INDR 450/550

Spring 2022

Lecture 20: Prescriptive analytics 2

April 27, 2022

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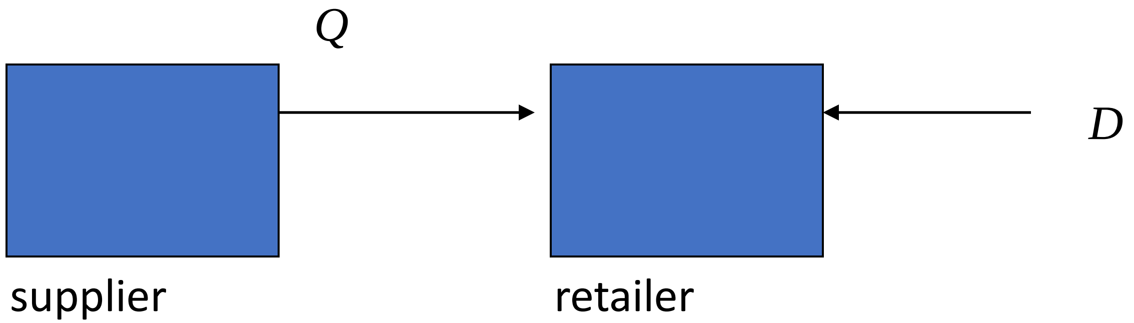
Announcements

- Class Exercise at the end of lecture today. If you are participating online, please upload your document under Course Contents/Class Exercises
- Lab 8 material (on trees, bagging, random forests, boosting) and a short video are available
- Exam scheduled for May 7 is postponed to May 13.
  - Review exercises are available
  - Make sure that you also review the class exercises and the homeworks
- HW 3 postponed to April 29

The Assumptions

- Short selling season
- Decision made in advance of the season
- No replenishments or capacity additions during the season (purchasing in advance is required)
- Demand forecasts have considerable uncertainty
- Items lose value significantly after the season

The Newsvendor Problem



(Predicted Consumer Demand)

D : demand (random variable)
Q : quantity ordered from supplier

w : wholesale price (of supplier)
r : retail price ($r>w$)
s : salvage value ($s<w$)
m: unit manufacturing cost of supplier ($m<w$)

The Newsvendor Problem

The profit as a function of Q :

$$\prod_R(Q) = r\min(Q,D) + s(Q-D)^+ - wQ$$

Because D is a r.v., we choose to maximize:

$$E[\prod_R(Q)] = E[r\min(Q,D) + s(Q-D)^+ - wQ]$$

$$(Q-D)^+ = \begin{cases} Q-D & \text{if } Q > D \\ 0 & \text{otherwise} \end{cases}$$

The Newsvendor Problem: the result

Solving for the optimal Q :

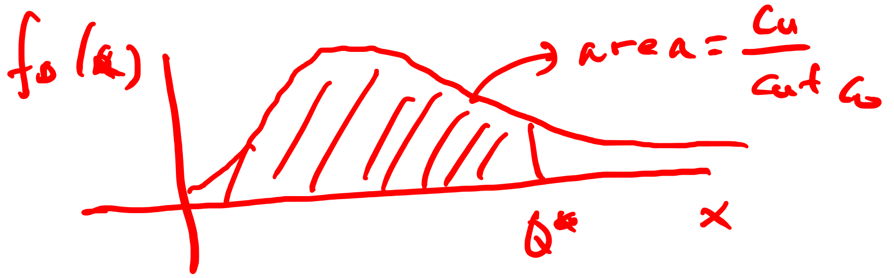
$Q^{**} = E[D]$?

