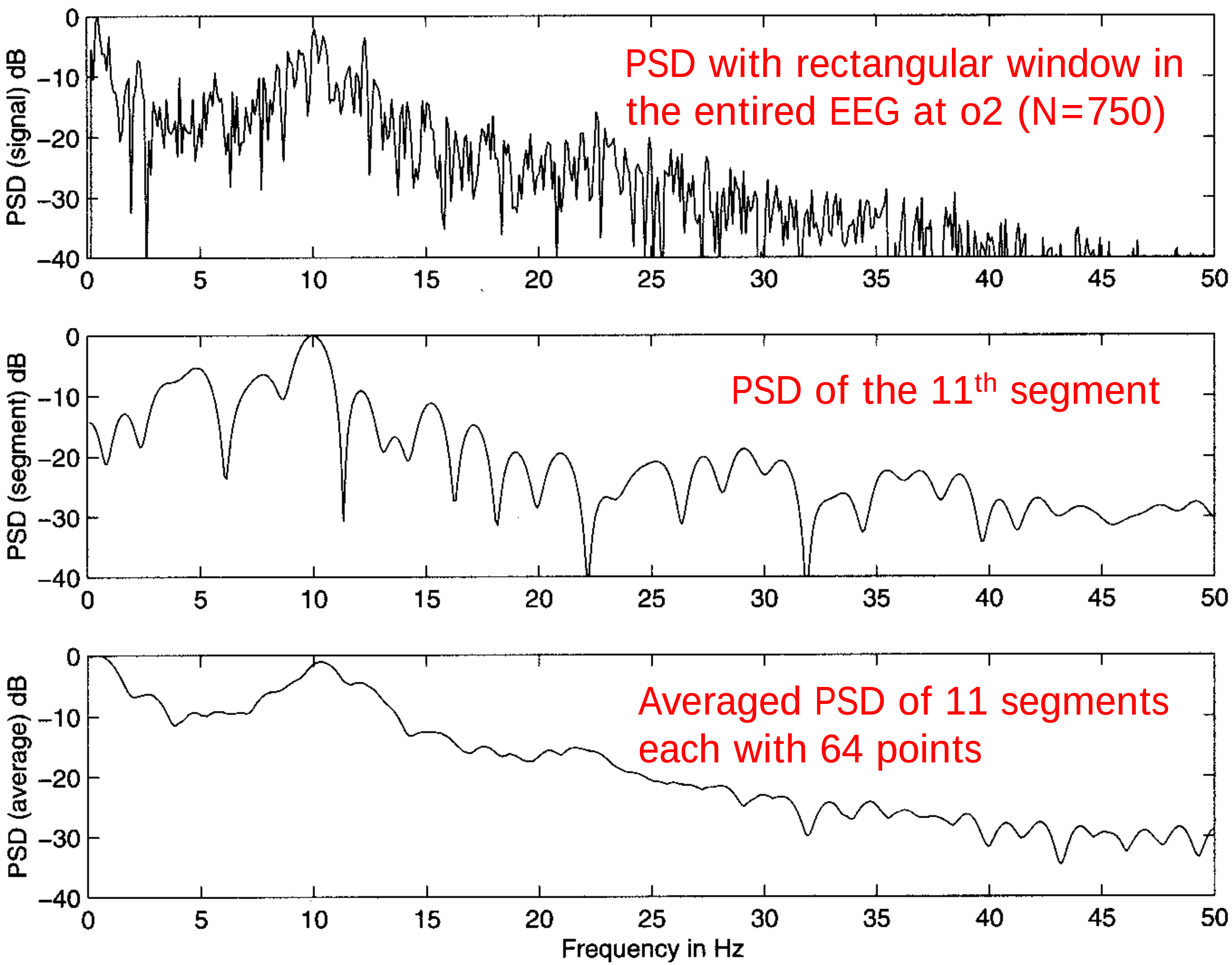
From .J.Roberts, Signals and Systems, M cGraw-Hill, 2003.

