INDR 422/522 - Spring 2023 Fikri Karaesmen

CLASS EXERCISE, March 14, 2023

1. Let X and Y be independent random variables with equal variances $\sigma^2 > 0$.

   (a) $Var(X+Y) =$
   Solution: Since X and Y are independent:
   $$Var(X+Y) = Var(X) + Var(Y) = 2\sigma^2.$$

   (b) $Var(X-Y) =$
   Solution: Since X and Y are independent:
   $$Var(X-Y) = Var(X) + (-1)^2 Var(Y) = 2\sigma^2.$$

   (c) $Var(X+X) =$
   Solution: We can have two different arguments. The first one is direct:
   $$Var(X+X) = Var(2X) = (2)^2 Var(X) = 4\sigma^2.$$
   The other one is more general. We note that X and X are not independent (in fact $Corr(X,X) = 1$). Then:
   $$Var(X+X) = Var(X) + Var(X) + 2Cov(X,X)$$
   and
   $$Cov(X,X) = E[XX] - E[X][X] = Var(X) = \sigma^2$$
   Therefore, $Var(X+X) = 2\sigma^2 + 2\sigma^2$

   (d) $Var(X-X)$
   Solution: Once again, let us review two different arguments. The first one:
   $$Var(X-X) = Var(0) = 0.$$
   For the second one, we note that X and $-X$ are not independent (in fact $Corr(X,X) = -1$). Then:
   $$Var(X-X) = Var(X)+Var(X)+2Cov(X,-X) = 2\sigma^2-2\sigma^2 = 0.$$

2. Assume that $Y_t = f(t) + \epsilon_t$ where ϵ_t are i.i.d random variables with mean zero and variance σ^2. Let $Z_t = Y_t - Y_{t-1}$. Which of the following are unbiased estimators for $f'(t)$? (*Note: This is a badly posed question because $f'(t)$ is not random but we can give an approximate answer.*)

   Solution: We note that

$$E[Z_t] = f(t) - f(t-1)$$

   In general, $f(t+1) - f(t) \neq f'(t)$

   For instance let us take:

$$f(t) = a_2 t^2$$
$$E[Z_t] = a_2 t^2 - a_2(t-1)^2 = 2a_2 t - a_2 \neq f'(t)$$

3. Let $f(t) = a_2 t^2$. Which one is an unbiased forecast for y_{t+1}? Recall that $f(x+h) = f(x) + f'(x)h + f''(x)h^2/2 + ...$

   Solution: We can follow the Taylor expansion and guess:

$$\hat{Y}_{t+1} = y_t + (y_t - y_{t-1}) + ((y_t - y_{t-1})) - ((y_{t-1} - y_{t-2}))/2.$$

   We can then check:

$$\begin{aligned}
E[\hat{Y}_{t+1}] &= a_2 t^2 + (a_2 t^2 - a_2(t-1)^2) \\
&\quad +((a_2 t^2 - a_2(t-1)^2) - ((a_2(t-1)^2 - a_2(t-2)^2)/2 \\
&= a_2 t^2 + 2(2a_2 t - a_2) - (2a_2(t-1) - a_2) \\
&= a_2 t^2 + 2a_2 t + a_2 \\
&= a_2(t+1)^2
\end{aligned}$$

   Therefore, $E[\hat{Y}_{t+1}] = a_2(t+1)^2 = E[Y_{t+1}]$.