Inference :

Simple Linear Regression

회귀직선의 유의성 검정

- Model : $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \;\; i = 1, 2, \ldots, n, \;\; \epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$

- 회귀직선의 유의성 검정 (F-test)
  - 가설 : $H_0 : \beta_1 = 0 \; vs. \; H_1 : \beta_1 \neq 0$
  - 검정통계량 : $F = \dfrac{MSR}{MSE} = \dfrac{SSR/1}{SSE/(n-2)} \sim_{H_0} F(1, n-2)$
  - 검정통계량의 관측값 : F_0
  - 유의수준 α에서의 기각역 : $F_0 \geq F_\alpha(1, n-2)$
  - 유의확률 $= P(F \geq F_0)$

회귀직선의 유의성 검정

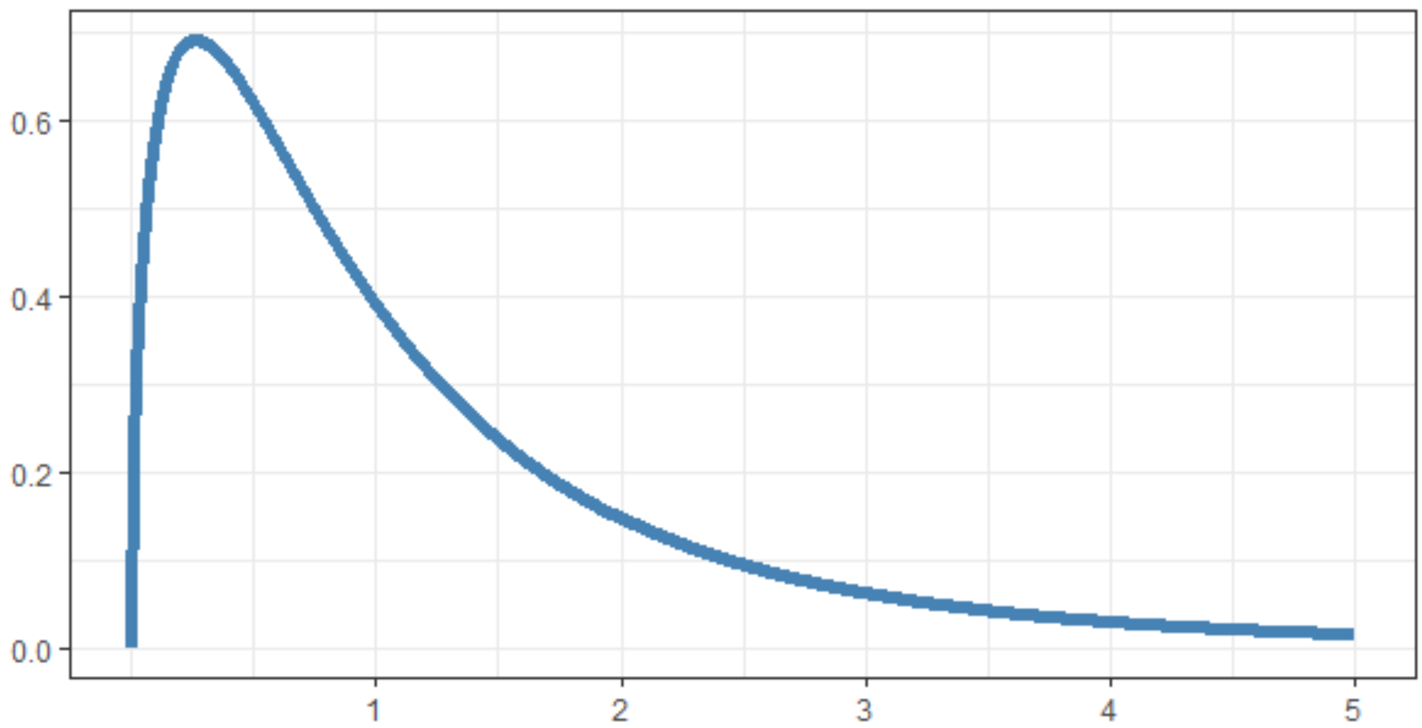
$F = \frac{\chi^2}{\chi^2}$

■ 회귀직선의 유의성 검정을 위한 분산분석표

요인	제곱합(SS)	자유도(df)	평균제곱(MS)	F_0	유의확률
회귀	SSR	1	$MSR = \frac{SSR}{1}$	$\frac{MSR}{MSE}$	$P(F \geq F_0)$
잔차	SSE	$n-2$	$MSE = \frac{SSE}{n-2}$		
계	SST	$n-1$			

Example

Figure: F분포의 확률밀도함수 그림 : $F(3,8)$



Example

광고료과 총판매액

- 회귀직선의 유의성 검정 : $H_0 : \beta_1 = 0 \ vs. \ H_1 : \beta_1 \neq 0$

요인	제곱합	자유도	평균제곱	f	유의확률
회귀	313.043	1	313.043	45.24	0.0001487
잔차	55.357	8	$6.92 (= \hat{\sigma}^2)$		
계	368.4	9			

- $F_{0.05}(1, 8) = 5.3177$

회귀계수에 대한 추론

- 모회귀계수(기울기) β_1 에 대한 추론
  - β_1 의 최소제곱추정량 : $\hat{\beta}_1 = \dfrac{S_{(xy)}}{S_{(xx)}}$
  - $\mathrm{E}(\hat{\beta}_1) = \beta_1, \ \mathrm{Var}(\hat{\beta}) = \dfrac{\sigma^2}{S_{(xx)}}$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{S_{(xx)}}\right)$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{S_{xx}}$$

$$= \Sigma \frac{(x_i-\bar{x})y_i - (x_i-\bar{x})\bar{y}}{S_{xx}}$$

$$= \Sigma \frac{x_i-\bar{x}}{S_{xx}} \cdot y_i + \frac{\bar{y}}{S_{xx}} \underbrace{\Sigma(x_i-\bar{x})}_{=0}$$

