Universidade Federal de Minas Gerais

Projective Geometry Again

TEAM:

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1 The Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{1}$$

$$B = X_1X_4 \cap X_3X_6 \tag{2}$$

$$C = X_2X_4 \cap X_3X_5 \in AB \tag{3}$$

$$A = (0, 0) \tag{4}$$

$$B = (0, a) \tag{5}$$

$$X_1 = (x_1, bx_1) \tag{6}$$

$$X_5 = (x_5, bx_5) \tag{7}$$

$$X_2 = (x_2, cx_2) \tag{8}$$

$$X_6 = (x_6, cx_6) \tag{9}$$

$$X_4 = (x_4, d(x_4 - a)) \tag{10}$$

$$bx_1 = d(x_1 - a) \Rightarrow d = \frac{bx_1}{x_1 - a} \tag{11}$$

$$X_3 = (x_3, e(x_3 - a)) \tag{12}$$

$$cx_6 = e(x_6 - a) \Rightarrow e = \frac{cx_6}{x_6 - a} \tag{13}$$

$$\alpha x_1^2 + \alpha(bx_1)^2 = 1 \Rightarrow x_1^2 = \frac{1}{\alpha + \alpha b^2} = \lambda \tag{14}$$

$$\alpha x_2^2 + \alpha(cx_2)^2 = 1 \Rightarrow x_2^2 = \frac{1}{\alpha + \alpha c^2} = \xi \tag{15}$$

$$\alpha x_5^2 + \alpha(bx_5)^2 = 1 \Rightarrow x_5^2 = \frac{1}{\alpha + \alpha b^2} = \lambda \therefore x_5 = \epsilon_1 x_1 \tag{16}$$

$$\alpha x_6^2 + \alpha(cx_6)^2 = 1 \Rightarrow x_6^2 = \frac{1}{\alpha + \alpha c^2} = \xi \therefore x_2 = \epsilon_6 x_6 \tag{17}$$

$$\alpha x_3^2 + \alpha[e(x_3 - a)]^2 = 1 \Rightarrow e^2 = \frac{1 - \alpha x_3^2}{\alpha(x_3 - a)^2} \tag{18}$$

$$\alpha x_4^2 + \alpha[d(x_4 - a)]^2 = 1 \Rightarrow d^2 = \frac{1 - \alpha x_4^2}{\alpha(x_4 - a)^2} \tag{19}$$

Therefore:

$$\frac{-y_3x_5+y_5x_3}{x_3-x_5}=\frac{-y_2x_4+y_4x_2}{x_2-x_4} \tag{20}$$

$$\frac{-\dfrac{cx_6}{x_6-a}(x_3-a)x_5+bx_5x_3}{x_3-x_5}=\frac{-cx_2x_4+\dfrac{bx_1}{x_1-a}(x_4-a)x_2}{x_2-x_4} \tag{21}$$

Below, we try to reduce that to $0x=0$:

The first intention is to eliminate x_1 and x_6.

