INDR 450/550

Spring 2022

Lecture 3: Simple forecasts

Feb. 21, 2022

Fikri Karaesmen

Announcements

- The TA for the course is Tufail Ahmad (tahmad20@ku.edu.tr)
- Blackboard page is becoming active
  - Slides of first week’s lectures
  - Links to books
- We may start looking at some data today
  - This week, we’ll upload a video on basic analysis of some forecasting models

Reminder: estimators and properties

- Let us note that sample based estimators are themselves random variables. Each time we draw a new random sample, we'll get a different value for our estimator.
- Unbiasedness: A desirable property for an estimator is that it does not have a systematic error on the average (in expectation). The sample mean $\bar{X}$ is an unbiased estimator of the population mean since:

$$E[\bar{X}] = \mu.$$

- Note that there are many unbiased estimators: X_1 and $(2X_1 + X_2)/3$ are also unbiased. Since:

$$E[X_1] = E[(2X_1 + X_2)/3] = \mu.$$

Reminder: estimators and properties

- Variance of the Estimator: Among unbiased estimators, it makes sense to prefer one with a lower variance.
- Assuming that our sample has variance σ^2:

$$Var[\bar{X}] = \frac{\sigma^2}{n}$$

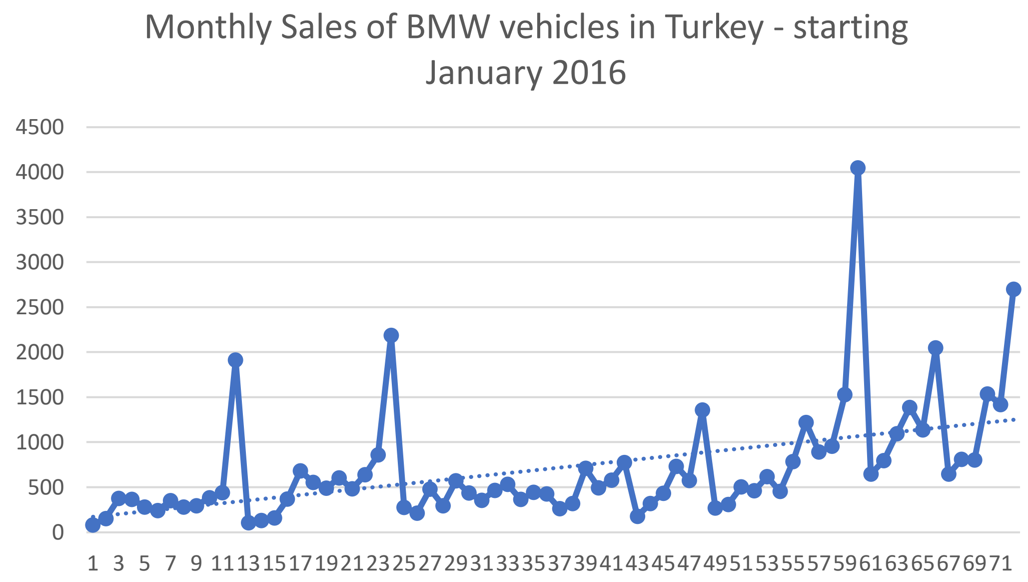
- whereas for the other estimators:

$$Var[X_1] = \sigma^2 \text{ and } Var[(2X_1 + X_2)/3] = \frac{5\sigma^2}{9}.$$

- We will see that for demand forecasting there is a trade-off between responsiveness and low variance.

Time series

- Demand data is typically in the form of a time series: an observation per period
- Note that we typically have sales data which may be different than the actual demand
- We'll forecast future values time series to eventually solve planning problems



Some more real data – Makridakis, Wheelwright, Hyndman (1997)

