

UNIVERSIDADE FEDERAL DE MINAS GERAIS



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# Projective Geometry Again

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TEAM:

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# 1 The Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \quad (1)$$

$$B = X_1X_4 \cap X_3X_6 \quad (2)$$

$$C = X_2X_4 \cap X_3X_5 \in AB \quad (3)$$

$$A = (0, 0) \quad (4)$$

$$B = (0, a) \quad (5)$$

$$X_1 = (x_1, bx_1) \quad (6)$$

$$X_5 = (x_5, bx_5) \quad (7)$$

$$X_2 = (x_2, cx_2) \quad (8)$$

$$X_6 = (x_6, cx_6) \quad (9)$$

$$X_4 = (x_4, d(x_4 - a)) \quad (10)$$

$$bx_1 = d(x_1 - a) \Rightarrow d = \frac{bx_1}{x_1 - a} \quad (11)$$

$$X_3 = (x_3, e(x_3 - a)) \quad (12)$$

$$cx_6 = e(x_6 - a) \Rightarrow e = \frac{cx_6}{x_6 - a} \quad (13)$$

$$\alpha x_1^2 + \beta(bx_1)^2 = 1 \Rightarrow x_1^2 = \frac{1}{\alpha + \beta b^2} = \lambda \quad (14)$$

$$\alpha x_2^2 + \beta(cx_2)^2 = 1 \Rightarrow x_2^2 = \frac{1}{\alpha + \beta c^2} = \xi \quad (15)$$

$$\alpha x_5^2 + \beta(bx_5)^2 = 1 \Rightarrow x_5^2 = \frac{1}{\alpha + \beta b^2} = \lambda \therefore x_5 = \epsilon_1 x_1 \quad (16)$$

$$\alpha x_6^2 + \beta(cx_6)^2 = 1 \Rightarrow x_6^2 = \frac{1}{\alpha + \beta c^2} = \xi \therefore x_6 = \epsilon_6 x_2 \quad (17)$$

$$\alpha x_3^2 + \beta[e(x_3 - a)]^2 = 1 \Rightarrow e^2 = \frac{1 - \alpha x_3^2}{\beta(x_3 - a)^2} \quad (18)$$

$$\alpha x_4^2 + \beta[d(x_4 - a)]^2 = 1 \Rightarrow d^2 = \frac{1 - \alpha x_4^2}{\beta(x_4 - a)^2} \quad (19)$$

Therefore:

$$\frac{-y_3x_5 + y_5x_3}{x_3 - x_5} = \frac{-y_2x_4 + y_4x_2}{x_2 - x_4} \quad (20)$$

$$\frac{-\frac{cx_6}{x_6 - a}(x_3 - a)x_5 + bx_5x_3}{x_3 - x_5} = \frac{-cx_2x_4 + \frac{bx_1}{x_1 - a}(x_4 - a)x_2}{x_2 - x_4} \quad (21)$$

Below, we try to reduce that to  $0x = 0$ :

The first intention is to eliminate  $x_1$  and  $x_6$ .

$$\zeta = bx_3x_5(x_2 - x_4) + cx_2x_4(x_3 - x_5) = \zeta_{51}x_5 + \zeta_{50} \quad (22)$$

