1. (a) The set $A := \{x \in \mathbb{R}^2 | x_1 + i^2 x_2 \leq 1, i = 1, ..., 10\}$ is convex.

Proof. $\partial A \subset \mathbb{R}^2$ is a line so A must be convex. Here's the brute force proof: Take points x and y in A. A is convex if the convex combination of these points,

$$xt + (1-t)y, \quad t \in [0,1]$$

remains in A $\forall x, y \in A$. Impose the constraints

$$\begin{aligned} tx_1 + i^2 tx_2 &\leq t \\ (t-1)y_1 + i^2(t-1)y_2 &\leq (1-t) \end{aligned}$$

and add:

$$\begin{aligned} tx_1 + ty_1 - y_1 + i^2(tx_2 + ty_2 - y_2) &\leq 1 \\ tx_1 + i^2 tx_2 - y_1 - i^2 y_2 + ty_1 + i^2 ty_2 &\leq 1 \\ tx - y + ty &\leq 1 \\ tx + (1-t)y &\leq 1 \end{aligned}$$

This is a special case of the following (also obvious) fact $\{x \in \mathbb{R}^n | a_i x^i \leq b\}$ $\forall a \in \mathbb{R}^n$, $b \in \mathbb{R}$. □

(b) The set $B := \{x \in \mathbb{R}^2 | x_1^2 + 2ix_1 + i^2 x_2^2 \leq 1, i = 1, ..., 10\}$ is convex.

Proof. The constraint can be written as

$$(x_1 + ix_2)^2 \leq 1 \implies -1 \leq (x_1 + ix_2) \leq 1.$$

Either constraint leads to a convex set due to part (a). Since the intersection of two convex sets is convex, B is convex. □

(c) The set $C := \{x \in \mathbb{R}^2 | x_1^2 + 5x_1 x_2 + 4x_2^2 \leq 1, i = 1, ..., 10\}$ is not convex.

Proof. $\partial C = \{x \in \mathbb{R}^2 | \underbrace{x_1^2 + 5x_1x_2 + 4x_2^2 - 1}_{f} = 0\}$.

Recall that $f \in C^2(\Omega)$ is convex in Ω iff $\text{Hess}f \geq 0$. Further, recall that a convex set is bounded by a convex curve. Then, we only need to prove that the boundary of C is not a convex curve by showing $\text{Hess}f < 0$.

$$\text{Hess}f = \begin{pmatrix} 2 & 5 \\ 5 & 4 \end{pmatrix}$$

has eigenvalues $\{4 + \sqrt{34}, 4 - \sqrt{34}\}$, the latter of which is negative. □

(d) The set $D := \{x \in \mathbb{R}^d | \sum_i x_i^2 = 1\}$ is not convex.

Proof. This is $S^{d-1} \hookrightarrow \mathbb{R}^d$. The sphere is obviously not convex in $\mathbb{R}^d$. By counterexample (or triangle inequality in general),

$$xt + (1-t)y\Big|_{t=1/2} = \frac{x+y}{2}$$

which has norm $\sqrt{2}/2$ and hence is outside D.

$\square$

2. (a) We have

$$\operatorname{Hess}\frac{x^2}{y} = \begin{pmatrix} \frac{2}{y} & -\frac{2x}{y} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{pmatrix}$$

which has eigenvalues

$$\left\{0, \frac{2(x^2+y^2)}{y^3}\right\} \geq 0 \text{ on } y > 0$$

so the function is convex.

(b) We have

$$\operatorname{Hess}\log\left(e^x + e^y\right) = \begin{pmatrix} \frac{e^{x+y}}{(e^x+e^y)^2} & -\frac{e^{x+y}}{(e^x+e^y)^2} \\ -\frac{e^{x+y}}{(e^x+e^y)^2} & \frac{e^{x+y}}{(e^x+e^y)^2} \end{pmatrix}$$

which has eigenvalues

$$\left\{0, \frac{2e^{x+y}}{(e^x+e^y)^2}\right\} \geq 0 \text{ on } \mathbb{R}^2$$

so the function is convex.