CMPS 101
Homework Assignment 4 Solutions

1. Consider the function $T(n)$ defined by the recurrence formula

$$T(n)=\begin{cases}6 & 1\le n<3\\ 2T(\lfloor n/3\rfloor)+n & n\ge 3\end{cases}$$

   a. Use the iteration method to write a summation formula for $T(n)$.

   Solution:

$$\begin{aligned}T(n)&=n+2T(\lfloor n/3\rfloor)\\ &=n+2(\lfloor n/3\rfloor+2T(\lfloor\lfloor n/3\rfloor/3\rfloor))\\ &=n+2\lfloor n/3\rfloor+2^2T(\lfloor n/3^2\rfloor)\\ &=n+2\lfloor n/3\rfloor+2^2\lfloor n/3^2\rfloor+2^3T(\lfloor n/3^3\rfloor)\quad \text{etc..}\end{aligned}$$

   After substituting the recurrence into itself k times, we get

$$T(n)\ =\ \sum_{i=0}^{k-1}2^i\left\lfloor\frac{n}{3^i}\right\rfloor\ +\ 2^kT(\lfloor n/3^k\rfloor).$$

   This process terminates when the recursion depth k is chosen so that $1\le\lfloor n/3^k\rfloor<3$, which is equivalent to $1\le n/3^k<3$, whence $3^k\le n<3^{k+1}$, so $k\le\log_3(n)<k+1$, and hence $k=\lfloor\log_3(n)\rfloor$. With this value of k we have $T(\lfloor n/3^k\rfloor)=T(1\,\text{or}\,2)=6$. Therefore

$$T(n)\ =\ \sum_{i=0}^{\lfloor\log_3(n)\rfloor-1}2^i\left\lfloor\frac{n}{3^i}\right\rfloor\ +\ 6\cdot 2^{\lfloor\log_3(n)\rfloor}.$$

   b. Use the summation in (a) to show that $T(n)=O(n)$

   Solution:

   Using the above summation, we have

$$\begin{aligned}T(n)&\le n\left(\sum_{i=0}^{\lfloor\log_3(n)\rfloor-1}(2/3)^i\right)+6\cdot 2^{\log_3(n)} && \text{since } \lfloor x\rfloor\le x \text{ for any } x\\ &\le n\left(\sum_{i=0}^{\infty}(2/3)^i\right)+6n^{\log_3(2)} && \text{adding } \infty\text{-many positive terms}\\ &=n\left(\frac{1}{1-(2/3)}\right)+6n^{\log_3(2)} && \text{by a well known formula}\\ &=3n+6n^{\log_3(2)}=O(n) && 2<3\Rightarrow\log_3(2)<1\Rightarrow n^{\log_3(2)}=o(n)\end{aligned}$$

   Therefore $T(n)=O(n)$.

   c. Use the Master Theorem to show that $T(n)=\Theta(n)$

   Solution:

Let $\varepsilon = 1 - \log_3(2) > 0$. Then $\log_3(2) + \varepsilon = 1$, and $n = n^{\log_3(2)+\varepsilon} = \Omega(n^{\log_3(2)+\varepsilon})$. Also for any c in the range $2/3 \le c < 1$, and any positive n, we have $2(n/3) = (2/3)n \le cn$, so the regularity condition holds. By case (3) of the Master Theorem $T(n) = \Theta(n)$.

2. Use the Master theorem to find asymptotic solutions to the following recurrences.

a. $T(n) = 7T(n/4) + n$

Solution:

$4 < 7 \Rightarrow 1 < \log_4(7) \Rightarrow \log_4(7) - 1 > 0$. Let $\varepsilon = \log_4(7) - 1$. Then $\varepsilon > 0$, and $1 = \log_4(7) - \varepsilon$, whence $n = n^{\log_4(7)-\varepsilon} = O(n^{\log_4(7)-\varepsilon})$. By case (1) we have $T(n) = \Theta(n^{\log_4(7)})$.

b. $T(n) = 9T(n/3) + n^2$

Observe that $n^2 = n^{\log_3(9)} = \Theta(n^{\log_3(9)})$, and therefore $T(n) = \Theta(n^2 \log(n))$ by case (2).

c. $T(n) = 6T(n/5) + n^2$

Solution:

Observe $6 < 25 \Rightarrow \log_5(6) < 2 \Rightarrow 2 - \log_5(6) > 0$. Let $\varepsilon = 2 - \log_5(6)$. Then $\log_5(6) + \varepsilon = 2$, and $n^2 = \Omega(n^{\log_5(6)+\varepsilon})$. Also for any c in the range $6/25 \le c < 1$, and for any positive n, we have $6(n/5)^2 = (6/25)n^2 \le cn^2$, so the regularity condition holds. Therefore $T(n) = \Theta(n^2)$ by case (3) of the Master Theorem.

d. $T(n) = 6T(n/5) + n\log(n)$

Solution:

Observe $\log_5(6) > 1$, so letting $\varepsilon = \dfrac{\log_5(6) - 1}{2}$, we have $\varepsilon > 0$ and $1 + \varepsilon = \log_5(6) - \varepsilon$. Therefore by l'Hopital's rule

$$\lim_{n\to\infty} \frac{n\log(n)}{n^{\log_5(6)-\varepsilon}} = \lim_{n\to\infty} \frac{n\log(n)}{n^{1+\varepsilon}} = \lim_{n\to\infty} \frac{\log(n)}{n^{\varepsilon}} = 0,$$

showing that $n\log(n) = o(n^{\log_5(6)-\varepsilon}) \subseteq O(n^{\log_5(6)-\varepsilon})$. Case (1) now gives $T(n) = \Theta(n^{\log_5(6)})$.

e. $T(n) = 7T(n/2) + n^2$

Solution:

Observe that $7 > 4 \Rightarrow \log_2(7) > 2$, so upon setting $\varepsilon = \log_2(7) - 2$ we have $\varepsilon > 0$. It follows that $2 = \log_2(7) - \varepsilon$, whence $n^2 = n^{\log_2(7)-\varepsilon} = O(n^{\log_2(7)-\varepsilon})$. Case 1 now gives $T(n) = \Theta(n^{\log_2(7)})$.

f. $S(n) = aS(n/4) + n^2$ (Note: your answer will depend on the parameter a.)

Solution:

We have three cases to consider corresponding to the three cases of the Master Theorem:

Case 1: $a > 16 \Rightarrow \log_4(a) > 2 \Rightarrow \varepsilon = \log_4(a) - 2 > 0 \Rightarrow n^2 = O(n^{\log_4(a)-\varepsilon})$, so $S(n) = \Theta(n^{\log_4(a)})$.

Case 2: $a = 16 \Rightarrow \log_4(a) = 2 \Rightarrow n^2 = \Theta(n^{\log_4(a)})$, whence $S(n) = \Theta(n^2 \log(n))$.

Case 3: $1 \le a < 16 \Rightarrow \log_4(a) < 2 \Rightarrow \varepsilon = 2 - \log_4(a) > 0 \Rightarrow n^2 = \Omega(n^{\log_4(a)+\varepsilon})$. Further, for any c in the range $a/16 \le c < 1$ we have $a(n/4)^2 = (a/16)n^2 \le cn^2$, showing that the regularity condition holds. Therefore $S(n) = \Theta(n^2)$.

3. p.75: 4.3-2

The recurrence $T(n) = 7T(n/2) + n^2$ describes the running time of an algorithm *A*. A competing algorithm *B* has a running time of $S(n) = aS(n/4) + n^2$. What is the largest integer value for *a* such that *B* is a faster algorithm than *A* (asymptotically speaking)? In other words, find the largest integer *a* such that $S(n) = o(T(n))$.

Solution:

We seek the largest integer *a* for which $S(n) = o(T(n))$. Using parts (e) and (f) of the previous problem, we find that $S(n) = o(T(n))$ in cases 2 and 3 since $4 < 7 \Rightarrow 2 < \log_2(7)$ and hence $n^2 = o(n^{\log_2(7)})$, and $n^2 \log(n) = o(n^{\log_2(7)})$. In case 1 we have $S(n) = o(T(n))$ if and only if $n^{\log_4(a)} = o(n^{\log_2(7)})$, i.e. if and only if $\log_4(a) < \log_2(7)$. Thus we seek the largest integer *a* such that $a < 4^{\log_2(7)} = 7^{\log_2(4)} = 7^2 = 49$. The largest such integer is $a = 48$.

4. Let *G* be an acyclic graph with *n* vertices, *m* edges, and *k* connected components. Show that $m = n - k$. (Hint: use the fact that $|E(T)| = |V(T)| - 1$ for any tree *T*, from the induction handout.)

