Universidade Federal de Minas Gerais

Projective Geometry Again

TEAM:

Vinicius Claudino Ferraz PPGEE-UFMG.

1 The ConicSection Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{1}$$
$$B = X_1X_4 \cap X_3X_6 \tag{2}$$
$$C = X_2X_4 \cap X_3X_5 \in AB \tag{3}$$
$$X = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t_1 \begin{pmatrix} x_5 - x_1 \\ y_5 - y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + t_2 \begin{pmatrix} x_6 - x_2 \\ y_6 - y_2 \end{pmatrix} \tag{4}$$
$$\begin{pmatrix} x_5 - x_1 & x_2 - x_6 \\ y_5 - y_1 & y_2 - y_6 \end{pmatrix} \cdot \begin{pmatrix} t_{1A} \\ t_{2A} \end{pmatrix} = X_2 - X_1 \Rightarrow t_A = M_{15}^{26}(X_2 - X_1) \tag{5}$$
$$M_{15}^{26} = \frac{1}{(x_5 - x_1)(y_2 - y_6) - (x_2 - x_6)(y_5 - y_1)} \begin{pmatrix} y_2 - y_6 & -x_2 + x_6 \\ -y_5 + y_1 & x_5 - x_1 \end{pmatrix} \tag{6}$$
$$t_B = M_{14}^{36}(X_3 - X_1) \tag{7}$$
$$t_C = M_{24}^{35}(X_3 - X_2) \tag{8}$$
$$C = A + \lambda(B - A) = X_2 + t_{1C}(X_4 - X_2) \tag{9}$$
$$\lambda = \frac{x_C - x_A}{x_B - x_A} = \frac{y_C - y_A}{y_B - y_A} \tag{10}$$

Therefore:

$$\frac{x_2 + t_{1C}(x_4 - x_2) - M_{15}^{36}(x_2 - x_1)}{M_{14}^{36}(x_3 - x_1) - M_{15}^{36}(x_2 - x_1)} = \frac{y_2 + t_{2C}(y_4 - y_2) - M_{15}^{36}(y_2 - y_1)}{M_{14}^{36}(y_3 - y_1) - M_{15}^{36}(y_2 - y_1)} \tag{11}$$

For all conic section in $x\hat{O}y$, there is a circle in $x\hat{O}z$. The theorem is simplified because the inverse projection of 3 collinear points are 3 collinear points too.

1.1 Parallel Straight Lines

$$y_1 = \epsilon_1 a \tag{12}$$
$$y_2 = \epsilon_2 a \tag{13}$$
$$y_3 = \epsilon_3 a \tag{14}$$
$$y_4 = \epsilon_4 a \tag{15}$$
$$y_5 = \epsilon_5 a \tag{16}$$
$$y_6 = \epsilon_6 a \tag{17}$$

Below, we try to reduce that to $0x = 0$.

$$\frac{x_2 + t_{1C}(x_4 - x_2) - M_{15}^{36}(x_2 - x_1)}{M_{14}^{36}(x_3 - x_1) - M_{15}^{36}(x_2 - x_1)} = \frac{\epsilon_2 a + t_{2C}(\epsilon_4 a - \epsilon_2 a) - M_{15}^{36}(\epsilon_2 a - \epsilon_1 a)}{M_{14}^{36}(\epsilon_3 a - \epsilon_1 a) - M_{15}^{36}(\epsilon_2 a - \epsilon_1 a)} \tag{18}$$

1.2 Concurrent Straight Lines

$$y_1 = \epsilon_1 a x_1 \tag{19}$$
$$y_2 = \epsilon_2 a x_2 \tag{20}$$
$$y_3 = \epsilon_3 a x_3 \tag{21}$$
$$y_4 = \epsilon_4 a x_4 \tag{22}$$
$$y_5 = \epsilon_5 a x_5 \tag{23}$$
$$y_6 = \epsilon_6 a x_6 \tag{24}$$

Below, we try to reduce that to $0x = 0$.

$$\frac{x_2 + t_{1C}(x_4 - x_2) - M_{15}^{36}(x_2 - x_1)}{M_{14}^{36}(x_3 - x_1) - M_{15}^{36}(x_2 - x_1)} = \frac{\epsilon_2 a x_2 + t_{2C}(\epsilon_4 a x_4 - \epsilon_2 a x_2) - M_{15}^{36}(\epsilon_2 a x_2 - \epsilon_1 a x_1)}{M_{14}^{36}(\epsilon_3 a x_3 - \epsilon_1 a x_1) - M_{15}^{36}(\epsilon_2 a x_2 - \epsilon_1 a x_1)} \tag{25}$$