$$\zeta_{51} = bx_2x_3 - bx_3x_4 - cx_2x_4 \quad (23)$$

$$\zeta_{50} = cx_2x_3x_4 \quad (24)$$

$$\eta = (x_4 - a)(x_3 - x_5) = \eta_{51}x_5 + \eta_{50} \quad (25)$$

$$\eta_{51} = a - x_4 \quad (26)$$

$$\eta_{50} = -x_3(a - x_4) \quad (27)$$

$$\kappa = (x_3 - a)^2(x_2 - x_4)^2 = \kappa_{21}x_2 + \kappa_{20} \quad (28)$$

$$\kappa_{21} = -(x_3 - a)^2x_4 \quad (29)$$

$$\kappa_{20} = (x_3 - a)^2x_4^2 + \xi(x_3 - a)^2 \quad (30)$$

$$\eta^2 = (\eta_{51}x_5 + \eta_{50})^2 = \hat{A}x_5 + \hat{B} \quad (31)$$

$$\hat{A} = 2\eta_{51}\eta_{50} \quad (32)$$

$$\hat{B} = \lambda\eta_{51}^2 + \eta_{50}^2 \quad (33)$$

$$\zeta^2 = (\zeta_{51}x_5 + \zeta_{50})^2 = \hat{C}x_5 + \hat{D} \quad (34)$$

$$\hat{C} = 2\zeta_{51}\zeta_{50} \quad (35)$$

$$\hat{D} = \lambda\zeta_{51}^2 + \zeta_{50}^2 \quad (36)$$

$$\mu = 4\beta(x_3 - a)^4(x_4 - a)^4(1 - \alpha x_4^2) \quad (37)$$

$$\nu = (1 - \alpha x_4^2)(x_3 - a)^2(x_4 - a) \quad (38)$$

$$\pi = \beta(x_3 - a)^2(x_4 - a)^3 \quad (39)$$

$$\rho = (1 - \alpha x_3^2)(x_4 - a)^3 \quad (40)$$

$$\kappa x_5^2 \underbrace{e^2}_{\text{}} = (\zeta - \eta x_2 \underbrace{d}_{\text{}})^2 \quad (41)$$

$$\kappa x_5^2 \frac{1 - \alpha x_3^2}{\beta(x_3 - a)^2} = \eta^2 x_2^2 \frac{1 - \alpha x_4^2}{\beta(x_4 - a)^2} - 2\zeta \eta x_2 \underbrace{d}_{\text{}} + \zeta^2 \quad (42)$$

$$\mu \zeta^2 \eta^2 x_2^2 = (\eta^2 x_2^2 \nu + \pi \zeta^2 - \kappa x_5^2 \rho)^2 \quad (43)$$

They are already eliminated:  $(d, e)$ . Let us eliminate  $x_5$  and, only partially,  $x_2$ .

$$\xi \mu (\hat{A} x_5 + \hat{B})(\hat{C} x_5 + \hat{D}) = \left[ \xi \nu (\hat{A} x_5 + \hat{B}) + \pi (\hat{C} x_5 + \hat{D}) - \lambda \rho \kappa \right]^2 \quad (44)$$

$$E x_5 + F = (G x_5 + H)^2 = 2GH x_5 + \lambda G^2 + H^2 \quad (45)$$

$$\lambda(E - 2GH)^2 = (\lambda G^2 + H^2 - F)^2 \quad (46)$$

$$E = \xi \mu (\hat{A} \hat{D} + \hat{B} \hat{C}) \quad (47)$$

$$F = \lambda \xi \mu [\hat{A} \hat{C} + \hat{B} \hat{D}(\alpha + \beta b^2)] \quad (48)$$

$$G = \xi \nu \hat{A} + \pi \hat{C} \quad (49)$$

$$H = \xi \nu \hat{B} + \pi \hat{D} - \lambda \rho \kappa \quad (50)$$

Let us eliminate  $x_2$  in  $(\zeta, \kappa, \hat{C}, \hat{D})$ .

$$\begin{aligned} & \lambda \left[ \xi \mu (\hat{A} \hat{D} + \hat{B} \hat{C}) - 2 \left( \xi \nu \hat{A} + \pi \hat{C} \right) \left( \xi \nu \hat{B} + \pi \hat{D} - \lambda \rho \kappa \right) \right]^2 = \\ & = \left\{ \lambda \left( \xi \nu \hat{A} + \pi \hat{C} \right)^2 + \left( \xi \nu \hat{B} + \pi \hat{D} - \lambda \rho \kappa \right)^2 - \lambda \xi \mu [\hat{A} \hat{C} + \hat{B} \hat{D}(\alpha + \beta b^2)] \right\}^2 \end{aligned} \quad (51)$$

