Midterm 2

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CMPS 130 Spring — Tantalo

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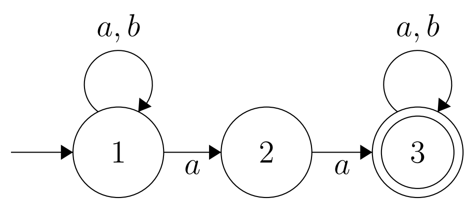
These are problems include some of the examples and exercises stated in class. Homework problems from hw4 through hw6 should also be considered good review problems.

Problem 1. Let $L \subseteq \{a, b\}^*$ correspond to the regular expression $(a + b)^*aa(a + b)^*$.

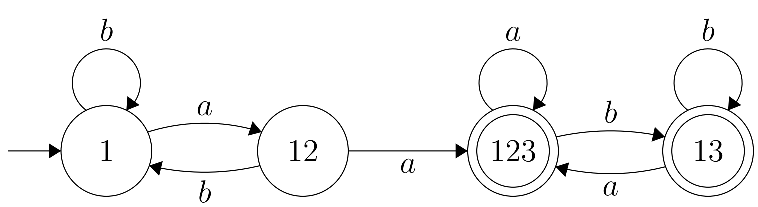
a. Give a verbal description of L.

   All strings containing the substring aa.

b. Draw an NFA with no λ-transitions accepting L.



c. Use the subset construction to obtain a DFA accepting L. Label the states in your DFA so as to make its relation to the NFA clear.



d. Show that $\{\lambda, a, aa\}$ is a pairwise L-distinguishable set. Explain why no DFA with fewer than three states will accept L.

   Proof. $\square$

Two strings are L-distinguishable if there is a string z that can be concatenated to make one included in the language and the other not. For the pair $\{\lambda, a\}$, let $z = a$, then $\lambda z = \lambda a \notin L$ and $az = aa \in L$, therefore the pair is L-distinguishable. For the pair $\{\lambda, aa\}$, let $z = \lambda$, then $\lambda z = \lambda\lambda \notin L$ and $aaz = aa\lambda \in L$, therefore the pair is L-distinguishable. For the pair $\{a, aa\}$, let $z = \lambda$, then $az = a\lambda \notin L$ and $aaz = aa\lambda \in L$, therefore the pair is L-distinguishable. Since all three strings are L-distinguishable, we have shown that L has 3 equivalences classes. We claim that the equivalence classes of I_L are the set $\{\lambda, a, aa\}$. The class of $[\lambda]$ includes all strings not ending in a and not containing aa. The class of $[a]$ contains all strings ending in a that do not contain aa. The class of $[aa]$ contains all the strings that are accepted by the language. Each equivalence class corresponds to a state in the minimal DFA. Since there are three equivalence classes, there must be at least as many states.

No 2 of these strings can go to the same state by the function

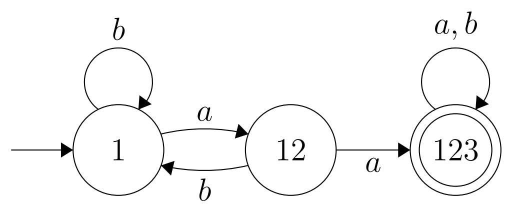
$$\delta^*(q_0, \cdot) : \{a, b\}^* \to Q$$

Suppose: $\delta^*(q_0, \lambda) = \delta^*(q_0, a)$
$\therefore \delta^*(\delta^*(q_0, \lambda), a) = \delta^*(\delta^*(q_0, a), a)$
$\therefore \delta^*(q_0, a) = \delta^*(q_0, aa)$
which is a contradiction because the LHS$\notin A$ and RHS$\in A$. $\therefore \delta^*(q_0, \lambda) \neq \delta^*(q_0, a)$.]]

e. Show that λ and b are L-indistinguishable.

