CMPS 101
Homework Assignment 3 **Solutions**

1. The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

   Proof: By Stirling's formula

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta(1/2n)\right)}{\left(\sqrt{2\pi\, n}\cdot\left(\frac{n}{e}\right)^{n}\cdot\left(1+\Theta(1/n)\right)\right)^2}$$

$$= \frac{2^{2n}}{\sqrt{\pi\, n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} = \frac{1}{\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}$$

so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad n\to\infty$$

The result now follows since $0 < \frac{1}{\sqrt{\pi}} < \infty$. ///

2. Exercise 1 from the induction handout

   Prove that for all $n \ge 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

   a. Using form **I**Ia of the induction step.
   b. Using form IIb of the induction step.

   Proof: Let $P(n)$ be the equation $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

   I. Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\frac{1\cdot(1+1)}{1}\right)^2$, whence $P(1)$ is true.

   IIa. Let $n \ge 1$ and assume $P(n)$ is true, i.e. for this n, we assume that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $P(n+1)$ holds: $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$. Thus

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{(by the induction hypothesis)}$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

showing that $P(n+1)$ is true. ///

IIb. Let $n>1$ and assume $P(n-1)$ is true, i.e. for this *n*, we assume that $\sum_{i=1}^{n-1} i^3 = \left(\frac{(n-1)n}{2}\right)^2$. We must show that $P(n)$ holds: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Thus

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n-1} i^3 + n^3$$

$$= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \qquad \text{(by the induction hypothesis)}$$

$$= \frac{(n-1)^2n^2+4n^3}{4} = \frac{n^2\left[(n^2-2n+1)+4n\right]}{4}$$

$$= \frac{n^2\left[n^2+2n+1\right]}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

showing that $P(n)$ is true. ///

3. Exercise 2 from the induction handout
   Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n=1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

Proof: Let $P(n)$ be the inequality $S(n) \geq \lg(n)$.

I. The inequality $S(1) \geq \lg(1)$ reduces to $0 \geq 0$, which is obviously true, so $P(1)$ holds.

IId. Let $n>1$ and assume for all *k* in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. Then

$$\begin{aligned} S(n) &= S(\lceil n/2 \rceil)+1 && \text{(by the definition of } S(n)\text{)} \\ &\geq \lg\lceil n/2 \rceil + 1 && \text{(by the induction hypothesis with } k = \lceil n/2 \rceil\text{)} \\ &\geq \lg(n/2)+1 && \text{(since } \lceil x \rceil \geq x \text{ for any } x\text{)} \\ &= \lg(n) - \lg(2) + 1 \\ &= \lg(n) \end{aligned}$$

showing that $P(n)$ holds. Therefore $S(n) \geq \lg(n)$ for all $n \geq 1$, as claimed. ///

4. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2)=\Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i)=\Theta(nf(n))$$

(Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k=\Theta(n^{k+1})$ for any positive integer k.)

Proof: Since $f(n)$ is increasing we have $\sum_{i=1}^{n} f(i) \;\le\; \sum_{i=1}^{n} f(n) \;=\; nf(n) \;=\; O(nf(n))$. Note also that

$$\begin{aligned}
\sum_{i=1}^{n} f(i) &\ge \sum_{i=\lceil n/2\rceil}^{n} f(i) && \text{by discarding some positive terms}\\
&\ge \sum_{i=\lceil n/2\rceil}^{n} f(\lceil n/2\rceil) && \text{since } f(n) \text{ is increasing}\\
&= (n-\lceil n/2\rceil+1)\, f(\lceil n/2\rceil) && \text{by counting terms}\\
&= (\lfloor n/2\rfloor+1)\, f(\lceil n/2\rceil) && \text{since } n=\lfloor n/2\rfloor+\lceil n/2\rceil\\
&> ((n/2)-1+1)\, f(n/2) && \text{since } f(n) \text{ is increasing, } \lceil x\rceil\ge x\text{, and } \lfloor x\rfloor> x-1\\
&= (n/2)\, f(n/2) &&\\
&= \Omega(nf(n)) && \text{since } f(n/2)=\Theta(f(n))\text{, whence } f(n/2)=\Omega(f(n))
\end{aligned}$$

It follows from an exercise in the handout on Asymptotic Growth rates that $\sum_{i=1}^{n} f(i)=\Theta(nf(n))$, as claimed. ///

5. Use the result of problem 4 above to give an alternate proof of $\log(n!)=\Theta(n\log(n))$ that does not use Stirling's formula.

Proof:
Observe that $\log(n/2)=\log(n)-\log(2)=\Theta(\log(n))$. We may therefore apply the result of problem 4 with $f(n)=\log(n)$, and properties of logarithms to get

$$\log(n!)=\sum_{i=1}^{n}\log(i)=\Theta(n\log(n))$$

as claimed. ///

6. Let $T(n)$ be defined by the recurrence formula

$$T(n) \;=\begin{cases} 1 & n=1\\ T(\lfloor n/2\rfloor)+n^2 & n\ge 2\end{cases}$$

Show that $\forall n\ge 1$: $T(n)\le \frac{4}{3}n^2$, and hence $T(n)=O(n^2)$. (Hint: follow example 3 on page 3 of the handout on induction proofs.)

Proof: Let $P(n)$ be the statement $T(n) \le (4/3)n^2$. Then $P(1)$ is true, since $T(1) = 1 \le 4/3 = (4/3) \cdot 1^2$, and the base case is satisfied. Let $n > 1$ be chosen arbitrarily, and suppose for all k in the range $1 \le k < n$ that $T(k) \le (4/3)k^2$. We must show as a consequence that $T(n) \le (4/3)n^2$. Observe

$$\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\
&\le (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\
&\le (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\
&= n^2/3 + n^2 \\
&= (4/3)n^2,
\end{aligned}$$

as required. ///