- $E(a_1X_1 + a_2X_2) = a_1E(X_1) + a_2E(X_2)$
- $Var(a_1X_1 + a_2X_2) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + 2a_1a_2 Cov(X_1, X_2)$

$\Rightarrow$

* $X_1 \sim N(\mu_1, \sigma_1^2)$ $X_2 \sim N(\mu_2, \sigma_2^2)$ 독립
a_1, a_2 : 상수

$$a_1X_1 + a_2X_2 \sim N(a_1\mu_1 + a_2\mu_2,\ a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$$

$$y_i = \beta_0 + \beta_1 x_i + \underbrace{\varepsilon_i}_{N(0,\sigma^2)} \quad : \quad y_i \mid (X = x_i) \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$\hat{\beta}_1 = \Sigma \frac{x_i-\bar{x}}{S_{xx}} \cdot y_i = \Sigma a_i y_i \sim N(\_\_, \_\_)$$

$$\boxed{E(\hat{\beta}_1)} = \Sigma a_i E(y_i) = \Sigma a_i(\beta_0 + \beta_1 x_i) = \beta_0 \Sigma a_i + \beta_1 \Sigma a_i x_i = \boxed{\beta_1}$$

$$\Sigma a_i = \Sigma \frac{x_i-\bar{x}}{S_{xx}} = \frac{1}{S_{xx}} \Sigma(x_i-\bar{x}) = 0$$

$$\Sigma a_i x_i = \Sigma \frac{(x_i-\bar{x})x_i}{S_{xx}} = \frac{1}{S_{xx}} \Sigma(x_i-\bar{x})(x_i-\bar{x}+\bar{x})$$

$$= \frac{1}{S_{xx}} \underbrace{\Sigma(x_i-\bar{x})^2}_{S_{xx}} = 1$$

$$Var\left(\sum a_i y_i\right) = \sum a_i^2 \, Var(y_i) = \sigma^2 \sum a_i^2$$

$$\sum a_i^2 = \sum \frac{(x_i-\bar{x})^2}{S_{xx}^2} = \frac{1}{S_{xx}^2} \underbrace{\sum (x_i-\bar{x})^2}_{S_{xx}} = \frac{1}{S_{xx}}$$

$$\Rightarrow Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$$

$$S_{xx} = \sum (x_i-\bar{x})^2$$

$$\Rightarrow \hat{\sigma}_{\hat{\beta}_1} = \sqrt{\frac{\sigma^2}{S_{xx}}} = \frac{\sigma}{\sqrt{S_{xx}}} = s.e(\hat{\beta}_1)$$

- $X_1, \cdots X_n$: $\underline{\underline{r.v}}$ $N(\mu, \sigma^2)$

$$\Rightarrow \boxed{\bar{X} = \frac{1}{n}\sum X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)} \quad \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

- $Z_1, \cdots, Z_n \sim N(0,1)$ indep

$$\Rightarrow \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

$$S = \sqrt{S^2}$$

$$\Rightarrow \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2/\sigma^2}{n-1}}} = t(n-1)$$

$$\Rightarrow V = Z_1^2 + \cdots + Z_n^2 \sim \chi^2(n)$$

$X_1, \cdots, X_n \sim N(\mu, \sigma^2)$ indep

$$\Rightarrow Z_1 = \frac{X_1-\mu}{\sigma}, \quad \cdots, \quad Z_n = \frac{X_n-\mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow \sum Z_i^2 \sim \chi^2(n)$$

$x_1, \cdots, x_n$: $\bar{x}$, $S^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} = \sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 = \sim \chi^2(n-1)$$

$$\Rightarrow Z \sim N(0,1), \quad V \sim \chi^2(k)$$

$$\Rightarrow \frac{Z}{\sqrt{V/k}} \sim t(k)$$

$\hat{\theta}$: 점추정 ⇒ (신뢰구간 / 가설검정) ⇒ 분포

추정량

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$\Rightarrow \bar{X} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$

표준오차.

$\cancel{\bar{X} \sim} \quad \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$

$\Rightarrow \bar{X} \pm t_{\alpha/2}(n-1) \cdot \sqrt{\frac{s^2}{n}}$

회귀계수에 대한 추론

■ 모회귀계수(기울기) β_1 에 대한 추론

- $\widehat{\mathrm{Var}}(\hat{\beta}_1) = \dfrac{MSE}{S_{(xx)}}$, $\hat{\sigma}_{\hat{\beta}_1} = \sqrt{\dfrac{MSE}{S_{(xx)}}}$

- studentized $\hat{\beta}_1$ 의 분포 :

$$\boxed{\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}/\sqrt{S_{(xx)}}} \sim t(n-2),}\quad \hat{\sigma} = \sqrt{MSE}$$

