Geometric Ergodicity Proof Structure

$$
\begin{tikzcd}
\text{Assumptions 1} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[ddd, bend right=60, Rightarrow] & \text{Assumptions 2} \arrow[d, Rightarrow] \\
\text{Geometric Ergodicity} & \text{lemma 2} \quad P(S_i = s \mid S_{i+1} = s, Y, T, \theta) \geq \underline{\underline{\delta_i}}_{\text{free of } \theta} \arrow[d, Rightarrow] \\
 & \text{lemma 5} \arrow[d, Rightarrow] \quad E[|J| \mid Y, T, \theta] \leq \underline{|T| - \sum \delta_i}_{\text{free of } \theta} \\
\text{Smallset} \arrow[uu, Rightarrow] & \text{drift condition} \arrow[uul, Rightarrow]
\end{tikzcd}
$$

$$P(S|T,Y,\theta) \propto P(S,T|\theta)\cdot L(Y|T,S)\quad P_0(\theta)$$

$$\propto V_0(S_0)\, g_0(S_0;\theta) \prod_{i=1}^{N} P(S_{i-1},S_i;\theta)\cdot g_i(S_i;\theta)\quad P_0(\theta)$$

$$g_i(S;\theta) = R(\theta)\, e^{-(T_{i+1}-T_i)R(\theta)} \times \prod_{j:\, T_i < t_j^{obj} < T_{i+1}} L_j(Y_j|S)$$

$$g_N(S;\theta) = e^{-(t^{max}-t_N)R(\theta)} \times \prod_{t_N < t_j^{obj} < t^{max}} L_j(Y_j|S)$$

Stochastic matrix $P(\theta)$:

$$P(\theta) = I + \frac{Q(\theta)}{R(\theta)}$$

$$P_{i-n_0:i}(S_{i-n_0},S;\theta) = \sum_{\{S_{i-n_0+1}\cdots S_{i-1}\}} \left(\prod_{l=i-n_0+1}^{i-1} P(S_{l-1},S_l;\theta)\right)\cdot P(S_{i-1},S;\theta)$$

$$P^{g}_{i-n_0:i}(S_{i-n_0},S;\theta) = \sum_{\{S_{i-n_0+1}\cdots S_{i-1}\}} \prod_{l=i-n_0+1}^{i-1} P(S_{l-1},S_l;\theta)\, g_l(S_l;\theta)$$

$$\cancel{= P(S_{i-n_0}=S_{i-n_0}, S_i = S | Y,T,\theta)}$$

$$P(S_{i-n_0}=S_{i-n_0}, S_i=S|Y,T,\theta) = C\cdot \sum_{\substack{S_0 S_1 \cdots \\ \text{except for} \\ S_{i-n_0}, S_i}} V_0(S_0)\, g_0(S_0;\theta) \prod_{l=1}^{i-n_0-1} P(S_{l-1},S_l;\theta)\, g_l(S_l;\theta) \cdot P^{g}_{i-n_0:i}(S_{i-n_0},S;\theta)\cdot\left(\prod_{l=i}^{N}\cdots\right)$$

Assumptions 1.

i) $\exists$ an irreducible matrix Q^{min} s.t.

$$Q(\theta; s, s') \geq Q^{min}(s, s') \quad \forall\, s \neq s' \in S$$

ii) $\exists\, \eta > 0$ s.t.

$$\sup_{\theta, s} \frac{Q(\theta; s)}{R(\theta)} \leq 1 - \eta \quad \forall s \in S$$

iii) $\exists\, r^{max} < +\infty$ s.t. $\sup_{\theta} R(\theta) \leq r^{max}$

iv) $q(\theta|\theta') \geq \kappa\, q_0(\theta)$. q_0 is ~~the~~ prior of θ.

e.g.

$$\begin{pmatrix} -\alpha & \alpha & 0 & \cdots & & \\ \beta & -\alpha-\beta & \alpha & 0 & \cdots & \\ 0 & \beta & -\alpha-\beta & \alpha & 0 & \cdots \end{pmatrix}$$

