Assignment 7: The Earth Temperature

Marta Casado Carrasquer

May 21, 2025

1 Introduction

The objective of this assignment is to construct a model for estimating the average surface temperature of the Earth by simulating radiative energy transfer through a vertical atmospheric column. The model includes both incoming visible radiation and outgoing infrared radiation, incorporating absorption and re-emission by atmospheric layers.

This model aims to capture the greenhouse effect, in which infrared radiation emitted by the Earth's surface is trapped by the atmosphere, leading to a surface temperature higher than the one without an atmosphere.

2 Theory and Methods

2.1 Assumptions

In order to build our model, the following assumptions are made:

- We model the atmosphere as a vertical column, with N horizontal layers of equal height h.
- We assume constant temperature with height in order to compute density and pressure.
- Radiative transfer is the only energy transport mechanism considered (convection, for instance, is ignored).
- Each layer absorbs, emits, and transmits radiation.
- The Earth behaves as a blackbody.
- A fraction of the radiation from the sun is reflected at the top of the atmosphere due to Earth's albedo, and an additional small fraction is reflected at the surface.

2.2 Equations used

For each atmospheric layer i, we track the following:

- $T_i^{\text{in,V}}$: Downward visible radiation reaching layer i
- $T_i^{\text{in,IR}}$: Downward infrared radiation reaching layer i
- T_i^{out}: Upward infrared radiation leaving layer i
- E_i: Emission by layer i

The visible radiation at the top of the atmosphere that comes from the sun is:

$$F_{\text{in}} = F_{\text{sun}}(1 - \text{albedo}) \tag{1}$$

where $F_{\text{sun}} = 344$ W/m^2 and albedo $= 0.3$. This radiation propagates downward according to:

$$T_i^{\text{in,V}} = T_{i+1}^{\text{in,V}} \exp(-\sigma_{\text{vis}}\rho_i h) \tag{2}$$

where σ_{vis} is the absorption cross section for visible light, ρ_i is the air density in layer i, and h is the height of each layer.

At the surface, a small fraction R_{surf} of visible light is reflected into the atmosphere:

$$T_1^{\text{in,V}} \mathrel{+}= T_0^{\text{in,V}} \cdot R_{\text{surf}} \cdot \exp(-\sigma_{\text{vis}}\rho_1 h) \tag{3}$$

Infrared radiation is taken into account in a similar way:

$$T_i^{\text{in,IR}} = \left(T_{i+1}^{\text{in,IR}} + \frac{E_{i+1}}{2}\right) \exp(-\sigma_{\text{IR}}\rho_i h) \tag{4}$$

$$T_i^{\text{out}} = \left(T_{i-1}^{\text{out}} + \frac{E_{i-1}}{2}\right) \exp(-\sigma_{\text{IR}}\rho_i h) \tag{5}$$

The emission E_i of layer i is the sum of absorbed visible and infrared radiation:

$$E_i = \left[\frac{E_{i-1} + E_{i+1}}{2} + T_{i-1}^{\text{out}} + T_{i+1}^{\text{in,IR}}\right](1 - \exp(-\sigma_{\text{IR}}\rho_i h)) + T_{i+1}^{\text{in,V}}(1 - \exp(-\sigma_{\text{vis}}\rho_i h)) \tag{6}$$

2.3 Numerical Method

These equations form a system that we solve iteratively:

1. Initialize all fluxes to zero except the incoming solar radiation at the top.

2. Go from top to bottom to compute $T_i^{\text{in,V}}$ and $T_i^{\text{in,IR}}$.
3. Go from bottom to top to compute T_i^{out}.
4. Update E_i in each layer.
5. Check for convergence: stop when $\max|E_i^{(n+1)} - E_i^{(n)}| < \varepsilon$.

Once our system converges, we use the surface flux $F_{\text{ground}} = E_0$ to estimate the surface temperature:

$$T_{\text{surface}} = \left(\frac{F_{\text{ground}}}{\sigma_{\text{SB}}}\right)^{1/4} \tag{7}$$

2.4 Density and Pressure

To compute the density as a function of altitude z, we use the formula:

$$\rho(z) = \rho_0 \exp\left(-\frac{g_0 M z}{R T_0}\right) \tag{8}$$

where:

- $\rho_0 = \dfrac{p_0}{(R/M)T_0}$ is the air density at sea level
- $p_0 = 101325$ Pa is the atmospheric pressure
- $M = 0.029$ kg/mol is the molar mass of air
- $R = 8.314$ J/(mol·K) is the universal gas constant
- $T_0 = 288$ K is the temperature at sea level
- $g_0 = 9.81$ m/s is the gravitational acceleration

The pressure is then computed as:

$$p(z) = \frac{\rho(z) R T_0}{M} \tag{9}$$

2.5 Model Validation

In order to validate our model, we check if the flux leaving the Earth is the same as the flux reaching the top of the atmosphere:

$$F_{\text{out}} = T_{N-1}^{\text{out}} + \frac{1}{2} E_{N-1} \tag{10}$$

so that $F_{\text{out}} \approx F_{\text{in}}$.

