Midterm 1

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CMPS 130 Spring — Tantalo

Problem 1. Write down all relations on the set $A = \{a, b\}$ and determine which of them are reflexive, symmetric or transitive.

$R = \emptyset$ symmetric, transitive
$R = \{(a, a)\}$ symmetric, transitive
$R = \{(a, b)\}$ transitive
$R = \{(b, a)\}$ transitive
$R = \{(b, b)\}$ symmetric, transitive
$R = \{(a, a), (a, b)\}$ transitive
$R = \{(a, a), (b, a)\}$ transitive
$R = \{(a, a), (b, b)\}$ reflexive, symmetric, transitive
$R = \{(a, b), (b, a)\}$ symmetric
$R = \{(a, b), (b, b)\}$ transitive
$R = \{(b, a), (b, b)\}$ transitive
$R = \{(a, a), (a, b), (b, a)\}$ symmetric
$R = \{(a, a), (a, b), (b, b)\}$ reflexive, transitive
$R = \{(a, a), (b, a), (b, b)\}$ reflexive, transitive
$R = \{(a, b), (b, a), (b, b)\}$ symmetric
$R = \{(a, a), (a, b), (b, a), (b, b)\}$ reflexive, symmetric, transitive

Problem 2. Let $f : A \to B$ and define a relation R on A by: xRy if and only if $f(x) = f(y)$.

a. Prove that R is an equivalence relation.

   Proof. R is reflexive:
   Reason: For any $x \in A$ we have $f(x) = f(x)$.
   R is symmetric:
   Reason: For any $x, y \in A$ we have $f(x) = f(y)$ implies $f(y) = f(x)$.
   R is transitive:
   Reason: For any $x, y, z \in A$ if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$. $\square$

b. What is $[x]$ under this relation?
   Let $x \in A$. The equivalence class of x is the set $[x] = \{y \in A | f(y) = f(x)\}$.

c. Let $A = B = \mathbf{R}$ and define $f : R \to R$ by $f(x) = x(x - 1)(x + 1)$. Determine $[0]$.
   $[0] = \{-1, 0, 1\}$

Problem 3. Let $L_1 = \{ab, ba\}$ and $L_2 = \{\lambda, aab, bab\}$.

a. Determine L_1L_2.
$L_1L_2 = \{ab, ba, abaab, abbab, baaab, babab\}$

b. Determine L_1^3.
$L_1^2 = \{abab, abba, baab, baba\}$
$L_1^3 = \{ababab, ababba, abbaab, abbaba, baabab, baabba, babaab, bababa\}$

Problem 4. Show that if $L_1L = L$ for every $L \subseteq \{a, b\}^*$, then $L_1 = \{\lambda\}$.

Proof. Assume $L_1L = L$ for all $L \subseteq \{a, b\}^*$. In particular $L_1\{\lambda\} = \{\lambda\}$. Pick an arbitrary $x \in L_1$. Then $x\lambda$, being a member of $L_1\{\lambda\}$, is also a member of $\{\lambda\}$. Therefore $x\lambda = \lambda$, hence $x = \lambda$. Since x was arbitrary, we've shown that $x \in L_1 \implies x = \lambda$, and thus $L_1 = \{\lambda\}$. □

Problem 5. Let S be a set. Prove that no function $f : S \to 2^S$ is onto. (Note 2^S denotes the power set of S, i.e. the set of all subsets of S.)

Proof. Suppose $f : S \to 2^S$ is onto. Define $A \subseteq S$ by

$$A = \{x \in S | x \notin f(x)\}$$

Since $A \in 2^S$ and A is onto there must exist $y \in S$ such that $f(y) = A$.
Either $y \in A$ or $y \notin A$. If $y \in A$ then $y \in f(y)$, hence $y \notin A$. =><=
If $y \notin A$ then $y \notin f(y)$, hence $y \in A$. =><= □

Problem 6. Recall the language $Expr \subseteq \{a, (,), +, \cdot\}^*$ was defined recursively as

(1) $a \in Expr$

(2) If $x \in Expr$, then $(x) \in Expr$.

(3) If $x, y \in Expr$, then $x + y \in Expr$.

(4) If $x, y \in Expr$, then $x \cdot y \in Expr$.

Prove that $\forall x \in Expr : |x|$ is odd. (Hint: use structural induction on x.)