$\boxed{\lambda \leq \alpha, \beta \leq \gamma}$

$\Downarrow$

Assumption 1

Assumptions 2.

for a fixed n_0, i, s.t. $1 \leq n_0 \leq i \leq n-1$

i) for some $\xi > 0$ $P_{i-n_0:i}(s_{i-n_0}, s; \theta) \geq \xi$ for $\forall\, s_{i-n_0}, s \in S$

ii) for some $\eta > 0$ $P(s, s; \theta) \geq \eta \quad \forall s \in S$

iii) for some g_l^{min} and g_l^{max}, we have $g_l^{min} \leq g_l(s; \theta) \leq g_l^{max}$

$\forall s \in S \quad l \in [i - n_0 + 1, i]$

Assumption 1 $\Rightarrow$ Assumptions 2.

i) $P^{min} \triangleq I + \frac{Q^{min}}{r^{max}}$

$\because$ Q^{min} is irreducible and aperiodic

$\therefore$ P^{min} is also irreducible and aperiodic

$\therefore$ $\exists\ n_0 \in \mathbb{Z}^+$ $\xi > 0$ s.t. $(P^{min})^{n_0}(s,s') \geq \xi$ $\forall s,s' \in S$

ii) $P(s,s;\theta) = 1 + \frac{Q(s,s;\theta)}{R(\theta)} \geq \eta$

iii) $g_e^{max} = g^{max} = r^{max}$

For lower bound:

for $g_e(s;\theta)$ including likelihood factors, there are at most k. We use 0 as lower bound.

For the remaining, we have $g_e^{min} = g^{min}$
$= q^{min} e^{-r^{max}(t^{max} - t^{min})}$

$q^{min} \triangleq \min_s Q^{min}(s) = \min_s \sum_{s' \neq s} Q^{min}(s,s')$

$\Rightarrow R(\theta) \geq q^{min} > 0$. $\square$

Lemma 2.

Under Assumptions 2

$$P(S_i = s \mid S_{i+1} = s, Y, T, \theta) \geq \delta_i \text{ and } \delta_i = \xi\eta \cdot \prod_{i=i-n_0+1}^{i} \frac{g_l^{min}}{g_l^{max}}$$

proof: $\left(\frac{a_i}{b_i} > c \Rightarrow \frac{\Sigma a_i}{\Sigma b_i} > c\right)$

We additionally Condition on $S_{i-n_0} = s_{i-n_0}$

δ_i is free of θ.

$$P(S_i = s \mid S_{i-n_0} = s_{i-n_0}, S_{i+1} = s, T, Y, \theta)$$

$$= \frac{P(S_i = s, S_{i+1} = s, S_{i-n_0} = s_{i-n_0} \mid T, Y, \theta)}{P(S_{i+1} = s, S_{i-n_0} = s_{i-n_0} \mid T, Y, \theta)}$$

$$= \frac{P^g_{i-n_0:i}(s_{i-n_0}, s;\theta)\, g_i(s;\theta)\, P(s,s;\theta)}{\sum_{s'} P^g_{i-n_0:i}(s_{i-n_0}, s;\theta) \cdot g_i(s';\theta) \cdot P(s',s;\theta)}.$$

$$\geq \frac{\prod_{l=i-n_0+1}^{i} g_l^{min}}{\prod_{l=i-n_0+1}^{i} g_l^{max}} \cdot \frac{P_{i-n_0:i}(s_{i-n_0}, s;\theta)\quad P(s,s;\theta)}{\sum_{s'} P_{i-n_0:i}(s_{i-n_0}, s';\theta)\, P(s',s;\theta)} \geq \xi\eta \prod_{l=i-n_0+1}^{i} \frac{g_l^{min}}{g_l^{max}} \quad \square$$

