Contents

Chapter 1

Introduction

1.1 Motivation

The original motivation for this work comes from partial differential equations and fluid mechanics. In this section we will describe how geometric approach can be used to answer these analytic questions. The original example for this approach is the Euler equation [23] for incompressible, non-viscous fluids which is given by

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p$$

$$\mathrm{div} u = 0$$

where u is the velocity field of the fluid and p is the scalar pressure.

Given a Lie group with a right-invariant Riemannian metric, Arnold created framework [3] in which its geodesic equation can be written in terms of the flow equation

$$\frac{d\eta}{dt} = u \circ \eta$$

and the Euler-Arnold equation can be written as

$$\frac{du}{dt} + \mathrm{ad}_u^* u = 0$$

where ad^* is given by the adjoint of the Riemannain metric as

$$\langle \mathrm{ad}_X^* Y, Z \rangle = \langle Y, \mathrm{ad}_X Z \rangle$$

and $\mathrm{ad}_X Z = -[X, Z]$ is the Lie bracket. Here the Euler-Arnold equation is obtained by optimization of the induced distance function on curves connecting two points on the original Lie group. In particular we obtain an analytic object in a PDE through a purely geometric technique.

Arnold then showed that when the group in consideration was $\mathcal{D}_\mu$, the group of volume preserving diffeomorphisms, the resulting Euler-Arnold equation was precisely the Euler equation for incompressible, non-viscous fluids. Later, Ebin and Marsden [12] showed that this approach was rigorous and that the geodesic equation was in fact an ODE in a Sobolev space. This approach lead to finding other classical differential equations as the Euler-Arnold equation to particular Lie groups.

Another fundamental equation in PDE is Burgers' equation which has great importance in fluid mechanics, in particular in nonlinear acoustics, gas dynamics, and gas flows. It is also related to the heat equation. The relevant Burgers' equation [10] here in this thesis is

$$u_t + 3uu_x = 0. \tag{1.1}$$

Another fundamental equation in PDE is the Camassa-Holm equation. This is a model for shallow water waves. Its importance lies in the fact that is an example of an integrable, non-linear PDE which has peakon solutions and also a Hamiltonian structure [10, 20, 25]. The equation we will be concerned with is

$$u_t - u_{xxt} + u(u_x - u_{xxx}) + 2(u - u_{xx})u_x = 0. \tag{1.2}$$

Next we consider the Lie group $\mathcal{D}(S^1)$ whose Lie algebra is $C^\infty(S^1)$ with right invariant metrics

$$\langle f, g\rangle_{L^2} = \int_{S^1} fg\,d\mu \tag{1.3}$$

and

$$\langle f, g\rangle_{H^1} = \int_{S^1} (f - f_{xx})g\,d\mu \tag{1.4}$$

and if we compute the Euler-Arnold equations we generate the Burgers' and the Camassa-Holm equations respectively. The diffeomorphism group of the circle [10] is a well studied object with extremely rich theory so its natural to try to generalize its results.

In the case of the circle, S^1, we can consider its coordinate x, and the standard 1-form on S^1 is $\theta = dx$. The Reeb field is $\frac{d}{dx}$. The contact operator which gives the identification between functions and vector fields of $\mathcal{D}(S^1)$ is trivial in this case and is given by $S_\theta f = f\frac{d}{dx}$. In this case, we have that the contactomorphism group $\mathcal{D}_\theta(S^1)$ coincides with the diffeomorphism group $\mathcal{D}(S^1)$ so that's why we see this work as a natural generalization.

We can also consider the $\dot{H}^{1/2}$ metric on $\mathcal{D}(S^1)$ [5,27,38]. The Euler-Arnold equation which is induced is the Wunsch equation

$$Hu_{xt} + uHu_x + 2u_xHu_x = 0$$

where H is the Hilbert transform. We can then consider the homogeneous space $\mathcal{D}(S^1)/S^1$ which becomes a Kähler manifold [18,19] under the foundations of Kirillov. Teo also discovered that this space also has applications to conformal welding on the universal Teichmüller curve [37].

1.2 Summary of results

Let (M, g, θ) be a smooth, compact, oriented K-contact manifold [7] of odd dimension $2n+1$, and we let $\mathcal{D}_\theta(M)$ denote the space of contactomorphisms of M. We consider two different metrics on $\mathcal{D}_\theta(M)$. One is the "natural" metric which we will refer to in this section as the H^1 metric on stream functions. The other is the L^2 metric on stream functions. There are four main results which we will summarize here.

The first result deals with curvature of the contactomorphism group endowed with the two different metrics mention above. Curvature is an important geometric object as the contactomorphism group has direct applications to partial differential equations. The two most notable examples being in fluid flows such as gas dynamics and shallow water waves which are described by Burgers' equation and the Camassa-Holm equation [10,17,20,25]. The result goes as follows:

Theorem 1.2.1. *$\mathcal{D}_\theta(M)$ with the H^1 metric has non-negative sectional curvature when one of the directions is a multiple of the Reeb vector field. In the L^2 case, $\mathcal{D}_\theta(M)$ always has non-negative* *sectional curvature.*

The next result is:

Theorem 1.2.2. *[9] With the H^1 metric on $\mathcal{D}_\theta(M)$, there exists conjugate points along geodesics in the direction of the Reeb vector field. They do not accumulate and are monoconjugate of finite order.*

Once we find a solution to the Euler-Arnold equation, a natural question to ask is whether conjugate points exists along this geodesic. In this theorem we find that the flow of the Reeb vector field is geodesic in $\mathcal{D}_\theta(M)$. In the next main result we find that the sectional curvature in every section which contains the Reeb vector field is non-negative so we suspect that there exists conjugate points along its geodesic flow. It is in fact interesting that we investigate the nature of these conjugate points and find in this result that they are all monoconjugate of finite order which demonstrates properties of its exponential map which we will discuss later.

Above we discussed the nature of the conjugate points along the flow of the Reeb vector field in $\mathcal{D}_\theta(M)$ with the H^1 metric. Here the two concepts are related as a conjugate point occurs when the differential of the exponential stops being invertible as a map from one tangent space to another. This phenomenon is easily seen in the sphere where conjugate points are the bipolar points. In the next result, we show that the exponential map is weakly Fredholm in the H^1 case. In particular we have that the exponential map is called *Fredholm* if $(d\exp_p)_v$ is a Fredholm linear operator for every point p and vector field v. Here the map is invertible if and only if it is and injection and its kernel is trivial.

Next we see that in the case of the L^2 topology on stream functions, the results is vastly different:

Theorem 1.2.3. *[8, 9] The exponential mapping onto $\mathcal{D}_\theta(M)$ is a C^1 not local diffeomorphism in the L^2 case. In the H^1 case, the exponential mapping is a C^1 local diffeomorphism, and furthermore it is weakly Fredholm.*

In the last main result we briefly discuss the quantomorphism group. These are contactomorphisms which exactly preserve the contact form and are volume preserving. The quantomorphism group is an important object of study as it is the odd dimensional analog to the symplectomorphism group. In the last part of Section 4, we briefly discuss the ability to between quantomorphism and symplectomorphism through Boothby-Wang fibration. The last main result is as follows:

Theorem 1.2.4. *[8] The quantomorphism group is closed and totally geodesic in $\mathcal{D}_\theta(M)$ with the L^2 metric.*

In the last part of the thesis, we programmed in Python with the goal of finding examples of negative curvature in the group $\mathcal{D}(S^1)/S^1$ endowed with the $\dot{H}^{1/2}$ metric [5]. $(S^1, \theta = dx)$ is a contact manifold and $\mathcal{D}(S^1)$ coincides with the group of contactomorphisms $\mathcal{D}_\theta(S^1)$ so this fits into the context of the other work. Also the L^2 is sometimes referred to as the H^0 metric so this can be seen as an in between case of the previous work. We consider the homogeneous space $\mathcal{D}(S^1)/S^1$ so that the $\dot{H}^{1/2}$ metric is not degenerate on this space. There are interesting applications of $\mathcal{D}(S^1)/S^1$ as it is a Kähler manifold [18, 19] and also to Teichmüler theory [37] as it related to conformal welding on the universal Teichmüller curve. In all simulations done so far, we have not found an example of negative sectional curvature on this space.

1.3 Previous work

As we said in previous section, Arnold laid out the framework for the geometry of infinite-dimensional Lie groups [3]. He discusses how this approach can be used to analyze hydrodynamics. He minimizes the length functional induced by the Riemannian metric to get the Euler-Arnold equation mentioned above.

Later Ebin and Marsden [3, 12] investigate diffeomorphism groups and construct weak Riemannian metrics on these diffeomorphism groups(citation). They also the geodesic flows which are associated to these groups endowed with the weak Riemannian structures. They justifiy Arnold's example of $\mathcal{D}_\mu$, the group of volume preserving diffeomorphisms, in which he shows that the Euler-

Arnold equation induced is precisely the classical Euler equation for incompressible, non-viscous fluids.

Next Constantin and Kolev investigate the geometry of the diffeomorphism group of the circle $\mathcal{D}(S^1)$ including giving an explicit atlas of charts [10]. They construct a Levi-Civita connection for the H^s metric and then us that to prove the existence of geodesics which are length minimizing. They then show that the exponential map is a smooth local diffeomorphism for H^s with $s > 0$.

Before commencing on this work, Ebin and Preston constructed the theory for studying the geometry of the contactomorphism group [14]. They consider the "padded" contactomorphism group which is the subgroup of elements of the semidirect product $\mathcal{D}^s(M) \ltimes H^s(M)$ which preserve the contact structure, in particular

$$\mathcal{D}^s_\theta(M) = \{(\eta, \Lambda) \in \mathcal{D}^s(M) \ltimes H^s(M) : \eta^*\theta = e^\Lambda \theta\}.$$

As with Constantin and Kolev, they proved the existence of geodesics and the locally diffeomorphic properties of the exponential map.

In the last part of this thesis, we discuss the homogeneous space $\mathcal{D}(S^1)/S^1$. Kirillov and Yur'ev study the space $\mathcal{D}_+(S^1)/S^1$ [18, 19], the space of orientation-preserving diffeomorphisms on the circle modulo its rotations, with his main motivation being Teichmüller theory. They investigates the Kähler structure of this infinite dimensional space with the almost complex structure being the Hilbert transform.

Next Teo describes conformal welding for the universal Teichmüler curve $\mathcal{T}(1)$ [37] as the set of maps

$$\tilde{\Gamma} = \tilde{\mathcal{C}} \to \mathcal{D}(S^1)/S^1$$

where $\tilde{\mathcal{C}}$ is the set of smooth, closed, simple curves which have conformal radius 1 centered at the origin. In this work the Velling-Kirilov metric on the universal Teichmüller curve induces the Wunsch equation.

This work relies heavily on the foundations built by the authors above, and the work in this thesis would not be possible without their contributions and inspiration they've provided.

1.4 Outline of thesis

In this section we go over how this thesis is organized. In Chapter 2, we summarized the preliminary geometric and analytic knowledge necessary to understand the thesis. We hope that the background provided is sufficient. We discuss contact manifolds, Sobolev theory, Fréchet manifolds and recall the fundamental results. We then describe weak Riemannian manifolds and breifly mention the difficulties of working with infinite-dimensional manifolds and the results which do no carry over from the finite-dimensional theory.

In Chapter 3, we present joint work which is published with my thesis advisor Stephen C. Preston. We investigate the geometry of the contactomorphism group with its "natural" induced metric. We prove that the sectional curvature is non-negative in all sections which contain the Reeb vector field. We show the existence of conjugate points and show that the conjugate points along the geodesic flow of the Reeb vector feild are all monoconjugate and of finite order. Lastly we show that the exponential map is a weak Fredholm. We also discuss how this result can be strengthen to strong Fredholmness.

Chapter 4 studies the contactomorphism group with a different weak Riemannian metric, the L^2 metric on stream functions. We show that sectional curvature is always non-negative. Next we show that the exponential map is not a local C^1 diffeomorphism. Lastly, we demonstrate that the quantomorphism group is a closed, totally geodesic submanifold of $\mathcal{D}_\theta(M)$.

Here Chapters 3 and 4 are reprints of publications [8, 9].

Finally in Chapter 5, we write a program in Python which computes the sectional curvature for the group $\mathcal{D}(S^1)/S^1$ with the $\dot{H}^{1/2}$ metric. We are able to do this quickly in Python since the Hilbert transform and derivatives are very easy to compute when the vectors are written in the basis functions in terms of$\{\sin nx\}$ and $\{\cos mx\}$. We have conducted a search for examples of negative curvature by plugging in random inputs and also random inputs with certain decay conditions but none have been found so far.

Chapter 2

Preliminaries

Here in this chapter we will give definitions and explain concepts necessary to carry out the main results of this thesis. This chapter will be structured as follows:

We go over basic material in Section 2.1 where we will recall definitions of contact and symplectic manifolds and their fundamental properties. We will also briefly discuss how they are related and the motivation for studying these types of manifolds.

In Section 2.2 we will build up all of the necessary background in order to define Sobolev manifolds including Sobolev spaces on $\mathbb{R}^n$ and integration on manifolds. Although we mostly will work with smooth mappings, it will be very important to develop these concepts as these are very fundamental objects and some of our arguments will in fact rely on the use of Sobolev manifolds.

Here in Section 2.3, we give the geometric framework for the manifold of mappings.

Next in Section 2.4, we will generalize the notion of a Riemannian metric. In particular, we will investigate these *weak Riemannian metrics* on these Sobolev manifolds. These are fundamental objects in Global Analysis although the increase in generality and lack of theory make these objects much more difficult to study than strong Riemannian metrics. We will discuss the differences between weak and strong Riemannian metrics and briefly discuss the difficulties with working with the former. Lastly we discuss Fredholmness as it relates to one of the main results and it is also an infinite-dimensional phenomenon in manifold theory.

Lastly in Section 2.5 we construct the group of contactomorphisms. We will briefly discuss its motivation, the group of volume preserving diffeomorphisms. At the end we will give examples of groups of contactomorphisms.

2.1 Contact Manifolds

Given (M, g) an orientable, compact Riemannian manifold of odd dimension $2n+1$. We call M a *contact manifold* [7] if there exists a 1-form θ which satisfies the non-degeneracy condition that

$$\theta \wedge d\theta^n \neq 0$$

everywhere where

$$d\theta^n = \underbrace{d\theta \wedge \ldots \wedge d\theta}_{n-\text{times}}$$

. In this case we call θ the contact form and (M, g, θ) our contact manifold. We can see from this definition that the contact form is not necessarily unique. Given the contact structure θ, we can define the *Reeb vector field* E, which is uniquely determined by the conditions that $\theta(E) = 1$ and $\iota_E d\theta = 0$. We would also like there be compatibility between the Riemannian metric g and the contact form θ. This leads to the following definition.

Definition. *If (M, θ) is a contact manifold and E is its Reeb vector field, a Riemannian metric $(\cdot, \cdot)_g$ is associated if it satisfies the following conditions:*

(1) $\theta(u) = (u, E)_g$ for for all $u \in TM$, and

(2) there exists a $(1,1)$-tensor field ϕ such that $\phi^2(u) = -u + \theta(u)E$ and $d\theta(u, v) = (u, \phi v)_g$ for all $u, v \in TM$

If in addition E is a Killing vector field, we say that (M, g, θ) is a K-contact manifold

Next we will discuss the even dimensional cousin of contact manifolds.

