Quarta prova de Análise no $\mathbb{R}^n$

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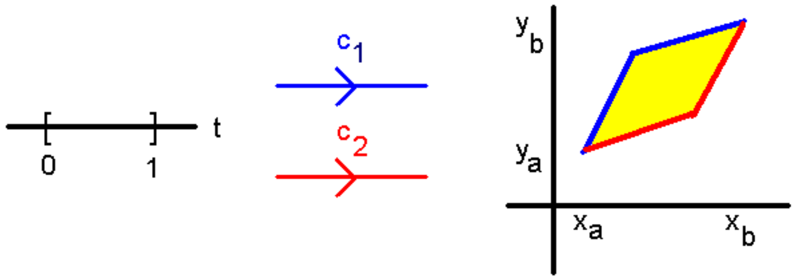
1 Questão 1

1.1 Questão 1.a



A figura acima aproxima as imagens por poligonais, mas sem perda de generalidade suponha aplicações de classe C^∞. Seja ω uma 1-forma em $\mathbb{R}^2$.

$$\omega = u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y \tag{1}$$
$$u, v : \mathbb{R}^2 \to \mathbb{R} \tag{2}$$
$$c_i : [0,1] \to A \times B \subset \mathbb{R}^2, \forall i \in \{1,2\} \tag{3}$$
$$c_i(0) = (x_a, y_a) \tag{4}$$
$$c_i(1) = (x_b, y_b) \tag{5}$$
$$c_i(t) = (x_i(t), y_i(t)) \tag{6}$$
$$\forall c_1, \forall c_2, \text{ temos } \int_{c_1} \omega = \int_{c_2} \omega \tag{7}$$

Queremos mostrar que ω é exata, isto é, seja θ uma 0-forma em $\mathbb{R}^2$.

$$\theta = f(x,y) \tag{8}$$
$$f : \mathbb{R}^2 \to \mathbb{R} \tag{9}$$
$$\omega = \mathrm{d}\theta = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \tag{10}$$

Ou seja, é necessário que exista f tal que

$$u = \frac{\partial f}{\partial x}, v = \frac{\partial f}{\partial y} \tag{11}$$
$$\text{Mas (7)} \Rightarrow \int_{c_1} \omega - \int_{c_2} \omega = 0 \tag{12}$$
$$c_1 - c_2 = \partial R, \text{ região amarela da figura} \tag{13}$$
$$\oint_{\partial R} \omega = 0 \tag{14}$$

Pelo teorema de Stokes,

$$\int_R \mathrm{d}\omega = 0 \tag{15}$$

$$\int_R \left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \mathrm{d}x \wedge \mathrm{d}y = 0 \tag{16}$$

Qual é a função cuja integral se anula para qualquer região R? A função identicamente nula.

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{17}$$

Utilizamos os problemas[1] 3.34 e 2.21 (a) para construir f, com $g_1 = u, g_2 = v$:

$$\mathrm{D}_1 g_2 = \mathrm{D}_2 g_1 \tag{18}$$

$$f(x,y) = \int_0^x g_1(t,0)\,\mathrm{d}t + \int_0^y g_2(x,t)\,\mathrm{d}t \tag{19}$$

$$\Leftrightarrow f(x,y) = \int_0^x u(t,0)\,\mathrm{d}t + \int_0^y v(x,t)\,\mathrm{d}t \tag{20}$$

$$\mathrm{D}_1 f = g_1, \mathrm{D}_2 f = g_2 \Leftrightarrow (11), \tag{21}$$

como queríamos demonstrar.

1.2 Questão 1.b

Toda forma exata é fechada.

$$\omega = \mathrm{d}\theta \Rightarrow \mathrm{d}\omega = \mathrm{d}(\mathrm{d}\theta) = \mathrm{d}^2\theta = 0 \tag{22}$$

Entretanto, a recíproca não é verdadeira. Construímos o contra-exemplo abaixo.