$$\zeta = bx_3x_5(x_2-x_4)+cx_2x_4(x_3-x_5)=\zeta_{51}x_5+\zeta_{50} \tag{22}$$

$$\zeta_{51} = bx_2x_3-bx_3x_4-cx_2x_4 \tag{23}$$

$$\zeta_{50} = cx_2x_3x_4 \tag{24}$$

$$\eta = (x_4-a)(x_3-x_5)=\eta_{51}x_5+\eta_{50} \tag{25}$$

$$\eta_{51} = a-x_4 \tag{26}$$

$$\eta_{50} = -x_3(a-x_4) \tag{27}$$

$$\kappa = (x_3-a)^2(x_2-x_4)^2=\kappa_{21}x_2+\kappa_{20} \tag{28}$$

$$\kappa_{21} = -(x_3-a)^2x_4 \tag{29}$$

$$\kappa_{20} = (x_3-a)^2x_4^2+\xi(x_3-a)^2 \tag{30}$$

$$\eta^2 = (\eta_{51}x_5+\eta_{50})^2=\hat{A}x_5+\hat{B} \tag{31}$$

$$\hat{A} = 2\eta_{51}\eta_{50} \tag{32}$$

$$\hat{B} = \lambda\eta_{51}^2+\eta_{50}^2 \tag{33}$$

$$\zeta^2 = (\zeta_{51}x_5+\zeta_{50})^2=\hat{C}x_5+\hat{D} \tag{34}$$

$$\hat{C} = 2\zeta_{51}\zeta_{50} \tag{35}$$

$$\hat{D} = \lambda\zeta_{51}^2+\zeta_{50}^2 \tag{36}$$

$$\mu = 4\alpha(x_3-a)^4(x_4-a)^4(1-\alpha x_4^2) \tag{37}$$

$$\nu = (1-\alpha x_4^2)(x_3-a)^2(x_4-a) \tag{38}$$

$$\pi = \alpha(x_3-a)^2(x_4-a)^3 \tag{39}$$

$$\rho = (1-\alpha x_3^2)(x_4-a)^3 \tag{40}$$

$$\kappa x_5^2 \underbrace{e^2} = (\zeta - \eta x_2 \underbrace{d})^2 \tag{41}$$

$$\kappa x_5^2 \frac{1-\alpha x_3^2}{\alpha(x_3-a)^2} = \eta^2 x_2^2 \frac{1-\alpha x_4^2}{\alpha(x_4-a)^2} - 2\zeta\eta x_2 \underbrace{d} + \zeta^2 \tag{42}$$

$$\mu\zeta^2\eta^2 x_2^2 = (\eta^2 x_2^2 \nu + \pi\zeta^2 - \kappa x_5^2 \rho)^2 \tag{43}$$

They are already eliminated: (d,e). Let us eliminate x_5 and, only partially, x_2.

$$\xi\mu(\hat{A}x_5+\hat{B})(\hat{C}x_5+\hat{D}) = \left[\xi\nu(\hat{A}x_5+\hat{B}) + \pi(\hat{C}x_5+\hat{D}) - \lambda\rho\kappa\right]^2 \tag{44}$$

$$Ex_5 + F = (Gx_5+H)^2 = 2GHx_5 + \lambda G^2 + H^2 \tag{45}$$

$$\lambda(E-2GH)^2 = \left(\lambda G^2 + H^2 - F\right)^2 \tag{46}$$

$$E = \xi\mu(\hat{A}\hat{D} + \hat{B}\hat{C}) \tag{47}$$

$$F = \lambda\xi\mu[\hat{A}\hat{C} + \hat{B}\hat{D}(\alpha + \alpha b^2)] \tag{48}$$

$$G = \xi\nu\hat{A} + \pi\hat{C} \tag{49}$$

$$H = \xi\nu\hat{B} + \pi\hat{D} - \lambda\rho\kappa \tag{50}$$

Let us eliminate x_2 in $(\zeta,\kappa,\hat{C},\hat{D})$.

$$\begin{aligned}&\lambda\left[\xi\mu(\hat{A}\hat{D}+\hat{B}\hat{C}) - 2\left(\xi\nu\hat{A}+\pi\hat{C}\right)\left(\xi\nu\hat{B}+\pi\hat{D}-\lambda\rho\kappa\right)\right]^2 = \\ &= \left\{\lambda\left(\xi\nu\hat{A}+\pi\hat{C}\right)^2 + \left(\xi\nu\hat{B}+\pi\hat{D}-\lambda\rho\kappa\right)^2 - \lambda\xi\mu[\hat{A}\hat{C}+\alpha\hat{B}\hat{D}(1+b^2)]\right\}^2\end{aligned} \tag{51}$$

$$\kappa(x_2) = \kappa_{21}x_2 + \kappa_{20} \tag{52}$$

$$\hat{C}(x_2) = 2(bx_2x_3 - bx_3x_4 - cx_2x_4)cx_2x_3x_4 = Ix_2 + J \tag{53}$$

$$\hat{D}(x_2) = \lambda(bx_2x_3 - bx_3x_4 - cx_2x_4)^2 + c^2x_2^2x_3^2x_4^2 = Kx_2 + L \tag{54}$$

