Terceira prova de Análise II

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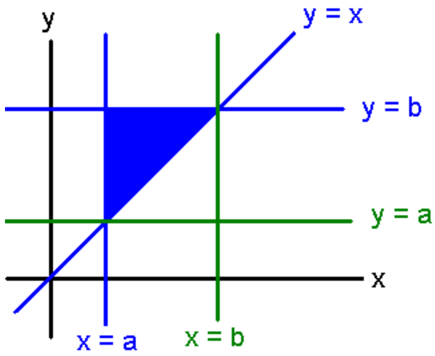
1 Questão 1

Seja $R \subset \mathbb{R}^2$ a região delimitada pelas retas $y = b$, $x = a$ e $y = x$.



$$I_1 = \iint_R f = \iint_{[a,b]^2} f \cdot \chi_R = \iint_{[a,b]^2} g \tag{1}$$

$$\chi_R(x, y) = 1, \text{ se } (x, y) \in R \tag{2}$$

$$\chi_R(x, y) = 0, \text{ se } (x, y) \notin R \tag{3}$$

f é integrável $\Rightarrow$ o conjunto D_1 dos pontos de descontinuidade de f tem medida nula.

O conjunto D_2 dos pontos de descontinuidade de χ_R é formado pelos pontos de ∂R, a fronteira de R. D_2 é composto de três segmentos, ou seja, as arestas do triângulo ΔABC, em que $A = (a, a), B = (a, b), C = (b, b)$.

$\dim D_2 = 1 < 2 = \dim[a, b]^2 \Rightarrow D_2$ tem medida nula em $[a, b]^2$.

Logo, o produto $f \cdot \chi_R = g$ é integrável (uma vez que os pontos de continuidade de cada fator tem medida nula). Se f é limitada, então g é limitada.

Pelo teorema de Fubini, existem as integrais iteradas e elas se igualam a I_1.

$$I_1 = \int_{[a,b]} \int_{[a,b]} g = \int_{[a,b]} \left(\int_{[a,b]} f(x, y) \cdot \chi_R(x, y)\, \mathrm{d}x \right) \mathrm{d}y = \int_{y=a}^{y=b} \left(\int_{x=a}^{x=y} f(x, y)\, \mathrm{d}x \right) \mathrm{d}y = I_2 \tag{4}$$

$$I_1 = \int_{[a,b]} \int_{[a,b]} g = \int_{[a,b]} \left(\int_{[a,b]} f(x, y) \cdot \chi_R(x, y)\, \mathrm{d}y \right) \mathrm{d}x = \int_{x=a}^{x=b} \left(\int_{y=x}^{y=b} f(x, y)\, \mathrm{d}y \right) \mathrm{d}x = I_3 \tag{5}$$

Portanto, $I_2 = I_3$.

2 Questão 2

$$f : A \subset \mathbb{R}^n \to \mathbb{R} \tag{6}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \Rightarrow f(x) \in \mathbb{R} \tag{7}$$

$$\mathrm{D}f : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}) \tag{8}$$

$$p \in \mathbb{R}^n \Rightarrow \mathrm{D}f(p) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}) \tag{9}$$

$$\mathrm{D}f(p) : \mathbb{R}^n \to \mathbb{R} \tag{10}$$

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \Rightarrow \mathrm{D}f(p)(v) \in \mathbb{R} \tag{11}$$

$$\mathrm{D}f(p)(v) = \frac{\partial f}{\partial x_1}(p)v_1 + ... + \frac{\partial f}{\partial x_n}(p)v_n \tag{12}$$

$$x_i : \mathbb{R}^n \to \mathbb{R}; \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_i \tag{13}$$

$$\mathrm{D}x_i : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}) \tag{14}$$

$$\mathrm{D}x_i(p) = \varphi_i : \mathbb{R}^n \to \mathbb{R} \tag{15}$$

$$\mathrm{D}x_i(p)(v) = \varphi_i(v) = v_i \tag{16}$$

$$\varphi_i = [\delta_{ij}]_{1\times n} \in \mathcal{M}_{1\times n} \Leftrightarrow \varphi_i(v) = [0, ..., 1, ..., 0]_{1\times n} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{n\times 1} = v_i \tag{17}$$

