1. (a) The set $A := \{x \in \mathbb{R}^2 | x_1 + i^2 x_2 \leq 1, i = 1, ..., 10\}$ is convex.

Proof. $\partial A \subset \mathbb{R}^2$ is a line so A must be convex. Here's the brute force proof: Take points x and y in A. A is convex if the convex combination of these points,

$$xt + (1-t)y, \quad t \in [0, 1]$$

remains in A $\forall x, y \in A$. Impose the constraints

$$\begin{aligned} tx_1 + i^2 tx_2 &\leq t \\ (t-1)y_1 + i^2(t-1)y_2 &\leq (1-t) \end{aligned}$$

and add:

$$\begin{aligned} tx_1 + ty_1 - y_1 + i^2(tx_2 + ty_2 - y_2) &\leq 1 \\ tx_1 + i^2 tx_2 - y_1 - i^2 y_2 + ty_1 + i^2 ty_2 &\leq 1 \\ tx - y + ty &\leq 1 \\ tx + (1-t)y &\leq 1 \end{aligned}$$

This is a special case of the following (also obvious) fact $\{x \in \mathbb{R}^n | a_i x^i \leq b\}$ $\forall a \in \mathbb{R}^n$, $b \in \mathbb{R}$. □

(b) The set $B := \{x \in \mathbb{R}^2 | x_1^2 + 2ix_1 + i^2 x_2^2 \leq 1, i = 1, ..., 10\}$ is convex.

Proof. The constraint can be written as

$$(x_1 + ix_2)^2 \leq 1 \implies -1 \leq (x_1 + ix_2) \leq 1.$$

Either constraint leads to a convex set due to part (a). Since the intersection of two convex sets is convex, B is convex. □

(c) The set $C := \{x \in \mathbb{R}^2 | x_1^2 + 5x_1x_2 + 4x_2^2 \leq 1, i = 1, ..., 10\}$ is not convex.

Proof. $\partial C = \{x \in \mathbb{R}^2 | \underbrace{x_1^2 + 5x_1x_2 + 4x_2^2 - 1}_{f} = 0\}$.

Recall that $f \in C^2(\Omega)$ is convex in Ω iff $\text{Hess}f \geq 0$. Further, recall that a convex set is bounded by a convex curve. Then, we only need to prove that the boundary of C is not a convex curve by showing $\text{Hess}f < 0$.

$$\text{Hess}f = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$$

has eigenvalues $\{4 + \sqrt{34}, 4 - \sqrt{34}\}$, the latter of which is negative. □

(d) The set $D := \{x \in \mathbb{R}^d | \sum_i x_i^2 = 1\}$ is not convex.

Proof. This is $S^{d-1} \hookrightarrow \mathbb{R}^d$. The sphere is obviously not convex in $\mathbb{R}^d$. By counterexample (or triangle inequality in general),

$$xt + (1-t)y\Big|_{t=1/2} = \frac{x+y}{2}$$

which has norm $\sqrt{2}/2$ and hence is outside D.

$\square$

2. (a) We have

$$\text{Hess}\frac{x^2}{y} = \begin{pmatrix} \frac{2}{y} & -\frac{2x}{y} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{pmatrix}$$

which has eigenvalues

$$\left\{0, \frac{2(x^2+y^2)}{y^3}\right\} \geq 0 \text{ on } y > 0$$

so the function is convex.

(b) We have

$$\text{Hess}\log\left(e^x + e^y\right) = \begin{pmatrix} \frac{e^{x+y}}{(e^x+e^y)^2} & -\frac{e^{x+y}}{(e^x+e^y)^2} \\ -\frac{e^{x+y}}{(e^x+e^y)^2} & \frac{e^{x+y}}{(e^x+e^y)^2} \end{pmatrix}$$

which has eigenvalues

$$\left\{0, \frac{2e^{x+y}}{(e^x+e^y)^2}\right\} \geq 0 \text{ on } \mathbb{R}^2$$

so the function is convex.

3. (a) I'm having a hard time interpreting the parameters of the problem. I'm going to assume what you have in mind is some sort of affine ReLu with a penalty term. Here's my formulation of the problem:
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the map

$$f(x;b) = \max(0, Ax+b) + \rho\|x\|^2$$

The function $\max(0, g(x))$ is non-differentiable at $0 = g(x)$. Otherwise, the unique subgradients are either 0 or ∇g. At the non-differentiable point

$$\partial \max(0, g(x)) = \text{conv}(0 \cup \nabla g) = \{\alpha \nabla g : \alpha \in [0,1]\}$$

The function $\rho\|x\|^2$ is differentiable so the subgradient is simply

$$\begin{aligned} \rho\nabla\|x\|^2 &= \rho\nabla_x(x^T \mathbb{I} x) \\ &= 2\rho\mathbb{I}x. \end{aligned}$$

Putting these together, the sub-differential wrt x is

$$\partial_x f(x;b) = \begin{cases} \{\alpha A^T + 2\rho\mathbb{I}x : \alpha \in [0,1]\} & Ax+b=0 \\ \{2\rho\mathbb{I}x\} & Ax+b<0 \\ \{A^T + 2\rho\mathbb{I}x\} & Ax+b>0 \end{cases}$$

Similarly,

$$\partial_b f(x;b) = \begin{cases} \{\alpha\vec{1} : \alpha \in [0,1]\} & Ax + b = 0 \\ \{\vec{0}\} & Ax + b < 0 \\ \{\vec{1}\} & Ax + b > 0 \end{cases}$$

(b) $\partial\|x\| = \{x/\|x\|\}$ if $x \neq 0$. Else $\partial\|x\| = \{\|x\| \leq 1\}$

Proof. When $x \neq 0$ the function is differentiable. Then, the subgradient is determined uniquely by the gradient:

$$\partial f(x \neq 0) = \nabla\|x\| = \frac{x}{\|x\|} = \text{sign } x.$$

When $x =$ the function is non-differentiable. Thus, we seek the set of $g \in \mathbb{R}^n$ s.t.

$$\|y\| - \|x\| \geq (g, y - x) \quad \forall y \in \mathbb{R}^n$$

which can be logically decomposed into

$$\|y\| \geq (g, y) \text{ and } \|x\| \geq (g, x).$$

A subgradient, g, satisfies these inequalities if $\|g\| \leq 1$. Hence, the subdifferential of $f(x) = \|x\|$ is

$$\partial f(x) = \begin{cases} \{x \in \mathbb{R}^n : \|x\| \leq 1\} & x = 0 \\ \{\frac{x}{\|x\|}\} & x \neq 0 \end{cases}$$

□