$$I = -2bcx_3^2x_4^2 \tag{55}$$

$$J = \xi(2bcx_3^2x_4 - 2c^2x_3x_4^2) \tag{56}$$

$$K = \lambda(2bcx_3x_4^2 - 2b^2x_3^2x_4) \tag{57}$$

$$L = \lambda b^2 x_3^2 x_4^2 + \xi c^2 x_3^2 x_4^2 + \lambda\xi(bx_3 - cx_4)^2 \tag{58}$$

$$M = \xi\mu(\hat{A}K + \hat{B}I) \tag{59}$$

$$N = \xi\mu(\hat{A}L + \hat{B}J) \tag{60}$$

$$P = \pi I \tag{61}$$

$$Q = \xi\nu\hat{A} + \pi J \tag{62}$$

$$R = \pi K - \lambda\rho\kappa_{21} \tag{63}$$

$$S = \xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20} \tag{64}$$

$$T = -\lambda\xi\mu[\hat{A}I + \alpha\hat{B}K(1 + b^2)] \tag{65}$$

$$U = -\lambda\xi\mu[\hat{A}J + \alpha\hat{B}L(1 + b^2)] \tag{66}$$

$$\lambda[Mx_2 + N - 2(Px_2 + Q)(Rx_2 + S)]^2 = [\lambda\left(Px_2 + Q\right)^2 + \left(Rx_2 + S\right)^2 + Tx_2 + U]^2 \tag{67}$$

$$V = M - 2PS - 2QR \tag{68}$$

$$W = N - 2\xi PR - 2QS \tag{69}$$

$$Y = 2\lambda VW \tag{70}$$

$$Z = \lambda\xi V^2 + \lambda W^2 \tag{71}$$

$$\hat{E} = 2\lambda PQ + 2RS + T \tag{72}$$

$$\hat{F} = \lambda\xi P^2 + \xi R^2 + \lambda Q^2 + S^2 + U \tag{73}$$

$$Yx_2 + Z = (\hat{E}x_2 + \hat{F})^2 = \xi\hat{E}^2 + \hat{F}^2 + 2\hat{E}\hat{F}x_2 \tag{74}$$

$$\xi(Y - 2\hat{E}\hat{F})^2 = \left(\xi\hat{E}^2 + \hat{F}^2 - Z\right)^2 \tag{75}$$

We have the Equation (75) in $\alpha, x_3, x_4, a, b, c$.

The main question is: is it an identity? Let us express it as a function of $(x_3, x_4) \equiv (x, y, z = x - a, w = y - a)$.

$$\begin{aligned} V &= \xi\mu(\hat{A}K + \hat{B}I) - 2\pi I(\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20}) \\ &\quad - 2\xi\nu\pi\hat{A}K + 2\lambda\xi\nu\rho\kappa_{21}\hat{A} - 2\pi^2 JK + 2\lambda\pi\rho\kappa_{21}J \end{aligned} \tag{76}$$

$$\begin{aligned} W &= \xi\mu(\hat{A}L + \hat{B}J) - 2\xi\pi I(\pi K - \lambda\rho\kappa_{21}) \\ &\quad - 2\xi^2\nu^2\hat{A}\hat{B} - 2\xi\nu\pi\hat{A}L + 2\lambda\xi\nu\rho\kappa_{20}\hat{A} - 2\xi\nu\pi\hat{B}J - 2\pi^2 JL + 2\lambda\pi\rho\kappa_{20}J \end{aligned} \tag{77}$$

Let us substitute from L and above, until η.

$$
\begin{aligned}
\nu^2 &= z^4(w - \alpha y^3 + \bar{k}y^2)^2 \\
&= z^4(w^2 + \alpha^2 y^6 + \bar{k}^2 y^4 - 2\alpha y^3 w + 2\bar{k}y^2 w - 2\alpha\bar{k}y^5)
\end{aligned} \tag{78}
$$

At this moment, we realize that we are next to a polynomial of a single variable, say x. Let us eliminate z.