$$F_D(Q^*) = \frac{c_u}{c_u + c_o}$$

$\Rightarrow Q^* = F_D^{-1}\left(\frac{c_u}{c_u+c_o}\right)$

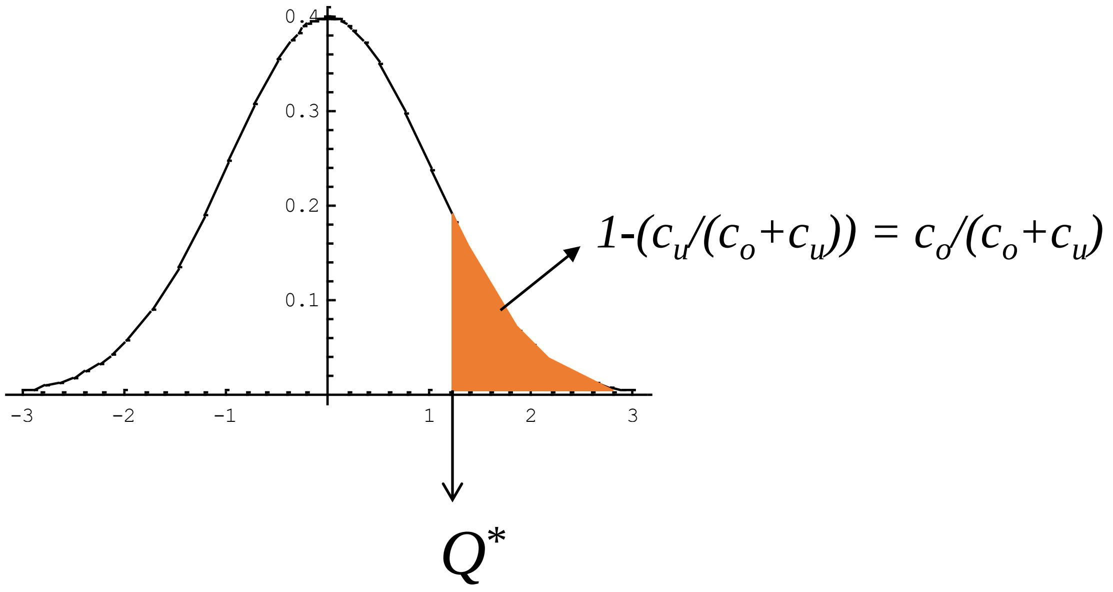
$c_u \nearrow \quad Q^* \nearrow \qquad c_o \searrow \quad Q^*$

Q^* is such that the probability of satisfying all the demand $P(D \leq Q^*)$ is equal to the critical fraction : $c_u/(c_u+c_o)$



if $c_u = c_o \Rightarrow Q^* = F_D^{-1}(1/2) = \tilde{M}$ median

The Newsvendor Problem: the optimal order quantity



A visual interpretation

Ex: Uniform $(0,a)$, $f_D(x) = \begin{cases} 1/a & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

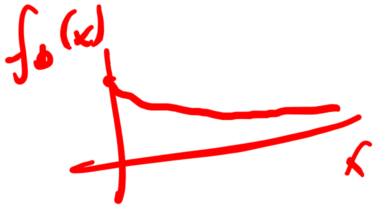
$$F_D(x) = \int_0^x 1/a \, dy = \frac{x}{a} \quad 0 < x < a$$

$$\Rightarrow F_D(Q^*) = \frac{c_u}{c_u + c_o} \Rightarrow Q^* = \left(\frac{c_u}{c_u + c_o}\right) a$$

$$\Rightarrow E[\pi(Q^*)] = (r-w) \cdot E[D] - c_u E[(D-Q^*)] - c_o E[(Q^*-D)]$$

Ex 2: $D \sim \text{expo}(\lambda)$ $\qquad f_D(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$F_D(x) = \int_0^x \lambda e^{-y} \, dy = 1 - e^{-\lambda x}$$



$$\Rightarrow \quad 1 - e^{-\lambda Q^*} = \frac{c_u}{c_u + c_o}$$

$$e^{-\lambda Q^*} = 1 - \frac{c_u}{c_u + c_o} = \frac{c_o}{c_u + c_o}$$

$$-\lambda Q^* = \ln\left(\frac{c_o}{c_u + c_o}\right) \Rightarrow Q^* = \frac{\ln\left(\frac{c_o}{c_u + c_o}\right)}{-\lambda}$$

$$\Rightarrow Q^* = -\ln\left(\frac{c_o}{c_u + c_o}\right) \cdot E[D]$$

but $F_D(x)$ may not be in closed form

or may not lead to an easy solution

for $F_D(Q^*) = \frac{cu}{cu + co}$

$$f_D(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Newsvendor Problem: normally distributed demand