2.1.1 Symplectic manifolds

Let (M, g) be an orientable, compact Riemannian manifold of even dimension $2n$. If there exists a closed, non-degenerate 2 form ω on (M, g), we call (M, g) a *symplectic manifold* [7]. The non-degeneracy condition on ω is that for each $p \in M$, it there exists an $X \in T_pM$ such that

$$\omega(X, Y) = 0$$

for all $Y \in T_pM$, it is necessary that $X = 0$. The closed condition requires that $d\omega = 0$ where d is the exterior derivative. In this case we call ω the *symplectic form.*

Both of these examples, contact and symplectic manifolds, arise in classical mechanics. The symplectic form ω allows us to obtain a vector field describing the flow of a system using the differential dH of a Hamiltonian function H. The non-degeneracy of ω allows us to find a unique corresponding vector field V_H such that

$$dH(\cdot) = \omega(V_H, \cdot).$$

Additionally, if we impose the condition that ω not change under flow lines, Cartan's formula [22] gives us

$$0 = \mathcal{L}_{V_H}(\omega) = d(\iota_{V_H}\omega) + \iota_{V_H} d\omega$$

$$0 = \mathcal{L}_{V_H}(\omega) = d(dH) + d\omega(V_H)$$

which equivalently gives that ω is closed.

One important application in symplectic geometry is almost complex structures. That is, given an even dimensional smooth manifold M, if there exists a smooth $(1, 1)$-tensor J such that $J^2 = -1$, we say that M admits an almost complex structure and we call J the *almost complex structure.* If we also require the compatibility

$$\omega(Ju, v) = (u, v)_g,$$

we call M a *Kähler manifold.*

Notice that ϕ in condition (2) in the definition of associated metric for contact manifolds is analagous to an almost-complex structure in the case of symplectic manifolds. Also using that same comparison, we see that associated metrics are analogous to Kähler metrics.

2.1.2 Boothby-Wang fibration

Now given a symplectic manifold (M, ω) with even dimension $2n$, there is a natural way to obtain a contact manifold of dimension $2n+1$. This is called the Boothby-Wang fibration [7]. More precisely, if we have the projection

$$\pi : P \to M$$

where P is the circle bundle over M, we get that $(P, \pi^*\omega)$ is a contact manifold where π^* is the pull-back of this projection map.

2.2 Sobolev Functions on Manifolds

Here in this section we will explore the theory of Sobolev functions. Although most of the work in this thesis is on smooth mappings, some of our arguments require the generality of Sobolev mappings. This will also provide sufficient background without unnecessary abstraction. First we will discuss Sobolev spaces on Euclidean space, next we will discuss integration on manifolds, and then extend these two notions to define Sobolev manifolds.

2.2.1 Sobolev spaces

In this section we will look at classical Sobolev spaces on $\mathbb{R}^n$ [16, 36]. Let $1 \leq p \leq \infty$ and let $k \geq 0$ be a natural number. A function is said to live in the Sobolev space $W^{k,p}$ if its weak derivatives $\partial^\alpha f$ exists and live in L^p for $|\alpha| \leq k$. We the define the Sobolev norm to be

$$|f|_{W^{k,p}} := \sum_{|\alpha| \leq k} |\partial^\alpha f|_{L^p}.$$

Next we introduce some notation

$$C_0^k = \{f \in C^k : \partial^\alpha f \in C_0 \text{ for } |\alpha| \leq k\}$$

where the space C_0 consists of continuous functions which tend to zero. Also we will denote the space $W^{s,2}$ as H^s. This will be the most common Sobolev space we will be working with in this thesis. H^s is particularly convenient as they are Hilbert spaces with the following inner product

$$(f,g)^{H^s} = \sum_{|\alpha|\leq k} (\partial^\alpha f)(\partial^\alpha g).$$

Here f and g are real valued functions. With this information we get the following theorem:

Theorem 2.2.1 (Sovolev Embedding). *[11, 16, 23, 36] $H^s \subset C_0^k$, and the inclusion map is continuous given $s > k + \frac{1}{2}n$.*

This theorem gives us sufficient conditions so we can say that distributions of high enough Sobolev class have certain regularity and is fundamental to all of the constructions of groups of mappings in this thesis.

2.2.2 Lebesgue measure on manifolds

We can extend the notion of Lebesgue measurability from $\mathbb{R}^n$ to manifolds very easily. Let M be an n dimensional manifold with a maximal atlas of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ where $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. We call $E \subset M$ *Lebesgue measurable* if we can find a covering of charts $\{(U_\beta, \phi_\beta)\}$ such that $\phi_\beta(E \cap U_\beta)$ is Lebesgue measurable in $\mathbb{R}^n$ for each β. In addition if we have another covering of E by charts $\{(U_\gamma, \phi_\gamma)\}$, we need to check that the transition function

$$\phi_{\beta\gamma} = \phi_\gamma|_{U_\gamma \cap U_\beta} \circ \phi_\beta^{-1}|_{\phi_\beta(U_\beta \cap U_\gamma)}$$

is a smooth diffeomorphism and thus maps Lebesgue measurable sets to Lebesgue measurable sets. The transition functions won't necessarily preserve the measure, but it will map null sets to null sets. Thus the notion of null sets will make sense on manifolds and is independent of charts. The Lebesgue measure will depend on the choice of maximal atlas so it is possible to get many different Lebesgue measures. Now we can define Lebesgue measurable functions on manifolds as functions whose preimage of a measurable(Lebesgue) set is also measurable(Borel).

From this above definition, we can see that if we have a volume form μ on M that has measurable coefficients, we can get an induced Lebesgue measure on M. This construction also works for any non-negative form on M.

We won't get into the details of the construction of Lebesgue measure by not mentioning notions such as Borel sets, outer measure, and measure space completions but we will mention two important results from measure theory:

Theorem 2.2.2 (Lebesgue dominated convergence). *[16] Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_k\}$ be a sequence of measurable functions on X converging almost everywhere to a function f. suppose further that there exists and L^1 function g so that $|f| \leq g$ almost everywhere. Then f is also in L^1 and*

$$\int_X f d\mu = \lim_{k\to\infty} \int_X f_k d\mu$$

Theorem 2.2.3 (Fatou's lemma). *[16] Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_k\}$ be a sequence of nonnegative measurable functions on X converging almost everywhere to a function f. Then f is measurable and*

$$\int_X f d\mu \leq \liminf_{k\to\infty} \int_X f_k d\mu$$

2.2.3 Sobolev manifolds

We can now generalize the definition of smooth manifold to Sobolev manifolds. Let M be a Hausdorff topological space. We say that M is a *Sobolev manifold* modeled on the Sobolev space E with atlas of coordinates charts $\{(U_\alpha, \phi_\alpha)\}$ such that each U_α is open and $\phi_\alpha : U_\alpha \subseteq M \to E$ is a homeomorphism onto its image. Also if $U_\alpha \cap U_\beta \neq \emptyset$, we require the transition map

$$\phi_{\beta\alpha} = \phi_\alpha|_{U_\alpha \cap U_\beta} \circ \phi_\beta^{-1}|_{\phi_\beta(U_\beta \cap U_\alpha)}$$

is a smooth diffeomorphism between Sobolev spaces.

Geometric objects such as tangent space, tangent bundles, differentiable maps, fiber bundles, and so on are defined exactly the same as in the theory of finite-dimensional manifolds [22]. We also use this above discussion to define Sobolev functions on manifolds analagously.

2.3 Manifolds of Mappings

Let M be an oriented, smooth manifold. For the vector bundle $\pi : E \to M$ and integer $s > 0$, we get a Hilbert space of sections of E, $H^s(E)$ [12]. These are the sections whose derivatives up to order s are square integrable, the sections of Sobolev class s. We can think of this space as the completion of $C^\infty(E)$ under this Sobolev norm. Now let $C^k(E)$ be the sections which are smoothly differentiable up to order k. This is a Banach space using the uniform norm on the derivatives and is also the completion of $C^\infty(E)$ using this norm.

By the Sobolev embedding theorem [11,16,23,36], we have that the inclusion $H^s(E) \subset C^k(E)$ is continuous when $s > n/2 + k$ where n is the dimension of M. So when $s > n/2$ we have that the space of $H^s(M, N)$ is a Hilbert manifold and the manifold structure is as follows. Let $f \in H^s(M, N)$ and we can form the Hilbert space

$$T_f H^s(M, N) = \{g \in H^s(M, TN) : \pi \circ g = f\}$$

where $\pi : TN \to N$ is the canonical projection and this give us our desired charts.

2.3.1 Geometric structure

Let M be a compact manifold and consider $C^\infty(M, M)$, the space of smooth mappings from M to itself. $C^\infty(M, M)$ carries the structure of a Fréchet vector space [31]. Now since $\mathcal{D}(M)$ carries the structure of a Fréchet manifold. Since this is a very important underlying object of study in this thesis, we will quickly define Fréchet manifolds for completeness.

Let V be a vector space over $\mathbb{F}$. A *seminorm* is a function $\|\cdot\|$ on V such that for all $x, y \in V$ and $\lambda \in \mathbb{F}$ we have :

(1) $\|v\|\| \geq 0$,

(2) $\|x + y\| \leq \|x\| + \|y\|$, and

(3) $\|\lambda x\| = |\lambda| \|x\|$

Now given a collection of seminorms $\{\|\|_i\}_{i\in I}$, we can define a topology by saying $x_k \to x$ if $\|x - x_k\|_i \to 0$ for each $i \in I$. A *locally convex topological vector space* (LCTVS)is a vector space with a compatible topology defined this way. A LCTVS is metrizable if it is defined by a countable set of seminorms. In fact, a metric can be defined as

$$d(x, y) = \sum_{i\in I} \frac{\|x - y\|_i}{2^i}.$$

A LCTVS is Hausdorff if and only if $\|x\|_i = 0$ for all $i \in I$ implies that $v = 0$. Completeness is defined in the usual way.

Definition. *A Fréchet space is a LCTVS which is metrizable, Hausdorff, and complete.*

One interesting example of a Fréchet space is $C^\infty([0, 1])$. We can use both the C^k and H^s norms as the collection of seminorms to generate the topology on $C^\infty([0, 1])$. The case of using the C^k norms is straightforward but in the H^s case, we need to use Sobolev embedding. Both of these ways generate the same topology on $C^\infty([0, 1])$.

There are many important differences between Fréchet spaces and Banach spaces such as the failure of the implicit function theorem but we will not discuss these details [1]. But it is important to keep these distinctions in mind.

We can also do calculus on Fréchet spaces [21, 22]. Let V and W be Fréchet spaces over $\mathbb{F}$ and let $U \subseteq V$ be an open set. Let $f : U \to W$ be a continuous map. We now define the differential of f at the point $u \in U$ in the direction of $v \in V$ for $t \in \mathbb{F}$ as:

$$Df(u)v := \lim_{t\to 0} \frac{f(u + tv) - f(u)}{t}.$$

We say that f is differentiable at u in the direction of v if this limit exists. We say that f is C^1 on U if the limit exists for all $u \in U$ and all $v \in V$ and the map

$$Df : U \times V \to W$$

is continuous in both U and V.

Now to define the second derivative, we must take the derivative of the derivative as usual. In particular consider the map

$$D^2 f : U \times V \times V \to W$$

and we have the same definitions as above. We can keep iterating to define C^k and C^∞ maps between Fréchet spaces.

We define Fréchet manifolds in the analogous manner in which we defined Sobolev manifolds in Section 2.2.3 with the maximal atlas of charts and smooth transition functions.

2.4 Weak Riemannian Manifolds

Here in this section we will discuss some important distinctions between finite-dimensional manifolds and their infinite-dimensional counterparts. We'll start with a simple example. It is well known and easy to show that for finite-dimensional vector spaces, all positive-definite inner products are equivalent. This is not true in infinite dimensions. We can see this break down in the example of $C^\infty([0,1])$, smooth functions on the interval. The inner products

$$(f, g)_0 = \int_{[0,1]} f g dx$$

and

$$(f, g)_1 = \int_{[0,1]} f g + f_x g_x dx$$

generate the L^2 and H^1 topologies on $C^\infty([0,1])$ respectively.

The analogy for positive definite inner products on vector spaces for manifold theory is Riemannian metrics. But of course we have that manifolds are modeled on these linear spaces and Riemannian metrics are simply positive-definite inner products on these linearizations that vary smoothly as we move along the manifold.

Now the above discussion about all positive-definite inner products being equivalent on finite-dimensional inner product spaces can be applied to the example of Euclidean space $\mathbb{R}^n$. If we have an n dimensional Riemannian manifold (M, g), it is clear that for each $x \in M$, the Riemannian metric induces a positive-definite inner product on T_xM which is equivalent to the usual dot product on $\mathbb{R}^n$. Therefore it induces the standard topology on $\mathbb{R}^n$.

We can see that this is not always the case for infinte dimensions. Given an infinite dimensional manifold (M, g) modeled on a Hilbert space H [22]. There are many different topologies on H so it won't be necessarily true that the Riemannian metric on T_xM for $x \in M$ will generate the same topology on H. So we can distinguish these two types of Riemannian metrics. If it is the case that g generates the same topology than that on H, we call g a *strong Riemannian metric*. If it is the case that g generates a different topology than that on H, we call g a *weak Riemannian metric*. In the latter case, the language makes sense because the topology generated by a weak Riemannian metric will be necessarily weaker than that on H. To demonstrate this we can think of trying to apply the H^2 norm on the space $H^1([0,1])$, space of H^1 functions on the interval. For example, we can come up with a sequence of function $\{f_n\}$ so that $f_n \to 0$ in L^2 but not in H^1. In particular consider the distributions

$$f_n(x) = (2x)^n \chi_{[0,1/2)}(x)$$

so that

$$f_n'(x) = n(2x)^{n-1} + n\delta_{1/2}(x)$$

where $\delta_{1/2}$ is the Kronecker delta at the point $x = 1/2$.

We mention the difference between these two types of metrics because there are many important subtleties. In the case of strong Riemannian manifolds, we can usually produce most of the results from the finite dimensional theory such as the existence of the Levi-Civita connection, geodesics, and exponential mapping. From the strong Riemannian metric, we can get a distance function on the manifold and this distance function will induce a topology which will coincide with the manifold's topology.

The theory of weak Riemannian metrics is extremely rich as the results from the finite dimensional theory are harder to prove or may even be outright not true. For example, it is possible to have a weak Riemannian metric which induces a distance function which is degenerate everywhere.

2.4.1 Strong and weak metrics and distance function

In this section we will go over the formalities of defining weak Riemannian manifolds [2, 3, 12, 21].

Let (M, g) be a Riemannian manifold. Just as in the case of finite dimensional manifold, we can define a distance function between points of M by considering the infima of the length functional [2, 3, 11].

