12pt

INDR 450/550 Lecture Notes 1

Fikri Karaesmen

Februay 26, 2022

1 Simple independent stationary demand process: unbiasedness and variance of Forecasts

Let us take the following simple demand process:

$$Y(t) = c + \epsilon_t$$

where ϵ_t are iid random variables with mean zero and standard deviation σ^2. Note that Y_t are also iid random variables with mean c and standard deviation σ^2.

1.1 Naive Forecast

For the naive forecast, the forecast is equal to the last observation:

$$\hat{y}_{t+1|t} = y_t.$$

We have:

$$E[\hat{Y}_{t+1|t}] = E[Y_t] = E[c + \epsilon_t] = c.$$

Therefore, the naive forecast is unbiased for this demand model. We can compute its variance as:

$$Var[\hat{Y}_{t+1|t}] = Var[Y_t] = \sigma^2.$$

1.2 Moving Average Forecast

For the k-period MA forecast, the forecast is equal to the average of the last k observation:

$$\hat{y}_{t+1|t} = (y_t + y_{t-1} + ... + y_{t-k+1+})/k.$$

We have:

$$E[\hat{Y}_{t+1|t}] = E[(Y_t + Y_{t-1} + ... + Y_{t-k+1+})/k] = ck/k = c.$$

Therefore, the MA-forecast is also unbiased. Its variance is:

$$Var[\hat{Y}_{t+1|t}] = Var[(Y_t + Y_{t-1} + ... + Y_{t-k+1+})/k] = \frac{k\sigma^2}{k^2} = \frac{\sigma^2}{k}.$$

Note that the variance of the forecast is decreasing in k.

1.3 Exponential Smoothing Forecast

For the ES forecast, the forecast is equal to a weighted average of the most recent observation with the most recent forecast. The weight α $(0 \leq \alpha \leq 1)$ is called a smoothing parameter

$$\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}$$

We can expand the recursion and obtain an equivalent expression:

$$\hat{y}_{t+1|t} = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + ... + \alpha(1-\alpha)^{t-1}y_1.$$

Letting $t \to \infty$ we can compute the asymptotic expected value of the forecast as:

$$E[\hat{Y}_{t+1|t}] = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i c = \frac{\alpha c}{1-(1-\alpha)} = c$$

Therefore, the ES-forecast is asymptotically unbiased. We can also compute its asymptotic variance as:

$$Var[\hat{Y}_{t+1|t}] = \sum_{i=0}^{\infty} \alpha^2(1-\alpha)^{2i}\sigma^2 = \frac{\alpha^2\sigma^2}{1-((1-\alpha)^2)} = \frac{\alpha\sigma^2}{2-\alpha}.$$

We can now note that the asymptotic variance of the forecast is increasing in the smoothing parameter α. To minimize the asymptotic variance, we select $\alpha = 0$ which leads to a forecast variance of zero.

2 Simple trend demand process with iid error terms: unbiasedness and variance of Forecasts

Let us now take the following simple demand process:

$$Y(t) = c + bt + \epsilon_t$$

where ϵ_t are iid random variables with mean zero and standard deviation σ^2. Note that Y_t are independent random variables with mean $c + bt$ and standard deviation σ^2.

2.1 Simple Forecasts are not Unbiased

Take the naive forecast: we have

$$E[\hat{Y}_{t+1|t}] = E[Y_t] = c + bt.$$

but

$$E[Y_{t+1}] = c + b(t+1) \neq E[\hat{Y}_{t+1|t}].$$

Therefore, the naive forecast of a bias of b.

If we consider a 3-period moving average, we have:

$$\begin{aligned}
E[\hat{Y}_{t+1|t}] &= E[Y_t + Y_{t-1} + Y_{t-2}]/3 \\
&= (3c + b((t-2+(t-1)+(t-3)))/3 \\
&= c + b(t-2)
\end{aligned}$$

Note that the forecast has a bias of: $2b$. It systematically lags behind the demand.

2.2 Trend forecasts

To have an unbiased forecast of a trend process, we need to involve a trend estimator. Here are some ways to do that:

$$\hat{y}_{t+1|t} = y_t + (y_{t-1} - y_{t-2})$$

$$\hat{y}_{t+1|t} = y_t + ((y_{t-1} - y_{t-3}))/2$$

or

$$\hat{y}_{t+1|t} = y_t + ((y_{t-1} - y_1))/(t-1).$$

All of the above can be verified to be unbiased. Let's check this for the first one:

$$\begin{aligned}
E[\hat{Y}_{t+1|t}] &= E[Y_t] + E[Y_{t-1} - Y_{t-2}] \\
&= c + bt + (c + b(t-1) - c - b(t-2)) \\
&= c + b(t+1) \\
&= E[Y_{t+1}].
\end{aligned}$$

We can now note that we are using an estimator of trend $(y_{t-1} - y_{t-2})$ in addition an estimator of the last position (y_t) of the process.

We can also compute the variance of such forecasts:

$$\begin{aligned}
Var[\hat{Y}_{t+1|t}] &= Var[Y_t] + Var[Y_{t-1} - Y_{t-2}] \\
&= 3\sigma^2
\end{aligned}$$

From the principles that we saw before, we know that we can reduce variances by averaging over multiple values of the estimators or taking a weighted average of the last observation with the last forecast.