Averaged periodogram

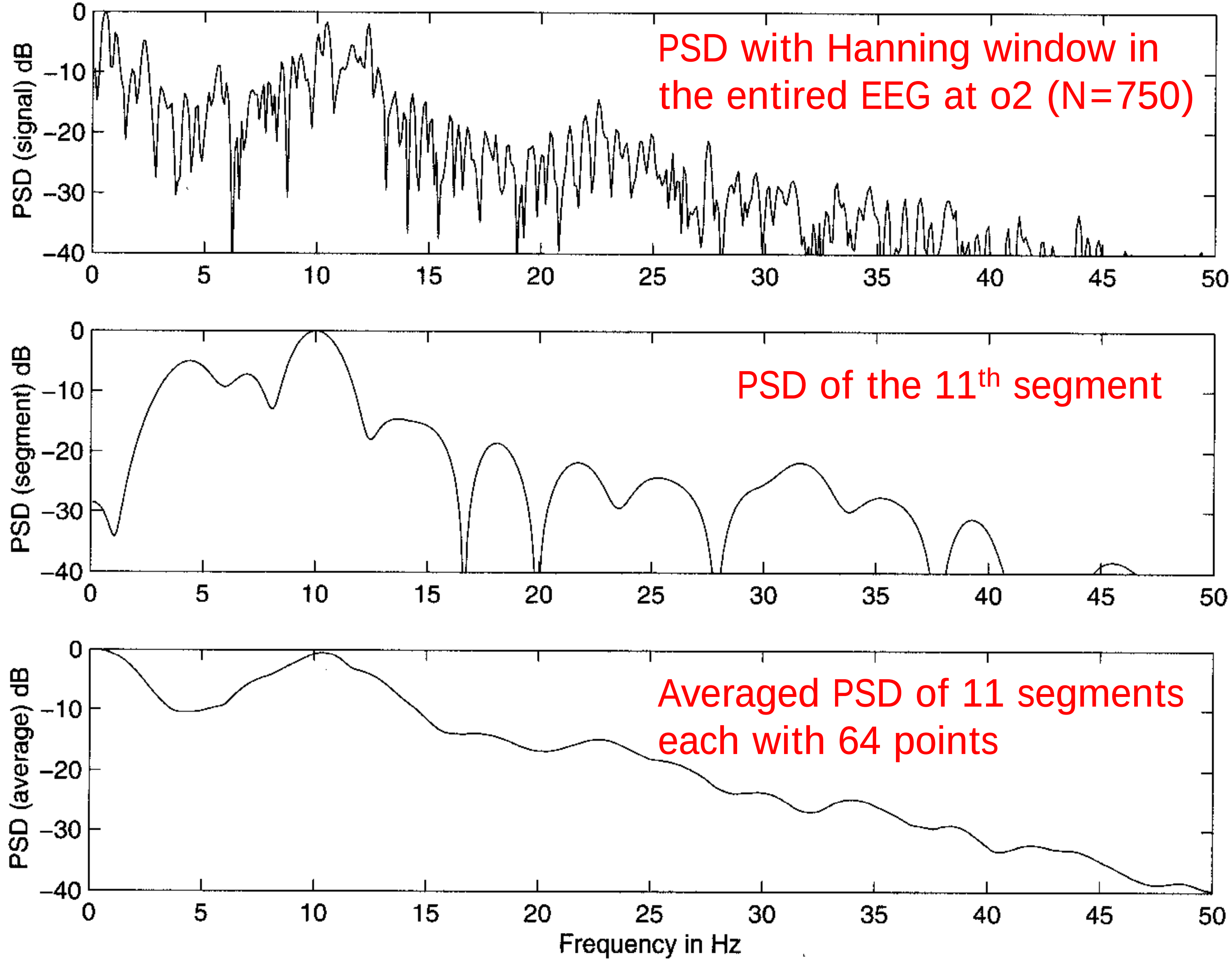
- A signal is divided into non-overlapping segments
  - Reducing variance of spectral analysis at the sacrifice with frequency resolution
- Welch periodogram
  - Using overlapping segments.
  - Widely use in spectral estimation

Characterization of EEG Alpha Rhythms

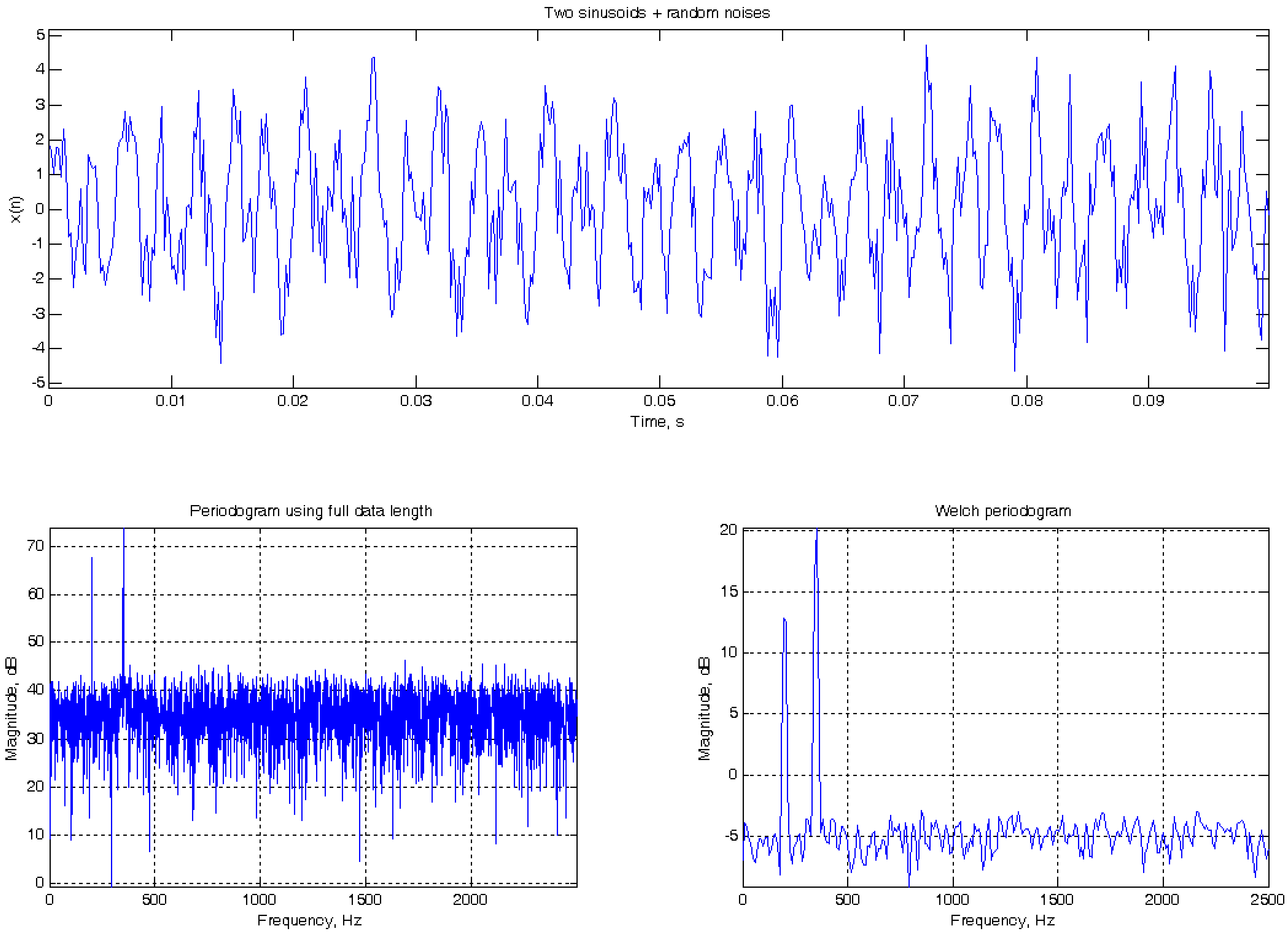




PSD with rectangular window in the entired EEG at o2 (N=750)
PSD (signal) dB
PSD of the 11th segment
PSD (segment) dB
Averaged PSD of 11 segments each with 64 points
PSD (average) dB
Frequency in Hz