In particular we let $a \leq b$, and let $\alpha : [a, b] \to M$ be a piecewise C^1 path. We can define

$$L(\alpha) := \int_{[a,b]} \sqrt{(\dot{\alpha}(t), \dot{\alpha}(t))_{g(t)}}$$

Then for any $x, y \in M$, we define

$$d_g(x, y) := \inf_{\alpha} L(\alpha)$$

where the infimum is taken over all piecewise C^1 paths which start at x and end at y.

It's easy to see that d_g is a *psuedometric*. The symmetry and the triangle inequality are clear. The only obstacle for d_g being a metric is the positive definiteness. It turns out that d_g is metric when g is a strong Riemannian metric. As we mentioned before, it can be possible to have weak Riemannian metrics in which the distance function is 0 is any two points of the manifold so that we have that $d_g(x, y) = 0$ for all points $x, y \in M$. This pathological example can be found in [24].

Definition. *Let M be a manifold modeled on the space H. A Riemannian metric g on M is a choice of scalar product $g(x)$ on each T_xM for each $x \in M$ such that the following hold:*

(1) g is smooth in the sense that for U a neighborhood of x and V, W are vector field defined on U, then $g(\cdot)(V, W) : U \to \mathbb{R}$ is a smooth local function;

(2) $g(x)$ is a continuous bilinear mapping;

(3) $g(x)$ is a positive definite operator.

And also, g is called

(1) strong if the topology generated by g coincides with the topology of the model space H;

(2) weak if the topology generated by g is strictly weaker than the topology of the model space H.

2.4.2 Levi-Civita connection and the exponential mapping

Let (M, g) be a finite dimensional Riemannian manifold [22]. There exists a unique connection ∇ that is both

(1) *metric*, in particular $X(Y,Z)_g = (\nabla_X Y, Z)_g + (Y, \nabla_X Z)_g$ for all vector fields X, Y, Z, and

(2) *torsion-free*, in particular $\nabla_X Y - \nabla_Y X = [X, Y]$

Both the existence and uinquness of this *Levi-Civita Connection* relies on Koszul formula. The Koszul formula states that a connection is both metric and torsion-free if the following formula holds for all vector fields X, Y, Z:

$$\begin{aligned}(\nabla_X Y, Z)_g =& X(Y,Z)_g + Y(X,Z)_g - Z(X,Y)_g \\ &- (X,[Y,Z])_g - (Y,[X,Z])_g + (Z,[X,Y])_g\end{aligned}$$

The existence and uniqueness of $\nabla_X Y$ for a point $x \in M$ follows from the Riesz representation theorem on $T_x M \cong \mathbb{R}^n$.

We know that the Riesz representation theorem holds for Hilbert spaces [16, 36]. Therefore we can loosen the assumptions a bit to only requiring (M, g) to be a strong Hilbert manifold. Here we need the metric to be strong so that its distance function generates the same topology of the model space.

We can then use the Levi-Civita connection to define geodesics as the paths α such that the equation $\nabla_{\dot\alpha}\dot\alpha$ is satisfied. We can then use the geodesics to define the exponential mapping as the time 1 geodesic. In particular, consider a tangent vector v at the point p and let α_v be the geodesic such that $\alpha_v(0) = p$ and $\alpha_v'(0) = v$, then we can define the *exponential mapping* as $\exp_p(v) = \alpha_v(1)$.

Now we can see how this breaks down once we consider weak Riemannian manifolds. Kozsul formula will not guarantee the existence and uniqueness of a Levi-Civita connection. This occurs because on weak Riemannian manifolds, the topology on T_pM which is generated by the metric g will not be complete and the Riesz representation theorem does not hold for incomplete spaces. But it is the case that if a connection does in fact exist which satisfies Kozsul formula, it will necessarily be unique. This follows simply by linearity.

So no we summarize by saying that for weak Riemannian manifolds, a Levi-Civita connection does not exist in general. As a result, geodesics and the exponential mapping might not exist in general.

We have to take a different approach when dealing with weak Riemannian manifolds as we cannot take for granted the results which follow automatically when dealing with strong Riemannian manifolds. In particular, in results or properties we would like, we must show directly. For example, one can explicitly construct a connection which satisfies Kozsul formula [11]; or one can compute geodesics explicitly by solving the Euler-Arnold formula [2, 3].

2.4.3 Fredholmness

Recall that a (strong)Fredholm operator is a bounded linear operator $T : X \to Y$ between Banach spaces if both $\ker T$ and $Y/\mathrm{im}T = \text{cokernel}$ are both finite dimensional [31]. In this case we can define the *index* as

$$\mathrm{index}(T) = \dim(\ker T) - \dim(\mathrm{cokernel} T).$$

We will call a bounded linear operator $S : X \to Y$ between Banach spaces *weakly Fredholm* [26] if

$$\dim(\ker S) < \infty \text{ and } \dim(Y/\overline{\mathrm{im}S}) < \infty$$

where $\overline{\mathrm{im}S}$ is the closure of S. In this case we have that a weakly Fredholm operator extends to a Fredholm operator. A map is Fredholm(or weakly Fredholm) if its Fréchet derivative is a

Fredholm(or weakly Fredholm) operator.

As we can see when we are investigating whether the exponential map is Fredholm or not, this is only an infinte-dimensional phenomenon. As we will see in this thesis and in the following examples, it depends heavily on the metric on the group.

It was shown that for a surface without boundary M, that $\mathcal{D}^s_\mu(M)$, the volume preserving diffeomorphism group of Sobolev index s is strong Fredholm when given the L^2 metric. With M being an n dimensional manifold, $\mathcal{D}^s(M)$ given the H^r metric is weakly Fredholm when $r \geq 1$ and $s > 3r + \frac{n}{2}$. In the case of the Bott-Virasoro group with Sobolev index s with the H^r metric, it is weakly Fredholm when $r \geq 2$ and $s > 3r + \frac{1}{2}$ [26].

2.5 Contactomorphisms

Now we have sufficient background in order to be able to talk about the group of contactomorphisms on a compact, oriented, finite-dimensional contact manifold. This is one of the main objects of study in this thesis. As we stated in Chapter 1, the main motivation for studying this group is the Euler equation for incompressible, non-viscous fluids [23] given by

$$\frac{\partial u}{\partial t} + \nabla_u u = \nabla p$$

$$\mathrm{div} u = 0$$

where u is the velocity field of the fluid and p is the scalar pressure. This arises as the Euler-Arnold equation on the group of volume preserving diffeomorphism $\mathcal{D}_\mu$.

Let (M, g, θ) be an orientable, compact, K-contact manifold of odd dimension $2n+1$. If we let $\mathcal{D}(M)$ be the group of diffeomorphisms of M, we will say that $\eta \in \mathcal{D}(M)$ is a *contactomorphism* if $\eta^*\theta$ is some positive functional multiple of θ. Here $\eta^*\theta$ is the pullback of θ by η. We will denote the group of contactomorphisms by $\mathcal{D}_\theta(M)$. We can keep track of this multiple so that we can write $\eta^*\theta = e^{\Sigma}\theta$ where $\Sigma : M \to \mathbb{R}$ is some function [8, 9, 14].

We can also impoose a manifold structure on $\mathcal{D}_\theta(M)$ by considering the Lie algebra $T_e\mathcal{D}_\theta(M)$. It contains vector fields u such that $\mathcal{L}_u\theta = \lambda\theta$ for some function $\lambda : M \to \mathbb{R}$. Any vector field is then uniquely determined by some function $f = \theta(u)$ so we can write $u = S_\theta f$. Now we have that we can identify the vector fields of $\mathcal{D}_\theta(M)$ with smooth functions on M. In particular we have that

$$T_e\mathcal{D}_\theta(M) = \{S_\theta f : f \in C^\infty(M)\}.$$

For more details, refer to (citation ebin preston). Here we call S_θ the *contact operator*. The Lie bracket on $T_e\mathcal{D}_\theta(M)$ is given by

$$[S_\theta f, S_\theta g] = S_\theta\{f, g\}, \text{ where } \{f, g\} = S_\theta f(g) - gE(f)$$

where E is the Reeb vector field which is uniquely determined by θ. We call $\{\cdot,\cdot\}$ the "contact Poisson bracket", in quotes as it does not satisfy Leibniz's rule.

We can define a right-invariant metric $\langle\cdot,\cdot\rangle$ on $\mathcal{D}_\theta(M)$ by

$$\langle S_\theta f, S_\theta g\rangle := \int_M \Lambda f g d\mu$$

where $S_\theta f, S_\theta g \in T_e\mathcal{D}_\theta(M)$ and μ is the volume form on M. The two cases we consider in this thesis are when $\Lambda_0 = 1$ and $\Lambda_1 = 1 - \Delta$, where Δ is the Laplacian. We will call these the L^2 and H^1 cases respectively.

Arnold showed that for any Lie group with a right-invariant Riemannian metric, the geodesic equation can be written in terms of the flow equation

$$\frac{d\eta}{dt} = u \circ \eta$$

and the Euler-Arnold equation

$$\frac{du}{dt} + \mathrm{ad}_u^* u = 0.$$

Recall that we have that $\mathrm{ad}_X^* Y$ is defined in terms of the following:

$$\langle \mathrm{ad}_X^* Y, Z\rangle = \langle Y, \mathrm{ad}_X Z\rangle$$

where $\mathrm{ad}_X Z = [X, Z]$

In the case of the diffeomorphism group, it is straightforward to compute ad^*.

Lemma 2.5.1. *For $X = S_\theta f$ and $Y = S_\theta g$ we have that*

$$\mathrm{ad}_X^* Y = S_\theta \Lambda^{-1}[S_\theta f(\Lambda g) + (n+2)E(f)\Lambda g].$$

Proof. Let $X = \S_\theta f, Y = S_\theta g$ and $Z = S_\theta h$ for some functions f, g, and h on M. Then we have

$$\begin{aligned}\langle \mathrm{ad}_X^* Y, Z\rangle &= \langle Y, \mathrm{ad}_X Z\rangle = -\langle Y, [X, Z]\rangle \\ &= -\int_M \Lambda g\{f, h\} d\mu = -\int_M \Lambda g(S_\theta f(h), hE(f)) d\mu \\ &= \int_M h[S_\theta f(\Lambda g) + \Lambda g(\mathrm{div}(S_\theta f) + E(f))].\end{aligned}$$

Now using a fact from (ebin preston) we have that $\mathrm{div}(S_\theta f) = (n+1)E(f)$ for K-contact manifolds, we get the desired result. □

Using this, we can compute various geodesic equations which depend on the metric on $\mathcal{D}_\theta(M)$ which depends on various Λ. We can then find solutions to this equation to obtain geodesics and thus obtain an exponential map.

Another object of study which is related to the contactomorphism group is the quantomorphism group $\mathcal{D}_q(M)$. These are contactomorphisms which exactly preserve the contact form, in particular

$$\mathcal{D}_q(M) := \{\eta \in \mathcal{D}_\theta(M) : \eta^*\theta = \theta\}.$$

These are exactly the contactomorphims which are volume preserving.

One example is the standard contact form $\theta = dz - ydx$ on $\mathbb{R}^3$. Here we have that the Reeb vector field is given by $\frac{\partial}{\partial z}$ and the contact operator is $S_\theta f = f\frac{\partial}{\partial z}$. If we equip $\mathcal{D}_\theta(\mathbb{R}^3)$ with the L^2 metric, we get that the Euler-Arnold equation becomes

$$\frac{\partial u}{\partial t} + 4uu_z = 0.$$

Given an initial condition u_0 we are able to solve this PDE implicitly to get

$$u(t, x, y, z - 4tf_0(x, y, z)) = u_0(x, y, z).$$

Another important example is (S^1, dx). Here the Reeb vector field is $\frac{d}{dx}$ and the contact operator is given by $S_\theta f = f\frac{d}{dx}$. We can endow $\mathcal{D}_\theta(S^1)$ with the metric

$$\langle S_\theta f, S_\theta g\rangle_0 = \int_{S^1} fg dx$$

and the corresponding Euler-Arnold equation is

$$\frac{du}{dt} + 3uu_x = 0$$

which is the right-invariant Burgers' equation. We can also give it the metric

$$\langle S_\theta f, S_\theta g\rangle_1 = \int_{S^1} (f - f_{xx}) g dx$$

and the corresponding Euler-Arnold equation becomes

$$(u - u_{xx})_t + u(u_x - u_{xxx}) + 2(u - u_{xx})u_x = 0$$

which is the Camassa-Holm equation. Notice here that in this case $\mathcal{D}_\theta(S^1)$ is the same as $\mathcal{D}(S^1)$ since the contact form is trivial.

Chapter 3

$\mathcal{D}_\theta(M)$ with the Natural Metric

This paper was published in *Differential Geomety and its Applications* in 2015. It appears in volume **40**, pp .223-231.

In this paper we investigate the "natural" metric on the group of contactomorphism, the H^1 metric on stream functions. We show that its sectional curvature is non-negative in every section which contains the Reeb vector field. Next we show that conjugate points exists along the geodesic flow of the Reeb vector field and show that they are all monoconujugate of finte order. Lastly we show that the Riemannian exponential map is weakly Fredholm.

3.1 Geometry of the Contactomorphism Group

3.1.1 Introduction

Let M be an orientable compact manifold (without boundary) of odd dimension $2n+1$. Recall that M is called a **contact manifold** if there is a 1-form θ on M satisfying the non-degeneracy condition that $\theta \wedge d\theta^n \neq 0$ everywhere. If $\mathcal{D}(M)$ denotes the group of diffeomorphisms of M, we say that $\eta \in \mathcal{D}(M)$ is a contactomorphism if $\eta^*\theta$ is some positive functional multiple of θ; the group of contactomorphisms is denoted by $\mathcal{D}_\theta(M)$. Keeping track of this multiple we may write $\eta^*\theta = e^\Sigma\theta$ where Σ is some function $\Sigma\colon M \to \mathbb{R}$, and we define the group of "padded contactomorphisms" to be the group

$$\widetilde{\mathcal{D}}_\theta(M) = \{(\eta, \Sigma) \mid \eta^*\theta = e^\Sigma\theta\}.$$

For details on these constructions, see [14].

We will be working primarily on the Lie algebras of these groups, and we will use the following well-known fact that the Lie algebra $T_e\mathcal{D}_\theta(M)$ can be identified with the space of functions $f\colon M \to \mathbb{R}$.

Proposition 3.1.1 (EP)**.** *The Lie algebra $T_e\mathcal{D}_\theta(M)$ consists of vector fields u such that $\mathcal{L}_u\theta = \lambda\theta$ for some function $\lambda : M \to \mathbb{R}$. Any such field is uniquely determined by the function $f = \theta(u)$, and we write $u = S_\theta f$. Thus we have that*

$$T_e\mathcal{D}_\theta(M) = \{S_\theta f : f \in C^\infty(M)\}.$$

Here we call S_θ the contact operator. The Lie bracket on $T_e\mathcal{D}_\theta(M)$ is given by

$$[S_\theta f, S_\theta g] = S_\theta\{f, g\}, \text{ where } \{f, g\} = S_\theta f(g) - gE(f); \tag{3.1}$$

here E denotes the Reeb vector field, uniquely specified by the conditions $\theta(E) = 1$, $\iota_E d\theta = 0$. We call $\{\cdot,\cdot\}$ the "contact Poisson bracket"; it is not a true Poisson bracket since it does not satisfy Leibniz's rule.