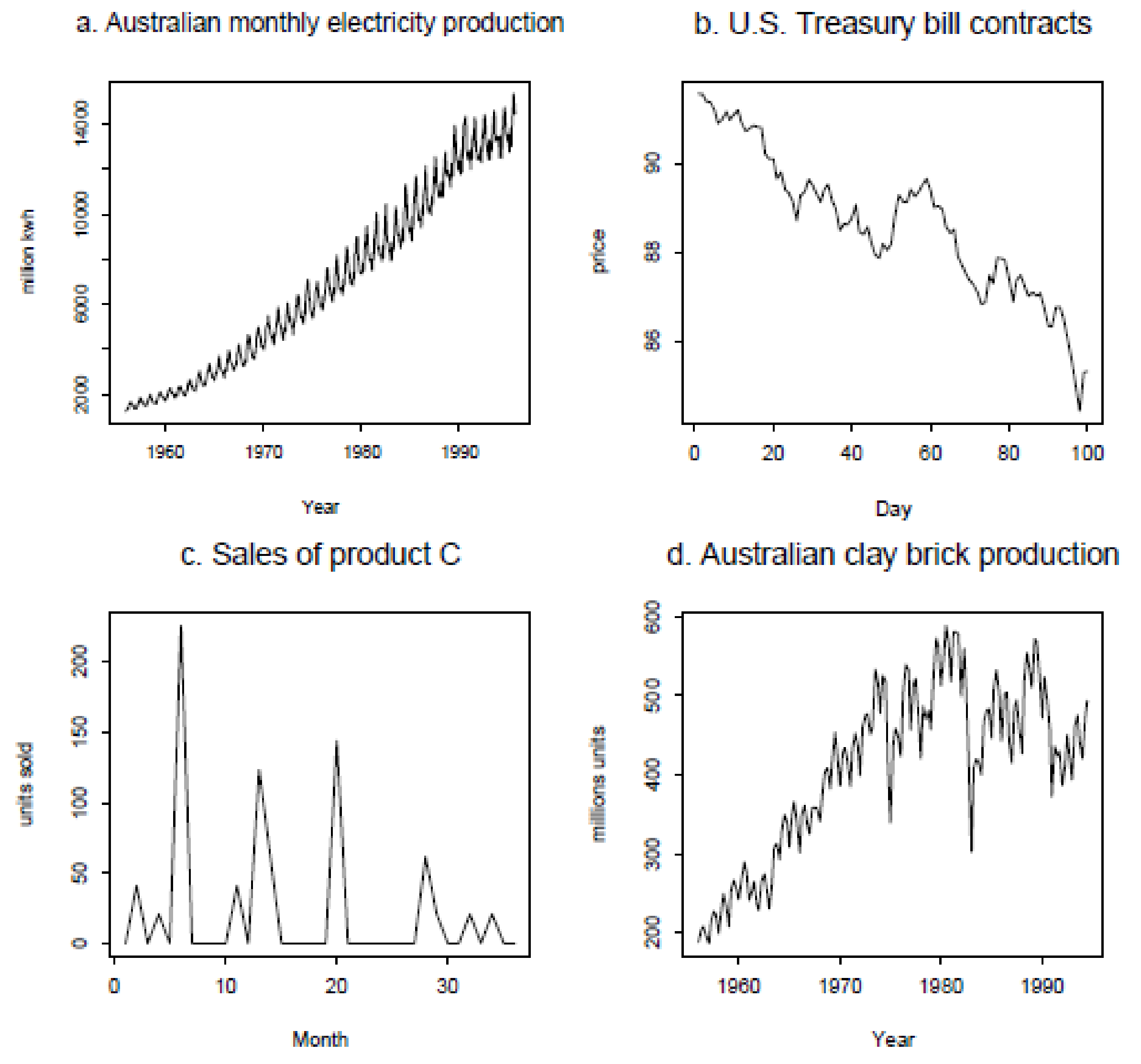


Figure 1-2: *Historical data on four variables for which forecasts might be required.*

Time series: simple forecasts

- We have data corresponding to a time series $y_1, y_2, ..y_T$. For our purposes, we can assume that y_t corresponds to demand in period t.The goal is to forecast the demand in period $T + h$ $h = 1, 2, ..$ given the observations.
- Here are some simple ideas: i) average

$$\hat{y}_{T+h|T} = \frac{\sum_{t=1}^{T} y_t}{T}$$

- ii) naive method

$$\hat{y}_{T+h|T} = y_T$$

- iii) seasonal naive method (where m is the seasonal period)

$$\hat{y}_{T+h|T} = y_{T+h-m} \text{ if } T + h - m \leq T.$$

Time series: simple forecasts

- iv.) Drift (trend) estimation

$$\hat{y}_{T+h|T} = y_T + h\left(\frac{y_T - y_1}{T - 1}\right)$$

- v) moving average over k periods

$$\hat{y}_{T+h|T} = \left(\frac{y_{T-k+1} + y_{T-k+2} + .. + y_T}{k}\right)$$

Time series: simple forecasts

- vi.) Exponential smoothing

$$\hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}$$

where $0 \leq \alpha \leq 1$. Note that since $\hat{y}_{T|T-1} = \alpha y_{T-1} + (1-\alpha)\hat{y}_{T-1|T-2}$ we can recursively write:

$$\begin{aligned} \hat{y}_{T+1|T} &= \alpha y_T + \alpha(1-\alpha)y_{T-1} + (1-\alpha)^2\hat{y}_{T-1|T-2} \\ &= \sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} y_t \end{aligned}$$

