INDR 450/550

Spring 2022

Lecture 19: Prescriptive analytics

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Announcements

- Class Exercise at the end of lecture today. If you are participating online, please upload your document under Course Contents/Class Exercises
- Lab 7 material (on lasso and ridge) and a short video are available
- Exam on May 7.
  - Review exercises are available
  - Make sure that you also review the class exercises and the homeworks
- The first seven labs were uploaded. Please follow them.

Prescriptive Analytics:

- Consider:

$$\min_{\mathbf{z}} E[c(\mathbf{Y}, \mathbf{z})]$$

- We focused in the first part of the course on estimating $E[\mathbf{Y}]$.
- Life would be easy if the following were true.

$$E[c(\mathbf{Y}, \mathbf{z})] = c(E[\mathbf{Y}], \mathbf{z})$$

- This would lead to a deterministic optimization problem:

$$\min_{\mathbf{z}} c(E[\mathbf{Y}], \mathbf{z})$$

Prescriptive Analytics:

- But in general:

$$E[c(\mathbf{Y}, \mathbf{z})] \neq c(E[\mathbf{Y}], \mathbf{z})$$

- And $c(\mathbf{Y}, \mathbf{z})$ is itself a random variable. This leads to more challenging and interesting optimization problems where we have to take into account the probability distribution of the random variable $\mathbf{Y}$.

Prescriptive Analytics:

- This requires obtaining more than point estimates for the predictions because we need the complete probability distribution. We have seen that sometimes the prediction errors can be easily characterized (i.e. normally distributed with an estimated standard deviation).
- For other implementations (i.e. KNN, tree-based regressions etc.), we'll try to find creative ways of obtaining estimators for the probability distribution of the prediction.

A simple operational problem

- We are considering investing k TL take part in some business (i.e. sell skateboards). We can sell the skateboards at p TL each and they cost c TL per unit each.
- Unfortunately, demand for skateboards is random. We can predict D using methods from the first part. Should we make the investment to enter the business?
- The expected profit if we make the investment is:

$$E[\Pi] = E[-k + (p - c)D] = -k + (p - c)E[D]$$

- This is a very easy special case where only $E[D]$ matters.

A simple operational problem: improvements

- But even in this special case, things get more complicated if we add a little bit of generality.
- If the decision-maker is risk averse, her criterion might be a mean-variance trade-off.

$$\alpha E[\Pi] + (1 - \alpha) Var[\Pi]$$

- If she risk averse (and more sophisticated), her criterion might be an expected utility with a concave non-linear utility function u.

$$E[u(\Pi)]$$

- If she is risk averse (and concerned with down-side risk, her criterion might be to take the expected loss with respect to a target minimum profit τ.

$$E[(\tau - \Pi)^+]$$

A simple operational problem: improvements

- Even under very simple assumptions, we would need to deal with the probability distribution of D.
- In more realistic operational problems, there is the following trade-off: if you invest more you can possibly earn more but you also risk to lose more.
  - You optimize the level of investment to manage the trade-off.
- We'll focus on one such problem: a single period inventory or capacity problem with random demand.

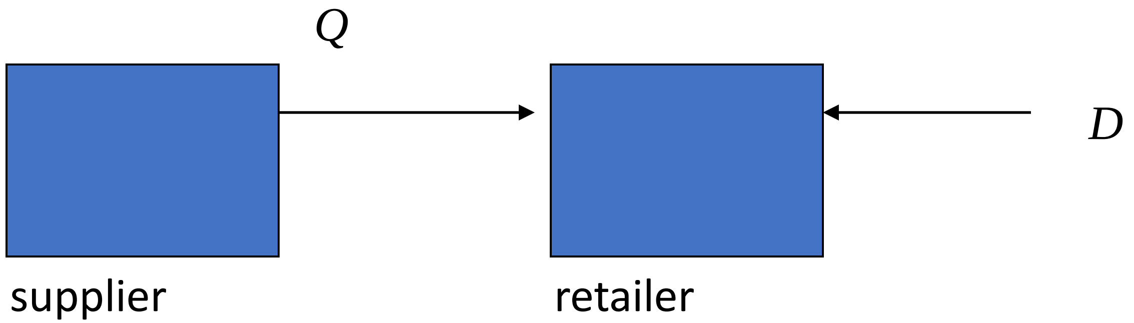
The Assumptions

- Short selling season
- Decision made in advance of the season
- No replenishments or capacity additions during the season (purchasing in advance is required)
- Demand forecasts have considerable uncertainty
- Items lose value significantly after the season

