INDR 422/522

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Spring 2023

ARIMA Forecasts- 3

March 23, 2023

Reminders

- Blackboard page is becoming active
- Second lab available, please take a look and work on the exercises
- Third lab will be available this Friday
- Participation taken. Please participate in polls.
- HW 1 (due-date March 31, 2023)

Class Exercise from last lecture

CLASS EXERCISE, March 21, 2023

1. Consider the process $D_t = 100 - \epsilon_{t-1}/2 + \epsilon_t$ where ϵ_t are i.i.d with mean zero and variance σ^2. Which of the following is true?

   (a) 1-lag AC is negative : True. We have an MA(1) process: $D_t = \theta_0 + \theta_1 \epsilon_{t-1} + \epsilon_t$. The AC at lag 1 has the same sign as the coefficient θ_1 (and $\theta_1 = -1/2$).

   (b) 1-lag AC is positive : False

   (c) The theoretical value of the one-lag AC is -1/2: False. The first lag AC for the MA(1) case is given by:

   $$\rho_1 = \frac{\theta_1}{1+\theta_1^2} = -2/5$$

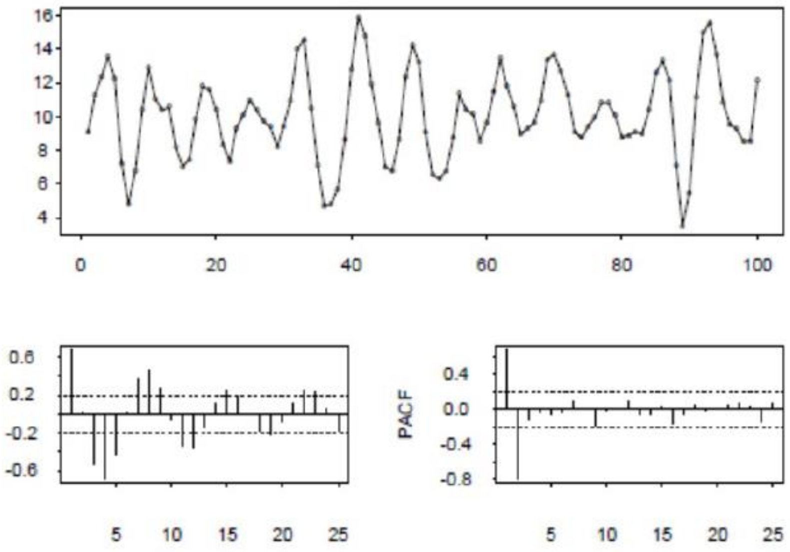
   (d) 2-lag AC is negative: This is false. For an MA(1) process, the second lag AC equals 0. In fact, the AC equals zero at all higher lags than one.

   (e) 2-lag AC is zero: True.

   (f) If D_{t-1} is below average, then D_t is more likely to be above average: True because of negative correlation at lag 1.

   (g) If D_{t-2} is below average, then D_t is more likely to be above average: False, D_{t-2} and D_t are independent.

Class Exercise from last lecture



2. Consider the data from the slides. Which of the following are true?

(a) The data may come from an i.i.d process: False. The ACF and PACF show that there is clear auto-correlation at multiple lags.

(b) The data may come from an AR(1) process: False. The ACF does not have a geometrically decreasing pattern and the PACF has spikes at multiple lags. In an AR(1) process, in the PACF we would see a single spike on the first lag only.

(c) The data may come from an MA(1) process: False. With an MA(1) we would see a single spike at lag 1 on the ACF.

(d) The spikes on the PACF for the first two lags may correspond to AR terms on the first two lags: This may be true and the PACF should show single spikes corresponding to each AR term and this is the case here.

(e) The first spike on the PACF may correspond to a positive AR coefficient on the first term: True

(f) The first spike on the PACF may correspond to a positive AR coefficient on the first term: True

(g) The data may come from an AR(2) process with a positive coefficient at the first lag and a negative coefficient at the second lag:True. We can combine the above evidence to conclude that an AR(2) model would be a good fit.

Summary last lectures

- Auto-regressive processes generate auto-correlation
- Our goal is to understand how AC is generated by particular models so that we can do predictions which require the reverse direction:
  - Given data with AC properties, what is the model that is likely to generate this data
- ARIMA: includes differencing in the ARMA framework
- SARIMA: further improves seasonal differencing
- Forecasting: fitting a model (estimate ARMA coefficients from data) and using the model to generate future instances from the series which correspond to predictions

ARIMA Framework

- Finally, we incorporate the basic transformations that are needed to convert the original series to a stationary series. One basic operation is differencing (multiple times if necessary).
- ARMA processes that require differencing are called ARIMA (Auto Regressive Integrated Moving Averages). Integration in this context is viewed as undoing the differencing (i.e. summation).
- We use the convention ARIMA(p, d, q) to denote that the original series was differenced d times, and then p AR and q MA terms were used on the differenced series.
- The ARIMA class is a broad and useful class.