PSD with Hanning window in the entired EEG at o2 (N=750)
PSD of the 11th segment
Averaged PSD of 11 segments each with 64 points
PSD (signal) dB
PSD (segment) dB
PSD (average) dB
0
−10
−20
−30
−40
0 5 10 15 20 25 30 35 40 45 50
Frequency in Hz

```
fs=5000;
t=0:1/fs:1;
x = sin(2*pi*200*t) + 2*sin(2*pi*350*t) + randn(size(t));

Xf=fft(x);
resolution=fs/length(Xf);
f=(0:length(Xf)-1)*resolution;
Xf_power = Xf.*conj(Xf);  % power spectral density
Xf_dB=10*log10(Xf_power);
index=1:length(Xf_power)/2;
plot(f(index),Xf_dB(index))

NFFT=512;  % length of window
[P1,f] = pwelch(x,hanning(NFFT),NFFT/2,NFFT);
% PWELCH(X,WINDOW,NOVERLAP,NFFT)
f=f/pi*fs/2;
P1_dB=10*log10(P1);
plot(f,P1_dB)
```


Two sinusoids + random noises
x(n)
Time, s
Periodogram using full data length
Magnitude, dB
Frequency, Hz
Welch periodogram
Magnitude, dB
Frequency, Hz

Cross spectral density and coherence

Cross spectrum

$$S_{xy}(f) = X(f) \cdot Y^*(f)$$

Transfer function

$$H(f) = Y(f) / X(f)$$

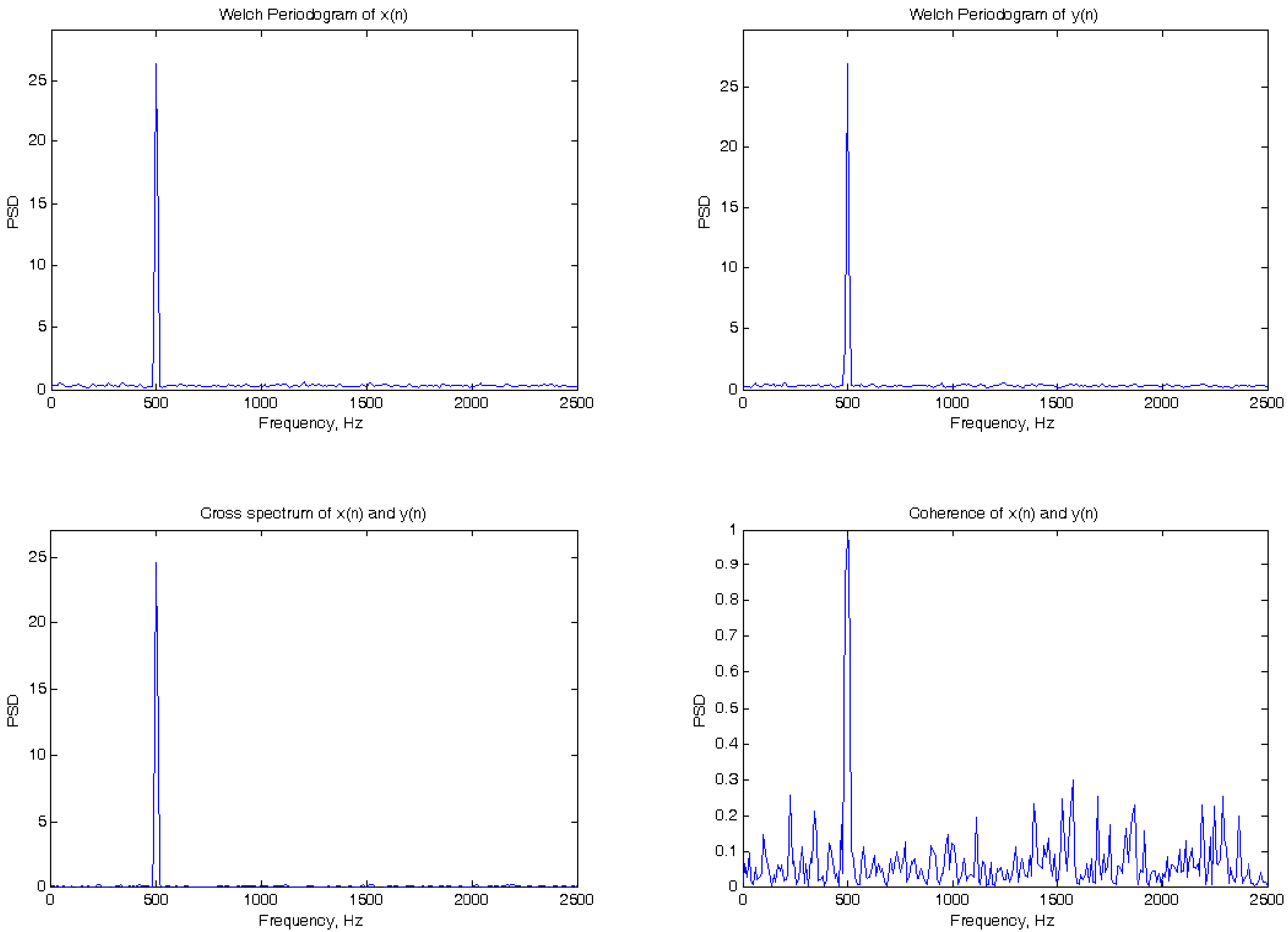
Coherence function

$$\gamma_{xy}^2(f) = \frac{\left| \frac{1}{K} \sum_{k=1}^{K} X_k(f) Y_k^*(f) \right|^2}{\frac{1}{K} \sum_{k=0}^{K} X_k(f) \cdot X_k^*(f) \times \frac{1}{K} \sum_{k=0}^{K} Y_k(f) \cdot Y_k^*(f)}$$

