# Assignment 1: Alpha Decay Model

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## 1 Introduction

In this report, we aim to model $\alpha$-decay by solving the time-independent Schrödinger equation across a piecewise potential representing Gamov's potential model.

By computing the transmission coefficient $T$, we are able to estimate the half-lives of the following nuclei: $^{212}$Po, $^{238}$U and $^{222}$Rn.

## 2 Theory and Methods

### 2.1 Quantum Tunnelling and $\alpha$-Decay

Alpha decay is a process in which an $\alpha$-particle (a $^{4}$He nucleus) escapes a heavier nucleus via tunnelling through a potential barrier. Despite the $\alpha$-particle not having enough energy to overcome the Coulomb barrier classically, quantum tunnelling allows it to escape the nucleus with a finite probability.

We can describe this process in three regions:

- **Region I:** The nuclear interior ($r < R$), where the potential is modelled as a deep square well.
- **Region II:** The Coulomb barrier ($R < r < r_{\text{turn}}$), where the $\alpha$-particle has insufficient energy to classically escape.
- **Region III:** The exterior ($r > r_{\text{turn}}$), where the particle is free to escape again.

The turning point $r_{\text{turn}}$ is defined by equating the potential and the kinetic energy:

$$r_{\text{turn}} = \frac{\alpha Z_D Z_\alpha \hbar c}{E} \tag{1}$$

where $Z_D$ and $Z_\alpha$ are the charges of the daughter nucleus and $\alpha$-particle respectively, $\alpha$ is the fine structure constant, and $E$ is the kinetic energy.

## 2.2 Mathematical Implementation

We solve the time-independent Schrödinger equation using the following ansatz in each region:

$$\psi_i(x) = A_i e^{ik_i x} + B_i e^{-ik_i x} \tag{2}$$

where $k_i$ is the local wave number:

$$k_i = \begin{cases} \sqrt{2\mu(E - V_i)}/\hbar, & \text{if } E > V_i \\ i\sqrt{2\mu(V_i - E)}/\hbar, & \text{if } E < V_i \end{cases} \tag{3}$$

Here $\mu$ is the reduced mass of the $\alpha$-daughter system.

At each interface between regions, we need to impose continuity of the wave function and its derivative:

$$\psi_i(x_{\text{int}}) = \psi_{i+1}(x_{\text{int}}) \tag{4}$$

$$\psi_i'(x_{\text{int}}) = \psi_{i+1}'(x_{\text{int}}) \tag{5}$$

With $2(N-1)$ conditions for $N$ regions, we have a linear system for the unknown coefficients: $A_i$ and $B_i$. We fix $A_1 = 1$ (the incoming wave) and $B_N = 0$ (no incoming wave from infinity).

## 2.3 Numerical Implementation

The domain is divided into $N$ regions and the potential is defined as:

$$V(r) = \begin{cases} -V_0, & r < R \\ \frac{\alpha Z_D Z_\alpha \hbar c}{r}, & r \geq R \end{cases} \tag{6}$$

The potential for each region is evaluated at the midpoint between its boundaries.

Furthermore, we solve our linear system of equations as: $\mathbf{A}\vec{X} = \vec{B}$. Where $\vec{X}$ has the unknown coefficients we are looking for. The system is solved using NumPy's linear solver.

### 2.4 Transmission Coefficient and Lifetime

Once the coefficients are obtained, we compute the transmission $T$ and reflection $R$ coefficients as follows:

$$T = \left|\frac{A_N}{A_1}\right|^2 \frac{k_N}{k_1} \quad R = \left|\frac{B_1}{A_1}\right|^2 \tag{7}$$

and check if the probability: $R + T = 1$ is conserved.

We assume that the $\alpha$-particle hits back and forth the walls of the nucleus of size $R$ with frequency: $v/(2R)$. The lifetime $\tau$ is then:

$$\frac{1}{\tau} = T\frac{v_{\mathrm{rel}}}{2R}; \quad v_{\mathrm{rel}} = v_\alpha \left(1 + \frac{M_\alpha}{M_{\mathrm{D}}}\right) \tag{8}$$

where $v_{\mathrm{rel}}$ is the relative velocity.

The half-life is computed as:

$$t_{1/2} = \tau \ln 2 \tag{9}$$

### 2.5 Wave Function

We reconstruct the wave function $\psi(r)$ using the found coefficients. We plot the probability density $|\psi(r)|^2$ alongside with the potential $V(r)$ in order to visualize the tunnelling behaviour.