ARIMA Framework: example

- ARIMA(1,1,0) refers to a process which was differenced once, and has an AR term on the difference:

$$
\begin{aligned}
i)\ W_t &= Y_t - Y_{t-1} \\
ii)\ W_t &= c + \phi_1 W_{t-1} + \epsilon_t
\end{aligned}
$$

We can revert the transformations to recover the original process:

$$Y_t = Y_{t-1} + W_t = Y_{t-1} + c + \phi_1 W_{t-1} + \epsilon_t$$

Finally replacing W_{t-1} by $Y_{t-1} - Y_{t-2}$, we have:

$$Y_t = Y_{t-1} + W_t = c + Y_{t-1} + \phi_1 (Y_{t-1} - Y_{t-2}) + \epsilon_t$$

SARIMA Framework: taking into account seasonality

- In addition to differencing (to remove trend), another common transformation is seasonal differencing.
- This leads to the bigger framework of Seasonal ARIMA (SARIMA).
- The convention is SARIMA$(p, d, q)(P, D, Q, m)$. The second parenthesis refers to the seasonal terms: P is the number of seasonal AR terms, D refers to the degree of seasonal differencing, Q to the number of seasonal MA terms and m the length of the season.

SARIMA Framework: Example

- SARIMA(1,0,1)(1,1,0,12) refers to a process which has an one regular AR term and was seasonally differenced once and has an AR term on the seasonal difference. The length of the season is 12.

This gets quite messy to write in terms of the original series and is very difficult to do without the backshift notation. We'll take a look at next.

Forecasting in the ARIMA Framework

- Once we pick a model such as ARIMA(1,0,1), the data is estimated from the parameters by Maximum Likelihood Estimation (i.e. find the parameters ϕ_1 and ϵ_1 that would make the observed series most probable).
- This is the hard part of the task but is done by numerical optimization and software has become reliable. item Let us assume that the MLE estimators of the parameters are $\phi_1 = 0.2$ and $\theta_1 = -0.5$.
- The model is then:

$$Y_t = 0.2Y_{t-1} - 0.5\epsilon_{t-1} + \epsilon_t$$

- To 'forecast' from the above process we simply plug in the observed values in the above evolution equation. Assume that $y_{t-1} = 20$

$$\hat{y}_t = 0.2(20) - 0.5(20 - \hat{y}_{t-1})$$

Forecasting in the ARIMA Framework

- Note that $y_1, y_2, ...y_{t-1}$ are observable to us but $\epsilon_1, \epsilon_2, ...\epsilon_{t-1}$ are not observable, we therefore estimate

$$\hat{\epsilon}_{t-1} = y_{t-1} - \hat{y}_{t-1}$$

- If we are forecasting for time $t + 1$ using data up to time $t - 1$. We proceed with:

$$\hat{y}_{t+1|t-1} = 0.2\hat{y}_t - 0.5\hat{\epsilon}_t$$

where we replaced the observation y_t with its estimator from the model $\hat{y}_t$. Since we have not observed y_t, our best estimator for $\hat{\epsilon}_t = 0$. The multi-step look ahead forecast simply reduces to:

$$\hat{y}_{t+h|t-1} = 0.2\hat{y}_{t+h-1} \text{ for } h = 1, 2, 3...$$

Forecasting in the ARIMA Framework

- Things get messier if we take more complicated models,but the principles are the same. Let us take ARIMA(1,2,1).
- Let us assume that this time the MLE estimators of the parameters are $c = 100$, $\phi_1 = 0.4$ and $\theta_1 = 0.6$.
- The model is then:

$$Z_t = 100 + 0.4Z_{t-1} + 0.6\epsilon_{t-1} + \epsilon_t$$

  where $Z_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})$.
- For this model, we need the last three observations $y_{t-1}, y_{t-2}, y_{t-3}$ to forecast for period t.

