Universidade Federal de Minas Gerais

Projective Geometry Again

TEAM:

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1 The Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{1}$$

$$B = X_1X_4 \cap X_3X_6 \tag{2}$$

$$C = X_2X_4 \cap X_3X_5 \in AB \tag{3}$$

$$A = (0,0) \tag{4}$$

$$B = (0,a) \tag{5}$$

$$X_1 = (x_1, bx_1) \tag{6}$$

$$X_5 = (x_5, bx_5) \tag{7}$$

$$X_2 = (x_2, cx_2) \tag{8}$$

$$X_6 = (x_6, cx_6) \tag{9}$$

$$X_4 = (x_4, d(x_4 - a)) \tag{10}$$

$$bx_1 = d(x_1 - a) \Rightarrow d = \frac{bx_1}{x_1 - a} \tag{11}$$

$$X_3 = (x_3, e(x_3 - a)) \tag{12}$$

$$cx_6 = e(x_6 - a) \Rightarrow e = \frac{cx_6}{x_6 - a} \tag{13}$$

$$\alpha x_1^2 + \beta(bx_1)^2 = 1 \Rightarrow x_1^2 = \frac{1}{\alpha + \beta b^2} = \lambda \tag{14}$$

$$\alpha x_2^2 + \beta(cx_2)^2 = 1 \Rightarrow x_2^2 = \frac{1}{\alpha + \beta c^2} = \xi \tag{15}$$

$$\alpha x_5^2 + \beta(bx_5)^2 = 1 \Rightarrow x_5^2 = \frac{1}{\alpha + \beta b^2} = \lambda \therefore x_5 = \epsilon_1 x_1 \tag{16}$$

$$\alpha x_6^2 + \beta(cx_6)^2 = 1 \Rightarrow x_6^2 = \frac{1}{\alpha + \beta c^2} = \xi \therefore x_2 = \epsilon_6 x_6 \tag{17}$$

$$\alpha x_3^2 + \beta[e(x_3 - a)]^2 = 1 \Rightarrow e^2 = \frac{1 - \alpha x_3^2}{\beta(x_3 - a)^2} \tag{18}$$

$$\alpha x_4^2 + \beta[d(x_4 - a)]^2 = 1 \Rightarrow d^2 = \frac{1 - \alpha x_4^2}{\beta(x_4 - a)^2} \tag{19}$$

Therefore:

$$\frac{-y_3x_5+y_5x_3}{x_3-x_5}=\frac{-y_2x_4+y_4x_2}{x_2-x_4} \tag{20}$$

$$\frac{-\dfrac{cx_6}{x_6-a}(x_3-a)x_5+bx_5x_3}{x_3-x_5}=\frac{-cx_2x_4+\dfrac{bx_1}{x_1-a}(x_4-a)x_2}{x_2-x_4} \tag{21}$$

Below, we try to reduce that to $0x=0$:

The first intention is to eliminate x_1 and x_6.