$$\mathrm{d}f : A \subset \mathbb{R}^n \to \Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^* = \mathcal{L}(\mathbb{R}^n; \mathbb{R}) \tag{18}$$

$$\mathrm{d}f(p)(v) = \mathrm{D}f(p)(v) \tag{19}$$

$$\mathrm{d}x_i : \mathbb{R}^n \to \Lambda^1(\mathbb{R}^n) \tag{20}$$

$$\mathrm{d}x_i(p)(v) = \mathrm{D}x_i(p)(v) = v_i \tag{21}$$

Pela definição de $\mathrm{D}f(p)(v)$:

$$\mathrm{d}f(p)(v) = \frac{\partial f}{\partial x_1}(p)v_1 + ... + \frac{\partial f}{\partial x_n}(p)v_n \tag{22}$$

Pela definição de $\mathrm{d}x_i(p)(v)$:

$$\mathrm{d}f(p)(v) = \frac{\partial f}{\partial x_1}(p)\mathrm{d}x_1(p)(v) + ... + \frac{\partial f}{\partial x_n}(p)\mathrm{d}x_n(p)(v) \tag{23}$$

Como a linha acima vale $\forall v \in \mathbb{R}^n$, temos igualdade de aplicações de $\mathbb{R}^n$ em $\mathbb{R}$:

$$\mathrm{d}f(p) = \frac{\partial f}{\partial x_1}(p)\varphi_1 + ... + \frac{\partial f}{\partial x_n}(p)\varphi_n \tag{24}$$

$$\Leftrightarrow \mathrm{d}f(p) = \frac{\partial f}{\partial x_1}(p)\mathrm{d}x_1(p) + ... + \frac{\partial f}{\partial x_n}(p)\mathrm{d}x_n(p) \tag{25}$$

Como a linha acima vale $\forall p \in \mathbb{R}^n$, temos igualdade de aplicações de $\mathbb{R}^n$ em $\Lambda^1(\mathbb{R}^n)$:

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + ... + \frac{\partial f}{\partial x_n}\mathrm{d}x_n \tag{26}$$

3 Questão 3

$$\omega = f\,\mathrm{d}x \wedge\, \mathrm{d}y \tag{27}$$

$$c : [0,1]^2 \to \mathbb{R}^2; \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} c_1(u,v) \\ c_2(u,v) \end{pmatrix} \tag{28}$$

Pela definição de ω:

$$c^*\omega = c^*(f\,\mathrm{d}x \wedge\, \mathrm{d}y) \tag{29}$$

Utilizamos a propriedade 1: $f^*(g\omega) = (g \circ f)f^*\omega$, válida para toda k-forma ω em $\mathbb{R}^n$, para toda $f : \mathbb{R}^a \to \mathbb{R}^b$ e para toda $g : \mathbb{R}^b \to \mathbb{R}^p$.

$$c^*\omega = (f \circ c)c^*(\mathrm{d}x \wedge\, \mathrm{d}y) \tag{30}$$

Utilizamos a propriedade 2: $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$, válida para toda k-forma ω em $\mathbb{R}^n$, para toda ℓ-forma θ em $\mathbb{R}^n$ e para toda $f : \mathbb{R}^a \to \mathbb{R}^b$.

$$c^*\omega = (f \circ c)c^*(\mathrm{d}x) \wedge c^*(\mathrm{d}y) \tag{31}$$

Utilizamos a propriedade 3: $f^*(\mathrm{d}x_i) = \sum_{j=1}^m \frac{\partial f_i}{\partial t_j}\,\mathrm{d}t_j$, válida para toda $f : \mathbb{R}^m \to \mathbb{R}^n$, para todo $(x_1, ..., x_n) \in \mathbb{R}^n$, e para todo $(t_1, ..., t_m) \in \mathbb{R}^m$.

$$c^*\omega = (f \circ c)\left(\frac{\partial c_1}{\partial u}\mathrm{d}u + \frac{\partial c_1}{\partial v}\mathrm{d}v\right) \wedge \left(\frac{\partial c_2}{\partial u}\mathrm{d}u + \frac{\partial c_2}{\partial v}\mathrm{d}v\right) \tag{32}$$