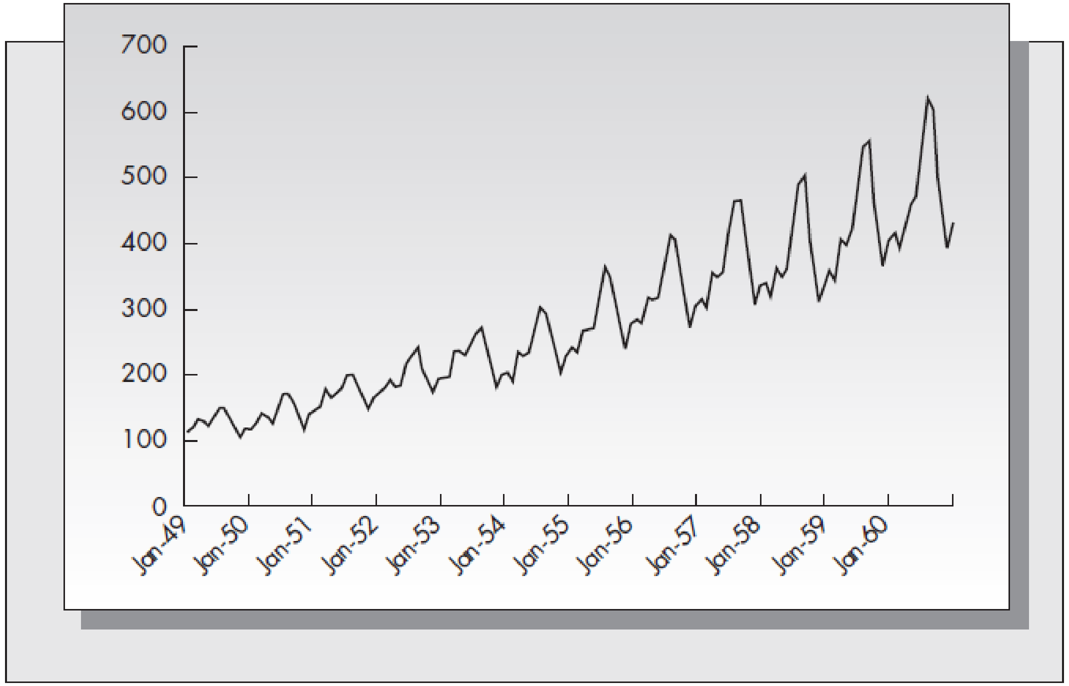
Forecasting in the ARIMA Framework

- With some guidance from our part, software enables us to fit models and perform comparisons across models.
- We can also perform exhaustive searches of a large class of models.
- It is then critical to validate and interpret the results.

Case: Forecasting Airline Passenger Demand

- Airline passenger demand from 1949 to 1960.

FIGURE 2–17
International airline passengers (thousands)

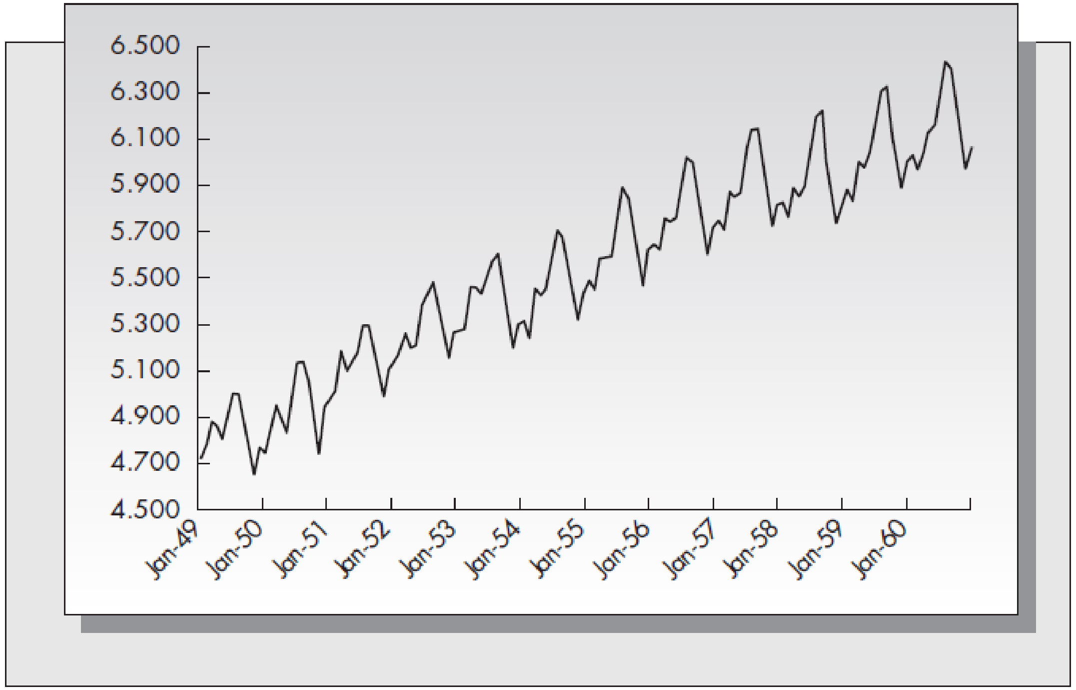


- There is trend and seasonality and increasing variance (fluctuations are increasing).

Step 1: Logarithmic transformation

- To stabilize variance take $Z_t = \ln(Y_t)$

FIGURE 2–18
Natural log of international airline passengers



- The fluctuations now appear stable.
- Trend and seasonality still remain.

Step 2: Detrend and deasonalize by differencing

- First order differencing to remove linear trend.
- 12 month differencing to remove annual seasonality.
- Check autocorrelations after differencing:

TABLE 2–4
Autocorrelations for the Transformed Airline Data Pictured in Figure 2–19 (after taking logarithms and two levels of differencing)

Lag	Autocorrelation	Lag	Autocorrelation	Lag	Autocorrelation
1	−0.34	13	0.15	25	−0.10
2	0.11	14	−0.06	26	0.05
3	−0.20	15	0.15	27	−0.03
4	0.02	16	−0.14	28	0.05
5	0.06	17	0.07	29	−0.02
6	0.03	18	0.02	30	−0.05
7	−0.06	19	−0.01	31	−0.05
8	0.00	20	−0.12	32	0.20
9	0.18	21	0.04	33	−0.12
10	−0.08	22	−0.09	34	0.08
11	0.06	23	0.22	35	−0.15
12	−0.39	24	−0.02	36	−0.01

- Still significant auto-correlation at 1 lag and 12 lag (additional MA terms are needed) but no other significant AC left.