Proof. For every element x of $Expr$, $|x|$ is odd.
Basis step. We wish to show that $|a|$ is odd. This is true because $|a| = 1$.
Induction hypothesis. x and y are in *Expr*, and $|x|$ and $|y|$ are odd.
Statement to be proved in the induction step. $|x + y|$, $|x \cdot y|$, and $|(x)|$ are odd.
Proof of induction step. The numbers $|x + y|$ and $|x \cdot y|$ are both $|x| + |y| + 1$, because the symbols of $x + y$ include those in x, those in y, and the additional operator symbol. The number $|(x)|$ is $|x| + 2$, because two parentheses have been added to the symbols of x. The first number is odd because the induction hypothesis implies that it is the sum of two odd numbers plus 1, and the second number is odd because the induction hypothesis implies that it is an odd number plus 2. □

Problem 7. Recall the language $Bal \subseteq \{(,)\}^*$.

a. Write down the recursive definition of Bal.

1. $\lambda \in Bal$
2. $x, y \in Bal \rightarrow xy \in Bal$
3. $x \in Bal \rightarrow (x) \in Bal$

b. Prove that $\forall x \in Bal : |x|$ is even.

Proof. $\forall x \in Bal : |x|$ is even

I. $|\lambda|$ is even

II. if $y, z \in Bal$ and $|y|$, $|z|$ are even, then
$|yz| = |y| + |z|$ is even
and $|(y)| = 1 + |y| + 1$ is even
$\therefore x \in Bal \implies |x|$ is even.

$\square$

c. Prove that $\forall x \in Bal : n_((x) = n_)(x)$.

Proof. $n_((x) = n_)(x)$

I. $n_((\lambda) = 0 = n_)(\lambda)$

II. Assume $n_((x) = n_)(x)$ true for some $x, y \in Bal$. We must show

- $n_((x) = n_)(x)$ holds for xy
- $n_((x) = n_)(x)$ holds for (x)

Since $n_((x) = n_)(x)$ we have $n_(((x)) = 1 + n_((x) = 1 + n_)(x) = n_)((x))$ (see page 31)

Induction hypothesis. $k \in \mathbb{N}$, and for every string x of parentheses, if $|x| \leq k$ and $B(x)$, then $x \in$ *Balanced*. (Writing for every string x of parentheses, if $|x| \leq k$ says the same thing as for every $m \leq k$, and every string x of parentheses with $|x| = m$ but involves one fewer variable and sounds a little simpler.)

Statement to be proved in induction step. For every string x of parentheses, if $|x| = k + 1$ and $B(x)$, then $x \in$ *Balanced*.

Proof of induction step. We suppose that x is a string of parentheses with $|x| = k + 1$ and $B(x)$. We must show that $x \in$ *Balanced*, which means that x can be obtained from statement 1 or statement 2 in the definition. Since $|x| > 0$, statement 1 wont help; we must show x can be obtained from statement 2. The trick here is to look at the two cases in statement 2 and work backward. If we want to show that $x = yz$ for two shorter strings y and z in *Balanced*, the way to do it is to show that $x = yz$ for two shorter strings y and z satisfying $B(y)$ and $B(z)$; for then the induction hypothesis will tell us that y and z are in *Balanced*. However, this may not be possible, because the statements $B(y)$ and $B(z)$ require that y and z both have equal numbers of left and right parentheses. The string ((())), for example, cannot be expressed as a concatenation like this. We must show that these other strings can be obtained from statement 2. It seems reasonable, then, to consider two cases. Suppose first that $x = yz$, where y and z are both shorter than x and have equal numbers of left and right parentheses. No prefix of y can have more right parentheses than left, because every prefix of y is a prefix of x and $B(x)$ is true. Because y has equal numbers of left and right, and because no prefix of x can have more right than left, no prefix of z can have more right than left. Therefore, both the statements $B(y)$ and $B(z)$ are true. Since $|y| \leq k$ and $|z| \leq k$, we can apply the induction hypothesis to both strings and conclude that y and z are both elements of Balanced. It follows from statement 2 of the definition that $x = yz$ is also. In the other case, we assume that $x = (y)$ for some string y of parentheses and that x cannot be written as a concatenation of shorter strings with equal numbers of left and right parentheses. This second assumption is useful, because it tells us that no prefix of y can have more right parentheses than left. (If some prefix did, then some prefix y_1 of y would have exactly one more right than left, which would mean that the prefix $(y_1$ of x had equal numbers; but this would contradict the assumption.) The string y has equal numbers of left and right parentheses, because x does, and so the statement $B(y)$ is true. Therefore, by the induction hypothesis, $y \in \mathit{Balanced}$, and it follows from statement 2 that $x \in \mathit{Balanced}$.