- $\hat{\beta}_1$ 의 $100(1-\alpha)\%$ 신뢰구간

$$\hat{\beta}_1 \pm t_{\alpha/2}(n-2)\frac{\hat{\sigma}}{\sqrt{S_{(xx)}}}$$

* $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

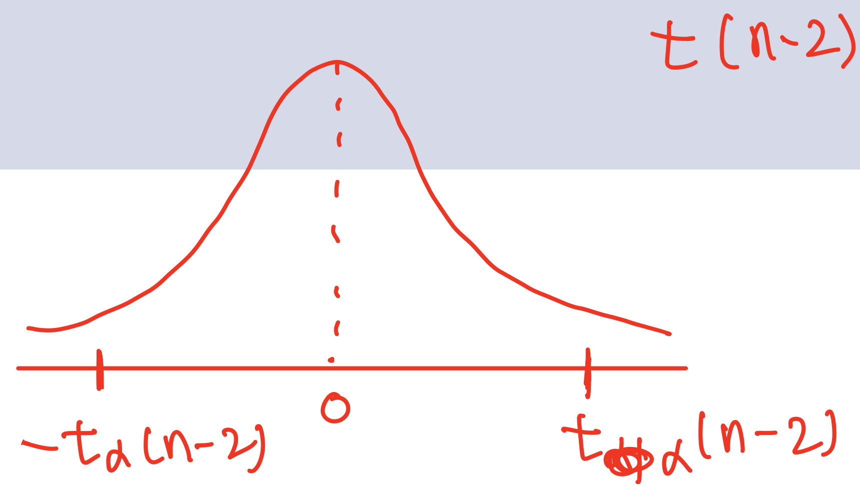
표준화 (standardization)

$\Rightarrow \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

studentization

$\Rightarrow \dfrac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \sim t(n-1)$

회귀계수에 대한 추론



- 모회귀계수(기울기) β_1 에 대한 추론
  - 가설검정 : $H_0 : \beta_1 = \beta_1^0$ 0
  - 검정통계량 : $T = \dfrac{\hat{\beta}_1 - \beta_1^0}{\hat{\sigma}/\sqrt{S_{(xx)}}} \sim_{H_0} t(n-2)$, 관측값 : t

대립가설	유의확률	유의수준 α 기각역
$H_1 : \beta_1 > \beta_1^0$	$P(T \geq t)$	$t \geq t_\alpha(n-2)$
$H_1 : \beta_1 < \beta_1^0$	$P(T \leq t)$	$t \geq t_\alpha(n-2)$
$H_1 : \beta_1 \neq \beta_1^0$	$P(\lvert T\rvert \geq \lvert t\rvert)$	$\lvert t\rvert \geq t_{\alpha/2}(n-2)$

회귀계수에 대한 추론

- 모회귀계수(절편) β_0에 대한 추론
  - β_0의 최소제곱추정량 : $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$
  - $\mathrm{E}(\hat{\beta}_0) = \beta_0, \quad \mathrm{Var}(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{(xx)}} \right)$

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{(xx)}} \right) \right)$$

모회귀계수(절편) β_0에 대한 추론

- studentized $\hat{\beta}_0$의 분포 :

$$\frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}_{\hat{\beta}_0}} \sim t(n-2), \quad \hat{\sigma}_{\hat{\beta}_0} = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{(xx)}}}$$

- $\hat{\beta}_0$의 $100(1-\alpha)\%$ 신뢰구간

$$\hat{\beta}_0 \pm t_{\alpha/2}(n-2)\hat{\sigma}\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{(xx)}}}$$

회귀계수에 대한 추론

- 모회귀계수(절편) β_0 에 대한 추론
  - 가설검정 : $H_0 : \beta_0 = \beta_0^0$
  - 검정통계량 : $T = \dfrac{\hat{\beta}_0 - \beta_0^0}{\hat{\sigma}_{\hat{\beta}_0}} \sim_{H_0} t(n-2)$, 관측값 : t

대립가설	유의확률	유의수준 α 기각역
$H_1 : \beta_0 > \beta_0^0$	$P(T \geq t)$	$t \geq t_\alpha(n-2)$
$H_1 : \beta_0 < \beta_0^0$	$P(T \leq t)$	$t \geq t_\alpha(n-2)$
$H_1 : \beta_0 \neq \beta_0^0$	$P(\lvert T\rvert \geq \lvert t\rvert)$	$\lvert t\rvert \geq t_{\alpha/2}(n-2)$

Example

- 광고료와 판매총액
  - $\hat{\beta}_0$ 와 $\hat{\beta}_1$ 의 95% 신뢰구간 ($t_{0.05/2}(8) = 2.306$)

$$
\begin{aligned}
\hat{\beta}_0 \pm t_{\alpha/2}\hat{\sigma}_{\hat{\beta}_0} &= -2.270 \pm 2.306 \times \sqrt{6.92} \times \sqrt{\frac{1}{10} + \frac{8^2}{46}} \\
&= -2.270 \pm 2.306 \times 3.21 = (-9.672, 5.132) \\
\hat{\beta}_1 \pm t_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{S_{(xx)}}} &= 2.609 \pm 2.306 \times \frac{\sqrt{6.92}}{\sqrt{46}} \\
&= 2.61 \pm 2.306 \times 0.388 = (1.714, 3.504)
\end{aligned}
$$