Utilizamos a propriedade distributiva do produto exterior.
$(\omega + \theta) \wedge \varphi = \omega \wedge \varphi + \theta \wedge \varphi$, para toda k-forma ω, θ em $\mathbb{R}^n$ e para toda ℓ-forma φ em $\mathbb{R}^n$.
$\omega + (\theta \wedge \varphi) = \omega \wedge \theta + \omega \wedge \varphi$, para toda k-forma ω em $\mathbb{R}^n$ e para toda ℓ-forma θ, φ em $\mathbb{R}^n$.

$$c^*\omega = (f \circ c)\left[\left(\frac{\partial c_1}{\partial u}\mathrm{d}u\right) \wedge \left(\frac{\partial c_2}{\partial u}\mathrm{d}u\right) + \left(\frac{\partial c_1}{\partial u}\mathrm{d}u\right) \wedge \left(\frac{\partial c_2}{\partial v}\mathrm{d}v\right)\right. \tag{33}$$

$$\left. + \left(\frac{\partial c_1}{\partial v}\mathrm{d}v\right) \wedge \left(\frac{\partial c_2}{\partial u}\mathrm{d}u\right) + \left(\frac{\partial c_1}{\partial v}\mathrm{d}v\right) \wedge \left(\frac{\partial c_2}{\partial v}\mathrm{d}v\right)\right] \tag{34}$$

Utilizamos a multiplicação por escalar: $(a\omega) \wedge \theta = \omega \wedge (a\theta) = a(\omega \wedge \theta)$, válida para toda k-forma ω em $\mathbb{R}^n$, para toda ℓ-forma θ em $\mathbb{R}^n$ e para todo $a \in \mathbb{R}$.

$$c^*\omega = (f \circ c)\left[\frac{\partial c_1}{\partial u}\frac{\partial c_2}{\partial u}\,\mathrm{d}u \wedge \mathrm{d}u + \frac{\partial c_1}{\partial u}\frac{\partial c_2}{\partial v}\,\mathrm{d}u \wedge \mathrm{d}v + \frac{\partial c_1}{\partial v}\frac{\partial c_2}{\partial u}\,\mathrm{d}v \wedge \mathrm{d}u + \frac{\partial c_1}{\partial v}\frac{\partial c_2}{\partial v}\,\mathrm{d}v \wedge \mathrm{d}v\right] \tag{35}$$

Utilizamos a identidade $\omega \wedge \omega = 0$, válida para 1-formas ω em $\mathbb{R}^n$.

$$c^*\omega = (f \circ c)\left[\frac{\partial c_1}{\partial u}\frac{\partial c_2}{\partial v}\,\mathrm{d}u \wedge \mathrm{d}v + \frac{\partial c_1}{\partial v}\frac{\partial c_2}{\partial u}\,\mathrm{d}v \wedge \mathrm{d}u\right] \tag{36}$$

Utilizamos a identidade $\omega \wedge \theta = (-1)^{k\ell}\theta \wedge \omega$, válida para toda k-forma ω em $\mathbb{R}^n$ e para toda ℓ-forma θ em $\mathbb{R}^n$.

$$c^*\omega = (f \circ c)\left[\frac{\partial c_1}{\partial u}\frac{\partial c_2}{\partial v} - \frac{\partial c_1}{\partial v}\frac{\partial c_2}{\partial u}\right]\mathrm{d}u \wedge \mathrm{d}v \tag{37}$$

4 Questão 4

Escrever a demonstração do teorema de Stokes para uma 2-forma e um 3-cubo de $\mathbb{R}^3$ e comparar com o teorema da divergência.

Sejam as funções $F, G, H : \mathbb{R}^3 \to \mathbb{R}$ de classe C^∞.
Seja $p = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.
Seja ω uma 2-forma em $\mathbb{R}^3$:

$$\omega(p) = F(p)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + G(p)\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + H(p)\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{38}$$

$$\omega_1 = F\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \Rightarrow \mathrm{d}\omega_1 = \frac{\partial F}{\partial x_3}\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{39}$$

$$\omega_2 = G\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 \Rightarrow \mathrm{d}\omega_2 = \frac{\partial G}{\partial x_2}\,\mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 \tag{40}$$

$$\omega_3 = H\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 \Rightarrow \mathrm{d}\omega_3 = \frac{\partial H}{\partial x_1}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{41}$$

$$\text{Logo, } \omega = \omega_1 + \omega_2 + \omega_3 \tag{42}$$

$$\mathrm{d}\omega = \left(\frac{\partial F}{\partial x_3} - \frac{\partial G}{\partial x_2} + \frac{\partial H}{\partial x_1}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{43}$$

