Exercícios

Reconstituir 76.4 e fz recíproca

(6) Ni = 0 <=> alfa é plana

76.6 contida em esfera => k >= 1/r
76.2 retas normais formam pencil => circunferência

43.2, integral sinh

Provar que comprimento tem cota mínima (ler Paulo Ventura – Cauchy-Schwarz no comprimento)
Provar que comprimento independe de parametrização.
Provar que torção independe de parametrização.

Matéria

Teorema 1 : $\left|\vec{w}(t)\right| = c \Leftrightarrow \vec{w}' \perp \vec{w}$

α é regular em $t_0 \Leftrightarrow$ diferenciável e $\vec{\alpha}'(t_0) \neq \mathbf{0}$ {ou $v(t_0) \neq 0$}; $v \equiv \left|\alpha'(t)\right|$; $\ell(\vec{\alpha}) = \int_a^b v(t)\, dt$

$\vec{\alpha} \to I = [a,b] \to \Re^3, t \in I$

$h : J \to I, s \in J$

$\beta : J \to \Re^3; \beta(s) = \vec{\alpha} \circ h(s)$

função comprimento : $L(t) = \int_{t_0}^{t} v(u)\, du$

Teorema da reparametrização pelo comprimento de arco : $\left|\boldsymbol{\alpha}' \circ L^{-1}(s)\right| = 1$

$\boldsymbol{\tau}, \mathbf{n}, \mathbf{b}$ são campos : tangente unitário, normal principal, binormal.

$$\boldsymbol{\tau} \equiv \frac{\vec{\alpha}'}{v}; \mathbf{n} \equiv \frac{\vec{\tau}'}{\tau'}; \mathbf{b} \equiv \boldsymbol{\tau} \times \mathbf{n}; \text{Curvatura} : \begin{cases} v(t) = 1 \Rightarrow k(t) \equiv \left|\vec{\alpha}''(t)\right| \\ k \equiv \dfrac{\left|\vec{\tau}'\right|}{v} \\ \text{Centro} : \vec{c}(t) \equiv \vec{\alpha}(t) + \dfrac{1}{k(t)}\mathbf{n} \end{cases}; \text{Torção } (\boldsymbol{V}) : \frac{1}{v}\vec{b}' = \boldsymbol{V}\mathbf{n}$$

$$\begin{cases} \dfrac{1}{v}\vec{\tau}' = k\mathbf{n} \\ \dfrac{1}{v}\vec{n}' = -k\boldsymbol{\tau} - \boldsymbol{V}\mathbf{b} \\ \dfrac{1}{v}\vec{b}' = \boldsymbol{V}\mathbf{n} \end{cases} \Rightarrow \frac{1}{v}\begin{pmatrix} \vec{\tau}' \\ \vec{n}' \\ \vec{b}' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\boldsymbol{V} \\ 0 & \boldsymbol{V} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

$$\vec{\alpha}' = v\boldsymbol{\tau} \Rightarrow \vec{\alpha}'' = v'\boldsymbol{\tau} + v\vec{\tau}' \Rightarrow \begin{pmatrix} \vec{\alpha}' \\ \vec{\alpha}'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v' & kv^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{n} \end{pmatrix}$$

$T1 \Rightarrow \mathbf{n} \perp \vec{n}'; \vec{\alpha}' // \boldsymbol{\tau} \perp \vec{\tau}' // \mathbf{n} // \vec{b}' \perp \mathbf{b}$

π é plano osculador a $\vec{\alpha}$ no ponto $t \Leftrightarrow \pi \ni \vec{\alpha}(t), \mathbf{n}(t) // \pi // \boldsymbol{\tau}(t)$

$\mathbf{n}, \boldsymbol{\tau}, \vec{\tau}', \vec{b}', \vec{\alpha}', \vec{\alpha}'' \in \pi$