$$\zeta = bx_3x_5(x_2-x_4)+cx_2x_4(x_3-x_5)=\zeta_{51}x_5+\zeta_{50} \tag{22}$$

$$\zeta_{51}=bx_2x_3-bx_3x_4-cx_2x_4 \tag{23}$$

$$\zeta_{50}=cx_2x_3x_4 \tag{24}$$

$$\eta=(x_4-a)(x_3-x_5)=\eta_{51}x_5+\eta_{50} \tag{25}$$

$$\eta_{51}=a-x_4 \tag{26}$$

$$\eta_{50}=-x_3(a-x_4) \tag{27}$$

$$\kappa=(x_3-a)^2(x_2-x_4)^2=\kappa_{21}x_2+\kappa_{20} \tag{28}$$

$$\kappa_{21}=-(x_3-a)^2x_4 \tag{29}$$

$$\kappa_{20}=(x_3-a)^2x_4^2+\xi(x_3-a)^2 \tag{30}$$

$$\eta^2=(\eta_{51}x_5+\eta_{50})^2=\hat{A}x_5+\hat{B} \tag{31}$$

$$\hat{A}=2\eta_{51}\eta_{50} \tag{32}$$

$$\hat{B}=\lambda\eta_{51}^2+\eta_{50}^2 \tag{33}$$

$$\zeta^2=(\zeta_{51}x_5+\zeta_{50})^2=\hat{C}x_5+\hat{D} \tag{34}$$

$$\hat{C}=2\zeta_{51}\zeta_{50} \tag{35}$$

$$\hat{D}=\lambda\zeta_{51}^2+\zeta_{50}^2 \tag{36}$$

$$\mu=4\beta(x_3-a)^4(x_4-a)^4(1-\alpha x_4^2) \tag{37}$$

$$\nu=(1-\alpha x_4^2)(x_3-a)^2(x_4-a) \tag{38}$$

$$\pi=\beta(x_3-a)^2(x_4-a)^3 \tag{39}$$

$$\rho = (1-\alpha x_3^2)(x_4-a)^3 \tag{40}$$

$$\kappa x_5^2 \underbrace{e^2} = (\zeta - \eta x_2 \underbrace{d})^2 \tag{41}$$

$$\kappa x_5^2 \frac{1-\alpha x_3^2}{\beta(x_3-a)^2} = \eta^2 x_2^2 \frac{1-\alpha x_4^2}{\beta(x_4-a)^2} - 2\zeta\eta x_2 \underbrace{d} + \zeta^2 \tag{42}$$

$$\mu\zeta^2\eta^2 x_2^2 = (\eta^2 x_2^2 \nu + \pi\zeta^2 - \kappa x_5^2 \rho)^2 \tag{43}$$

They are already eliminated: (d, e). Let us eliminate x_5 and, only partially, x_2.

$$\xi\mu(\hat{A}x_5 + \hat{B})(\hat{C}x_5 + \hat{D}) = \left[\xi\nu(\hat{A}x_5 + \hat{B}) + \pi(\hat{C}x_5 + \hat{D}) - \lambda\rho\kappa\right]^2 \tag{44}$$

$$Ex_5 + F = (Gx_5 + H)^2 = 2GHx_5 + \lambda G^2 + H^2 \tag{45}$$

$$\lambda(E - 2GH)^2 = \left(\lambda G^2 + H^2 - F\right)^2 \tag{46}$$

$$E = \xi\mu(\hat{A}\hat{D} + \hat{B}\hat{C}) \tag{47}$$

$$F = \lambda\xi\mu[\hat{A}\hat{C} + \hat{B}\hat{D}(\alpha + \beta b^2)] \tag{48}$$

$$G = \xi\nu\hat{A} + \pi\hat{C} \tag{49}$$

$$H = \xi\nu\hat{B} + \pi\hat{D} - \lambda\rho\kappa \tag{50}$$

Let us eliminate x_2 in $(\zeta, \kappa, \hat{C}, \hat{D})$.

$$\begin{aligned} &\lambda\left[\xi\mu(\hat{A}\hat{D} + \hat{B}\hat{C}) - 2\left(\xi\nu\hat{A} + \pi\hat{C}\right)\left(\xi\nu\hat{B} + \pi\hat{D} - \lambda\rho\kappa\right)\right]^2 = \\ &= \left\{\lambda\left(\xi\nu\hat{A} + \pi\hat{C}\right)^2 + \left(\xi\nu\hat{B} + \pi\hat{D} - \lambda\rho\kappa\right)^2 - \lambda\xi\mu[\hat{A}\hat{C} + \hat{B}\hat{D}(\alpha + \beta b^2)]\right\}^2 \end{aligned} \tag{51}$$

$$\kappa(x_2) = \kappa_{21}x_2 + \kappa_{20} \tag{52}$$

$$\hat{C}(x_2) = 2(bx_2x_3 - bx_3x_4 - cx_2x_4)cx_2x_3x_4 = Ix_2 + J \tag{53}$$

$$\hat{D}(x_2) = \lambda(bx_2x_3 - bx_3x_4 - cx_2x_4)^2 + c^2x_2^2x_3^2x_4^2 = Kx_2 + L \tag{54}$$