$$\kappa(x_2) = \kappa_{21} x_2 + \kappa_{20} \quad (52)$$

$$\hat{C}(x_2) = 2(bx_2 x_3 - bx_3 x_4 - cx_2 x_4)cx_2 x_3 x_4 = Ix_2 + J \quad (53)$$

$$\hat{D}(x_2) = \lambda(bx_2 x_3 - bx_3 x_4 - cx_2 x_4)^2 + c^2 x_2^2 x_3^2 x_4^2 = Kx_2 + L \quad (54)$$

$$I = -2bcx_3^2 x_4^2 \quad (55)$$

$$J = \xi(2bcx_3^2 x_4 - 2c^2 x_3 x_4^2) \quad (56)$$

$$K = \lambda(2bcx_3 x_4^2 - 2b^2 x_3^2 x_4) \quad (57)$$

$$L = \lambda b^2 x_3^2 x_4^2 + \xi c^2 x_3^2 x_4^2 + \lambda \xi (bx_3 - cx_4)^2 \quad (58)$$

$$M = \xi \mu (\hat{A}K + \hat{B}I) \quad (59)$$

$$N = \xi \mu (\hat{A}L + \hat{B}J) \quad (60)$$

$$P = \pi I \quad (61)$$

$$Q = \xi \nu \hat{A} + \pi J \quad (62)$$

$$R = \pi K - \lambda \rho \kappa_{21} \quad (63)$$

$$S = \xi \nu \hat{B} + \pi L - \lambda \rho \kappa_{20} \quad (64)$$

$$T = -\lambda \xi \mu [\hat{A}I + \hat{B}K(\alpha + \beta b^2)] \quad (65)$$

$$U = -\lambda \xi \mu [\hat{A}J + \hat{B}L(\alpha + \beta b^2)] \quad (66)$$

$$\lambda [Mx_2 + N - 2(Px_2 + Q)(Rx_2 + S)]^2 = [\lambda (Px_2 + Q)^2 + (Rx_2 + S)^2 + Tx_2 + U]^2 \quad (67)$$

$$V = M - 2PS - 2QR \quad (68)$$

$$W = N - 2\xi PR - 2QS \quad (69)$$

$$Y = 2\lambda VW \quad (70)$$

$$Z = \lambda \xi V^2 + \lambda W^2 \quad (71)$$

$$\hat{E} = 2\lambda PQ + 2RS + T \quad (72)$$

$$\hat{F} = \lambda \xi P^2 + \xi R^2 + \lambda Q^2 + S^2 + U \quad (73)$$

$$Yx_2 + Z = (\hat{E}x_2 + \hat{F})^2 = \xi \hat{E}^2 + \hat{F}^2 + 2\hat{E}\hat{F}x_2 \quad (74)$$

$$\xi(Y - 2\hat{E}\hat{F})^2 = \left( \xi \hat{E}^2 + \hat{F}^2 - Z \right)^2 \quad (75)$$

We have an Equation in  $\alpha, \beta, x_3, x_4, a, b, c$ .

The question is: is it an identity? Let us express it as a function of  $(x_3, x_4) \equiv (x, y, z = x - a, w = y - a)$ .

$$V = \xi \mu (\hat{A}K + \hat{B}I) - 2\pi I (\xi \nu \hat{B} + \pi L - \lambda \rho \kappa_{20}) \quad (76)$$

$$- 2\xi \nu \pi \hat{A}K + 2\lambda \xi \nu \rho \kappa_{21} \hat{A} - 2\pi^2 JK + 2\lambda \pi \rho \kappa_{21} J \quad (77)$$

$$W = \xi \mu (\hat{A}L + \hat{B}J) - 2\xi \pi I (\pi K - \lambda \rho \kappa_{21}) \quad (78)$$

$$- 2\xi^2 \nu^2 \hat{A}\hat{B} - 2\xi \nu \pi \hat{A}L + 2\lambda \xi \nu \rho \kappa_{20} \hat{A} - 2\xi \nu \pi \hat{B}J - 2\pi^2 JL + 2\lambda \pi \rho \kappa_{20} J \quad (79)$$