1.3 Circle

$$X_1 = (\cos t_1, \sin t_1) \tag{26}$$
$$X_2 = (\cos t_2, \sin t_2) \tag{27}$$
$$X_3 = (\cos t_3, \sin t_3) \tag{28}$$
$$X_4 = (\cos t_4, \sin t_4) \tag{29}$$
$$X_5 = (\cos t_5, \sin t_5) \tag{30}$$
$$X_6 = (\cos t_6, \sin t_6) \tag{31}$$

Below, we try to reduce that to $0x = 0$.

$$\frac{\cos t_2 + t_{1C}(\cos t_4 - \cos t_2) - M_{15}^{36}(\cos t_2 - \cos t_1)}{M_{14}^{36}(\cos t_3 - \cos t_1) - M_{15}^{36}(\cos t_2 - \cos t_1)} = \frac{\sin t_2 + t_{2C}(\sin t_4 - \sin t_2) - M_{15}^{36}(\sin t_2 - \sin t_1)}{M_{14}^{36}(\sin t_3 - \sin t_1) - M_{15}^{36}(\sin t_2 - \sin t_1)} \tag{32}$$

2 The Converse ConicSection Theorem

$$A = X_1X_5 \cap X_2X_6 \tag{33}$$
$$B = X_1X_4 \cap X_3X_6 \tag{34}$$
$$C = X_2X_4 \cap X_3X_5 \in AB \Leftrightarrow (11) \tag{35}$$

Therefore, $\exists \lambda \in \mathbb{R}^6 - \{0\}$, such that:

$$\begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & y_6^2 & x_6y_6 & x_6 & y_6 & 1 \end{pmatrix} \cdot \lambda = 0 \tag{36}$$

We want to show that:

$$0 = \begin{vmatrix} x_2^2 - x_1^2 & y_2^2 - y_1^2 & x_2y_2 - x_1y_1 & x_2 - x_1 & y_2 - y_1 \\ x_3^2 - x_1^2 & y_3^2 - y_1^2 & x_3y_3 - x_1y_1 & x_3 - x_1 & y_3 - y_1 \\ x_4^2 - x_1^2 & y_4^2 - y_1^2 & x_4y_4 - x_1y_1 & x_4 - x_1 & y_4 - y_1 \\ x_5^2 - x_1^2 & y_5^2 - y_1^2 & x_5y_5 - x_1y_1 & x_5 - x_1 & y_5 - y_1 \\ x_6^2 - x_1^2 & y_6^2 - y_1^2 & x_6y_6 - x_1y_1 & x_6 - x_1 & y_6 - y_1 \end{vmatrix} \tag{37}$$