$$I = -2bcx_3^2x_4^2 \tag{55}$$

$$J = \xi(2bcx_3^2x_4 - 2c^2x_3x_4^2) \tag{56}$$

$$K = \lambda(2bcx_3x_4^2 - 2b^2x_3^2x_4) \tag{57}$$

$$L = \lambda b^2 x_3^2 x_4^2 + \xi c^2 x_3^2 x_4^2 + \lambda\xi(bx_3 - cx_4)^2 \quad (58)$$

$$M = \xi\mu(\hat{A}K + \hat{B}I) \quad (59)$$

$$N = \xi\mu(\hat{A}L + \hat{B}J) \quad (60)$$

$$P = \pi I \quad (61)$$

$$Q = \xi\nu\hat{A} + \pi J \quad (62)$$

$$R = \pi K - \lambda\rho\kappa_{21} \quad (63)$$

$$S = \xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20} \quad (64)$$

$$T = -\lambda\xi\mu[\hat{A}I + \hat{B}K(\alpha + \beta b^2)] \quad (65)$$

$$U = -\lambda\xi\mu[\hat{A}J + \hat{B}L(\alpha + \beta b^2)] \quad (66)$$

$$\lambda[Mx_2 + N - 2(Px_2 + Q)(Rx_2 + S)]^2 = [\lambda\left(Px_2 + Q\right)^2 + \left(Rx_2 + S\right)^2 + Tx_2 + U]^2 \quad (67)$$

$$V = M - 2PS - 2QR \quad (68)$$

$$W = N - 2\xi PR - 2QS \quad (69)$$

$$Y = 2\lambda VW \quad (70)$$

$$Z = \lambda\xi V^2 + \lambda W^2 \quad (71)$$

$$\hat{E} = 2\lambda PQ + 2RS + T \quad (72)$$

$$\hat{F} = \lambda\xi P^2 + \xi R^2 + \lambda Q^2 + S^2 + U \quad (73)$$

$$Yx_2 + Z = (\hat{E}x_2 + \hat{F})^2 = \xi\hat{E}^2 + \hat{F}^2 + 2\hat{E}\hat{F}x_2 \quad (74)$$

$$\xi(Y - 2\hat{E}\hat{F})^2 = \left(\xi\hat{E}^2 + \hat{F}^2 - Z\right)^2 \quad (75)$$

We have an Equation in $\alpha, \beta, x_3, x_4, a, b, c$.

The question is: is it an identity? Let us express it as a function of $(x_3, x_4) \equiv (x, y, z = x - a, w = y - a)$.

$$Y = 2\lambda(\xi\mu(\hat{A}K + \hat{B}I) - 2\pi I(\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20}) \quad (76)$$

$$- 2\xi\nu\pi\hat{A}K + 2\lambda\xi\nu\rho\kappa_{21}\hat{A} - 2\pi^2 JK + 2\lambda\pi\rho\kappa_{21}J)(\xi\mu(\hat{A}L + \hat{B}J) \quad (77)$$

$$- 2\xi\pi I(\pi K - \lambda\rho\kappa_{21}) \quad (78)$$

$$- 2\xi^2\nu^2\hat{A}\hat{B} - 2\xi\nu\pi\hat{A}L + 2\lambda\xi\nu\rho\kappa_{20}\hat{A} - 2\xi\nu\pi\hat{B}J - 2\pi^2 JL + 2\lambda\pi\rho\kappa_{20}J) \quad (79)$$

Let us substitute from L and above, until η.

$$Y = 2\lambda\{-2\bar{e}x^2y^2w^2(\xi\bar{\ell}z^4w^4 + \xi\bar{m}z^4w^6 + \xi\bar{n}z^4w^5) \tag{80}$$

$$- 2\bar{f}x^3yw^2(\xi\bar{\ell}z^4w^4 + \xi\bar{m}z^4w^6 + \xi\bar{n}z^4w^5) \tag{81}$$

$$+ \lambda\bar{\imath}x^2y^2w^2(\xi\bar{\ell}z^4w^4 + \xi\bar{m}z^4w^6 + \xi\bar{n}z^4w^5) \tag{82}$$

$$+ \bar{\imath}x^4y^2w^2(\xi\bar{\ell}z^4w^4 + \xi\bar{m}z^4w^6 + \xi\bar{n}z^4w^5) \tag{83}$$