For all conic section in $x\hat{O}y$, there is a circle in $x\hat{O}z$. The theorem is simplified because the inverse projection of 3 collinear points are 3 collinear points too.

$$V = x(x-a)^4 y w^2(\bar{o}_0 + \bar{o}_1 x + o_2 x^2 + o_3 x^3) \tag{79}$$

$$W = x(x-a)^4(\bar{p}_0 + \bar{p}_1 x + \bar{p}_2 x^2 + \bar{p}_3 x^3) \tag{80}$$

$$Y = 2\lambda x^2(x-a)^8 y w^2(\bar{o}_0 + \bar{o}_1 x + o_2 x^2 + o_3 x^3)(\bar{p}_0 + \bar{p}_1 x + \bar{p}_2 x^2 + \bar{p}_3 x^3) \tag{81}$$

If we did not want to give it all up, we “would” have:

$$
\begin{aligned}
Z &= x^2(x-a)^8[\lambda\xi y^2 w^4(\bar{o}_0 + \bar{o}_1 x + o_2 x^2 + o_3 x^3)^2 + \lambda(\bar{p}_0 + \bar{p}_1 x + \bar{p}_2 x^2 + \bar{p}_3 x^3)^2] \\
&= \cdots
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
S(\nu) &= \xi(z^2 w - \alpha y^3 z^2 + \bar{k}y^2 z^2)(\lambda w^2 + x^2 w^2) + \alpha z^2 w^3(\bar{a}x^2 y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy) \\
&\quad - \lambda(w^3 - \alpha x^2 w^3)(y^2 z^2 + \xi z^2) \qquad (83) \\
&= \cdots \qquad (84)
\end{aligned}
$$

$$Q(\nu) = -2\xi x w^2(z^2 w - \alpha y^3 z^2 + \bar{k} y^2 z^2) + \alpha z^2 w^3(\bar{g} x y^2 + \bar{h} x^2 y) \tag{85}$$

$$\begin{aligned}\hat{E} &= 2\lambda\pi I\,\overbrace{(\xi\nu\hat{A} + \pi J)}^{Q} + 2(\pi K - \lambda\rho\kappa_{21})\,\overbrace{(\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20})}^{S} \\ &\quad - \xi\mu(\lambda\hat{A}I + \hat{B}K)\end{aligned} \tag{86}$$

$$\begin{aligned}\hat{F} &= \lambda\xi\pi^2 I^2 + \xi(\pi K - \lambda\rho\kappa_{21})^2 + \lambda(\xi\nu\hat{A} + \pi J)^2 + (\xi\nu\hat{B} + \pi L - \lambda\rho\kappa_{20})^2 \\ &\quad - \xi\mu(\lambda\hat{A}J + \hat{B}L)\end{aligned} \tag{87}$$

$$L = \bar{a}x^2y^2 + \bar{b}x^2 + \bar{c}y^2 + \bar{d}xy\,;\ \bar{a} = \lambda b^2 + \xi c^2\,;\ \bar{b} = \lambda\xi b^2\,;\ \bar{c} = \lambda\xi c^2\,;\ \bar{d} = -2\lambda\xi bc \tag{88}$$

$$K = \bar{e}xy^2 + \bar{f}x^2y\,;\ \bar{e} = 2\lambda bc\,;\ \bar{f} = -2\lambda b^2 \tag{89}$$

$$J = \bar{g}xy^2 + \bar{h}x^2y\,;\ \bar{g} = -2\xi c^2\,;\ \bar{h} = 2\xi bc \tag{90}$$

$$I = \bar{\iota}x^2y^2\,;\ \bar{\iota} = -2bc \tag{91}$$

$$(Y - 2\hat{E}\hat{F})^2 = \cdots \tag{92}$$

$$\hat{E}^2 = \cdots \tag{93}$$

$$\hat{F}^2 = \cdots \tag{94}$$

$$\kappa_{21} = -yz^2 \tag{95}$$

$$\kappa_{20} = y^2z^2 + \xi z^2 \tag{96}$$

$$\rho = w^3 - \alpha x^2 w^3 \tag{97}$$

$$\pi = \alpha z^2 w^3 \tag{98}$$

$$\nu = z^2 w - \alpha y^3 z^2 + \bar{k} y^2 z^2 \,;\, \bar{k} = \alpha \cdot a \tag{99}$$