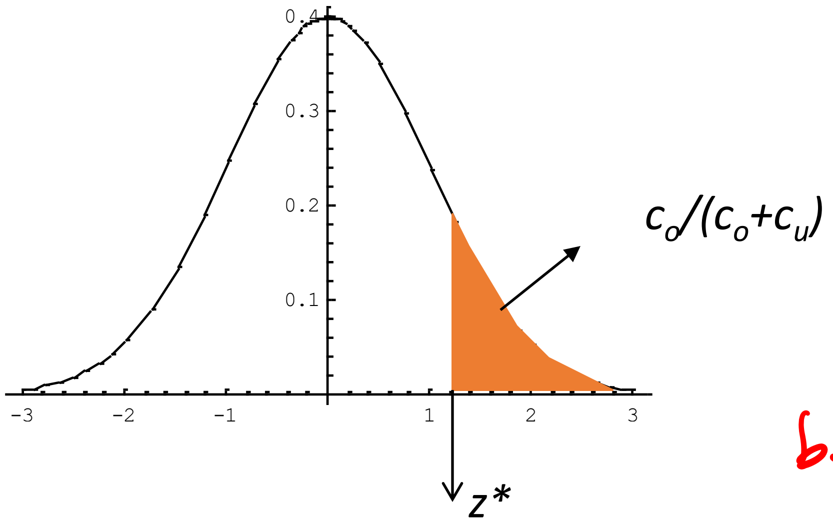
D : Normal (μ,σ)

$$F_D(Q^*) = \frac{c_u}{c_u + c_o}$$

Define $Z=(D-\mu)/\sigma$, we are looking for z^* such that:

$$F_Z(z^*) = \frac{c_u}{c_u + c_o}$$

The Newsvendor Problem: normally distributed demand



but if
$c_o > c_u$
$\Rightarrow z^* < 0$

$$\Rightarrow Q^* = \mu + z^* \sigma$$

Interpretation : z^* – safety factor (depends only on the financial parameters)

optimal order quantity = mean demand (μ)+ safety stock ($z^*\sigma$)

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

$$\Phi(z^*) = \int_{-\infty}^{z^*} f_Z(z)\,dz \qquad \Phi(z^*) = 1 - \Phi(z^*)$$

some properties

1. $f_Z(z) = f_Z(-z)$

2. $\Phi(-z) = 1 - \Phi(z) = \Phi(z)$

3. $f_Z'(z) = -z \cdot f_Z(z)$

let's now calculate

$$E\left[(Z-z^*)^+\right] = \int_{z^*}^{\infty} (z-z^*) f_Z(z)\, dz$$

$$= \underbrace{\int_{z^*}^{\infty} z f_Z(z)\, dz}_{} - z^* \underbrace{\int_{z^*}^{\infty} f_Z(z)\, dz}_{\bar{\Phi}(z^*)}$$

$$\int_{z^*}^{\infty} -f'(z)\, dz = f_Z(z^*)$$

$$= f_Z(z^*) - z^* \bar{\Phi}(z^*)$$

$$E[(z^* - Z)] = \int_{-\infty}^{z^*} (z^* - z) f_z(z) dz$$

$$= z^* F(z^*) + f_z(z^*)$$

Typically we take $D \sim N(\mu, \sigma)$

and $\mu > 4\sigma$

now, let's go for $E[(D - Q^*)^+]$

but first

$$P(D > d) = \int_d^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma dz$$

$$= \int_{(d-\mu)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2} \sigma dz = P\left(Z > \frac{d-\mu}{\sigma}\right)$$

now $E[(D-Q^*)]^+ = \int_{Q^*}^{\infty} (x-Q^*) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$

