1. $\vec{E}_0 = \hat{x} E_0$

$\vec{E}_{in} = \vec{E} - \vec{E}_{out}$

$\vec{P}_{in} = -3\varepsilon_0 \vec{E}_{in}$

$\vec{D}_{in} = \varepsilon_0 \vec{E}_{in} + \vec{P}_{in}$

2. infinity에서 potential 이 0 이라면,

point charge Q의 에너지는 $W = QV(\vec{r})$ 이다.

$$U = q V(\vec{r}+\vec{d}) - q V(\vec{r}) = q\left(V(\vec{r}+\vec{d}) - V(\vec{r})\right)$$

$$= q\left[-\int_{\vec{r}}^{\vec{r}+\vec{d}} \vec{E}\cdot d\vec{l}\right]$$

for an ideal dipole,

$$U = -q\vec{E}\cdot\vec{d} = -\vec{p}\cdot\vec{E} \quad (\because \vec{p} = q\vec{d})$$

infinity에서 potential 이 0이 아니라하면,

$$W = Q\left(V(\vec{r}) - V(\vec{r}_0)\right)$$

$$U = q\left(V(\vec{r}+\vec{d}) - V(\vec{r}_0)\right) - q\left(V(\vec{r}) - V(\vec{r}_0)\right)$$

$$= q\left(V(\vec{r}+\vec{d}) - V(\vec{r})\right) = -q\vec{E}\cdot\vec{d} = -\vec{p}\cdot\vec{E}$$

3. $U = -\vec{p}_1\cdot\vec{E}_2$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\vec{p}_2\cdot\hat{r})\hat{r} - \vec{p}_2\right)$$

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})\right].$$

4.

i) $\rho_b = -\nabla \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2}$

$\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} +\vec{P} \cdot \hat{r} = k/b & (\text{at } r=b) \\ -\vec{P} \cdot \hat{r} = -k/a & (\text{at } r=a) \end{cases}$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$

for $r<a$ $\quad Q_{enc}=0$, $\vec{E}=0$.

for $r>b$ $\quad Q_{enc}=0$ $\quad \vec{E}=0$.

for $a<r<b$. $\quad Q_{enc} = -\frac{k}{a}(4\pi a^2) + \int_a^r \left(-\frac{k}{\bar{r}^2}\right) 4\pi \bar{r}^2 \, d\bar{r}$

$= -4\pi k a - 4\pi k (r-a) = -4\pi k r$.

$\vec{E} = -\frac{k}{\varepsilon_0 r}\hat{r}$

ii) $\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}} = 0$

$\Rightarrow \vec{D} = 0$

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0 \quad \Rightarrow \quad \vec{E} = -\frac{1}{\varepsilon_0}\vec{P}$

for $r<a$ and $r>b$, $\vec{E}=0$

for $a<r<b$ $\quad \vec{E} = -\frac{k}{\varepsilon_0 r}\hat{r}$.

5.

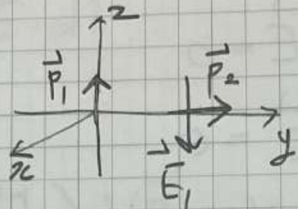
(a) $\vec{F}_2 = (\vec{p}_2 \cdot \nabla)\vec{E}_1 = p_2 \frac{\partial}{\partial y}\vec{E}_1$

$$\vec{E}_1 = \frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\theta} = -\frac{p_1}{4\pi\varepsilon_0 y^3}\hat{z}$$

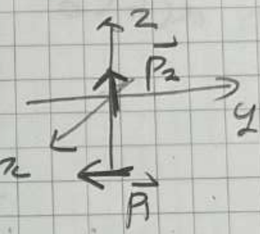
$$\vec{F}_2 = -\frac{p_1 p_2}{4\pi\varepsilon_0}\left(\frac{d}{dy}\left(\frac{1}{y^3}\right)\right)\hat{z} = \frac{3p_1 p_2}{4\pi\varepsilon_0 y^4}\hat{z}$$