Proof:

Let $T_1, T_2, \ldots, T_k$, be the connected components of *G*, each of which is necessarily a tree. Let $n_i = |V(T_i)|$, and $m_i = |E(T_i)|$, for $1 \le i \le k$. By a theorem proved in the induction handout, we have $m_i = n_i - 1$, for $1 \le i \le k$. Therefore

$$m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

///

5. Appendix B.4 problem 3

Show that any connected graph *G* satisfies $|E(G)| \ge |V(G)| - 1$. (Hint: use induction on the number of edges.)

Proof:

Let $G = (V, E)$ be a connected graph, and suppose $|E| = 0$. Since *G* is connected we must have $|V| = 1$, whence $|E| \ge |V| - 1$, and so the base case is satisfied.

Now suppose $|E| > 0$ and assume the result holds for any graph with fewer than $|E|$ edges. In other words, we assume that for all graphs $G' = (V', E')$ with $|E'| < |E|$ that $|E'| \ge |V'| - 1$. Now pick any edge *e* in *G* and remove it, and let $G - e$ denote the resulting graph. We have two cases to consider.

Case 1: $G-e$ is connected. In this case we apply the induction hypothesis to $G-e=(V,E-e)$ which has fewer edges than G. We conclude that $|E-e|\geq|V|-1$, so that $|E|-1\geq|V|-1$, and therefore $|E|\geq|V|>|V|-1$, as required.

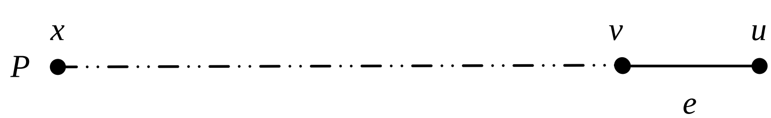
Case 2: $G-e$ is disconnected. In this case $G-e$ consists of two connected components. (**See the claim and proof below, which are not necessary for full credit on this problem.) Call them $H_1=(V_1,E_1)$ and $H_2=(V_2,E_2)$. Note that both H_1 and H_2 have fewer edges than G, so we may apply the induction hypothesis to obtain $|E_1|\geq|V_1|-1$ and $|E_2|\geq|V_2|-1$. Therefore

$$
\begin{aligned}
|E| &= |E_1|+|E_2|+1 \\
&\geq \left(|V_1|-1\right)+\left(|V_2|-1\right)+1 && \text{(by the induction hypothesis)} \\
&= |V_1|+|V_2|-1 \\
&= |V|-1 && \text{(no vertices were removed so } |V_1|+|V_2|=|V|\text{).}
\end{aligned}
$$

The result now holds for all connected graphs by induction. ///

Claim**: Let G be a connected graph and $e\in E(G)$, and suppose that $G-e$ is disconnected. (Such an edge e is called a *bridge*). Then $G-e$ has exactly two connected components.

Proof: Since $G-e$ is disconnected, it has at least two components. We must show that it also has at most two components. Let e have end vertices u, and v. Let C_u and C_v be the connected components of $G-e$ that contain u and v respectively. Choose $x\in V(G)$ arbitrarily, and let P be an x-u path in G. Either P includes the edge e, or it does not. If P does not contain e, then P remains intact after the removal of e, and hence P is an x-u path in $G-e$, whence $x\in C_u$. If on the other hand P does contain the edge e, then e must be the last edge along P from x to u.



In this case $P-e$ is an x-v path in $G-e$, whence $x\in C_v$. Since x was arbitrary, every vertex in $G-e$ belongs to either C_u or C_v, and therefore $G-e$ has at most two connected components. ///

6. Show that the number vertices of odd degree in any graph must be even. (Hint: suppose G contains an odd number of odd vertices. Use the Handshake Lemma mentioned in the Graph Theory handout. Argue that the left hand side is odd, while the right hand side is clearly even.)

Proof:
Recall the handshake lemma says: $\sum_{x\in V(G)}\deg(x) = 2|E(G)|$. Let $E=\{x\in V(G)\,|\,\deg(x) \text{ is even}\}$ and $O=\{y\in V(G)\,|\,\deg(y) \text{ is odd}\}$. The handshake lemma can then be written as:

$$\sum_{x\in E}\deg(x) + \sum_{y\in O}\deg(y) = 2|E(G)|$$

The right hand side of the above equation is obviously even and the first term on the left hand side is also even, being the sum of even numbers. Therefore the second term on the left hand side is even as well. Now the sum of an odd number of odd numbers is necessarily odd, while the sum of an even number of odd numbers is even. It follows that the sum

$$\sum_{y \in O} \deg(y)$$

contains an even number of terms. Therefore G contains an even number of odd-degree vertices. //