$$\omega : \mathbb{R}^2 - 0 \to \Lambda^1(\mathbb{R}^2) \tag{23}$$

$$\omega = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \tag{24}$$

$$A = \{(x,y) \in \mathbb{R}^2; x < 0\} \cup \{(x,y) \in \mathbb{R}^2; x \geq 0, y \neq 0\} \subset \mathbb{R}^2 \tag{25}$$

$$\theta : A \to (0, 2\pi) \subset \mathbb{R} \tag{26}$$

$$\theta(x,y) = \begin{cases} \arctan\frac{y}{x}, x > 0, y > 0 \\ \pi + \arctan\frac{y}{x}, x < 0 \\ 2\pi + \arctan\frac{y}{x}, x > 0, y < 0 \\ \frac{\pi}{2}, x = 0, y > 0 \\ \frac{3\pi}{2}, x = 0, y < 0 \end{cases} \tag{27}$$

$$\omega = \mathrm{d}\theta \,\text{em}\, A \tag{28}$$

$$\text{Suponha a extensão } \omega = \mathrm{d}f, f : \mathbb{R}^2 - \{0\} \to \mathbb{R} \tag{29}$$

$$\text{Então } \frac{\partial f}{\partial x} = \frac{\partial \theta}{\partial x}, \frac{\partial f}{\partial y} = \frac{\partial \theta}{\partial y} \tag{30}$$

$$f = \theta + k, k \in \mathbb{R} \text{ Quod Erat Absurdum} \tag{31}$$

2 Questão 2

$$f(t) = c_1^A(t) = \begin{bmatrix} f_1(t) = 1 + \cos(2\pi t) \\ f_2(t) = \sin(2\pi t) \end{bmatrix} \tag{32}$$

$$g(t) = c_\pi^A(t) = \begin{bmatrix} g_1(t) = 1 + \pi\cos(2\pi t) \\ g_2(t) = \pi\sin(2\pi t) \end{bmatrix} \tag{33}$$

$$f, g : [0,1] \subset \mathbb{R} \to \mathbb{R}^2 \tag{34}$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{35}$$

$$F_{11}(x,y) = -\frac{y}{(x-1)^2 + y^2} \tag{36}$$

$$F_{12}(x,y) = +\frac{x-1}{(x-1)^2 + y^2} \tag{37}$$

$$F_{21}(x,y) = -\frac{y}{(x+1)^2 + y^2} \tag{38}$$

$$F_{22}(x,y) = +\frac{x+1}{(x+1)^2 + y^2} \tag{39}$$

$$F_{ij} : \mathbb{R}^2 \to \mathbb{R} \tag{40}$$

$$\omega_A = F_{11}\,\mathrm{d}x + F_{12}\,\mathrm{d}y \tag{41}$$

$$\omega_B = F_{21}\,\mathrm{d}x + F_{22}\,\mathrm{d}y \tag{42}$$

$$\omega = \omega_A - \omega_B \tag{43}$$

$$\int_f \omega = \int_f \omega_A - \int_f \omega_B = \int_f F_{11}\,\mathrm{d}x + \int_f F_{12}\,\mathrm{d}y - \int_f F_{21}\,\mathrm{d}x - \int_f F_{22}\,\mathrm{d}y \tag{44}$$

$$\int_g \omega = \int_g \omega_A - \int_g \omega_B = \int_g F_{11}\,\mathrm{d}x + \int_g F_{12}\,\mathrm{d}y - \int_g F_{21}\,\mathrm{d}x - \int_g F_{22}\,\mathrm{d}y \tag{45}$$

$$\int_f F_{11}\,\mathrm{d}x = \int_0^1 f^*(F_{11}\,\mathrm{d}x) = \int_0^1 (F_{11} \circ f) f^*(\mathrm{d}x) = \int_0^1 (F_{11} \circ f)\frac{\partial f_1}{\partial t}\,\mathrm{d}t = \tag{46}$$

$$= \int_0^1 F_{11}\begin{pmatrix} 1 + \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}(-\sin(2\pi t))2\pi\,\mathrm{d}t \tag{47}$$

$$= +2\pi \int_0^1 \frac{\sin(2\pi t)}{\cos^2(2\pi t) + \sin^2(2\pi t)} \sin(2\pi t)\,\mathrm{d}t \tag{48}$$

$$= 2\pi \int_0^1 \sin^2(2\pi t)\,\mathrm{d}t \tag{49}$$

$$\int_f F_{12}\,\mathrm{d}y = \int_0^1 f^*(F_{12}\,\mathrm{d}y) = \int_0^1 (F_{12} \circ f) f^*(\mathrm{d}y) = \int_0^1 (F_{12} \circ f)\frac{\partial f_2}{\partial t}\,\mathrm{d}t = \tag{50}$$