Proof. Let $z = aa$. $\lambda z = aa \in L$. $bz = baa \in L$. Since there exists a substring which allows both strings to be accepted, they are indistinguishable in L. □

f. Draw a DFA with only three states accepting L. (Either run the state minimization algorithm on your DFA in (c), or just eyeball it.)



Problem 2. Let L_1 and L_2 be regular languages over Σ. Show that $L_1 \cap L_2$ is also regular. (Hint: This is hard to prove directly from the definition of regular language. Instead use some of the theorems we have proved in chapters 2 and 3.)

Proof. Kleene's Theorem says that every every regular language can be accepted by an FA. By Theorem 2.15, suppose $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$ are finite automata accepting L_1 and L_2, respectively. For every $p \in Q_1$, every $q \in Q_2$, if $A = \{(p, q) | p \in A_1 \wedge q \in A_2\}$, then there is an FA that accepts $L_1 \cap L_2$, and if a language is accepted by an FA, then it is regular, by Kleene's Theorem. □

Problem 3. Let $M = (Q, \Sigma, q_0, A, \delta)$ be an NFA, and let $S_1 \subseteq S_2 \subseteq Q$.

a. Write the recursive definitions of $\lambda(S_1)$ and $\lambda(S_1)$.
   1. $S_1 \subseteq \lambda(S_1)$.
   2. For every $q \in \lambda(S_1)$, $\delta(q, \lambda) \subseteq \lambda(S_1)$.
   1. $S_2 \subseteq \lambda(S_2)$.
   2. For every $q \in \lambda(S_2)$, $\delta(q, \lambda) \subseteq \lambda(S_2)$.

b. Show that $\lambda(S_1) \subseteq \lambda(S_2)$ using structural induction.

   Proof. $\lambda(S_1) \subseteq \lambda(S_2)$

   I. If there are no λ-transitions, $S_1 = \lambda(S_1)$, $S_2 = \lambda(S_2)$

   II. For every $q \in S_1$, $\delta(q, \lambda) \subseteq S_1$.
   For every $p \in S_2$, $\delta(p, \lambda) \subseteq S_2$.

   Therefore, $\lambda(S_1) \subseteq \lambda(S_2)$. □

Problem 5. State the Pumping Lemma and use it to prove that $L = \{x \in \{a, b\}^* | n_a(x) = n_b(x)\}$ is not accepted by any DFA.
Suppose L is a language over the alphabet Σ. If L is accepted by a finite automaton $M = (Q, \Sigma, q_0, A, \delta)$, and if n is the number of states of M, then for every $x \in L$ satisfying $|x| \geq n$, there are three strings u, v, and w such that $x = uvw$ and the following three conditions are true:
1. $|uv| \leq n$.
2. $|v| > 0$ (i.e., $v \neq \Lambda$).
3. For every $i \geq 0$, the string uv^iw also belongs to L.

Proof. $AeqB$ is not accepted by an FA.
Let $x = a^nb^n$. Assume for sake of contradiction that $x \in L$ is accepted by a DFA. Since $|uv| \leq n$, there are only a's in uv, and all b's are in w. Because $|v| > 0$, v must contain at least 1 a and no b's. Therefore, if $i = 0$, then uv^0w removes at least one a and no b's, in which case $n_a(x) < n_b(x)$. If $i > 1$, then uv^iw adds a's and no b's, in which case $n_a(x) > n_b(x)$. This is a contradiction. Therefore, $L = \{x \in \{a, b\}^* | n_a(x) = n_b(x)\}$ is not accepted by any DFA. □

Problem 6. (This is problem 3.31 with the typo in the last sentence fixed.) Let $M = (Q, \Sigma, q_0, A, \delta)$ be a DFA and let $M_1 = (Q, \Sigma, q_0, A, \delta_1)$ be the NFA with no λ-transitions for which $\delta_1(q,\sigma) = \{\delta(q, \sigma)\}$ for every $q \in Q$ and $\sigma \in \Sigma$. Show for all $x \in \Sigma^*$ and $q \in Q$ that $\delta_1^*(q,x) = \{\delta^*(q, x)\}$. In addition also show that $L(M) = L(M_1)$.