Sejam $t_1, t_2, t_3 \in \mathbb{R}$. E seja c um 3-cubo de $\mathbb{R}^3$ tal que

$$c : [0,1]^3 \to \mathbb{R}^3; \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \mapsto \begin{bmatrix} c_1(t_1,t_2,t_3) \\ c_2(t_1,t_2,t_3) \\ c_3(t_1,t_2,t_3) \end{bmatrix} \tag{44}$$

4.1 Enunciado do Teorema de Stokes

Seja ω uma $(k-1)$-forma em $\mathbb{R}^n$ e seja c um k-cubo em $\mathbb{R}^n$. Então:

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega \tag{45}$$

Em particular, fixado ω como na linha (38),

$$\int_c \left(\frac{\partial F}{\partial x_3} - \frac{\partial G}{\partial x_2} + \frac{\partial H}{\partial x_1}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 = \int_{\partial c} F\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + G\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + H\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{46}$$

Em geral, o teorema de Stokes vale para uma k-cadeia C.
Sejam os k-cubos $c_i, i \in \{1, 2, ..., \ell\}, \ell \in \mathbb{N}, z_i \in \mathbb{Z}$.

$$C = \sum_{i=1}^{\ell} z_i c_i \tag{47}$$

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{48}$$

4.2 Enunciado do Teorema da Divergência

Seja Φ uma aplicação de classe C^∞ tal que $\Phi : \mathbb{R}^3 \to \mathbb{R}^3; (p) \mapsto \begin{bmatrix} \Phi_1(p) \\ \Phi_2(p) \\ \Phi_3(p) \end{bmatrix}$.
Seja n vetor normal a c, orientado de forma a apontar sempre para fora. Então:

$$\int_c \langle \nabla, \Phi \rangle \, \mathrm{d}V = \int_{\partial c} \langle \Phi, n \rangle \, \mathrm{d}A \tag{49}$$

$$\langle \nabla, \Phi \rangle = \langle (\mathrm{D}x_1, \mathrm{D}x_2, \mathrm{D}x_3), (\Phi_1, \Phi_2, \Phi_3) \rangle \tag{50}$$

$$\langle \Phi, n \rangle \, \mathrm{d}A = \Phi_1(n_1 \, \mathrm{d}A) + \Phi_2(n_2 \, \mathrm{d}A) + \Phi_3(n_3 \, \mathrm{d}A) \tag{51}$$

$$\text{Ou seja, } \int_c \mathrm{d}\theta = \int_{\partial c} \theta \tag{52}$$

$$\theta = \Phi_1 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \Phi_2 \, \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \Phi_3 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{53}$$

Para passar do enunciado do exercício para o teorema da divergência, basta fazermos

$$\Phi_1 = H, \Phi_2 = -G, \Phi_3 = F \tag{54}$$

em (46) e assim obteremos

$$\int_c \left(\frac{\partial \Phi_3}{\partial x_3} + \frac{\partial \Phi_2}{\partial x_2} + \frac{\partial \Phi_1}{\partial x_1} \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 = \int_{\partial c} \Phi_3 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \Phi_2 \, \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \Phi_1 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3 \tag{55}$$

Redigiremos agora a demonstração requisitada.

$$\int_c \mathrm{d}\omega \overset{56.1}{=} \int_{[0,1]^3} c^*(\mathrm{d}\omega) \overset{56.2}{=} \underbrace{\int_{I^3} \mathrm{d}(c^*\omega)}_{T} \overset{56.3}{=} \underbrace{\int_{\partial I^3} c^*\omega}_{S} \overset{56.4}{=} \sum_{i,\alpha} (-1)^{i+\alpha} \int_{I^3_{(i,\alpha)}} c^*\omega \overset{56.5}{=} \tag{56}$$

$$\overset{56.5}{=} \sum_{i,\alpha} (-1)^{i+\alpha} \int_{[0,1]^2} c^*_{(i,\alpha)}\omega \overset{56.6}{=} \sum_{i,\alpha} (-1)^{i+\alpha} \int_{c(i,\alpha)} \omega \overset{56.7}{=} \int_{\partial c} \omega \tag{57}$$