Let us substitute from  $L$  and above, until  $\eta$ .

$$\nu^2 = z^4(w - \alpha y^3 + \bar{k}y^2)^2 \quad (80)$$

$$= z^4(w^2 + \alpha^2 y^6 + \bar{k}^2 y^4 - 2\alpha y^3 w + 2\bar{k}y^2 w - 2\alpha\bar{k}y^5) \quad (81)$$

At this moment, we realize that we are next to a polynomial of a single variable, say  $x$ . Let us eliminate  $z$ .

By inverse central projection, we “should” eliminate  $\beta = \alpha > 0$ .

$$V[17] = (x^5 - 4ax^4 + 6a^2x^3 - 4a^3x^2 + a^4x)yw^2 \left[ -2\bar{e}xy(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) \right. \quad (82)$$

$$- 2\bar{f}x^2(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) \quad (83)$$

$$+ \lambda\bar{t}xy(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) \quad (84)$$

$$+ \bar{t}x^3y(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) \quad (85)$$

$$- 2\lambda\beta\bar{t}xyw^2(\xi w - \xi\alpha y^3 + \xi\bar{k}y^2) \quad (86)$$

$$- 2\beta\bar{t}x^3yw^3(\xi w - \xi\alpha y^3 + \xi\bar{k}y^2) \quad (87)$$

$$- 2\beta\bar{t}xy\beta w^4(\bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy) \quad (88)$$

$$+ 2\lambda\beta\bar{t}xyw^4(y^2 + \xi) \quad (89)$$

$$- 2\beta\bar{t}\lambda\alpha x^3yw^4(y^2 + \xi) \quad (90)$$

$$+ 4\beta\xi\bar{e}xyw^3(w - \alpha y^3 + \bar{k}y^2) \quad (91)$$

$$+ 4\beta\xi\bar{f}x^2w^3(w - \alpha y^3 + \bar{k}y^2) \quad (92)$$

$$+ 4\lambda\xi w^3(w - \alpha y^3 + \bar{k}y^2) \quad (93)$$

$$- 4\lambda\xi\alpha x^2w^3(w - \alpha y^3 + \bar{k}y^2) \quad (94)$$

$$- 2\beta^2\bar{g}yw^4(\bar{e}xy^2 + \bar{f}x^2y) \quad (95)$$

$$- 2\beta^2\bar{h}xw^4(\bar{e}xy^2 + \bar{f}x^2y) \quad (96)$$

$$- 2\lambda\beta w^4(\bar{g}y^2 + \bar{h}xy) \quad (97)$$

$$+ 2\lambda\beta\alpha xw^4(\bar{g}xy^2 + \bar{h}x^2y) \quad (98)$$

$$W[26] = (x^5 - 4ax^4 + 6a^2x^3 - 4a^4x + a^4x) \left[ -2\xi w^2 \bar{a} x^2 y^2 (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \right. \quad (99)$$

$$- 2\xi x w^2 \bar{b} x (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (100)$$

$$- 2\xi w^2 \bar{c} y^2 (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (101)$$

$$- 2\xi w^2 \bar{d} x y (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (102)$$

$$+ \xi \lambda w^2 \bar{g} y^2 (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (103)$$

$$+ \xi \lambda w^2 \bar{h} x y (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (104)$$

$$+ \xi x w^2 \bar{g} x y^2 (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (105)$$

$$+ \xi x w^2 \bar{h} x^2 y (\bar{\ell} w^4 + \bar{m} w^6 + \bar{n} w^5) \quad (106)$$