$$- 2\beta z^2w^3\bar{\imath}x^2y^2\Big[\xi(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(\lambda w^2 + x^2w^2) + \beta z^2w^3(\bar{a}x^2y^2 \tag{84}$$

$$+ \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy) - \lambda(w^3 - \alpha x^2w^3)(y^2z^2 + \xi z^2)\Big] \tag{85}$$

$$+ 4\beta\xi xz^2w^5(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(\bar{e}xy^2 + \bar{f}x^2y) \tag{86}$$

$$+ 4\lambda\xi xyz^2w^2(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(w^3 - \alpha x^2w^3) \tag{87}$$

$$- 2\beta^2z^4w^6(\bar{g}xy^2 + \bar{h}x^2y)(\bar{e}xy^2 + \bar{f}x^2y) \tag{88}$$

$$- 2\lambda\beta yz^4w^3(w^3 - \alpha x^2w^3)(\bar{g}xy^2 + \bar{h}x^2y)\}\cdot \tag{89}$$

$$\cdot\ \{\xi(\bar{\ell}z^4w^4 + \bar{m}z^4w^6 + \bar{n}z^4w^5)\Big[-2xw^2(\bar{a}x^2y^2 + \bar{b}x^2 \tag{90}$$

$$+ \bar{c}y^2 + \bar{d}xy) + (\lambda w^2 + x^2w^2)(\bar{g}xy^2 + \bar{h}x^2y)\Big] \tag{91}$$

$$- 2\xi\beta z^2w^3\bar{\imath}x^2y^2\Big[\beta z^2w^3(\bar{e}xy^2 + \bar{f}x^2y) + \lambda yz^2(w^3 - \alpha x^2w^3)\Big] \tag{92}$$

$$+ 4\xi^2xw^2(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)^{2}(\lambda w^2 + x^2w^2) \tag{93}$$

$$+ 4\xi\beta xz^2w^5(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(\bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy) \tag{94}$$

$$- 4\lambda\xi xw^2(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(w^3 - \alpha x^2w^3)(y^2z^2 + \xi z^2) \tag{95}$$

$$- 2\xi\beta z^2w^3(z^2w - \alpha y^3z^2 + \bar{k}y^2z^2)(\lambda w^2 + x^2w^2)(\bar{g}xy^2 + \bar{h}x^2y) \tag{96}$$

$$- 2\beta^2z^4w^6(\bar{g}xy^2 + \bar{h}x^2y)(\bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy) \tag{97}$$

$$+ 2\lambda\beta z^2w^3(w^3 - \alpha x^2w^3)(y^2z^2 + \xi z^2)(\bar{g}xy^2 + \bar{h}x^2y)\} \tag{98}$$

If we did not want to give it all up, we "would" have:

$$Z = \lambda\xi[\xi\mu(\hat{A}K + \hat{B}I) - 2\pi I(\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20}) - 2\xi\nu\pi\hat{A}K \tag{99}$$

$$+ 2\lambda\xi\nu\rho\kappa_{21}\hat{A} - 2\pi^2 JK + 2\lambda\pi\rho\kappa_{21}J]^2 + \lambda[\xi\mu(\hat{A}L + \hat{B}J) \tag{100}$$

$$- 2\xi\pi I(\pi K - \lambda\rho\kappa_{21}) \tag{101}$$

$$- 2\xi^2\nu^2\hat{A}\hat{B} - 2\xi\nu\pi\hat{A}L + 2\lambda\xi\nu\rho\kappa_{20}\hat{A} - 2\xi\nu\pi\hat{B}J - 2\pi^2 JL + 2\lambda\pi\rho\kappa_{20}J]^2 \tag{102}$$

$$\hat{E} = 2\lambda\pi I(\xi\nu\hat{A} + \pi J) + 2(\pi K - \lambda\rho\kappa_{21})(\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20}) \tag{103}$$

$$- \xi\mu(\lambda\hat{A}I + \hat{B}K) \tag{104}$$

$$\hat{F} = \lambda\xi\pi^2 I^2 + \xi(\pi K - \lambda\rho\kappa_{21})^2 + \lambda(\xi\nu\hat{A} + \pi J)^2 + (\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20})^2 \tag{105}$$

