CMPS 130
Spring 2016
Homework Assignment 1 Solutions

Chapter 1 (p.34): 18, 19ab, 20a-e, 23a-c, 25a-c, 26, 27a-c, 28a-c, 30

1. Problem 1.18
   Find a formula for a function from $\mathbb{Z}$ to $\mathbb{N}$ that is a bijection.
   Solution:
   Define $f:\mathbb{Z}\to\mathbb{N}$ by

$$f(k)=\begin{cases} 2k-1 & \text{if } \ k>0 \\ -2k & \text{if } \ k\le 0 \end{cases}$$

   f is one-to-one: Assume $f(k_1)=f(k_2)$. Then this common value is either even or odd. If it is even then $f(k_1)=-2k_1$ and $f(k_2)=-2k_2$. We have $-2k_1=-2k_2$, and hence $k_1=k_2$. If this common value is odd then $f(k_1)=2k_1-1$ and $f(k_2)=2k_2-1$. In this case $2k_1-1=2k_2-1$, hence $2k_1=2k_2$ and therefore $k_1=k_2$. In either case $k_1=k_2$, showing that f is one-to-one.

   f is onto: Let $n\in\mathbf{N}$. If n is even let $k=-n/2$. Then $k\le 0$, hence $f(k)=-2(-n/2)=n$. If n is odd then put $k=\dfrac{n+1}{2}$. Then $k>0$ and $f(k)=2\cdot\dfrac{n+1}{2}-1=n+1-1=n$. In either case there exists $k\in\mathbf{Z}$ such that $f(k)=n$, showing that f is onto. ///

2. Problem 1.19
   In each case say whether the function is one-to-one and whether it is onto.

   a. $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ be defined by $f(a,\ b)=(a+b, a-b)$.
   b. $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ be defined by $f(a,b)=(a+b, a-b)$.

   Solution:
   We can solve both parts simultaneously by noticing that the function f is a linear transformation of the plane given by matrix multiplication

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}a+b\\ a-b\end{pmatrix}$$

   The matrix $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ is invertible with inverse $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^{-1}=\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}$. The function f in part (b) is therefore invertible with inverse $f^{-1}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ given by

$$f^{-1}(x,\ y)=(\,(x+y)/2,\ (x-y)/2\,).$$

   Being invertible, the function in (b) is a bijection (both one-to-one and onto.) The function in (a) is nothing but the restriction of the function in (b) to the integer lattice $\mathbb{Z}\times\mathbb{Z}$. The restriction of a one-to-one function is one-to-one, so the function in (a) is one-to-one. It is not onto however since by the above formula $f^{-1}(1,\ 0) = (1/2,\ 1/2)$ which is not part of the integer lattice. Therefore there is no $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(a,\ b) = (1, 0)$. ///

3. Problem 1.20
Suppose A and B are sets and $f : A \to B$. For a subset S of A, we use the notation $f(S)$ to denote the set $\{ f(x) \mid x \in S\}$. Let S and T be subsets of A.

a. Is the set $f(S \cup T)$ a subset of $f(S) \cup f(T)$? If so, give a proof; if not, give a counterexample (i.e., say what the sets A, B, S, and T are and what the function f is).
b. Is the set $f(S) \cup f(T)$ a subset of $f(S \cup T)$? Give either a proof or a counterexample.
c. Repeat part (a) with intersection instead of union.
d. Repeat part (b) with intersection instead of union.
e. In each of the first four parts where your answer is no, what extra assumption on the function f would make the answer yes? Give reasons for your answer.