$$- 2\xi \beta \bar{\ell} x y^2 \beta w^6 (\bar{e} x y^2 + \bar{f} x^2 y) \quad (107)$$

$$- 2\xi \beta w^3 \bar{\ell} x \lambda y^3 (w^3 - \alpha x^2 w^3) \quad (108)$$

$$+ 4\lambda \xi^2 w^4 (w^2 + \alpha^2 y^6 + \bar{k}^2 y^4 - 2\alpha y^3 w + 2\bar{k} y^2 w - 2\alpha y^5 \bar{k}) \quad (109)$$

$$+ 4\xi^2 x^2 w^4 (w^2 + \alpha^2 y^6 + \bar{k}^2 y^4 - 2\alpha y^3 w + 2\bar{k} y^2 w - 2\alpha y^5 \bar{k}) \quad (110)$$

$$+ 4\xi \beta w^6 (\bar{a} x^2 y^2 + \bar{b} x^2 + \bar{c} y^2 + \bar{d} x y) \quad (111)$$

$$- 4\xi \beta w^5 \alpha y^3 (\bar{a} x^2 y^2 + \bar{b} x^2 + \bar{c} y^2 + \bar{d} x y) \quad (112)$$

$$+ 4\xi \beta w^5 \bar{k} y^2 (\bar{a} x^2 y^2 + \bar{b} x^2 + \bar{c} y^2 + \bar{d} x y) \quad (113)$$

$$+ w (-4\lambda \xi w^5 y^2 - 4\lambda \xi w^5 \xi + 4\lambda \xi \alpha x^2 w^5 y^2 + 4\lambda \xi \alpha x^2 w^5 \xi) \quad (114)$$

$$- \alpha y^3 (-4\lambda \xi w^5 y^2 - 4\lambda \xi w^5 \xi + 4\lambda \xi \alpha x^2 w^5 y^2 + 4\lambda \xi \alpha x^2 w^5 \xi) \quad (115)$$

$$+ \bar{k} y^2 (-4\lambda \xi w^5 y^2 - 4\lambda \xi w^5 \xi + 4\lambda \xi \alpha x^2 w^5 y^2 + 4\lambda \xi \alpha x^2 w^5 \xi) \quad (116)$$

$$- 2\xi \beta w^3 \bar{g} y^2 \lambda w^2 (w - \alpha y^3 + \bar{k} y^2) \quad (117)$$

$$- 2\xi \beta w^3 \bar{g} y^2 x^2 w^2 (w - \alpha y^3 + \bar{k} y^2) \quad (118)$$

$$- 2\xi \beta w^3 \bar{h} x y \lambda w^2 (w - \alpha y^3 + \bar{k} y^2) \quad (119)$$

$$- 2\xi \beta w^3 \bar{h} x^3 y w^2 (w - \alpha y^3 + \bar{k} y^2) \quad (120)$$

$$- 2\beta^2 z^4 w^6 \bar{g} y^2 (\bar{a} x^2 y^2 + \bar{b} x^2 + \bar{c} y^2 + \bar{d} x y) \quad (121)$$

$$- 2\beta^2 z^4 w^6 \bar{h} x y (\bar{a} x^2 y^2 + \bar{b} x^2 + \bar{c} y^2 + \bar{d} x y) \quad (122)$$

$$+ 2\lambda \beta w^6 (y^2 \bar{g} y^2 + y^2 \bar{h} x y + \xi \bar{g} y^2 + \xi \bar{h} x y) \quad (123)$$

$$- 2\lambda \beta \alpha x w^6 (y^2 \bar{g} x y^2 + y^2 \bar{h} x^2 y + \xi \bar{g} x y^2 + \xi \bar{h} x^2 y) \quad (124)$$

