ETH Zürich
D-ITET
Biomedical Engineering

Master Studies

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Biomedical Imaging

# Homework #6 - X-ray Imaging 2

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# 1 TASK 1

The reconstruction is shown in figure 1.1.

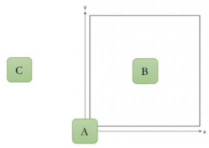


**Figure 1.1:** Reconstruction

The sinogram is shown in figure 1.2.

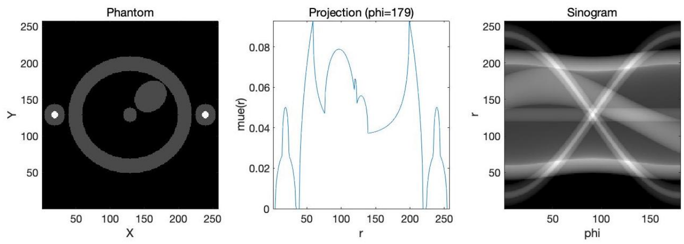


**Figure 1.2:** Sinogram

## 2 TASK 2

The Radon Transform of an image is given as:

$$g(\omega,\theta) = \mathscr{R}I(x,y) = \iint_D I(x,y)\delta(xcos\theta + ysin\theta - \rho)dxdy \tag{2.1}$$

Where I(x,y) is the original image.
Consider a fixed angle $\theta$'s projection, and take 1-D Fourier Transform with respect to $\rho$:

$$G(\omega,\theta) = \int g(\omega,\theta)e^{-j2\pi\omega\rho}d\rho = \int[\iint I(x,y)\delta(xcos\theta + ysin\theta - \rho)dxdy]d\rho \tag{2.2}$$

By exchanging the order of integration, we obtain:

$$G(\omega,\theta) = \iint I(x,y)[\int \delta(xcos\theta+ysin\theta-\rho)d\rho]dxdy = \iint I(x,y)e^{-j2\pi\omega(xcos\theta+ysin\theta)}dxdy \tag{2.3}$$

Substituting $\omega cos\theta$ with u and $\omega sin\theta$ with v we get:

$$G(\omega,\theta) = \iint I(x,y)e^{-j2\pi\omega(ux+vy)}dxdy\Big|_{u=\omega cos\theta, v=\omega sin\theta} \tag{2.4}$$

From equation 2.4 we can draw the conclusion that that the 1-D Fourier Transform of Radon Projection of an image is a slice through the 2-D Fourier Transform of that image. Moreover, the angle from which the projection is obtained is equal to the slicing angle in 2-D Fourier Transform. This is called the Fourier Slicing Theorem.
According to the Fourier Slicing Theorem, we can obtain the 2-D Fourier Transform of an image by summing up the 1-D Fourier Transform of the Radon Projection of that image under different angle $\theta$:

$$\mathscr{F}[I(x,y)] = \sum_{\theta=0}^{\pi} G(\omega,\theta) \tag{2.5}$$

Since summation in the image domain equals to summation in the frequency domain, the simple back-projection of the original image is given by:

$$I(x,y) = \mathscr{F}^{-1}[\sum_{\theta=0}^{\pi} G(\omega,\theta)] = \sum_{\theta=0}^{\pi}\mathscr{F}^{-1}[G(\omega,\theta)] = \sum_{\theta=0}^{\pi} g(xcos\theta + ysin\theta,\theta) = \sum_{\theta=0}^{\pi} F_\theta(x,y) \tag{2.6}$$

The result of simple back-projection is show in the figure 2.1.