Solution

a. $f(S \cup T) \subseteq f(S) \cup f(T)$? **Yes**.
Proof: Let $y \in f(S \cup T)$. We must show that $y \in f(S) \cup (T)$. Since $y \in f(S \cup T)$, there exists $x \in S \cup T$ such that $y = f(x)$. Either $x \in S$ or $x \in T$. If $x \in S$ then $y = f(S)$. If $x \in T$ then $y = f(T)$. Thus either $y \in f(S)$ or $y \in f(T)$, and hence $y \in f(S) \cup (T)$. We've shown that $f(S \cup T) \subseteq f(S) \cup f(T)$, as required.
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b. $f(S) \cup f(T) \subseteq f(S \cup T)$? **Yes**.
Proof: Let $y \in f(S) \cup f(T)$. We must show that $y \in f(S \cup T)$. Since $y \in f(S) \cup f(T)$, either $y \in f(S)$ or $y \in f(T)$. If $y \in f(S)$, then there exists $x_1 \in S$ such that $y = f(x_1)$. But $S \subseteq S \cup T$, so $x_1 \in S \cup T$, whence $y \in f(S \cup T)$. If on the other hand, $y \in f(T)$, then there exists $x_2 \in T$ such that $y = f(x_2)$. But again since $S \subseteq S \cup T$, we have $x_2 \in S \cup T$, whence $y \in f(S \cup T)$. In both cases we have $y \in f(S \cup T)$. We've shown that $f(S) \cup f(T) \subseteq f(S \cup T)$, as required.
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c. $f(S \cap T) \subseteq f(S) \cap f(T)$? **Yes**.
Proof: Let $y \in f(S \cap T)$. We must show that $y \in f(S) \cap f(T)$. Since $y \in f(S \cap T)$, there exists $x \in S \cap T$ such that $y = f(x)$. Since $x \in S \cap T$, we have $x \in S$ and $x \in T$, hence $y \in f(S)$ and $y \in f(T)$, showing that $y \in f(S) \cap f(T)$. We've shown that $f(S \cap T) \subseteq f(S) \cap f(T)$, as required.
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d. $f(S) \cap f(T) \subseteq f(S \cap T)$? **No**.
Counter example: Let $A = \{-1, 0, 1\}$, $B = \{0, 1\}$ and define $f: A \to B$ by $f(x) = x^2$. Let $S = \{0, 1\}$ and $T = \{0, -1\}$. Then $\{0, 1\} = f(S) = f(T) = f(S) \cap f(T)$. Also $f(S \cap T) = f(\{0\}) = \{0\}$. But $\{0, 1\} \nsubseteq \{0\}$ so $f(S) \cap f(T) \nsubseteq f(S \cap T)$.
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e. If the function $f: A \to B$ is injective, then the answer in (d) is **Yes**.
Proof: Assume $f: A \to B$ is injective and let $y \in f(S) \cap f(T)$. We must show that $y \in f(S \cap T)$. We have that $y \in f(S)$ and $y \in f(T)$. Since $y \in f(S)$, there exists $x_1 \in S$ such that $y = f(x_1)$. Since $y \in f(T)$, there exists $x_2 \in T$ such that $y = f(x_2)$. Therefore $f(x_1) = y = f(x_2)$. Since f is injective, this implies that $x_1 = x_2$. Let x denote this common value, i.e. $x = x_1 = x_2$. Thus $x = x_1 \in S$ and $x = x_2 \in T$, whence $x \in S \cap T$ and $y = f(x)$. Therefore $y \in f(S \cap T)$, showing that $f(S) \cap f(T) \subseteq f(S \cap T)$, as required.
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4. Problem 1.23
   In each case below, a relation on the set {1, 2, 3} is given. Of the three properties, reflexivity, symmetry, and transitivity, determine which ones the relation has. Give reasons.

   a. R = {(1, 3), (3, 1), (2, 2)}
   b. R = {(1, 1), (2, 2), (3, 3), (1, 2)}
   c. $R = \emptyset$

   Solution:

