Please note that the questions are out of order, sorry about that!

Discussion 1 Solutions

2. Recall that the Taylor series for cosine and sine for small x are

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

   (a) Since a is small, and x in the integrand is between 0 and a, x is also small. Thus, we may substitute the Taylor series for cosine and sine into the integrand.

$$\begin{aligned} 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ \frac{1 - \cos x}{x^2} &= \frac{1}{2!} - \frac{x^2}{4!} + \cdots \\ &= \frac{1}{2} - \frac{x^2}{24} + \cdots \end{aligned}$$

$$\begin{aligned} I_1(a) &= \int_0^a \frac{1 - \cos x}{x^2} dx \\ &= \frac{a}{2} - \frac{a^3}{72} + \cdots \end{aligned}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \cdots$$

$$\begin{aligned} I_2(a) &= \int_0^a \frac{\sin x}{x} dx \\ &= a - \frac{a^3}{18} + \cdots \end{aligned}$$

   (b) Substituting $a = 1$ and approximating with only the first two non-vanishing terms, we obtain

$$I_1(1) \approx \frac{1}{2} - \frac{1}{72} = \frac{35}{72} = 0.49$$
$$I_2(1) \approx 1 - \frac{1}{18} = \frac{17}{18} = 0.94$$

4. Let $g(x, y) = \eta(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$.

   (a) The local maximum (or minimum) of this function satisfies $\nabla g = 0$, *i.e.* $\dfrac{\partial g}{\partial x} = 0$ and $\dfrac{\partial g}{\partial y} = 0$.

$$0 = \frac{\partial g}{\partial x} = \eta(2y - 6x - 18)$$
$$0 = \frac{\partial g}{\partial y} = \eta(2x - 8y + 28)$$

   Solving this system of equations give $x = -2$, $y = 3$. Since there is only one point for which $\nabla g = 0$, we do not need to check second derivatives to decide whether it is a maximum. Furthermore, we're considering the function for the whole xy-plane, so there is no boundary. Thus, the point $(-2, 3)$

is either a global maximum or a global minimum. We simply need to compare the values of g at $(-2,3)$ and at any other point, for example $(0,0)$.

$$\begin{aligned} g(-2,3) &= \eta\big[2(-2)3 - 3(-2)^2 - 4\cdot 3^2 - 18(-2) + 28\cdot 3 + 12\big] \\ &= \eta(-12 - 12 - 36 + 36 + 84 + 12) = 72\eta = 1.44\,\mathrm{m} \\ g(0,0) &= \eta(0+12) = 12\eta = 0.24\,\mathrm{m} < g(-2,3) \end{aligned}$$

Thus, $(-2,3)$ is indeed the top of the hill, and the height is $1.44\,\mathrm{m}$.

(b) For this problem, we interpret the equation $z = g(x,y)$ as a surface in 3D, where a function of 3 variables takes a constant value. To do this, we move $g(x,y)$ to the LHS of the equation.

$$\begin{aligned} z - g(x,y) &= 0 \\ f(x,y,z) &= 0 \end{aligned}$$

where $f(x,y,z) = z - g(x,y)$ takes the value 0 along the "hill".
Now, from multivariable calculus, we know that ∇f, which is the 3-dimensional gradient, represents the direction perpendicular to the hill.

$$\begin{aligned} \nabla f(x,y,z) &= -\frac{\partial g}{\partial x}\hat{\imath} - \frac{\partial g}{\partial y}\hat{\jmath} + \hat{k} \\ &= -\eta(2y - 6x - 18)\hat{\imath} - \eta(2x - 8y + 28)\hat{\jmath} + \hat{k} \\ \nabla f(1,1,g(1,1)) &= 22\eta\hat{\imath} - 22\eta\hat{\jmath} + \hat{k} \end{aligned}$$

Denote this vector by $\vec{v}$. We want to find the angle between this vector and $\hat{k}$. The easiest way is to use dot product.

$$\begin{aligned} \vec{v}\cdot\hat{k} &= v|\hat{k}|\cos\theta \\ (22\eta\hat{\imath} - 22\eta\hat{\jmath} + \hat{k})\cdot\hat{k} &= \sqrt{1 + (22\eta)^2 + (22\eta)^2}\cdot 1\cdot\cos\theta \\ \cos\theta &= \frac{1}{\sqrt{1 + 968\eta^2}} \end{aligned}$$

Plugging in $\eta = 0.02\,\mathrm{m}$, we obtain $\theta = 31.9°$.

(c) The vector ∇g is the "uphill" direction (steepest increase in g). Hence, the direction directly downhill is $-\nabla g$.

$$\begin{aligned} -\nabla g(x,y) &= -\eta(2y - 6x - 18)\hat{\imath} - \eta(2x - 8y + 28)\hat{\jmath} \\ -\nabla g(1,1) &= 22\eta(\hat{\imath} - \hat{\jmath}) \end{aligned}$$

Since $\eta > 0$, this is the SE direction.

1. Acceleration vector is calculated using $\dfrac{d^2\vec{r}}{dt^2}$.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = \omega b\cos\omega t\,\hat{\imath} - \omega b\sin\omega t\,\hat{\jmath} + 2ct\hat{k} \\ \vec{a} &= \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = -\omega^2 b\sin\omega t\,\hat{\imath} - \omega^2 b\cos\omega t\,\hat{\jmath} + 2c\hat{k} \\ a = |\vec{a}| &= \sqrt{\omega^4 b^2(\sin^2\omega t + \cos^2\omega t) + (2c)^2} = \sqrt{\omega^4 b^2 + 4c^2}, \end{aligned}$$

which is a constant.

5. We are given

$$\begin{aligned} \vec{b} \cdot \vec{v} &= \lambda \\ \vec{b} \times \vec{v} &= \vec{c} \end{aligned}$$

In order to use the vector identity that's given, take the cross product of the second equation with $\vec{b}$ from the *left*.

$$\begin{aligned} \vec{b} \times (\vec{b} \times \vec{v}) &= \vec{b} \times \vec{c} \\ \underbrace{(\vec{b} \cdot \vec{v})}_{=\lambda}\vec{b} - (\vec{b} \cdot \vec{b})\vec{v} &= \vec{b} \times \vec{c} \\ \lambda\vec{b} - \vec{b} \times \vec{c} &= b^2\vec{v} \\ \vec{v} &= \frac{\lambda}{b^2}\vec{b} - \frac{(\vec{b} \times \vec{c})}{b^2} \end{aligned}$$

3. (a)

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial}{\partial x}(x^3 z^4) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(x^2 y^2) \\ &= 3x^2 z^4 + xz^2 \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{curl} \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 z^4 & xyz^2 & x^2 y^2 \end{vmatrix} \\ &= (2x^2 y - 2xyz)\hat{\imath} + (4x^3 z^3 - 2xy^2)\hat{\jmath} + yz^2\hat{k} \end{aligned}$$

(c)

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) &= \vec{\nabla} \cdot \left[(2x^2 y - 2xyz)\hat{\imath} + (4x^3 z^3 - 2xy^2)\hat{\jmath} + yz^2\hat{k}\right] \\ &= (4xy - 2yz) + (-4xy) + (2yz) = 0 \end{aligned}$$