$$- \xi\mu(\lambda\hat{A}J + \hat{B}L) \tag{106}$$

$$L = \bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy\,;\ \bar{a} = \lambda b^2 + \xi c^2\,;\ \bar{b} = \lambda\xi b^2\,;\ \bar{c} = \lambda\xi c^2\,;\ \bar{d} = -2\lambda\xi bc \tag{107}$$

$$K = \bar{e}xy^2 + \bar{f}x^2y\,;\ \bar{e} = 2\lambda bc\,;\ \bar{f} = -2\lambda b^2 \tag{108}$$

$$J = \bar{g}xy^2 + \bar{h}x^2y\,;\ \bar{g} = -2\xi c^2\,;\ \bar{h} = 2\xi bc \tag{109}$$

$$I = \bar{\imath}x^2y^2\,;\ \bar{\imath} = -2bc \tag{110}$$

$$\kappa_{21} = -yz^2 \tag{111}$$

$$\kappa_{20} = y^2z^2 + \xi z^2 \tag{112}$$

$$\rho = w^3 - \alpha x^2w^3 \tag{113}$$

$$\pi = \beta z^2w^3 \tag{114}$$

$$\nu = z^2w - \alpha y^3z^2 + \bar{k}y^2z^2\,;\ \bar{k} = \alpha\cdot a \tag{115}$$

$$\mu = \bar{\ell}z^4w^4 + \bar{m}z^4w^6 + \bar{n}z^4w^5\,;\ \bar{\ell} = 4\beta - 4\alpha\beta a^2\,;\ \bar{m} = -4\alpha\beta\,;\ \bar{n} = -8\alpha\beta a \tag{116}$$

$$\hat{B} = \lambda w^2 + x^2w^2 \tag{117}$$

$$\hat{A} = -2xw^2 \tag{118}$$

$$\bar{o} = \tag{119}$$

Below, let us explain the Equation (20), expliciting C on Equation (3).

$$\begin{pmatrix} x_2 & 1 \\ x_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{x_2 - x_4} \begin{pmatrix} 1 & -1 \\ -x_4 & x_2 \end{pmatrix} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} \tag{120}$$

$$\begin{pmatrix} x_3 & 1 \\ x_5 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} y_3 \\ y_5 \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{x_3 - x_5} \begin{pmatrix} 1 & -1 \\ -x_5 & x_3 \end{pmatrix} \begin{pmatrix} y_3 \\ y_5 \end{pmatrix} \tag{121}$$

$$ax + b = cx + d \Rightarrow x = \frac{d - b}{a - c} = 0 \therefore d = b \tag{122}$$

2 The Converse Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{123}$$

$$B = X_1X_4 \cap X_3X_6 \tag{124}$$

$$C = X_2X_4 \cap X_3X_5 \in AB \tag{125}$$

$$A = (0,0) \tag{126}$$

$$B = (0,a) \tag{127}$$

$$X_1 = (x_1, bx_1) \tag{128}$$

$$X_5 = (x_5, bx_5) \tag{129}$$

$$X_2 = (x_2, cx_2) \tag{130}$$

$$X_6 = (x_6, cx_6) \tag{131}$$

$$X_4 = (x_4, d(x_4 - a) \tag{132}$$

$$bx_1 = d(x_1 - a) \tag{133}$$

$$X_3 == e(x_3 - a) \tag{134}$$

$$cx_6 = e(x_6 - a) \tag{135}$$

$$\frac{-y_3x_5 + y_5x_3}{x_3 - x_5} = \frac{-y_2x_4 + y_4x_2}{x_2 - x_4} \tag{136}$$

Então $\exists P, \theta, \alpha, \beta, \gamma, \delta \in \{0,1\}$ tais que:

$$\alpha x_i^2 + \beta y_i^2 + x(y_i - \gamma x_i) = \delta, \forall i \in \{1, 2, \cdots, 6\} \tag{137}$$

■

Out of charity, there is no salvation at all. With charity, we evolve.

June, the 17th, 2024.