CMPS 101
Summer 2010
Homework Assignment 4 Solutions

1. (3 Points)
Consider the function $T(n)$ defined by the recurrence formula

$$T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

a. (1 Points) Use the iteration method to write a summation formula for $T(n)$.
Solution:

$$\begin{aligned} T(n) &= n + 2T(\lfloor n/3 \rfloor) \\ &= n + 2(\lfloor n/3 \rfloor + 2T(\lfloor \lfloor n/3 \rfloor /3 \rfloor)) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 T(\lfloor n/3^2 \rfloor) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 \lfloor n/3^2 \rfloor + 2^3 T(\lfloor n/3^3 \rfloor) \quad \text{etc..} \end{aligned}$$

After substituting the recurrence into itself k times, we get

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k T(\lfloor n/3^k \rfloor).$$

This process terminates when the recursion depth k is chosen so that $1 \le \lfloor n/3^k \rfloor < 3$, which is equivalent to $1 \le n/3^k < 3$, whence $3^k \le n < 3^{k+1}$, so $k \le \log_3(n) < k+1$, and hence $k = \lfloor \log_3(n) \rfloor$. With this value of k we have $T(\lfloor n/3^k \rfloor) = T(1 \text{or} 2) = 6$. Therefore

$$T(n) = \sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 6 \cdot 2^{\lfloor \log_3(n) \rfloor}.$$

b. (1 Points) Use the summation in (a) to show that $T(n) = O(n)$
Solution:
Using the above summation, we have

$$\begin{aligned} T(n) &\le n\left(\sum_{i=0}^{\lfloor \log_3(n) \rfloor - 1} (2/3)^i \right) + 6 \cdot 2^{\log_3(n)} && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\ &\le n\left(\sum_{i=0}^{\infty} (2/3)^i \right) + 6n^{\log_3(2)} && \text{adding } \infty\text{-many positive terms} \\ &= n\left(\frac{1}{1-(2/3)} \right) + 6n^{\log_3(2)} && \text{by a well known formula} \\ &= 3n + 6n^{\log_3(2)} = O(n) && 2 < 3 \Rightarrow \log_3(2) < 1 \Rightarrow n^{\log_3(2)} = o(n) \end{aligned}$$

Therefore $T(n) = O(n)$.

c. (1 Points) Use the Master Theorem to show that $T(n) = \Theta(n)$

Solution:

Let $\varepsilon = 1 - \log_3(2) > 0$. Then $\log_3(2) + \varepsilon = 1$, and $n = n^{\log_3(2)+\varepsilon} = \Omega(n^{\log_3(2)+\varepsilon})$. Also for any c in the range $2/3 \le c < 1$, and any positive n, we have $2(n/3) = (2/3)n \le cn$, so the regularity condition holds. By case (3) of the Master Theorem $T(n) = \Theta(n)$.

2. (6 Points)

Use the Master theorem to find asymptotic solutions to the following recurrences.

a. (1 Point) $T(n) = 7T(n/4) + n$

Solution:

$4 < 7 \Rightarrow 1 < \log_4(7) \Rightarrow \log_4(7) - 1 > 0$. Let $\varepsilon = \log_4(7) - 1$. Then $\varepsilon > 0$, and $1 = \log_4(7) - \varepsilon$, whence $n = n^{\log_4(7)-\varepsilon} = O(n^{\log_4(7)-\varepsilon})$. By case (1) we have $T(n) = \Theta(n^{\log_4(7)})$.

b. (1 Point) $T(n) = 9T(n/3) + n^2$

Observe that $n^2 = n^{\log_3(9)} = \Theta(n^{\log_3(9)})$, and therefore $T(n) = \Theta(n^2 \log(n))$ by case (2).

c. (1 Point) $T(n) = 6T(n/5) + n^2$

Solution:

