Solution to coupled linear oscillator

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Coupled linear oscillators revisited

Let us consider again the chain of coupled oscillators with Lagrangian

$$L = \frac{1}{2}\sum_{i}^{N} \dot{x}_i^2 m_i - \frac{1}{2}\sum_{i,j,|i-j|=1}^{N} k_i(x_i - x_j)^2.$$

The constrained sum in the potential asserts that only nearest neighbors. We saw that the equations of motion can be written in matrix form:

$$(\mathbf{M}\partial_t^2 - \mathbf{K})\vec{x}(t) = 0$$

We can render this EOM algebraic by a Fourier transform:

$$(\mathbf{M}\omega^2 + \mathbf{K})\vec{x}(\omega) = 0.$$

Let's take a more direct approach than last time and try to obtain the modes of oscillation and the trajectories. First, notice that any physically meaningful mass matrix must be invertible (it is always diagonal and positive, actually). Then,

$$\begin{aligned}
\mathbf{M}^{-1}(\mathbf{M}\omega^2 + \mathbf{K})\vec{x}(\omega) &= \mathbf{M}^{-1}0 \\
(\mathbb{I}\omega^2 + \underbrace{\mathbf{M}^{-1}\mathbf{K}}_{\bar{\mathbf{K}}})\vec{x}(\omega) &= 0 \\
\bar{\mathbf{K}}\vec{x}(\omega) &= -\omega^2\vec{x}(\omega)
\end{aligned}$$

We have therefore formulated the EOM as an eigenvalue equation: $-\omega^2$ is the eigenvalue of eigenvector $\vec{x}(\omega)$. For small matrices, the eigenvalues can be computed by hand using

$$\det(\omega^2\mathbb{I} - \bar{\mathbf{K}}) = 0 \text{ (look familiar?)}.$$

This will yield a set of eigenvalues, $\{\omega_i\}$ which are, in general, complex [1]. We then plug this back into the equation and solve for the i-th eigenvector:

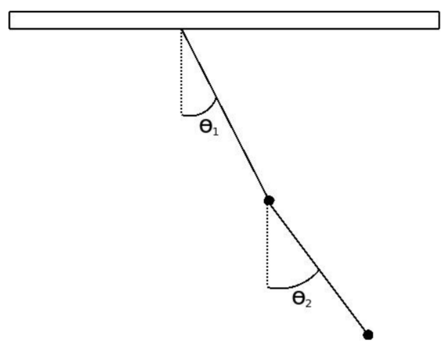
[1] *All* square matrices over $\mathbb{C}$ have (complex) eigenvalues. The same is not true for the reals (because $\mathbb{R}$ is not algebraically closed)

$\bar{\mathbf{K}}\vec{x}_i(\omega_i) = -\omega_i^2 x_i(\omega_i)$. Of course, in real life such things are never computed by hand. One would either diagonalize numerically or use some form of symbolic math software like Mathematica or Sage.

We will walk through a specific computation in class where you will get to practice this technique.

The double pendulum

The double pendulum is an important example so let's go through the construction of its Lagrangian.



We have two generalized coordinates, θ_1, θ_2 which relate to the Cartesion coordinates as

$$q_1 = (L_1 \sin\theta_1, -L_1 \cos\theta_1)$$
$$q_2 = q_1 + (L_2 \sin\theta_1, -L_2 \cos\theta_2)$$

We can then construct the Lagrangian as

$$\mathcal{L} = \frac{1}{2}m_1 L_1^2 \dot{\theta}_i^2 + \frac{1}{2}m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 L_2^2 \dot{\theta}_2^2 + m_1 g L_1 \cos\theta_1 + m_2(L_1 \cos\theta_1 + L_2 \cos\theta_2)$$

You can, and should, show that the EOM is given by the matrix equation

$$(\mathbf{M}\partial_t^2 + \mathbf{K})\vec{\theta} = 0$$

with $\mathbf{M}$ and $\mathbf{K}$ suitably defined. If $m_1 = m_2$ and $L_1 = L_2$, you obtain what is given to you in the problem sheet.

Here is a numerical implementation in python. They use odeint to integrate the EOM directly. You'll notice that they hard code the number of degrees of freedom. If instead of odeint you numerically compute the eigenvectors and eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ (then inverse Foureri transform), you can generalize to the N-pendulum problem.