The theorem just proved is useful in studying the question of extending the domain of definition of an analytic function. More precisely, given two domains D_1 and D_2, consider the *intersection* $D_1 \cap D_2$, consisting of all points that lie in both D_1 and D_2. If D_1 and D_2 have points in common (see Fig. 34) and a function f_1 is analytic in D_1, there *may* exist a function f_2, which is analytic in D_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $D_1 \cap D_2$. If so, we call f_2 an ***analytic continuation*** of f_1 into the second domain D_2.

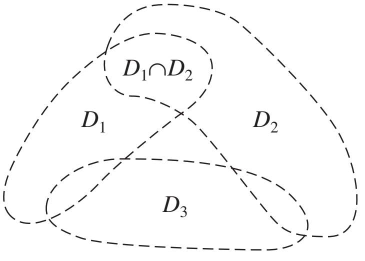

FIGURE 34

Whenever that analytic continuation exists, it is unique, according to the theorem just proved. That is, not more than one function can be analytic in D_2 and assume the value $f_1(z)$ at each point z of the domain $D_1 \cap D_2$ interior to D_2. However, if there is an analytic continuation f_3 of f_2 from D_2 into a domain D_3 which intersects D_1, as indicated in Fig. 34, it is not necessarily true that $f_3(z) = f_1(z)$ for each z in $D_1 \cap D_3$. Exercise 2, Sec. 29, illustrates this.

If f_2 is the analytic continuation of f_1 from a domain D_1 into a domain D_2, then the function F defined by means of the equations

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } D_1, \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{cases}$$

is analytic in the *union* $D_1 \cup D_2$, which is the domain consisting of all points that lie in either D_1 or D_2. The function F is the analytic continuation into $D_1 \cup D_2$ of either f_1 or f_2; and f_1 and f_2 are called ***elements*** of F.

29. REFLECTION PRINCIPLE

The theorem in this section concerns the fact that some analytic functions possess the property that $\overline{f(z)} = f(\overline{z})$ for all points z in certain domains, while others do not. We note, for example, that the functions $z + 1$ and z^2 have that property when D is the entire finite plane; but the same is not true of $z + i$ and iz^2. The theorem here, which is known as the ***reflection principle,*** provides a way of predicting when $\overline{f(z)} = f(\overline{z})$.

Theorem. *Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then*

$$\overline{f(z)} = f(\overline{z}) \tag{1}$$