Time series: simple forecasts

- To get some insight, let us consider some models that will generate data. Assume that ϵ_t are iid random variables with mean zero and standard deviation σ.

  i) stationary i.i.d model

  $$Y_t = c + \epsilon_t$$

  ii) stationary seasonal model

  $$Y_t = c_{t(mod\ m)} + \epsilon_t$$

  iii) a model with linear trend

  $$Y_t = bt + c + \epsilon_t$$

  iv) a model with quadratic trend

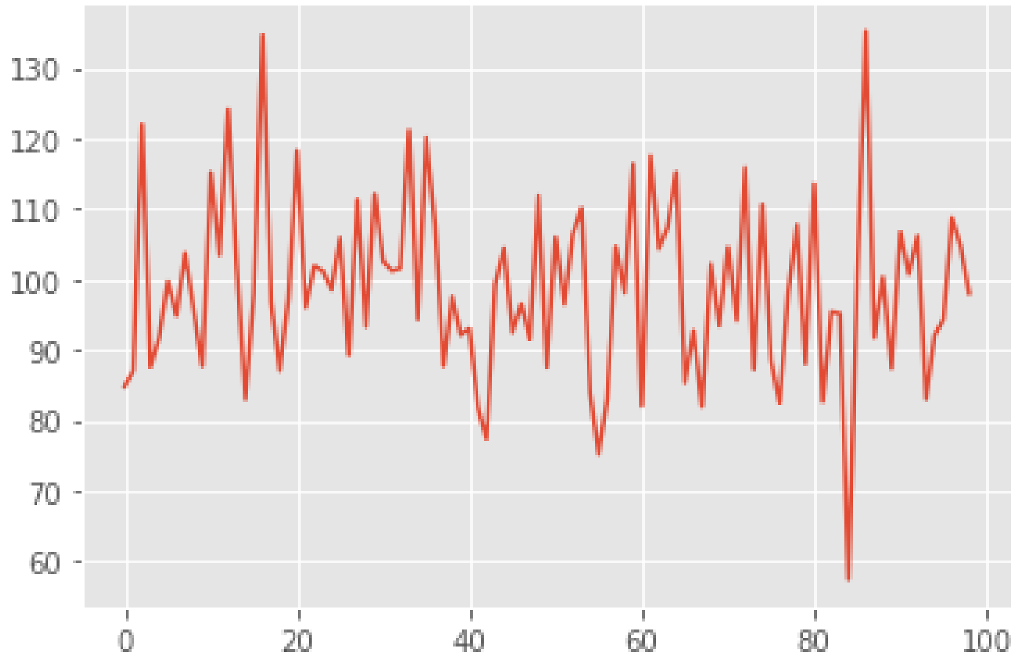
  $$Y_t = at^2 + bt + c + \epsilon_t$$

Time series: simple forecasts

i) stationary i.i.d model

$$Y_t = c + \epsilon_t$$

```
c=100; sigma=15;
y=[0]*101;
for i in range(1,100):
  y[i] =c + sigma*random.normalvariate(0, 1)
```