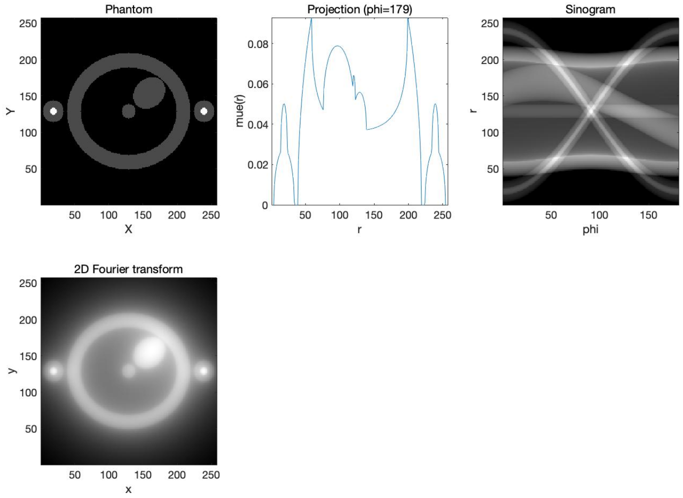


**Figure 2.1:** Simple Back-projection

If we consider the simple back-projection in the frequency domain, it is done by directly adding up slices in different angle. This will leads to the oversampling of the middle of the 2-D Fourier Transform which contains the low frequency information of the image. In contrast, the high frequency information which is contained in the periphery of the 2-D Fourier Transform is undersampled due to the fact that the slicing is discontinuous. Since the low frequencies take care of the smooth surfaces of an image and high frequencies take care of the details and the sharp edges, the simple back-projected image is blurred.

# 3 TASK 3

We want:

$$I(x,y) = \mathscr{F}^{-1}[F(u,v)] = \iint F(u,v)e^{j2\pi(ux+vy)}dudv \tag{3.1}$$

If we change the Cartesian coordinates to Polar coordinates, we can obtain that:

$$I(x,y) = \int_0^{2\pi}\int_{-\infty}^{\infty} F(\omega cos\theta, \omega sin\theta)e^{j2\pi(x\omega cos\theta + y\omega sin\theta)}\omega d\omega d\theta \tag{3.2}$$

According to Fourier Slicing theory, we can easily derive that:

$$I(x,y) = \int_0^{2\pi}\int_{-\infty}^{\infty} G(\omega,\theta)e^{j2\pi(x\omega cos\theta + y\omega sin\theta)}\omega d\omega d\theta \tag{3.3}$$

Since $G(\omega,\theta+\pi) = G(-\omega,\theta)$, we can simplify the equation 3.3 to the following equation:

$$I(x,y) = \int_0^{\pi}[\int_{-\infty}^{\infty} G(\omega,\theta)e^{j2\pi\omega\rho}|\omega|d\omega]\Big|_{\rho=xcos\theta+ysin\theta} d\theta \tag{3.4}$$

$\int_{-\infty}^{\infty} G(\omega,\theta)e^{j2\pi\omega\rho}|\omega|d\omega$ is the inverse Fourier Transform of $G(\omega,\theta)$ multiplied by a high pass filter $|\omega|$.

This is called Filtered Back-projection which takes the place of $F_\theta$ from simple back-projection.

Now we consider the differences between the simple back-projection and Filtered Back-projection. In simple back-projection, we did not reconstruct the image from the 2-D Fourier Transform but the 2-D Fourier Transform approximated by the summation of 1-D Fourier Transform of the Radon Projection of the image. Thus it leads to the oversampling and undersmapling problems which yields the blurring in the reconstructed image. However, in the Filtered Back-projection, we reconstruct the image directly from the 2-D Fourier Transform of the image so we do not encounter the problem of oversampling.

Since multiplying in frequency domain equals to convolution in image domain, we can obtain the following equation:

$$I(x,y) = \int_0^{\pi}(s(\rho) * g(\rho,\theta))\Big|_{\rho=xcos\theta+ysin\theta} d\theta \tag{3.5}$$

Where $s(\rho) = \mathscr{F}^{-1}[|\omega|]$.

In terms of signal processing, Filtered Back-projection is like adding a high-pass filter to simple back-projection.

The result of Filtered Back-projection is shown in figure 3.1.