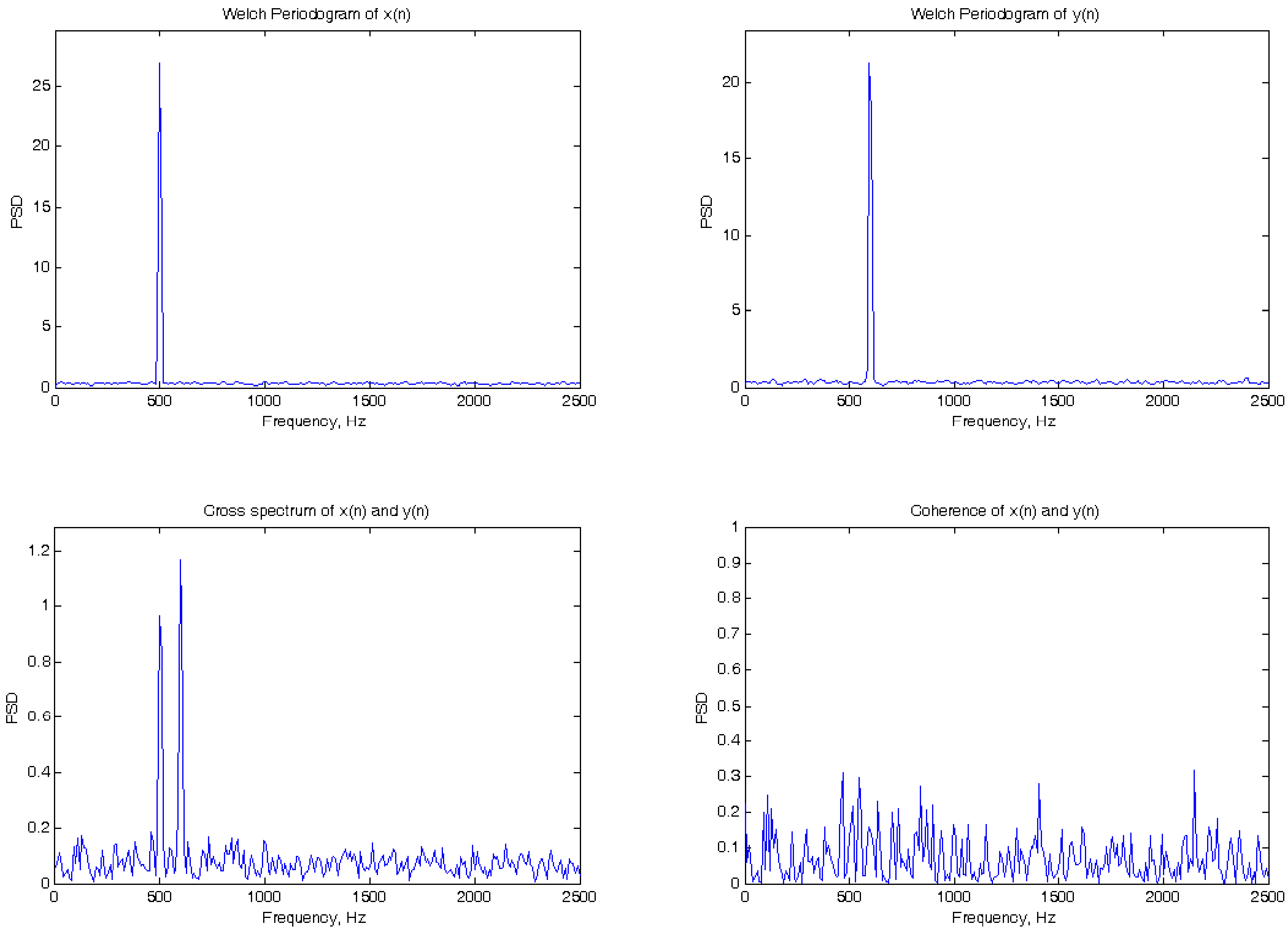
$$where \quad 0 \le \gamma_{xy}^2(f) \le 1$$

```
% Generate two sinudoids with the same frequency
fs=5000;
t=0:1/fs:1;
x = sin(2*pi*500*t) + randn(size(t));
y = sin(2*pi*500*t+pi/10) + randn(size(t));

% Welch periodogram
NFFT=512;
[P1,f] = pwelch(x,hanning(NFFT),NFFT/2,NFFT);
f=f/pi*fs/2;
plot(f,P1)
[P2,f] = pwelch(y,hanning(NFFT),NFFT/2,NFFT);
f=f/pi*fs/2;
plot(f,P2)

% Cross spectral analysis
[Pxy,f] = cpsd(x,y,hanning(NFFT),NFFT/2,NFFT);
f=f/pi*fs/2;
plot(f,Pxy)

% Coherence
[Cxy,f] = mscohere(x,y,hanning(NFFT),NFFT/2,NFFT);
f=f/pi*fs/2;
plot(f,Cxy)
```