$$= \int_0^1 F_{12}\begin{pmatrix} 1 + \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}\cos(2\pi t)2\pi\,\mathrm{d}t \tag{51}$$

$$= 2\pi \int_0^1 \frac{\cos(2\pi t)}{\cos^2(2\pi t) + \sin^2(2\pi t)} \cos(2\pi t)\,\mathrm{d}t \tag{52}$$

$$= 2\pi \int_0^1 \cos^2(2\pi t)\,\mathrm{d}t \tag{53}$$

$$-\int_f F_{21}\,\mathrm{d}x = -\int_0^1 f^*(F_{21}\,\mathrm{d}x) = -\int_0^1 (F_{21} \circ f) f^*(\mathrm{d}x) = +\int_0^1 (F_{21} \circ f)\sin(2\pi t)2\pi\,\mathrm{d}t = \tag{54}$$

$$= -2\pi \int_0^1 \frac{\sin(2\pi t)}{(2 + \cos(2\pi t))^2 + \sin^2(2\pi t)} \sin(2\pi t)\,\mathrm{d}t \tag{55}$$

$$= -2\pi \int_0^1 \frac{\sin^2(2\pi t)}{4\cos(2\pi t) + 5}\,\mathrm{d}t \tag{56}$$

$$-\int_f F_{22}\,\mathrm{d}y = -\int_0^1 f^*(F_{22}\,\mathrm{d}y) = -\int_0^1 (F_{22}\circ f)f^*(\mathrm{d}y) = -\int_0^1 (F_{22}\circ f)\cos(2\pi t)2\pi\,\mathrm{d}t \tag{57}$$

$$= -2\pi\int_0^1 \frac{2+\cos(2\pi t)}{(2+\cos(2\pi t))^2+\sin^2(2\pi t)}\cos(2\pi t)\,\mathrm{d}t \tag{58}$$

$$= -2\pi\int_0^1 \frac{2\cos(2\pi t)+\cos^2(2\pi t)}{4\cos(2\pi t)+5}\,\mathrm{d}t \tag{59}$$

$$(44) \Rightarrow \int_f \omega = 2\pi\int_0^1 \left[\sin^2(2\pi t)+\cos^2(2\pi t) - \frac{\sin^2(2\pi t)+2\cos(2\pi t)+\cos^2(2\pi t)}{4\cos(2\pi t)+5}\right]\mathrm{d}t \tag{60}$$

$$= 2\pi\int_0^1 \left[1-\frac{2\cos(2\pi t)+1}{4\cos(2\pi t)+5}\right]\mathrm{d}t \tag{61}$$

$$= 2\pi\int_0^1 \left[1-\frac{1}{2}+\frac{3}{2}\frac{1}{4\cos(2\pi t)+5}\right]\mathrm{d}t \tag{62}$$

$$= \pi\int_0^1 \left[1+3\frac{1}{4\cos(2\pi t)+5}\right]\mathrm{d}t \tag{63}$$

$$= \pi+3\pi\int_0^1 \frac{\mathrm{d}t}{4\cos(2\pi t)+5} = \pi+3\pi\frac{1}{3} = 2\pi \tag{64}$$

$$\int_g F_{11}\,\mathrm{d}x = \int_0^1 g^*(F_{11}\,\mathrm{d}x) = \int_0^1 (F_{11}\circ g)g^*(\mathrm{d}x) = \int_0^1 (F_{11}\circ g)\frac{\partial g_1}{\partial t}\,\mathrm{d}t = \tag{65}$$

$$= \int_0^1 F_{11}\begin{pmatrix}1+\pi\cos(2\pi t)\\ \pi\sin(2\pi t)\end{pmatrix}\pi(-\sin(2\pi t))2\pi\,\mathrm{d}t \tag{66}$$

$$= +2\pi^2\int_0^1 \frac{\pi\sin(2\pi t)}{\pi^2\cos^2(2\pi t)+\pi^2\sin^2(2\pi t)}\sin(2\pi t)\,\mathrm{d}t \tag{67}$$

$$= 2\pi\int_0^1 \sin^2(2\pi t)\,\mathrm{d}t \tag{68}$$

$$\int_g F_{12}\,\mathrm{d}y = \int_0^1 g^*(F_{12}\,\mathrm{d}y) = \int_0^1 (F_{12}\circ g)\frac{\partial g_2}{\partial t}\,\mathrm{d}t = \tag{69}$$