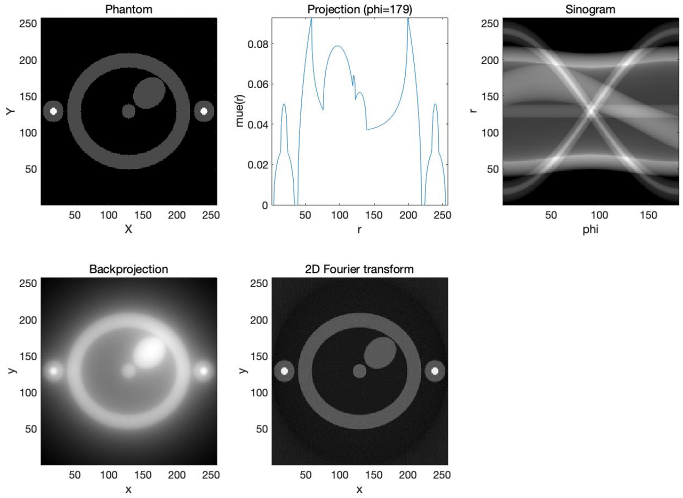


**Figure 3.1:** Filtered Back-projection

As is shown in figure 3.1, "streak" artifacts are seen in the result of Filtered Back-projection. This is due to the fact the number of projection is limited so we cannot take the integral(which requires infinity numbers of projection) but take the summation of different angle $\theta$. In another word, we still have the problem of undersampling. So part of the information contains in the high frequencies are somehow still missing.
This effect can only be diminished but not removed due to the nature of limited times of projection.
In 2-D Fourier Transform, the further a point is away from the center, the more densely sampling(i.e. $\Delta\theta = \theta_{k+1} - \theta_k$ is smaller) is needed to capture its information. That is to say, the higher frequency information we want to capture, the more densely sampling is needed to better reconstruct the image. Thus, if we simply cut off the higher frequencies, we can better reconstruct the image with comparably less densely sampling.
We can imperfectly remove the "streak" artifacts by combing the ideal high-pass with a low-pass filter. However this comes with a cost that part of high frequency information(usually contains details and sharp edge information of the image) gets lost. So the image we get is blurred.

Figure 3.2 shows the filter multiplied by Hamming window function in frequency domain and the corresponding depiction in spatial domain.

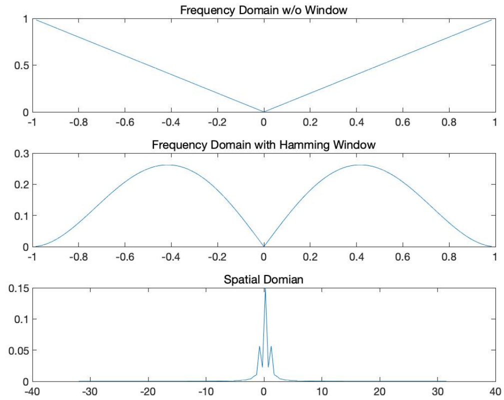


**Figure 3.2:** Filter with Hamming Window

The result of Filtered Back-projection with Hamming window function is shown in figure 3.3.

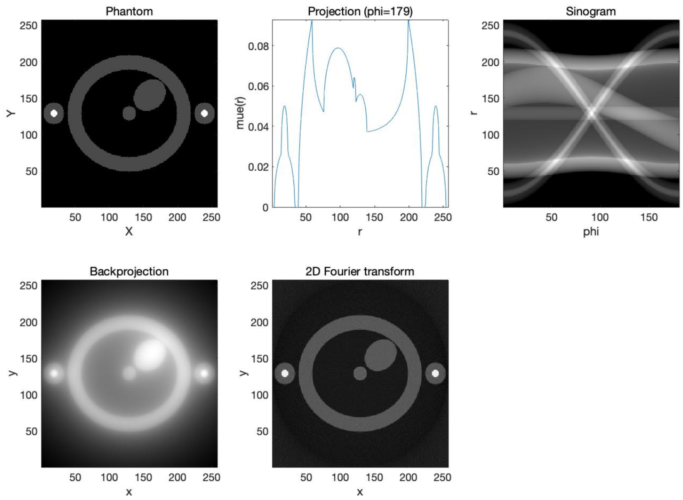


**Figure 3.3:** Filtered Back-projection with Hamming Window

# 4 Task 4

The Fourier Slicing Theory is that the 1-D Fourier Transform of Radon Projection of an image is a slice through the 2-D Fourier Transform of that image.
According to the Fourier Slicing Theory, if we take 1-D Fourier Transform of Radon Projection at different angle $\theta$(i.e. different slice) and change the coordinates from Polar Coordinates to Cartesian Coordinates, we can obtain the 2-D Fourier Transform of the original image along a certain line indicated by $xcos\theta + ysin\theta = \rho$. Thus, we can obtain the whole 2-D Fourier Transform by integration of the angle.