Observe $6 < 25 \Rightarrow \log_5(6) < 2 \Rightarrow 2 - \log_5(6) > 0$. Let $\varepsilon = 2 - \log_5(6)$. Then $\log_5(6) + \varepsilon = 2$, and $n^2 = \Omega(n^{\log_5(6)+\varepsilon})$. Also for any c in the range $6/25 \le c < 1$, and for any positive n, we have $6(n/5)^2 = (6/25)n^2 \le cn^2$, so the regularity condition holds. Therefore $T(n) = \Theta(n^2)$ by case (3) of the Master Theorem.

d. (1 Point) $T(n) = 6T(n/5) + n\log(n)$

Solution:

Observe $\log_5(6) > 1$, so letting $\varepsilon = \dfrac{\log_5(6) - 1}{2}$, we have $\varepsilon > 0$ and $1 + \varepsilon = \log_5(6) - \varepsilon$. Therefore by l'Hopital's rule

$$\lim_{n\to\infty} \frac{n\log(n)}{n^{\log_5(6)-\varepsilon}} = \lim_{n\to\infty} \frac{n\log(n)}{n^{1+\varepsilon}} = \lim_{n\to\infty} \frac{\log(n)}{n^{\varepsilon}} = 0,$$

showing that $n\log(n) = o(n^{\log_5(6)-\varepsilon}) \subseteq O(n^{\log_5(6)-\varepsilon})$. Case (1) now gives $T(n) = \Theta(n^{\log_5(6)})$.

e. (1 Point) $T(n) = 7T(n/2) + n^2$

Solution:

Observe that $7 > 4 \Rightarrow \log_2(7) > 2$, so upon setting $\varepsilon = \log_2(7) - 2$ we have $\varepsilon > 0$. It follows that $2 = \log_2(7) - \varepsilon$, whence $n^2 = n^{\log_2(7)-\varepsilon} = O(n^{\log_2(7)-\varepsilon})$. Case 1 now gives $T(n) = \Theta(n^{\log_2(7)})$.

f. (1 Point) $S(n) = aS(n/4) + n^2$ (Note: your answer will depend on the parameter a.)

Solution:

We have three cases to consider corresponding to the three cases of the Master Theorem:

Case 1: $a > 16 \Rightarrow \log_4(a) > 2 \Rightarrow \varepsilon = \log_4(a) - 2 > 0 \Rightarrow n^2 = O(n^{\log_4(a)-\varepsilon})$, so $S(n) = \Theta(n^{\log_4(a)})$.

Case 2: $a = 16 \Rightarrow \log_4(a) = 2 \Rightarrow n^2 = \Theta(n^{\log_4(a)})$, whence $S(n) = \Theta(n^2 \log(n))$.

Case 3: $1 \le a < 16 \Rightarrow \log_4(a) < 2 \Rightarrow \varepsilon = 2 - \log_4(a) > 0 \Rightarrow n^2 = \Omega(n^{\log_4(a)+\varepsilon})$. Further, for any c in the range $a/16 \le c < 1$ we have $a(n/4)^2 = (a/16)n^2 \le cn^2$, showing that the regularity condition holds. Therefore $S(n) = \Theta(n^2)$.

3. (1 Point) p.75: 4.3-2

The recurrence $T(n) = 7T(n/2) + n^2$ describes the running time of an algorithm *A*. A competing algorithm *B* has a running time of $S(n) = aS(n/4) + n^2$. What is the largest integer value for *a* such that *B* is a faster algorithm than *A* (asymptotically speaking)? In other words, find the largest integer *a* such that $S(n) = o(T(n))$.

Solution:

We seek the largest integer a for which $S(n) = o(T(n))$. Using parts (e) and (f) of the previous problem, we find that $S(n) = o(T(n))$ in cases 2 and 3 since $4 < 7 \Rightarrow 2 < \log_2(7)$ and hence $n^2 = o(n^{\log_2(7)})$, and $n^2 \log(n) = o(n^{\log_2(7)})$. In case 1 we have $S(n) = o(T(n))$ if and only if $n^{\log_4(a)} = o(n^{\log_2(7)})$, i.e. if and only if $\log_4(a) < \log_2(7)$. Thus we seek the largest integer a such that $a < 4^{\log_2(7)} = 7^{\log_2(4)} = 7^2 = 49$. The largest such integer is $a = 48$.