or

$$\vec{F}_2 = \frac{3p_1 p_2}{4\pi\varepsilon_0 r^4}\hat{z}$$



To calculate $\vec{F}_1$, put $\vec{p}_1$ and $\vec{p}_2$ like this →



$$\vec{F}_1 = (\vec{p}_1 \cdot \nabla)\vec{E}_2 = -p_1 \frac{\partial \vec{E}_2}{\partial y}\Big|_{x=y=0,\ z=-r}$$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3(\vec{p}_2 \cdot \vec{r})\vec{r}}{r^2} - \vec{p}_2\right)$$ where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{p}_2 = p_2\hat{z}$

$$\vec{E}_2 = \frac{p_2}{4\pi\varepsilon_0}\left(\frac{3z(x\hat{x} + y\hat{y} + z\hat{z}) - (x^2+y^2+z^2)\hat{z}}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right)$$

$$\frac{\partial \vec{E}_2}{\partial y} = \frac{p_2}{4\pi\varepsilon_0}\left(-\frac{5}{2}\frac{2y}{r^7}\left(3xz\hat{x} + 3yz\hat{y} - (x^2+y^2-2z^2)\hat{z}\right) + \frac{1}{r^5}(3z\hat{y} - 2y\hat{z})\right)$$

$$\frac{\partial E_2}{\partial y}\Big|_{(0,0)} = \frac{p_2}{4\pi\varepsilon_0}\frac{3z}{r^5}\hat{y}, \quad \vec{F}_1 = -p_1\left(\frac{p_2}{4\pi\varepsilon_0}\frac{-3r}{r^5}\hat{y}\right) = \frac{3p_1 p_2}{4\pi\varepsilon_0 r^4}\hat{y}$$

$\hat{y}$ is coorespond to $-\hat{z}$ in the original system, so

these results are consistent with $\vec{F}_1 = -\vec{F}_2$ (Netwon's third law)

(b) $\vec{N}_2 = (\vec{p}_2 \times \vec{E}_1) + (\vec{r} \times \vec{F}_2)$

$\vec{p}_2 \times \vec{E}_1 = \frac{p_1 p_2}{4\pi\varepsilon_0 r^3}(-\hat{x})$ (in Prob. 4.5)

$\vec{r} \times \vec{F}_2 = (r\hat{y}) \times \left(\frac{3p_1 p_2}{4\pi\varepsilon_0 r^4} \hat{z} \right) = \frac{3p_1 p_2}{4\pi\varepsilon_0 r^3} \hat{x}$

$\vec{N}_2 = \frac{2p_1 p_2}{4\pi\varepsilon_0 r^3} \hat{x}$

this is equal and opposite direction to the torque on $\vec{p}_1$ due to $\vec{p}_2$

6. (a) $\vec{F} = \alpha (\vec{E} \cdot \nabla) \vec{E}$

$\nabla \vec{E}^2 = \nabla(\vec{E} \cdot \vec{E}) = 2\vec{E} \times (\nabla \times \vec{E}) + 2(\vec{E} \cdot \nabla)\vec{E}$

$\nabla \times \vec{E} = 0 \Rightarrow (\vec{E} \cdot \nabla)\vec{E} = \frac{1}{2}\nabla(E^2)$

$\vec{F} = \frac{1}{2}\alpha \nabla(E^2)$

(b) Suppose E^2 has a local maximum at point P.

Then, there's a ~~sphere~~ sphere about P such that $E^2(p') < E^2(P)$

and $|\vec{E}(p')| < |\vec{E}(P)|$ for all points on the surface.

But, if there's no charge inside sphere, the average field is equal to field at center,

$\frac{1}{4\pi R^2} \int \vec{E}\, da = \vec{E}(P)$

$\frac{1}{4\pi R^2} \int E_z\, da = E(P)$ (choosing the z axis to lie along $\vec{E}(P)$)

E^2 is max at P,

$\int E_z\, da \leq \int |\vec{E}|\, da < \int |\vec{E}(P)|\, da = 4\pi R^2 E(P) \Rightarrow E(P) < E(P)$ contradiction!

