Скаларно произведение

1 зад.

$\vec{a},\vec{b},\vec{c}: |\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=\sqrt{2}$

$\sphericalangle(\vec{a},\vec{b})=\frac{\pi}{2}, \sphericalangle(\vec{b},\vec{c})=\frac{\pi}{2}, \sphericalangle(\vec{a},\vec{c})=\frac{\pi}{4}$

$\vec{a},\vec{b},\vec{c}$ – ЛНЗ

$\vec{p}=\vec{a}+\vec{b}-\vec{c}$

$\vec{q}=2\vec{a}-3\vec{b}+\vec{c}$

$\vec{r}=\vec{a}+\lambda.\vec{b}-\vec{c}$

$\vec{a}^2=|\vec{a}|^2=1$

$\vec{b}^2=2^2=4$

$\vec{c}^2=(\sqrt{2})^2=2$

$(\vec{a}.\vec{b})=1.2.\cos\frac{\pi}{2}=0$

$(a.c)=1.\sqrt{2}.\frac{\sqrt{2}}{2}=1$

$(\vec{b}.\vec{c})=0$

а) $|\vec{p}|=?,\ |\vec{q}|=?$

$|\vec{q}|^2=\vec{q}^2=\underbrace{(2\vec{a}-3\vec{b}+\vec{c})^2}=$

$=4\vec{a}^2+9.\vec{b}^2+\vec{c}^2-2.(2\vec{a}).(3\vec{b})+2.(2\vec{a}).\vec{c}-2.(3\vec{b}).\vec{c}=$

$=4.\underset{1}{\vec{a}^2}+9.\underset{4}{\vec{b}^2}+\underset{2}{\vec{c}^2}-12.\underset{0}{(\vec{a}.\vec{b})}+4.\underset{1}{(\vec{a}.\vec{c})}-6.\underset{0}{(\vec{b}.\vec{c})}=$

$=4+36+2+4=46 \Rightarrow |\vec{q}|=\sqrt{46}$

$|\vec{p}|=?$ Упр.

$(\vec{a}.\vec{a})=|\vec{a}|.|\vec{a}|.\cos 0=|\vec{a}|^2$

$\boxed{\overset{\text{число}}{|\vec{a}|^2}=\overset{\text{число}}{\vec{a}^2}}$

$\underset{\text{число}}{|\vec{a}|}\neq\underset{\text{вектор}}{\vec{a}}$ НЕ

б) $(\vec{p}.\vec{q})=(\vec{a}+\vec{b}-\vec{c}).(2.\vec{a}-3\vec{b}+\vec{c})=$

$=2.\vec{a}^2-3.\underset{0}{(\vec{a}.\vec{b})}+(\vec{a}.\vec{c})+2.\underset{0}{(\vec{a}.\vec{b})}-3.\vec{b}^2+\underset{0}{(\vec{b}.\vec{c})}-2.(\vec{a}.\vec{c})+3.\underset{0}{(\vec{b}.\vec{c})}-\vec{c}^2=$

$=2.1+1-3.4-2.1-2=-13 \Rightarrow (\vec{p}.\vec{q})=-13<0 \Rightarrow \sphericalangle(\vec{p},\vec{q})\in(\frac{\pi}{2};\pi)$

$\vec{a}=|\vec{a}|^2=1$

$\vec{b}^2=2^2=4$

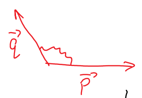
$\vec{c}^2=(\sqrt{2})^2=2$

$(\vec{a}.\vec{b})=1.2.\cos\frac{\pi}{2}=0$

$(a.c)=1.\sqrt{2}.\frac{\sqrt{2}}{2}=1$

$(\vec{b}.\vec{c})=0$

$\cos\sphericalangle(\vec{p},\vec{q})=\frac{(\vec{p}.\vec{q})}{|\vec{p}|.|\vec{q}|}=\frac{-13}{|\vec{p}|.\sqrt{46}}<0$



в) $\lambda=?: \vec{p}\perp\vec{r}$ (Упр.) Отг. $\lambda=-\frac{1}{4}$ {Упътване. $\vec{p}\perp\vec{r} \Leftrightarrow (\vec{p}.\vec{r})=0$ }

г) $\lambda=?: |\vec{r}|=\sqrt{5}$

Решение:

$\vec{r}=\vec{a}+\lambda.\vec{b}-\vec{c}\neq\sqrt{5}$ НЕ

$|\vec{r}|=\sqrt{5}$

$|\vec{r}|^2=\vec{r}^2$

$5=(\vec{a}+\lambda.\vec{b}-\vec{c})^2$

$\vec{a}^2+\lambda^2.\vec{b}^2+\vec{c}^2+2.\lambda.\underset{0}{(\vec{a}.\vec{b})}-2.(\vec{a}.\vec{c})-2\lambda.\underset{0}{(\vec{b}.\vec{c})}=5$