$$= \int_0^1 F_{12}\begin{pmatrix}1+\pi\cos(2\pi t)\\ \pi\sin(2\pi t)\end{pmatrix}\pi\cos(2\pi t)2\pi\,\mathrm{d}t \tag{70}$$

$$= 2\pi^2\int_0^1 \frac{\pi\cos(2\pi t)}{\pi^2\cos^2(2\pi t)+\pi^2\sin^2(2\pi t)}\cos(2\pi t)\,\mathrm{d}t \tag{71}$$

$$= 2\pi\int_0^1 \cos^2(2\pi t)\,\mathrm{d}t \tag{72}$$

$$-\int_g F_{21}\,\mathrm{d}x = -\int_0^1 g^*(F_{21}\,\mathrm{d}x) = -\int_0^1 (F_{21}\circ g)g^*(\mathrm{d}x) = -\int_0^1 (F_{21}\circ g)\pi(-\sin(2\pi t))2\pi\,\mathrm{d}t \tag{73}$$

$$= -2\pi^2\int_0^1 \frac{\pi\sin(2\pi t)}{(2+\pi\cos(2\pi t))^2+\pi^2\sin^2(2\pi t)}\sin(2\pi t)\,\mathrm{d}t \tag{74}$$

$$= -2\pi^2\int_0^1 \frac{\pi\sin^2(2\pi t)}{4+4\pi\cos(2\pi t)+\pi^2}\,\mathrm{d}t \tag{75}$$

$$-\int_g F_{22}\,\mathrm{d}y = -\int_0^1 g^*(F_{22}\,\mathrm{d}y) = -\int_0^1 (F_{22}\circ g)g^*(\mathrm{d}y) = -\int_0^1 (F_{22}\circ g)\pi\cos(2\pi t)2\pi\,\mathrm{d}t \tag{76}$$

$$= -2\pi^2\int_0^1 \frac{2+\pi\cos(2\pi t)}{(2+\pi\cos(2\pi t))^2+\pi^2\sin^2(2\pi t)}\cos(2\pi t)\,\mathrm{d}t \tag{77}$$

$$= -2\pi^2\int_0^1 \frac{2\cos(2\pi t)+\pi\cos^2(2\pi t)}{4+4\pi\cos(2\pi t)+\pi^2}\,\mathrm{d}t \tag{78}$$

$$(45) \Rightarrow \int_g \omega = 2\pi \int_0^1 [\sin^2(2\pi t) + \cos^2(2\pi t)]\,\mathrm{d}t - 2\pi^2 \int_0^1 \frac{\pi \sin^2(2\pi t) + 2\cos(2\pi t) + \pi\cos^2(2\pi t)}{4 + 4\pi\cos(2\pi t) + \pi^2}\,\mathrm{d}t \tag{79}$$

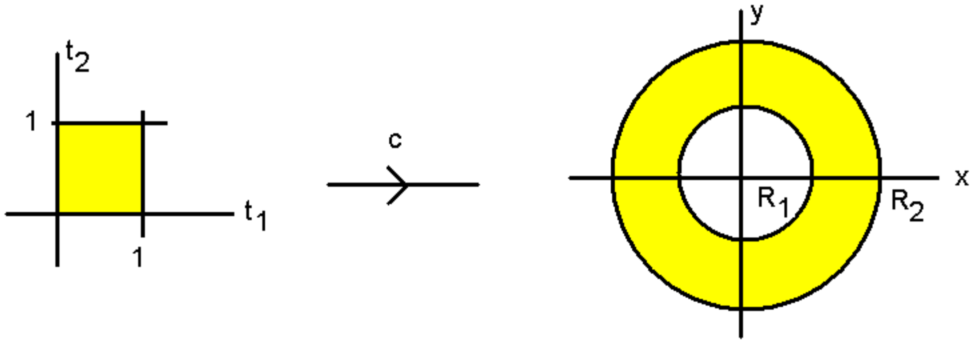
$$= 2\pi \int_0^1 \mathrm{d}t - 2\pi^2 \int_0^1 \frac{2\cos(2\pi t) + \pi}{4\pi\cos(2\pi t) + \pi^2 + 4}\,\mathrm{d}t \tag{80}$$