$$\begin{aligned} 0 = (y_2 - y_1) &\begin{vmatrix} x_3^2 - x_1^2 & y_3^2 - y_1^2 & x_3y_3 - x_1y_1 & x_3 - x_1 \\ x_4^2 - x_1^2 & y_4^2 - y_1^2 & x_4y_4 - x_1y_1 & x_4 - x_1 \\ x_5^2 - x_1^2 & y_5^2 - y_1^2 & x_5y_5 - x_1y_1 & x_5 - x_1 \\ x_6^2 - x_1^2 & y_6^2 - y_1^2 & x_6y_6 - x_1y_1 & x_6 - x_1 \end{vmatrix} - (y_3 - y_1) \begin{vmatrix} x_2^2 - x_1^2 & y_2^2 - y_1^2 & x_2y_2 - x_1y_1 & x_2 - x_1 \\ x_4^2 - x_1^2 & y_4^2 - y_1^2 & x_4y_4 - x_1y_1 & x_4 - x_1 \\ x_5^2 - x_1^2 & y_5^2 - y_1^2 & x_5y_5 - x_1y_1 & x_5 - x_1 \\ x_6^2 - x_1^2 & y_6^2 - y_1^2 & x_6y_6 - x_1y_1 & x_6 - x_1 \end{vmatrix} \\ + (y_4 - y_1) &\begin{vmatrix} x_2^2 - x_1^2 & y_2^2 - y_1^2 & x_2y_2 - x_1y_1 & x_2 - x_1 \\ x_3^2 - x_1^2 & y_3^2 - y_1^2 & x_3y_3 - x_1y_1 & x_3 - x_1 \\ x_5^2 - x_1^2 & y_5^2 - y_1^2 & x_5y_5 - x_1y_1 & x_5 - x_1 \\ x_6^2 - x_1^2 & y_6^2 - y_1^2 & x_6y_6 - x_1y_1 & x_6 - x_1 \end{vmatrix} - (y_5 - y_1) \begin{vmatrix} x_2^2 - x_1^2 & y_2^2 - y_1^2 & x_2y_2 - x_1y_1 & x_2 - x_1 \\ x_3^2 - x_1^2 & y_3^2 - y_1^2 & x_3y_3 - x_1y_1 & x_3 - x_1 \\ x_4^2 - x_1^2 & y_4^2 - y_1^2 & x_4y_4 - x_1y_1 & x_4 - x_1 \\ x_6^2 - x_1^2 & y_6^2 - y_1^2 & x_6y_6 - x_1y_1 & x_6 - x_1 \end{vmatrix} \\ + (y_6 - y_1) &\begin{vmatrix} x_2^2 - x_1^2 & y_2^2 - y_1^2 & x_2y_2 - x_1y_1 & x_2 - x_1 \\ x_3^2 - x_1^2 & y_3^2 - y_1^2 & x_3y_3 - x_1y_1 & x_3 - x_1 \\ x_4^2 - x_1^2 & y_4^2 - y_1^2 & x_4y_4 - x_1y_1 & x_4 - x_1 \\ x_5^2 - x_1^2 & y_5^2 - y_1^2 & x_5y_5 - x_1y_1 & x_5 - x_1 \end{vmatrix} \end{aligned} \tag{38}$$

2.1 Beyond ConicSections

More generally,

$$A = X_1X_5 \cap X_2X_6 \tag{39}$$
$$B = X_1X_4 \cap X_3X_6 \tag{40}$$
$$C = X_2X_4 \cap X_3X_5 \in AB \Leftrightarrow (11) \Rightarrow \forall f : \mathbb{R}^2 \to \mathbb{R}, \tag{41}$$
$$f(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j \Rightarrow \begin{cases} f(x_1, y_1) = 0 \\ f(x_2, y_2) = 0 \\ f(x_3, y_3) = 0 \\ f(x_4, y_4) = 0 \\ f(x_5, y_5) = 0 \\ f(x_6, y_6) = 0 \end{cases} \tag{42}$$

(43)

3 From Circle to Parabola

$$S' : x'^2 + z'^2 = R^2, y = 0 \tag{44}$$
$$v = (\mathbb{R}, 0, R) \tag{45}$$
$$P = (0, -p, R) \tag{46}$$
$$A' = (x', 0, z') \in x\hat{O}z \tag{47}$$
$$A'P : (x, y, z) = (x' + \lambda x', \lambda p, z' + \lambda(z' - R)) \tag{48}$$
$$B = \pi(A')\,;\; \pi : x\hat{O}z \to x\hat{O}y \tag{49}$$
$$= A'P \cap x\hat{O}y = (x_b, y_b, z_b = 0) \tag{50}$$
$$x_b = \frac{Rx'}{R - z'}\,;\; y_b = \frac{pz'}{R - z'} \tag{51}$$
$$(R - z')y_b = pz' \Rightarrow z' = \frac{Ry_b}{p + y_b} \tag{52}$$
$$x' = \frac{R - z'}{R} \cdot x_b = \frac{px_b}{p + y_b} \tag{53}$$
$$\pi(S') : p^2 x_b^2 + \cancel{R^2 y_b^2} = R^2(p + y_b)^2 = p^2 R^2 + 2pR^2 y_b + \cancel{R^2 y_b^2}\,;\; z_b = 0 \tag{54}$$
$$\therefore y_b = \frac{px_b^2 - pR^2}{2R^2} \tag{55}$$

That is a parabola that intercepts $\hat{O}y$ at $V = \left(0, -\frac{p}{2}, 0\right)$, and intercepts $\hat{O}x$ at $(\pm R, 0, 0)$.