Cross spectrum and coherence of two sinusoids with identical frequency

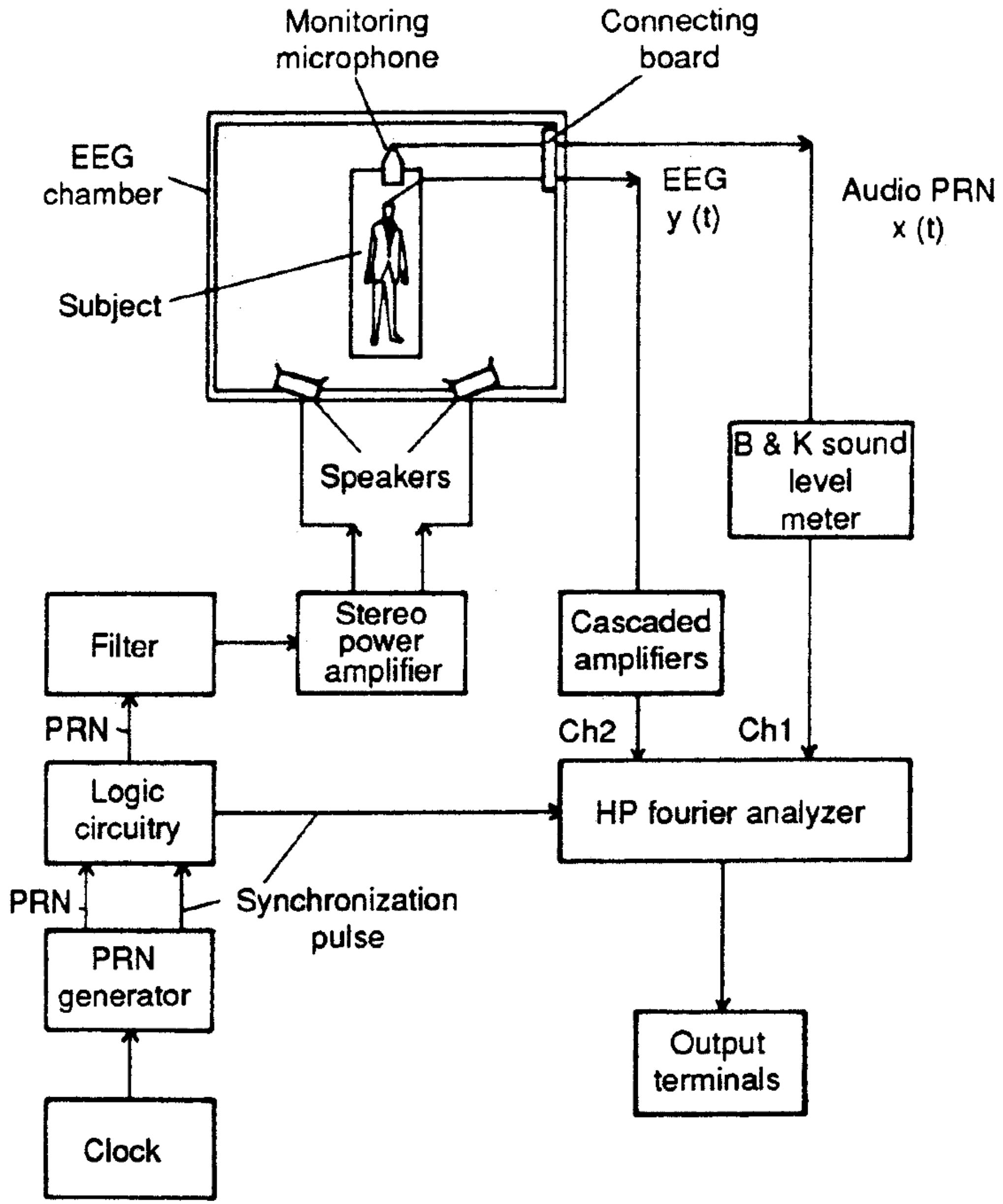


Cross spectrum and coherence of two sinusoids with different frequency

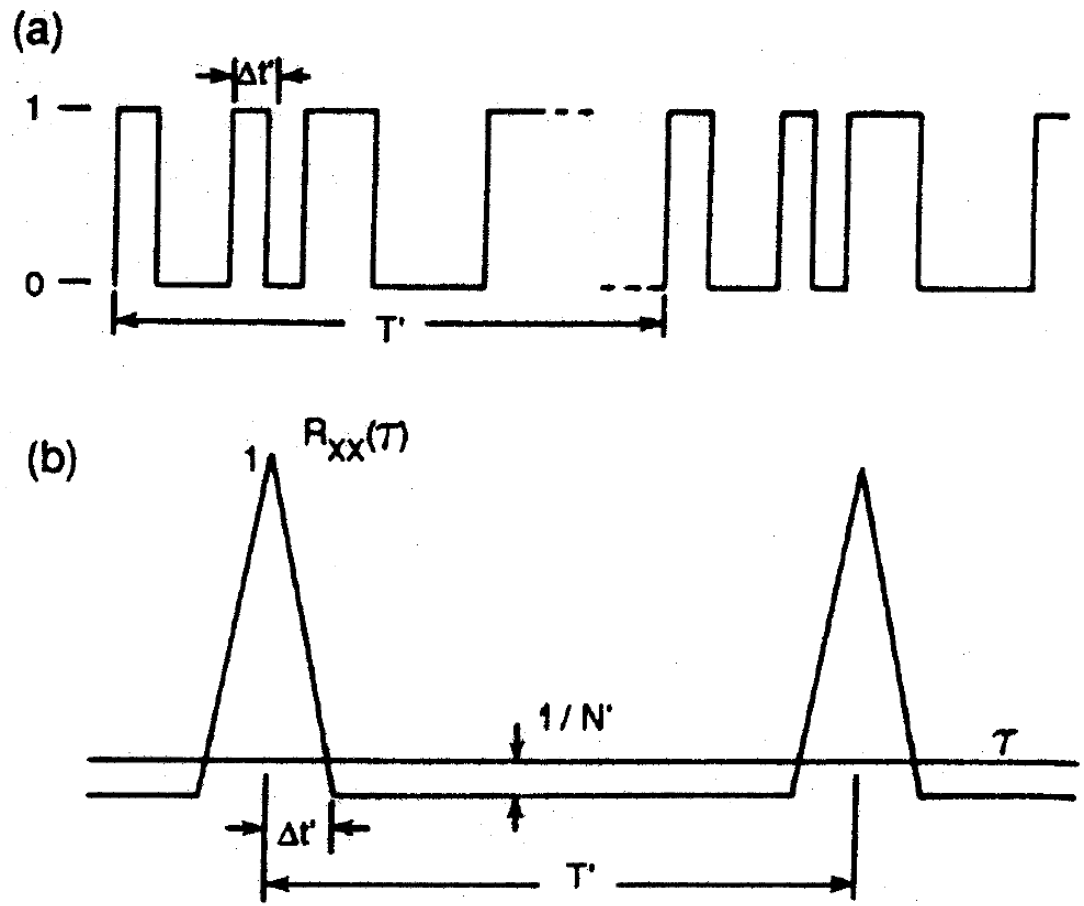


Analysis of auditory evoked potentials

SM. Reddy, RL. Kirlin, Spectral analysis of auditory evoked potentials with pseudorandom random noise excitation, IEEE Trans. Biomed Eng, 26(8):479-487, 1979.