Example

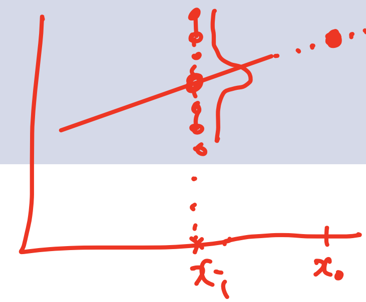
■ 광고료와 판매총액

- $H_0 : \beta_0 = 0 \; vs. \; H_1 : \beta_0 \neq 0$ 에 대한 가설검정 ($\alpha = 0.05$)
  - ▹ 검정통계량 관측값 : $t = \dfrac{\hat{\beta}_0 - 0}{\hat{\sigma}_{\hat{\beta}_0}} = \dfrac{-2.270}{3.21} = -0.707$
  - ▹ 기각역 : $|t| \geq t_{0.05/2}(8) = 2.306$
  - ▹ 결과 : 기각 못함, 유의확률 $= 0.5$
- $H_1 : \beta_1 = 0 \; vs. \; H_1 : \beta_1 \neq 0$ 에 대한 가설검정 ($\alpha = 0.05$)
  - ▹ 검정통계량 관측값 : $t = \dfrac{\hat{\beta}_1 - 0}{\hat{\sigma}_{\hat{\beta}_1}} = \dfrac{2.61}{0.388} = 6.72$
  - ▹ 기각역 : $|t| \geq t_{0.05/2}(8) = 2.306$
  - ▹ 결과 : 기각!, 유의확률 $= < 0.001$

평균반응예측

$$y_i = \underbrace{\beta_0 + \beta_1 x_i}_{} + \varepsilon_i$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \overbrace{E(y_i \mid X = x_i)}^{}$$



- $x = x_0$가 주어졌을 때 평균반응의 예측
  - 평균반응 (mean response) : $\mu_0 = E(Y|x_0) = \beta_0 + \beta_1 x_0$
  - 평균반응 추정량 : $\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$
  - $\mathrm{E}(\hat{\mu}_0) = \mu_0, \quad \mathrm{Var}(\hat{\mu}_0) = \sigma^2 \left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}} \right)$

$$\hat{\mu}_0 \sim N\left(\mu_0, \left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}} \right) \sigma^2 \right)$$

평균반응예측

- $x = x_0$가 주어졌을 때 평균반응의 예측
  - sutentized $\hat{\mu}_0$ 의 분포

$$\frac{\hat{\mu}_0 - \mu_0}{\hat{\sigma}_{\hat{\mu}_0}} \sim t(n-2), \quad \hat{\sigma}_{\hat{\mu}_0} = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}}$$

  - $\hat{\mu}_0$의 $100(1-\alpha)\%$ 신뢰구간

$$\hat{\mu}_0 \pm t_{\alpha/2}(n-2)\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}}$$

  - 신뢰대 (confidence band)

Example

- 광고료와 판매총액
  - $\hat{\mu}_0$ 의 $100(1-\alpha)\%$ 신뢰구간

$$\hat{\mu}_0 \pm t_{\alpha/2}(n-2)\hat{\sigma}_{\hat{\mu}_0}$$

  - $x_0 = 4$ 인 경우

$$\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0 = -2.270 + (2.609)(4) = 8.17$$

$$\widehat{\mathrm{Var}}(\hat{\mu}_0) = \hat{\sigma}\left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}\right) = (6.92)\left(\frac{1}{10} + \frac{(4-8)^2}{46}\right) = 3.10$$

$$\hat{\sigma}_{\hat{\mu}_0} = \sqrt{3.10} = 1.76$$

$$\Rightarrow \hat{\mu}_0 \pm t_{\alpha/2}(n-2)\hat{\sigma}_{\hat{\mu}_0} = 8.17 \pm (2.306)(1.76) = (4.11, 12.23)$$

Example

- 광고료와 판매총액

$$
\begin{aligned}
x = 6: &\quad 13.38 \pm (2.306)(1.14) = 13.38 \pm 2.63 = (10.75, 16.01) \\
x = 8: &\quad 18.60 \pm (2.306)(0.83) = 18.60 \pm 1.94 = (16.69, 20.51) \\
x = 9: &\quad 21.21 \pm (2.306)(0.92) = 21.21 \pm 2.12 = (19.09, 23.33) \\
x = 10: &\quad 23.82 \pm (2.306)(1.14) = 23.82 \pm 2.63 = (21.19, 26.45) \\
x = 12: &\quad 29.04 \pm (2.306)(1.76) = 29.04 \pm 4.59 = (24.45, 33, 63)
\end{aligned}
$$