Consideremos inicialmente a parcela ω_1 na decomposição em (42).
56.1 segue da definição de integral:

$$c^*(\mathrm{d}\omega_1) = c^*\left(\frac{\partial F}{\partial x_3} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \right) = \left(\frac{\partial F}{\partial x_3} \circ c \right) c^*(\mathrm{d}x_1) \wedge c^*(\mathrm{d}x_2) \wedge c^*(\mathrm{d}x_3) \tag{58}$$

$$c^*(\mathrm{d}\omega_1) = \left(\frac{\partial F}{\partial x_3} \circ c \right) \left(\frac{\partial c_1}{\partial t_1} \mathrm{d}t_1 + \frac{\partial c_1}{\partial t_2} \mathrm{d}t_2 + \frac{\partial c_1}{\partial t_3} \mathrm{d}t_3 \right) \wedge \left(\frac{\partial c_2}{\partial t_1} \mathrm{d}t_1 + \frac{\partial c_2}{\partial t_2} \mathrm{d}t_2 + \frac{\partial c_2}{\partial t_3} \mathrm{d}t_3 \right) \wedge \tag{59}$$

$$\wedge \left(\frac{\partial c_3}{\partial t_1} \mathrm{d}t_1 + \frac{\partial c_3}{\partial t_2} \mathrm{d}t_2 + \frac{\partial c_3}{\partial t_3} \mathrm{d}t_3 \right) \tag{60}$$

$$c^*(\mathrm{d}\omega_1) = \left(\frac{\partial F}{\partial x_3} \circ c \right) \left(\frac{\partial c_1}{\partial t_1} \frac{\partial c_2}{\partial t_2} \frac{\partial c_3}{\partial t_3} - \frac{\partial c_1}{\partial t_1} \frac{\partial c_3}{\partial t_2} \frac{\partial c_2}{\partial t_3} - \frac{\partial c_2}{\partial t_1} \frac{\partial c_1}{\partial t_2} \frac{\partial c_3}{\partial t_3} + \frac{\partial c_2}{\partial t_1} \frac{\partial c_3}{\partial t_2} \frac{\partial c_1}{\partial t_3} \right. \tag{61}$$

$$\left. + \frac{\partial c_3}{\partial t_1} \frac{\partial c_1}{\partial t_2} \frac{\partial c_2}{\partial t_3} - \frac{\partial c_3}{\partial t_1} \frac{\partial c_2}{\partial t_2} \frac{\partial c_1}{\partial t_3} \right) \mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \mathrm{d}t_3 \tag{62}$$

56.2 segue da seguinte propriedade: pull-back de diferencial é igual a diferencial de pull-back.

$$c^*\omega_1 = c^*(F\,\mathrm{d}x_1 \wedge \mathrm{d}x_2) = (F\circ c)c^*(\mathrm{d}x_1)\wedge c^*(\mathrm{d}x_2) \tag{63}$$

$$c^*\omega_1 = (F\circ c)\left(\frac{\partial c_1}{\partial t_1}\,\mathrm{d}t_1 + \frac{\partial c_1}{\partial t_2}\,\mathrm{d}t_2 + \frac{\partial c_1}{\partial t_3}\,\mathrm{d}t_3\right)\wedge\left(\frac{\partial c_2}{\partial t_1}\,\mathrm{d}t_1 + \frac{\partial c_2}{\partial t_2}\,\mathrm{d}t_2 + \frac{\partial c_2}{\partial t_3}\,\mathrm{d}t_3\right) \tag{64}$$

$$c^*\omega_1 = (F\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_2}{\partial t_2} - \frac{\partial c_2}{\partial t_1}\frac{\partial c_1}{\partial t_2}\right)\mathrm{d}t_1\wedge\mathrm{d}t_2 + (F\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_2}{\partial t_3} - \frac{\partial c_2}{\partial t_1}\frac{\partial c_1}{\partial t_3}\right)\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{65}$$

$$+ (F\circ c)\left(\frac{\partial c_1}{\partial t_2}\frac{\partial c_2}{\partial t_3} - \frac{\partial c_2}{\partial t_2}\frac{\partial c_1}{\partial t_3}\right)\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{66}$$