## 3 Results and Discussion

### 3.1 Overview

We have applied our implementation to simulate the $\alpha$-decay of three heavy nuclei:

- $^{212}$Po, with daughter $^{208}$Pb.
- $^{238}$U, with daughter $^{234}$Th.
- $^{222}$Rn, with daughter $^{218}$Po.

For each case, we compute the transmission coefficient $T$, estimate the lifetime $\tau$, and determine the half-life $t_{1/2}$. The reduced mass $\mu$, nuclear radius $R$, and turning point $r_{\mathrm{turn}}$ are calculated accordingly for each nucleus.

Table 1: Computed and experimental $\alpha$-decay values.

| **Nucleus** | $E_\alpha$ **(MeV)** | $T$ **(dimensionless)** | $t_{1/2}^{\mathbf{(comp)}}$ **(s)** | $t_{1/2}^{\mathbf{(exp)}}$ **(s)** |
|---|---|---|---|---|
| $^{212}$Po | 8.95 | $1.01 \times 10^{-15}$ | $5.43 \times 10^{-7}$ | $2.99 \times 10^{-7}$ |
| $^{238}$U | 4.27 | $2.19 \times 10^{-39}$ | $3.90 \times 10^{17}$ | $1.41 \times 10^{17}$ |
| $^{222}$Rn | 5.59 | $1.91 \times 10^{-27}$ | $3.86 \times 10^{5}$ | $3.30 \times 10^{5}$ |

## 3.2 Transmission Coefficients and Half-Lives

Table 1 shows the results obtained for each nuclei. We compare them with experimental values for validation.

We observe how our results deviate slightly from the known experimental values. Nevertheless, they remain in the same order of magnitude. $^{222}$Rn is the closest to the experimental result.

It is worth mentioning that tuning the values of $R_0 = 1.40 - 1.47$ fm in the formula for the Nuclear radius:

$$R = R_0 A^{1/3} \tag{10}$$

where $A$ is the respective mass number, helped achieving a better value for $t_{1/2}$. Thus noting the sensitivity of our model to the width of the potential well.

The depth of the potential is also of interest since setting $V_0 = 0$ MeV also yielded different results for $t_{1/2}$. For instance, $t_{1/2} = 3.67 \times 10^{-7}$ s for $^{212}$Po when $V_0 = 0$ MeV. This is a more accurate result than the one obtained with $V_0 = -134$ MeV. Nevertheless, our model already makes some assumptions and approximations. Thus, we decide to set $V_0 = -134$ MeV and maintain a strongly negative potential in order to achieve as realistic an approach as possible.

## 3.3 Probability Density and Potentials

The plots below show the potentials and probability densities for each computed nuclei.

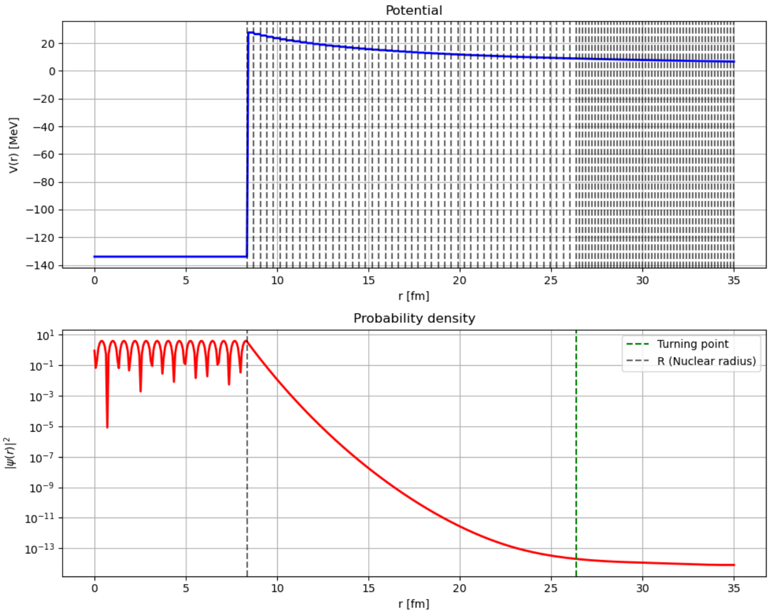


Figure 1: Potential and Probability density for $\alpha-$decay of $^{212}$Po where $R \approx$ 8.35 fm and $r_{\text{turn}} \approx 26.39$ fm.

