Non-intertial motion - Rotating frames

November 2, 2020

We start with noting a direct corollary of one of the axioms of classical mechanics:

Thm. *Given a motion* $\vec{x} : \mathbb{R} \to \mathbb{R}^3$*, one can always find a frame in which* $\vec{F} = m\ddot{\vec{x}}$*. Such frames are inertial frames.*

That is, a motion specifies a system of *inertial* forces and vice versa. Note, however, that it is not always easy to work with inertial frames. As an example, try to solve the first discussion problem in inertial coordinates.

Thm. *Let* $\vec{x}_{NI}$ *be a non-inertial motion. Then, in this non-inertial frame there is an effective system of forces such that*

$$\vec{F}_{eff} = m\ddot{\vec{x}}_{NI}.$$

where $\vec{F}_{eff} = \vec{F}_{real} + \vec{F}_{fake}$.

Proof. $\vec{x}_{NI}$ is a trajectory and has the usual nice properties one expects of a solution to a second order ODE. So, (by Picard's theorem), such an $\vec{x}_{NI}$ solves *some* second order ODE which we may massage in to the form of Newton's second law. Clearly, the system of forces is not just the physical/real forces as this would contradict the previous theorem. Hence, the given decomposition. $\square$

Let us specialize to rotating frames and rework this idea more explicitly.

Prop. *Let* x *denote motion in the non-inertial rotating frame and let the primed coordinates be relative to the inertial frame. These coordinates are related through relation* $x' = x + x_0$ *(*x_0 *denoting origin of rotating frame). We have infinitesimally*

$$\dot{x}' = \dot{x} + \dot{x}_0 + \omega \times x$$

This is true in general for any vector $\vec{x}$*, not just positions/motion.*

Proof. I'm sure you'll see this result derived in class. $\square$

For reference, here is this result expressed more cleanly:

$$\boxed{\vec{v}_I = \vec{v}_0 + \vec{v}_{NI} + \vec{\omega} \times \vec{x}}$$

$\vec{\omega}$ is the rotation frequency of the frame.

Thm. *Let $\vec{x}_{NI}$ be a motion in a rotating frame. Then, in this non-inertial frame there is an effective system of forces such that*

$$\vec{F}_{eff} = m\ddot{\vec{x}}_{NI}.$$

where

$$\boxed{\vec{F}_{eff} = \vec{F}_{real} - m\dot{v}_0 - m\dot{\omega} \times x_{NI} - m\omega \times (\omega \times x_{NI}) - 2m\omega \times v_{NI}}$$

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Proof. Start with $F = m\dot{v}_I$ and carefully differentiate in inertial frame to obtain F_{fake}. □

Remark. Note that we can write $\dot{v}_0 = \omega \times v_0 = \omega \times (\omega \times x_0)$ by our proposition applied twice.

Defn. The force of the form $2m\omega \times v_{NI}$ felt by a rotating observer is the *Coriolis force*. $-m\omega \times (\omega \times x_{NI})$ is the *centrifugal force*.

Motion relative to rotating Earth

First, note that the rotation of the Earth about its center is almost constant in time, $\dot{\omega} = 0$. Also note in general that $\vec{F}_{real}$ is very complicated. We often write $\vec{F}_{real} = \vec{f} + mg_I$ and we don't worry about the $\vec{f}$. Then,

$$F_{eff} = f + mg_I - m\dot{v}_0 - m\omega \times (\omega \times x_{NI}) - 2m\omega \times v_{NI}.$$

We can replace the acceleration of the origin $\dot{v}_0$ with $\omega \times (\omega \times x_0)$ and get

$$F_{eff} = f + mg_I - m\dot{v}_0 - m\omega \times (\omega \times (x_{NI} + x_0)) - 2m\omega \times v_{NI}.$$

The action of the centrifugal force is usually absorbed into an effective non-inertial gravitational acceleration

$$g_{NI} = g_I - \omega \times (\omega \times (x_{NI} + x_0)).$$

Finally,

$$\boxed{F_{eff} = f + mg_{NI} - 2m\omega \times v_{NI}}$$

In practice, you will usually only care about the Coriolis part of this expression.