Proof. $\delta_1^*(q, x) = \{\delta^*(q, x)\}$

Let $x = y\sigma$. Since there are no λ-transitions, $\delta_1^*(q, \lambda) = \{q\}$ and

$$\begin{aligned}
\delta_1^*(q, x) &= \delta_1^*(q, y\sigma) \\
&= \lambda(\cup\{\delta_1(p, \sigma) | p \in \delta_1^*(q, y)\}) \qquad \text{(by definition of } \delta_1^*) \\
&= \cup\{\delta_1(p, \sigma) | p \in \delta_1^*(q, y)\} \\
&= \{\delta^*(q, x)\}
\end{aligned}$$

Therefore for every $q \in Q$ and $x \in \Sigma^*$, $\delta_1^*(q, x) = \{\delta^*(q, x)\}$. $\square$

Proof. $L(M) = L(M_1)$

$$\begin{aligned} x \in L(M) &\text{ iff } \delta^*(q_0, x) \in A \\ &\text{ iff } \{\delta^*(q_0, x)\} \cap A \neq \emptyset \\ &\text{ iff } \delta_1^*(q_0, x) \cap A \neq \emptyset \\ &\text{ iff } x \in L(M_1) \end{aligned}$$

Therefore, $L(M) = L(M_1)$. $\square$

Problem 7. (This is problem 3.53 on p. 127). Let Σ_1 and Σ_2 be finite alphabets. A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a *homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in \Sigma_1^*$. Let f be such a function.

a. Show that $f(\lambda) = \lambda$

Proof. $f(\lambda) = \lambda$
$f(x) = f(x\lambda) = f(x)f(\lambda)$ for any $x \in \Sigma^*$
$\therefore |f(x)| = |f(x)f(\lambda)| = |f(x)| + |f(\lambda)|$
$\therefore |f(\lambda)| = 0$
$\therefore f(\lambda) = \lambda$ $\square$

b. The *image* of $L_1 \subseteq \Sigma_1^*$ under f is the set $f(L_1) = \{f(x) | x \in L_1\} \subseteq \Sigma_2^*$. Show that if L_1 is regular, then $f(L_1)$ is regular. (Hint: use structural induction and the recursive definition of the set of regular languages over Σ_1.)

Proof. b) Show $L_1 \in \mathcal{R}_{\Sigma_1} \Rightarrow f(L_1) \in \mathcal{R}_{\Sigma_2}$
I. Base lang in $\mathcal{R}_{\Sigma_1}$ are: $\emptyset$ and $\{\sigma\}$ for $\sigma \in \Sigma$
$f(\emptyset) = \emptyset \in \mathcal{R}_{\Sigma_2}$
$f(\{\sigma\}) = \{f(\sigma)\}$ is a finite lang. $\therefore f(\{\sigma\}) \in \mathcal{R}_{\Sigma_2}$
II. Let $L_1, L_2, L \in \mathcal{R}_{\Sigma_1}$. Assume $f(L_1), f(L_2), f(L) \in \mathcal{R}_{\Sigma_2}$
We must show $f(L_1 \cup L_2), f(L_1L_2), f(L^*) \in \mathcal{R}_{\Sigma_2}$
Lemma $f(L_1 \cup L_2) = f(L_1) \cup f(L_2) \therefore f(L_1 \cup L_2) \in \mathcal{R}_{\Sigma_2}$ by ind. hyp.
Lemma $f(L_1L_2) = f(L_1)f(L_2) \therefore f(L_1L_2) \in \mathcal{R}_{\Sigma_2}$ by ind. hyp.
Lemma $f(L_1^*) = f(L_1)^* \therefore f(L_1^*) \in \mathcal{R}_{\Sigma_2}$ by ind. hyp. $\square$

c. The *preimage* of $L_2 \subseteq \Sigma_2^*$ under f is the set $f^{-1}(L_2) = \{x \in L_1 | f(x) \in L_2\} \subseteq \Sigma_1^*$. Show that if L_2 is regular, then $f^{-1}(L_2)$ is regular. (Hint: start with a DFA accepting L_2 then show how to construct from it a DFA accepting $f^{-1}(L_2)$.)