We also need a Riemannian structure on (M, θ), and we will require that the Riemannian metric be **associated** to the contact form. It will also be convenient to assume that E is a Killing field (i.e., its flow consists of isometries).

Definition. *If (M, θ) is a contact manifold and E is the Reeb field, a Riemannian metric $(\cdot,\cdot)_g$ is **associated** if it satisfies the following conditions:*

(1) $\theta(u) = (u, E)_g$ for all $u \in TM$, and

(2) there exists a $(1, 1)$-tensor field ϕ such that $\phi^2(u) = -u + \theta(u)E$ and $d\theta(u, v) = (u, \phi v)_g$ for all u and v.

*If in addition E is a Killing field, we say that that (M, θ, g) is K-**contact**.*

Now if we have a K-contact manifold (M, θ) with an associated metric $(\cdot,\cdot)_g$, we define a right-invariant metric $\langle\cdot,\cdot\rangle$, on $\mathcal{D}_\theta(M)$ by

$$\langle S_\theta f, S_\theta g\rangle = \int_M (S_\theta f, S_\theta g)_g + E(f)E(g)\, d\mu = \int_M (f - \Delta f) g d\mu,$$

where the latter formula applies since the metric is associated [14]. This is the natural metric induced on $\widetilde{\mathcal{D}}_\theta(M)$ as a submanifold of the semidirect product $\mathcal{D}(M) \ltimes C^\infty(M, \mathbb{R})$.

On any Lie group with a right-invariant Riemannian metric, the geodesic equation can be written in terms of the flow equation

$$\frac{d\eta}{dt} = u \circ \eta$$

and the Euler-Arnold equation, given by

$$\frac{du}{dt} + \mathrm{ad}_u^* u = 0.$$

For $\mathcal{D}_\theta(M)$ with an associated Riemannian metric, the Euler-Arnold equation is given by

$$(f - \Delta f)_t + S_\theta f(f - \Delta f) + (n+2)(f - \Delta f)E(f) = 0. \tag{3.2}$$

Example 3.1.2. In the case of the circle $M = S^1$ with coordinate α, the standard 1-form is $\theta = d\alpha$, and every diffeomorphism is a contactomorphism. The Reeb field is given by $\frac{\partial}{\partial\alpha}$, and the contact operator is given by $S_\theta f = fE$. Hence the Euler-Arnold equation is given by

$$(f - f_{\alpha\alpha})_t + f(f_\alpha - f_{\alpha\alpha\alpha}) + 2(f - f_{\alpha\alpha})f_\alpha = 0,$$

which is the Camassa-Holm equation [**?**, 17, 20].

Here we prove three results. First, we demonstrate that the flow of the Reeb field is a geodesic, and we show that the sectional curvature is non-negative in all sections containing the Reeb field. It is then natural to ask whether there are conjugate points along the corresponding Reeb flow geodesic. We compute the Jacobi fields along this geodesic explicitly and find all such conjugate points. Having obtained conjugate points, it is natural to ask whether such points must be isolated and of finite order; we prove the answer is affirmative by showing that the differential of the exponential map is Fredholm. For simplicity of exposition we demonstrate only "weak" Fredholmness, though we show how one can use the technique of [26] to demonstrate strong Fredholmness in the context of Sobolev manifolds.

3.1.2 Sectional Curvature

The curvature of a Lie group G with right-invariant metric in the section determined by a pair of vectors X, Y in the Lie algebra $\mathfrak{g}$ is given by the following formula [2].

$$C(X,Y) = \langle d,d\rangle + 2\langle a,b\rangle - 3\langle a,a\rangle - 4\langle B_X, B_Y\rangle \tag{3.3}$$

where

$$2d = B(X,Y) + B(Y,X), \qquad 2b = B(X,Y) - B(Y,X),$$
$$2a = \mathrm{ad}_X Y, \qquad 2B_X = B(X,X), \qquad 2B_Y = B(Y,Y),$$

where B is the bilinear operator on $\mathfrak{g}$ given by the relation $\langle B(X,Y), Z\rangle = \langle X, \mathrm{ad}_Y Z\rangle$, i.e., $B(X,Y) = \mathrm{ad}_Y^* X$. Note that in terms of the usual Lie bracket of vector fields, we have $\mathrm{ad}_X Y = -[X,Y]$; see [2]. The sectional curvature is given by the normalization $K(X,Y) = C(X,Y)/|X \wedge Y|^2$, but since we only care about the sign, we will work with C instead of K.

In this section we will show that the curvature takes on both signs; in fact we will show that $C(E,Y) \geq 0$ for all Y, and that there are many sections such that $C(X,Y) < 0$.

Lemma 3.1.3. *If $X = S_\theta f$ and $Y = S_\theta g$, then*

$$\mathrm{ad}_X^* Y = S_\theta (1-\Delta)^{-1}\big[S_\theta f(g - \Delta g) + (n+2)E(f)(g-\Delta g)\big]. \tag{3.4}$$

Proof. Let $Z = S_\theta h$ for some function h. Then we have

$$\begin{aligned}
\langle \mathrm{ad}_X^* Y, Z\rangle = \langle Y, \mathrm{ad}_X Z\rangle &= -\int_M \langle S_\theta g, S_\theta\{f,h\}\rangle \, d\mu \\
&= -\int_M (g-\Delta g)\{f,h\}\, d\mu = -\int_M (g-\Delta g)\big(S_\theta f(h) - hE(f)\big)\, d\mu \\
&= \int_M h\big[S_\theta f(g-\Delta g) + (g-\Delta g)\big(\mathrm{div}(S_\theta f) + E(f)\big)\big]\, d\mu.
\end{aligned}$$

Now using the fact from [14] that $\mathrm{div}(S_\theta f) = (n+1)E(f)$ for an associated metric, we obtain

$$\langle \mathrm{ad}_X^* Y, Z\rangle = \int_M h\big[S_\theta f(g-\Delta g) + (n+2)E(f)(g-\Delta g)\big]\, d\mu.$$

Since this is true for every h, we conclude formula (4.4). $\square$

Combining Lemma 3.1.3 with the general formula (4.5), we obtain the following formula.

Theorem 3.1.4. *Suppose M is a contact manifold with associated Riemannian metric and a Killing Reeb field. Then the sectional curvature of $\mathcal{D}_\theta(M)$ is non-negative when one of the directions is the Reeb Field.*

Proof. Write $X = S_\theta(1) = E$ and $Y = S_\theta g$ for some function g. Then we have $\mathrm{ad}_X Y = -[X, Y] = -S_\theta\{f, g\} = -S_\theta(E(g))$ using (4.1).

Furthermore, we find that

$$B(X,Y) = \mathrm{ad}_Y^* X = S_\theta(1-\Delta)^{-1}\big[(n+2)E(g)\big],$$
$$B(Y,X) = \mathrm{ad}_X^* Y = S_\theta(1-\Delta)^{-1}\big[E(g-\Delta g)\big],$$

and we conclude that $B(X, X) = 0$. Note that there is no need to calculate $B(Y, Y)$ as it only appears in the curvature formula coupled with $B(X, X) = 0$.

Formula (4.5) yields

$$\begin{aligned} C(X,Y) = {}& \tfrac{1}{4}\langle S_\theta E(g) + S_\theta(1-\Delta)^{-1}[(n+2)E(g)], S_\theta E(g) + S_\theta(1-\Delta)^{-1}[(n+2)E(g)]\rangle \\ &+ \tfrac{1}{2}\langle S_\theta E(g), S_\theta E(g) - S_\theta(1-\Delta)^{-1}[(n+2)E(g)]\rangle - \tfrac{3}{4}\langle S_\theta E(g), S_\theta E(g)\rangle, \end{aligned}$$

and thus we have that

$$C(X,Y) = \tfrac{(n+2)^2}{4}|S_\theta(1-\Delta)^{-1}[E(g)]|^2.$$

In particular we have that $K(X, Y)$ is non-negative. $\square$

Observe that the sectional curvature $K(E, S_\theta g)$ is zero if and only if $E(g) \equiv 0$. If this is the case, $S_\theta g$ actually preserves the contact form (not just the contact structure); that is, if η is the flow of $S_\theta g$ then $\eta^*\theta = \theta$, and η is called a quantomorphism.

It would be interesting to determine whether there are any other velocities $X \in T_e\mathcal{D}_\theta(M)$ for which $C(X, Y) \geq 0$ for all Y. On the volumorphism group $\mathcal{D}_\mu(M)$ of a manifold M of dimension two or higher, for example, this is only true when X is a Killing field; see [17].

To demonstrate negative curvature, it is sufficient to work on the quantomorphism group, using the following result from [14].

Theorem 3.1.5 (EP). *If M is a contact manifold with a regular Reeb field E for which the orbits are all closed and of the same length, then the group of quantomorphisms $\mathcal{D}_q(M)$ consisting of those contactomorphisms η such that $\eta^*\theta = \theta$ is a closed and totally geodesic submanifold.*

As a consequence, the second fundamental form $B(X,Y)$ of the quantomorphism group in the contactomorphism group is zero, which means by the Gauss-Codazzi formula that

$$C_q(X,Y) = C_\theta(X,Y) + \langle B(X,X), B(Y,Y)\rangle - \langle B(X,Y), B(X,Y)\rangle = C_\theta(X,Y)$$

whenever X and Y are tangent to the quantomorphism group; that is, $X = S_\theta f$ and $Y = S_\theta g$ where $E(f) = E(g) = 0$. in other words, the curvature can be computed using only the quantomorphism group formulas, and these were worked out by Smolentsev [34].

Theorem 3.1.6 (S). *Let M be a contact manifold with a regular contact form; then the group of quantomorphisms has tangent space $T_e\mathcal{D}_q(M) = \{S_\theta f \mid E(f) \equiv 0\}$, and the curvature in the section spanned by $X = S_\theta f$ and $Y = S_\theta g$ is given by the formula*

$$\begin{aligned} C_q(X,Y) = \tfrac{1}{4}\int_M \{f,g\}^2\,d\mu + \tfrac{3}{4}\int_M \{f,g\}\Delta\{f,g\}\,d\mu + \tfrac{1}{2}\int_M \{f,g\}\big(\{f,\Delta g\} - \{g,\Delta f\}\big)\,d\mu \\ - \int_M \{f,\Delta f\}(1-\Delta)^{-1}\{g,\Delta g\}\,d\mu \\ + \tfrac{1}{4}\int_M \big(\{f,\Delta g\} + \{g,\Delta f\}\big)(1-\Delta)^{-1}\big(\{f,\Delta g\} + \{g,\Delta f\}\big)\,d\mu. \end{aligned} \quad (3.5)$$

Note that since the functions f and g are Reeb-invariant, we may equivalently think of them as defined on the Boothby-Wang quotient N, and use the Laplacian on N instead of the one on M. Note also that Smolentsev uses the opposite sign convention for the Laplacian Δ.

Example 3.1.7. Let f_k and g_k be two distinct eigenfunctions of the Laplacian on N which share the eigenvalue $-\lambda_k$, and let $h_k = \{f_k, g_k\}$. Smolentsev's formula (3.5) then reduces to

$$C_q(S_\theta f_k, S_\theta g_k) = \tfrac{1}{4}\int_M h_k^2 d\mu + \tfrac{3}{4}\int_M h_k\Delta h_k d\mu - \lambda_k\int_M h_k^2 d\mu$$

using the anti-symmetry of the bracket. Letting $-\lambda$ be an upper bound for the eigenvalues of the Laplacian we get the inequality

$$C_q(S_\theta f_k, S_\theta g_k) \leq \left(\tfrac{1}{4} - \tfrac{3}{4}\lambda - \lambda_k\right) \int_M h_k^2 \, d\mu.$$

Thus we get that the curvature in the section spanned by $X = S_\theta f_k$ and $Y = S_\theta g_k$ is negative whenever $\frac{1}{4} - \frac{3}{4}\lambda - \lambda_k < 0$. For example if $M = S^3$ with the round metric of radius 2, then the Boothby-Wang quotient is $N = S^2$ with the round metric of radius 1. All eigenvalues of the Laplacian on S^2 are of the form $n(n+1)$ for an integer n with multiplicity at least two, and thus this construction always gives infinitely many sections of negative curvature on $\mathcal{D}_\theta(S^3)$.

3.1.3 Conjugate Points

It is easy to see that the function $f \equiv 1$ is a steady solution of the Euler-Arnold contactomorphism equation (3.2), and hence the flow of the Reeb field E is a geodesic in $\mathcal{D}_\theta(M)$, with non-negative curvature in every section containing it. It is natural to ask whether there are conjugate points along this geodesic. We answer this question by solving the Jacobi equation explicitly and locating all the conjugate points; we find that they are all monoconjugate of finite order, an illustration of the fact that the exponential map is Fredholm (as we will show in the next section).

Recall that the exponential map $\exp_p$ on a Riemannian manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$ is the map $\exp_p(v) = \gamma(1)$ where γ is the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Its differential determines the conjugate points on $\mathcal{M}$: a point q is called conjugate to p along γ if $q = \gamma(\tau)$ for some τ and if $(d\exp_p)_{\tau\gamma'(0)}$ is not invertible as a map from $T_p\mathcal{M}$ to $T_q\mathcal{M}$. The exponential map is called Fredholm if $(d\exp_p)_v$ is a Fredholm linear operator for every p and $v \in T_p\mathcal{M}$; in this case the map is invertible if and only if it is one-to-one, and the nullspace is finite-dimensional. The following proposition ends up being the most convenient way to both compute conjugate points and to prove Fredholmness if $\mathcal{M}$ is a Lie group with right-invariant Riemannian metric.

Proposition 3.1.8. *[13, 26] Suppose we have a Lie group G with a right-invariant metric and a smooth geodesic $\eta(t)$ with $\eta(0) = e$ and $\dot\eta(0) = u_0$. Define linear operators $\Lambda(t)$ and K_{u_0} on T_eG by* *the formulas*

$$\Lambda(t)(v) = \mathrm{Ad}^*_{\eta(t)}\mathrm{Ad}_{\eta(t)}(v)$$

and

$$K_{u_0}(v) = \mathrm{ad}^*_v u_0.$$

Then the Jacobi equation solution operator

$$\Phi(t) = t dL_{\eta(t)^{-1}}(d\exp_e)_{tu_0}$$

satisfies the equation

$$\Phi(t) = \Omega(t) + \int_0^t \Lambda(\tau)^{-1} K_{u_0}\Phi(\tau)d\tau \tag{3.6}$$

where

$$\Omega(t) = \int_0^t \Lambda(\tau)^{-1} d\tau.$$

Proof. The proof follows from rewriting the Jacobi equation using left-translation: if $J(t)$ is a Jacobi field and we write $J(t) = dL_{\eta(t)}w(t)$, then $w(t)$ satisfies the equation

$$\frac{d}{dt}\left(\Lambda(t)\,\frac{dw}{dt}\right) + \mathrm{ad}^*_{dw/dt}u_0 = 0. \tag{3.7}$$

With initial conditions $w(0) = 0$ and $w'(0) = v_0$, we write $w(t) = \Phi(t)(v_0)$ and find that Φ satisfies (3.6). □

This proposition shows that a point $\eta(\tau)$ is conjugate to the identity if and only if $\Phi(\tau)$ is non-invertible. The operator Φ is particularly easy to compute in the case when the curve η is a family of isometries of the underlying manifold. First we demonstrate that the operator K_{u_0} is compact on the contactomorphism manifold.