$$\mu = \bar{\ell} z^4 w^4 + \bar{m} z^4 w^6 + \bar{n} z^4 w^5 \,;\, \bar{\ell} = 4\alpha - 4\alpha^2 a^2 \,;\, \bar{m} = -4\alpha^2 \,;\, \bar{n} = -8\alpha^2 a \tag{100}$$

$$\hat{B} = \lambda w^2 + x^2 w^2 \tag{101}$$

$$\hat{A} = -2xw^2 \tag{102}$$

$$\bar{o}_0 = 4\lambda\xi w^3(w - \alpha y^3 + \bar{k} y^2) - 2\lambda\alpha w^4 \bar{g} y^2 \tag{103}$$

$$\begin{aligned} \bar{o}_1 = & -2\bar{e}y(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) + \lambda\bar{\imath}y(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) \\ & - 2\lambda\alpha\bar{\imath}yw^2(\xi w - \xi\alpha y^3 + \xi\bar{k}y^2) - 2\alpha^2\bar{\imath}yw^4\bar{c}y^2 + 2\lambda\alpha\bar{\imath}yw^4(y^2 + \xi) \\ & + 4\alpha\xi\bar{e}yw^3(w - \alpha y^3 + \bar{k}y^2) - 2\alpha^2\bar{g}yw^4\bar{e}y^2 - 2\lambda\alpha w^4\bar{h}y \end{aligned} \tag{104}$$

$$\begin{aligned} \bar{o}_2 = & -2\bar{f}(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) - 2\alpha^2\bar{\imath}yw^4\bar{d}y + 4\alpha\xi\bar{f}w^3(w - \alpha y^3 + \bar{k}y^2) \\ & - 4\lambda\xi\alpha w^3(w - \alpha y^3 + \bar{k}y^2) - 2\alpha^2\bar{g}yw^4\bar{f}y - 2\alpha^2\bar{h}w^4\bar{e}y^2 + 2\lambda\alpha^2w^4\bar{g}y^2 \end{aligned} \tag{105}$$

$$\begin{aligned} \bar{o}_3 = & \ \bar{\imath}y(\xi\bar{\ell}w^4 + \xi\bar{m}w^6 + \xi\bar{n}w^5) - 2\alpha\bar{\imath}yw^3(\xi w - \xi\alpha y^3 + \xi\bar{k}y^2) \\ & - 2\alpha^2\bar{\imath}\lambda yw^4(y^2 + \xi) - 2\alpha^2\bar{h}w^4\bar{f}y + 2\lambda\alpha^2w^4\bar{h}y - 2\alpha^2\bar{\imath}yw^4(\bar{a}y^2 + \bar{b}) \end{aligned} \tag{106}$$

$$\begin{aligned} \bar{p}_0 = & -2\xi w^2\bar{c}y^2(\bar{\ell}w^4 + \bar{m}w^6 + \bar{n}w^5) + \xi\lambda w^2\bar{g}y^2(\bar{\ell}w^4 + \bar{m}w^6 + \bar{n}w^5) \\ & + 4\lambda\xi^2w^4(w^2 + \alpha^2y^6 + \bar{k}^2y^4 - 2\alpha y^3w + 2\bar{k}y^2w - 2\alpha y^5\bar{k}) + 4\xi\alpha w^6\bar{c}y^2 \\ & + 4\xi\alpha w^5\bar{k}y^2\bar{c}y^2 + w(-4\lambda\xi w^5y^2 - 4\lambda\xi w^5\xi) - \alpha y^3(-4\lambda\xi w^5y^2 - 4\lambda\xi w^5\xi) \\ & + \bar{k}y^2(-4\lambda\xi w^5y^2 - 4\lambda\xi w^5\xi) - 2\xi\alpha w^3\bar{g}y^2\lambda w^2(w - \alpha y^3 + \bar{k}y^2) \\ & + 2\lambda\alpha w^6(\bar{g}y^4 + \xi\bar{g}y^2) - 2\alpha^2w^6\bar{g}y^2\bar{c}y^2 - 4\xi\alpha w^5\alpha y^3\bar{c}y^2 \end{aligned} \tag{107}$$