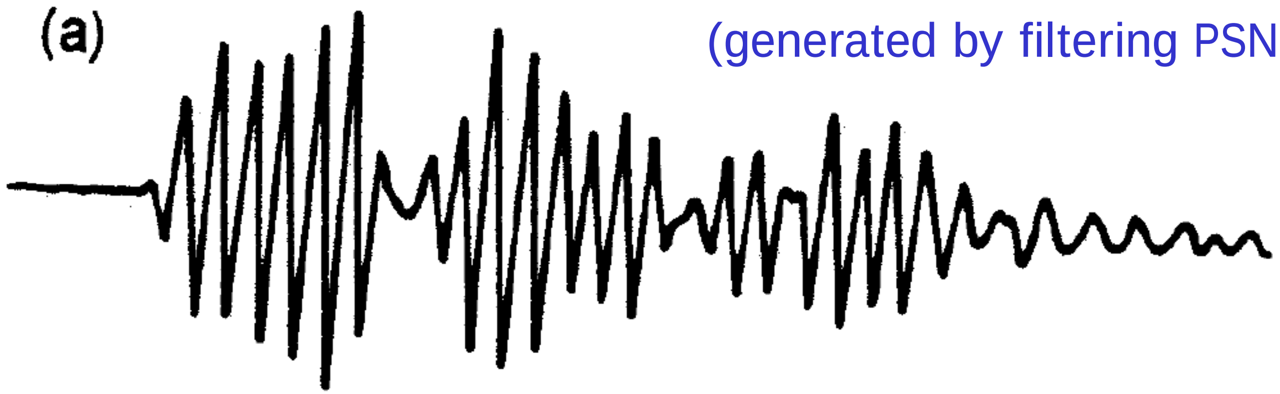
Pseudorandom noise (PSN)



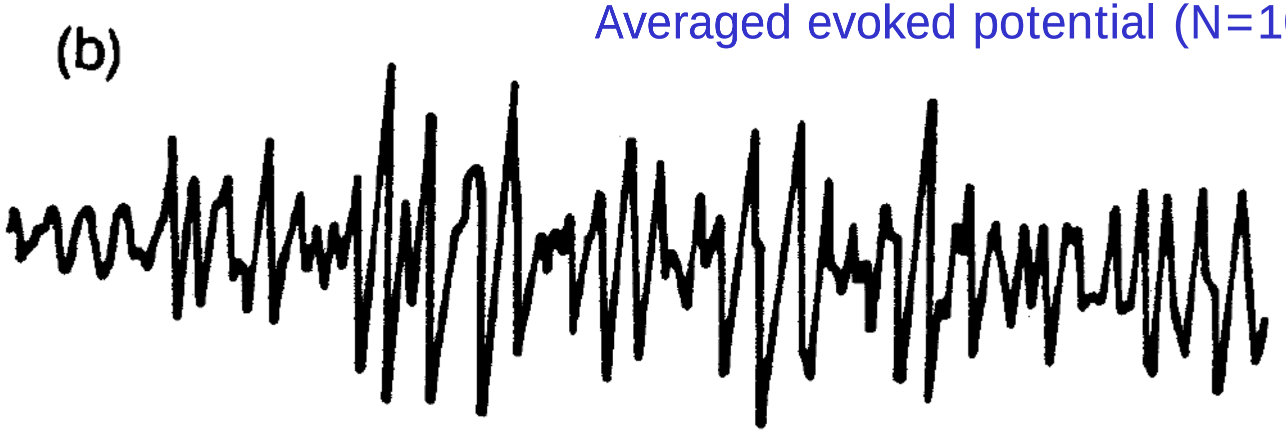
Narrow-band burst stimulus at 600 Hz in one normal subject

Burst Stimulus

(generated by filtering PSN)