□

d. Prove that $\forall x \in Bal$: if z is any prefix of x, then $n_)(z) \leq n_((z)$

Proof. if z is a prefix of x, then $n_)(z) \leq n_((z)$ i.e. no prefix has more right than left parens.

I. λ is the only prefix of λ, so $n_)(z) \leq n_((z)$ reduces to $0 \leq 0$.

II. Assume $n_)(z) \leq n_((z)$ true for some $x, y \in Bal$. We must show

- $n_)(z) \leq n_((z)$ holds for xy
- $n_)(z) \leq n_((z)$ holds for (x)

Book proves 1st bullet. (P. 31) We do 2nd bullet which is part (3) of recursion.
(a) Since $n_((x) = n_)(x)$ (ind. hyp.) we have $n_(((x)) = 1 + n_)(x) = n_)((x))$
(b) Let z be a prefix of (x). We must show
$n_)(z) \leq n_((z)$.
If $z = \lambda$ inequality is $0 \leq 0$.
if $z = (x)$ then $n_)(z) = n_((z)$ by Part (a). Otherwise $z = (w$ where w is a prefix of x. By ind. hyp. on x we have $n_)(w) \leq n_((w)$. Thus
$n_)(z) = n_)(w) \leq n_((w) < 1 + n_((w) = n_((z)$.

□

Problem 8. Let Σ be a finite alphabet.

a. Write down the recursive definition of Σ^*.

1. $\lambda \in \Sigma^*$
2. $x \in \Sigma^*$ and $\sigma \in \Sigma$, then $x\sigma \in \Sigma^*$

b. Write down the recursive definition of the reversal function $r : \Sigma^* \to \Sigma^*$.

1. $r(\lambda) = \lambda$
2. $\forall x, \sigma \in \Sigma^* \to r(x\sigma) = \sigma r(x)$

c. Prove that $\forall x, y \in \Sigma^* : r(xy) = r(y)r(x)$.

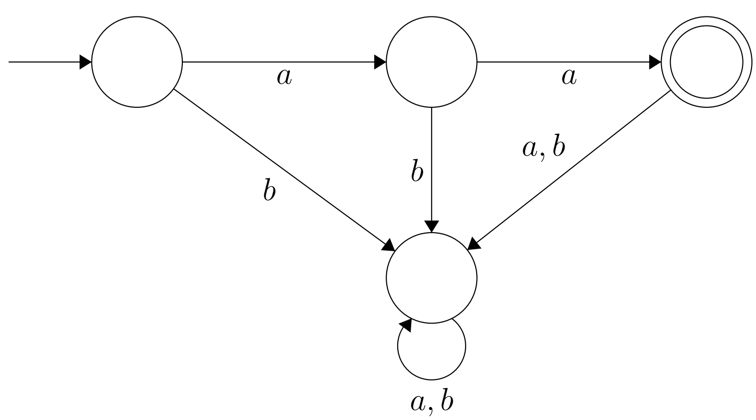
Proof. $\forall x \forall y \in \Sigma^* : r(xy) = r(y)r(x) = P(x)$

I. $P(\lambda)$ says: $\forall y : (y\lambda)^r = \lambda^r y^r$
This is true since $(y\lambda)^r = y^r = \lambda y^r = \lambda^r y^r$ (rec. defn. of r pt.(1))

II. Assume $P(x)$ is true for some $x \in \{a,b\}^*$, i.e. for this $x : \forall y : (yx)^r = x^r y^r$
We must show $P(x\sigma)$ is true for any $\sigma \in \{a,b\}$, i.e. we must show
$\forall y : (y(x\sigma))^r = (x\sigma)^r y^r$.
Let $y \in \{a,b\}^*$. Then
$(y(x\sigma))^r = ((yx)\sigma)^r$ assoc.
$= \sigma(yx)^r$ defn. of r
$= \sigma(x^r y^r)$ ind. hyp.
$= (\sigma x^r)y^r$ assoc.
$= (x\sigma)^r y^r$ defn. of r

