CMPS 101
Algorithms and Abstract Data Types
Summer 2015
Midterm Exam 1 Solutions

1. (20 Points) Determine whether the following statements are true or false. Prove or disprove each statement accordingly.

a. (10 Points) If $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(g(n))$, then $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$. **True.**

Proof:
By hypothesis there exist positive constants n_1, n_2, a_1, b_1, a_2, and b_2 such that

$$\forall\, n \ge n_1: \quad 0 \le a_1 f(n) \le h_1(n) \le b_1 f(n)$$

and

$$\forall\, n \ge n_2: \quad 0 \le a_2 g(n) \le h_2(n) \le b_2 g(n)$$

Let $c = a_1 a_2$, $d = b_1 b_2$ and $n_0 = \max(n_1, n_2)$. Then c, d and n_0 are positive, and for any $n \ge n_0$ both of the above inequalities are true. Multiplying them together gives

$$\forall\, n \ge n_0: \quad 0 \le c\, f(n) g(n) \le h_1(n) h_2(n) \le d\, f(n) g(n),$$

and hence $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$ as required. ///

b. (10 Points) $2^{\ln(n)} = o(n)$. **True.**

Proof:
$2^{\ln n} = n^{\ln(2)}$ by an identity proved in class ($a^{\log_b(x)} = x^{\log_b(a)}$). Since $2 < e$, we have $\ln(2) < 1$, and therefore $n^{\ln(2)} = o(n^1)$, whence $2^{\ln(n)} = o(n)$. ///

2. (20 Points) Use Stirling's formula to prove that $\log(n!) = \Theta\left(n\log(n)\right)$.

Proof:
Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get

$$\begin{aligned}
\log(n!) &= \log\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta(1/n)\right)\right) \\
&= \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log\left(1+\Theta(1/n)\right) \\
&= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\left(1+\Theta(1/n)\right).
\end{aligned}$$

Therefore

$$\frac{\log(n!)}{n\log(n)} = \frac{\log(2\pi)}{2n\log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log\left(1+\Theta(1/n)\right)}{n\log(n)},$$

and hence $\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1$. Thus $\log(n!) = \Theta(n\log(n))$ as claimed. ///

3. (20 Points) Consider the following algorithm that wastes time.

<u>WasteTime(n)</u> (pre: $n \geq 1$)

1. if $n = 1$
2.   waste 2 units of time
3. else
4.   WasteTime$\left(\lceil n/2 \rceil\right)$
5.   WasteTime$\left(\lfloor n/2 \rfloor\right)$
6.   waste 5 units of time

a. (10 Points) Write a recurrence relation for the number of units of time $T(n)$ wasted by this algorithm.

Solution:

$$T(n) = \begin{cases} 2 & n = 1 \\ T\left(\lceil n/2 \rceil\right) + T\left(\lfloor n/2 \rfloor\right) + 5 & n \geq 2 \end{cases}$$

b. (10 Points) Show that $T(n) = 7n - 5$ is the solution to this recurrence. (Hint: you may use without proof the fact that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.)

Proof:
First observe that if $T(n) = 7n - 5$, then $T(1) = 7 - 5 = 2$. Second, if $n \geq 2$ we have

$$\begin{aligned} \text{RHS} &= T\left(\lceil n/2 \rceil\right) + T\left(\lfloor n/2 \rfloor\right) + 5 \\ &= \left(7\lceil n/2 \rceil - 5\right) + \left(7\lfloor n/2 \rfloor - 5\right) + 5 \\ &= 7\left(\lceil n/2 \rceil + \lfloor n/2 \rfloor\right) - 5 \\ &= 7n - 5 = T(n) = \text{LHS}, \end{aligned}$$

showing that $T(n) = 7n - 5$ solves the recurrence. ///

4. (20 Points) Prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$. (Hint: use weak induction.)

Proof:

Let $P(n)$ be the formula $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Base step

$P(1)$ says that $\sum_{i=1}^{1} i^3 = \left(\frac{1(1+1)}{2}\right)^2$, i.e. that $1^3 = 1^2$, i.e. $1 = 1$, which is true.

Induction Step (IIa)

Let $n \geq 1$ be chosen arbitrarily. Assume for this n that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$, i.e. $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$. Now observe that

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \qquad \text{(by the induction hypothesis)} \\
&= (n+1)^2\left(\frac{n^2}{2^2} + (n+1)\right) \\
&= (n+1)^2\left(\frac{n^2+4(n+1)}{4}\right) \\
&= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}
$$

as required. It follows from the first principle of mathematical induction that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$. ///

5. (20 Points) Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

a. (4 Points) Determine the values $T(2)$, $T(3)$, $T(4)$, and $T(5)$.

Solution:

$T(2) = T(1) + 2^2 = 1 + 4 = 5$
$T(3) = T(1) + 3^3 = 1 + 9 = 10$
$T(4) = T(2) + 4^2 = 5 + 16 = 21$
$T(5) = T(2) + 5^2 = 5 + 25 = 30$

b. (16 Points) Prove that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. (Hint: use strong induction.)

Proof:
Base Step
Observe that $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, which establishes the base case.

Induction Step (IId)
Let $n > 1$ be chosen arbitrarily. Assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. It follows from the second principle of mathematical induction that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. ///