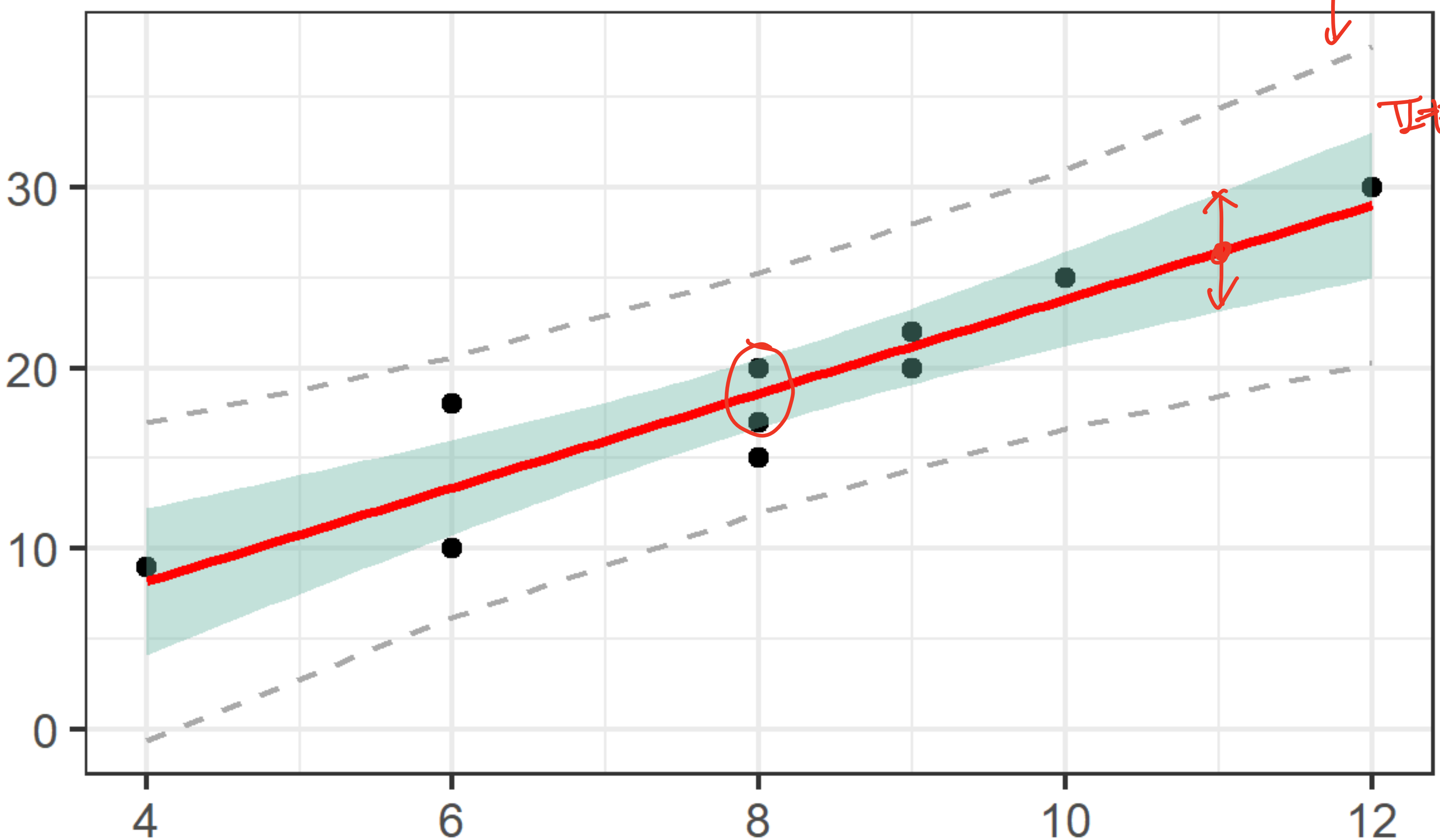
Example

$\hat{y}_i$

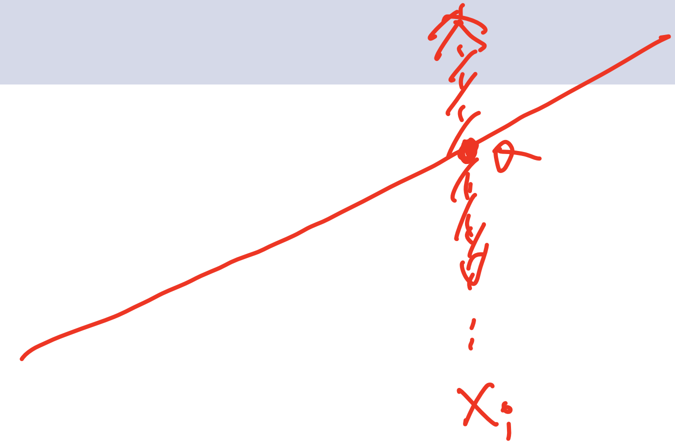
y_0
개별관측값

평균반응 μ_0

Figure: Confidence Band (신뢰대)



개별적인 y값 예측



- $x = x_0$가 주어졌을 때 $y = y_0$ 예측
  - $y_0 = \beta_0 + \beta_1 x_0 + \varepsilon_0$
  - 예측값 : $\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$
  - $\mathrm{E}(\hat{y}_0) = \mu_0, \quad \mathrm{Var}(\hat{y}_0) = \sigma^2 \left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}\right)$

$$\hat{y}_0 \sim N\left(\mu_0, \left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}\right)\sigma^2\right)$$

개별적인 y값 예측

- $x = x_0$가 주어졌을 때 $y = y_0$ 예측
  - studentized $\hat{y}_0$ 의 분포 :

$$\frac{\hat{y}_0 - y_0}{\hat{\sigma}_{\hat{y}_0}} \sim t(n-2), \quad \hat{\sigma}_{\hat{y}_0} = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{(xx)}}}$$

  - $\hat{y}_0$의 $100(1-\alpha)\%$ 신뢰구간

$$\hat{y}_0 \pm t_{\alpha/2}(n-2)\hat{\sigma}_{\hat{y}_0}$$

모형의 타당성

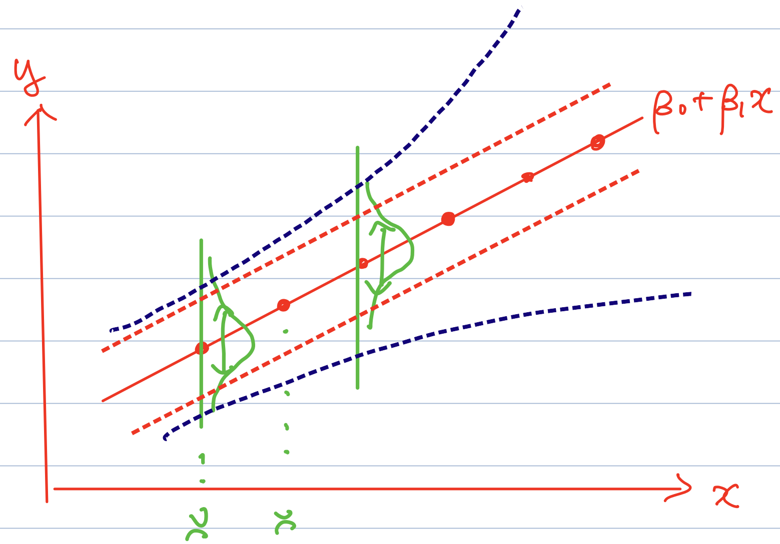
$\varepsilon_i \sim N(0, \sigma^2)$. iid.