Proof. Show $L_2 \in \mathcal{R}_{\Sigma_2} \Rightarrow f^{-1}(L_2) \in \mathcal{R}_{\Sigma_2}$
By Kleene's Theorem it is sufficient to show that L_2 is accepted by a DFA $\Rightarrow f^{-1}(L_2)$ accepted by DFA
Let $M_2 = (Q_2, \Sigma_2, q_2, A_2, \delta_2)$ accept L_2

Define $M_1 = (Q_1, \Sigma_1, q_1, A_1, \delta_1)$ where
$Q_1 = Q_2$, $q_1 = q_2$, $A_1 = A_2$, $\delta_1(q, \sigma) = \delta_2^*(q, f(\sigma))$
Show: M_1 accepts $f^{-1}(L_2)$
Proof: $x \in L(M_1)$ iff $\delta_1^*(q, x) \in A_1$ (definition of accept)
iff $\delta_2^*(q, f(x)) \in A_2$ (lemma & definition of accept)
iff $f(x) \in L(M_2) = L_2$ (definition of accept)
iff $x \in f^{-1}(L_2)$ $\square$

Problem 8. Let $L \subseteq \Sigma$ and $x, y \in \Sigma$. Write the definitions of xI_Ly and L/x. Prove that xI_Ly if and only if $L/x = L/y$.

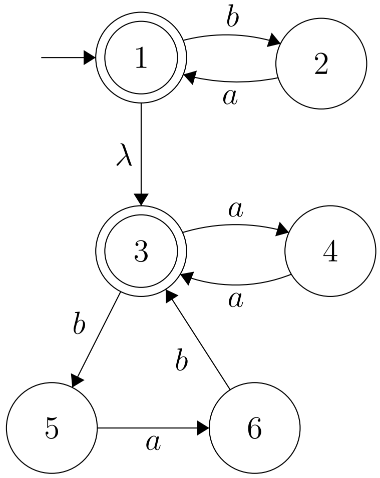
Proof. xI_Ly iff $L/x = L/y$

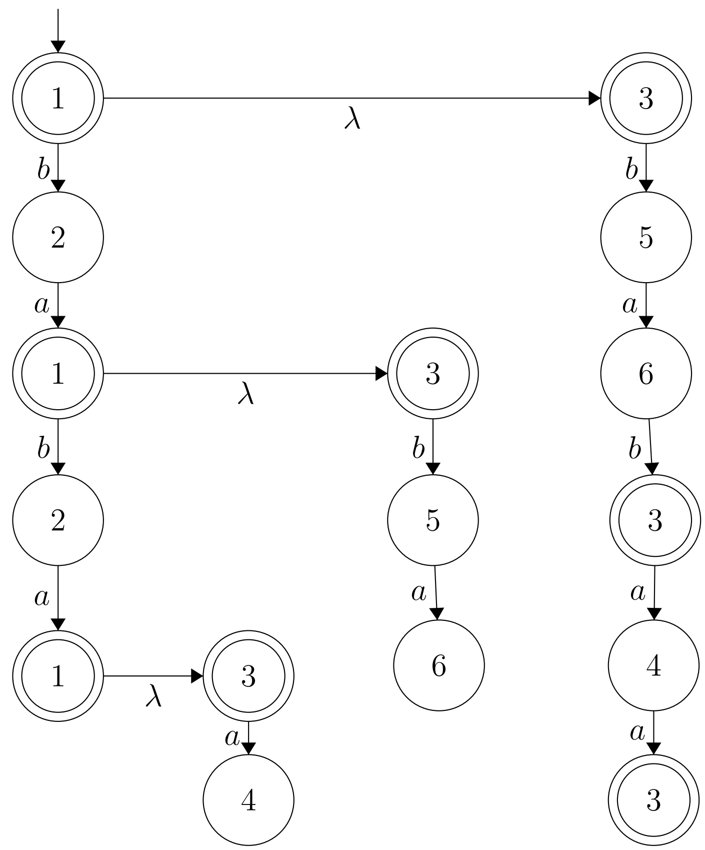
$$\begin{aligned} xI_Ly \text{ iff } & \forall z \in \Sigma^* : xz \in L \leftrightarrow yz \in L \\ \text{iff } & \forall z \in \Sigma^* : z \in L/x \leftrightarrow z \in L/y \\ \text{iff } & L/x = L/y \end{aligned}$$