If we did not want to give it all up, we “would” have:

$$VW = 17 \cdot 26 = 442 \tag{125}$$

$$Y > 442^2 = 195364 \tag{126}$$



$$V, W \Rightarrow Z = \quad (127)$$

$$S(\nu) = \quad (128)$$

$$Q(\nu) = \quad (129)$$

$$\hat{E} = 2\lambda\pi I \overbrace{(\xi \nu \hat{A} + \pi J)}^Q + 2(\pi K - \lambda\rho\kappa_{21}) \overbrace{(\xi \nu \hat{B} + \pi L - \lambda\rho\kappa_{20})}^S \quad (130)$$

$$- \xi\mu(\lambda\hat{A}I + \hat{B}K) \quad (131)$$

$$\hat{F} = \lambda\xi\pi^2 I^2 + \xi(\pi K - \lambda\rho\kappa_{21})^2 + \lambda(\xi \nu \hat{A} + \pi J)^2 + (\xi \nu \hat{B} + \pi L - \lambda\rho\kappa_{20})^2 \quad (132)$$

$$- \xi\mu(\lambda\hat{A}J + \hat{B}L) \quad (133)$$

$$L = \bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy; \bar{a} = \lambda b^2 + \xi c^2; \bar{b} = \lambda\xi b^2; \bar{c} = \lambda\xi c^2; \bar{d} = -2\lambda\xi bc \quad (134)$$

$$K = \bar{e}xy^2 + \bar{f}x^2y; \bar{e} = 2\lambda bc; \bar{f} = -2\lambda b^2 \quad (135)$$

$$J = \bar{g}xy^2 + \bar{h}x^2y; \bar{g} = -2\xi c^2; \bar{h} = 2\xi bc \quad (136)$$

$$I = \bar{t}x^2y^2; \bar{t} = -2bc \quad (137)$$

$$\kappa_{21} = -yz^2 \quad (138)$$

$$\kappa_{20} = y^2z^2 + \xi z^2 \quad (139)$$

$$\rho = w^3 - \alpha x^2w^3 \quad (140)$$

$$\pi = \beta z^2w^3 \quad (141)$$

$$\nu = z^2w - \alpha y^3z^2 + \bar{k}y^2z^2; \bar{k} = \alpha \cdot a \quad (142)$$

$$\mu = \bar{\ell}z^4w^4 + \bar{m}z^4w^6 + \bar{n}z^4w^5; \bar{\ell} = 4\beta - 4\alpha\beta a^2; \bar{m} = -4\alpha\beta; \bar{n} = -8\alpha\beta a \quad (143)$$

$$\hat{B} = \lambda w^2 + x^2w^2 \quad (144)$$

$$\hat{A} = -2xw^2 \quad (145)$$

$$\bar{o} = \quad (146)$$

Below, let us explain the Equation (20), expliciting  $C$  on Equation (3).

$$\begin{pmatrix} x_2 & 1 \\ x_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{x_2 - x_4} \begin{pmatrix} 1 & -1 \\ -x_4 & x_2 \end{pmatrix} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} \quad (147)$$

$$\begin{pmatrix} x_3 & 1 \\ x_5 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} y_3 \\ y_5 \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{x_3 - x_5} \begin{pmatrix} 1 & -1 \\ -x_5 & x_3 \end{pmatrix} \begin{pmatrix} y_3 \\ y_5 \end{pmatrix} \quad (148)$$

$$ax + b = cx + d \Rightarrow x = \frac{d - b}{a - c} = 0 \therefore d = b \quad (149)$$

## 2 The Converse Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \quad (150)$$

$$B = X_1X_4 \cap X_3X_6 \quad (151)$$

$$C = X_2X_4 \cap X_3X_5 \in AB \quad (152)$$

$$A = (0, 0) \quad (153)$$

$$B = (0, a) \quad (154)$$

$$X_1 = (x_1, bx_1) \quad (155)$$

$$X_5 = (x_5, bx_5) \quad (156)$$

$$X_2 = (x_2, cx_2) \quad (157)$$

$$X_6 = (x_6, cx_6) \quad (158)$$

$$X_4 = (x_4, d(x_4 - a)) \quad (159)$$

$$bx_1 = d(x_1 - a) \quad (160)$$

$$X_3 == e(x_3 - a) \quad (161)$$

$$cx_6 = e(x_6 - a) \quad (162)$$

$$\frac{-y_3x_5 + y_5x_3}{x_3 - x_5} = \frac{-y_2x_4 + y_4x_2}{x_2 - x_4} \quad (163)$$