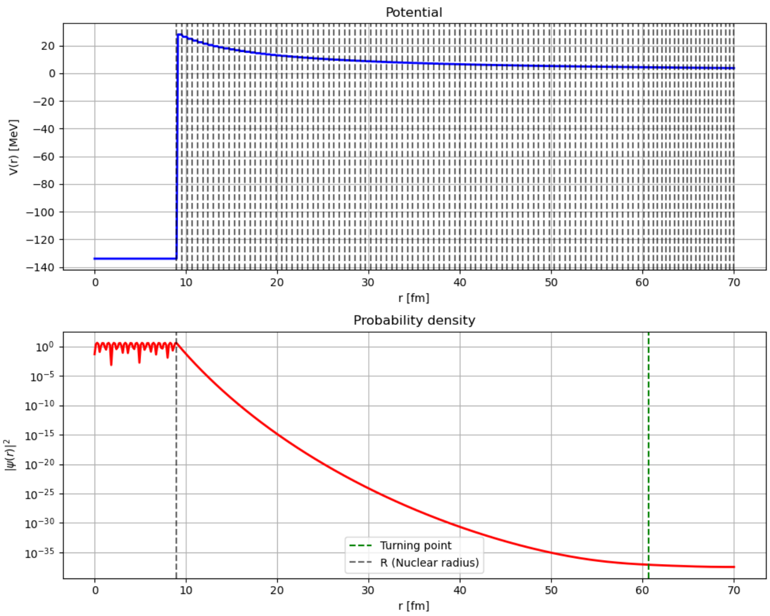


Figure 2: Potential and Probability density for $\alpha-$decay of $^{238}$U where $R \approx$ 8.99 fm and $r_{\text{turn}} \approx 60.70$ fm.

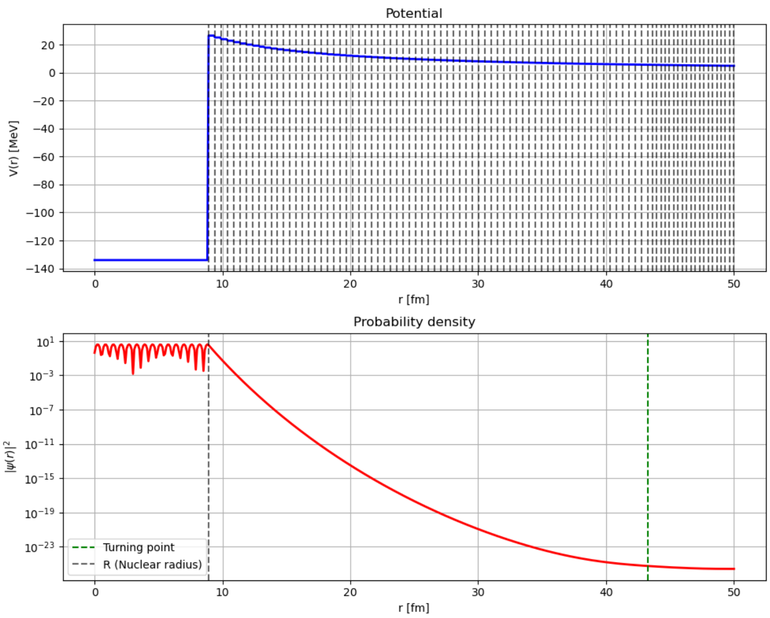


Figure 3: Potential and Probability density for $\alpha-$decay of $^{222}$Rn where $R \approx 8.90$ fm and $r_{\text{turn}} \approx 43.28$ fm.

All three plots show the expected behaviour for the probability density, due to the corresponding ansatz, in each region:

- **Region I** ($r < R$): an oscillatory wave.
- **Region II** ($R < r < r_{\text{turn}}$): exponential decay in the in the classically forbidden region.
- **Region III** ($r > r_{\text{turn}}$): an outgoing wave.

Furthermore, as we can see, the heavier the nuclei, the wider the Coulomb barrier results. This translates in a much lower probability density, a shorter transmission coefficient $T$ and a much larger half-life $t_{1/2}$.

## 3.4 Model Simplifications

In our model, some simplifications have been made: we approximate the potential as a square well, we neglect any nuclear structure effects, we reduce the problem to one dimension, etc.

# 4 Conclusion

The results for $^{212}$Po, $^{238}$U, and $^{222}$Rn show a relative agreement with known experimental half-lives. Furthermore, we are able to observe the great impact on half-lives that nuclear size has. We also observe the sensitivity of our model to some potential parameters.

In summary, despite the approximations made, our model provides some good insight in order to better understand $\alpha$-decay.