$z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma \, dz$

$$= \int_{\underbrace{\frac{Q^*-\mu}{\sigma}}_{z^*}}^{\infty} (\mu + z\sigma - \mu + z^*\sigma) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \sigma E[(z-z^*)^+]$$

similarly: $E[(Q^*-D)^+] = \sigma \cdot E[(z^*-z)^+]$

And finally

$$E[C(Q^*)] = c_u E[(D-Q^*)^+] + c_o E[(Q^*-D)^+]$$

and because $F(Q^*) = \frac{c_u}{c_u + c_o}$

$$\Rightarrow E[C(Q^*)] = (c_u + c_o)\,\sigma f_z(z^*)$$

$E[C(Q^*)]$ increasing in σ !

recall that

$$Q^* = \mu + z^* \sigma$$

$E[C(Q^*)]$ does not depend on μ

but recall that

$$E[\pi(Q^*)] = c_u \mu - E[C(Q^*)]$$

The Newsvendor Problem: normally distributed demand

Let f_z be the pdf of Z (normal (0,1)):

$$Q^* = \mu + z^*\sigma,$$

The optimal cost :

$$E[C(Q^*)] = E[c_u(D - Q^*)^+ + c_o(Q^* - D)^+] = (c_u + c_o)f_Z(z^*)\sigma$$

This is a simplification that only works for a normal distribution.
The optimal cost does not depend on the average demand !

The Newsvendor Problem: normally distributed demand

How can we improve performance (reduce costs) ?

Reduce variance of demand prediction !
The MSE of the predictor plays a crucial role.

Expected Profit

The expected optimal profit is:

$$E[\Pi(Q^*)] = (c_u)\mu \; - \; E[C(Q^*)]$$
$$= (c_u)\mu - E[c_u(D - Q^*)^+ + c_o(Q^* - D)^+]$$
$$= (c_u)\mu - \; (c_u + c_o)f(z^*)\sigma$$

Note that the expected profit depends on the average demand.
Moreover, it can be negative if:

$$(c_u)\mu < \; (c_u + c_o)f(z^*)\sigma$$

Prediction based on a regression

- Assume that the prediction of demand is based on a regression model:

$$d_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + ... + \beta_p x_{pt} + \epsilon_t$$

- Assume that the most recent values of the predictors are: $x_1', x_2'..., x_p'$. If the regression errors are normally distributed with standard deviation $\hat{\sigma}$ We then have:

$$Q^* = (\beta_0 + \beta_1 x_1' + \beta_2 x_2' + ... + \beta_p x_p') + z^* \hat{\sigma}$$

- If we let $\beta_{00} = \beta_0 + z^* \hat{\sigma}$, we can note that optimal order quantity is linear in the predictors:

$$Q^* = \beta_{00} + \beta_1 x_1' + \beta_2 x_2' + ... + \beta_p x_p'$$

Prediction based on a regression

- We can also get a closed-form expression for the optimal profit:

$$E[\Pi(Q^*)] = c_u(\beta_0 + \beta_1 x_1' + \beta_2 x_2' + ... + \beta_p x_p') - (c_u + c_o) f_Z(z^*)\hat{\sigma}$$

- We can the see that there are some general statements to make:
  1. If $\beta_i > 0$, then when predictor x_i' increases, expected optimal profit increases
  2. If $\beta_i < 0$, then when predictor x_i' increases, expected optimal profit decreases
  3. Expected optimal profit always decreases in $\hat{\sigma}$

Discrete Demand

- When demand arrives in small discrete quantities, the continuous approximation is not reasonable
- There are many practically relevant discrete distributions
  - Poisson (discrete analogue to Normal distribution)
  - Negative binomial (known to be a good model of retail demand)
  - And of course, discrete empirical distributions (each value from a sample of n past demand observations has equal probability $1/n$).

The Newsvendor Problem: discrete demand

Now assume that D is a discrete random variable taking values restricted to integers $Z^{+} = \{0,1,2,\ldots\}$.

$$Q^* = \min\left\{F_D(Q) \geq \frac{c_u}{c_u + c_0},\ Q \in Z^+\right\}$$