Proposition 3.1.9. *Suppose M is a contact Riemannian manifold with associated Riemannian metric. Then for any fixed f with $u = S_\theta f$, the operator K_u is compact. Since it is also anti-selfadjoint, it has a basis of complex eigenvectors $v_k = S_\theta g_k$ such that $K_u(v_k) = i\lambda_k v_k$, with $\lambda_k \in \mathbb{R}$ and $\lambda_k \to 0$ as $k \to \infty$.*

Proof. From Lemma 3.1.3 we conclude that with $u = S_\theta f$ and $v = S_\theta g$,

$$K_u(S_\theta g) = S_\theta(1-\Delta)^{-1}\big[S_\theta g(\phi) + (n+2)E(g)\phi\big], \tag{3.8}$$

where $\phi = f - \Delta f$. From this equation we see that K_u gains two derivatives from the inverse Laplacian $(1-\Delta)^{-1}$ but only loses one derivative because of the contact operator S_θ, overall gaining a derivative. Thus K_u is a compact operator.

The fact that K_u is anti-selfadjoint follows from the equation

$$\langle K_u v, v\rangle = \langle \mathrm{ad}_v^* u, v\rangle = \langle u, \mathrm{ad}_v v\rangle = 0,$$

for any v. Finally the statement about eigenvalues follows from the fact that iK_u is a self-adjoint compact operator and general spectral theory. □

Theorem 3.1.10. *Suppose M is a contact manifold with an associated Riemannian metric and a regular Reeb field E that is also a Killing field. Let $\eta(t)$ be a geodesic on $\mathcal{D}_\theta(M)$ with initial condition $\eta(0) = e$ and $\dot{\eta}(0) = E$. Then $\eta(T)$ is conjugate to $\eta(0)$ for $T > 0$ if and only if*

$$T = \frac{2\pi m}{|\lambda|},$$

where λ is one of the real eigenvalues of iK_E as in Proposition 3.1.9.

Proof. Since E is a steady solution of (3.2), its flow $\eta(t)$ is a geodesic in $\mathcal{D}_\theta(M)$, and since E is a Killing field, every $\eta(t)$ is an isometry of M. We therefore have that the operator $\Lambda(t)$ defined in Proposition 3.1.8 is the identity, since we have for every vector field v on M that

$$\langle \Lambda(t)v, v\rangle = |\mathrm{Ad}_{\eta(t)} v|^2 = \int_M |D\eta(t)(v)|^2 \circ \eta(t)^{-1}\, d\mu.$$

The fact that $\eta(t)$ is an isometry implies that $|D\eta(t)v_x|^2 = |v_x|^2$ for every point $x \in M$, and in addition that the Jacobian of η is one, so that $\langle \Lambda(t)v, v\rangle = \langle v, v\rangle$ for every vector field v, and hence $\Lambda(t)$ is the identity for all t.

Since K_E is diagonalizable by Proposition 3.1.9, equation (3.7) diagonalizes as well: if $K_{u_0}(v_k) = i\lambda_k v_k$ for some k with $\lambda_k \in \mathbb{R}$, then equation (3.7) for $w(t) = f(t)v_k$ where $f\colon \mathbb{R} \to \mathbb{C}$

takes the form $f''(t) + i\lambda f'(t) = 0$, whose solution with $f(0) = 0$ and $f'(0) = 1$ is obviously

$$f(t) = \frac{i}{\lambda}(e^{-i\lambda t} - 1).$$

We therefore get a conjugate point at time $T = 2\pi/|\lambda|$, and at all integer multiples thereof. $\square$

In general we can write

$$K_E(S_\theta g) = (n+2)S_\theta(1-\Delta)^{-1}E(g).$$

Example 3.1.11. On the 3-sphere where E is a left-invariant vector field, the operators Δ and $\mathcal{L}_E := g \mapsto E(g)$ commute (since E is Killing) and have a basis of simultaneous eigenfunctions g_{pq} such that $\Delta g_{pq} = -q(q+2)g_{pq}$ and $\mathcal{L}_E(g_{pq}) = ipg_{pq}$ whenever q is a positive integer and p is an integer in the set $\{-q, -q+2, -q+4, \cdots, q-4, q-2, q\}$; see [28]. In this case we get

$$K_E(S_\theta(g_{pq})) = \frac{3ip}{(q+1)^2} S_\theta g_{pq},$$

and we obtain conjugate points along the Reeb geodesic at times $T = \frac{2\pi m(q+1)^2}{3p}$ for q any positive integer, p any positive integer with $p \leq q$ and $q - p$ even, and m any positive integer.

3.1.4 The Exponential Map

Now we would like to show Fredholmness of the exponential map. In some sense the three-dimensional contactomorphism equation is a hybrid of the Camassa-Holm equation and the two-dimensional Euler equation for ideal fluids as discussed in [14], and both of these diffeomorphism groups have strongly Fredholm exponential maps [13, 26]. Hence intuitively we would expect the same on the contactomorphism group. To prove this, we use Proposition 3.1.8.

The point is that by Proposition 3.1.8, we can essentially decompose the differential of the exponential map into the sum of operators $\Omega(t)$ (which is positive-definite and thus invertible) and a remainder expressed as a composition with K_{u_0}. Since K_{u_0} is compact by Proposition 3.1.9, we know $(d\exp_e)_{tu_0}$ will be a Fredholm operator, and hence the exponential map will be a non-linear Fredholm map. As a consequence [26] we obtain that conjugate points are of finite multiplicity and form a discrete set along any geodesic, along with various other analogues of theorems in global Riemannian geometry which would otherwise fail in infinite dimensions.

Theorem 3.1.12. *The Riemannian exponential map on $\mathcal{D}_\theta(M)$ is weakly Fredholm; that is, the differential of the exponential map extends to a Fredholm operator in the closure of $T_e\mathcal{D}_\theta(M)$ in the L^2 topology generated by the Riemannian metric.*

Proof. Using the notation from Proposition 3.1.8, we would like to show that the solution operator Φ is the sum of an invertible operator and a compact operator, thus making the exponential map Fredholm. By Proposition 3.1.9, we know K_{u_0} is compact, so we just need to know that $\Omega(t)$ is invertible for every t. It is sufficient to show that $\Omega(t)$ is positive-definite.

For $v = S_\theta g$ and $\eta(t)$ a geodesic, we have that $\eta(t)^*\theta = e^{\Sigma(t)}\theta$ for some $\Sigma(t)\colon M \to \mathbb{R}$, and we compute that

$$\mathrm{Ad}_\eta S_\theta g = S_\theta\left((e^{\Sigma}g)\circ\eta^{-1}\right).$$

Now we have that

$$\begin{aligned}
\langle v, \mathrm{Ad}^*_{\eta(t)}\mathrm{Ad}_{\eta(t)}v\rangle &= \langle \mathrm{Ad}_{\eta(t)}v, \mathrm{Ad}_{\eta(t)}v\rangle \\
&= \int_M (1-\Delta)\left((e^{\Sigma}g)\circ\eta^{-1}(t)\right)\left((e^{\Sigma}g)\circ\eta^{-1}(t)\right)\,d\mu \\
&= \int_M \left(e^{\Sigma}g\right)^2\circ\eta^{-1}(t)d\mu + \int_M \left|\nabla\cdot\left((e^{\Sigma}g)\circ\eta^{-1}(t)\right)\right|^2\,d\mu \\
&= \int_M \left(e^{\Sigma}g\right)^2\mathrm{Jac}(\eta(t))d\mu + \int_M \left|D\eta^{-1}(t)\circ\eta(t)\nabla\cdot(e^{\Sigma}g)\right|^2\mathrm{Jac}(\eta(t))\,d\mu
\end{aligned} \tag{3.9}$$

where the second to last line is justified by integration by parts and the last line is justified by a change of variables.

Now consider $D\eta^{-1}(t)\circ\eta(t) = (D\eta(t))^{-1}$, since we would like to bound the quantity

$$\langle v, \mathrm{Ad}^*_{\eta(t)}\mathrm{Ad}_{\eta(t)}v\rangle / \langle v, v\rangle$$

uniformly above by some positive number. In order to do this, we look at the eigenvalues of $D\eta^\dagger D\eta(t)$ and take the supremum over all of M; we will call this supremum $\alpha(t)$, which is finite since M is compact. Thus we have

$$\int_M \left|D\eta^{-1}(t)\circ\eta(t)\nabla\cdot(e^{\Sigma}g)\right|^2\mathrm{Jac}(\eta(t))\,d\mu \le \alpha(t)\int_M \left|\nabla\cdot(e^{\Sigma}g)\right|^2\mathrm{Jac}(\eta(t))d\mu.$$

Finally using the fact that $\mathrm{Jac}(\eta(t)) = e^{(n+1)\Sigma(t)}$ by [14] and spatial smoothness of $\Sigma(t)$, we obtain the desired upper bound in terms of the suprema of $\Sigma(t)$ and $\alpha(t)$, which depend on the C^1 norm of η.

Thus we get for each t, we have that $\langle v, \mathrm{Ad}^*_{\eta(t)}\mathrm{Ad}_{\eta(t)}v\rangle/\langle v, v\rangle$ is bounded above by a positive number independent of v. Now integrating in time we see that $\Omega(t)$ is also positive-definite, thus invertible. $\square$

We can prove strong Fredholmness using essentially the same techniques as [26]: approximate $\eta \in \mathcal{D}^s_\theta(M)$ by $\tilde{\eta} \in \mathcal{D}_\theta(M)$ (i.e., a C^∞ geodesic with initial velocity $\tilde{u}_0$) so that Proposition 3.1.8 (which loses derivatives) makes sense in H^s. Then apply commutator estimates to show that $\Lambda(t)$ is the sum of a positive-definite and a compact operator, so that $\Omega(t)$ is as well, and conclude that $(d\exp_e)_{t\tilde{u}_0}$ is a Fredholm operator. Then the fact that Fredholm operators are open in the space of all operators implies that $(d\exp_e)_{tu_0}$ is also Fredholm for $u_0 \in H^s$ sufficiently close to $\tilde{u}_0$. We omit the details, which are very similar to those of [26] due to the fact that (3.9) is so similar to the corresponding operator for the Camassa-Holm equation.

Chapter 4

$\mathcal{D}_\theta(M)$ with L^2 Metric on Stream Functions

This paper was published in *Annals of Global Analysis and Geometry* in 2016. It appears in volume **49**, pp. 205-2011.

Here in this paper we study the contactomorphism group $\mathcal{D}_\theta(M)$ with the L^2 metric on its stream functions. We show that the sectional curvature is always non-negative. Next we show that the Riemannian exponential map is not a local C^1 diffeomorphism of the identity. Lastly we show that the quatomorphism group under this same metric is a closed, totally geodesic submanifold of $\mathcal{D}_\theta(M)$.

4.1 Contactomorphism Group with the L^2 Metric on Stream Functions

4.1.1 Introduction

Let M be an orientable compact, contact manifold (without boundary) of odd dimension $2n+1$. Recall that a manifold M is a **contact manifold** if there exists a 1-form θ which satisfies the non-degeneracy condition that $\theta \wedge d\theta^n \neq 0$ everywhere [7]. We call θ the **contact form**. If we let $\mathcal{D}(M)$ be the group of diffeomorphisms of M, we say that $\eta \in \mathcal{D}(M)$ is a **contactomorphism** if $\eta^*\theta$ is some positive functional multiple of θ. We will denote the contactomorphism group by $\mathcal{D}_\theta(M)$. $\mathcal{D}_\theta(M)$ can be thought of as an infinite dimensional Riemannian manifold using the framework of Arnold [3].

The diffeomorphism group of the circle, $\mathcal{D}(S^1)$, has been heavily studied and has interesting applications to fluid mechanics. Depending on the metric, some classical PDE arise as the geodesic equation on $\mathcal{D}(S^1)$ such as the right-invariant Burgers' equation and the Camassa-Holm equation. It was shown that the Riemannian exponential map is not a local C^1 map for the L^2 metric [10]. This is not the case when they considered the H^1 metric. Later it was shown that $\mathcal{D}(S^1)$ has vanishing geodesic distance for the H^s metric if and only if $s \leq 1/2$ [4, 24].

The contactomorphism group has been studied before but in many different contexts. Smolentsev [33, 35] worked with the quantomorphism group $\mathcal{D}_q(M)$, which is the group of diffeomorphisms which exactly preserve the contact form, with the bi-invariant L^2 metric on stream functions. In [9, 14], $\mathcal{D}_\theta(M)$ was studied with the L^2 metric on velocity fields which in turn becomes the full H^1 metric on stream functions. In this paper, we consider $\mathcal{D}_\theta(M)$ with the L^2 metric on its stream functions. The geometric differences of $\mathcal{D}_\theta(M)$ with these two metrics are apparent just as in the case of $\mathcal{D}(S^1)$. As $\mathcal{D}(S^1)$ coincides with $\mathcal{D}_\theta(S^1)$ trivially, we view $\mathcal{D}_\theta(M)$ as a natural generalization to $\mathcal{D}(S^1)$. In [32], Shelukhin considers the L^∞ norm on the contactomorphisms isotopic to the identity and shows how that induces a bi-invariant distance function on the full $\mathcal{D}_\theta(M)$.

We summarize the results of this paper as follows. First we show that $\mathcal{D}_\theta(M)$ has non-negative sectional curvature. Next we prove that the Riemannian exponential map is not a local C^1 map. Lastly, we show that the quantomorphism group $\mathcal{D}_q(M)$ is a totally geodesic submanifold of $\mathcal{D}_\theta(M)$.

I would like to thank Martin Bauer for suggesting this idea and also my advisor Stephen C. Preston for the helpful discussion. I gratefully acknowledge the support of ESI in Vienna and the Simons Foundation Collaboration Grant #318969.

4.1.2 Geometric Background

We will be working primarily on the Lie algebra of $\mathcal{D}_\theta(M)$, and we will use the following well-known fact that the Lie algebra $T_e\mathcal{D}_\theta(M)$ can be identified with the space of smooth functions $f\colon M \to \mathbb{R}$.

Proposition 4.1.1. *[14] The Lie algebra $T_e\mathcal{D}_\theta(M)$ consists of vector fields u such that $\mathcal{L}_u\theta = \lambda\theta$ for some function $\lambda : M \to \mathbb{R}$. Any such field is uniquely determined by the function $f = \theta(u)$, and* *we write $u = S_\theta f$. Thus we have that*

$$T_e\mathcal{D}_\theta(M) = \{S_\theta f : f \in C^\infty(M)\}.$$

Here we call S_θ the contact operator. The Lie bracket on $T_e\mathcal{D}_\theta(M)$ is given by

$$[S_\theta f, S_\theta g] = S_\theta\{f,g\}, \text{ where } \{f,g\} = S_\theta f(g) - gE(f); \tag{4.1}$$

here E denotes the Reeb vector field, uniquely specified by the conditions $\theta(E) = 1$, $\iota_E d\theta = 0$. We call $\{\cdot,\cdot\}$ the "contact Poisson bracket"; it is not a true Poisson bracket since it does not satisfy Leibniz's rule.