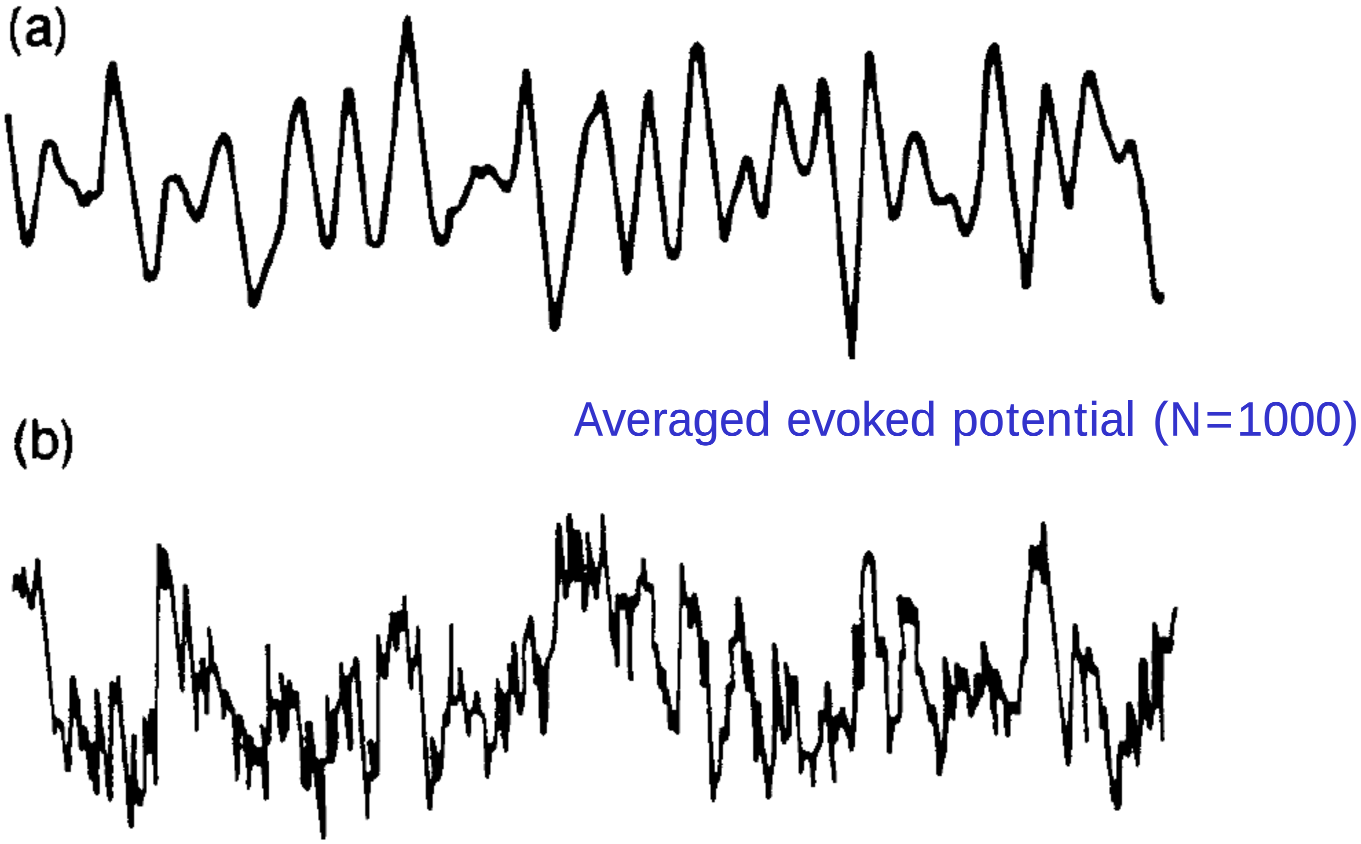


Averaged evoked potential (N=1000)



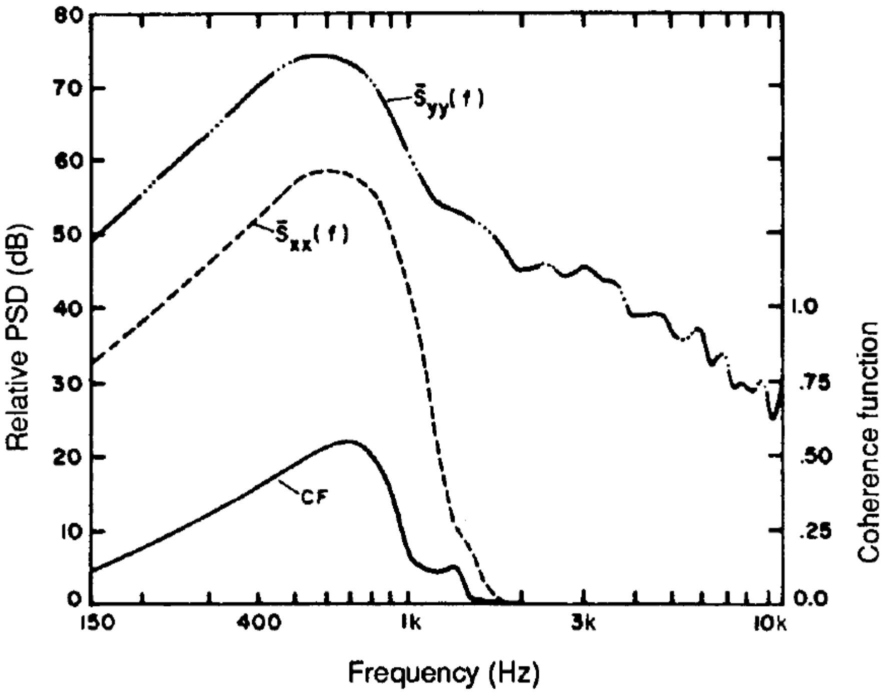
Narrow-band burst stimulus at 600 Hz in one patient with hearing loss