Notations. $P(S \mid T, Y, \theta) \propto \nu_0(s_0)\, g_0(s_0) \prod_{i=1}^{N} P(s_{i-1}, s_i;\theta) \cdot g(s_i;\theta) \cdot P(\theta)$

~~Pss~~ $P(\theta) = I + \frac{Q(\theta)}{R(\theta)}$

Lemma 5 $\quad |T| = n+1$.

if Assumptions 2 are true for $i \in [n_0, n-1]$ then

$$\mathbb{E}\left[|J| \mid Y, T, \theta\right] \leq n+1 - \sum_{i=n_0}^{n-1} \delta_i$$

$$J = \{i \in [1, n] : S_{i-1} \neq S_i\} \cup \{0\}$$

proof: $|J| = 1 + \sum_{i=0}^{n-1} \mathbb{1}\{S_i \neq S_{i+1}\}$

Apply Lemma 2 for $\forall i \in [n_0, n-1]$

$$P(S_i = s \mid S_{i+1}, Y, T, \theta) \geq \delta_i$$

$$\mathbb{E}\left[\mathbb{1}\{S_i \neq S_{i+1}\} \mid Y, T, \theta\right]$$

$$= \mathbb{E}\left[P(S_i \neq S_{i+1} \mid S_{i+1}, Y, T, \theta) \mid Y, T, \theta\right]$$

$$\leq 1 - \delta_i$$

for $i < n_0$ we use upper bound 1.

$$\mathbb{E}\left[|J| \mid T, Y, \theta\right] \leq n+1 - \sum_{i=n_0}^{n-1} \delta_i = |T| - \sum_{i=n_0}^{n-1} \delta_i \quad \square$$

drift Condition:

Under Assumptions 1, $\exists\ \delta > 0$ and $c < +\infty$

s.t. $\mathbb{E}\left[|J(X')|\ \middle|\ X\right] \le (1-\delta)|J(X)| + c$

$X \triangleq (\text{path}, \theta)$

$$X \xrightarrow{\text{Step 1}} (S, T', \theta) \xrightarrow{\text{Step 2}} (S, T', \theta') \xrightarrow{\text{Step 3}} (S', T', \theta') \longrightarrow X'$$

Proof:

in Step 1, we add potential jumps V.

$|V|$ has rate $\cancel{\xi}\,(t^{max} - t^{min}) R(\theta) - \int_{t_{min}}^{t_{max}} \mathcal{U}(s_t; \theta)\,dt$

$\le \mu^{max}(t^{max} - t^{min}) \triangleq \mathcal{M}$.

$\therefore \mathbb{E}[|T'|\,|X] \le |J(X)| + \mathcal{M}$,

$|T'| = n+1$

use lemma 2.

There are at most $(k+1)n_0$ s.t. $\delta_i = 0$

at least $n-(k+1)n_0$ $\delta_i = \xi\,\eta\left(\frac{g_{min}}{g_{max}}\right)^{n_0} \triangleq \delta > 0$

$\therefore$ lemma 5 $\Rightarrow \mathbb{E}\left[|J'|\,\middle|\,T'\right] \le (1-\delta)|T'| + (k+1)n_0\delta + \delta$

$\mathbb{E}\left[|J(X')|\,\middle|\,X\right] = \mathbb{E}\left[\mathbb{E}\left[|J(X')|\,\middle|\,T'\right]\middle|\,X\right]$

$\le (1-\delta)(|J(X)| + \mathcal{M}) + (k+1)n_0\delta + \delta$ □

Small set Condition.

The small set $\{x : |J(x)| \le h\}$ is 1-small set for $\forall h > 0$.