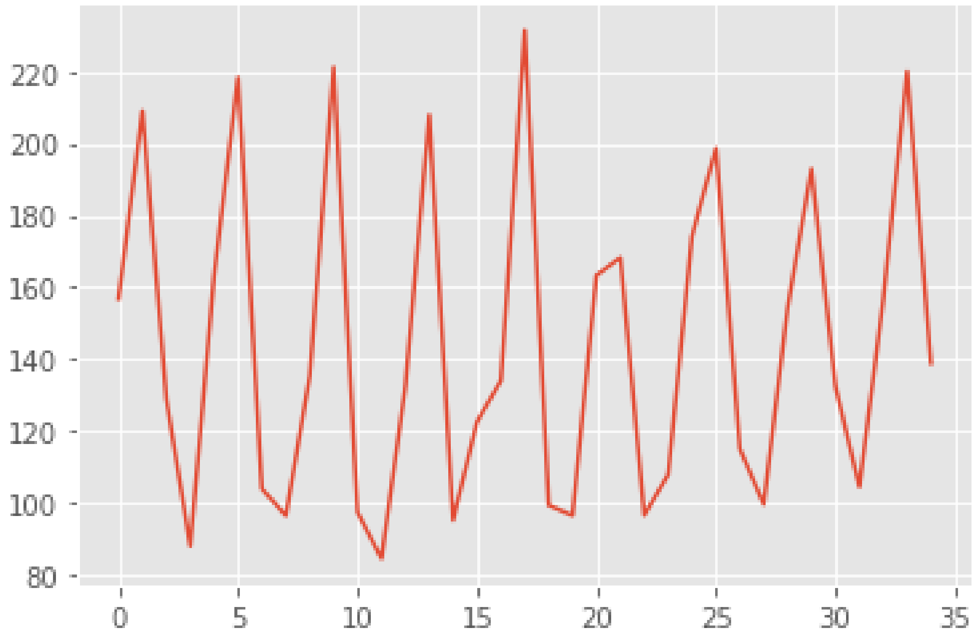


Time series: simple forecasts

ii) stationary seasonal model

$$Y_t = c_{t(mod\ m)} + \epsilon_t$$

```
ssc=[0]∗4;
ssc[0]=100; ssc[1]=150; ssc[2]=200; ssc[3]=120; sigma=15; b=2;
yss=[0]∗101;
for i in range(1,100):
  yss[i] =ssc[i % 4] + sigma∗random.normalvariate(0, 1)
```

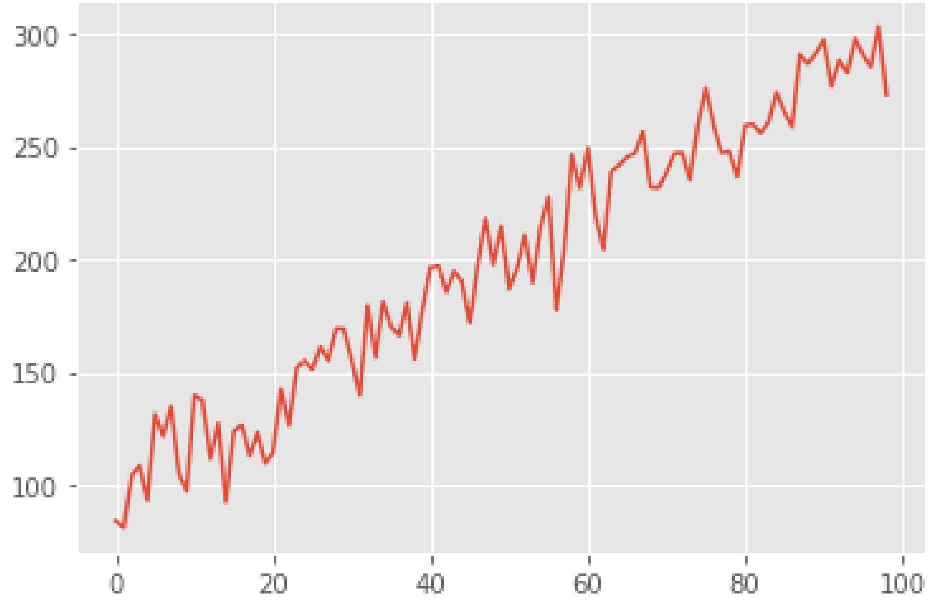


Time series: simple forecasts

iii) a model with linear trend

$$Y_t = bt + c + \epsilon_t$$

```
c=100; sigma=15; b=2;
ytr=[0]*101;
for i in range(1,100):
 ytr[i] =c + b*i+ sigma*random.normalvariate(0, 1)
```