   a. R is **not reflexive**.
      Reason: $1R1$ and $3R3$ are both false, so it is not the case that xRx is true for all $x \in \{1, 2, 3\}$.
      R is **symmetric**.
      Reason: When x and y range independently over {1, 2, 3}, there are only three cases in which xRy is true. In the case $1R3$, we also have $3R1$, so the implication $1R3 \rightarrow 3R1$ is true. In the case $3R1$ we also have $1R3$, so the implication $3R1 \rightarrow 1R3$ is true as well. In the case $2R2$ we again have the true implication $2R2 \rightarrow 2R2$. If xRy is false then the implication $xRy \rightarrow yRx$ is automatically true. Thus $xRy \rightarrow yRx$ is true for all $x, y \in \{1, 2, 3\}$.
      R is **not transitive**.
      Reason: $1R3$ and $3R1$ are both true, so $1R3 \wedge 3R1$ is true. But $1R1$ is false, so the implication $1R3 \wedge 3R1 \rightarrow 1R1$ is false. Thus $xRy \wedge yRz \rightarrow xRz$ is not true for all $x, y, z \in \{1, 2, 3\}$.
   b. R **is reflexive**.
      Reason: $1R1$, $2R2$ and $3R3$ are all true.
      R is **not symmetric**.
      Reason: $1R2$ is true but $2R1$ is false.
      R is **transitive**.
      Reason: Of the 27 instances of the statement $xRy \wedge yRz \rightarrow xRz$, there are only 5 in which the statement $xRy \wedge yRz$ is true, namely $1R1 \wedge 1R1 \rightarrow 1R1$, $1R1 \wedge 1R2 \rightarrow 1R2$, $1R2 \wedge 2R2 \rightarrow 1R2$, $2R2 \wedge 2R2 \rightarrow 2R2$ and $3R3 \wedge 3R3 \rightarrow 3R3$, which are all true. The remaining implications are all true since in those cases, $xRy \wedge yRz$ is false.
   c. R is **reflexive**.
      Reason: The statement $\forall x \in \emptyset: xRx$ can be false only if there exists an $x \in \emptyset$ for which xRx is false. But obviously no such x exists in the empty set.
      R is **symmetric**.
      Reason: The statement $\forall x, y \in \emptyset: xRy \rightarrow yRx$ is false only if there exists $x, y \in \emptyset$ for which xRy is true and yRx is false. No such x, y exists in the empty set.
      R is **transitive**.
      Reason: The statement $\forall x, y, z \in \emptyset: xRy \wedge yRz \rightarrow xRz$ is false only if there exists $x, y, z \in \emptyset$ for which both xRy and yRz are true and xRz is false. No such x, y, z exist in the empty set.

5. Problem 1.25
   Each case below gives a relation on the set of all nonempty subsets of $\mathbb{N}$. In each case, say whether the relation is reflexive, whether it is symmetric, and whether it is transitive.

   a. R is defined by: ARB if and only if $A \subseteq B$.
   b. R is defined by: ARB if and only if $A \cap B = \emptyset$.
   c. R is defined by: ARB if and only if $1 \in A \cap B$.

   Solution:

a. <u>R is **reflexive**</u>.
Reason: $A \subseteq A$ for any $A \subseteq \mathbb{N}$.
<u>R is **not symmetric**</u>.
Reason: $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \nsubseteq \{1\}$.
<u>R is **transitive**</u>.
Reason: It is a general fact of set theory that $(A \subseteq B) \wedge (B \subseteq C) \rightarrow (A \subseteq C)$, however we provide a proof here for the sake of completeness. (The following is not necessary for full credit.)
Proof: Assume $(A \subseteq B) \wedge (B \subseteq C)$ we must show that $A \subseteq C$. Let $x \in A$ be chosen arbitrarily. We must now show that $x \in C$. Since $x \in A$ and $A \subseteq B$ we have that $x \in B$. Now since $B \subseteq C$ we have $x \in C$, as required.

b. <u>R is **not reflexive**</u>.
Reason: If A is a non-empty subset of $\mathbb{N}$, then $A \cap A = A \neq \emptyset$.
<u>R is **symmetric**</u>.
Reason: If $A \cap B = \emptyset$ then $B \cap A = A \cap B = \emptyset$.
<u>R is **not transitive**</u>.
Reason: Let $A = \{1\}$, $B = \{2\}$, $C = \{1,3\}$. Then $A \cap B = \emptyset$ and $B \cap C = \emptyset$ but $A \cap C \neq \emptyset$.

c. <u>R is **not reflexive**</u>.
Reason: $1 \notin \{2\} = \{2\} \cap \{2\}$ so $\{2\}R\{2\}$ is false, hence $\forall A \subseteq \mathbb{N}$: ARA is false.
<u>R is **symmetric**</u>.
Reason: Since $A \cap B = B \cap A$, we have $1 \in A \cap B$ implies $1 \in B \cap A$.
<u>R is **transitive**</u>.
Reason: Suppose $1 \in A \cap B$ and $1 \in B \cap C$, so $1 \in A$, $1 \in B$ and $1 \in C$. Therefore $1 \in A \cap C$.

