

**CMPS 101**  
**Algorithms and Abstract Data Types**  
**Fall 2016**  
**Midterm Exam 1      Solutions**

1. (20 Points) Determine whether the following statements are true or false. Prove or disprove each statement accordingly.

a. (10 Points) If  $h_1(n) = \Theta(f(n))$  and  $h_2(n) = \Theta(g(n))$ , then  $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$ . **True.**

**Proof:**

By hypothesis there exist positive constants  $n_1$ ,  $n_2$ ,  $a_1$ ,  $b_1$ ,  $a_2$ , and  $b_2$  such that

$$\begin{aligned} \forall n \geq n_1: \quad 0 \leq a_1 f(n) \leq h_1(n) \leq b_1 f(n) \\ \text{and} \quad \forall n \geq n_2: \quad 0 \leq a_2 g(n) \leq h_2(n) \leq b_2 g(n) \end{aligned}$$

Let  $c = a_1 a_2$ ,  $d = b_1 b_2$  and  $n_0 = \max(n_1, n_2)$ . Then  $c$ ,  $d$  and  $n_0$  are positive, and for any  $n \geq n_0$  both of the above inequalities are true. Multiplying them together gives

$$\forall n \geq n_0: \quad 0 \leq c f(n) g(n) \leq h_1(n) h_2(n) \leq d f(n) g(n),$$

and hence  $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$  as required. ///

b. (10 Points)  $2^{\ln(n)} = o(n)$ . **True.**

**Proof:**

$2^{\ln n} = n^{\ln(2)}$  by an identity proved in class ( $a^{\log_b(x)} = x^{\log_b(a)}$ ). Since  $2 < e$ , we have  $\ln(2) < 1$ , and therefore  $n^{\ln(2)} = o(n^1)$ , whence  $2^{\ln(n)} = o(n)$ . ///

2. (20 Points) Use Stirling's formula to prove that  $\log(n!) = \Theta(n \log(n))$ .

**Proof:**

Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get

$$\begin{aligned}\log(n!) &= \log\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot (1 + \Theta(1/n))\right) \\ &= \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log(1 + \Theta(1/n)) \\ &= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log(1 + \Theta(1/n)).\end{aligned}$$

Therefore

$$\frac{\log(n!)}{n\log(n)} = \frac{\log(2\pi)}{2n\log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log(1 + \Theta(1/n))}{n\log(n)},$$

and hence  $\lim_{n \rightarrow \infty} \left( \frac{\log(n!)}{n\log(n)} \right) = 1$ . Thus  $\log(n!) = \Theta(n\log(n))$  as claimed.

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3. (20 Points) Consider the following algorithm that wastes time.

WasteTime( $n$ ) (pre:  $n \geq 1$ )

1. if  $n=1$
2.     waste 2 units of time
3. else
4.     WasteTime( $\lceil n/2 \rceil$ )
5.     WasteTime( $\lfloor n/2 \rfloor$ )
6.     waste 5 units of time

- a. (10 Points) Write a recurrence relation for the number of units of time  $T(n)$  wasted by this algorithm.

**Solution:**

$$T(n) = \begin{cases} 2 & n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 & n \geq 2 \end{cases}$$

- b. (10 Points) Show that  $T(n) = 7n - 5$  is the solution to this recurrence. (Hint: you may use without proof the fact that  $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ .)

**Proof:**

First observe that if  $T(n) = 7n - 5$ , then  $T(1) = 7 - 5 = 2$ . Second, if  $n \geq 2$  we have

$$\begin{aligned} \text{RHS} &= T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 \\ &= (7\lceil n/2 \rceil - 5) + (7\lfloor n/2 \rfloor - 5) + 5 \\ &= 7(\lceil n/2 \rceil + \lfloor n/2 \rfloor) - 5 \\ &= 7n - 5 = T(n) = \text{LHS}, \end{aligned}$$

showing that  $T(n) = 7n - 5$  solves the recurrence.

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4. (20 Points) Prove that  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$  for all  $n \geq 1$ . (Hint: use weak induction.)

**Proof:**

Let  $P(n)$  be the formula  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ .

**Base step**

$P(1)$  says that  $\sum_{i=1}^1 i^3 = \left(\frac{1(1+1)}{2}\right)^2$ , i.e. that  $1^3 = 1^2$ , i.e.  $1 = 1$ , which is true.

**Induction Step (IIa)**

Let  $n \geq 1$  be chosen arbitrarily. Assume for this  $n$  that  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ . We must show that

$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$ , i.e.  $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$ . Now observe that

$$\begin{aligned}
 \sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^n i^3\right) + (n+1)^3 \\
 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 && \text{(by the induction hypothesis)} \\
 &= (n+1)^2 \left(\frac{n^2}{2^2} + (n+1)\right) \\
 &= (n+1)^2 \left(\frac{n^2 + 4(n+1)}{4}\right) \\
 &= (n+1)^2 \left(\frac{n^2 + 4n + 4}{4}\right) \\
 &= \frac{(n+1)^2 (n+2)^2}{4} \\
 &= \left(\frac{(n+1)(n+2)}{2}\right)^2
 \end{aligned}$$

as required. It follows from the first principle of mathematical induction that  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$  for all  $n \geq 1$ . ///

5. (20 Points) Let  $T(n)$  be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

a. (4 Points) Determine the values  $T(2)$ ,  $T(3)$ ,  $T(4)$ , and  $T(5)$ .

**Solution:**

$$T(2) = T(1) + 2^2 = 1 + 4 = 5$$

$$T(3) = T(1) + 3^2 = 1 + 9 = 10$$

$$T(4) = T(2) + 4^2 = 5 + 16 = 21$$

$$T(5) = T(2) + 5^2 = 5 + 25 = 30$$

b. (16 Points) Prove that  $T(n) \leq \frac{4}{3}n^2$  for all  $n \geq 1$ . (Hint: use strong induction.)

**Proof:**

**Base Step**

Observe that  $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$ , which establishes the base case.

**Induction Step (IId)**

Let  $n > 1$  be chosen arbitrarily. Assume for all  $k$  in the range  $1 \leq k < n$  that  $T(k) \leq (4/3)k^2$ . We must show as a consequence that  $T(n) \leq (4/3)n^2$ . Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. It follows from the second principle of mathematical induction that  $T(n) \leq \frac{4}{3}n^2$  for all  $n \geq 1$ . ///