Plano normal : $\vec{\alpha}(t) \in \pi_n \perp \boldsymbol{\tau} \Rightarrow \vec{\tau}', \vec{b}' \in \pi_n : \vec{X} = \vec{\alpha} + \lambda\mathbf{n} + \mu\mathbf{b}$

$\vec{n}', \vec{\alpha}' \in pl(\vec{\tau}, \mathbf{b})$

Teorema : $\vec{w} = \vec{\alpha}' \times \vec{\alpha}'' \Rightarrow k(t) = \dfrac{\left|\vec{w}\right|}{v^3}, \boldsymbol{V}(t) = -\dfrac{\vec{w} \cdot \vec{\alpha}'''}{\left|\vec{w}\right|^2}$

O'Neil acrescenta : $\mathbf{b} = \dfrac{\vec{w}}{\left|\vec{w}\right|}, \mathbf{n} = \mathbf{b} \times \boldsymbol{\tau}$

(1B)

$\vec{\alpha}:[a,b]\to\Re^3$

$\ell(\vec{\alpha})\geq|\vec{\alpha}(b)-\vec{\alpha}(a)|$

Pelo Paulo Ventura,

$|u\cdot v|\leq|u|\cdot|v|$

Cauchy - Schwarz

(2) parametrização

$\alpha(t)=A+t\overrightarrow{AB}, t\in[0,1]$

$\beta(t)=A+f(t)\overrightarrow{AB}, f(t)\in[0,1], t\in[p,q]$

$\Rightarrow\ell(\alpha)=\ell(\beta)$

$\alpha'=B-A\Rightarrow\ell(\alpha)=|AB|\int_0^1 dt=|AB|$

$y=f'(t);\beta'=(B-A)y\Rightarrow\ell(\beta)=|AB|\int_p^q|y|dt$

$\int_p^q|y|dt=F(1)-F(0)=1?$

$f(q)-f(p)=1-0=1$

(4)

$\mathcal{V}$ independe de parametrização

(6)

$\mathcal{V}\equiv 0\Leftrightarrow\alpha$ é plana

$\Leftarrow$

$\alpha\subset\pi: Ax+By+Cz+D=0$

$z=ax+by+c\Rightarrow z'=ax'+by'$

$$w=\alpha'\times\alpha''\Rightarrow\mathcal{V}=\frac{1}{|w|^2}\begin{vmatrix}x' & y' & ax'+by'\\ x'' & y'' & ax''+by''\\ x''' & y''' & ax'''+by'''\end{vmatrix}=\xrightarrow{\text{por escalonamento}}=0,\underline{\underline{cqd}}$$

???

$$\boldsymbol{\nu} = 0 \Rightarrow \alpha \subset \pi : ax + by + cz + d = 0$$

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0 = x''' \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} - y''' \begin{vmatrix} x' & z' \\ x'' & z'' \end{vmatrix} + z''' \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}$$

Álgebra linear

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = 0 \Rightarrow aer + bfp + \underline{\underline{cd}}q = \underline{\underline{af}}q + bdr + cep$$

$$q = \frac{bdr + cep - aer - bfp}{af - cd}, \begin{vmatrix} a & c \\ d & f \end{vmatrix} \neq 0$$

$$(a,c) \vee (d,f) \vee (a,d) \vee (c,f) = 0 \vee \frac{a}{c} = \frac{d}{f} \Rightarrow \begin{vmatrix} a & c \\ d & f \end{vmatrix} = 0 \Rightarrow \underline{\underline{ae}}r + bfp = \underline{\underline{bd}}r + cep$$

$$r = \frac{cep - bfp}{ae - bd}, \begin{vmatrix} a & b \\ d & e \end{vmatrix} \neq 0$$

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \Rightarrow (a,b) \vee (d,e) \vee (a,d) \vee (b,e) = 0 \vee \frac{a}{d} = \frac{b}{e} \Rightarrow q = \frac{cep - bfp}{af - cd}; p = 0 \vee \begin{vmatrix} b & c \\ e & f \end{vmatrix} = 0$$