Step 3: Final model and parameter estimation

- Final model (parameters optimized in Statistical software):

$$z_t = z_{t-1} + z_{t-12} - z_{t-13} + \epsilon_t - 0.333\epsilon_{t-1} - 0.544\epsilon_{t-12} + 0.181\epsilon_{t-13}.$$

- And don't forget:

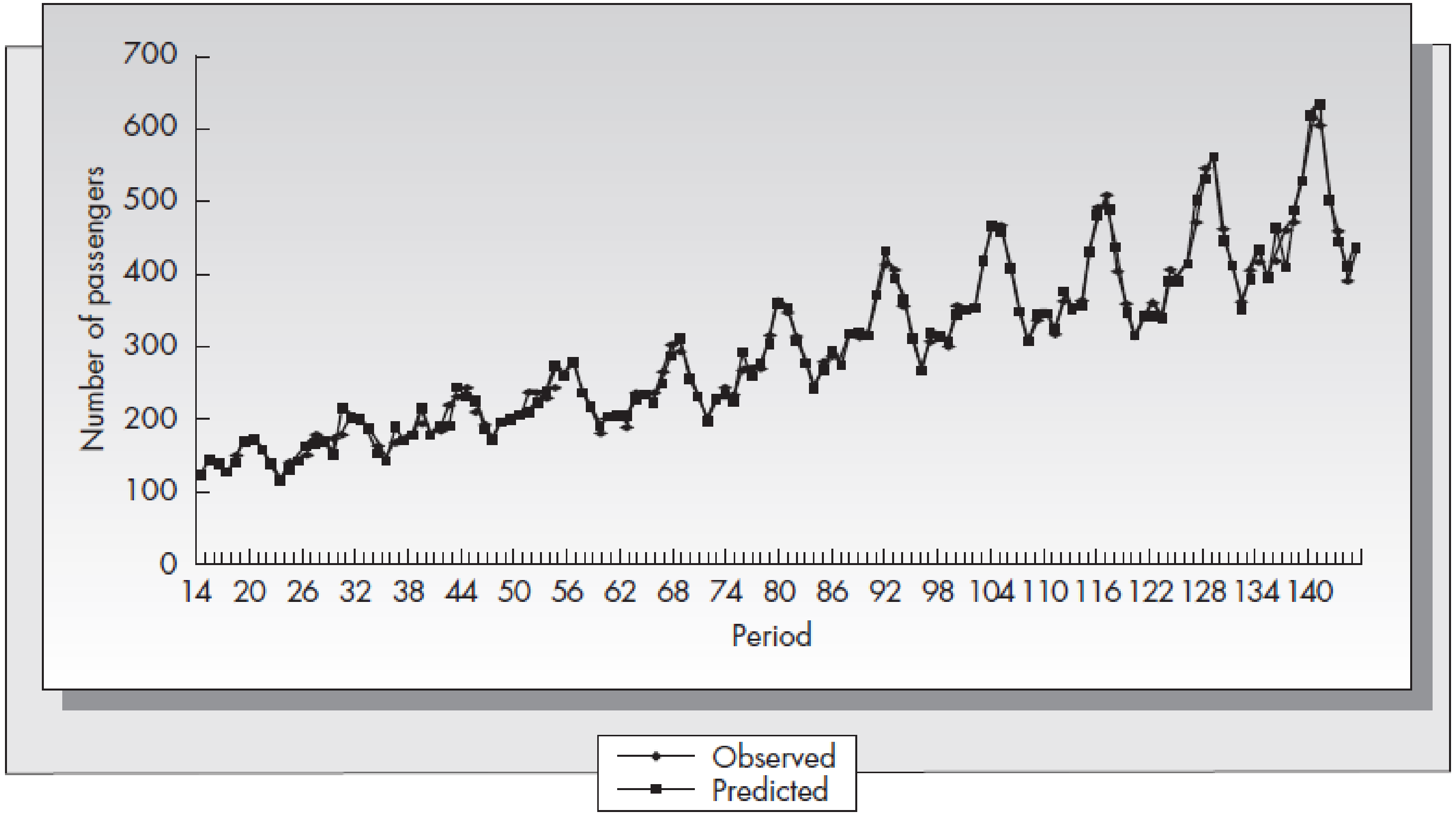
$$Z_t = \ln(Y_t), \quad Y_t = e^{Z_t}$$

- Because of the first order differencing and the yearly differencing, the first forecast can be made for period 14.

Results

- From period 14 on:

FIGURE 2–20
Observed versus predicted number of airline sales



- The resulting forecasts are excellent!