Burst Stimulus

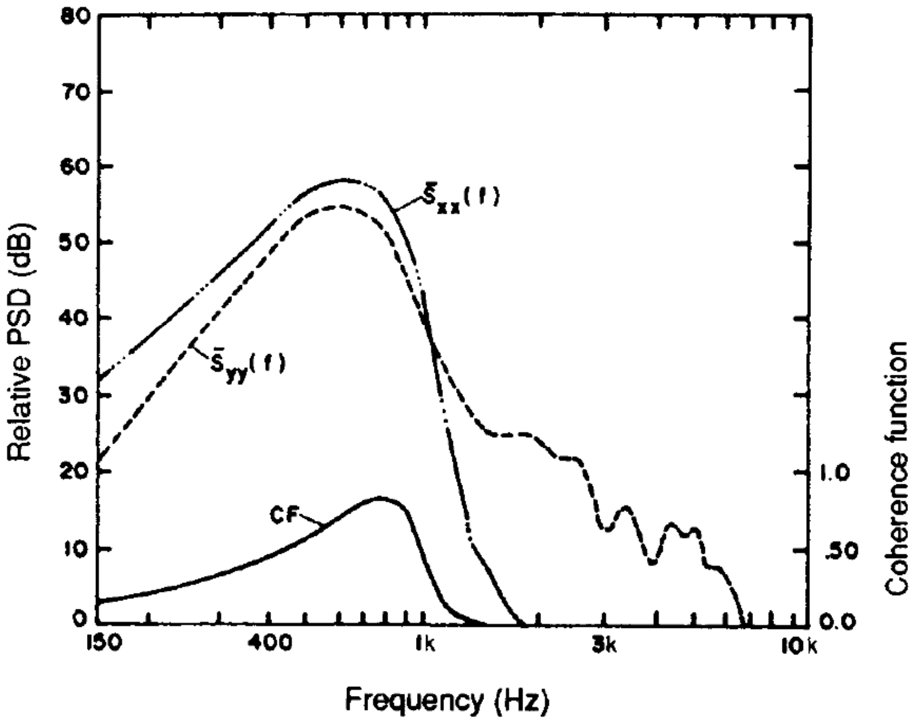


Power spectral density and coherence using 600-Hz stimulus

Normal Subject

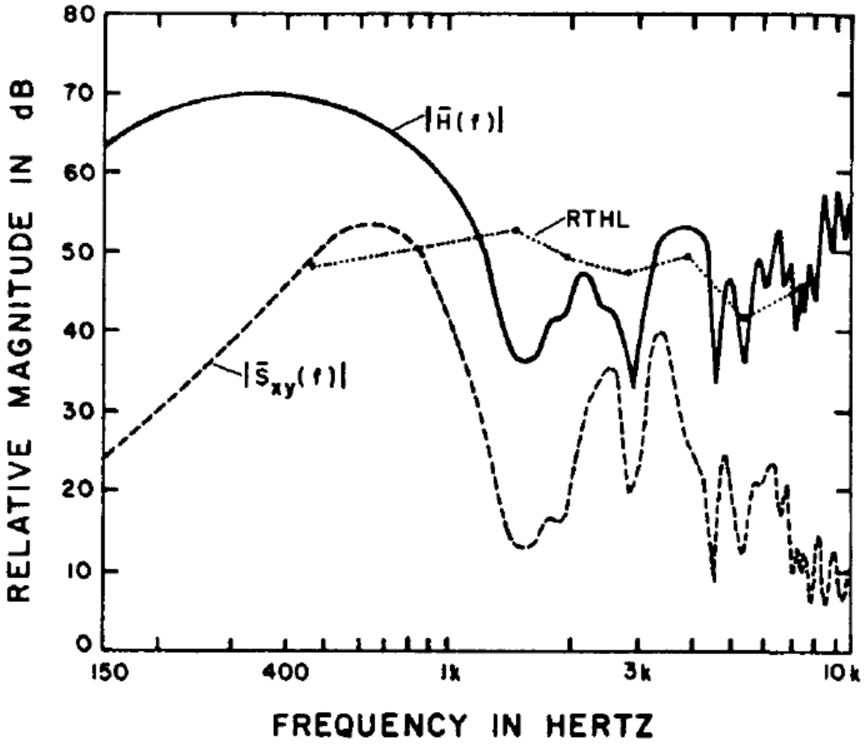


Patient with Hearing Loss

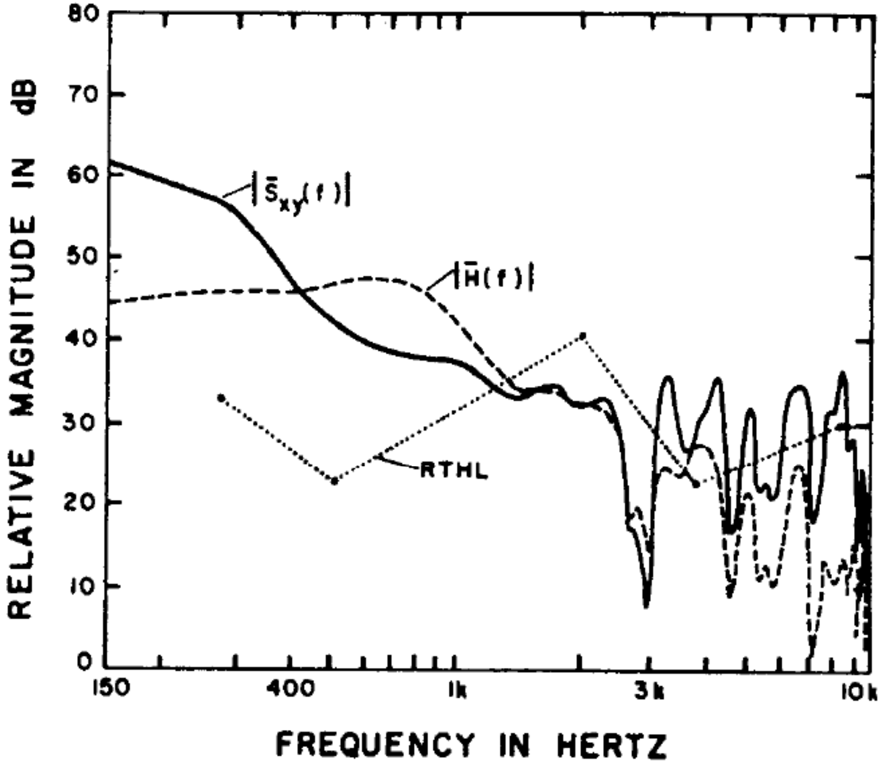


Cross spectral density and transfer function

Normal subject



Patient with hearing loss



Reference

- J.H. McClellan, R.W. Schafer, M.A. Yoder, Signal Processing First, Prentice Hall, 2003.
- R. Rangayyan, Biomedical Signal Analysis, John Wiely & Sons, 2002.
- J. Semmlow, Circuits, Signals, and Systems for Bioengineers: A MATLAB-Based Introduction, Academic Press, 2005.
- M.J. Roberts, Signals and Systems: Analysis of Signals Through Linear Systems, McGraw-Hill, 2003.