CMPS 101
Homework Assignment 2
Solutions

1. p.50: 3.1-1
   Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $f(n)+g(n)=\Theta(\max(f(n),g(n)))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically non-negative, there exists a positive constant n_0 such that $f(n)\geq 0$ and $g(n)\geq 0$ for all $n\geq n_0$. For such n we have

$$\begin{aligned}0 &\leq \max(f(n),g(n))\\ &\leq \min(f(n),g(n))+\max(f(n),g(n))\\ &\leq 2\cdot\max(f(n),g(n)).\end{aligned}$$

   But $f(n)+g(n)=\min(f(n),g(n))+\max(f(n),g(n))$, so for all $n\geq n_0$ we have

$$0\leq 1\cdot\max(f(n),g(n))\leq f(n)+g(n)\leq 2\cdot\max(f(n),g(n)).$$

   Thus $f(n)+g(n)=\Theta(\max(f(n),g(n)))$, as required. ///

2. p.50: 3.1-3
   Explain why the statement "The running time of algorithm A is at least $O(n^2)$" is meaningless.

   Solution:
   This statement is true under all circumstances, hence it conveys no useful information, and is therefore meaningless. To illustrate, let $T(n)$ be the running time of algorithm A. To say that $T(n)$ is "at least $O(n^2)$" is to say that $T(n)$ is bounded below by a function that is bounded above (asymptotically) by n^2. If $T(n)$ in the class $O(n^2)$, then $T(n)$ is bounded below by itself, which is bounded above asymptotically by n^2, and hence the statement is true. If on the other hand, $T(n)$ is in the class $\Omega(n^2)$, then $T(n)$ is bounded below by cn^2 (for sufficiently large n), which is bounded above asymptotically by n^2, and again the statement is true. Even if $T(n)$ is not comparable to n^2, $T(n)$ is bounded below by some positive constant, which is bounded above by n^2. (This is true since even if algorithm A performs no operations, it takes some time to start and stop, so that $T(n)$ is greater than some, possibly small, positive number.) In all cases the statement is true, and therefore devoid of meaning. ///

3. p. 50: 3.1-4
Determine whether the following statements are true or false.

a. $2^{n+1} = O(2^n)$

Solution: True since $2^{n+1} = 2 \cdot 2^n = \text{const} \cdot 2^n = O(2^n)$.

b. $2^{2n} = O(2^n)$

Solution: False since $2^{2n} = (2^2)^n = 4^n = \omega(2^n)$ and since $\omega(2^n) \cap O(2^n) = \varnothing$.

4. p.58: 3-2abcdef
Indicate, for each pair of expressions (*A*, *B*) in the table below, whether *A* is *O*, *o*, Ω, ω, or Θ of *B*. Assume that $k \geq 1$, $\varepsilon > 0$, and $c > 1$ are constants. Place 'yes' or 'no' in each of the empty cells below, and justify your answers.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^{ε}	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

Justifications:

a. Applying l'Hopital's rule $\lceil k \rceil$ times, we get $\lim_{n\to\infty} \frac{(\lg n)^k}{n^{\varepsilon}} = 0$, whence $\lg^k n = o(n^{\varepsilon})$.

b. Again, by $\lceil k \rceil$ applications of l'Hopital's rule, $\lim_{n\to\infty} \frac{n^k}{c^n} = 0$, whence $n^k = o(c^n)$.

c. Let $c > 0$, and observe that $n^{\sin n} > c\sqrt{n}$ for $n = 2\pi k + \frac{\pi}{2}$, where *k* is a sufficiently large positive integer. Therefore the inequality $n^{\sin n} \leq c\sqrt{n}$ is false for arbitrarily large *n*, and hence $n^{\sin n} \neq O(\sqrt{n})$. Also note that $n^{\sin n} < c\sqrt{n}$ for $n = 2\pi k + \frac{3\pi}{2}$, where *k* is sufficiently large, and therefore the inequality $n^{\sin n} \geq c\sqrt{n}$ is false for arbitrarily large *n*, whence $n^{\sin n} \neq \Omega(\sqrt{n})$.

d. Observe $\lim_{n\to\infty}(2^n / 2^{n/2}) = \lim_{n\to\infty}(\sqrt{2})^n = \infty$, and therefore $2^n = \omega(2^{n/2})$.

e. Recall it was shown in class that $n^{\lg c} = c^{\lg n}$.

f. Note $\lg(n^n) = n\lg(n)$, and it was shown using Stirling's formula that $\lg(n!) = \Theta(n\lg(n))$.