Parametrização

$$a < s < b \in J \xrightarrow{h} h(a) < t < h(b) \in I \xrightarrow{\alpha} R^3$$

$$L_{\alpha(h(s))} = L_{\alpha(t)}$$

$$\int_a^b \left| \frac{d\alpha}{dh} \frac{dh}{ds} \right| ds = \int_{h(a)}^{h(b)} |\alpha'(t)|\, dt = \int_{h(a)}^{h(b)} \left| \frac{d\alpha}{dh} \frac{dh}{dt} \right| dt$$

$$\boldsymbol{\nu}_{\alpha(h(s))} = \boldsymbol{\nu}_{\alpha(t)}$$

$$\nu = \frac{w \cdot \alpha'''}{|w|^2} =$$

Prontos

(1A) Comprimento

Seja $\vec{\alpha}:[a,b]\to\Re^3$. $\ell(\vec{\alpha})=|\vec{\alpha}(b)-\vec{\alpha}(a)|\Leftrightarrow\vec{\alpha}=\vec{p}+t\vec{q}$

$\Leftarrow$

$\ell=\int_a^b|\vec{\alpha}'|\,dt=\int_a^b|\vec{q}|\,dt=|\vec{q}|(b-a)=|\vec{q}(b-a)|=|\vec{p}+b\vec{q}-\vec{p}-a\vec{q}|=|\vec{\alpha}(b)-\vec{\alpha}(a)|$

$\Rightarrow$

Seja $F(t)=\int|\vec{\alpha}'|\,dt=\int\sqrt{(x')^2+(y')^2}\,dt$

$\ell=F(b)-F(a)=|\vec{\alpha}(b)-\vec{\alpha}(a)|=\left|\begin{pmatrix}x(b)-x(a)\\y(b)-y(a)\end{pmatrix}\right|=\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}$

$F^2(b)+F^2(a)-2F(a)F(b)=(x_b-x_a)^2+(y_b-y_a)^2=x_b^2+y_b^2+x_a^2+y_a^2-2x_ax_b-2y_ay_b$

Parece que $F(t)=\sqrt{x_t^2+y_t^2}$. Vamos demostrar.

$F(a)F(b)=\sqrt{x_a^2+y_a^2}\sqrt{x_b^2+y_b^2}=x_ax_b+y_ay_b\Leftrightarrow(x_ax_b)^2+(y_ax_b)^2+(x_ay_b)^2+(y_ay_b)^2=$

$=(x_ax_b)^2+(y_ay_b)^2+2x_ax_by_ay_b\Rightarrow(y_ax_b)^2+(x_ay_b)^2=2x_ax_by_ay_b\Rightarrow y_ax_b-x_ay_b=0=\begin{vmatrix}x_a & y_a\\x_b & y_b\end{vmatrix}$

Se algum termo é zero $\Rightarrow\vec{\alpha}(a)\vee\vec{\alpha}(b)\vee(x_a,x_b)\vee(y_a,y_b)=0$; Caso contrário, $\dfrac{x_b}{x_a}=\dfrac{y_b}{y_a}=k\Rightarrow\vec{\alpha}(b)=k\vec{\alpha}(a)$

Em todos os casos, uma reta que passa pela origem liga os extremos.

Sempre podemos escolher o sistema de coordenadas dessa forma.

Logo, $\int\sqrt{(x')^2+(y')^2}\,dt=\sqrt{x^2+y^2}\Rightarrow\sqrt{(x')^2+(y')^2}=\dfrac{2xx'+2yy'}{2\sqrt{x^2+y^2}}\Rightarrow\sqrt{(x')^2+(y')^2}\sqrt{x^2+y^2}=xx'+yy'$

$(xx')^2+(xy')^2+(x'y)^2+(yy')^2=(xx')^2+(yy')^2+2xx'yy'\Rightarrow(xy')^2+(x'y)^2=2xy'x'y\Rightarrow\begin{vmatrix}x & y\\x' & y'\end{vmatrix}=0$