$$\mathscr{F}[I(x, y)] = \int \mathscr{F}[R(\rho, \theta)]d\theta \tag{4.1}$$

Where $R(\rho, \theta)$ is the Radon Transform of the image(i.e. $R(\rho, \theta) = \mathscr{R}I(x, y)$).
Compared with the simple back-projection approach in task-2, we can avoid the problem of oversampling the central part of 2-D Fourier Transform because we, in this approach, calculate the intensity distribution of 2-D Fourier Transform in Cartesian coordinates. In another word, we do not summing up the slices like we do in simple back-projection but first convert the slice into an intensity distribution with respect to Cartesian Coordinates(i.e (x,y)) along a certain slice and then do the summation over different slices to obtain the whole distribution.
The result of image reconstruction from projections using the 2-D Fourier Transform is shown in figure 4.1.

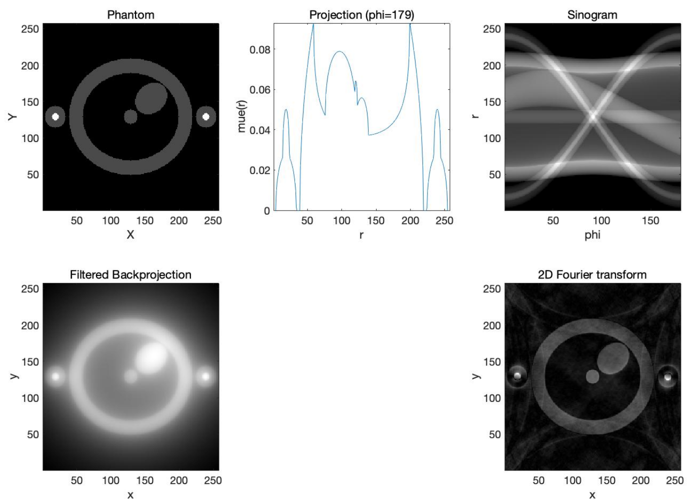


**Figure 4.1:** Back-projection of 2-D Fourier Transform

As is shown in figure 4.1, compared to the result of simple back-projection, the reconstructed image from 2-D Fourier Transform have more streak artifacts but less blurring which results from the oversampling of the central part of the 2-D Fourier Transform.
However, in practical, we cannot integrate over $\theta$ because the number of slices is limited. Thus, we still have the problem of undersampling. In another word, we cannot obtain the whole intensity distribution of 2-D Fourier Transform so there are intensity values of some unsampled points missed. This will lead to the artifact in the recon-

structed image. We can apply a simple interpolation by adding 0 to those points whose intensity values are missing. The result is shown in figure 4.2.

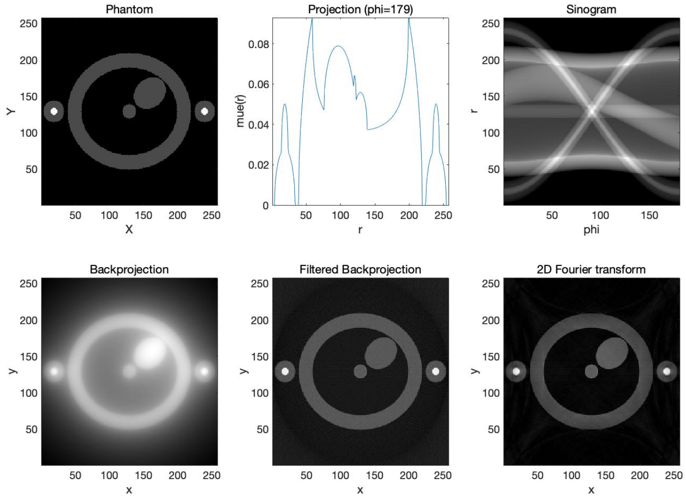


**Figure 4.2:** Back-projection of 2-D Fourier Transform with Interpolation