CMPS 101
Algorithms and Abstract Data Types
Summer 2013
Midterm Exam 1

Solutions

1. (20 Points) Determine whether the following statements are true or false. Give a proof if the statement is true, give a counter-example if the statement is false.

   a. (10 Points) If $f(n) = o(n)$ and $g(n) = o(n^2)$, then $f(n) \cdot g(n) = o(n^3)$.

   Solution: The statement is true.

   Proof: We know from the hypothesis that $\lim_{n\to\infty}\left(\frac{f(n)}{n}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{g(n)}{n^2}\right) = 0$. Therefore

$$\lim_{n\to\infty}\left(\frac{f(n)\cdot g(n)}{n^3}\right) = \lim_{n\to\infty}\left(\frac{f(n)}{n}\cdot\frac{g(n)}{n^2}\right) = \lim_{n\to\infty}\left(\frac{f(n)}{n}\right)\cdot\lim_{n\to\infty}\left(\frac{g(n)}{n^2}\right) = 0\cdot 0 = 0$$

   whence $f(n)\cdot g(n) = o(n^3)$. ///

   b. (10 Points) If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

   Solution: The statement is false.
   Counter-example: Observe that $2n = \Theta(n)$ but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{2n} \neq \Theta(2^n)$. ///

2. (20 Points) Use Stirling's formula to prove that $\log(n!) = \Theta\big(n\log(n)\big)$.
   Proof:
   Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get

$$\begin{aligned}
\log(n!) &= \log\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\big(1+\Theta(1/n)\big)\right)\\
&= \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log\big(1+\Theta(1/n)\big)\\
&= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\big(1+\Theta(1/n)\big).
\end{aligned}$$

   Therefore

$$\frac{\log(n!)}{n\log(n)} = \frac{\log(2\pi)}{2n\log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log\big(1+\Theta(1/n)\big)}{n\log(n)},$$

   hence $\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1$ and $\log(n!) = \Theta(n\log(n))$ as claimed. ///

3. (20 Points) Consider the following algorithm that wastes time.

<u>WasteTime(n)</u> (pre: $n \geq 1$)
1. if $n = 1$
2. waste 1 unit of time
3. else
4. WasteTime$\left(\lceil n/2 \rceil\right)$
5. WasteTime$\left(\lfloor n/2 \rfloor\right)$
6. waste 3 units of time

a. (10 Points) Write a recurrence relation for the number of units of time $T(n)$ wasted by this algorithm.
Solution:

$$T(n) = \begin{cases} 1 & n = 1 \\ T\left(\lceil n/2 \rceil\right) + T\left(\lfloor n/2 \rfloor\right) + 3 & n \geq 2 \end{cases}$$

b. (10 Points) Show that $T(n) = 4n - 3$ is the solution to this recurrence. (Hint: you may use without proof the fact that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.)
Proof:
First observe that if $T(n) = 4n - 3$, then $T(1) = 4 - 3 = 1$. If $n \geq 2$ then

$$\begin{aligned} \text{RHS} &= T\left(\lceil n/2 \rceil\right) + T\left(\lfloor n/2 \rfloor\right) + 3 \\ &= \left(4\lceil n/2 \rceil - 3\right) + \left(4\lfloor n/2 \rfloor - 3\right) + 3 \\ &= 4\left(\lceil n/2 \rceil + \lfloor n/2 \rfloor\right) - 3 \\ &= 4n - 3 \\ &= T(n) \\ &= \text{LHS}, \end{aligned}$$

showing that $T(n) = 4n - 3$ solves the recurrence. ///

4. (20 Points) Use weak induction to prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$.

Proof: Let $P(n)$ be the equation $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

I. Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\frac{1\cdot(1+1)}{1}\right)^2$, whence $P(1)$ is true.

IIa. Let $n \geq 1$ and assume that $P(n)$ is true, i.e. for this *n*, we assume $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $P(n+1)$ also holds: $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$. Therefore

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{(by the induction hypothesis)} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2\left[n^2 + 4n + 4\right]}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}
$$

showing that $P(n+1)$ is true. ///

5. (20 Points) Let $T(n)$ be defined by the following recurrence relation.

$$T(n) = \begin{cases} 5 & n = 1 \\ T(\lfloor n/2 \rfloor) + 3 & n \geq 2 \end{cases}$$

a. (10 Points) Determine the values $T(2)$, $T(3)$, $T(4)$, and $T(5)$.

Solution: $T(2) = T(1) + 3 = 5 + 3 = 8$

$T(3) = T(1) + 3 = 8$

$T(4) = T(2) + 3 = 8 + 3 = 11$

$T(5) = T(2) + 3 = 11$

b. (10 Points) Prove that $T(n) \leq 8\lg(n)$ for all $n \geq 2$. (Hint: use strong induction with two base cases.)

Proof:

<u>I. Two base cases:</u>

From part (a) we have $T(2) = 8 = 8 \cdot 1 = 8 \cdot \lg(2)$, and $T(3) = 8 \leq 8 \cdot \lg(3)$, so the first two cases of the inequality are true.

<u>II. Strong Induction:</u> $\forall n \geq 4 : \left(\forall k \in [2, n) : T(k) \leq 8\lg(k) \right) \rightarrow T(n) \leq 8\log(n)$

Pick $n \geq 4$ arbitrarily. Assume for all integers k in the range $2 \leq k < n$ that $T(k) \leq 8\lg(k)$. In particular for $k = \lfloor n/2 \rfloor$ we have $T(\lfloor n/2 \rfloor) \leq 8\lg(\lfloor n/2 \rfloor)$. Then

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + 3 && \text{by the recurrence formula} \\ &\leq 8\lg(\lfloor n/2 \rfloor) + 3 && \text{by the induction hypothesis} \\ &\leq 8\lg(n/2) + 3 && \text{since } \lfloor x \rfloor \leq x \\ &= 8(\lg(n) - \lg(2)) + 3 && \text{using laws of logarithms} \\ &= 8\lg(n) - 8 + 3 \\ &= 8\lg(n) - 5 \\ &\leq 8\lg(n) \end{aligned}$$

The result follows for all $n \geq 2$ by induction. ///