3.1 From Circle to Hiperbola

$$S'' : x'^2 + z'^2 = (R + q)^2, y = 0 \tag{56}$$
$$\pi(S'') : p^2 x_b^2 + R^2 y_b^2 = (R + q)^2(p^2 + 2py_b + y_b^2)\,;\; z_b = 0 \tag{57}$$
$$p^2 x_b^2 = (2Rq + q^2)y_b^2 + 2p(R + q)^2 y_b + p^2(R + q)^2 \tag{58}$$
$$p^2(R + q)^4 - p^2 q(R + q)^2(2R + q) = \left[(2R + q)qy_b + p(R + q)^2\right]^2 - (2R + q)p^2 q x_b^2 \tag{59}$$
$$\frac{1}{A^2}\left[y_b + \frac{p(R + q)^2}{q(2R + q)}\right]^2 - \frac{x_b^2}{B^2} = 1\,;\; A = \frac{pR(R + q)}{q(2R + q)}\,;\; B = \frac{R(R + q)}{\sqrt{q(2R + q)}} \tag{60}$$

Whenever $x_b = 0$, we have $y_b + C(q) = \pm A$. Therefore, that's a vertical hyperbola. We want to prove that while $q \to \infty$, the projection is a degenerated hyperbola.

$$\frac{(y+c)^2}{a^2}-\frac{x^2}{b^2}=1 \Rightarrow y=-c\pm a\sqrt{1+\frac{x^2}{b^2}} \Rightarrow y'=\pm\frac{ax}{\sqrt{1+\frac{x^2}{b^2}}} \xrightarrow{x\to\infty} \pm\frac{a}{b} \tag{61}$$

$$y=\pm\frac{ax}{b}+y_0\,;\;y_0=-c\pm\cancel{a} \tag{62}$$

$$\frac{A}{B}=\frac{p}{\sqrt{q(2R+q)}} \xrightarrow{q\to\infty} 0 \tag{63}$$

$$-C\pm A=-\frac{p(R+q)^2}{q(2R+q)}\pm\frac{pR(R+q)}{q(2R+q)} \xrightarrow{q\to\infty} -p \tag{64}$$

3.2 From Circle to Ellipsis

$$S''' : x'^2+z'^2=(R-q)^2, y=0 \tag{65}$$

$$\pi(S''') : \frac{1}{A^2}\left[y_b-\frac{p(R-q)^2}{q(2R-q)}\right]^2+\frac{x_b^2}{B^2}=1\,;\;A=\frac{pR(R-q)}{q(2R-q)}\,;\;B=\frac{R(R-q)}{\sqrt{q(2R-q)}} \tag{66}$$

Whenever $y_b+C(-q)=0$, we have $x_b=\pm B$.

3.3 Invariant Straight Lines

$$t' : Ax'+Bz'=C \tag{67}$$

$$\pi(t') : Apx_b+BRy_b=C(p+y_b), \text{ which is a straight line.} \tag{68}$$

$$r' : x'=0\,;\;s' : z'=Ax' \tag{69}$$

$$\pi(r') : x_b=0\,;\;\pi(s') : Ry_b=Apx_b \tag{70}$$

$$\therefore 0=r'\cap s' \Rightarrow 0=\pi(r')\cap\pi(s') \tag{71}$$

3.4 Parallelism

$$s'' : z'=Ax'+R \tag{72}$$

$$\pi(s'') : \cancel{Ry_b}=Apx_b+R(p+\cancel{y_b}) \Rightarrow x_b=-\frac{R}{A} \tag{73}$$

$$\therefore r'\cap s''=(0,R)\in v \Rightarrow \pi(r')/\!\!/\pi(s'') \tag{74}$$

4 Degenerated Section

A degenerated conicSection are 2 straight lines, but we do not want 2 distinct proofs. How do we merge a theorem T_1 on a circle and a theorem T_2 about 2 straight lines?

We want to prove that: $T(S) \Leftrightarrow$ "three points are collinear" holds in a conicSection if and only if $T(r,s) \Leftrightarrow$ "three points are collinear" holds degeneratedly in 2 straight lines too.

Our way is to intercept a cone $K : z^2 = a^2(x^2+y^2)$ by a plane.

$L = K \cap \pi_R : y = 0$, for two straight lines $\Rightarrow z = \pm ax$;

$H = K \cap \pi_H : y = c$, for a hyperbola;

$E = K \cap \pi_E : z = bx + c$; $b < b_E$ for an ellipsis;

$P = K \cap \pi_P : z = bx + c$; $b_E < b < b_H$ for a parabola. Here, it suffices that π_P has a slope greater than for an ellipsis and less than for an hyperbola.