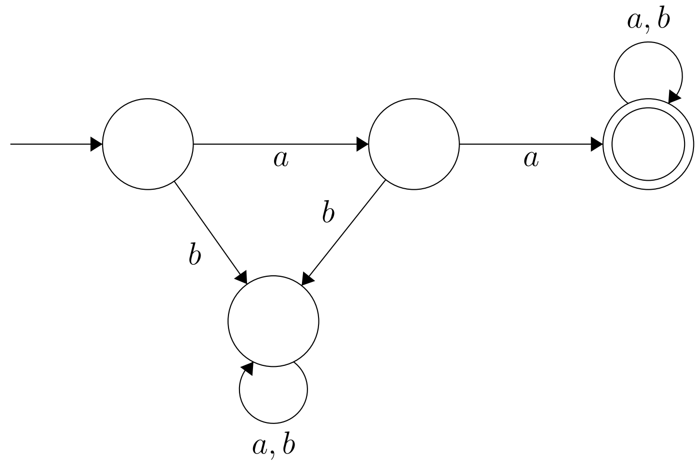
$\therefore P(x\sigma)$ is true. We conclude $P(x)$ for all $x \in \{a,b\}^*$, i.e. $\forall x \forall y (yx)^r = x^r y^r$. □

Problem 9. Write a simple description of the language $L \subseteq \{a,b\}^*$ accepted by the following finite automata. No justification is required.



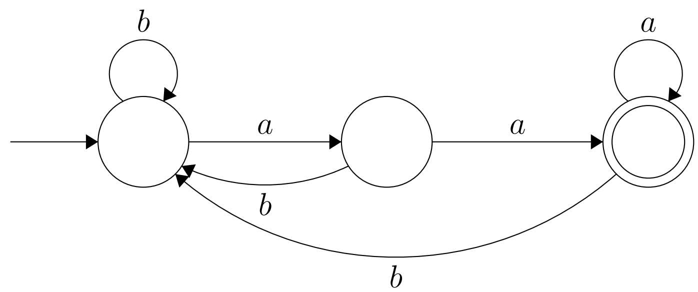
a.

The language of containing only the string aa.



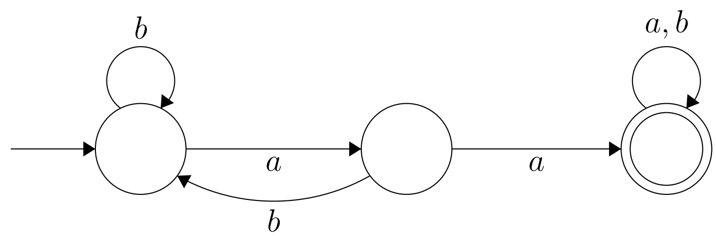
b.

The language of all strings beginning with the string aa.



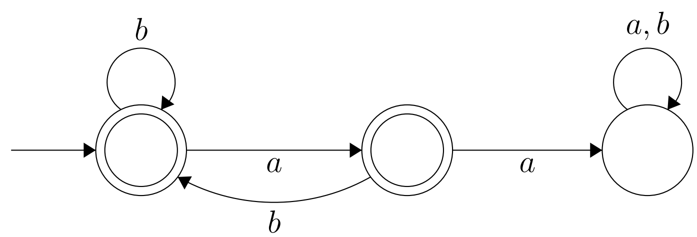
c.

The language of all strings ending with the string aa.



d.

The language of all strings containing the string aa.

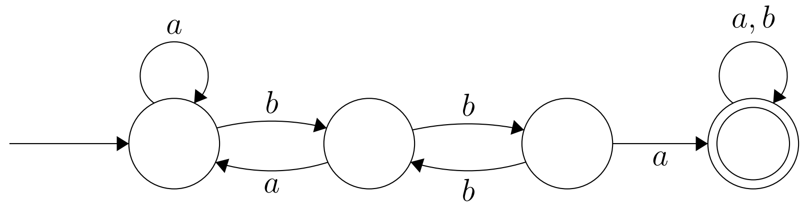


e.

The language of all strings not containing the string aa.