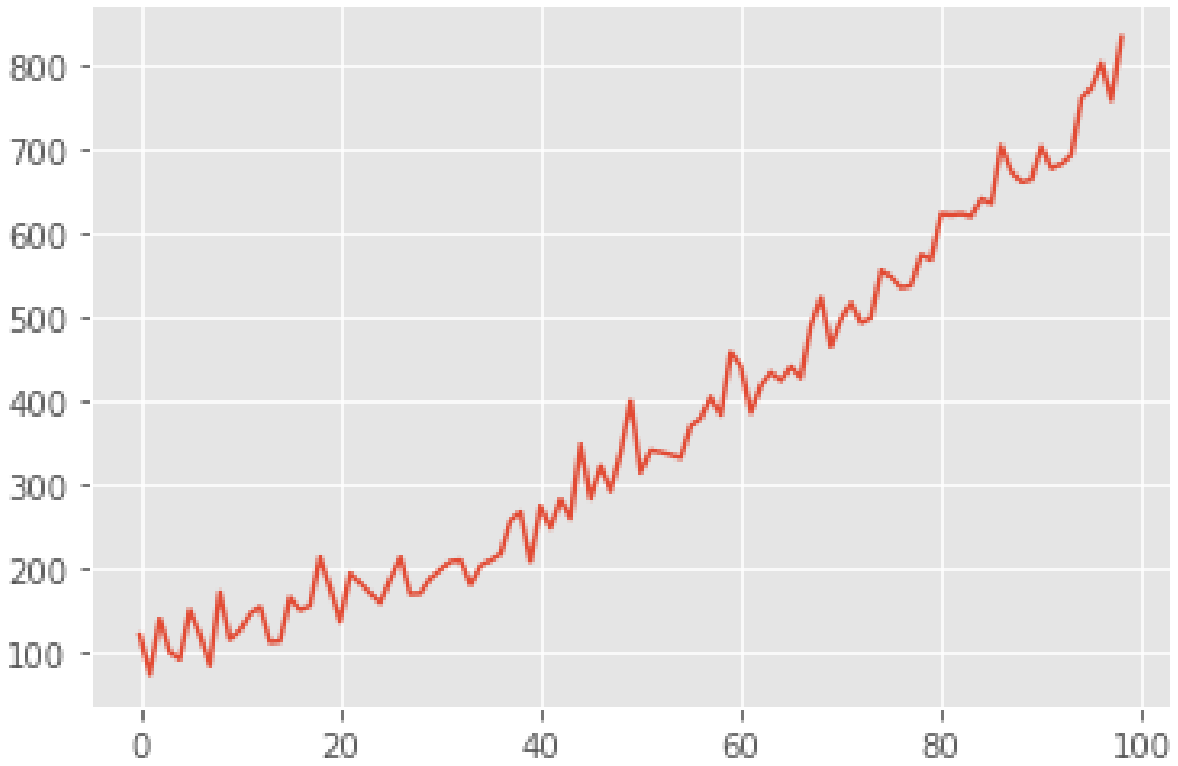


Time series: simple forecasts

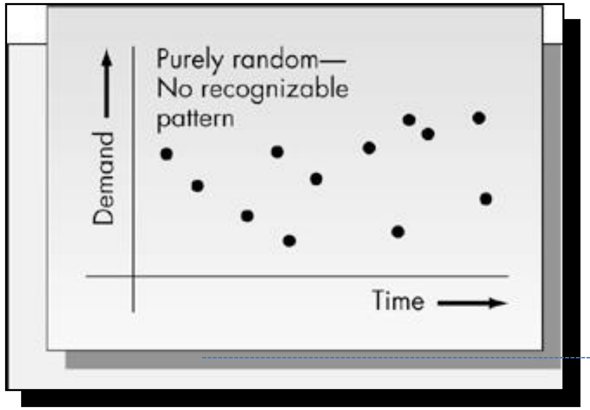
iv) a model with quadratic trend

$$Y_t = at^2 + bt + c + \epsilon_t$$

```
c=100; sigma=25; b=2; a=0.05;
yqr=[0]*101;
for i in range(1,100):
 yqr[i] =c + b*i+ a*i*i+ sigma*random.normalvariate(0, 1)
```



Time series: simple forecasts



- We can now test the properties of the simple estimators:
  i) stationary i.i.d model
  $$Y_t = c + \epsilon_t$$
- The average method and the naive method are both unbiased estimators: $E[\hat{Y}_T] = E[Y_T] = c$.
- The variance of the estimator $Var[\hat{Y}_T]$ is σ^2 for the naive method and σ^2/T for the average method.
- The drift method is also unbiased. The estimator of the drift term is zero in expectation.