Seja agora mais uma decomposição:

$$f = (F\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_2}{\partial t_2} - \frac{\partial c_2}{\partial t_1}\frac{\partial c_1}{\partial t_2}\right) \tag{67}$$

$$g = (F\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_2}{\partial t_3} - \frac{\partial c_2}{\partial t_1}\frac{\partial c_1}{\partial t_3}\right) \tag{68}$$

$$h = (F\circ c)\left(\frac{\partial c_1}{\partial t_2}\frac{\partial c_2}{\partial t_3} - \frac{\partial c_2}{\partial t_2}\frac{\partial c_1}{\partial t_3}\right) \tag{69}$$

$$\text{Logo, } c^*\omega_1 = f\,\mathrm{d}t_1\wedge\mathrm{d}t_2 + g\,\mathrm{d}t_1\wedge\mathrm{d}t_3 + h\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{70}$$

$$\mathrm{d}(c^*\omega_1) = \left(\frac{\partial f}{\partial t_3} - \frac{\partial g}{\partial t_2} + \frac{\partial h}{\partial t_1}\right)\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{71}$$

Na linha acima está exibido o integrando em 56.2 (T). A região de integração é o cubo Identidade, definido por $I^3 = I$, por simplicidade. Seja $I : [0,1]^3 \to \mathbb{R}^3; p \mapsto p = I(p) = I^3(p)$.

Vamos agora provar 56.3, ou seja, $S = T$. Por linearidade, temos que

$$S = S_1 + S_2 + S_3 \tag{72}$$

$$S_1 = \int_{\partial I^3} f\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \tag{73}$$

$$S_2 = \int_{\partial I^3} g\,\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{74}$$

$$S_3 = \int_{\partial I^3} h\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{75}$$

$$S_1 = \int_{\partial I^3} f\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \overset{76.1}{=} \sum_{j,\alpha}(-1)^{j+\alpha}\int_{I^3_{(j,\alpha)}} f\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \overset{76.2}{=} \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(I^3_{(j,\alpha)})^*(f\,\mathrm{d}t_1\wedge\mathrm{d}t_2) \tag{76}$$

$$\overset{76.3}{=} \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(f\circ I_{(j,\alpha)})I^*_{(j,\alpha)}(\mathrm{d}t_1)\wedge I^*_{(j,\alpha)}(\mathrm{d}t_2) \tag{77}$$

76.1 segue da definição de bordo de I. 76.2 segue da definição de integral. 76.3 segue da definição de pull-back.

Seja (e_1, e_2, e_3) a base canônica de $\mathbb{R}^3$. Sabemos que $I^*_{(j,\alpha)}(\mathrm{d}t_\ell) = 0$, se $\ell \neq j$, uma vez que $\mathrm{d}t_\ell(p)(e_j) = \delta_{j\ell}$.

Sabemos também que S_1 é integral sem o termo $\mathrm{d}t_3$, que está faltando para completar 3-forma em $\mathbb{R}^3$. E assim só aparecem os cubos bidimensionais $c(3,\alpha)$, que são faces do cubo c definido no início.

$$S_1 = \sum_{\alpha=0}^{1}(-1)^{3+\alpha}\int_{[0,1]^2}(f\circ I^3_{(3,\alpha)})\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \tag{78}$$
$$= \sum_{\alpha=0}^{1}(-1)^{3+\alpha}\int_{[0,1]^2} f(t_1,t_2,\alpha)\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \tag{79}$$
$$= (-1)^{3+1}\int_{[0,1]^2}(f(t_1,t_2,1)-f(t_1,t_2,0))\,\mathrm{d}t_1\wedge\mathrm{d}t_2 \tag{80}$$
$$= +\int_{[0,1]^2}\left[\int_{[0,1]}\frac{\partial f}{\partial t_3}(t_1,t_2)\,\mathrm{d}t_3\right]\mathrm{d}t_1\wedge\mathrm{d}t_2 \tag{81}$$
$$= +\int_{[0,1]^3}\frac{\partial f}{\partial t_3}\,\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3 = T_1 \tag{82}$$

A linha acima fez coincidir exatamente com uma parcela de (71). Por sua vez, T_1 no último membro é parte de T em (56), restringindo-nos ainda a ω_1.