$1+\lambda^2.4+2-2.1=5$

$4\lambda^2=4$

$\lambda^2=1$

$\lambda=\pm1$ $\nearrow \vec{r}_1=\vec{a}+\vec{b}-\vec{c}=\vec{p}$; $\searrow \vec{r}_2=\vec{a}-\vec{b}-\vec{c}$

$\vec{a}=|\vec{a}|^2=1$

$\vec{b}^2=2^2=4$

$\vec{c}^2=(\sqrt{2})^2=2$

$(\vec{a}.\vec{b})=1.2.\cos\frac{\pi}{2}=0$

$(a.c)=1.\sqrt{2}.\frac{\sqrt{2}}{2}=1$

$(\vec{b}.\vec{c})=0$

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2 зад.

$\vec{a},\vec{b},\vec{c}$ – ЛНЗ

$\vec{p}: \vec{p}\perp\vec{a},\ \vec{p}\perp\vec{b},\ \vec{p}\perp\vec{c}$

Да се док., че $\vec{p}=\vec{0}$

Д-во:

$(\vec{a}.\vec{p})=(\vec{b}.\vec{p})=(\vec{c}.\vec{p})=0$

$\vec{p}=\alpha.\vec{a}+\beta.\vec{b}+\gamma.\vec{c}\ \ |.\vec{p}$

$\vec{p}^2=\alpha.(\vec{a}.\vec{p})+\beta.(\vec{b}.\vec{p})+\gamma.(\vec{c}.\vec{p})=0 \Rightarrow \vec{p}=\vec{o}$

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3 зад.

$\vec{a},\vec{b},\vec{c}$ – ЛНЗ

$|\vec{a}|=2, |\vec{b}|=1, |\vec{c}|=3$

$\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{b},\vec{c})=\sphericalangle(\vec{c},\vec{a})=\frac{\pi}{3}$

$\vec{OA}=\vec{a},\ \vec{OB}=\vec{b},\ \vec{OC}=\vec{c}$

$\vec{a}^2=4,\ \vec{b}^2=1,\ \vec{c}^2=9$

$(\vec{a}.\vec{b})=2.1.\frac{1}{2}=1$

$(\vec{b}.\vec{c})=1.3.\frac{1}{2}=\frac{3}{2}$

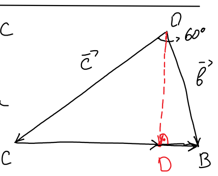
$(\vec{a}.\vec{c})=2.3.\frac{1}{2}=3$

коефициенти на метрикаta

а) Нека т. $D \in BC$, $OD\perp BC$

$\vec{OD}=?$ чрез $\vec{a}, \vec{b}, \vec{c}$

$\vec{OD}, \vec{b}, \vec{c}$ са компланарни



$\vec{OD}=\vec{OC}+\vec{CD}$

$\vec{CD}\parallel\vec{CB}=\vec{OB}-\vec{OC}=\vec{b}-\vec{c}$

$\vec{CD}=x.\vec{CB}$

$\vec{OD}=\vec{OC}+x.\vec{CB}\ \ |.\vec{CB}$, защото $\vec{OD}\perp\vec{CB}$

$(\vec{OD}.\vec{CB})=0$

$(\vec{OC}+x.\vec{CB}).\vec{CB}=0$

$(\vec{OC}.\vec{CB})+x.\vec{CB}^2=0$

$(\vec{OC}.\vec{CB})=\vec{c}.(\vec{b}-\vec{c})=(\vec{c}.\vec{b})-\vec{c}^2=\frac{3}{2}-9=-\frac{15}{2}$

$\vec{CB}^2=(\vec{b}-\vec{c})^2=\vec{b}^2-2(\vec{b}.\vec{c})+\vec{c}^2=1-2.\frac{3}{2}+9=7$

$(\vec{OD}.\vec{CB})=0,\ \vec{OD}\perp\vec{CB}$

$-\frac{15}{2}+x.7=0$

$7x=\frac{15}{2}$

$x=\frac{15}{14} \Rightarrow \vec{OD}=\vec{OC}+\frac{15}{14}.\vec{CB}=\vec{c}+\frac{15}{14}.(\vec{b}-\vec{c})$

