A very specific review and some coupled oscialltions

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Things to recall:

- $E = T + U$, the total energy, is a constant of motion (in closed mechanical systems). The dynamical data under this constraint is encoded in $\dot{E} = 0$.

  Ex (Simple pendulum)**.** *We have the total energy* $\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$. *The constancy of this quantity implies*

  $$\dot{E} = 0 \implies ml^2\ddot{\theta}\dot{\theta} + mgl\sin\theta\dot{\theta} = 0$$

  which yields the familiar

  $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

  as the equation of motion. In this case the statement that total energy is conserved contains all the dynamical information of the system.

- A rolling object is said to not slip if its transnational velocity is matched by the tangential velocity of rolling: $\dot{q}_{center} = r\dot{\phi}$.

- What we mean when we say *solving* a classical mechanical system (in the Lagrangian formalism): For N degrees of freedom identify N generalized coordinates. Work in coordinate systems that are convenient for the problem. Once the coordinate system is identified, $\{q_i, \dot{q}_i\}_{i=1,\ldots,N}$, construct the Lagrangian

  $$L[q(t), \dot{q}(t); t] = T - U$$

  and solve

  $$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

  for each coordinate.

- When we say frequency of oscillation, we specifically mean the oscillation frequency of the harmonic oscillator. This is the ω that appears in

  $$L_{SHO} = \frac{1}{2}m\dot{q}^2 - \underbrace{\frac{1}{2}m\omega^2 q^2}_{U_{SHO}}$$

When you are asked to find the frequency of small oscillations, you need to figure out the parameters for which

$$U_{\text{your system}} = U_{SHO} + \mathcal{O}(q^3)$$

Binary oscillator

Consider N *linearly* coupled bodies with Lagrangian

$$\frac{1}{2}\sum_{i}^{N} \dot{x}_i^2 m_i - \frac{1}{2}\sum_{i,j}^{N} k_{ij}(x_i - x_j)^2.$$

We have a notion of "$F = ma$" though in matrix form (tensorial if you want to be more precise):

$$M_{ij}\partial_t^2 x_j = -K_{ij}x_j \tag{1}$$

where the mass matrix entries are given by

$$M_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \partial x_j}$$

and correspondingly the matrix entries of the stiffness

$$K_{ij} = \frac{\partial^2 V}{\partial \dot{x}_i \partial x_j}.$$

You might imagine K_{ij} to be read as the stiffness of the spring coupling degrees of freedom i and j. Keep in mind, however, that this intuition is basis dependent.

Since our EOM (Eq. 1) is linear, we can go ahead and Fourier transform:

$$-M_{ij}\omega^2 x_j(\omega) = -K_{ij}x_j(\omega) \implies (M_{ij}\omega^2 - K_{ij})x_j(\omega) = 0$$

Here is a theorem you will come across in liner algebra:

Thm. *The following are equivalent*

1. $\dim Ker(M_{ij}\omega^2 - K_{ij}) > 0$
2. $\det(M_{ij}\omega^2 - K_{ij}) = 0$

This tells you that motion will be *non trivial* only if $\det(M_{ij}\omega^2 - K_{ij}) = 0$ so we must find ωs that satisfy this equation. In practice, you will be able to construct these $\mathbf{K}$ and $\mathbf{M}$ matrices explicitly.

Formal details

In my more general example, I proceed like so:

First, assume $\mathbf{M}$ is diagonal, as it almost always is, and write $M_{ij} = m_j\delta_{ij} = m_i$. Let us take Λ to the matrix of ordered and orthonormalized eigenvectors of $\mathbf{K}$. Then, recalling that $\det \mathbf{A} = \prod_i \text{Eigenvalues}(A)_i$

$$\prod_i (\omega^2 m_i - (\mathbf{\Lambda} \cdot \mathbf{K}\mathbf{\Lambda}^T)_{ii}) = 0$$

Thus, opereationally, we just need to diagonalize the stiffness matrix to determin the ω's.