$$= 2\pi - 2\pi^2 \int_0^1 \left[\frac{1}{2\pi} + \frac{\pi^2 - 4}{2\pi}\frac{1}{4\pi\cos(2\pi t) + \pi^2 + 4}\right]\mathrm{d}t \tag{81}$$

$$= 2\pi - \pi \int_0^1 \mathrm{d}t - \pi(\pi^2 - 4)\int_0^1 \frac{\mathrm{d}t}{4\pi\cos(2\pi t) + \pi^2 + 4} \tag{82}$$

$$= 2\pi - \pi - \pi(\pi^2 - 4)\frac{1}{\pi^2 - 4} = 0 \tag{83}$$

3 Questão 3



Na imagem de c, t_1 representa o raio, t_2 representa o ângulo. Suponha $R_2 > R_1$.

$$c_{R,n}(t) = \begin{bmatrix} R\cos(2\pi n t) \\ R\sin(2\pi n t) \end{bmatrix} \tag{84}$$

$$R > 0, n \neq 0 \tag{85}$$

$$c_{R,n} : [0,1] \to \mathbb{R}^2 - \{0\} \tag{86}$$

$$\exists c \in C^\infty((0,1)^2 \to \mathbb{R}^2 - \{0\}); \partial c = c_{R_1,n} - c_{R_2,n} \tag{87}$$

$$c(t_1,t_2) = \begin{bmatrix} \rho(t_1)\cos(2\pi n t_2) \\ \rho(t_1)\sin(2\pi n t_2) \end{bmatrix} \text{ tal que } \rho(0) = R_1, \rho(1) = R_2 \tag{88}$$

$$\rho(t_1) = at_1 + b \Rightarrow b = R_1, a = \frac{R_2 - R_1}{1 - 0} \tag{89}$$

$$c(t_1,t_2) = \begin{bmatrix} [(R_2 - R_1)t_1 + R_1]\cos(2\pi n t_2) \\ [(R_2 - R_1)t_1 + R_1]\sin(2\pi n t_2) \end{bmatrix} \tag{90}$$

O período de $f(x) = \cos(ax) = f(x+T)$ ou $g(x) = \sin(ax) = g(x+T)$ é:

$$|a|T = 2\pi \Rightarrow T = \frac{2\pi}{|a|} \tag{91}$$

em que T equivale ao número de voltas sobre S^1 quando x percorre o domínio $[0,1] \subset \mathbb{R}$. Logo, o período de $c_{R,n}$ é

$$\frac{2\pi}{|2\pi n|} = \frac{1}{|n|} \tag{92}$$

Portanto, o bordo de c dá $|n|$ voltas sobre o círculo maior e $|n|$ voltas sobre o círculo menor. Seja $m \neq n$. Sem perda de generalidade, suponhamos dois círculos completos: $|m|, |n| \geq 1$. Podemos variar o período com o raio:

$$\partial C = c_{R_1,m} - c_{R_2,n} \tag{93}$$

$$C(t_1,t_2) = \begin{bmatrix} [(R_2 - R_1)t_1 + R_1]\cos(2\pi t_2 N(t_1)) \\ [(R_2 - R_1)t_1 + R_1]\sin(2\pi t_2 N(t_1)) \end{bmatrix} \text{tal que } N(0) = m, N(1) = n \tag{94}$$

$$N(t_1) = At_1 + B \Rightarrow B = m, A = \frac{n-m}{1-0} \tag{95}$$

$$C(t_1,t_2) = \begin{bmatrix} [(R_2 - R_1)t_1 + R_1]\cos(2\pi t_2((n-m)t_1 + m)) \\ [(R_2 - R_1)t_1 + R_1]\sin(2\pi t_2((n-m)t_1 + m)) \end{bmatrix} \tag{96}$$

O bordo de C dá $|n|$ voltas sobre o círculo maior e $|m|$ voltas sobre o círculo menor. Portanto, por construção, sim, podemos dizer o mesmo:

$$\exists C \in C^\infty((0,1)^2 \to \mathbb{R}^2 - \{0\}); \partial C = c_{R_1,m} - c_{R_2,n} \tag{97}$$

4 Referência

[1] SPIVAK, M. Calculus on Manifolds. Addisson-Wesley Publishing Company, 1965