Para provar que $S = T$, basta repetir o raciocínio e lembrar que $T = T_1 + T_2 + T_3$.

$$S_2 = \int_{\partial I^3} g\,\mathrm{d}t_1\wedge\mathrm{d}t_3 = \sum_{j,\alpha}(-1)^{j+\alpha}\int_{I^3_{(j,\alpha)}} g\,\mathrm{d}t_1\wedge\mathrm{d}t_3 = \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(I^3_{(j,\alpha)})^*(g\,\mathrm{d}t_1\wedge\mathrm{d}t_3) \tag{83}$$
$$= \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(g\circ I_{(j,\alpha)})I^*_{(j,\alpha)}(\mathrm{d}t_1)\wedge I^*_{(j,\alpha)}(\mathrm{d}t_3) \tag{84}$$

Sabemos que S_2 é integral sem o termo $\mathrm{d}t_2$, que está faltando para completar 3-forma em $\mathbb{R}^3$. E assim só aparecem os cubos bidimensionais $c(2,\alpha)$, que são faces do cubo c definido no início.

$$S_2 = \sum_{\alpha=0}^{1}(-1)^{2+\alpha}\int_{[0,1]^2}(g\circ I^3_{(2,\alpha)})\,\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{85}$$
$$= \sum_{\alpha=0}^{1}(-1)^{2+\alpha}\int_{[0,1]^2} g(t_1,\alpha,t_3)\,\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{86}$$
$$= (-1)^{2+1}\int_{[0,1]^2}(g(t_1,1,t_3)-g(t_1,0,t_3))\,\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{87}$$
$$= -\int_{[0,1]^2}\left[\int_{[0,1]}\frac{\partial g}{\partial t_2}(t_1,t_3)\,\mathrm{d}t_2\right]\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{88}$$
$$= -\int_{[0,1]^3}\frac{\partial g}{\partial t_2}\,\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3 = T_2 \tag{89}$$

$$S_3 = \int_{\partial I^3} h\,\mathrm{d}t_2\wedge\mathrm{d}t_3 = \sum_{j,\alpha}(-1)^{j+\alpha}\int_{I^3_{(j,\alpha)}} h\,\mathrm{d}t_2\wedge\mathrm{d}t_3 = \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(I^3_{(j,\alpha)})^*(h\,\mathrm{d}t_2\wedge\mathrm{d}t_3) \tag{90}$$
$$= \sum_{j,\alpha}(-1)^{j+\alpha}\int_{[0,1]^2}(h\circ I_{(j,\alpha)})I^*_{(j,\alpha)}(\mathrm{d}t_2)\wedge I^*_{(j,\alpha)}(\mathrm{d}t_3) \tag{91}$$

Sabemos que S_3 é integral sem o termo $\mathrm{d}t_1$, que está faltando para completar 3-forma em $\mathbb{R}^3$. E assim só aparecem os cubos bidimensionais $c(1,\alpha)$, que são faces do cubo c definido no início.

$$S_3 = \sum_{\alpha=0}^{1}(-1)^{1+\alpha}\int_{[0,1]^2}(h\circ I^3_{(1,\alpha)})\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{92}$$
$$= \sum_{\alpha=0}^{1}(-1)^{1+\alpha}\int_{[0,1]^2}h(\alpha,t_2,t_3)\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{93}$$
$$= (-1)^{1+1}\int_{[0,1]^2}(h(1,t_2,t_3)-h(0,t_2,t_3))\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{94}$$
$$= +\int_{[0,1]^2}\left[\int_{[0,1]}\frac{\partial h}{\partial t_1}(t_2,t_3)\,\mathrm{d}t_1\right]\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{95}$$
$$= +\int_{[0,1]^3}\frac{\partial h}{\partial t_1}\,\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3 = T_3 \tag{96}$$

Logo, $S(\omega_1) = T(\omega_1)$.
Para completar nosso trabalho, resta considerar ω_2 e ω_3 a partir de (63).