Proof: First define a sequence $X \triangleq (\text{path}, \theta)$

$$\text{path}^* = \begin{matrix} S^* \\ T^* \end{matrix} \begin{pmatrix} s_0^*, s_1^*, \cdots, s_n^* \\ t_0^*, t_1^*, \cdots, t_n^* \end{pmatrix} \qquad n \ge k$$

Then the regeneration measure $\Phi(dx')$ is described as follows

$$T_i' \sim \text{unif}(t_{i-1}^*, t_i^*) \quad \text{for } i = 1, 2, \cdots, n$$
$$S_i' = S_i^* \quad \text{for } i = 0, 1, 2, \cdots n$$
$$\theta' \sim P_0(\theta') \quad (\text{prior})$$

The random time vector $(T_1', T_2', T_3', \cdots, T_n')$ has the uniform distribution on the set $T = \{(t_1, t_2, \cdots t_n) \mid t_{i-1}^* \le t_i \le t_i^*, \text{ for } i = 1, 2, \cdots, n\}$

The path' is determined by (T', S')

We first construct $S^+ (S_1^+, S_2^+, S_3^+, \cdots, S_k^+)$ s.t. $\prod_{j=1}^{k} L_j(Y_j | S_j^+) \triangleq L^+ > 0$

~~Under assumption 1~~

define $S^* = (S_0^*, S_1^*, S_2^*, \cdots, S_n^*)$ s.t. S^+ is a subsequence of S^* $(S_{i-1}^* \ne S_i^*)$

Also, embed $t^{obs} = (t_1^{obs}, t_2^{obs}, \cdots, t_k^{obs})$ in a longer sequence $(t_1^*, t_2^*, \cdots t_n^*)$.

$$\nu_0(S_0^*) \cdot \prod_{i=1}^{n-1} P(\theta; S_i^*, S_{i+1}^*) \ge \nu_0(S_0^*) \cdot \prod_{i=1}^{n-1} \frac{Q^{min}(S_i^*, S_{i+1}^*)}{h^{max}} \triangleq \beta_1^* > 0$$

fix $X = (\text{path}, \theta)$ with $|\mathcal{J}(X)| \le h$.

There is another way in Which the algorithm can be executed.

Note that $\dfrac{Q(\theta; S)}{R(\theta)} \le 1 - \eta \Rightarrow R(\theta) - Q(\theta; S) \ge \eta R(\theta) \ge \eta q^{\min} \triangleq \varepsilon > 0$

in step 1. We independently Sample two poisson processes on the interval $[t^{\min}, t^{\max}]$, say V^0 and V^{rest} with rate ε and $R(\theta) - Q(\theta; S_t) - \varepsilon$

let $V = V^0 \cup V^{rest}$

$$T' = \mathcal{J}(X) \cup V$$

$$\mathbb{P}(V_0 \in T) \triangleq \beta_0 > 0$$

$$\mathbb{P}(V^{rest} = \phi) \ge e^{-r^{\max}(t^{\max} - t^{\min})} \triangleq \beta^{rest} > 0.$$

in step 2. We update θ using MH. Assume $q(\theta'|\theta) \ge \cancel{K P_0(\theta')}$ $\exists K$ $K P_0(\theta')$

i) propose $\theta' \sim q(\theta'|\theta)$

ii) Accept θ' with probability $\alpha = \dfrac{q(\theta|\theta')\, P(\theta'|T', Y)}{q(\theta'|\theta)\, P(\theta|T', Y)} \wedge 1$
$\triangleq \underline{\alpha} \wedge 1$

$$P(\theta, T', Y) = \int_S P(\theta, S, T, Y)\, ds$$

$$= \int_S P_0(\theta) \cdot P(S, T|\theta) \cdot P(Y|S, T')\, ds$$

$$\in [L_{\min} \cdot P_0(\theta) \cdot P(T'|\theta),\ L_{\max} \cdot P_0(\theta) \cdot P(T'|\theta)]$$

$$\therefore \underline{\alpha} \ge \frac{q(\theta|\theta')}{q(\theta'|\theta)} \cdot \frac{L_{\min}}{L_{\max}} \cdot \frac{P(T'|\theta')}{P(T'|\theta)} \cdot \frac{P_0(\theta')}{P_0(\theta)} = (*)$$

$$\because P(T'|\theta) = R(\theta)^n (t^{\max} - t^{\min})^n e^{-R(\theta)(t^{\max} - t^{\min})}$$