We also need a Riemannian structure on (M,θ), and we will require that the Riemannian metric be **associated** to the contact form. It will also be convenient to assume that E is a Killing field (i.e., its flow consists of isometries).

Definition. *If (M,θ) is a contact manifold and E is the Reeb field, a Riemannian metric $(\cdot,\cdot)_g$ is* ***associated*** *if it satisfies the following conditions:*

(1) $\theta(u) = (u,E)_g$ for all $u \in TM$, and

(2) there exists a $(1,1)$-tensor field ϕ such that $\phi^2(u) = -u + \theta(u)E$ and $d\theta(u,v) = (u,\phi v)_g$ for all u and v.

If in addition E is a Killing field, we say that that (M,θ,g) is K***-contact****.*

Now if we have a K-contact manifold (M,θ,g), we define a right-invariant metric $\langle\cdot,\cdot\rangle$ on $\mathcal{D}_\theta(M)$ by

$$\langle S_\theta f, S_\theta g\rangle = \int_M fg\,d\mu \tag{4.2}$$

Lemma 4.1.2. *With $X = S_\theta f$ and $Y = S_\theta g$ we have that*

$$\mathrm{ad}^*_X Y = S_\theta[S_\theta f(g) + g(n+2)E(f)]$$

Proof. Let $X = S_\theta f$, $Y = S_\theta g$, and $Z = S_\theta h$ so we have

$$\langle \mathrm{ad}^*_X Y, Z\rangle = \langle ad^*_{S_\theta f} S_\theta g, S_\theta h\rangle = \langle S_\theta g, \mathrm{ad}_{S_\theta f} S_\theta h\rangle = -\int_M g S_\theta f(h) d\mu + \int_M gE(f)hd\mu =$$
$$\int_M hS_\theta f(g) + hg\mathrm{div} S_\theta f + hgE(f)d\mu = \int_M (S_\theta f(g) + g(n+1)E(f) + gE(f))hd\mu. \quad (4.3)$$

Thus we have that

$$\mathrm{ad}^*_X Y = S_\theta[S_\theta f(g) + g(n+2)E(f)] \quad (4.4)$$

□

On any Lie group with a right-invariant Riemannian metric, the geodesic equation [3] can be written in terms of the flow equation

$$\frac{d\eta}{dt} = u \circ \eta$$

and the Euler-Arnold equation

$$\frac{du}{dt} + \mathrm{ad}^*_u u = 0.$$

In this case, the Euler-Arnold equation becomes

$$\tfrac{df}{dt} + f(n+3)E(f) = 0.$$

Example 4.1.3. *For $M = S^1$ with the coordinate being α and the standard $1-$form being $d\alpha$ we get that the Reeb field is $E = \frac{d}{d\alpha}$ and the contact operator is $S_\theta f = fE$. Thus the geodesic equation on the circle becomes*

$$\tfrac{df}{dt} + 3ff_\alpha = 0.$$

This is the right-invariant Burgers' equation which is studied in [10]. It is usual Euler-Arnold equation on $\mathcal{D}(S^1)$, the diffeomorphism group of the circle.

In the above example of the circle, the diffeomorphism group, which is studied in [10, 24], coincides with the contactomorphism group. As we will see later in this paper, the contactomorphism group shares many properties with the diffeomorphism group of the circle with the L^2 right-invariant metric. Thus, we view $D_\theta(M)$ as a generalization of $\mathcal{D}(S^1)$.

4.1.3 The Curvature

In [21] it was shown that the contactomorphism group is a regular smooth Lie group. The curvature of a Lie group G with right-invariant metric in the section determined by a pair of vectors X, Y in the Lie algebra $\mathfrak{g}$ is given by the following formula [2].

$$C(X,Y) = \langle d, d\rangle + 2\langle a, b\rangle - 3\langle a, a\rangle - 4\langle B_X, B_Y\rangle \tag{4.5}$$

where

$$2d = B(X,Y) + B(Y,X), \qquad 2b = B(X,Y) - B(Y,X),$$
$$2a = \mathrm{ad}_X Y, \qquad 2B_X = B(X,X), \qquad 2B_Y = B(Y,Y),$$

where B is the bilinear operator on $\mathfrak{g}$ given by the relation $\langle B(X,Y), Z\rangle = \langle X, \mathrm{ad}_Y Z\rangle$, i.e., $B(X,Y) = \mathrm{ad}_Y^* X$. Note that in terms of the usual Lie bracket of vector fields, we have $\mathrm{ad}_X Y = -[X,Y]$. The sectional curvature is then given by the normalization $K(X,Y) = C(X,Y)/|X \wedge Y|^2$. But here we only care about the sign so we will work with C only.

Next we will show that the sectional curvature will always be non-negative.

Theorem 4.1.4. *The sectional curvature is nonnegative.*

Proof. With $X = S_\theta f$ and $Y = S_\theta g$, we use the above formula and (4.4) to compute

$$\begin{aligned} C(X,Y) = \tfrac{1}{4}\int_M &[(n+3)(fE(g) + gE(f))]^2 \\ &- 2[(\{f,g\})(S_\theta g(f) - S_\theta f(g) + (n+2)(fE(g) - gE(f))] \\ &- 3[\{f,g\}^2] - 4[(n+3)^2 fE(f)gE(g)]d\mu \\ = \tfrac{1}{4}\int_M &[(n+3)(fE(g) + gE(f))]^2 \\ &- 2(n+3)\{f,g\}(fE(g) - gE(f)) + \{f,g\}^2 \\ &- 4[(n+3)^2 fE(f)gE(g)]d\mu \end{aligned}$$

because $S_\theta g(f) - S_\theta f(g) + fE(g) - gE(f) = 2\{f,g\}$ by antisymmetry of the contact Poisson bracket. Now since

$$(fE(g) + gE(f))^2 - 4fE(f)gE(g) = (fE(g) - gE(f))^2,$$

we have that the non-normalized sectional curvature is given by

$$C(X,Y) = \tfrac{1}{4}\int_M [\{f,g\} - (n+3)(fE(g) - gE(f))]^2 d\mu.$$

□

Here we can immediately see how the geometry of $\mathcal{D}_\theta(M)$ changes when we consider the L^2 metric on stream functions rather than the H^1 metric where it was shown in [9] that the curvature can take on any sign.

4.1.4 Geodesics

From (4.4) we have that the flow equation and Euler-Arnold equation are given by

$$\tfrac{\partial \varphi}{\partial t} = u \circ \varphi \tag{4.6}$$

$$\tfrac{\partial f}{\partial t} + 3fE(f) = 0 \tag{4.7}$$

where $u = S_\theta f$.

Let $(x,z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ be Darboux coordinates for our contact manifold and thus the contact form is given by

$$\alpha = dz - \sum y_i dx_i$$

and the Reeb field is given by

$$E = \tfrac{\partial}{\partial z}.$$

Now given an initial condition, f_0, we are able to solve this first order PDE implicitly in these coordinates to get

$$f(t, x, z - 3tf_0(x,z)) = f_0(x,z).$$

Note that this solution does not describe trajectories.

In [32], it was shown that given the metric (4.2), the energy functional is in fact degenerate. Thus, just as in the case of the diffeomorphism group of the circle with the L^2 metric [24], $\mathcal{D}_\theta(M)$ has vanishing geodesic distance.

4.1.5 The Exponential Map

Let $\varphi(t; v)$ be the geodesic starting at the identity and in the direction of v. Recall that the exponential map $\exp_p$ on a Riemannian manifold M at a point $p \in M$ is defined by the geodesic flow at time 1. Explicitly, it is defined as $\exp_p(v) = \varphi(1, v)$. Next we will show that as with the case of the diffeomorphism group of the circle [10]; $\mathcal{D}_\theta(M)$ with the L^2 metric on stream functions has an exponential map which is not locally C^1.

Theorem 4.1.5. *The Riemannian exponential map of the L^2 right invariant metric on stream functions of $\mathcal{D}_\theta(M)$ is not a C^1 map from a neighborhood of zero in $T_e\mathcal{D}_\theta(M)$ to $\mathcal{D}_\theta(M)$.*

Proof. Let's assume for a contradiction that exp is a C^1 map.

Consider the curve given by $t \mapsto tu_0$ with $t > 0$ and $u_0 \in T_e\mathcal{D}_\theta(M)$. For t small enough we have that $\exp(tu_0) = \varphi(1; tu_0) = \varphi(t; u_0)$ we compute

$$\frac{d}{dt}\exp(tu_0)\bigg|_{t=0} = \frac{d}{dt}\varphi(t; u_0)\bigg|_{t=0} = u_0$$

so we have that $D\exp(0)$ is the identity.

Now we would like to show that exp is not invertible in a neighborhood of $u_0 \in T_e\mathcal{D}_\theta(M)$ so we consider the Jacobi fields. Let $\eta(t)$ be a smooth geodesic with $\eta(0) = e$ and $\dot{\eta}(0) = u_0$ so that every Jacobi field satisfies

$$\tfrac{\partial}{\partial t}\left(\mathrm{Ad}_\eta^*\mathrm{Ad}_\eta \tfrac{\partial v}{\partial t}\right) + \mathrm{ad}^*_{\frac{\partial v}{\partial t}} u_0 = 0$$

with $J(t) = dL_\eta v$. This equation is obtained by left translating the Jacobi equation [13, 26]. Now since E is a steady state solution to the Euler-Arnold equation, we have that its flow is geodesic in $\mathcal{D}_\theta(M)$, and since E is a Killing field, we have that $\eta(t)$ is an isometry of M. Thus $\mathrm{Ad}_\eta^*\mathrm{Ad}_\eta$ is the identity for all time. Now with $v = S_\theta g$ and $u_0 = S_\theta f_0$ for $g, f_0 \in C^\infty(M)$ and setting $f_0 = c > 0$, we can rewrite the Jacobi equation as

$$\tfrac{\partial^2 g}{\partial t} + c(n+2)E(\tfrac{\partial g}{\partial t}) = 0.$$

We set $w = \frac{\partial g}{\partial t}$ with initial condition w_0 and locally, in Darboux coordinates (x, z) we have that the above equation becomes

$$\tfrac{\partial w}{\partial t} + c(n+2)\tfrac{\partial w}{\partial z} = 0$$

thus solving for g we get

$$g(x, z, t) = \tfrac{1}{(c(n+2))^2} \int_z^{z-c(n+2)t} w_0(x, s)ds.$$

So letting $c_m = \frac{1}{m}$, we have that $w_0 = \sin\left(\frac{2\pi m}{n+2} z\right)$ gets annihilated at the points $S_\theta c_m = c_m E$. Thus we have that the $D\exp(c_m E)$ fails to be invertible at points near zero. That is because c_m is a sequence going to zero so in any topology, $c_m E$ also approaches zero. This violates the Inverse Function Theorem which gives us our desired contradiction. □

4.1.6 The Quantomorphism Group

In this section we will be considering the group of quantomorphisms. That is, the contactomorphisms which exactly preserve the contact form, not just the structure. This can be written as

$$\mathcal{D}_q(M) = \{\eta \in \mathcal{D}_\theta(M) : \eta^*\theta = \theta\}.$$

A contact form is said to be **regular** if the Reeb field induces a free action of the unit circle on M. If a contact form is regular, we are able to define the Boothby-Wang quotient [7] manifold $M/S^1 = N$ and the 2-form $d\theta$ can be then used to define a symplectic structure ω on N by

$$\pi^*\omega = d\theta,$$

where $\pi : M \to N$.

Theorem 4.1.6. *If (M, θ, g) is a K-contact manifold with Reeb field E. If θ is a regular contact form, then $D_q(M)$ is a closed and totally geodesic submanifold of $D_\theta(M)$.*

Proof. In order to show that a submanifold is totally geodesic, it is equivalent to show that the second fundamental form vanishes identically. To do so, it suffices to show that $\langle \nabla_u u, v\rangle = 0$ whenever u is tangent and v is orthogonal to the submanifold. For a right-invariant metric on a Lie group, we have that $\nabla_u u = \mathrm{ad}_u^* u$. Thus we would like to show that

$$\langle u, \mathrm{ad}_u v\rangle = 0$$

whenever $u \in T_e\mathcal{D}_q(M)$ and $v \in T_e\mathcal{D}_\theta(M)$ with v orthogonal to $T_e\mathcal{D}_q(M)$.

So let $u = S_\theta f \in T_e\mathcal{D}_q(M)$ and $v = S_\theta g \in T_e\mathcal{D}_\theta(M)$ orthogonal to $T_e\mathcal{D}_q(M)$.

$$\begin{aligned}\langle \nabla_u u, v\rangle &= \langle \mathrm{ad}_u^* u, v\rangle = \langle u, \mathrm{ad}_u v\rangle \\ &= -\int_M (S_\theta f, S_\theta\{f,g\})_g d\mu = -\int_M f\{f,g\} d\mu \\ &= -\int_M f S_\theta f(g) d\mu = \int_M g(E(f) + f\mathrm{div} S_\theta f) d\mu = 0\end{aligned} \tag{4.8}$$

□

From Smolentsev [33, 35], we can see that the quantomorphism group admits a Riemannian submersion onto the symplectomorphism group of the Boothby-Wang quotient. Let (M, θ, g) be a K-contact manifold with regular contact form θ. The vector fields of the quantomorphism group, $T_e\mathcal{D}_q(M)$, are those $V \in T_e\mathcal{D}_\theta(M)$ such that $\mathcal{L}_V\theta = 0$. Now let $N = M/S^1$ with ω the induced symplectic structure by $\pi : M \to N$ and let $D_\omega(N)$ be the group of symplectomorphisms of N

$$\mathcal{D}_\omega(N) = \{\eta \in \mathcal{D}(N) : \eta^*\omega = \omega\}.$$

Now $T_e\mathcal{D}_\omega(N)$ consists of the vector fields V such that $\mathcal{L}_V\omega = 0$. We call a vector field V Hamiltonian if we can associate a function H such that $\omega(\cdot, V) = dH(\cdot)$. In order for this definition to be unambiguous, we require that the Hamiltonians have mean zero.

For $V \in T_e\mathcal{D}_q(M)$, we have that $[V, E] = 0$ and thus $T_e\mathcal{D}_q(M) \to T_e\mathcal{D}_\omega(N)$ is a projection. We can see that elements of $T_e\mathcal{D}_q(M)$ are of the form $V = hE + X$. These vector fields project onto $T_e\mathcal{D}_\omega(N)$ by $d\pi \circ X = Y \circ \pi$ with $Y \in T_e\mathcal{D}_\omega(N)$. Here we have that $\mathcal{L}_V\theta$ implies that $E(h) = 0$ so that h is constant in the Reeb direction. Now combined with the fact that we require our stream functions and Hamiltonians to have mean zero, we can see that the map

$$d\pi : \ker(d\pi)^{\perp} \to TD_{\omega}(N)$$

is an isometry by scaling the one of the volume forms by a constant. Thus the projection of $\mathcal{D}_q(M)$ onto $\mathcal{D}_{\omega}(N)$ is a Riemannian submersion.