6. Problem 1.26
Let *R* be a relation on a set *S*. Write three quantified statements (the domain being *S* in each case), which say, respectively, that *R* is not reflexive, *R* is not symmetric, and *R* is not transitive.

Solution:
<u>R is not reflexive:</u> $\exists x \in S$: $\neg xRx$ (here $\neg$ is the logical negation operator.)
<u>R is not symmetric:</u> $\exists x, y \in S$: $xRy \wedge \neg yRx$.
<u>R is not transitive:</u> $\exists x, y, z \in S$: $xRy \wedge yRz \wedge \neg xRz$.

7. Problem 1.27
Suppose *S* is a nonempty set, $A = 2^S$, and the relation *R* on *A* is defined as follows: For every *X* and every *Y* in *A*, *XRY* if and only if there is a bijection from *X* to *Y*.

a. Show that *R* is an equivalence relation.
b. If *S* is a finite set with *n* elements, how many equivalence classes does the equivalence relation *R* have?
c. Again assuming that *S* is finite, describe a function $f: A \rightarrow N$ so that for every *X* and *Y* in *A*, *XRY* if and only if $f(X) = f(Y)$.

Solution:
a. <u>R is reflexive:</u>
Reason: Let $X \subseteq S$. Then there is a bijection from *X* to *X*, namely the identity mapping. Thus XRX for any $X \subseteq S$.
<u>R is symmetric:</u>
Reason: If $g: X \rightarrow Y$ is a bijection, then g is invertible and its inverse $g^{-1}: Y \rightarrow X$ is also a bijection. (See remarks at the end of this document.)

R is transitive:
Reason: If $g: X \to Y$ and $h: Y \to Z$ are bijections, then so is their composition $h \circ g: X \to Z$. (See remarks at the end of this document.)

b. R has $n+1$ equivalence classes.
Proof: By part (c) below, XRY if and only X and Y have the same number of elements. Therefore the equivalence classes of R are exactly: {0-element subsets of S}, {1-element subsets of S}, {2-element subsets of S},…..,{n-element subsets of S}. Therefore R has $n+1$ equivalence classes.

c. Define $f: A \to N$ by $f(X) = |X| =$ the number of elements in X.
Reason: If $g: X \to Y$ is a bijection, then each element in X corresponds, via g, to exactly one element in Y, and each element in Y corresponds, via g^{-1}, to exactly one element in X. This is exactly what it means for X and Y to have the same number of elements. Thus that XRY if and only if $|X| = |Y|$, and hence XRY if and only if $f(X) = f(Y)$.

8. Problem 1.28
Suppose A and B are sets, $f : A \to B$ is a function, and R is the relation on A so that for $x, y \in A$, xRy if and only if $f(x) = f(y)$.

a. Show that R is an equivalence relation on A.
b. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, $B = N$, and $f(x) = (x-3)^2$ for every $x \in A$, how many equivalence classes are there, and what are the elements of each one?
c. Suppose A has p elements and B has q elements. If the function f is one-to-one (not necessarily onto), how many equivalence classes does the equivalence relation R have? If the function f is onto (not necessarily one-to-one), how many equivalence classes does R have?

Solution:

a. R is reflexive.
Reason: For any $x \in A$ we have $f(x) = f(x)$.
R is symmetric.
Reason: For any $x, y \in A$ we have $f(x) = f(y)$ implies $f(y) = f(x)$.
R is transitive.
Reason: For any $x, y, z \in A$, if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$.