$\vec{OD}=\frac{15}{14}.\vec{b}-\frac{1}{14}.\vec{c}$

б) Нека т. $A_1: \begin{cases} \in (BOC) \\ AA_1 \perp (BOC) \end{cases}$

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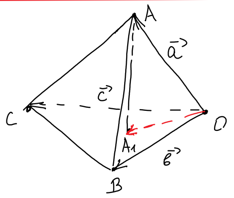

Да се изрази $\vec{OA_1} = ?$ чрез $\vec{a}, \vec{b}, \vec{c}$

$\vec{OA_1}$, $\vec{b}, \vec{c}$ са компланарни =>

=> $\exists!$ β и γ : $\vec{OA_1} = \beta.\vec{b} + \gamma.\vec{c}$, $\beta = ?$, $\gamma = ?$

$AA_1 \perp (BOC)$

$\vec{AA_1} = \vec{OA_1} - \vec{OA} = \beta.\vec{b} + \gamma.\vec{c} - \vec{a}$, $\vec{AA_1} \perp \vec{b}$ => , $\vec{AA_1} \perp \vec{c}$ => $\begin{vmatrix} (\vec{AA_1}.\vec{b}) = 0 \\ (\vec{AA_1}.\vec{c}) = 0 \end{vmatrix}$ =>

$$=> \begin{vmatrix} (\beta.\vec{b} + \gamma.\vec{c} - \vec{a}).\vec{b} = 0 \\ (\beta.\vec{b} + \gamma.\vec{c} - \vec{a}).\vec{c} = 0 \end{vmatrix} => \begin{vmatrix} \beta.(\vec{b}^2) + \gamma.(\vec{c}.\vec{b}) - (\vec{a}.\vec{b}) = 0 \\ \beta.(\vec{b}.\vec{c}) + \gamma.(\vec{c}^2) - (\vec{a}.\vec{c}) = 0 \end{vmatrix}$$

$\vec{a}^2 = 4$, $\vec{b}^2 = 1$, $\vec{c}^2 = 9$

$(\vec{a}.\vec{b}) = 2.1.\frac{1}{2} = 1$

$(\vec{b}.\vec{c}) = 1.3.\frac{1}{2} = \frac{3}{2}$

$(\vec{a}.\vec{c}) = 2.3.\frac{1}{2} = 3$

коефициенти на метриката

$$\begin{vmatrix} \beta.1 + \gamma.\frac{3}{2} - 1 = 0 \;/.2 \\ \beta.\frac{3}{2} + \gamma.9 - 3 = 0 \;/.\frac{2}{3} \end{vmatrix}$$

$$\begin{vmatrix} 2\beta + 3.\gamma = 2 & \mid.2 \\ \beta + 6.\gamma = 2 & (-) \end{vmatrix} => 3\beta = 2 => \beta = \frac{2}{3}$$

$6\gamma = 2 - \frac{2}{3} = \frac{4}{3}$

$\gamma = \frac{2}{9}$

$\vec{OA_1} = \frac{2}{3}.\vec{b} + \frac{2}{9}.\vec{c}$

Детерминанта на Грам

$$\Gamma(\vec{a}) = \vec{a}^2 = |\vec{a}|^2, \quad \Gamma(\vec{a}, \vec{b}) = \begin{vmatrix} \vec{a}^2 & (\vec{a}.\vec{b}) \\ (\vec{b}.\vec{a}) & \vec{b}^2 \end{vmatrix}, \quad \Gamma(a, b, c) = \begin{vmatrix} \vec{a}^2 & (\vec{a}.\vec{b}) & (\vec{a}.\vec{c}) \\ (\vec{b}.\vec{a}) & \vec{b}^2 & (\vec{b}.\vec{c}) \\ (\vec{c}.\vec{a}) & (\vec{c}.\vec{b}) & \vec{c}^2 \end{vmatrix}$$

Твърдение: $\vec{a_1}, \dots, \vec{a_n}$ са л.з. $\Leftrightarrow$ $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}) = 0$

$\vec{a_1}, \dots, \vec{a_n}$ са лнз $\Leftrightarrow$ $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}) \neq 0$

Пример:

$\vec{a}^2 = 4$, $\vec{b}^2 = 1$, $\vec{c}^2 = 9$

$(\vec{a}.\vec{b}) = 2.1.\frac{1}{2} = 1$

$(\vec{b}.\vec{c}) = 1.3.\frac{1}{2} = \frac{3}{2}$

$(\vec{a}.\vec{c}) = 2.3.\frac{1}{2} = 3$

коефициенти на метриката

$$\Gamma(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} 4 & 1 & 3 \\ 1 & 1 & \frac{3}{2} \\ 3 & \frac{3}{2} & 9 \end{vmatrix} = 36 + \frac{9}{2} + \frac{9}{2} - (9 + 9 + 9) = 45 - 27 = 18 \neq 0 =>$$

$\vec{a}, \vec{b}, \vec{c}$ са лнз

Твърдение: Векторите $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$ са линейно зависими $\Leftrightarrow$ $\Gamma(\vec{a_1}, \dots, \vec{a_n}) = 0$.