The Backshift Notation

The backward shift operator B is a useful notational device when working with time series lags:

$$By_t = y_{t-1} \,.$$

(Some references use L for "lag" instead of B for "backshift".) In other words, B, operating on y_t, has the effect of shifting the data back one period. Two applications of B to y_t shifts the data back two periods:

$$B(By_t) = B^2 y_t = y_{t-2} \,.$$

For monthly data, if we wish to consider "the same month last year," the notation is $B^{12} y_t$ = y_{t-12}.

The backward shift operator is convenient for describing the process of *differencing*. A first difference can be written as

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B) y_t \,.$$

Note that a first difference is represented by $(1 - B)$. Similarly, if second-order differences have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - 2B + B^2) y_t = (1 - B)^2 y_t \,.$$

In general, a dth-order difference can be written as

$$(1 - B)^d y_t.$$

The Backshift Formulation for ARIMA

ARIMA(p,d,q):

$$\underset{\substack{\uparrow \\ \mathrm{AR}(p)}}{(1-\phi_1 B-\cdots-\phi_p B^p)} \quad \underset{\substack{\uparrow \\ d \text{ differences}}}{(1-B)^d y_t} \quad = \quad c+\underset{\substack{\uparrow \\ \mathrm{MA}(q)}}{(1+\theta_1 B+\cdots+\theta_q B^q)\varepsilon_t}$$

ARMA: Backshift example

- Ex: An arbitrary ARMA process

$$Y_t = c + \phi_1 Y_{t-1} + \theta_4 \varepsilon_{t-4} + \varepsilon_t$$

- Using the backshift notation, we can write:

$$(1 - \phi_1 B) Y_t = c + (1 + \theta_4 B^4) \epsilon_t$$

ARIMA: Backshift example

- Consider ARIMA(1,2,1), we write

$$Y_t = c + \phi_1 Y_{t-1} + \theta_1 \varepsilon_{t-4} + \varepsilon_t$$

- Using the backshift notation, we can write:

$$(1 - \phi_1 B)(1 - B)^2 Y_t = c + (1 + \theta_1 B)\epsilon_t$$

- It's now easy to see that Y_t is related to terms up to$Y_{t\text{-}3}$.

SARIMA: Backshift example

- Consider SARIMA(1,1,1)(1,1,1,4) we write

- Using the backshift notation, we can write:

$$(1-\phi_1 B)\ (1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)\ (1+\Theta_1 B^4)\varepsilon_t.$$

- We can see that Y_t is related to terms up to $Y_{t\text{-}10}$.

SARIMA Framework: Example

- SARIMA(1,0,1)(1,1,0,12) refers to a process which has an one regular AR term and was seasonally differenced once and has an AR term on the seasonal difference. The length of the season is 12.

- Using the backshift notation, we can write:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})(1-B^{12})Y_t = c + (1+\theta_1 B^1)\epsilon_t$$

- We can now see that Y_t is related to terms upto$Y_{t\text{-}25}$.

Model Fitting Examples: Simulated AR(1)

- Example: Let us try the model fitting implementation on a synthetic case. Let's generate data from an AR-1 process.

$$Y_t = c + \phi_1 Y_{t-1} + \epsilon_t$$

- We would need to plot the ACF and PACF to have some guidance, but since we have synthesized data, we already know that the best fit is ARIMA(1,0,0).
- We can then let the software estimate the intercept and the first lag AR coefficients: $\hat{c}$ and $\hat{\phi}_1$ and assess the results.

The model that is simulated is $Y_t = 250 + 0.7\, Y_{t-1} + \varepsilon_t$ and $\sigma^2 = 100$.

