Spring 2023 Fikri Karaesmen

INDR 422/522 CLASS EXERCISE, March, 2, 2023

1. Consider two discrete random variables X and Y with the following joint probability mass function: $P(X = 0, Y = 0) = 1/4$, $P(X = 0, Y = 1) = 1/4$, $P(X = 1, Y = 0) = 0$, $P(X = 1, Y = 1) = 1/2$. Find:

   (a) $E[X]$

   Solution:

$$\begin{aligned} E[X] &= \sum_x \sum_y xp(x,y) \\ &= 0(p(0,0) + p(0,1)) + 1(p(1,0) + p(1,1)) \\ &= 1/2 \end{aligned}$$

   (b) $E[Y]$

   (c) $E[XY]$

   Solution:

$$\begin{aligned} E[XY] &= \sum_x \sum_y xyp(x,y) \\ &= 1p(1,1) \\ &= 1/2 \end{aligned}$$

   (d) $P(X = 0|Y = 1)$

   Solution:

$$\begin{aligned} P(X = 0|Y = 1) &= \frac{P(X = 0, Y = 1)}{P(Y = 1)} \\ &= \frac{p(0,1)}{p(0,1) + p(1,1)} \\ &= \frac{1/4}{1/4 + 1/2} \\ &= 1/3 \end{aligned}$$

   (e) $E[X|Y = 1]$

   Solution: We found that $P(X = 0|Y = 1) = 1/3$, therefore $P(X = 1|Y = 1) = 2/3$. Then,

$$E[X|Y = 1] = 0(1/3) + 1(2/3) = 2/3.$$

2. Consider a random sample (X_i, Y_i) from the joint pmf in the previous exercise.

   (a) Consider the following estimator for $E[Y]$: $\hat{Y} = 0.2Y_1 + 0.8Y_2$. Check whether it is unbiased.

   Solution: We need to check whether the below is true:

   $$E[Y] = E[\hat{Y}]$$

   But we know that $E[Y_1] = E[Y_2] = E[Y]$, therefore $E[\hat{Y}] = E[Y]$ so the estimator is unbiased.

   (b) Consider the following estimator for $E[Y]$: $\hat{Y} = Y_1^2$. Check whether it is unbiased.

   Solution: We have:

   $$E[Y_1^2] = E[Y^2] = \sum_y \sum_x y^2 p(x,y) = 1(1/4 + 1/2) = 3/4.$$

   We then have $E[Y] = E[\hat{Y}] = 3/4$.

   (c) Consider the following estimator for $E[Y]$: $\hat{Y} = (Y_1^2 + Y_3)/2$. Check whether it is unbiased.

   Solution: We have:

   $$E[(Y_1^2 + Y_3)/2] = E[Y^2] = 3/4.$$

   We then have $E[Y] = E[\hat{Y}] = 3/4$.