$$c^*\omega_2 = c^*(G\,\mathrm{d}x_1\wedge\mathrm{d}x_3) = (G\circ c)c^*(\mathrm{d}x_1)\wedge c^*(\mathrm{d}x_3) \tag{97}$$
$$c^*\omega_2 = (G\circ c)\left(\frac{\partial c_1}{\partial t_1}\mathrm{d}t_1+\frac{\partial c_1}{\partial t_2}\mathrm{d}t_2+\frac{\partial c_1}{\partial t_3}\mathrm{d}t_3\right)\wedge\left(\frac{\partial c_3}{\partial t_1}\mathrm{d}t_1+\frac{\partial c_3}{\partial t_2}\mathrm{d}t_2+\frac{\partial c_3}{\partial t_3}\mathrm{d}t_3\right) \tag{98}$$
$$c^*\omega_2 = \underbrace{(G\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_3}{\partial t_2}-\frac{\partial c_3}{\partial t_1}\frac{\partial c_1}{\partial t_2}\right)}_{f_2}\mathrm{d}t_1\wedge\mathrm{d}t_2 + \underbrace{(G\circ c)\left(\frac{\partial c_1}{\partial t_1}\frac{\partial c_3}{\partial t_3}-\frac{\partial c_3}{\partial t_1}\frac{\partial c_1}{\partial t_3}\right)}_{g_2}\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{99}$$
$$+\underbrace{(G\circ c)\left(\frac{\partial c_1}{\partial t_2}\frac{\partial c_3}{\partial t_3}-\frac{\partial c_3}{\partial t_2}\frac{\partial c_1}{\partial t_3}\right)}_{h_2}\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{100}$$
$$c^*\omega_2 = f_2\,\mathrm{d}t_1\wedge\mathrm{d}t_2 + g_2\,\mathrm{d}t_1\wedge\mathrm{d}t_3 + h_2\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{101}$$

Analogamente, $S(\omega_2) = T(\omega_2)$.

$$c^*\omega_3 = c^*(H\,\mathrm{d}x_2\wedge\mathrm{d}x_3) = (H\circ c)c^*(\mathrm{d}x_2)\wedge c^*(\mathrm{d}x_3) \tag{102}$$
$$c^*\omega_3 = (H\circ c)\left(\frac{\partial c_2}{\partial t_1}\mathrm{d}t_1+\frac{\partial c_2}{\partial t_2}\mathrm{d}t_2+\frac{\partial c_2}{\partial t_3}\mathrm{d}t_3\right)\wedge\left(\frac{\partial c_3}{\partial t_1}\mathrm{d}t_1+\frac{\partial c_3}{\partial t_2}\mathrm{d}t_2+\frac{\partial c_3}{\partial t_3}\mathrm{d}t_3\right) \tag{103}$$
$$c^*\omega_3 = (H\circ c)\left(\frac{\partial c_2}{\partial t_1}\frac{\partial c_3}{\partial t_2}-\frac{\partial c_3}{\partial t_1}\frac{\partial c_2}{\partial t_2}\right)\mathrm{d}t_1\wedge\mathrm{d}t_2 + (H\circ c)\left(\frac{\partial c_2}{\partial t_1}\frac{\partial c_3}{\partial t_3}-\frac{\partial c_3}{\partial t_1}\frac{\partial c_2}{\partial t_3}\right)\mathrm{d}t_1\wedge\mathrm{d}t_3 \tag{104}$$
$$+(H\circ c)\left(\frac{\partial c_2}{\partial t_2}\frac{\partial c_3}{\partial t_3}-\frac{\partial c_3}{\partial t_2}\frac{\partial c_2}{\partial t_3}\right)\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{105}$$
$$c^*\omega_3 = f_3\,\mathrm{d}t_1\wedge\mathrm{d}t_2 + g_3\,\mathrm{d}t_1\wedge\mathrm{d}t_3 + h_3\,\mathrm{d}t_2\wedge\mathrm{d}t_3 \tag{106}$$

Analogamente, $S(\omega_3) = T(\omega_3)$.
Portanto, por linearidade,

$$\int_{I^3}\mathrm{d}(c^*(\omega_1+\omega_2+\omega_3)) = \int_{\partial I^3}c^*(\omega_1+\omega_2+\omega_3) \tag{107}$$
$$T(\omega_1)+T(\omega_2)+T(\omega_3) = S(\omega_1)+S(\omega_2)+S(\omega_3) \tag{108}$$
$$T = S \tag{109}$$

Está provado 56.3.

56.4 segue da definição de bordo de I^3.
56.5 segue da definição de integral de $c^*\omega$.
56.6 segue da definição de integral de ω.
56.7 segue da definição de bordo de c.
Conclusão: por linearidade, passamos de 3-cubos a 3-cadeias e vale o Teorema de Stokes para $k = 3$.