Chapter 5

$\mathcal{D}(S^1)/S^1$ with $\dot{H}^{1/2}$ metric

5.1 Introduction

The last two chapters deal with the H^1 and L^2 metrics on the space of contactomorphisms. Here in this chapter we will discuss the an in between case. In particular we will investigate the $\dot{H}^{1/2}$ metric on the space of contactomorphisms of the circle [5, 8, 9]. This metric is given by

$$\langle S_\theta f, S_\theta g\rangle_{1/2} = \int_{S^1} \Lambda f g dx$$

where the operator Λ is given by

$$\Lambda(f) = H f_x$$

where H is the Hilbert transform. The circle is a contact manifold with the trivial contact form $\theta = dx$ Because of this fact have that the contactomorphism group $\mathcal{D}_\theta(S^1)$ coincides with the regular diffeomorphism group. Now instead of considering the space $\mathcal{D}(S^1)$ we will consider the homogeneous space $\mathcal{D}(S^1)/S^1$, the diffeomormphism group modulo rotations of the circle. The reason we do this is to make the $\dot{H}^{1/2}$ a non-degenerate metric.

Kirillov and Yur'ev studies with the space $\mathcal{D}(S^1)/S^1$ with the $\dot{H}^{1/2}$ metric and puts a Kähler structure on it [18, 19]. Here the symplectic form is given by

$$\omega(u, v) = \int_{S^1} u_x v dx$$

and the almost complex structure is the Hilbert transform. It is clear that under these conditions that $\mathcal{D}(S^1)/S^1$ is in fact a Kähler manifold. In particular, ω is a symplectic form, the Hilbert transform is an almost complex structure, and ω is compatible with the $\dot{H}^{1/2}$ metric.

This space under this metric also holds important applications in fluid mechanics. In the case of the circle we have that the Reeb vector field is $\frac{d}{dx}$ and the contact operator is $S_\theta f = f\frac{d}{dx}$. Using this we can compute the Euler-Arnold equation which becomes

$$Hu_{xt} + uHu_x + 2u_xHu_x = 0$$

which is the Wunsch equation [27, 38]. It is known that every solution of this equation has blowup [5, 29].

The space $\mathcal{D}(S^1)/S^1$ also has important applications to Teimüller theory. Teo describes conformal welding on the universal Teichmüller curve as the set of mappings

$$\tilde{C} \to \mathcal{D}(S^1)/S^1$$

where $\tilde{C}$ is the set of smooth, closed, simple curves whic have conformal radius 1 which are centered at the origin. Also the Euler-Arnold equation for the universal Teichmüller curve using the Velling-Kirillov metric [37] is the same as the Wunsch equation described above.

5.2 Curvature Computation

We can try to compute the curvature of $\mathcal{D}(S^1)/S^1$ using the Arnold formula by first writing each of the vector fields in terms of the basis functions $\{\sin nx\}$ and $\{\cos mx\}$ but we find that the computation gets complicated very quickly. Whether or not the curvature is always positive or not is an open problem. There exists partial results on this problem. For example, the sectional curvature between $\sin nx$ and $\sin mx$ has been computed for every $\dot{H}^s$ metric when $s > 0$ [5].

Also one of the facts we rely on is how easily it is to compute the Hilbert transform on these basis functions and the fact that the Hilbert transform is linear. In particular we have that

$$H(\sin nx) = -\cos nx \quad \text{and} \quad H(\cos mx) = \sin mx.$$

We can see in that in the Arnold formula for sectional curvature, the terms consist of combinations of

$$\mathrm{ad}_u v = -u_x v + uv + x$$

and

$$\mathrm{ad}_u^* v = 2u_x H v_x + u H v_{xx}.$$

It is easy to see that span$\{\sin nx, \cos mx\}$ is an algebra which is closed under taking derivatives and the Hilbert transform. The fact that it is closed under multiplication relies on the sum or difference of angles. With general indices, it is difficult to use the sum and difference formulas so with all of this in mind, we decided to write a program which computes the sectional curvature of $\mathcal{D}(S^1)/S^1$.

It is possible to use engines such as Mathematica or Maple which are powerful enough to directly compute Hilbert transforms as well as derivatives but these algorithms are extremely slow. In fact these programs fail to run when given inputs have more than a couple of non-zero coefficients. So we take advantage of the fact of how we can "trick" Python into computing both Hilbert transforms and derivatives when these vector fields are in terms of their basis functions.

We will spend the rest of this section explaining the curvature code. In the code we compute the non-normalized sectional curvature in the directions of u and v. Having those two vector fields written in the basis functions of $\{\sin nx\}$ and $\{\cos mx\}$, we then separate them into 4 arrays of coefficients corresponding to the sine and cosine components of u and v. Once we have these 4 arrays, it is straight forward to compute the first two derivatives and their Hilbert transforms. In particular, to compute the derivative of an array of sine coefficients, we put the coefficients in the cosine array and multiply the entry by its position in the array. To compute the Hilbert transform of an array of sine coefficients we use that fact that

$$H(\sin nx) = -\cos nx$$

and put the coefficients into the cosine array and multiply by (-1).

Now we use the Arnold formula for the non-normalized sectional curvature [2] given by

$$C(u,v) = \langle \delta, \delta \rangle - 2\langle \alpha, \beta \rangle - 3\langle \alpha, \alpha \rangle - 4\langle B_u, B_v \rangle$$

where

$$2\alpha = \mathrm{ad}_u v, \quad 2\beta = \mathrm{ad}_u^* v - \mathrm{ad}_v^* u, \quad 2\delta = \mathrm{ad}_u^* v + \mathrm{ad}_v^* u,$$

$$2B_u = \mathrm{ad}_u^* u, \quad \text{and} \quad 2B_v = \mathrm{ad}_v^* v.$$

Also we have that

$$\mathrm{ad}_u v = -u_x v + u v_x$$

and

$$\mathrm{ad}_u^* v = 2u_x H v_x + u H v_{xx}.$$

Next we compute all 5 terms of the Arnold formula by multiplying the appropriate terms together. We do the multiplication by putting the terms into a matrix. For example, when multiplying a cosine array with a sine array, we put the terms in an array and the (i, j)-th entry of the matrix is the coefficient of $\cos ix \sin jx$ in the multiplication. There will be 3 matrices for each of the terms of the Arnold formula: a matrix of coefficients of $\cos ix \cos jx$ terms, matrix of coefficients of $\sin ix \sin jx$ terms, and matrix of coefficients of $\cos ix \sin jx$ terms. Once that is complete, it is then easy to translate these matrices back into sin and cos arrays using the sum and difference of angles formulas. In particular we have that

$$\begin{aligned}
\cos nx \cos mx &= \frac{1}{2}\left(\cos(n+m)x + \cos(n-m)x\right) \\
\sin nx \sin mx &= \frac{1}{2}\left(\cos(n-m)x - \cos(n+m)x\right) \\
\cos nx \sin mx &= \frac{1}{2}\left(\sin(n+m)x - \sin(n-m)x\right)
\end{aligned}$$

Lastly we use the inner product on $\mathcal{D}(S^1)/S^1$ given by

$$\langle u, v\rangle = \int_{S^1} \Lambda u v dx$$

where $\Lambda u = H u_x$ and the orthonormality of the basis functions

$$\{\sin nx\} \quad \text{and} \quad \{\cos mx\}$$

to compute the "1", "2", "3", and "4" terms of the Arnold formula to then add the terms together. The code then prints the non-normalized sectional curvature and if the result is negative, the program stops and prints the inputs.

5.3 Examples

We computed these examples by hand in order to give us some intuition on how to approach the coding of the program and also as a method for debugging the code. We believe these 4 examples are sufficient as they demonstrate the complete use of the Arnold formula for sectional curvature. Here we will use J to denote the Hilbert transform. The code that we have programmed and discussed in the previous section is in the Appendix.

5.3.1 example 1

Let

$$u = \sin x \quad \text{and} \quad v = \sin 2x$$

So now we have that

$$\begin{array}{llll} u = \sin x & Ju = -\cos x & v = \sin 2x & Jv = -\cos 2x \\ u_x = \cos x & Ju_x = \sin x & v_x = 2\cos 2x & Jv_x = 2\sin 2x \\ u_{xx} = -\sin x & Ju_{xx} = \cos x & v_{xx} = -4\sin 2x & Jv_{xx} = 4\cos 2x \end{array}$$

$$2\alpha = -u_x v + u v_x = -\cos x \sin 2x + 2\sin x \cos 2x = -\frac{1}{2}(\sin 3x + \sin x) + (\sin 3x - \sin x) \quad (5.1)$$

so that we can divide and simplify to get

$$\alpha = \frac{1}{4}\sin 3x - \frac{3}{4}\sin x$$

and

$$\Lambda\alpha = \frac{3}{4}\sin 3x - \frac{3}{4}\sin x$$

so that we can integrate to get

$$-3\langle\alpha,\alpha\rangle = -3(\frac{3}{16}+\frac{9}{16})\pi/(2\pi) = -\frac{9}{8}.$$

Next

$$\begin{aligned}2\Lambda\beta &= 2u_xJv_x + uJv_{xx} - 2v_xJu_x - vJu_{xx}\\ &= 4\cos x\sin 2x + 4\cos 2x\sin x - 4\cos 2x\sin x - \cos x\sin 2x\\ &= 3\cos x sin 2x = \frac{3}{2}(\sin 3x+\sin x) \quad (5.2)\end{aligned}$$

so that

$$\Lambda\beta = \frac{3}{4}\sin 3x + \frac{3}{4}\sin x,$$

again we integrate to get

$$-2\langle\alpha,\beta\rangle = -2(\frac{3}{16}-\frac{9}{16})\pi/(2\pi) = \frac{3}{8}.$$

Now

$$\begin{aligned}2\Lambda\delta &= 2u_xJv_x + uJv_{xx} + 2v_xJu_x + vJu_{xx}\\ &= 4\cos x\sin 2x + 4\cos 2x\sin x + 4\cos 2x\sin x + \cos x\sin 2x\\ &= 8\cos 2x\sin x + 5\cos x\sin 2x = 4(\sin 3x-\sin x)+\frac{5}{2}(\sin 3x+\sin x) \quad (5.3)\end{aligned}$$

so that

$$\Lambda\delta = \frac{13}{4}\sin 3x - \frac{3}{4}\sin x$$

and

$$\delta = \frac{13}{12}\sin 3x - \frac{3}{4}\sin x$$

so that

$$\langle\delta,\delta\rangle = (\frac{13^2}{48}+\frac{9}{16})\pi/(2\pi)$$

Lastly

$$2\Lambda B_u = 2u_xJu_x + uJu_{xx} = 2\cos x\sin x + \sin x\cos x = \frac{3}{2}\sin 2x$$

so that

$$B_u = \frac{3}{4}\sin 2x,$$

and

$$\Lambda B_v = 2v_x Jv_x + vJv_{xx} = 8\cos 2x \sin 2x + 4\sin 2x \cos 2x = 6\sin 4x$$

so that we can integrate to get

$$-4\langle B_u, B_v\rangle = 0.$$

We now add the terms together so that the computation gives

$$(\frac{13^2}{48} + \frac{9}{16})/2 + \frac{3}{8} - \frac{9}{8} + 0 = \frac{31}{24} = 1.291\bar{6}.$$

While the code when given the same inputs gives

$$1.29166666667.$$

5.3.2 example 2

Let

$$u = \sin x + \sin 3x \quad \text{and} \quad v = \sin 2x$$

So now we have that

$$\begin{array}{llll} u = \sin x + \sin 3x & Ju = -\cos x - \cos 3x & v = \sin 2x & Jv = -\cos 2x \\ u_x = \cos x + 3\cos 3x & Ju_x = \sin x + 3\sin 3x & v_x = 2\cos 2x & Jv_x = 2\sin 2x \\ u_{xx} = -\sin x - 9\sin 3x & Ju_{xx} = \cos x + 9\cos 3x & v_{xx} = -4\sin 2x & Jv_{xx} = 4\cos 2x \end{array}$$

$$\begin{aligned} 2\alpha &= -u_x v + uv_x \\ &= -(\cos x + 3\cos 3x)\sin 2x + 2(\sin x + \sin 3x)\cos 2x \\ &= -\frac{1}{2}(\sin 3x + \sin x) - \frac{3}{2}(\sin 5x - \sin x) + (\sin 3x - \sin x) + (\sin 5x + \sin x) \end{aligned} \tag{5.4}$$

so that

$$\alpha = -\frac{1}{4}\sin 5x + \frac{1}{4}\sin 3x + \frac{1}{2}\sin x$$

and

$$\Lambda\alpha = -\frac{5}{4}\sin 5x + \frac{3}{4}\sin 3x + \frac{1}{2}\sin x$$

so that after integrating we get

$$-3\langle\alpha,\alpha\rangle = -3(\frac{5}{16} + \frac{3}{16} + \frac{1}{4})\pi/(2\pi) = -\frac{9}{8}.$$

Now

$$2\Lambda\beta = 2u_x Jv_x + uJv_{xx} - 2v_x Ju_x - vJu_{xx}$$

$$= 4(\cos x + 3\cos x)\sin 2x + 4\cos 2x(\sin x + \sin 3x)$$

$$- 4\cos 2x(\sin x + 3\sin 3x) - (\cos x + 9\cos 3x)\sin 2x$$

$$= 3\cos x sin 2x + 3\cos 3x\sin 2x - 8\cos 2x\sin 3x$$

$$= \frac{3}{2}(\sin 3x + \sin x) + \frac{3}{2}(\sin 5x - \sin x) - 4(\sin 5x + \sin x) \quad (5.5)$$

so that

$$\Lambda\beta = -\frac{5}{4}\sin 5x + \frac{3}{4}\sin 3x + -2\sin x$$

so that we can compute

$$2\langle\alpha,\beta\rangle = -2(\frac{5}{16} + \frac{3}{16} - 1)\pi/(2\pi) = \frac{1}{2}.$$

Now

$$2\Lambda\delta = 2u_x Jv_x + uJv_{xx} + 2v_x Ju_x + vJu_{xx}$$

$$= 4(\cos x + 3\cos x)\sin 2x + 4\cos 2x(\sin x + \sin 3x)$$

$$+ 4\cos 2x(\sin x + 3\sin 3x) + (\cos x + 9\cos 3x)\sin 2x$$

$$= 5\cos x\sin 2x + 21\cos 3x\sin 2x + 8\cos 2x\sin x + 16\cos 2x\sin 3x$$

$$= \frac{5}{2}(\sin 3x + \sin x) + \frac{21}{2}(\sin 5x - \sin x) + 4(\sin 3x - \sin x) + 8(\sin 5x + \sin x) \quad (5.6)$$

so that

$$\Lambda\delta = \frac{37}{4}\sin 5x + \frac{13}{4}\sin 3x - 2\sin x$$

and now we get

$$\delta = \frac{37}{20}\sin 5x + \frac{13}{12}\sin 3x - 2\sin x$$

so that integrating gives

$$\langle \delta, \delta \rangle = (\frac{37^2}{80} + \frac{13^2}{48} + 4)\pi/(2\pi).$$

Now

$$\begin{aligned}
2\Lambda B_u &= 2u_x J u_x + u J u_{xx} \\
&= 2(\cos x + 3\cos 3x)(\sin x + 3\sin 3x) + (\sin x + \sin 3x)(\cos x + 9\cos 3x) \\
&= 2\cos x \sin x + 6\cos 3x \sin x + 6\cos x \sin 3x + 18\cos 3x \sin 3x \\
&\quad + \cos x \sin x + \cos x \sin 3x + 9\cos 3x \sin x + 9\cos 3x \sin 3x \\
&= 3\cos x \sin x + 7\cos x \sin 3x + 15\cos 3x \sin x + 27\cos 3x \sin 3x \\
&= \frac{3}{2}\sin 2x + \frac{7}{2}(\sin 4x + \sin 2x) + \frac{15}{2}(\sin 4x - \sin 2x) + \frac{27}{2}(\sin 6x) \qquad (5.7)
\end{aligned}$$

so that simplifying gives

$$\Lambda B_u = \frac{27}{4}\sin 6x + \frac{11}{2}\sin 4x - \frac{5}{4}\sin 2x$$

so that

$$B_u = \frac{27}{24}\sin 6x + \frac{11}{8}\sin 4x - \frac{5}{10}\sin 2x$$

and

$$2\Lambda B_v = 2v_x J v_x + v J v_{xx} = 8\cos 2x \sin 2x + 4\sin 2x \cos 2x = 6\sin 4x.$$

So that

$$-4\langle Bu, Bv \rangle = -4(\frac{33}{8})\pi/(2\pi).$$

So by adding the terms, the computation gives

$$(\frac{37^2}{80} + \frac{13^2}{48} + 4)/2 + \frac{1}{2} - \frac{9}{8} - \frac{33}{4} = \frac{413}{120} = 3.441\bar{6}.$$

While the code outputs

3.44166666667.