Therefore,  $\exists \lambda \in \mathbb{R}^6$ , such that:

$$\begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & y_6^2 & x_6y_6 & x_6 & y_6 & 1 \end{pmatrix} \cdot \lambda = 0 \quad (164)$$

### 3 From Circle to Parabola

$$S' : x'^2 + z'^2 = R^2, y = 0 \quad (165)$$

$$v = (\mathbb{R}, 0, R) \quad (166)$$

$$P = (0, -p, R) \quad (167)$$

$$A' = (x', 0, z') \in x\hat{O}z \quad (168)$$

$$A'P : (x, y, z) = (x' + \lambda x', \lambda p, z' + \lambda(z' - R)) \quad (169)$$

$$B = \pi(A') ; \pi : x\hat{O}z \rightarrow x\hat{O}y \quad (170)$$

$$= A'P \cap x\hat{O}y = (x_b, y_b, z_b = 0) \quad (171)$$

$$x_b = \frac{Rx'}{R - z'} ; y_b = \frac{pz'}{R - z'} \quad (172)$$

$$(R - z')y_b = pz' \Rightarrow z' = \frac{Ry_b}{p + y_b} \quad (173)$$

$$x' = \frac{R - z'}{R} \cdot x_b = \frac{px_b}{p + y_b} \quad (174)$$

$$\pi(S') : p^2x_b^2 + R^2y_b^2 = R^2(p + y_b)^2 = p^2R^2 + 2pR^2y_b + R^2y_b^2 ; z_b = 0 \quad (175)$$

$$\therefore y_b = \frac{px_b^2 - pR^2}{2R^2} \quad (176)$$

That is a parabola that crosses  $\hat{O}y$  at  $V = \left(0, -\frac{p}{2}, 0\right)$ , and crosses  $\hat{O}x$  at  $(\pm R, 0, 0)$ .

#### 3.1 From Circle to Hiperbola

$$S'' : x'^2 + z'^2 = (R + q)^2, y = 0 \quad (177)$$

$$\pi(S'') : p^2x_b^2 + R^2y_b^2 = (R + q)^2(p^2 + 2py_b + y_b^2) ; z_b = 0 \quad (178)$$

$$p^2x_b^2 = (2Rq + q^2)y_b^2 + 2p(R + q)^2y_b + p^2(R + q)^2 \quad (179)$$

$$p^2(R + q)^4 - p^2q(R + q)^2(2R + q) = [(2R + q)qy_b + p(R + q)^2]^2 - (2R + q)p^2qx_b^2 \quad (180)$$

$$\frac{1}{A^2} \left[ y_b + \frac{p(R + q)^2}{q(2R + q)} \right]^2 - \frac{x_b^2}{B^2} = 1 ; A = \frac{pR(R + q)}{q(2R + q)} ; B = \frac{R(R + q)}{\sqrt{q(2R + q)}} \quad (181)$$

### 3.2 From Circle to Ellipsis

$$S''' : x'^2 + z'^2 = (R - q)^2, y = 0 \quad (182)$$

$$\pi(S''') : \frac{1}{A^2} \left[ y_b - \frac{p(R - q)^2}{q(2R - q)} \right]^2 + \frac{x_b^2}{B^2} = 1; A = \frac{pR(R - q)}{q(2R - q)}; B = \frac{R(R - q)}{\sqrt{q(2R - q)}} \quad (183)$$

■

Out of charity, there is no salvation at all. With charity, we evolve.

June, the 21th, 2024.