Model Fitting Examples

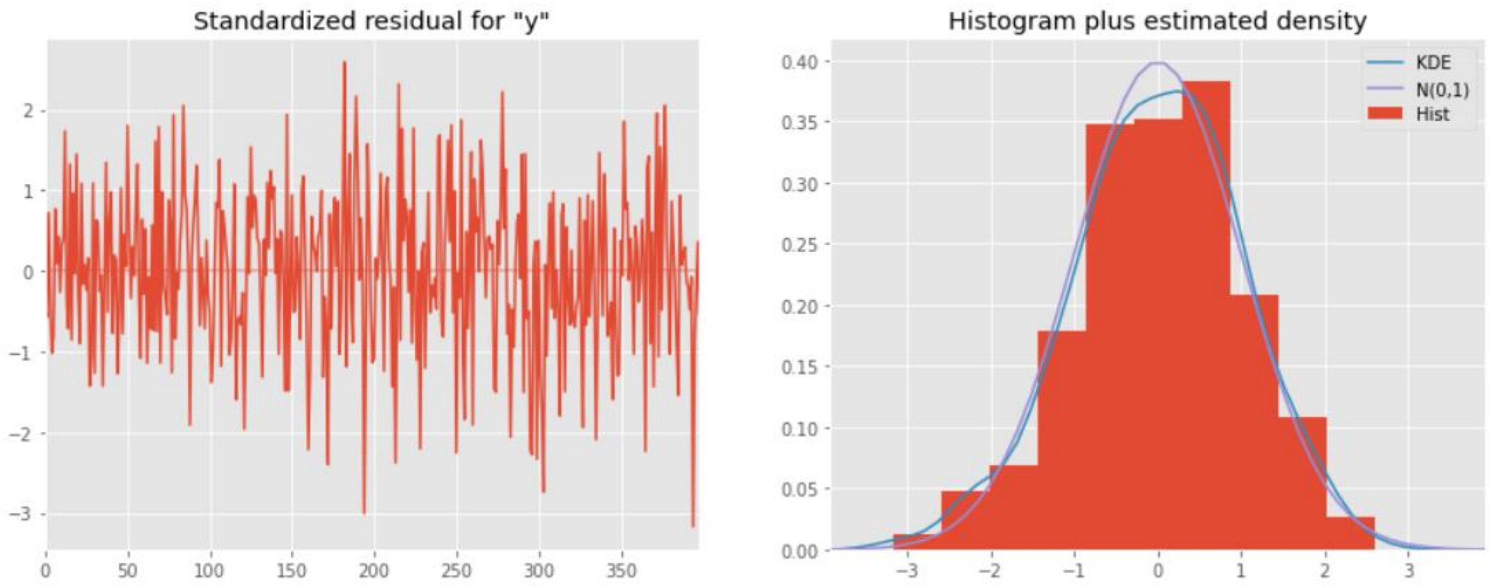
```
[7]: # Fit the model
modar = sm.tsa.statespace.SARIMAX(y_ar[100:499], trend='c', order=(1,0,0))
res = modar.fit(disp=False)
print(res.summary())
```

```
                               SARIMAX Results
==============================================================================
Dep. Variable:                      y   No. Observations:                  399
Model:               SARIMAX(1, 0, 0)   Log Likelihood               -1458.647
Date:                Tue, 01 Mar 2022   AIC                           2923.295
Time:                        10:09:29   BIC                           2935.261
Sample:                             0   HQIC                          2928.034
                                - 399
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
intercept    293.9478     33.949      8.658      0.000     227.408     360.487
ar.L1          0.6470      0.041     15.866      0.000       0.567       0.727
sigma2        87.7254      6.350     13.815      0.000      75.279     100.172
===================================================================================
Ljung-Box (L1) (Q):                   0.00   Jarque-Bera (JB):                 3.89
Prob(Q):                              0.95   Prob(JB):                         0.14
Heteroskedasticity (H):               1.44   Skew:                            -0.24
Prob(H) (two-sided):                  0.04   Kurtosis:                         3.01
===================================================================================
```

The fitted model is $Y_t = 293 + 0.64\, Y_{t-1} + \varepsilon_t$

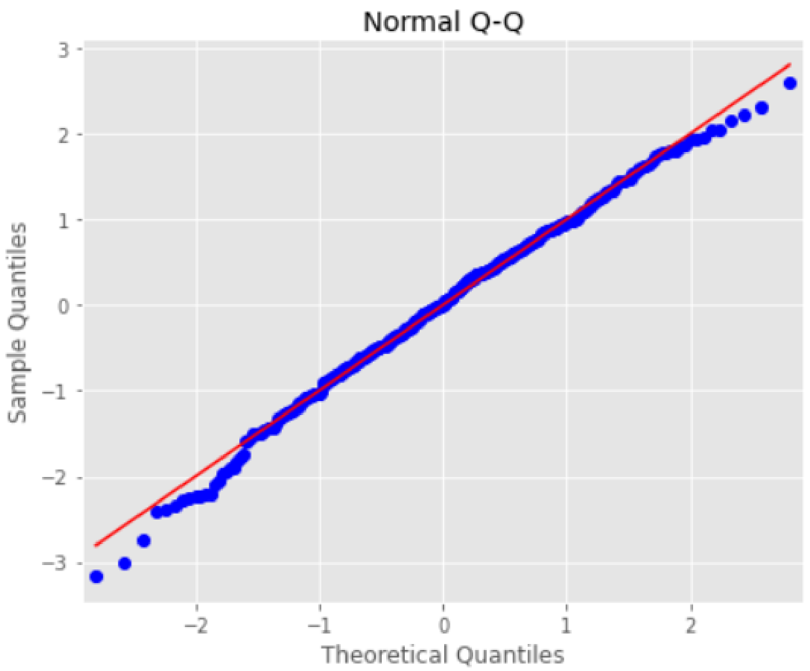
The true model was $Y_t = 250 + 0.7\, Y_{t-1} + \varepsilon_t$ and $\sigma^2 = 100$.