$$\therefore (*) \frac{P_0(\theta')\, q(\theta|\theta')}{P_0(\theta)\, q(\theta'|\theta)} \cdot \frac{L_{\min}}{L_{\max}} \cdot \left(\frac{R(\theta')}{R(\theta)}\right)^n e^{-(R(\theta') - R(\theta))(t^{\max} - t^{\min})} \ge \frac{q(\theta|\theta')}{q(\theta'|\theta)} \cdot \frac{L_{\min}}{L_{\max}} \cdot \left(\frac{q_{\min}}{r_{\max}}\right)^n e^{-r_{\max}(t_{\max} - t_{\min})} \cdot \frac{P_0(\theta')}{P_0(\theta)}$$

$$q(\theta'|\theta)\, d\theta' \cdot \underline{\alpha} \ge q(\theta|\theta') \cdot \text{Const} \cdot \frac{P_0(\theta')}{P_0(\theta)} \ge k\, \text{Const} \cdot P_0(\theta')\, d\theta'$$

$$q(\theta'|\theta)\, d\theta' \cdot 1 \ge K P_0(\theta')\, d\theta'$$

Step 3. $P(S' \mid T', \theta', Y)$

$$\prod_{i=0}^{|T'|} g_i(S_i') \le (h^{max})^{|T'|} L^{max}$$

$$L^{max} = \prod_{j=1}^{k} \max_{s} L_j(Y_j \mid S)$$

We can use rejection sampling to sample S' from $P(S' \mid T', \theta', Y)$

i) Simulate S' with transition matrix $P(\theta'; \cdot, \cdot)$ and initial ν_0

ii) Accepts S' with prob

$$\frac{\nu_0(S_0') \prod_{i=1}^{|T'|-2} P(\theta'; S_i', S_{i+1}') g_i^{\theta'}(S_i') \cdot g_0^{\theta'}(S_0') g_{|T'|-1}^{\theta'}(S_{|T'|-1}')}{\nu_0(S_0') \prod_{i=1}^{|T'|-2} P(\theta'; S_i', S_{i+1}') \, |h^{max}|^{|T'|} L^{max}}$$

$$= \frac{\prod_{i=0}^{|T'|} g_i^{\theta'}(S_i')}{|h^{max}|^{|T'|} \cdot L^{max}}$$

$X = (\text{path}, \theta) \longrightarrow X = (\text{path}', \theta')$

(E_1) in step 1, we obtain $T' = J(X) \cup V^0$ and $V^0 \in \Upsilon$

$$P(E_1) \ge \beta^0 \cdot \beta^{rest}$$

(E_2) update θ with θ'

$$P(E_2) \ge \text{Const} \cdot P_0(\theta') d\theta'$$

(E_3) All points in $J(X)$ ~~are~~ become virtual

$$P(E_3) \ge \beta_1^* \cdot y^{|J(X)|} = \beta_1^* \cdot y^h \triangleq \beta_1$$

(E_4) given $E_1 - E_3$, S' is accepted

$$P(E_4) \ge \left(\frac{g^{min}}{h^{max}}\right)^{h+n} \exp(-h^{max}(t^{max} - t^{min})) \frac{L^+}{L^{max}} = \beta_2$$

$E = E_1 \cap E_2 \cap E_3 \cap E_4$

$\therefore$

$$\left.\begin{aligned} & \cancel{A(x, dx') \geq \Phi(dpath') \cdot Const \cdot P_0(\theta')d\theta'} \\ & \cancel{= \Phi(dx') \cdot Const.} \end{aligned}\right\} ?$$

if E happens, then a path' is generated and it's independent with X. $path \sim \Phi\big|_{path}(\cdot)$

$\therefore$

$$A(X, dx') \geq C \cdot P_0(\theta')d\theta' \cdot \Phi_{path}(dpath')$$
$$= C \cdot \Phi(dx') \qquad \square$$