4. (1 Point)

Let *G* be an acyclic graph with *n* vertices, *m* edges, and *k* connected components. Show that $m = n - k$. (Hint: use the fact that $|E(T)| = |V(T)| - 1$ for any tree *T*, from the induction handout.)

Proof:

Let $T_1, T_2, \ldots, T_k$, be the connected components of *G*, each of which is necessarily a tree. Let $n_i = |V(T_i)|$, and $m_i = |E(T_i)|$, for $1 \le i \le k$. By a theorem proved in the induction handout, we have $m_i = n_i - 1$, for $1 \le i \le k$. Therefore

$$m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

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5. (1 Point) (Appendix B.4 problem 3)

Show that any connected graph *G* satisfies $|E(G)| \ge |V(G)| - 1$. (Hint: use induction on the number of **edges**. Note: this hint wrongly said "induction on the number of vertices" in the original statement of the problem.)

Proof:

Let $G = (V, E)$ be a connected graph, and suppose $|E| = 0$. Since G is connected we must have $|V| = 1$, whence $|E| \ge |V| - 1$, and so the base case is satisfied.

Now suppose $|E| > 0$ and assume the result holds for any graph with fewer than $|E|$ edges. In other words, we assume that for all graphs $G' = (V', E')$ with $|E'| < |E|$ that $|E'| \ge |V'| - 1$. Now pick any edge e in G and remove it, and let $G - e$ denote the resulting graph. We have two cases to consider.

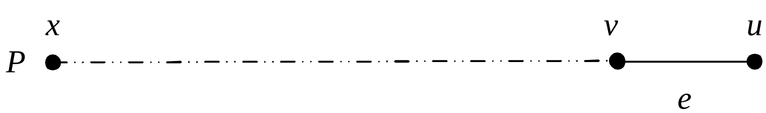
Case 1: $G-e$ is connected. In this case we apply the induction hypothesis to $G-e=(V,E-e)$ which has fewer edges than *G*. We conclude that $|E-e|\geq|V|-1$, so that $|E|-1\geq|V|-1$, and therefore $|E|\geq|V|>|V|-1$, as required.
Case 2: $G-e$ is disconnected. In this case $G-e$ consists of two connected components. (**See the claim and proof below, which are not necessary for full credit on this problem.) Call them $H_1=(V_1,E_1)$ and $H_2=(V_2,E_2)$. Note that both H_1 and H_2 have fewer edges than *G*, so we may apply the induction hypothesis to obtain $|E_1|\geq|V_1|-1$ and $|E_2|\geq|V_2|-1$. Therefore

$$\begin{aligned}
|E| &= |E_1|+|E_2|+1 \\
&\geq (|V_1|-1)+(|V_2|-1)+1 && \text{(by the induction hypothesis)} \\
&= |V_1|+|V_2|-1 \\
&= |V|-1 && \text{(no vertices were removed so } |V_1|+|V_2|=|V|\text{).}
\end{aligned}$$

The result now holds for all connected graphs by induction. ///

Claim**: Let *G* be a connected graph and $e\in E(G)$, and suppose that $G-e$ is disconnected. (Such an edge *e* is called a *bridge*). Then $G-e$ has exactly two connected components.

Proof: Since $G-e$ is disconnected, it has at least two components. We must show that it also has at most two components. Let *e* have end vertices *u*, and *v*. Let C_u and C_v be the connected components of $G-e$ that contain *u* and *v* respectively. Choose $x\in V(G)$ arbitrarily, and let *P* be an *x*-*u* path in *G*. Either *P* includes the edge *e*, or it does not. If *P* does not contain *e*, then *P* remains intact after the removal of *e*, and hence *P* is an *x*-*u* path in $G-e$, whence $x\in C_u$. If on the other hand *P* does contain the edge *e*, then *e* must be the last edge along *P* from *x* to *u*.



In this case $P-e$ is an *x*-*v* path in $G-e$, whence $x\in C_v$. Since *x* was arbitrary, every vertex in $G-e$ belongs to either C_u or C_v, and therefore $G-e$ has at most two connected components. ///