INDR 422/522 - Spring 2023 Fikri Karaesmen

CLASS EXERCISE, March 7, 2023

1. Consider two discrete random variables X and Y with the following joint probability mass function: $P(X = 0, Y = 0) = 1/4$, $P(X = 0, Y = 1) = 1/4$, $P(X = 1, Y = 0) = 0$, $P(X = 1, Y = 1) = 1/2$. Find:

   (a) $Var(X)$

   Solution:

   We can use the short cut formula:

   $$Var(X) = E[X^2] - E[X]^2 = 1/2 - (1/2)^2 = 1/4$$

   Note that $Var(X) = E[(X - E[X])^2]$ and is therefore always non-negative.

   (b) $Cov(X, Y)$

   Solution:

   We can use the short cut formula:

   $$Cov(X, Y) = E[XY] - E[X]E[Y] = (1/2) - (1/2)(3/4) = 1/8.$$

   Recall that $Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$. $Cov(X, Y) > 0$ implies that when X takes large (small) values with respect to its mean Y is also likely to take large (small) values with respect to its mean.

   (c) $Corr(X, Y)$

   Solution:

   $$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

   Since σ_X and σ_Y cannot be negative, $Corr(X, Y)$ has the same sign as $Cov(X, Y)$ but is normalized to the interval $[-1, 1]$.

2. Let $Y_t = \mu + \epsilon_t$ where ϵ_t are i.i.d random variables with mean zero and variance σ^2. Consider the forecast: $\hat{y}_{t+1} = (y_t + y_{t-1})/2$.

(a) Is this forecast unbiased?

Solution: Let us check:

$$E[\hat{Y}_{t+1}] = E[(Y_t + Y_{t-1})/2] = (2\mu)/2 = \mu$$

We therefore have: $E[\hat{Y}_{t+1}] = E[Y_{t+1}] = \mu$ so the forecast is unbiased.

(b) Variance of the above forecast?

Solution: Let us check:

$$Var[\hat{Y}_{t+1}] = Var[(Y_t + Y_{t-1})/2] = (2\sigma^2)/4 = \sigma^2/2.$$

(c) Now consider the forecast: $\hat{y}_{t+1} = (3/4)y_t + (1/4)y_{t-1}$. Is this unbiased.

Solution: We can see that : $E[\hat{Y}_{t+1}] = E[Y_{t+1}] = \mu$ so the forecast is unbiased.

(d) Variance of the above forecast?

$$Var[\hat{Y}_{t+1}] = Var[(3/4)Y_t + (1/4)Y_{t-1}] = (9/16)\sigma^2 + (1/16)\sigma^2 = (5/8)\sigma^2.$$