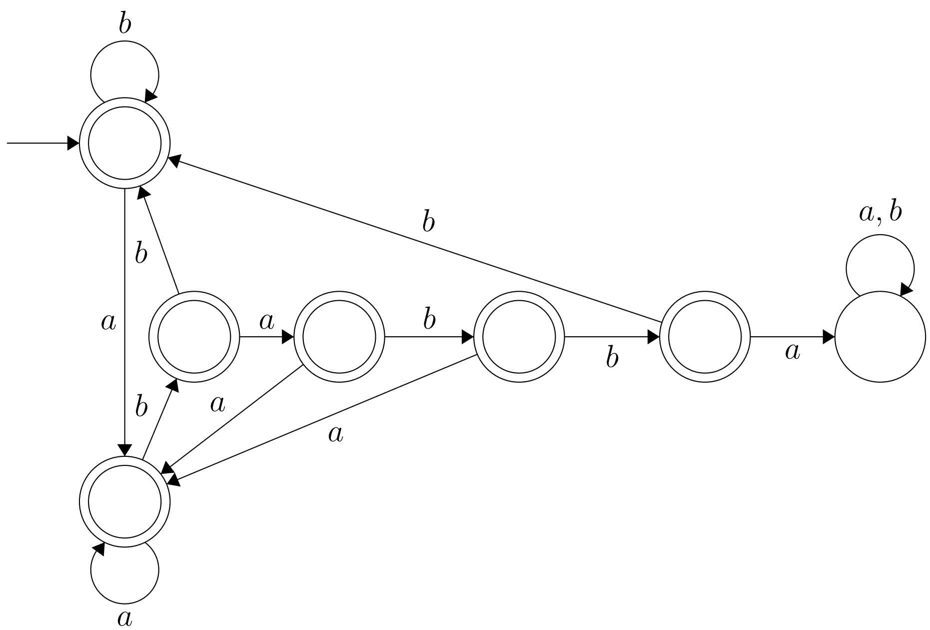
Problem 10. Draw a state transition diagram for a DFA accepting the language

$$L = \{x \in \{a, b\}^* | x \text{ contains the substring } bba\}$$



Problem 11. . Draw a state transition diagram for a DFA accepting the language

$$L = \{x \in \{a,b\}^* | x \text{ does not contain the substring } ababba\}$$



Problem 12. Given a DFA $M = (Q, \Sigma, q_0, A, \delta)$

a. Write the recursive definition of the extended transition function $\delta^* : Q \times \Sigma^* \to Q$.

1. $\forall q \in Q, \delta^*(q, \lambda) = q$
2. $\forall q \in Q, y \in \Sigma^*$ and $\sigma \in \Sigma^*$

$$\delta^*(q, y\sigma) = \delta(\delta^*(q, y), \sigma)$$

b. Prove that $\delta^*(q, \sigma) = \delta(q, \sigma)$ for any $q \in Q$ and $\sigma \in \Sigma$.

Proof. $\delta^*(q, \sigma) = \delta(q, \sigma)$

$$\begin{aligned} \delta^*(q,\sigma) &= \delta^*(q, \lambda\sigma) && (\sigma = \lambda\sigma) \\ &= \delta(\delta^*(q,\lambda), \sigma) && (\text{definition of } \delta^*) \\ &= \delta(q, \sigma) && (\delta^*(q,\lambda) = q) \end{aligned}$$

Therefore $\delta^*(q, \sigma) = \delta(q, \sigma)$ for any $q \in Q$ and $\sigma \in \Sigma$. $\square$

c. Prove that for all $x, y \in \Sigma^*$ and $q \in Q : \delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$

Proof. $P(y) = \delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$

I. $P(\lambda) = \forall y : \delta^*(q, y\lambda) = \delta^*(\delta^*(q, \lambda,\text{y})$
$\delta^*(\delta^*(q, x), \lambda) = \delta^*(q, x) = \delta^*(q, x\lambda)$

II. Assume $P(y)$ holds for some $y \in \Sigma^*$, we must show $\forall x | \delta^*(q, x(y\sigma)) = \delta^*(\delta^*(q, x), y\sigma)$

Let $x \in \Sigma^*$

$$\begin{aligned}
\delta^*(q, x(y\sigma)) &= \delta^*(q, (xy)\sigma) \\
&= \delta(\delta^*(q, xy), \sigma) \\
&= \delta(\delta^*(\delta^*(q, x), y), \sigma) \\
&= \delta(\delta^*(q, x), y\sigma)
\end{aligned}$$

Therefore $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$. $\square$