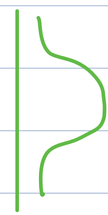
$y_i = \beta_0 + \beta_1 x_i + \underline{\varepsilon_i}$

$E(y_i \mid x_i) = \beta_0 + \beta_1 x_i + 0$

■ 기본 가정

- Linearity (선형성) : $E(Y|X=x) = \mu_{y\cdot x} = \beta_0 + \beta_1 x$
- Homoscedastic (등분산성) : $Var(Y|X=x) = \sigma^2$
- Normality (정규성) : $Y|X=x \sim N(E(Y|X=x), \sigma^2)$
- Independency (독립성) : ϵ's are mutually independent


y
$\beta_0 + \beta_1 x$
x
$\tilde{x}$
$\tilde{x}$



잔차의 검토

- 잔차(residual) : $e_i = y_i - \hat{y}_i$
- 잔차를 통한 모형의 가정 검토
- 잔차의 산점도 : (x_i, e_i) 또는 $(\hat{y}_i, e_i)$

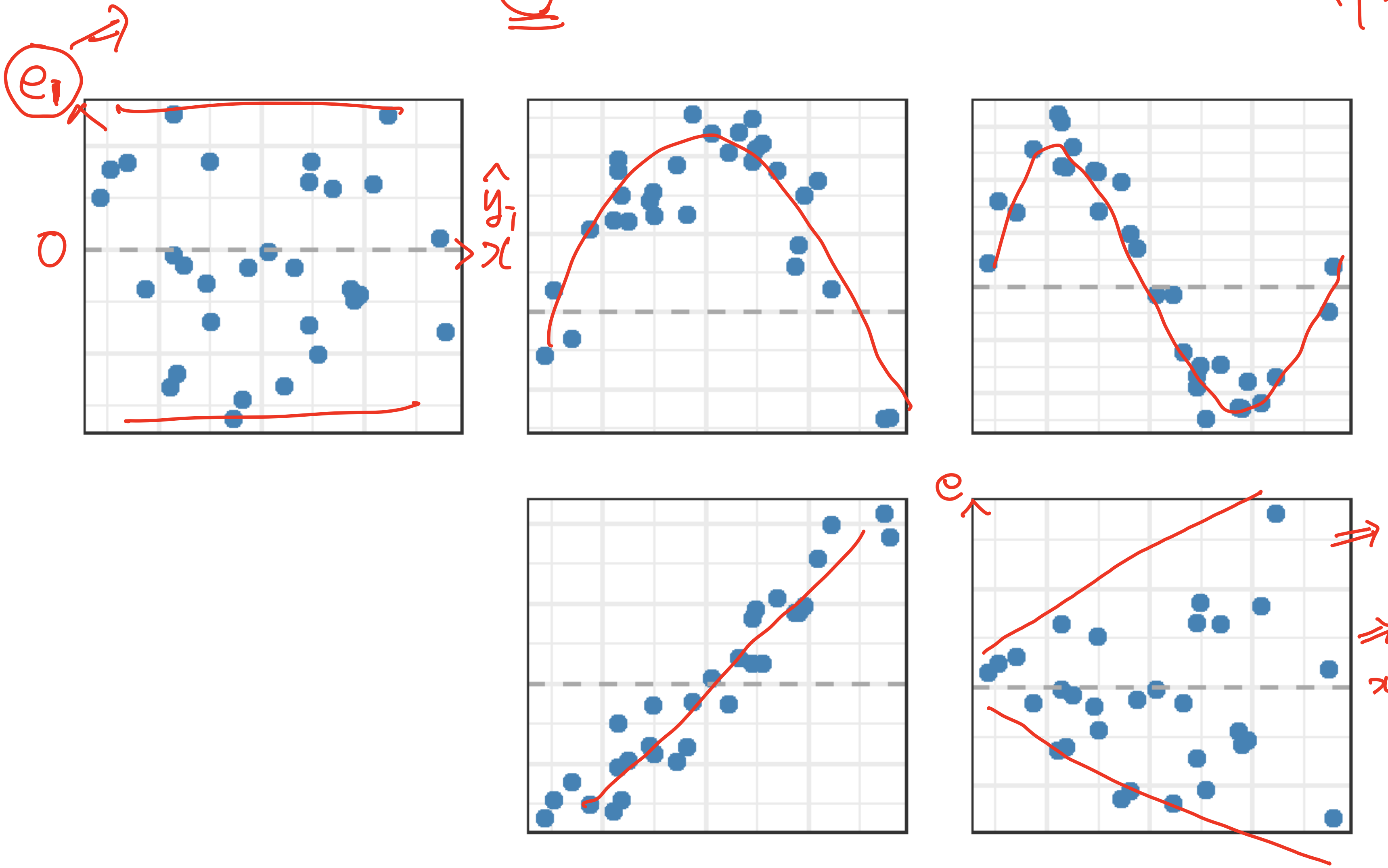
$$\because \left(\sum_i x_i e_i = 0, \sum_i e_i = 0\right), \left(\underline{\sum_i \hat{y}_i e_i = 0,} \sum_i e_i = 0\right)$$

