

1. (1 Point) The last exercise in the handout entitled *Some Common Functions*.

Use Stirling's formula to prove that  $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$ .

**Proof:** By Stirling's formula

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot (1 + \Theta(1/2n))}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot (1 + \Theta(1/n))\right)^2} \\ &= \frac{2^{2n}}{\sqrt{\pi n}} \cdot \frac{1 + \Theta(1/2n)}{(1 + \Theta(1/n))^2} = \frac{1}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot \frac{1 + \Theta(1/2n)}{(1 + \Theta(1/n))^2}\end{aligned}$$

so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1 + \Theta(1/2n)}{(1 + \Theta(1/n))^2} \rightarrow \frac{1}{\sqrt{\pi}} \quad \text{as } n \rightarrow \infty$$

The result now follows since  $0 < \frac{1}{\sqrt{\pi}} < \infty$ .

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2. (2 Points) (Exercise 1 from the induction handout)

Prove that for all  $n \geq 1$ :  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ . Do this twice:

- (1 Point) using form IIa of the induction step.
- (1 Point) using form IIb of the induction step.

**Proof:** Let  $P(n)$  be the equation  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ .

I. Observe that  $\sum_{i=1}^1 i^3 = 1^3 = 1^2 = \left(\frac{1 \cdot (1+1)}{2}\right)^2$ , whence  $P(1)$  is true.

IIa. Let  $n \geq 1$  and assume  $P(n)$  is true, i.e. for this  $n$ , we assume that  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ . We must

show that  $P(n+1)$  holds:  $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$ . Thus

$$\begin{aligned}\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^n i^3 + (n+1)^3 \\ &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad (\text{by the induction hypothesis})\end{aligned}$$

$$\begin{aligned}
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2[n^2 + 4n + 4]}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

showing that  $P(n+1)$  is true. ///

IIb. Let  $n > 1$  and assume  $P(n-1)$  is true, i.e. for this  $n$ , we assume that  $\sum_{i=1}^{n-1} i^3 = \left(\frac{(n-1)n}{2}\right)^2$ . We

must show that  $P(n)$  holds:  $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ . Thus

$$\begin{aligned}
\sum_{i=1}^n i^3 &= \sum_{i=1}^{n-1} i^3 + n^3 \\
&= \left(\frac{(n-1)n}{2}\right)^2 + n^3 && \text{(by the induction hypothesis)} \\
&= \frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{n^2[(n-1)^2 + 4n]}{4} \\
&= \frac{n^2[n^2 + 2n + 1]}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

showing that  $P(n)$  is true. ///

### 3. (1 Point) Exercise 2 from the induction handout)

Define  $S(n)$  for  $n \in \mathbb{Z}^+$  by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that  $S(n) \geq \lg(n)$  for all  $n \geq 1$ , and hence  $S(n) = \Omega(\lg n)$ .

**Proof:** Let  $P(n)$  be the inequality  $S(n) \geq \lg(n)$ .

I. The inequality  $S(1) \geq \lg(1)$  reduces to  $0 \geq 0$ , which is obviously true, so  $P(1)$  holds.

II. Let  $n > 1$  and assume for all  $k$  in the range  $1 \leq k < n$  that  $S(k) \geq \lg(k)$ . Then

$$\begin{aligned}
S(n) &= S(\lceil n/2 \rceil) + 1 && \text{(by the definition of } S(n) \text{)} \\
&\geq \lg \lceil n/2 \rceil + 1 && \text{(by the induction hypothesis with } k = \lceil n/2 \rceil \text{)} \\
&\geq \lg(n/2) + 1 && \text{(since } \lceil x \rceil \geq x \text{ for any } x \text{)} \\
&= \lg(n) - \lg(2) + 1 \\
&= \lg(n)
\end{aligned}$$

showing that  $P(n)$  holds. Therefore  $S(n) \geq \lg(n)$  for all  $n \geq 1$ , as claimed. ///

4. (1 Point)

Let  $f(n)$  be a positive, increasing function that satisfies  $f(n/2) = \Theta(f(n))$ . Show that

$$\sum_{i=1}^n f(i) = \Theta(nf(n))$$

(Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that  $\sum_{i=1}^n i^k = \Theta(n^{k+1})$  for any positive integer  $k$ .)

**Proof:** Since  $f(n)$  is increasing we have  $\sum_{i=1}^n f(i) \leq \sum_{i=1}^n f(n) = nf(n) = O(nf(n))$ . Note also that

$$\begin{aligned} \sum_{i=1}^n f(i) &\geq \sum_{i=\lceil n/2 \rceil}^n f(i) && \text{by discarding some positive terms} \\ &\geq \sum_{i=\lceil n/2 \rceil}^n f(\lceil n/2 \rceil) && \text{since } f(n) \text{ is increasing} \\ &= (n - \lceil n/2 \rceil + 1)f(\lceil n/2 \rceil) && \text{by counting terms} \\ &= (\lfloor n/2 \rfloor + 1)f(\lceil n/2 \rceil) && \text{since } n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \\ &> ((n/2) - 1 + 1)f(n/2) && \text{since } f(n) \text{ is increasing, } \lceil x \rceil \geq x, \text{ and } \lfloor x \rfloor > x - 1 \\ &= (n/2)f(n/2) \\ &= \Omega(nf(n)) && \text{since } f(n/2) = \Theta(f(n)), \text{ whence } f(n/2) = \Omega(f(n)) \end{aligned}$$

It follows from an exercise in the handout on Asymptotic Growth rates that  $\sum_{i=1}^n f(i) = \Theta(nf(n))$ , as claimed. ///

5. (1 Point)

Let  $T(n)$  be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that  $\forall n \geq 1: T(n) \leq \frac{4}{3}n^2$ , and hence  $T(n) = O(n^2)$ . (Hint: follow example 3 on page 3 of the handout on induction proofs.)

**Proof:** Let  $P(n)$  be the statement  $T(n) \leq (4/3)n^2$ . Then  $P(1)$  is true, since  $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$ , and the base case is satisfied. Let  $n > 1$  be chosen arbitrarily, and suppose for all  $k$  in the range  $1 \leq k < n$  that  $T(k) \leq (4/3)k^2$ . We must show as a consequence that  $T(n) \leq (4/3)n^2$ . Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. ///