Typical examples

- The newsvendor: how many newspapers to order
- Baker: how many pides to bake in the evening
- Garment manufacturer what quantity should be produced prior to selling season
- Supermarket manager: how much fresh produce to stock
- Florist: how many roses to order before Valentine’s day
- Turkeys for New Year’s, beach toys for the summer season
- Blood bank: how much blood inventory to keep, how to allocate between hospitals

The Newsvendor Problem



D : demand (random variable)
Q : quantity ordered from supplier

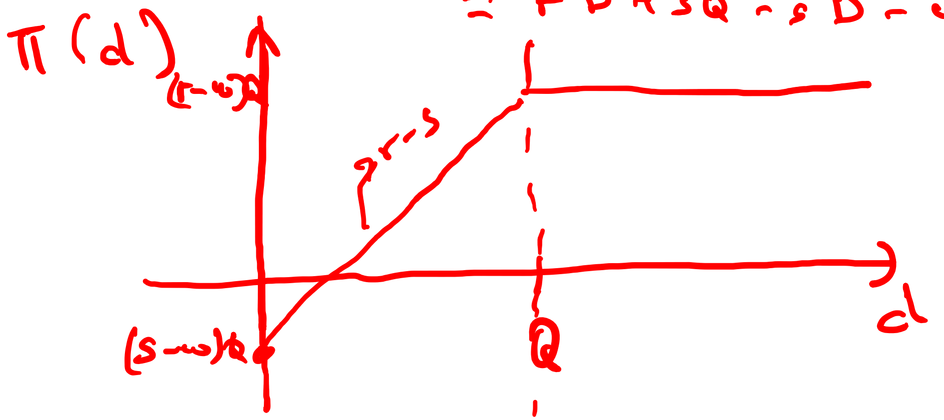
w : wholesale price (of supplier)
r : retail price ($r>w$)
s : salvage value ($s<w$)
m: unit manufacturing cost of supplier ($m<w$)

The Newsvendor Problem

The retailer's profit (random variable) :

If $Q < D$, profit= $(r\text{-}w)\ Q$
if $Q > D$, profit = $(r\text{-}w)\ D$+ $(s\text{-}w)\ (Q\text{-}D)$

$= rD + s(Q-D) - wQ$
$= rD + sQ - sD - wQ = (r-s)D + (r-w)Q$



loss $\to (r-s)D +$
$(s-w)Q < 0$

$\Rightarrow D < \dfrac{(w-s)Q}{(r-s)}$

The Newsvendor Problem

The profit as a function of Q :

$$\Pi_R(Q) = r\min(Q, D) + s(Q-D)^+ - wQ$$

Because D is a r.v., we choose to maximize:

$$E[\Pi_R(Q)] = E[r\min(Q, D) + s(Q-D)^+ - wQ]$$

$$(Q-D)^+ = \begin{cases} Q-D & \text{if } Q > D \\ 0 & \text{otherwise} \end{cases}$$

The Newsvendor Problem

if $Q < D \Rightarrow \min(Q, D) = Q$

and $D - (D - Q) = Q$

Two small tricks:

$$\min(Q, D) = D - (D - Q)^+$$

and

$$Q = D - (D - Q)^+ + (Q - D)^+$$

$$\Rightarrow E[\Pi_R(Q)] = \underbrace{(r - w)E[D]}_{\text{Sure profit}} - \underbrace{E[(\overset{C_u}{r - w})(D - Q)^+ + (\overset{C_o}{w - s})(Q - D)^+]}_{\text{Loss due to randomness}}$$

E[C(Q)]

The Newsvendor Problem

Now let $c_u = (r\text{-}w)$ the unit underage cost
and $c_o = (w\text{-}s)$ unit overage cost

$$E[C(Q)] = E[c_u(D - Q)^+ + c_o(Q - D)^+]$$

Expected Profit maximization is equivalent with minimizing $\mathrm{E}[C(Q)]$ (as a function of Q)