잔차의 산점도

N $\varepsilon_i \sim N(0, \sigma^2)$ iid.

e_i

$y = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3$



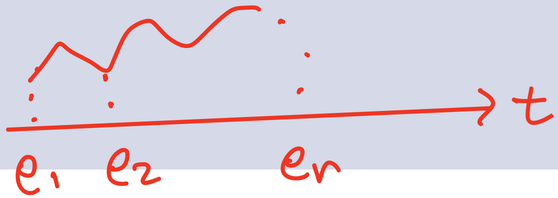
⇒ 변수 변환 log y_i ~

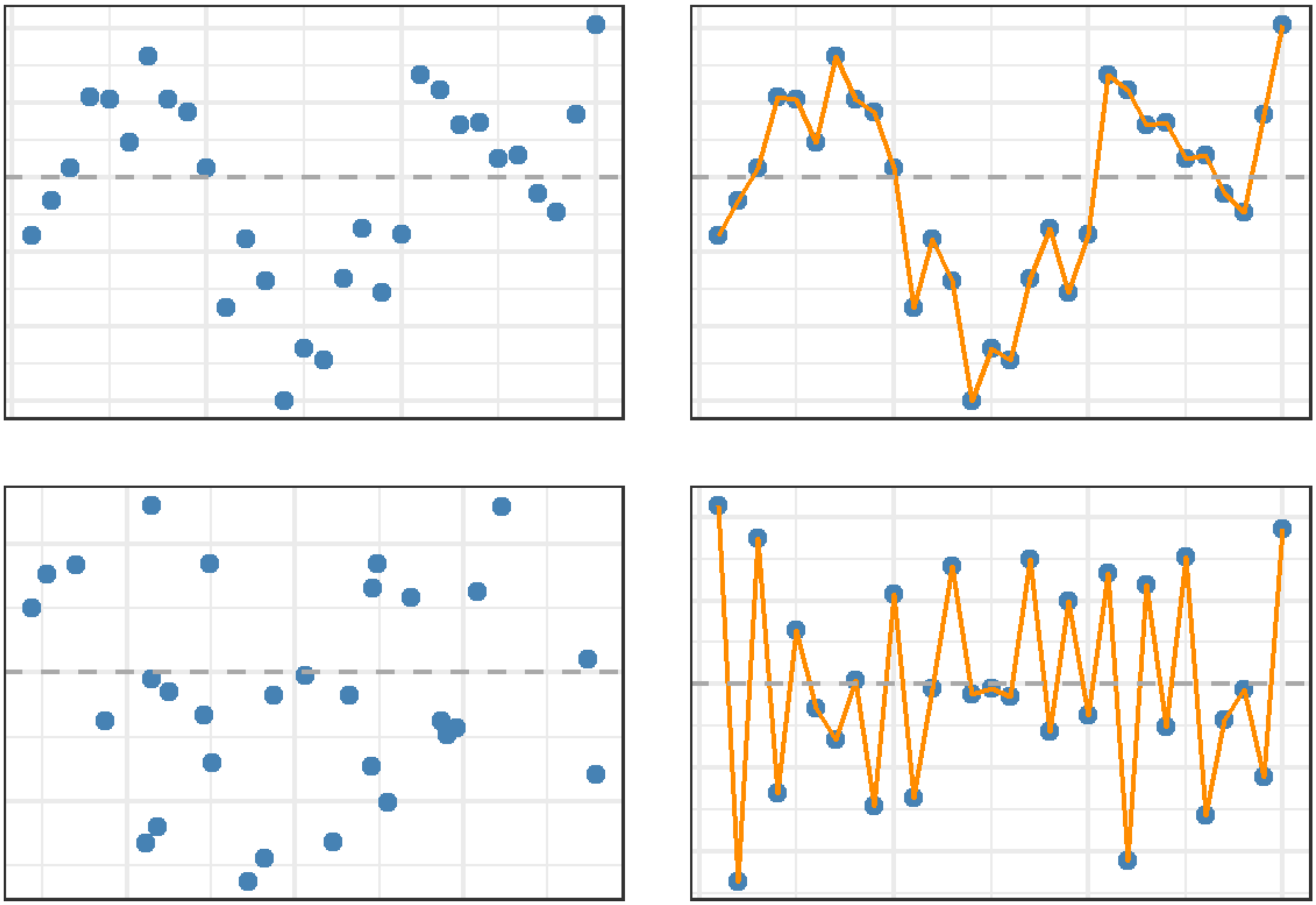
⇒ WLSE.

오차의 자기 상관

$e_1, e_2, \cdots, e_n$ 독립.

$Corr(\underline{X}, \underline{Y})$





$e_1, e_2, \cdots, e_n$ $\quad$ Corr. (X, Y)

AutoCorrelation
(자기상관)

$Corr(e_t, e_{t-1})$
(e_2, e_1)
(e_3, e_2)
$\vdots$
(e_n, e_{n-1})

$Corr(X, Y)$

$$\begin{pmatrix} x_1 - y_1 \\ x_2 \ \ y_2 \\ \vdots \\ x_n \ \ y_n \end{pmatrix}$$

오차의 자기 상관

ρ_ε : 오차의 1차 자기상관.
$= \mathrm{Corr}(e_t, e_{t-1})$

Durbin-Watson Test

- hypothesis

  $\rho_\varepsilon = 0$

  H_1 $\rho_\varepsilon > 0$, $\rho_\varepsilon < 0$ — 단
  $\rho_\varepsilon \neq 0$ — 양

  H_0 : 오차항들은 독립이다 vs. H_1 : 오차항들은 독립이 아니다.