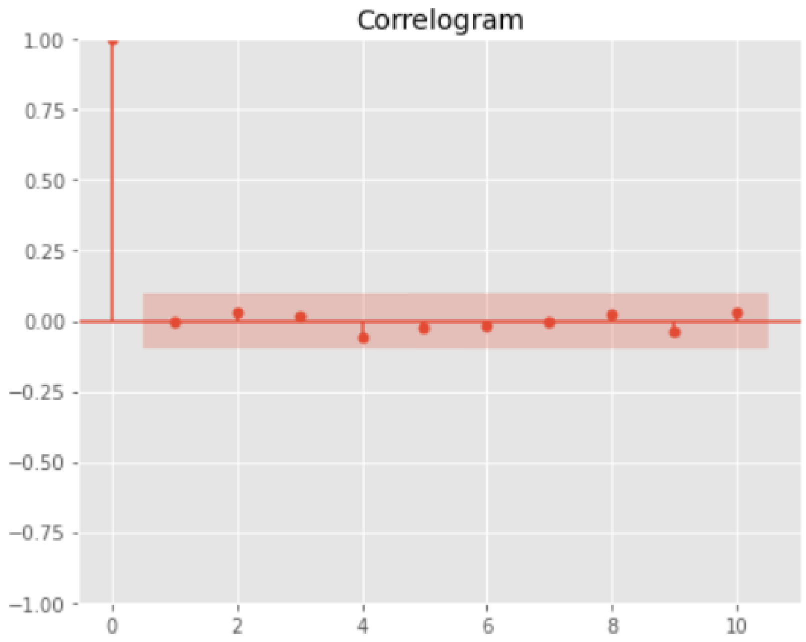
Model Fitting Examples: Residual Checks



The residuals seem random and the histogram looks like a normal distribution.

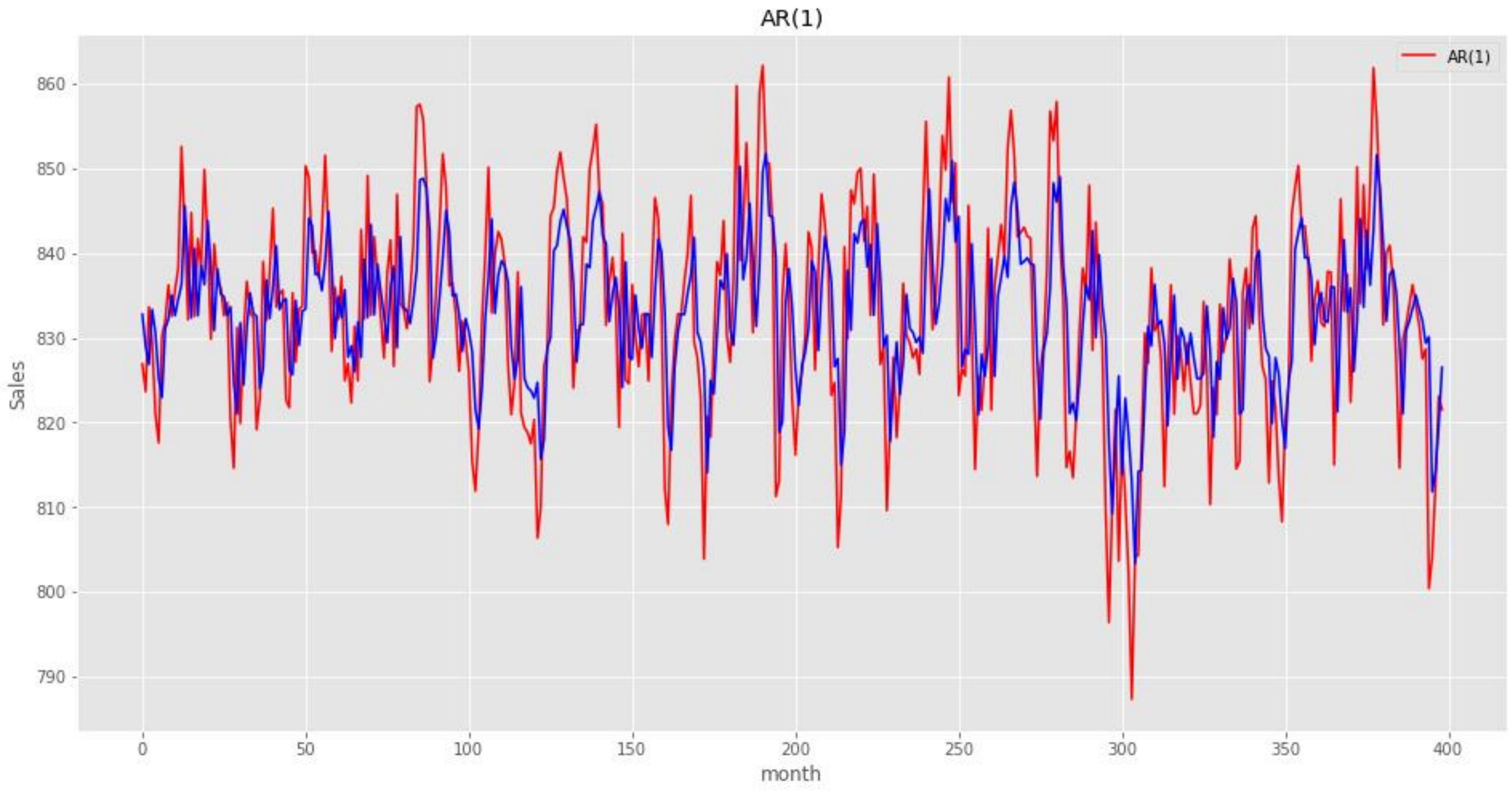
Model Fitting Examples: Residual Checks





The Q-Q plot confirms normality and there is no significant auto-correlation in the residuals. We can conclude that there errors are independent and normally distributed with mean zero and variance 87.72 (from the results table).