$$
\begin{aligned}
\bar{p}_1 &= -2\xi w^2\bar{d}y(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5)+\xi\lambda w^2\bar{h}y(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5)-2\xi\alpha w^3\bar{\imath}\lambda y^3w^3\\
&\quad+4\xi\alpha w^6\bar{d}y-4\xi\alpha w^5\alpha y^3\bar{d}y+4\xi\alpha w^5\bar{k}y^2\bar{d}y-2\lambda\xi\alpha\bar{h}yw^5(w-\alpha y^3+\bar{k}y^2)\\
&\quad-2\alpha^2w^6\bar{g}y^2\bar{d}y-2\alpha^2w^6\bar{h}y\bar{c}y^2+2\lambda\alpha w^6(\bar{h}y^3+\xi\bar{h}y) \qquad (108)\\
\bar{p}_2 &= -2\xi w^2\bar{a}y^2(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5)-2\xi\alpha\bar{\imath}y^2\alpha w^6\bar{e}y^2-2\xi w^2\bar{b}(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5)\\
&\quad+4\xi^2w^4(w^2+\alpha^2y^6+\bar{k}^2y^4-2\alpha y^3w+2\bar{k}y^2w-2\alpha y^5\bar{k})+4\xi\alpha w^6(\bar{a}y^2+\bar{b})\\
&\quad-4\xi\alpha w^5\alpha y^3(\bar{a}y^2+\bar{b})+4\xi\alpha w^5\bar{k}y^2(\bar{a}y^2+\bar{b})+w(4\lambda\xi\alpha w^5y^2+4\lambda\xi\alpha w^5\xi)\\
&\quad-\alpha y^3(4\lambda\xi\alpha w^5y^2+4\lambda\xi\alpha w^5\xi)+\bar{k}y^2(4\lambda\xi\alpha w^5y^2+4\lambda\xi\alpha w^5\xi)\\
&\quad-2\alpha^2z^4w^6\bar{g}y^2(\bar{a}y^2+\bar{b})-2\alpha^2z^4w^6\bar{h}y\bar{d}y-2\lambda\alpha^2w^6(\bar{g}y^4+\xi\bar{g}y^2)\\
&\quad-2\xi\alpha w^3\bar{g}y^2w^2(w-\alpha y^3+\bar{k}y^2)+\xi w^2\bar{g}y^2(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5) \qquad (109)\\
\bar{p}_3 &= \xi w^2\bar{h}y(\bar{\ell}w^4+\bar{m}w^6+\bar{n}w^5)-2\xi\alpha^2\bar{\imath}y^2w^6\bar{f}y-2\xi\alpha w^3\bar{\imath}\lambda y^3\alpha w^3\\
&\quad-2\xi\alpha\bar{h}yw^5(w-\alpha y^3+\bar{k}y^2)-2\alpha^2w^6\bar{h}y(\bar{a}y^2+\bar{b})-2\lambda\alpha^2w^6(\bar{h}y^3+\xi\bar{h}y) \qquad (110)
\end{aligned}
$$

Below, let us explain the Equation (20), expliciting C on Equation (3).