3 Results and Discussion

3.1 Model Parameters

We simulate the Earth's atmosphere with a vertical column of $N = 50$ layers over a total height of 100 km with $\sigma_{\mathrm{IR}} = 1.1 \times 10^{-3}$ m^{2}kg^{-1} and $\sigma_{\mathrm{vis}} = 1 \times 10^{-4}$ m^{2}kg^{-1}.

We have observed that the model is very sensitive to:

- **Infrared absorption cross-section $\boldsymbol{\sigma_{\mathrm{IR}}}$**: Small changes in σ_{IR} significantly impact the final surface temperature of the Earth where higher values of σ_{IR} lead to higher temperatures.
- **Visible absorption cross-section $\boldsymbol{\sigma_{\mathrm{vis}}}$**: Changing this parameter leads to drastic changes in the outgoing flux at the top of the atmosphere.
- **Atmosphere height and number of layers N**: Increasing the height or the number of layers affects the final output. While the model should be insensitive to N, we found that changing N alters the results, indicating that our model is not working correctly.

3.2 Energy Balance and Model Failure

A sign of the correct implementation of our model is the conservation of energy. The total flux leaving the top of the atmosphere should equal the incoming solar flux. However, our model violates this condition by yielding the following results:

$$F_{\mathrm{in}} = 240.80 \text{ W/m}^2 \tag{11}$$

$$F_{\mathrm{out}} = T^{\mathrm{out}}_{N-1} + \frac{1}{2}E_{N-1} = 127.98 \text{ W/m}^2 \tag{12}$$

$$F_{\mathrm{in}} - F_{\mathrm{out}} = 112.82 \text{ W/m}^2 \tag{13}$$

Individual checks for each layer were also performed:

Table 1: Radiative fluxes for each atmospheric layer.

Layer	**Height (km)**	$\boldsymbol{\rho}$ (kg/m^3)	$T_{\text{in}}^{\text{vis}}$	$T_{\text{in}}^{\text{IR}}$	T_{out}	E (W/m^2)	T (K)
0	0.00	1.2272	75.45	19.55	0.00	549.07	313.70
1	2.00	0.9676	99.55	33.95	32.66	513.86	308.54
2	4.00	0.7630	117.04	49.68	54.05	471.34	301.95
3	6.00	0.6016	136.33	63.56	77.12	405.26	290.76
4	8.00	0.4744	153.76	73.20	98.52	331.18	276.45
5	10.00	0.3740	169.07	77.65	115.99	260.41	260.33
6	12.00	0.2949	182.20	77.24	128.67	199.13	243.44
7	14.00	0.2325	193.27	73.08	136.84	149.42	226.57
8	16.00	0.1834	202.47	66.48	141.32	110.82	210.26
9	18.00	0.1446	210.03	58.65	143.13	81.73	194.85
10	20.00	0.1140	216.19	50.52	143.18	60.20	180.51
11	22.00	0.0899	221.18	42.70	142.19	44.44	167.32
12	24.00	0.0709	225.19	35.56	140.67	32.95	155.27
13	26.00	0.0559	228.41	29.27	138.96	24.58	144.29
14	28.00	0.0441	230.97	23.87	137.28	18.45	134.31
15	30.00	0.0347	233.02	19.33	135.72	13.94	125.22
16	32.00	0.0274	234.64	15.56	134.34	10.60	116.94
17	34.00	0.0216	235.93	12.48	133.16	8.11	109.36
18	36.00	0.0170	236.95	9.97	132.17	6.24	102.41
19	38.00	0.0134	237.76	7.94	131.35	4.82	96.00
20	40.00	0.0106	238.40	6.31	130.68	3.73	90.09
21	42.00	0.0084	238.91	5.00	130.13	2.90	84.60
22	44.00	0.0066	239.31	3.96	129.69	2.27	79.51
23	46.00	0.0052	239.62	3.14	129.34	1.77	74.76
24	48.00	0.0041	239.87	2.48	129.06	1.39	70.32
25	50.00	0.0032	240.07	1.96	128.83	1.09	66.17
26	52.00	0.0025	240.22	1.55	128.65	0.85	62.29
27	54.00	0.0020	240.35	1.22	128.51	0.67	58.64
28	56.00	0.0016	240.44	0.96	128.40	0.53	55.22
29	58.00	0.0012	240.52	0.76	128.31	0.41	52.01
30	60.00	0.0010	240.58	0.60	128.24	0.33	48.99
31	62.00	0.0008	240.63	0.47	128.18	0.26	46.15
32	64.00	0.0006	240.66	0.37	128.14	0.20	43.47
33	66.00	0.0005	240.69	0.29	128.11	0.16	40.96
34	68.00	0.0004	240.72	0.23	128.08	0.13	38.59
35	70.00	0.0003	240.73	0.18	128.06	0.10	36.36
36	72.00	0.0002	240.75	0.14	128.04	0.08	34.26
37	74.00	0.0002	240.76	0.11	128.03	0.06	32.28
38	76.00	0.0001	240.77	0.08	128.02	0.05	30.41
39	78.00	0.0001	240.78	0.06	128.01	0.04	28.66
40	80.00	0.0001	240.78	0.05	128.00	0.03	27.00
41	82.00	0.0001	240.79	0.04	128.00	0.02	25.45
42	84.00	0.0001	240.79	0.03	127.99	0.02	23.98
43	86.00	0.0000	240.79	0.02	127.99	0.01	22.59
44	88.00	0.0000	240.79	0.01	127.99	0.01	21.29
45	90.00	0.0000	240.80	0.01	127.98	0.01	20.06
46	92.00	0.0000	240.80	0.01	127.98	0.01	18.91
47	94.00	0.0000	240.80	0.00	127.98	0.01	17.81
48	96.00	0.0000	240.80	0.00	127.98	0.00	16.79
49	98.00	0.0000	240.80	0.00	127.98	0.00	15.21