5. p.58: 3-4cdeh

Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n)>0$ and $g(n)>0$ for sufficiently large *n*.) Prove or disprove the following statements.

c.

Assume $\lg(g(n))\ge 1$ and $f(n)\ge 1$ for all sufficiently large *n*. Then $f(n)=O(g(n))$ implies $\lg(f(n))=O(\lg(g(n)))$. **True**

Proof:

Assume $f(n)=O(g(n))$. Then there exists $c>0$ and $n_0>0$ such that $f(n)\le cg(n)$ for all $n\ge n_0$. The above hypotheses say we can take n_0 large enough to also guarantee $f(n)\ge 1$ and $\lg(g(n))\ge 1$ for $n\ge n_0$. Thus $1\le f(n)\le cg(n)$ for all $n\ge n_0$. Take $\log_2$ of this inequality to obtain

(*) $$0\le \lg(f(n))\le \lg(c)+\lg(g(n))=\left(\frac{\lg(c)}{\lg(g(n))}+1\right)\lg(g(n))\le(\lg(c)+1)\lg(g(n)).$$

The last inequality is a consequence of $\lg(g(n))\ge 1$, for this implies $1\ge\frac{1}{\lg(g(n))}$, which implies $\lg(c)\ge\frac{\lg(c)}{\lg(g(n))}$. Define the constant $b=\lg(c)+1$. Inequality (*) gives $0\le\lg(f(n))\le b\lg(g(n))$ for all $n\ge n_0$, showing that $\lg(f(n))=O(\lg(g(n)))$, as claimed. ///

d.

$f(n)=O(g(n))$ implies $2^{f(n)}=O(2^{g(n)})$. **False**

Counter-Example:

Let $f(n)=2n$ and $g(n)=n$. Then $2^{g(n)}=2^n$ and $2^{f(n)}=2^{2n}=4^n=\omega(2^n)$, so $2^{f(n)}=\omega(2^{g(n)})$, and therefore $2^{f(n)}\ne O(2^{g(n)})$.

e.

$f(n)=O((f(n))^2)$. **False**

Counter-Example:

Let $f(n)=1/n$. Then $f(n)=\omega((f(n))^2)$ since $\lim_{n\to\infty}\frac{f(n)}{f(n)^2}=\lim_{n\to\infty}\frac{1}{f(n)}=\lim_{n\to\infty}n=\infty$, and therefore $f(n)\ne O((f(n))^2)$. ///

h.

$f(n)+o(f(n))=\Theta(f(n))$. **True**

Proof:

In the above formula, $o(f(n))$ stands for some anonymous function $h(n)$ in the class $o(f(n))$, whence $\lim_{n\to\infty}\frac{h(n)}{f(n)}=0$. Thus $\lim_{n\to\infty}\frac{f(n)+h(n)}{f(n)}=\lim_{n\to\infty}\left(1+\frac{h(n)}{f(n)}\right)=1$, and $f(n)+h(n)=\Theta(f(n))$, as claimed. ///

6. Let $f(n)=\Theta(n)$. Prove that $\sum_{i=1}^{n} f(i)=\Theta(n^2)$. (See the hint at bottom of p.4 of the handout on asymptotic growth rates.)

Proof:
Since $f(n)=\Theta(n)$ there exist positive constants c_1, c_2, and n_0 such that for all $n \geq n_0$ the inequality $0 \leq c_1 n \leq f(n) \leq c_2 n$ holds. In particular, given $n_0 \leq i \leq n$ we have $c_1 i \leq f(i) \leq c_2 i$. Upon summing these inequalities from $i=n_0$ to $i=n$ we obtain

$$\sum_{i=n_0}^{n} c_1 i \;\leq\; \sum_{i=n_0}^{n} f(i) \;\leq\; \sum_{i=n_0}^{n} c_2 i$$

for any $n \geq n_0$. Define $A=\sum_{i=1}^{n_0-1} f(i)$. Then the above inequality can be rewritten as

$$\sum_{i=1}^{n} c_1 i \;-\; \sum_{i=1}^{n_0-1} c_1 i \;\leq\; \sum_{i=1}^{n} f(i) \;-\; A \;\leq\; \sum_{i=1}^{n} c_2 i \;-\; \sum_{i=1}^{n_0-1} c_2 i$$

Adding the constant A to all terms, and using the well known formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we obtain

$$c_1\left(\frac{n(n+1)}{2}\right) - c_1\left(\frac{n_0(n_0-1)}{2}\right) + A \leq \sum_{i=1}^{n} f(i) \leq c_2\left(\frac{n(n+1)}{2}\right) - c_2\left(\frac{n_0(n_0-1)}{2}\right) + A$$