- 검정통계량

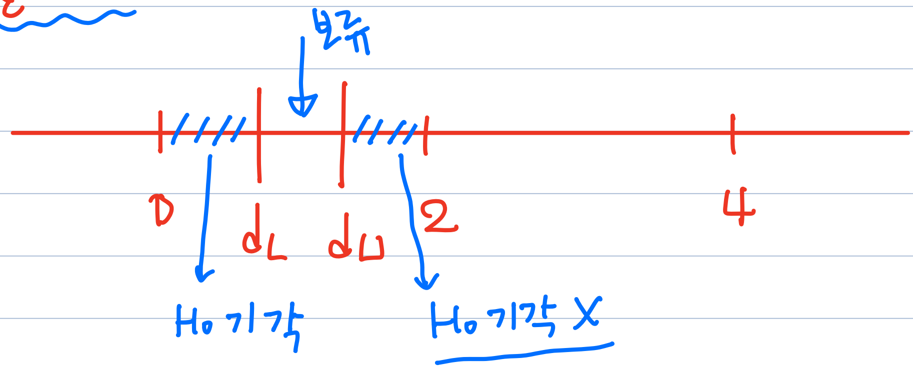
  $0 \le d \le 4$

$$d = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2} \approx 2(1-\rho_\varepsilon)$$

$\rho_\varepsilon \uparrow 1 \Rightarrow d \downarrow 0$
$\rho_\varepsilon \downarrow -1 \Rightarrow d \uparrow 4$
$\rho_\varepsilon \to 0 \Rightarrow d \to 2$

- 검정 $(d_L = d_L(n,p,\alpha), d_U = d_U(n,p,\alpha))$
  - d or $4-d < d_L \;\Rightarrow\; H_0$ 기각
  - d or $4-d > d_U \;\Rightarrow\; H_0$ 기각못함
  - $d_L < d$ or $4-d < d_L \;\Rightarrow\;$ 결정 보류

$H_0: \rho_\varepsilon = 0 \qquad H_1: \rho_\varepsilon > 0$

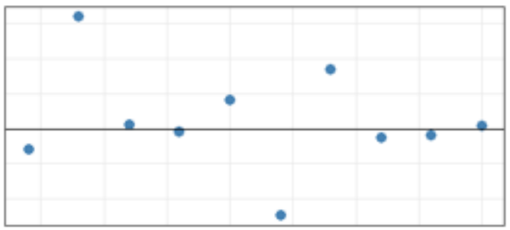


$H_0: \rho_\varepsilon = 0 \qquad H_1: \rho_\varepsilon < 0$

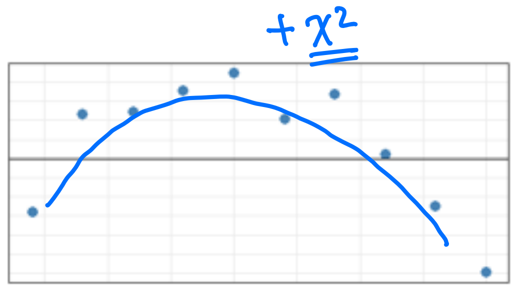
$4-d$

오차의 자기 상관

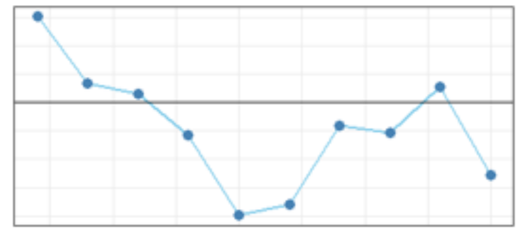
- Durbin-Watson Test



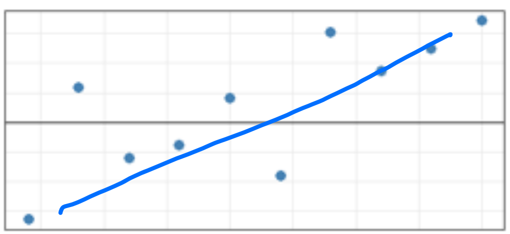
D = 3.013, p-value = 0.921



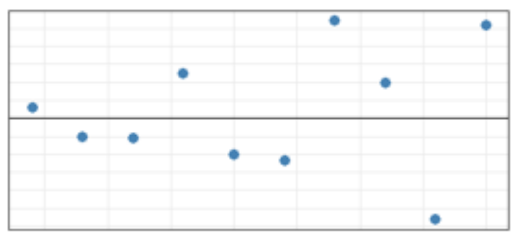
D = 0.734, p-value = 0.001



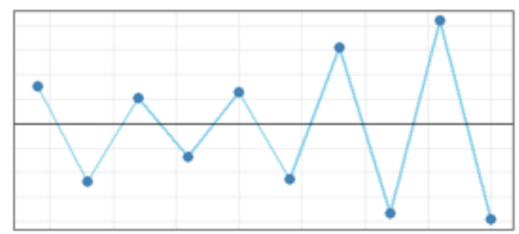
D = 1.066, p-value = 0.015



D = 3.013, p-value = 0.921



D = 2.661 p-value = 0.767



D = 3.507, p-value <0.005