INDR 422/522 - Spring 2023 Fikri Karaesmen

Notes on MA and AR processes: Summary Statistics (Mean, Variance, Auto-covariance, Auto-correlations)

1 Summary statistics of MA(1) processes

Let's first compute theoretical unconditional expected value and variance of a MA(1) process. We are assuming that we have not observed realizations $y_1, y_2, ..., y_{t-1}$ and the error terms $\epsilon_1, \epsilon_2, ..., \epsilon_{t-1}$. We can therefore not condition on the past observations.

$$Y_t = b_0 + \theta_1 \epsilon_{t-1} + \epsilon_t$$

where ϵ_t are i.i.d random variables with mean zero and variance σ_ϵ^2 and $|\theta_1| < 1$.

Note that:

$$E[Y_t] = E[b_0 + \theta_1 \epsilon_{t-1} + \epsilon_t] = E[b_0] + \theta_1 E[\epsilon_{t-1}] + E[\epsilon_t] = b_0$$

and

$$Var[Y_t] = Var[b_0 + \theta_1 \epsilon_{t-1} + \epsilon_t] = Var[b_0] + \theta_1^2 Var[\epsilon_{t-1}] + Var(\epsilon_t) = (\theta_1^2 + 1)\sigma_\epsilon^2.$$

Let's now compute the 1-lag autocovariance:

$$Cov(Y_t, Y_{t-1}) = E[Y_t Y_{t-1}] - E[Y_t]E[Y_{t-1}] = E[Y_t Y_{t-1}] - b_0^2$$

focusing on the first term, we obtain:

$$\begin{aligned}
E[Y_t Y_{t-1}] &= E[(b_0 + \theta_1 \epsilon_{t-1} + \epsilon_t)(b_0 + \theta_1 \epsilon_{t-2} + \epsilon_{t-1})] \\
&= E[b_0^2] + E[b_0 \theta_1 \epsilon_{t-2}] + E[b_0 \epsilon_{t-1}] + E[b_0 \theta_1 \epsilon_{t-1} \epsilon_{t-2}] + E[\theta_1^2 \epsilon_{t-1} \epsilon_{t-2}] \\
&\quad + E[\theta_1 \epsilon_{t-1}^2] + E[b_0 \epsilon_t] + E[\theta_1 \epsilon_t \epsilon_{t-2}] + E[\epsilon_t \epsilon_{t-1}] \\
&= b_0^2 + \theta_1 E[\epsilon_t^2] \\
&= b_0^2 + \theta_1 (\theta_1^2 + 1)\sigma_\epsilon^2
\end{aligned}$$

Note that $E[\epsilon_{t-1}\epsilon_{t-2}] = E[\epsilon_{t-1}]E[\epsilon_{t-2}] = 0$ because ϵ_t are independent (the same holds for $E[\epsilon_t \epsilon_{t-2}]$ and $E[\epsilon_t \epsilon_{t-1}]$). Finally $Var[\epsilon_t] = E[\epsilon_t^2] - E[\epsilon_t]^2$ so $E[\epsilon_t^2] = \sigma_\epsilon^2$.

So we have:

$$Cov(Y_t, Y_{t-1}) = b_0^2 + \theta_1\sigma_\epsilon^2 - b_0^2 = \theta_1\sigma_\epsilon^2$$

Finally,

$$Corr(Y_t, Y_{t-1}) = \frac{Cov(Y_t, Y_{t-1})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_t - 1)}} = \frac{\theta_1\sigma_\epsilon^2}{(1+\theta_1^2)\sigma_\epsilon^2} = \frac{\theta_1}{1+\theta_1^2}$$

We can also compute the k lag autocorrelation for $k \geq 2$ similarly. Let's investigate:

$$E[Y_tY_{t-k}] = E[(b_0 + \theta_1\epsilon_{t-1} + \epsilon_t)(b_0 + \theta_1\epsilon_{t-k-1} + \epsilon_{t-k})]$$

Note that, as in the one-lag case, because of independence of all ϵ_t, all expected values involving the products of ϵ_t terms such as $E[\epsilon_t\epsilon_{t-k}]$ equal zero. Therefore:

$$E[Y_tY_{t-k}] = b_0^2$$

So we obtain:

$$Cov(E[Y_tY_{t-k}]) = b_0^2 - b_0^2 = 0$$

and therefore $Corr(Y_tY_{t-k}) = 0$ for all $k \geq 2$.

2 Summary statistics of AR(1) processes

Let's now consider an AR(1) process:

$$Y_t = a_0 + \phi_1 Y_{t-1} + \epsilon_t$$

where ϵ_t are i.i.d random variables with mean zero and variance σ_ϵ^2 and $|\phi_1| < 1$.

We can compute the expected value and variance using stationarity of the process i.e. we must have $E[Y_t] = E[Y_{t-1}] = \mu$ and $Var[Y_t] = Var[Y_{t-1}] = \sigma_Y^2$ for all t. Note that:

$$E[Y_t] = a_0 + E[\phi_1 Y_{t-1}] + E[\epsilon_t] = a_0 + \phi_1 E[Y_{t-1}]$$

Using stationarity we must have $E[Y_t] = E[Y_{t-1}] = \mu_Y$. So:

$$E[Y_t] = \frac{a_0}{1-\phi_1}.$$

and similarly:

$$Var[Y_t] = Var[a_0] + Var[\phi_1 Y_{t-1}] + Var[\epsilon_t] = \phi_1^2 Var[Y_t] + \sigma_\epsilon^2$$

where we used $Var[Y_t] = Var[Y_{t-1}]$. This yields:

$$Var[Y_t] = \frac{\sigma_\epsilon^2}{1-\phi_1^2}.$$

Let's now focus on:

$$Cov(Y_t, Y_{t-1}) = E[Y_t Y_{t-1}] - E[Y_t]E[Y_{t-1}] = E[Y_t Y_{t-1}] - \frac{a_0^2}{(1-\phi_1)^2} \tag{1}$$

We need to compute:

$$E[Y_t Y_{t-1}] = E[(a_0 + \phi_1 Y_{t-1} + \epsilon_t)Y_{t-1}]$$

Note that all terms involving products of ϵ_t (and ϵ_{t-1}) will equal zero in expected value since ϵ_t are independent. Therefore, we can simplify:

$$E[Y_t Y_{t-1}] = E[a_0 Y_{t-1}] + E[\phi_1 Y_{t-1}^2] = \frac{a_0^2}{(1-\phi_1)} + \phi_1 E[Y_{t-1}^2].$$

Now noting that $E[Y_{t-1}^2] = E[Y_t^2] = Var(Y_t) + E[Y_t]^2$, we can write:

$$E[Y_t Y_{t-1}] = \frac{a_0^2}{(1-\phi_1)} + \phi_1 \left(\frac{\sigma_\epsilon^2}{1-\phi_1^2} + \frac{a_0^2}{(1-\phi_1)^2} \right).$$

After some algebra, combining the above with equation (1) we obtain:

$$Cov(Y_t, Y_{t-1}) = \phi_1 \frac{\sigma^2}{1-\phi_1^2}.$$

and

$$Corr(Y_t, Y_{t-1}) = \frac{Cov(Y_t, Y_{t-1})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-1})}} = \phi_1.$$

Higher lag autocorrelations can be obtained similarly (with more effort) and we find that the k-lag correlation is given by:

$$Corr(Y_t, Y_{t-k}) = \phi_1^k.$$