b. R has 6 equivalence classes: $\{0, 6\}, \{1, 5\}, \{2, 4\}, \{3\}, \{7\}, \{8\}$.
Proof: Let $x \in A$ and suppose $f(x) = m$. The equivalence class of x is the set $[x] = \{ y \in A \mid f(y) = m\}$. This set is usually called the *inverse image* of m under f, and is denoted $f^{-1}(m)$. (Note this notation does not require that f be invertible.) Therefore the equivalence classes of this relation are the sets $f^{-1}(m)$ where $m \in \text{range}(f)$. One readily checks that $f^{-1}(0) = \{3\}$, $f^{-1}(1) = \{2, 4\}$, $f^{-1}(4) = \{1, 5\}$, $f^{-1}(9) = \{0, 6\}$, $f^{-1}(16) = \{7\}$ and $f^{-1}(25) = \{8\}$.

c. If $f: A \to B$ is one-to-one, where $|A| = p$ and $|B| = q$, then R has p equivalence classes.
Proof: Since f is one-to-one, each inverse image under f contains exactly one element, hence the equivalence classes are the singleton subsets of A, of which there are p.
If $f: A \to B$ is onto, where $|A| = p$ and $|B| = q$, then R has q equivalence classes.
Proof: Since f is onto, $\text{range}(f) = B$, hence for each $b \in B$ there is exactly one equivalence class $f^{-1}(b)$. The number of such equivalence classes is therefore q.

9. Problem 1.30
For a positive integer n, find a function $f : N \to N$ so that the equivalence relation $\equiv_n$ on N can be described as in Exercise 1.28.

Solution:
Define $f: N \to N$ by $f(k) =$ the remainder of k on (integer) division by n. It was mentioned in class (and proved below, though that proof is not necessary for full credit) that $f(k_1) = f(k_2)$ if and only if $k_1 \equiv_n k_2$, as required.

Remarks
The following facts from CMPE 16 may be used without proof in this and subsequent assignments.

- A function $f: A \to B$ is a bijection if and only if it is *invertible*, i.e. if and only if there exists a function $g: B \to A$ such that $g \circ f(x) = x$ for all $x \in A$, and $f \circ g(y) = y$ for all $y \in B$. The function g is usually denoted f^{-1} and called the *inverse* of *f*.
- If f is a bijection, then so is f^{-1}, and in fact $(f^{-1})^{-1} = f$.
- The composition of two bijections is a bijection. To see this suppose $f: A \to B$ and $g: B \to C$ are bijections. Then $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \text{identity}_{B \to B} \circ g^{-1} = g \circ g^{-1}$ $= \text{identity}_{C \to C}$. Similarly $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{identity}_{A \to A}$. Hence $g \circ f$ is invertible, with inverse $f^{-1} \circ g^{-1}$, and therefore $g \circ f$ is a bijection.
- $a \equiv_n b$ if and only if *a* and *b* have the same remainder upon division by *n*.
  Proof: First suppose $a \equiv_n b$. Then the difference $a - b$ is divisible by *n*. This means $a - b = kn$ for some $k \in \mathbb{Z}$. Divide both a and b by n to get quotients and remainders: $a = q_1 n + r_1$ and $b = q_2 n + r_2$ where both $0 \leq r_1 < n$ and $0 \leq r_2 < n$. At most one of the numbers $r_1 - r_2$ and $r_2 - r_1$ is negative, hence at least one is non-negative. Suppose for definiteness that $r_1 - r_2 \geq 0$ (the other case being similar, and we omit it.) Together with the previous inequalities we get $0 \leq r_1 - r_2 < n$. Also

  $$kn = a - b = (q_1 n + r_1) - (q_2 n + r_2) = (q_1 - q_2)n + (r_1 - r_2)$$

  and hence $r_1 - r_2 = (k - q_1 + q_2)n$, showing that $r_1 - r_2$ is divisible by *n*. But this, together with $0 \leq r_1 - r_2 < n$ implies that $r_1 - r_2 = 0$, whence $r_1 = r_2$. We've shown that $a \equiv_n b$ implies that *a* and *b* have the same remainder when divided by *n*. For the converse, assume *a* and *b* have the same remainder upon division by *n*. Then $a = q_1 n + r$ and $b = q_2 n + r$ where $0 \leq r < n$. Therefore $a - b = (q_1 n + r) - (q_2 n + r) = (q_1 - q_2)n$, showing $a - b$ is divisible by *n*, and hence $a \equiv_n b$.