7. $\vec{E}^{\parallel}$ is continous, $D_{\perp}$ is continous,

$$E_{x_1} = E_{x_2}, \quad D_{y_1} = D_{y_2}$$

$$\varepsilon_1 E_{y_1} = \varepsilon_2 E_{y_2}$$

So, $$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \frac{E_{y_1}}{E_{y_2}} = \frac{\varepsilon_2}{\varepsilon_1}$$

8. the net dipole moment at the center is,

$$\vec{p}\,' = \vec{p} - \frac{\chi_e}{1+\chi_e}\vec{p} = \frac{1}{1+\chi_e}\vec{p} = \frac{1}{\varepsilon_r}\vec{p}$$

We want the potential produced by $\vec{p}\,'$ and σ_b

Using separation of variables,

$$\begin{cases} \text{Outside} & V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\ \text{Inside} & V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{\varepsilon_r r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \end{cases}$$

V continous at $R \Rightarrow \begin{cases} \frac{B_l}{R^{l+1}} = A_l R^l \\ \frac{B_1}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{p}{\varepsilon_r R^2} + A_1 R \end{cases}$

$$\left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} = -\sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) + \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{\varepsilon_r R^3}$$

$$-\sum l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma_b$$

$$= -\frac{1}{\varepsilon_0}\vec{P}\cdot\hat{r} = -\frac{1}{\varepsilon_0}(\varepsilon_0\chi_e\vec{E}\cdot\hat{r}) = \chi_e\left.\frac{\partial V}{\partial r}\right|_{R-}$$

$$= \chi_e\left(-\frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{\varepsilon_r R^3} + \sum l A_l R^{l-1} P_l(\cos\theta)\right)$$

$$-(l+1)\frac{B_l}{R^{l+2}} - l A_l R^{l-1} = \chi_e l A_l R^{l-1} (l \neq 1)$$

or

$$-(2l+1) A_l R^{l-1} = \chi_e l A_l R^{l-1} \Rightarrow A_l = 0 \quad (l \neq 1)$$

For $l=1$,

$$-2\frac{B_1}{R^3} + \frac{1}{4\pi\varepsilon_0}\frac{2p}{\varepsilon_r R^3} - A_1 = \chi_e\left(-\frac{1}{4\pi\varepsilon_0}\frac{2p}{\varepsilon_r R^3} + A_1\right) \quad -B_1 + \frac{p}{4\pi\varepsilon_0\varepsilon_r} - \frac{A_1 R^3}{2}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r} + \chi_e\frac{A_1 R^3}{2}$$

$$-\frac{p}{4\pi\varepsilon_0\varepsilon_r} - A_1 R^3 + \frac{p}{4\pi\varepsilon_0\varepsilon_r} - \frac{A_1 R^3}{2} = -\frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r} + \chi_e\frac{A_1 R^3}{2}$$

$$\Rightarrow \frac{A_1 R^3}{2}(3+\chi_e) = \frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r}$$

$$\Rightarrow A_1 = \frac{1}{4\pi\varepsilon_0}\frac{2\chi_e p}{R^3\varepsilon_r(3+\chi_e)} = \frac{1}{4\pi\varepsilon_0}\frac{2(\varepsilon_r-1)p}{R^3\varepsilon_r(\varepsilon_r+2)}$$

$$B_1 = \frac{p}{4\pi\varepsilon_0\varepsilon_r}\left(1+\frac{2(\varepsilon_r-1)}{\varepsilon_r+2}\right) = \frac{p}{4\pi\varepsilon_0\varepsilon_r}\frac{3\varepsilon_r}{\varepsilon_r+2}$$

$$V(r,\theta) = \left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right)\left(\frac{3}{\varepsilon_r+2}\right) \quad (r \geq R)$$

for $r \leq R$, $$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{\varepsilon_r r^2} + \frac{1}{4\pi\varepsilon_0}\frac{p r\cos\theta}{R^3}\frac{2(\varepsilon_r-1)}{\varepsilon_r(\varepsilon_r+2)}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2\varepsilon_r}\left(1+2\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)\frac{r^3}{R^3}\right) \quad (r \leq R)$$