Stability of EOM solutions

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Remark. Nonlinear ODEs typically have multiple solutions. A very common thing one has to get in the habit of thinking about is if particular solutions you found make sense physically. In mathematical terms, this usually means doing a stability analysis of the solutions because Nature tends to realize stable configurations in the space of all possible configurations.

Defn. A steady state solution y_s is said to be perturbatively stable if $y(t) \to y_s$ as $t \to \infty$ for y close to y_s (in some quantifiable way that we don't need to rigorously specify). If $y \to y_s$ for any y, then y_s is arbitrarily stable.

Defn. For an ODE of the form (I'll call this the standard form)

$$y'' + ay' + f(b, y) = 0$$

where f is a smooth function of its parameters, a point in the space (b, y) is a bifurcation point if the stability of solutions change at that point.

Defn. A solution branch to an ODE of the standard form is a solution to the steady state condition

$$f(b, y) = 0.$$

Prop. *For an ODE of the standard form one must have at a bifurcation point,* $(b^\star, y^\star)$, *that*

$$\boxed{\begin{aligned} f(b^\star, y^\star) &= 0 \\ \partial_y f(b^\star, y^\star) &= 0 \end{aligned}}$$

Proof. Fairly direct application of the implicit function theorem. The proof isn't very relevant to the discussion. □

Prop. *Let* y_s *be a steady state solution and let* y *be* ϵ*-close to* y_s *at some time* t*. Then, if*

$$\partial_y f(b, y_s) > 0$$

then y_s *is* perturbatively *asymptotically stable. Else, if*

$$\partial_y f(b, y_s) < 0$$

y_s *is unstable.*

Linearized stability

Proof. Starting with a solution ϵ-close to a steady state solution is written out as the initial data

$$y(0) = y_s + \epsilon\kappa_1, \;\; y'(0) = \epsilon\kappa_2$$

for some $\kappa_{1,2} \in \mathbb{R}$ and ϵ small. Hence, it makes sense to consider the deformation

$$y(t) \sim y_s + \epsilon y_1(t)$$

The ODE becomes

$$y_1'' + ay_1' + f(b, y_s + \epsilon y_1(t)) = 0$$

By Taylor's theorem, to first order in ϵ this is

$$y_1'' + ay_1' + f(b, y_s) + \epsilon\partial_y f(b, y)|_{y_s} = 0$$

with initial data

$$y_1(0) = \kappa_1, \;\; y'(0) = \kappa_2.$$

Notice that we have linearized the ODE and we may solve it directly. If $\partial_y f(b, y)|_{y_s} \neq a^2/4$ the solution is

$$y_1(t) = \alpha e^{\eta_+ t} + \beta e^{\eta_- t}$$

where

$$\eta_\pm = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \partial_y f(b, y)|_{y_s}}$$

Note that $\text{Re}\eta_-$ is always negative if $a > 0$. Looking at $\text{Re}\eta_+$ we find that we need

$$\partial_y f(b, y_s) > 0 \implies \eta_+ < 0$$

If $\eta_\pm < 0$ $y_1(t) \to 0$ as $t \to \infty$ and hence $y(t) \to y_s$

$\square$

Example 1

Consider the cubic oscillator ODE

$$y'' + ay' - by + y^3 = 0.$$

Suppose as $t \to \infty$ $y \to$ const meaning that a steady state solution exists (this is not always true but is in this case). Then, clearly we have

$$by = y^3$$

as $t \to \infty$. This (asymptotic) equality holds for all $b \in \mathbb{R}$ if $y = 0$. Otherwise, $y = \pm\sqrt{b}$ if $b > 0$. Thus, the steady state solutions are both branches of the square root. Notice that the neighborhood of the point $(y, b) \sim (0, 0)$ is common to both branches.

In this case $f_y(b, y_s) = -b + 3y_s^2 = 2b$. Thus, if $b > 0$, the $\pm\sqrt{b}$ branches are stable. The $y_s = 0$ branch has $f_y(b, 0) = -b$ which is unstable for all positive b and stable for all negative b. The point $(0, 0)$ is the bifurcation point of the three solution branches.

Example 2

Consider the pendulum EOM

$$\ddot{\theta} + \alpha \sin\theta = 0$$

where $\alpha = g/L > 0$ with L the length of the string holding the pendulum bob. In equilibrium (steady state)

$$\alpha \sin\theta = 0.$$

For $\theta \in [-\pi, \pi]$, there are three solution branches (which are points in this case):

$$\theta_s \in \{-\pi, 0, \pi\}.$$

Carrying out the stability analysis,

$$f_\theta(\alpha, \theta_s) = \alpha \cos\theta_s = \pm\alpha$$

Thus, $\theta_s = 0$ is stable while the other two configurations are not. That is to say, you don't see inverted pendula in equilibrium (unless you drive the pivot a particular way).