Now observe that since c_1, c_2, n_0, and A are constants, the left hand side is in the class $\Omega(n^2)$ and the right hand side is in $O(n^2)$, and this inequality holds for all $n \geq n_0$. In other words, the inequality $h_1(n) \leq \sum_{i=1}^{n} f(i) \leq h_2(n)$ holds for all sufficiently large n, where $h_1(n)=\Omega(n^2)$ and $h_2(n)=O(n^2)$. By an exercise in the handout on asymptotic growth of functions, we conclude that $\sum_{i=1}^{n} f(i)=\Theta(n^2)$, as required. ///

7. The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n}=\Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

   Proof: By Stirling's formula

$$\binom{2n}{n}=\frac{(2n)!}{n!(2n-n)!}=\frac{(2n)!}{(n!)^2}=\frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta(1/2n)\right)}{\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^{n}\cdot\left(1+\Theta(1/n)\right)\right)^2}$$

$$= \frac{2^{2n}}{\sqrt{\pi n}} \cdot \frac{1+\Theta(1/2n)}{(1+\Theta(1/n))^2} = \frac{1}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot \frac{1+\Theta(1/2n)}{(1+\Theta(1/n))^2}$$

so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1+\Theta(1/2n)}{(1+\Theta(1/n))^2} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad n \to \infty$$

The result now follows since $0 < \frac{1}{\sqrt{\pi}} < \infty$. ///

8. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

(Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer *k*.)

Proof: Since $f(n)$ is increasing we have $\sum_{i=1}^{n} f(i) \le \sum_{i=1}^{n} f(n) = nf(n) = O(nf(n))$. Note also that

$$\begin{aligned}
\sum_{i=1}^{n} f(i) &\ge \sum_{i=\lceil n/2 \rceil}^{n} f(i) && \text{by discarding some positive terms} \\
&\ge \sum_{i=\lceil n/2 \rceil}^{n} f(\lceil n/2 \rceil) && \text{since } f(n) \text{ is increasing} \\
&= (n - \lceil n/2 \rceil + 1) f(\lceil n/2 \rceil) && \text{by counting terms} \\
&= (\lfloor n/2 \rfloor + 1) f(\lceil n/2 \rceil) && \text{since } n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \\
&> ((n/2) - 1 + 1) f(n/2) && \text{since } f(n) \text{ is increasing, } \lceil x \rceil \ge x, \text{ and } \lfloor x \rfloor > x - 1 \\
&= (n/2) f(n/2) && \\
&= \Omega(nf(n)) && \text{since } f(n/2) = \Theta(f(n)), \text{ whence } f(n/2) = \Omega(f(n))
\end{aligned}$$

It follows from an exercise in the handout on Asymptotic Growth rates that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$, as claimed. ///

9. Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n\log(n))$ that does not use Stirling's formula.

Proof:
Observe that $\log(n/2) = \log(n) - \log(2) = \Theta(\log(n))$. We may therefore apply the result of the preceding problem with $f(n) = \log(n)$, and properties of logarithms to get

$$\log(n!) = \sum_{i=1}^{n} \log(i) = \Theta(n\log(n))$$

as claimed. ///

10. Let $g(n)$ be an asymptotically non-negative function. Prove that $o(g(n)) \cap \Omega(g(n)) = \varnothing$.

Proof:
Assume to get a contradiction that $f(n) \in o(g(n)) \cap \Omega(g(n))$. Then since $f(n) = \Omega(g(n))$ we have

(1) $\exists c_1 > 0,\ \exists n_1 > 0,\ \forall n \geq n_1: \quad 0 \leq c_1 g(n) \leq f(n)$

Also, since $f(n) = o(g(n))$ we have

(2) $\forall c_2 > 0,\ \exists n_2 > 0,\ \forall n \geq n_2: \quad 0 \leq f(n) < c_2 g(n)$

Let $c_2 = c_1$. Then $c_2 > 0$, and by (2) there exists $n_2 > 0$ such that $0 \leq f(n) < c_1 g(n)$ for all $n \geq n_2$. Pick any $m \geq \max(n_1, n_2)$. Then by (1) we have $0 \leq c_1 g(m) \leq f(m) < c_1 g(m)$, and hence $c_1 g(m) < c_1 g(m)$, a contradiction. Our assumption was therefore false, and no such function $f(n)$ can exist. We conclude that $o(g(n)) \cap \Omega(g(n)) = \varnothing$. ///