The plots obtained for the temperature, density, and pressure are shown below:

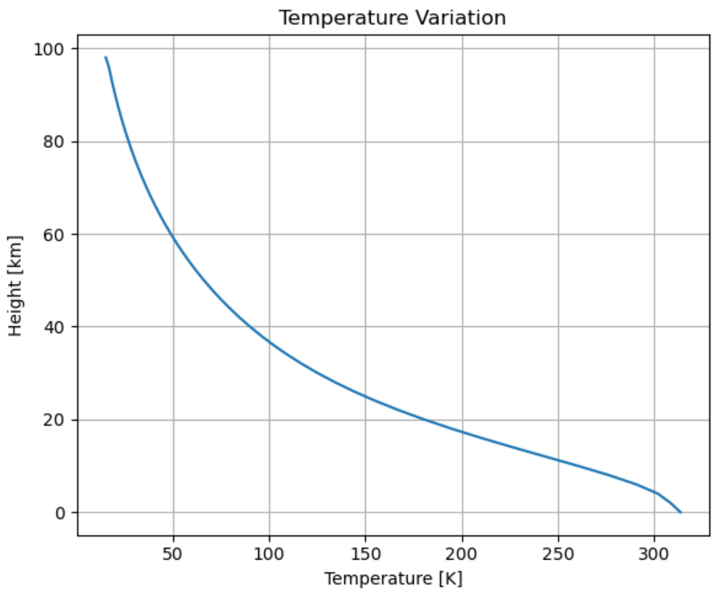

Figure 1: Temperature variation with height.

The final surface temperature was obtained to be:

$$T_{\text{surf}} = 313.70 \text{ K} = 40.55°\text{C} \tag{14}$$

with a surface flux of: $E_{\text{surf}} = 549.07 \text{ W/m}^2$.

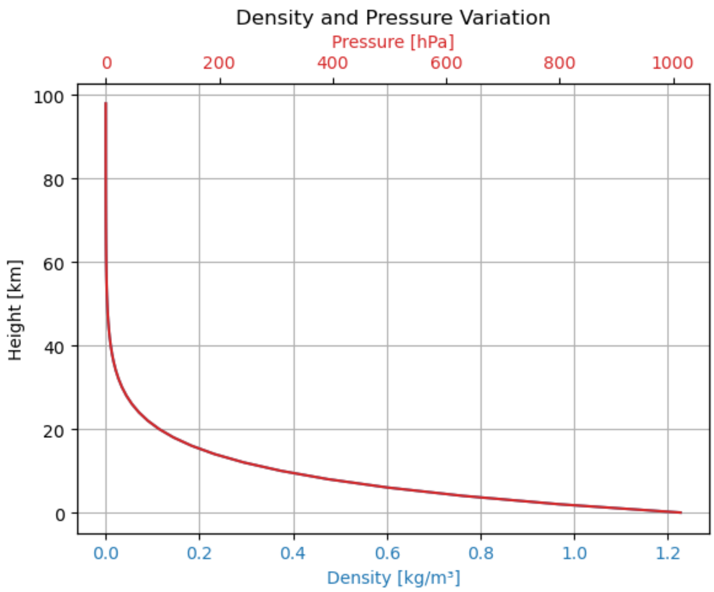

Figure 2: Density and pressure variation with height.

4 Conclusions

Despite, I think, following correctly the guidance for our implementation, the outputs of our simulation do not align with the expected behaviour. For instance, the model fails to conserve energy, the final surface temperature is very dependent on the height of the atmosphere and its number of layers, and adjusting the cross-section parameters does not lead to a physically plausible result.

In conclusion, we suspect that there may be a conceptual oversight that we are missing or a numerical error in our iteration scheme that we are not able to capture.

Without really knowing what is going wrong, we are forced to conclude that our model fails in its aim of estimating Earth's average surface temperature.