5 Higher Dimensions

$$A = X_1X_5 \cap X_2X_6X_7 \tag{75}$$
$$B = X_1X_4 \cap X_3X_6X_7 \tag{76}$$
$$C = X_2X_4 \cap X_3X_5X_7 \tag{77}$$
$$D = X_4X_7 \cap X_2X_3X_5 \in \langle ABC\rangle \tag{78}$$
$$X = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t_1 \begin{pmatrix} x_5 - x_1 \\ y_5 - y_1 \\ z_5 - z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + t_2 \begin{pmatrix} x_6 - x_2 \\ y_6 - y_2 \\ z_6 - z_2 \end{pmatrix} + t_3 \begin{pmatrix} x_7 - x_2 \\ y_7 - y_2 \\ z_7 - z_2 \end{pmatrix} \tag{79}$$
$$\begin{pmatrix} x_5 - x_1 & x_2 - x_6 & x_2 - x_7 \\ y_5 - y_1 & y_2 - y_6 & y_2 - y_7 \\ z_5 - z_1 & z_2 - z_6 & z_2 - z_7 \end{pmatrix} \cdot \begin{pmatrix} t_{1A} \\ t_{2A} \\ t_{3A} \end{pmatrix} = X_2 - X_1 \Rightarrow t_A = M_{15}^{267}(X_2 - X_1) \tag{80}$$
$$t_B = M_{14}^{367}(X_3 - X_1) \tag{81}$$
$$t_C = M_{24}^{357}(X_3 - X_2) \tag{82}$$
$$t_D = M_{47}^{235}(X_2 - X_4) \tag{83}$$
$$D = A + \lambda(B - A) + \mu(C - A) \tag{84}$$
$$\begin{pmatrix} x_B - x_A & x_C - x_A \\ y_B - y_A & y_C - y_A \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} x_D \\ y_D \end{pmatrix} \tag{85}$$
$$(z_B - z_A)\lambda + (z_C - z_A)\mu = z_D \tag{86}$$
$$0 = \langle (x^2, y^2, z^2, xy, xz, yz, x, y, z, 1), (\lambda_1, \cdots, \lambda_{10})\rangle \tag{87}$$

A conicSection in $\mathbb{R}^n$ is expressed as :

$$\sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} x_i x_j + \sum_{i=1}^{n} c_i x_i + d = 0 \tag{88}$$
$$\dim \lambda = n + \binom{n}{2} + n + 1 = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2} \tag{89}$$

Generalizing, as 2-dimensioned planes intersect at 1-dimensioned planes in $\mathbb{R}^3$, i.e., $ABC \cap ABD = AB$, also:

$$\langle X_1, \cdots, X_n\rangle \cap \langle X_1, \cdots, X_{n-1}, \widehat{X_n}, X_{n+1}\rangle = \langle X_1, \cdots, X_{n-1}\rangle \tag{90}$$

A third degree section in $\mathbb{R}^n$ is expressed as:

$$\sum_{i=1}^{n} a_i x_i^3 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} x_i^2 x_j + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} x_i x_j^2 + \sum_{i=1}^{n} d_i x_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} x_i x_j + \sum_{i=1}^{n} e_i x_i + f = 0 \quad (91)$$

$$\dim \lambda = 1 + 3n + 3 \binom{n}{2} = \frac{3n^2 + 3n + 2}{2} \quad (92)$$

A k-th degree section has:

$$\dim \lambda = 1 + nk + \binom{n}{2} \sum_{k=1}^{n-1} k = 1 + nk + \frac{n^2(n-1)^2}{2} \quad (93)$$

Let us distinguish between even and odd dimensions. In $\mathbb{R}^4$, $\langle e_1, e_2 \rangle \cap \langle e_3, e_4 \rangle = 0$. In $\mathbb{R}^5$, $\langle e_1, e_2, e_3 \rangle \cap \langle e_3, e_4, e_5 \rangle = \langle e_3 \rangle$. So, we define the least by $\ell = \left\lceil \frac{n}{2} \right\rceil$.

$\blacksquare$

Out of charity, there is no salvation at all. With charity, we evolve.
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