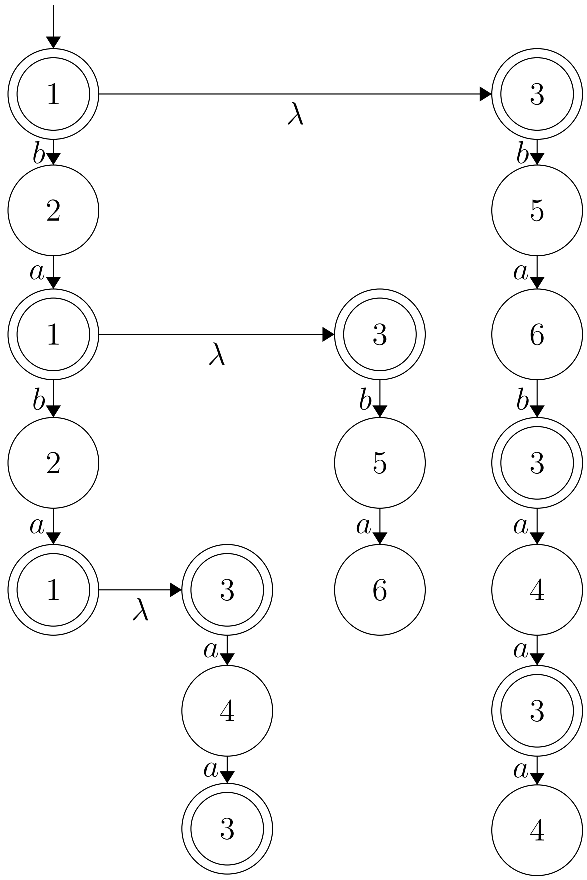
$\therefore xI_Ly$ if and only if $L/x = L/y$ $\square$

Problem 9. Draw an NFA corresponding to the regular expression $(ba)^*(bab + aa)^*$. Draw the computation trees corresponding to the strings *babaa*, *babaaa* and *babaaabab*. Which of these strings are accepted?

All the strings are accepted.



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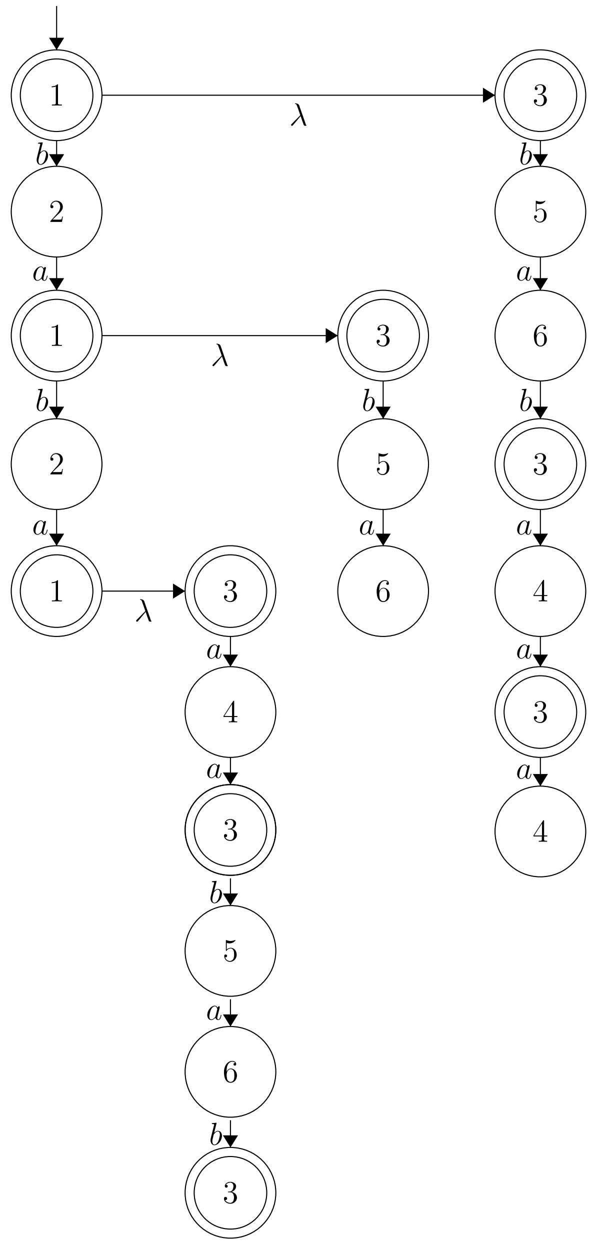