The Newsvendor Problem

Let f_D be the pdf of the r.v. D and F_D its cumulative distribution function:

$$c_o \int_0^{\infty} (Q-x)^+ f_D(x)\,dx + c_u \int_0^{\infty} (x-Q)^+ f_D(x)\,dx$$

$$\mathrm{E}[\mathrm{C}(Q)] = c_o \int_0^Q (Q-x) f_D(x)\,dx + c_u \int_Q^{\infty} (x-Q) f_D(x)\,dx$$

The Newsvendor Problem: the optimality condition

Minimize (in Q) :

$$\mathrm{E}[\mathrm{C}(Q)] = c_o \int_0^Q (Q - x) f(x) dx + c_u \int_Q^\infty (x - Q) f(x) dx$$

First order optimality condition: $\dfrac{d\mathrm{E}[C(Q)]}{dQ} = 0$

Reminder (Leibniz rule):

$$\frac{d}{dy} \int_{a_1(y)}^{a_2(y)} h(x, y)\, dx = \int_{a_1(y)}^{a_2(y)} \frac{\partial h(x, y)}{\partial y} dx + h(a_2(y), y) a_2'(y) - h(a_1(y), y) a_1'(y)$$

$$\frac{d\,E[(Q-D)^+]}{dQ} \qquad E[(Q-D)^+] = \int_0^Q (Q-x)\, f_D(x)\, dx$$

$$\Rightarrow \frac{d\,E[(Q-D)]^+}{dQ} = \int_0^Q 1 \cdot f_D(x)\, dx + 1\,(Q-Q) - 0$$

$$= F_D(Q)$$

$$\frac{d\,E[(D-Q)^+]}{dQ} \qquad \frac{d \int_Q^\infty (x-Q)^+ f_D(x)\, dx}{dQ} = -\underbrace{\int_Q^\infty f_D(x)\, dx}_{(1-F_D(Q))}$$

The Newsvendor Problem: the optimality condition

We also need to check whether the expected profit is concave,
Or equivalently whether $E[C(Q)]$ is convex.

$$\frac{d^2E[C(Q)]}{dQ^2} = c_u f_D(Q) + c_o f_D(Q) = (c_u + c_o) f_D(Q) > 0$$

and $f_D(Q) \geq 0$ because it's a density function

convex

$E[C(Q)]$ is convex, first order condition ensures optimality!

$$\Rightarrow c_o F_D(Q^*) - c_u(1 - F_D(Q^*)) = 0$$

The Newsvendor Problem: the optimality condition

$$\frac{d\mathrm{E}[\mathrm{C}(Q)]}{dQ} = c_o \int_0^Q \underbrace{f(x)dx}_{F_D(Q)} - c_u \int_Q^\infty \underbrace{f(x)dx}_{1\text{-}F_D(Q)} = 0$$

$$\Rightarrow c_o F_D(Q^*) - c_u(1 - F_D(Q^*)) = 0$$

$$\text{and } \frac{d^2\mathrm{E}[\mathrm{C}(Q)]}{dQ^2} \geq 0$$

$E[C(Q)]$ is convex, first order condition ensures optimality!

The Newsvendor Problem: the result

Solving for the optimal Q :

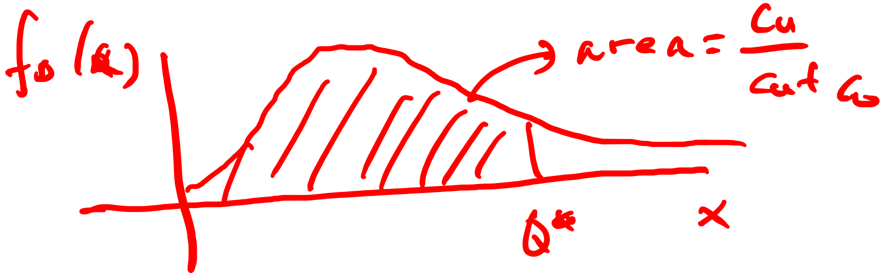
$Q^{**} = E[D]$?

$$F_D(Q^*) = \frac{c_u}{c_u + c_o}$$

$\Rightarrow Q^* = F_D^{-1}\left(\frac{c_u}{c_u + c_o}\right)$

$c_u \nearrow \quad Q^* \nearrow \qquad c_o \searrow \quad Q^*$

Q^* is such that the probability of satisfying all the demand $P(D \leq Q^*)$ is equal to the critical fraction : $c_u/(c_u+c_o)$



if $c_u = c_o \Rightarrow Q^* = F_D^{-1}(1/2) = \tilde{M}$ median