Algebraic Independence

Let $A^2 = \{(x,y) \in (\mathbb{R}-\mathbb{Q})^2; \exists p \in \mathbb{Z}[x,y]-\{0\}; p(x,y)=0\}$.
$\#A^2 = \#\mathbb{Q}^2$
We want a proof that A is dense in $\mathbb{R}^2$, i.e., $\forall p \in \mathbb{R}^2 - A^2, \forall \epsilon > 0, \exists q \in A^2; |q-p| < \epsilon$.
For $n=1$, we have the proof below.
There is a sequence of algebraic irrationals which converges to:
(i) algebraic numbers: $a_n = \sqrt{2}$ (constant sequence $c_n = c_0$);

(ii) rationals: $p(x) = nx^2 - 1 \Rightarrow a_n = \dfrac{1}{\sqrt{n}} \to 0$ and $q(x) = (nx^2 - 1 - a^2 n)^2 - 4a^2 n \Rightarrow b_n = \dfrac{1}{\sqrt{n}} + a \to a \in \mathbb{Q}$;

(iii) transcendentals: $a_n = \sqrt{2}\sum_{k=0}^{n} \dfrac{1}{k!} \to \sqrt{2}\exp 1 = e\sqrt{2}$. Each partial sum is an algebraic number y;

$y^2 = 2 \cdot \dfrac{r}{q} \in \mathbb{Q} \Leftarrow p(x) = qx^2 - 2r$. Analogously, we get any transcendental t. As the closure of $\mathbb{Q}$ is $\mathbb{R}$, there is a sequence of rationals $q_n; \lim q_n = \dfrac{t}{\sqrt{2}}$. Therefore $b_n = q_n\sqrt{2}$ converges to t.

$A^1 = \{x \in \mathbb{R}-\mathbb{Q}; \exists p \in \mathbb{Z}[x]-\{0\}; p(x)=0\}$ is countable and dense in $\mathbb{R}$.

Generalizing for $A^n, n \geq 2$
Fix $p = (t_1, t_2, \cdots, t_n), t_i \neq t_j \in \mathbb{R}-\mathbb{Q}-A^1 = \mathbb{T}$ (transcendentals) and $\epsilon > 0$.
Let the line α be $\alpha(x) = (t_1, t_2 + x, t_3, \cdots, t_n)$
$\exists y_0 = \dfrac{a}{b} \cdot t_1; a, b \in \mathbb{Z}-\{0\}; t_2 < y_0 < t_2 + \epsilon$

The hyperplane β of equation $p(x_1, x_2, \cdots, x_n) = x_2 - \dfrac{a}{b} \cdot x_1 = 0$ intersects α at $q = (t_1, y_0, t_3, \cdots, t_n) \in A^n$ because q is root of p.

$A^n = \{(x_1, \cdots, x_n) \in (\mathbb{R}-\mathbb{Q})^n; \exists p \in \mathbb{Z}[x_1, \cdots, x_n]-\{0\}; p(x_1, \cdots, x_n) = 0\}$ is countable and dense in $\mathbb{R}^n$, but it's not $\mathbb{Q}^n$.

Let $(e,e) \in A_e^2 = \{(x,e) \in (\mathbb{R}-\mathbb{Q})^2; \exists p \in \mathbb{Z}[x, y=e]-\{0\}; p(x,e)=0\}$.
Let $(\pi,\pi) \in A_\pi^2 = \{x \in \mathbb{R}-\mathbb{Q}; \exists p \in R[x]-\{0\}; p(x)=0\} \times \{\pi\}$.
$\deg p(x,\pi) = g$
$q_g \in \mathbb{Z}[x], \deg q_g = g$
$p(x) = \sum_{i=0}^{g} q_{g-i}(\pi)x^i$
$p(x) \in \mathbb{Z}(\pi)[x] = R[x] \subset \mathbb{R}[x]$
In particular, $g = 2 \Rightarrow p(x,\pi) = a_0 + a_1 x + a_2\pi + a_3 x^2 + a_4 x\pi + a_5\pi^2 = q_2(\pi) + q_1(\pi)x + q_0(\pi)x^2$.
$T_t^2 = [\mathbb{R}-\mathbb{Q}-A-\{q_1(t)\}-\{q_2(t)\}-\{q_3(t)\}-\cdots] \times \{t\}$
$T_{t_1,t_2}^3 = [\mathbb{R}-\mathbb{Q}-A-\{q_1(t_1,t_2)\}-\{q_2(t_1,t_2)\}-\{q_3(t_1,t_2)\}-\cdots] \times \{t_1\} \times \{t_2\}, q_g \in \mathbb{Z}[x,y], \deg q_g = g$
Exchange $t_i \neq t_j$ above by $t_2 \notin \mathbb{Z}(t_1); t_3 \notin \mathbb{Z}(t_1,t_2); \cdots; t_n \notin \mathbb{Z}(t_1, \cdots, t_{n-1})$.
$\dim T_{t_1,\cdots,t_n}^{n+1} = 1$. What about T^ω?
How many are the algebraically independent numbers? n, $\aleph_0$ or $\aleph_1$?

We want a proof that $(e,\pi) \notin A_\pi^2$, which is countable and dense in $\mathbb{R} \times \{\pi\}$. So, $(e,\pi) \in T_\pi^2$.
We want another proof that $(\pi,e) \notin A_e^2$, which is countable and dense in $\mathbb{R} \times \{e\}$. So, $(\pi,e) \in T_e^2$.