Доказателство:

I Нека $\vec{a_1}, \dots, \vec{a_n}$ са л.з. => $\exists (\alpha_1, \dots, \alpha_n) \neq (0, \dots, 0)$:

$$\alpha_1.\vec{a_1} + \alpha_2.\vec{a_2} + \dots + \alpha_n.\vec{a_n} = \vec{o} \quad \begin{matrix} /.\vec{a_1} => \\ /.\vec{a_2} => \\ \vdots \\ /.\vec{a_n} \end{matrix} \begin{vmatrix} \alpha_1.\vec{a_1}^2 + \alpha_2.(\vec{a_1}.\vec{a_2}) + \dots + \alpha_n.(\vec{a_1}.\vec{a_n}) = 0 \\ \alpha_1.(\vec{a_1}.\vec{a_2}) + \alpha_2.\vec{a_2}^2 + \dots + \alpha_n.(\vec{a_2}.\vec{a_n}) = 0 \\ \vdots \\ \alpha_1.(\vec{a_1}.\vec{a_n}) + \alpha_2.(\vec{a_2}.\vec{a_n}) + \dots + \alpha_n.\vec{a_n}^2 = 0 \end{vmatrix} (*)$$

Системата (*) е ХСЛУ с детерминанта $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n})$.

Тази система има решение $(\alpha_1, \alpha_2, \dots, \alpha_n) \neq (0, 0, \dots, 0)$.

Това е изпълнено точно, когато $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}) = 0$.

II Нека $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}) = 0$

Разглеждаме ХСЛУ

$$\begin{vmatrix} \alpha_1.\vec{a_1}^2 + \alpha_2.(\vec{a_1}.\vec{a_2}) + \dots + \alpha_n.(\vec{a_1}.\vec{a_n}) = 0 \\ \alpha_1.(\vec{a_1}.\vec{a_2}) + \alpha_2.\vec{a_2}^2 + \dots + \alpha_n.(\vec{a_2}.\vec{a_n}) = 0 \\ \vdots \\ \alpha_1.(\vec{a_1}.\vec{a_n}) + \alpha_2.(\vec{a_2}.\vec{a_n}) + \dots + \alpha_n.\vec{a_n}^2 = 0 \end{vmatrix} (**)$$

с неизвестни $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Детерминантата на системата (**) е точно $\Gamma(\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}) = 0$ =>

=> системата има решение $(\alpha_1^0, \alpha_2^0, \dots, \alpha_n^0) \neq (0, \dots, 0)$.

Разглеждаме линейната комбинация:

$\alpha_1^0 . \vec{a}_1 + \alpha_2^0 . \vec{a}_2 + \dots + \alpha_n^0 . \vec{a}_n = \vec{v} \ | . \vec{a}_1 \Rightarrow (\vec{v}.\vec{a}_1) = \alpha_1^0 . \vec{a}_1^{\,2} + \dots + \alpha_n^0 . (\vec{a}_1 . \vec{a}_n) = 0,$

от първото уравнение на $(*)$

$(\vec{v}.\vec{a}_2) = \alpha_1^0 . (\vec{a}_1 . \vec{a}_2) + \dots + \alpha_n^0 . (\vec{a}_n \vec{a}_2) = 0$

$\vdots$

$(\vec{v}.\vec{a}_n) = 0$

$(\vec{v}.\vec{a}_1) = (\vec{v}.\vec{a}_2) = \dots = (\vec{v}.\vec{a}_n) = 0$

Тогава $(\vec{v}.\vec{v}) = \alpha_1^0 . (\vec{a}_1 . \vec{v}) + \alpha_2^0 . (\vec{a}_2 . \vec{v}) + \dots + \alpha_n^0 . (\vec{a}_n . \vec{v}) = 0$

От $\vec{v}^{\,2} = 0 \Rightarrow \vec{v} = \vec{o} \Rightarrow \left| \begin{array}{l} \alpha_1^0 . \vec{a}_1 + \dots + \alpha_n^0 . \vec{a}_n = \vec{o} \\ (\alpha_1^0, \dots, \alpha_n^0) \neq (0, \dots, 0) \end{array} \right. \Rightarrow \vec{a}_1, \dots, \vec{a}_n$ са ЛЗ.