$$
\begin{pmatrix} x_2 & 1\\ x_4 & 1\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix}=\begin{pmatrix} y_2\\ y_4\end{pmatrix}\Rightarrow\begin{pmatrix} a\\ b\end{pmatrix}=\frac{1}{x_2-x_4}\begin{pmatrix} 1 & -1\\ -x_4 & x_2\end{pmatrix}\begin{pmatrix} y_2\\ y_4\end{pmatrix} \qquad (111)
$$

$$
\begin{pmatrix} x_3 & 1\\ x_5 & 1\end{pmatrix}\begin{pmatrix} c\\ d\end{pmatrix}=\begin{pmatrix} y_3\\ y_5\end{pmatrix}\Rightarrow\begin{pmatrix} c\\ d\end{pmatrix}=\frac{1}{x_3-x_5}\begin{pmatrix} 1 & -1\\ -x_5 & x_3\end{pmatrix}\begin{pmatrix} y_3\\ y_5\end{pmatrix} \qquad (112)
$$

$$
ax+b=cx+d\Rightarrow x=\frac{d-b}{a-c}=0\therefore d=b \qquad (113)
$$

2 The Converse Conic Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{114}$$

$$B = X_1X_4 \cap X_3X_6 \tag{115}$$

$$C = X_2X_4 \cap X_3X_5 \in AB \tag{116}$$

$$A = (0,0) \tag{117}$$

$$B = (0,a) \tag{118}$$

$$X_1 = (x_1, bx_1) \tag{119}$$

$$X_5 = (x_5, bx_5) \tag{120}$$

$$X_2 = (x_2, cx_2) \tag{121}$$

$$X_6 = (x_6, cx_6) \tag{122}$$

$$X_4 = (x_4, d(x_4 - a) \tag{123}$$

$$bx_1 = d(x_1 - a) \tag{124}$$

$$X_3 = e(x_3 - a) \tag{125}$$

$$cx_6 = e(x_6 - a) \tag{126}$$

$$\frac{-y_3x_5 + y_5x_3}{x_3 - x_5} = \frac{-y_2x_4 + y_4x_2}{x_2 - x_4} \tag{127}$$

Therefore, $\exists\lambda \in \mathbb{R}^6$, such that:

$$\begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & y_6^2 & x_6y_6 & x_6 & y_6 & 1 \end{pmatrix} \cdot \lambda = 0 \tag{128}$$

3 From Circle to Parabola

$$S' : x'^2 + z'^2 = R^2, y = 0 \tag{129}$$

$$v = (\mathbb{R}, 0, R) \tag{130}$$

$$P = (0, -p, R) \tag{131}$$

$$A' = (x', 0, z') \in x\hat{O}z \tag{132}$$

$$A'P : (x, y, z) = (x' + \lambda x', \lambda p, z' + \lambda(z' - R) \tag{133}$$

$$B = \pi(A')\,;\, \pi : x\hat{O}z \to x\hat{O}y \tag{134}$$

$$= A'P \cap x\hat{O}y = (x_b, y_b, z_b = 0) \tag{135}$$

$$x_b = \frac{Rx'}{R - z'};\, y_b = \frac{pz'}{R - z'} \tag{136}$$

$$(R - z')y_b = pz' \Rightarrow z' = \frac{Ry_b}{p + y_b} \tag{137}$$

$$x' = \frac{R - z'}{R} \cdot x_b = \frac{px_b}{p + y_b} \tag{138}$$

$$\pi(S') : p^2 x_b^2 + \cancel{R^2 y_b^2} = R^2(p + y_b)^2 = p^2 R^2 + 2pR^2 y_b + \cancel{R^2 y_b^2}\,;\, z_b = 0 \tag{139}$$

$$\therefore y_b = \frac{px_b^2 - pR^2}{2R^2} \tag{140}$$

That is a parabola that intercepts $\hat{O}y$ at $V = \left(0, -\frac{p}{2}, 0\right)$, and intercepts $\hat{O}x$ at $(\pm R, 0, 0)$.