Model Fitting Examples: In-sample predictions



Model Fitting Examples: (wrong) ARMA

Let's check the effect of fitting a wrong model. For instance, we might wrongfully think that MA terms are needed at lags 1 and 3.

- We can also attempt to fit a wrong (or superficial) model.For instance, we can attempt to fit:

$$Y_t = c + \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1} + \theta_3 \epsilon_{t-3} + \epsilon_t$$

- Note that the above is not exactly ARIMA(1,0,3) since it does not contain the MA-term at the second lag.
- We would need a more complete specification and use ARIMA(1,0,[1,0,1]).

Model Fitting Examples: (wrong) ARMA

```
In [4]: # Fit the model
        restest = modtest.fit(disp=False)
        print(restest.summary());
```

```
                               SARIMAX Results
==============================================================================
Dep. Variable:                      y   No. Observations:                  399
Model:            SARIMAX(1, 0, [1, 3])   Log Likelihood               -1483.013
Date:                Sun, 06 Mar 2022   AIC                           2976.026
Time:                        18:47:31   BIC                           2995.971
Sample:                             0   HQIC                          2983.926
                                - 399
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
intercept    275.9916     49.422      5.584      0.000     179.127     372.857
ar.L1          0.6693      0.059     11.292      0.000       0.553       0.786
ma.L1          0.0112      0.077      0.146      0.884      -0.140       0.162
ma.L3         -0.0270      0.061     -0.442      0.658      -0.147       0.093
sigma2        98.9059      7.420     13.329      0.000      84.362     113.450
===================================================================================
Ljung-Box (L1) (Q):                   0.00   Jarque-Bera (JB):                 0.36
Prob(Q):                              0.96   Prob(JB):                         0.84
Heteroskedasticity (H):               0.92   Skew:                             0.05
Prob(H) (two-sided):                  0.64   Kurtosis:                         2.89
===================================================================================
```

Model Fitting Examples: (wrong) ARMA

- The resulting model is very different than the theoretical model we simulated.
- MSE 1 = 99.62, MSE 2 = 99.55
- Second model has lower MSE but it looks very suspicious because the p-values of the the two MA terms are not statistically significant.
- These are all signs of overfitting due to the additional parameters.
- We'll do our best to avoid overfitting.

Overfitting: some introduction

- ARIMA models and their software implementation enable us to test and implement alternative models with different parameters.
- Using more parameters (more AR and MA terms) increases the degrees of freedom and therefore increases the model fitting error performance (MSE etc.) on the given data.
- But by using too many parameters we might be overfitting the model to the particular (training) sample.
- We should be aware of this and take caution.

Bias – Variance Tradeoff

- There are two types of errors when estimation is based on a sample of data using a mathematical model.
- Sampling error (variance) because the estimated model yields different results in a new sample.
- Model based error (bias) because the model that was fit is an inaccurate representation of reality.
- Unfortunately, the two errors are in conflict:
  - To reduce bias, we need a model that yields a closer fit to the training sample. This, in general, means more complicated models with a larger number of parameters.
  - But complicated models generate more sampling errors when tested out of sample. They are an excellent representation of the training sample but do not necessarily perform well in other samples from the same population. This is the problem of overfitting.
- There is a need to find the right trade-off between model complexity and variance.

Bias – Variance Tradeoff: Information Criteria

Information Criteria

Akaike's Information Criterion (AIC), which was useful in selecting predictors for regression, is also useful for determining the order of an ARIMA model. It can be written as

$$\text{AIC} = -2\log(L) + 2(p+q+k+1),$$

where L is the likelihood of the data, $k=1$ if $c \neq 0$ and $k=0$ if $c=0$. Note that the last term in parentheses is the number of parameters in the model (including σ^2, the variance of the residuals).

For ARIMA models, the corrected AIC can be written as

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2},$$

and the Bayesian Information Criterion can be written as

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p+q+k+1).$$

Good models are obtained by minimising the AIC, AICc or BIC. Our preference is to use the AICc.

from Hyndman and Athanasapoulos

Bias – Variance Tradeoff: Information Criteria

- We must always keep an eye on AIC and BIC measures.
- But they are not always conclusive.
- For the synthetic AR-1 example, model 1 has slightly lower AIC than model 2.
- We will also have to validate out-of-sample (more on this later).
  - Fit the model on training data
  - Evaluate performance on a separate test set.