Primeira Lista de I.E.D.P.

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1 Questão 1

$$u_t = u_{xx}; x \in (0, L); t \in (0, \infty) \tag{1}$$
$$u_x(t, 0) = 0, \forall t \geq 0 \tag{2}$$
$$u_x(t, L) = 0, \forall t \geq 0 \tag{3}$$
$$u(0, x) = f(x) \tag{4}$$
$$f \in C^1([0, L]) \tag{5}$$
$$f'(0) = 0 \tag{6}$$
$$f'(L) = 0 \tag{7}$$
$$u(x, t) = \varphi(x)\psi(t) \tag{8}$$
$$(1) \Rightarrow \varphi(x)\psi'(t) = \varphi''(x)\psi(t) \tag{9}$$
$$\frac{\psi'(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda \in \mathbb{R} \tag{10}$$
$$\psi'(t) = -\lambda\psi(t) \Rightarrow \psi(t) = Ce^{-\lambda t}, C \in \mathbb{R} \tag{11}$$
$$\varphi''(x) = -\lambda\varphi(x) \Rightarrow \varphi(x) = C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx) \tag{12}$$
$$a = 0, b = \sqrt{\lambda} \tag{13}$$
$$(8) \Rightarrow u(t, x) = e^{-\lambda t}(C_1\cos(x\sqrt{\lambda}) + C_2\sin(x\sqrt{\lambda})) \tag{14}$$
$$(4) \Rightarrow u(0, x) = C_1\cos(x\sqrt{\lambda}) + C_2\sin(x\sqrt{\lambda}) = f(x) \tag{15}$$
$$f'(x) = -C_1\sqrt{\lambda}\sin(x\sqrt{\lambda}) + C_2\sqrt{\lambda}\cos(x\sqrt{\lambda}) \tag{16}$$
$$f'(0) = C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0 \tag{17}$$
$$f'(L) = -C_1\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0 \Rightarrow L\sqrt{\lambda} = n\pi, n \in \mathbb{Z} \tag{18}$$
$$f(x) = \sum_{n=0}^{\infty} c_n \cos\frac{n\pi x}{L} \Rightarrow u(t, x) = \sum_{n=0}^{\infty} c_n e^{-\frac{n^2\pi^2}{L^2}t}\cos\frac{n\pi x}{L} \tag{19}$$

2 Questão 2

$$u_1 = \frac{1}{\sqrt{2}} \tag{20}$$
$$u_2 = \cos\frac{n\pi x}{L} \tag{21}$$
$$u_3 = \sin\frac{n\pi x}{L} \tag{22}$$
$$\langle u_1, u_2\rangle_2 = \int_{-L}^{L} u_1 u_2 \,\mathrm{d}x = 2u_1\int_0^L u_2 \,\mathrm{d}x = 2u_1[\alpha u_3]_0^L = 0 \tag{23}$$
$$\langle u_1, u_3\rangle_2 = \int_{-L}^{L} u_1 u_3 \,\mathrm{d}x = u_1\int_{-L}^{L} u_3 \,\mathrm{d}x = 0, \text{ porque o seno é ímpar.} \tag{24}$$
$$\langle u_2, u_3\rangle_2 = \int_{-L}^{L} u_2 u_3 \,\mathrm{d}x = 0, \text{pois o integrando é par vezes ímpar} = \text{ímpar} \tag{25}$$

$$\langle u_1, u_1\rangle_2 = \int_{-L}^{L} u_1^2 \,\mathrm{d}x = u_1^2 \cdot 2L = L \tag{26}$$

$$\langle u_2, u_2\rangle_2 = \int_{-L}^{L} u_2^2 \,\mathrm{d}x = \frac{L}{n\pi}\int_{-n\pi}^{n\pi} \cos^2 u \,\mathrm{d}u = \frac{L}{2n\pi}\int_{-n\pi}^{n\pi} (1+\cos 2u)\,\mathrm{d}u \tag{27}$$

$$\langle u_2, u_2\rangle_2 = \frac{L}{2n\pi}\left[u + \frac{1}{2}\sin 2u\right]_{-n\pi}^{n\pi} = L \tag{28}$$

$$\langle u_3, u_3\rangle_2 = \int_{-L}^{L} u_3^2 \,\mathrm{d}x = \frac{L}{n\pi}\int_{-n\pi}^{n\pi} \sin^2 u \,\mathrm{d}u = \frac{L}{2n\pi}\int_{-n\pi}^{n\pi} (1-\cos 2u)\,\mathrm{d}u \tag{29}$$

$$\langle u_3, u_3\rangle_2 = \frac{L}{2n\pi}\left[u - \frac{1}{2}\sin 2u\right]_{-n\pi}^{n\pi} = L \tag{30}$$

3 Questão 3

$$f \in CP[a,b]; a < b \in \mathbb{R} \tag{31}$$

Queremos mostrar que

$$\lim_{n\to\infty}\int_a^b f(x)\cos(nx)\,\mathrm{d}\,x = 0 \tag{32}$$

$$\lim_{n\to\infty}\int_a^b f(x)\sin(nx)\,\mathrm{d}\,x = 0 \tag{33}$$

O lema de Riemmann-Lebesgue afirma que, se $g \in CP[-L, L]$, então

$$\lim_{n\to\infty} a_n(g) = \frac{1}{L}\lim_{n\to\infty}\int_{-L}^{L} g(y)\cos\frac{n\pi y}{L}\,\mathrm{d}\,y = 0 \tag{34}$$

$$\lim_{n\to\infty} b_n(g) = \frac{1}{L}\lim_{n\to\infty}\int_{-L}^{L} g(y)\sin\frac{n\pi y}{L}\,\mathrm{d}\,y = 0 \tag{35}$$