De forma análoga, $\underbrace{(x,y)\vee(x',y')}_{\vec{\alpha}(t)=\vec{P}\,(\text{ponto})}\vee\underbrace{(x,x')\vee(y,y')}_{\text{eixos}}=0$ ou $\dfrac{y'}{y}=\dfrac{x'}{x}=k\Rightarrow y'=ky\Rightarrow\vec{\alpha}(t)=\underbrace{(c_1,c_2)e^{kt}}_{y=x}\subset$ reta, cqd

(3)

$(v\times w)'=v'\times w+v\times w'$

$v(t)=(x,y,z),w(t)=(f,g,h)\Rightarrow v\times w=\begin{vmatrix}i & j & k\\x & y & z\\f & g & h\end{vmatrix}=\begin{pmatrix}hy-gz\\fz-hx\\gx-fy\end{pmatrix};v\cdot w=fx+gy+hz$

$(v\times w)'=\begin{pmatrix}h'y+hy'-g'z-gz'\\f'z+fz'-h'x-hx'\\g'x+gx'-f'y-fy'\end{pmatrix}=\begin{pmatrix}hy'-gz'\\fz'-hx'\\gx'-fy'\end{pmatrix}+\begin{pmatrix}h'y-g'z\\f'z-h'x\\g'x-f'y\end{pmatrix}=v'\times w+v\times w'$

$(v\cdot w)'=f'x+fx'+g'y+gy'+h'z+hz'=(fx'+gy'+hz')+(f'x+g'y+h'z)=v'\cdot w+v\cdot w'$

(5) plano osculador da hélice

$\alpha'\perp\pi:-x\sin t+y\cos t+z+d=0$

$\alpha\in\pi\Rightarrow-\cos t\sin t+\sin t\cos t+t+d=0\Rightarrow d=-t$

$\therefore\pi:z=x\sin t-y\cos t+t$

(7)

$$\text{Teorema}: \vec{w} = \vec{\alpha}' \times \vec{\alpha}'' \Rightarrow k(t) = \frac{|\vec{w}|}{v^3}; \boldsymbol{V}(t) = -\frac{\vec{w}\cdot\vec{\alpha}'''}{|\vec{w}|^2}; \mathbf{b} = \frac{\vec{w}}{|\vec{w}|}$$

$$\vec{w} = v\boldsymbol{\tau} \times \left(v'\boldsymbol{\tau} + kv^2\mathbf{n}\right) = kv^3\mathbf{b} \Rightarrow \frac{|\vec{w}|}{v^3} = k; \frac{\vec{w}}{|\vec{w}|} = \mathbf{b}, \underline{\underline{cqd}}$$

$$\vec{\alpha}' = v\boldsymbol{\tau}$$

$$\vec{\alpha}'' = v'\boldsymbol{\tau} + v\vec{\tau}' = v'\boldsymbol{\tau} + kv^2\mathbf{n}$$

$$\alpha''' = v''\boldsymbol{\tau} + v'\vec{\tau}' + \left(k'v^2 + 2kvv'\right)\mathbf{n} + kv^2\vec{n}' =$$

$$\alpha''' = v''\boldsymbol{\tau} + v'kv\mathbf{n} + \left(k'v^2 + 2kvv'\right)\mathbf{n} + kv^2\left(-kv\boldsymbol{\tau} - \boldsymbol{V}v\mathbf{b}\right)$$

$$\begin{pmatrix} \vec{\alpha}' \\ \vec{\alpha}'' \\ \vec{\alpha}''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ v' & kv^2 & 0 \\ v'' - k^2v^3 & k'v^2 + 3kvv' & -\boldsymbol{V}kv^3 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

$$w \cdot a''' = -kv^3 \cdot \boldsymbol{V}kv^3 \Rightarrow \frac{w \cdot a'''}{|w|^2} = -\boldsymbol{V}, \underline{\underline{cqd}}$$