

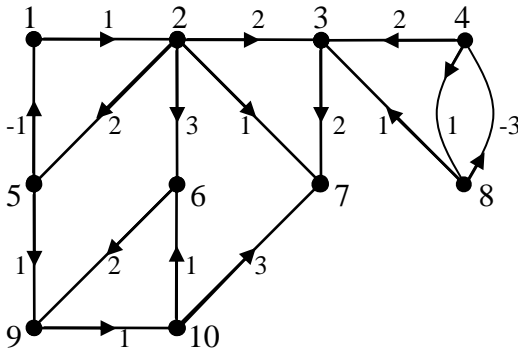
CMPS 101

Final Review Problems

Be sure to look at the problems on all previous review sheets and homework assignments.

1. Let $G = (V, E)$ be a graph with n vertices, m edges, and k connected components.
 - a. Show that if G is connected and acyclic, then $m = n - 1$. Use induction on either m or n .
 - b. Show that if G is acyclic, then $m = n - k$. Use part (a).
 - c. Show that if G is connected, then $m \geq n - 1$. Use induction on m .
 - d. Show that in any case $m \geq n - k$. Use part (c).
2. Let G be a digraph. Determine whether, at any point during a Depth First Search of G , there can exist an edge of the following kind.
 - a. A tree edge that joins a white vertex to a gray vertex.
 - b. A back edge that joins a black vertex to a white vertex.
 - c. A forward edge that joins a gray vertex to a black vertex.
 - d. A cross edge that joins a black vertex to a gray vertex.
 - e. A tree edge that joins a gray vertex to a gray vertex.
 - f. A forward edge that joins a black vertex to a black vertex.
 - g. A cross edge that joins a white vertex to a black vertex.
 - h. A back edge that joins a gray vertex to a white vertex.
3.
 - a. State the parenthesis theorem.
 - b. State the white path theorem.
 - c. State the max-Heap property.
 - d. State the min-Heap property.
 - e. State the Binary Search Tree properties.
 - f. State the Red Black Tree properties.
4. Let G be a directed graph. Prove that if G contains a directed cycle, then G contains a back edge. (Hint: use the white path theorem.)
5. Let T be a binary tree. Let $n(T)$ denote the number of nodes in T , and $h(T)$ denote the height of T . Show that $h(T) \geq \lfloor \lg(n(T)) \rfloor$.
6. Re-write the algorithms Heapify, and HeapIncreaseKey from the point of view of a min-Heap, rather than a max-Heap. (In particular, HeapIncreaseKey should be renamed HeapDecreaseKey.)
7. Trace HeapSort on the following arrays:
 - a. (9, 3, 5, 4, 8, 2, 5, 10, 12, 2, 7, 4)
 - b. (5, 3, 7, 1, 10, 12, 19, 24, 5, 7, 2, 6)
 - c. (9, 8, 7, 6, 5, 4, 3, 2, 1)
8. Let G be a directed graph, and let $s, x \in V(G)$. Suppose that after $\text{Initialize}(G, s)$ is executed, some sequence of calls to $\text{Relax}(,)$ results in $d[x]$ becoming finite. Show that G contains an s - x path of weight $d[x]$. (Use strong induction on the number of calls to $\text{Relax}(,)$.)
9. Let G be a directed graph, $s, x \in V(G)$, and suppose $\text{Initialize}(G, s)$ is executed. Show that the inequality $\delta(s, x) \leq d[x]$ is maintained over *any* sequence of calls to $\text{Relax}(,)$. (Use the result of problem 8.)

10. Perform $\text{BellmanFord}(G, s)$ on the weighted directed graph pictured below. Determine the $d[]$ and $p[]$ values for each vertex after each pass over the edge set in the main loop of $\text{BellmanFord}()$. Draw the resulting Shortest Paths tree, and determine the return value (true or false) of $\text{BellmanFord}()$.
- Use $s = 1$ as source vertex.
 - Use $s = 8$ as source vertex.



11. Change the sign on the edge weights for $(5, 1)$ and $(8, 4)$ in the digraph in problem 10 above. Perform $\text{Dijkstra}(G, s)$ on the resulting digraph. Trace the $d[]$ and $p[]$ values for each vertex after each call to $\text{Relax}()$, and draw the resulting Shortest Paths tree.
- Use $s = 1$ as source vertex.
 - Use $s = 2$ as source vertex.
12. Draw the Binary Search Tree resulting from inserting the keys: 5 8 3 4 6 1 9 2 7 (in that order) into an initially empty tree. Write pseudo-code for the following recursive algorithms, and write their output when run on this tree.
- $\text{InOrderTreeWalk}()$
 - $\text{PreOrderTreeWalk}()$
 - $\text{PostOrderTreeWalk}()$
13. Assign colors to the nodes in the above BST in such a way that it becomes a valid RBT. Note there is more than one way to do this. Find all such color assignments.
14. Let x be a node in a Red-Black Tree, and let $N(x)$ denote the number of internal nodes in the subtree rooted at x . Show that $N(x) \geq 2^{\text{bh}(x)} - 1$. (Hint: use strong induction on $\text{height}(x)$.)
15. Let T be a Red-Black Tree having n internal nodes, and height h . Show that $h \leq 2\lg(n+1)$. (Hint: use the result of problem 14 and RBT property (4).)
16. Insert the following keys (in order) into an initially empty Binary Search Tree: 11, 2, 13, 1, 3, 12, 4, 9, 7, 10, 6, 8, 5. Draw the resulting Binary Search Tree. Prove that it is not possible to assign colors Red and Black to the nodes of this tree in such a way that the Red-Black tree properties are satisfied. (Hint: use contradiction and the result of problem 15.)