Basta construirmos funções reais em compactos da seguinte forma:

$$\begin{tikzcd}
{[a,b]} \arrow[r, "\rho"] \arrow[d, "f"'] & {[p,q]} \arrow[d, "h"] \\
\mathbb{R} & {[-L,L]} \arrow[l, "g"']
\end{tikzcd} \tag{36}$$

$$y = \alpha z + \beta = h(z) \tag{37}$$

$$\alpha = \frac{2L}{q-p} \tag{38}$$

$$\beta = L - q\frac{2L}{q-p} \tag{39}$$

$$\lim_{n\to\infty} a_n(g) = \frac{1}{L}\lim_{n\to\infty}\int_p^q g(h(z))\cos\frac{n\pi(\alpha z+\beta)}{L}\alpha\,\mathrm{d}\,z = 0 \tag{40}$$

$$\lim_{n\to\infty} b_n(g) = \frac{\alpha}{L}\lim_{n\to\infty}\int_p^q g(h(z))\sin\frac{n\pi(\alpha z+\beta)}{L}\,\mathrm{d}\,z = 0 \tag{41}$$

$$x = \frac{\pi}{L}(\alpha z + \beta) \Leftrightarrow \rho(x) = z \tag{42}$$

$$\lim_{n\to\infty} a_n(g) = \frac{\alpha}{L} \lim_{n\to\infty} \int_a^b g(h(\rho(x))) \cos(nx) \cdot \frac{L}{\pi\alpha} \,\mathrm{d}\, x = 0 \tag{43}$$

$$\lim_{n\to\infty} b_n(g) = \frac{1}{\pi} \lim_{n\to\infty} \int_a^b g(h(\rho(x))) \sin(nx) \,\mathrm{d}\, x = 0 \tag{44}$$

$$f = g \circ h \circ \rho, \text{ por construção.} \tag{45}$$

$$\lim_{n\to\infty} a_n(g) = \frac{1}{\pi} \lim_{n\to\infty} \int_a^b f(x) \cos(nx) \,\mathrm{d}\, x = 0 \tag{46}$$

$$\lim_{n\to\infty} b_n(g) = \frac{1}{\pi} \lim_{n\to\infty} \int_a^b f(x) \sin(nx) \,\mathrm{d}\, x = 0 \tag{47}$$

4 Questão 4

De fato, temos

$$\frac{1}{L}\int_0^L f(x+\xi) D_N(\xi)\,\mathrm{d}\xi = \frac{1}{L}\int_0^L [f(x+\xi) - f(x+)] D_N(\xi)\,\mathrm{d}\xi \tag{48}$$

$$+ \frac{1}{L}\int_0^L f(x+) D_N(\xi)\,\mathrm{d}\xi. \tag{49}$$

Como

$$\frac{1}{L}\int_0^L f(x+) D_N(\xi)\,\mathrm{d}\xi = f(x+)\frac{1}{L}\int_0^L D_N(\xi)\,\mathrm{d}\xi = \frac{f(x+)}{2}, \tag{50}$$

Queremos mostrar que

$$\lim_{N\to\infty} \frac{1}{L}\int_0^L [f(x+\xi) - f(x+)] D_N(\xi)\,\mathrm{d}\xi = 0. \tag{51}$$

Assim, pela fórmula do núcleo,

$$\frac{1}{L}\int_0^L [f(x+\xi) - f(x+)] D_N(\xi)\,\mathrm{d}\xi = \frac{1}{L}\int_0^L [f(x+\xi) - f(x+)] \frac{\sin\left(\frac{\pi}{2L}(2N+1)\xi\right)}{2\sin\left(\frac{\pi}{2L}\xi\right)}\mathrm{d}\xi \tag{52}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} v(\eta) \sin[(2N+1)\eta]\,\mathrm{d}\eta, \tag{53}$$

em que

$$v(\eta) = \frac{f\left(x + \frac{2L}{\pi}\eta\right) - f(x+)}{2\sin\eta}. \tag{54}$$

Pelo exercício (3),

$$\lim_{N\to\infty} v(\eta)\sin[(2N+1)\eta]\,\mathrm{d}\eta = 0. \tag{55}$$

Uma vez que $v \in CP\left(0, \dfrac{\pi}{2}\right]$, basta agora mostrar que $v(0+)$ é finito.

$$\lim_{\eta\to 0+} v(\eta) = \lim_{\eta\to 0+} \frac{f\left(x + \dfrac{2L}{\pi}\eta\right) - f(x+)}{\frac{2L}{\pi}\eta} \cdot \frac{\dfrac{2L}{\pi}\eta}{2\sin\eta} \tag{56}$$

$$= f'(x+)\frac{L}{\pi}\lim_{\eta\to 0+}\frac{\eta}{\sin\eta} < \infty \tag{57}$$