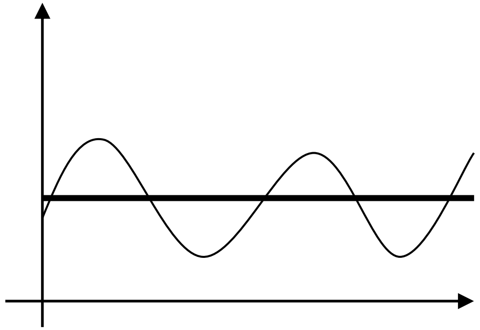
Time series: simple forecasts

- The k-period moving average is unbiased with variance σ^2/k.
- Exponential smoothing is unbiased with asymptotic variance (as $T \to \infty$): $(\alpha\sigma^2)/(2-\alpha)$.
- Note that there are many other unbiased forecasts for a simple stationary series, for instance

$$\hat{y}_{T+h} = y_{T_1},$$
$$\hat{y}_{T+h} = y_T + (y_{T-1} - y_{T-2}),$$
$$\hat{y}_{T+h} = \beta y_T + (1-\beta) y_{T-2} \ (0 \leq \beta \leq 1). \text{ etc.}$$

- These simple models (MA and ES) are basic but effective and frequently used in practice thanks to their responsiveness. Note that MA puts equal weight on the k most recent observations whereas ES puts geometrically decreasing weight on all past observations.

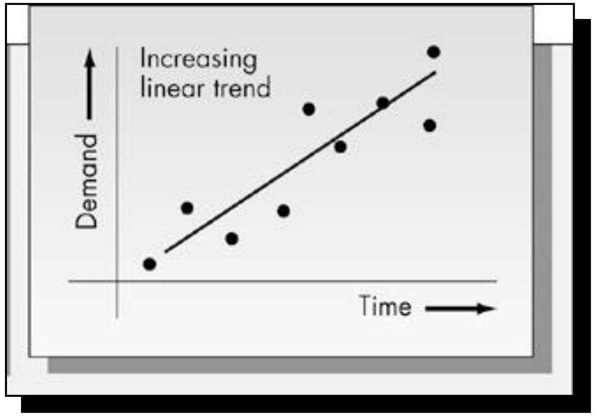
Time series: simple forecasts



- None of the simple forecasts are unbiased for a seasonal series except for the naive seasonal forecast.
- By using the same principles we can build other simple unbiased forecasts.
- For instance, inspired by moving averages we have:
- $$\hat{y}_{T+h|T} = \frac{y_{T+h-m} + y_{T+h-2m} + y_{T+h-3m}}{3} \text{ if } T+h-m \leq T.$$
- And inspired by exponential smoothing we have:

$$\hat{y}_{T+h|T} = \alpha y_{T+h-m} + (1-\alpha)\hat{y}_{T+h-2m} \text{ if } T+h-m \leq T.$$

Time series: simple forecasts



- Let's now check the simple trend model: $Y_t = bt + c + \epsilon_t$.
- The naive forecast is not unbiased: $\hat{y}_{T+h|T} = y_T$. Taking expectations: $E[\hat{Y}_{T+h}] = b(T+h) + c \neq E[Y_T] = bT + c$.
- Similarly, average, moving average, and exponential smoothing are not unbiased.
- This is expected because to capture the functional form (i.e. slope), we would need to estimate an additional term beyond the 'level' of the series.
- The trend forecast is unbiased:

$$E[\hat{Y}_{T+h|T}] = E\left[Y_T + h\left(\frac{Y_T - Y_1}{T-1}\right)\right] = c + bT + hb = c + (T+h)b.$$

Time series: simple forecasts

- We can of course develop other unbiased estimators. Inspired by the naive method:

$$\hat{y}_{T+h|T} = y_T + (y_T - y_{T_1})$$

- Inspired by moving averages:

$$\hat{y}_{T+h|T} = y_T + \frac{(y_T - y_{T-1}) + (y_{T-1} - y_{T-2}) + (y_{T-2} - y_{T-3})}{3}$$

- Inspired by exponential smoothing:

$$\hat{y}_{T+h|T} = y_T + \alpha(y_T - y_{T-1}) + (1 - \alpha)(\hat{y}_{T-1} - \hat{y}_{T-2})$$