Problem 10. Problem 3.49ab. Note this material will be covered Tuesday before the midterm. You can find it in the proof of Theorem 3.25, and following examples (page 111-114 in the text). It will be included in hw7, which will be due the Tuesday after the midterm.

Figure 3.39 shows FAs M_1 and M_2 accepting languages L_1 and L_2, respectively. Draw NFAs accepting each of the following languages, using the constructions in the proof of Theorem 3.25.

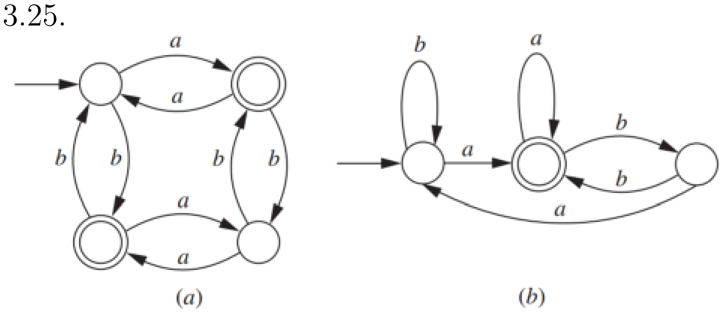
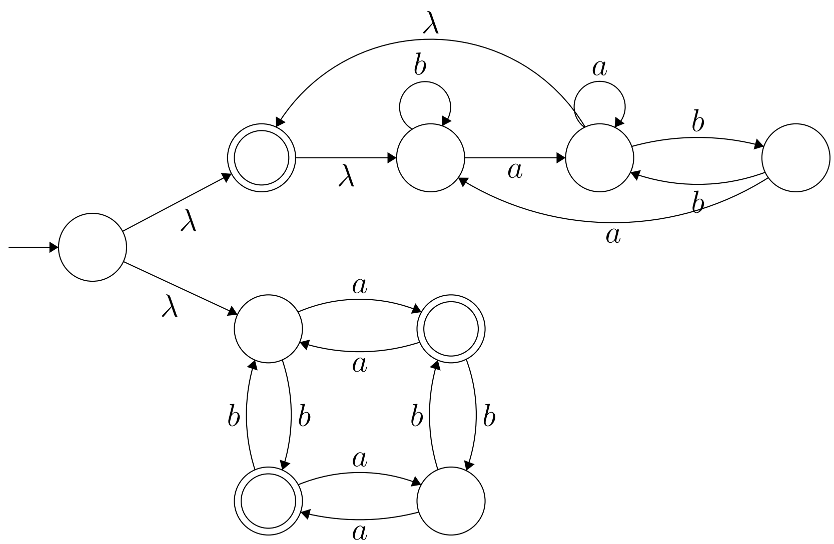


Figure 3.39 |

a. $L_2^* \cup L_1$



b. $L_2 L_1^*$