3.1 From Circle to Hiperbola

$$S'' : x'^2 + z'^2 = (R+q)^2, y = 0 \tag{141}$$

$$\pi(S'') : p^2 x_b^2 + R^2 y_b^2 = (R+q)^2 (p^2 + 2py_b + y_b^2)\,;\; z_b = 0 \tag{142}$$

$$p^2 x_b^2 = (2Rq + q^2) y_b^2 + 2p(R+q)^2 y_b + p^2 (R+q)^2 \tag{143}$$

$$p^2 (R+q)^4 - p^2 q (R+q)^2 (2R+q) = \left[(2R+q)qy_b + p(R+q)^2\right]^2 - (2R+q)p^2 q x_b^2 \tag{144}$$

$$\frac{1}{A^2}\left[y_b + \frac{p(R+q)^2}{q(2R+q)}\right]^2 - \frac{x_b^2}{B^2} = 1\,;\; A = \frac{pR(R+q)}{q(2R+q)}\,;\; B = \frac{R(R+q)}{\sqrt{q(2R+q)}} \tag{145}$$

Whenever $x_b = 0$, we have $y_b + C(q) = \pm A$. Therefore, that's a vertical hyperbola.

We want to prove that while $q \to \infty$, the projection is a degenerated hyperbola.

$$\frac{(y+c)^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow y = -c \pm a\sqrt{1 + \frac{x^2}{b^2}} \Rightarrow y' = \pm\frac{ax}{\sqrt{1+\frac{x^2}{b^2}}} \xrightarrow{x\to\infty} \pm\frac{a}{b} \tag{146}$$

$$y = \pm\frac{ax}{b} - c \because y(0) = -c \tag{147}$$

$$\frac{A}{B} = \frac{p}{\sqrt{q(2R+q)}} \xrightarrow{q\to\infty} 0 \tag{148}$$

3.2 From Circle to Ellipsis

$$S''' : x'^2 + z'^2 = (R-q)^2, y = 0 \tag{149}$$

$$\pi(S''') : \frac{1}{A^2}\left[y_b - \frac{p(R-q)^2}{q(2R-q)}\right]^2 + \frac{x_b^2}{B^2} = 1\,;\; A = \frac{pR(R-q)}{q(2R-q)}\,;\; B = \frac{R(R-q)}{\sqrt{q(2R-q)}} \tag{150}$$

Whenever $y_b + C(-q) = 0$, we have $x_b = \pm B$.

3.3 Invariant Straight Lines

$$t' : Ax' + Bz' = C \tag{151}$$

$$\pi(t') : Apx_b + BRy_b = C(p + y_b), \text{ which is a straight line.} \tag{152}$$

$$r' : x' = 0\,;\ s' : z' = Ax' \tag{153}$$

$$\pi(r') : x_b = 0\,;\ \pi(s') : Ry_b = Apx_b \tag{154}$$

$$\therefore 0 = r' \cap s' \Rightarrow 0 = \pi(r') \cap \pi(s') \tag{155}$$

3.4 Parallelism

$$s'' : z' = Ax' + R \tag{156}$$

$$\pi(s'') : \cancel{Ry_b} = Apx_b + R(p + \cancel{y_b}) \Rightarrow x_b = -\frac{R}{A} \tag{157}$$

$$\therefore r' \cap s'' = (0, R) \in v \Rightarrow \pi(r') /\!\!/ \pi(s'') \tag{158}$$

4 Degenerated Section

A degenerated conic section are 2 straight lines, but we do not want 2 distinct proofs.

How do we merge a theorem T_1 on a circle and a theorem T_2 about 2 straight lines?

We want to prove that: $T(S) \Leftrightarrow$ "three points are collinear" holds in a conic section if and only if $T(r, s) \Leftrightarrow$ "three points are collinear" holds degeneratedly in 2 straight lines too.

Our way is to intercept a cone $K : z^2 = a^2(x^2 + y^2)$ by a plane.

$L = K \cap \pi_R : y = 0$, for two straight lines $\Rightarrow z = \pm ax$;

$H = K \cap \pi_H : y = c$, for a hyperbola;

$E = K \cap \pi_E : z = bx + c$; $b < b_E$ for an ellipsis;

$P = K \cap \pi_P : z = bx + c$; $b_E < b < b_H$ for a parabola. Here, it suffices that π_P has a slope greater than for an ellipsis and less than for an hyperbola.

■

Out of charity, there is no salvation at all. With charity, we evolve.

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