5.3.3 example 3

Let

$$u = \sin x + \cos 3x \quad \text{and} \quad v = \sin 2x$$

so now we have that

$$\begin{array}{llll} u = \sin x + \cos 3x & Ju = -\cos x + \sin 3x & v = \sin 2x & Jv = -\cos 2x \\ u_x = \cos x - 3\sin 3x & Ju_x = \sin x - 3\cos 3x & v_x = 2\cos 2x & Jv_x = 2\sin 2x \\ u_{xx} = -\sin x - 9\cos 3x & Ju_{xx} = \cos x - 9\sin 3x & v_{xx} = -4\sin 2x & Jv_{xx} = 4\cos 2x \end{array}$$

which gives us

$$2\alpha = -u_x v + u v_x$$

$$= -(\cos x - 3\sin 3x)\sin 2x + 2(\sin x + \cos 3x)\cos 2x$$

$$= -\cos x \sin 2x + 3\sin 3x \sin 2x + 2\sin x \cos 2x + 2\cos 3x \cos 2x$$

$$= -\frac{1}{2}(\sin 3x + \sin x) + \frac{3}{2}(\cos x - \cos 5x) + (\sin 3x - \sin x) + (\cos 5x + \cos x) \tag{5.8}$$

so that

$$\alpha = \frac{1}{4}\sin 3x - \frac{3}{4}\sin x - \frac{1}{4}\cos 5x + \frac{5}{4}\cos x$$

and

$$\Lambda\alpha = \frac{3}{4}\sin 3x - \frac{3}{4}\sin x - \frac{5}{4}\cos 5x + \frac{5}{4}\cos x$$

so that integrating gives

$$-3\langle\alpha, \alpha\rangle = -3(\frac{3}{16} + \frac{9}{16} + \frac{5}{16} + \frac{25}{16})\pi/(2\pi) = -\frac{3(42)}{32}.$$

Now we compute

$$2\Lambda\beta = 2u_x Jv_x + uJv_{xx} - 2v_x Ju_x - vJu_{xx}$$

$$= 4(\cos x - 3\sin 3x)\sin 2x + 4\cos 2x(\sin x + \cos 3x)$$

$$- 4\cos 2x(\sin x + 3\cos 3x) - (\cos x - 9\sin 3x)\sin 2x$$

$$= 3\cos x \sin 2x - 8\cos 2x \cos 3x - 3\sin 3x \sin 2x$$

$$= \frac{3}{2}(\sin 3x + \sin x) - 4(\cos x + \cos 5x) - \frac{3}{2}(\cos x - \cos 5x) \tag{5.9}$$

so that we simplify to get

$$\Lambda\beta = \frac{3}{4}\sin 3x + \frac{3}{4}\sin x - \frac{5}{4}\cos 5x - \frac{11}{4}\cos x$$

so that

$$-2\langle\alpha,\beta\rangle = -2(\frac{3}{16} - \frac{9}{16} + \frac{5}{16} - \frac{55}{16})\pi/(2\pi) = \frac{7}{2}.$$

Next we compute

$$\begin{aligned}
2\Lambda\delta &= 2u_x Jv_x + uJv_{xx} + 2v_x Ju_x + vJu_{xx} \\
&= 4(\cos x - 3\sin 3x)\sin 2x + 4\cos 2x(\sin x + \cos 3x) \\
&\quad + 4\cos 2x(\sin x + 3\cos 3x) + (\cos x - 9\sin 3x)\sin 2x \\
&= 5\cos x\sin 2x + 8\cos 2x\sin x + 16\cos 2x\cos 3x - 21\sin 3x\sin 2x \\
&= \frac{5}{2}(\sin 3x + \sin x) + 4(\sin 3x - \sin 1x) + 8(\cos x + \cos 5x) - \frac{21}{2}(\cos x - \cos 5x) \qquad (5.10)
\end{aligned}$$

so that we simplify to get

$$\Lambda\delta = \frac{13}{4}\sin 3x - \frac{3}{4}\sin x + \frac{37}{4}\cos 5x - \frac{5}{4}\cos x$$

and

$$\delta = \frac{13}{12}\sin 3x - \frac{3}{4}\sin x + \frac{37}{20}\cos 5x - \frac{5}{4}\cos x$$

so that integrating yeilds

$$\langle\delta,\delta\rangle = (\frac{13^2}{48} + \frac{9}{16} + \frac{37^2}{80} + \frac{25}{16})\pi/(2\pi).$$

Lastly we compute

$$\begin{aligned}
2\Lambda B_u &= 2u_x Ju_x + uJu_{xx} \\
&= 2(\cos x - 3\sin 3x)(\sin x + 3\cos 3x) + (\sin x + \cos 3x)(\cos x - 9\sin 3x) \\
&= 2\cos x\sin x - 6\sin 3x\sin x + 6\cos x\cos 3x - 18\cos 3x\sin 3x \\
&\quad + \cos x\sin x + \cos x\cos 3x - 9\sin 3x\sin x - 9\cos 3x\sin 3x \\
&= 3\cos x\sin x - 27\cos 3x\sin 3x - 15\sin 3x\sin x + 7\cos 3x\cos x \\
&= \frac{3}{2}\sin 2x - \frac{27}{2}(\sin 6x) - \frac{15}{2}(\cos 2x - \cos 4x) + \frac{7}{2}(\cos 2x + \cos 4x) \qquad (5.11)
\end{aligned}$$

so we can combine terms and simplify to get

$$2\Lambda B_u = -\frac{27}{4}\sin 6x + \frac{3}{4}\sin 2x + \frac{11}{2}\cos 4x - 2\cos 2x$$

so that

$$B_u = -\frac{27}{24}\sin 6x + \frac{3}{8}\sin 2x + \frac{11}{8}\cos 4x - \cos 2x$$

and now

$$\Lambda B_v \quad = \quad 2v_x J v_x \;+\; vJv_{xx} \quad = \quad 8\cos 2x \sin 2x \;+\; 4\sin 2x\cos 2x \quad = \quad 6\sin 4x \tag{5.12}$$

So that by orthogonality we get

$$-4\langle Bu, Bv\rangle = 0).$$

Adding the terms of the computation gives

$$(\frac{13^2}{48} + \frac{9}{16} + \frac{37^2}{80} + \frac{25}{16})/2 + \frac{7}{2} - \frac{126}{32} + 0 = \frac{1313}{120} = 10.941\bar{6}.$$

And the code gives

$$10.9416666667.$$

5.3.4 example 4

Let

$$u = \sin x + \cos 3x \quad \text{and} \quad v = \sin 2x + \cos x$$

so now we have that

$$\begin{array}{llll} u = \sin x + \cos 3x & Ju = -\cos x + \sin 3x & v = \sin 2x + \cos x & Jv = -\cos 2x + \sin x \\ u_x = \cos x - 3\sin 3x & Ju_x = \sin x - 3\cos 3x & v_x = 2\cos 2x - \sin x & Jv_x = 2\sin 2x + \cos x \\ u_{xx} = -\sin x - 9\cos 3x & Ju_{xx} = \cos x - 9\sin 3x & v_{xx} = -4\sin 2x - \cos x & Jv_{xx} = 4\cos 2x - \cos x. \end{array}$$

First we compute

$$
\begin{aligned}
2\alpha &= -u_x v + u v_x \\
&= -(\cos x - 3\sin 3x)(\sin 2x + \cos x) + (\sin x + \cos 3x)(2\cos 2x - \sin x) \\
&= -\cos x \sin 2x + 3\sin 3x \sin 2x - \cos^2 x + 3\cos x \sin 3x \\
&\quad + 2\cos 2x \sin x + \cos 3x \cos 2x - \sin^2 x - \cos 3x \sin x \\
&= -\frac{1}{2}(\sin 3x + \sin x) + \frac{3}{2}(\cos x - \cos 5x) - \frac{1}{2}(1 + \cos 2x) + \frac{3}{2}(\sin 4x + \sin 2x) \\
&\quad + (\sin 3x - \sin x) + (\cos 5x + \cos x) - \frac{1}{2}(1 - \cos 2x) - \frac{1}{2}(\sin 4x - \sin 2x) \qquad (5.13)
\end{aligned}
$$

and simplify so that

$$\alpha = -\frac{1}{2} - \frac{3}{4}\sin x + \sin 2x + \frac{1}{4}\sin 3x + \frac{1}{2}\sin 4x + \frac{5}{4}\cos x - \frac{1}{4}\cos 5x$$

and computing $\Lambda\alpha$ we get

$$\Lambda\alpha = -\frac{3}{4}\sin x + 2\sin 2x + \frac{3}{4}\sin 3x + 2\sin 4x + \frac{5}{4}\cos x - \frac{5}{4}\cos 5x$$

so that we can integrate to get

$$-3\langle\alpha, \alpha\rangle = -3(\frac{9}{16} + 2 + \frac{3}{16} + 1 + \frac{25}{16} + \frac{5}{16})\pi/(2\pi) = -\frac{3(90)}{32}.$$

Next we compute

$$
\begin{aligned}
2\Lambda\beta &= 2u_x J v_x + uJv_{xx} - 2v_x J u_x - vJu_{xx} \\
&= 2(\cos x - 3\sin 3x)(2\sin 2x + \cos x) + (\sin x + \cos 3x)(4\cos 2x - \sin x) \\
&- 2(2\cos 2x - \sin x)(\sin x + 3\cos 3x) - (\sin 2x + \cos x)(\cos x - 9\sin 3x) \\
&= 4\cos x \sin 2x - 12\sin 3x \sin 2x + 2\cos^2 x - 6\cos x \sin 3x \\
&+ 4\cos 2x \sin x + 4\cos 3x \cos 2x - \sin^2 x - \cos 3x \sin x \\
&- 4\cos 2x \sin x + 2\sin^2 x - 12\cos 3x \cos 2x + 6\cos 3x \sin x \\
&- \cos x \sin 2x - \cos^2 x + 9\sin 3x \sin 2x + 9\cos x \sin 3x \\
&= 3\cos x \sin 2x + 3\cos x \sin 3x + \cos 3x \sin x \\
&- 3\sin 3x \sin 2x - 8\cos 3x \cos 2x + \cos^2 x + \sin^2 x \\
&= \frac{3}{2}(\sin 3x + \sin x) + \frac{3}{2}(\sin 4x + \sin 2x) + \frac{5}{2}(\sin 4x - \sin 2x) \\
&\qquad - \frac{3}{2}(\cos x - \cos 5x) - 4(\cos 5x + \cos x) + 1 \qquad (5.14)
\end{aligned}
$$

so that we simplify to get

$$
\Lambda\beta = \frac{1}{2} + \frac{3}{4}\sin x - \frac{1}{2}\sin 2x + \frac{3}{4}\sin 3x + 2\sin 4x - \frac{11}{4}\cos x - \frac{5}{4}\cos 5x
$$

and integrate to get

$$
-2\langle\alpha, \beta\rangle = -2(-\frac{1}{4} - \frac{9}{16} - \frac{1}{2} + \frac{3}{16} + 1 - \frac{55}{16} + \frac{5}{16})\pi/(2\pi) = \frac{13}{4}.
$$

Now

$$
\begin{aligned}
2\Lambda\delta &= 2u_x J v_x + uJv_{xx} + 2v_x J u_x + vJu_{xx} \\
&= 2(\cos x - 3\sin 3x)(2\sin 2x + \cos x) + (\sin x + \cos 3x)(4\cos 2x - \sin x) \\
&+ 2(2\cos 2x - \sin x)(\sin x + 3\cos 3x) + (\sin 2x + \cos x)(\cos x - 9\sin 3x) \\
&= 4\cos x \sin 2x - 12\sin 3x \sin 2x + 2\cos^2 x - 6\cos x \sin 3x \\
&+ 4\cos 2x \sin x + 4\cos 3x \cos 2x - \sin^2 x - \cos 3x \sin x \\
&+ 4\cos 2x \sin x - 2\sin^2 x + 12\cos 3x \cos 2x - 6\cos 3x \sin x \\
&+ \cos x \sin 2x + \cos^2 x - 9\sin 3x \sin 2x - 9\cos x \sin 3x \\
&= 5\cos x \sin 2x - 15\cos x \sin 3x + 8\cos 2x \sin x - 7\cos 2x \sin 3x \\
&+ 8\cos 3x \cos 2x - 21\sin 3x \sin 2x + 3\cos^2 x - 3\sin^2 x \\
&= \frac{5}{2}(\sin 3x + \sin x) + 4(\sin 3x - \sin 1x) + 8(\cos x + \cos 5x) - \frac{21}{2}(\cos x - \cos 5x) \qquad (5.15)
\end{aligned}
$$

so that

$$\Lambda\delta = -\frac{3}{4}\sin x - 2\sin 2x + \frac{13}{4}\sin 3x - \frac{11}{2}\sin 4x - \frac{5}{4}\cos x + \frac{3}{2}\cos 2x + \frac{37}{4}\cos 5x$$

and computing the inverse of Λ we get

$$\delta = -\frac{3}{4}\sin x - \sin 2x + \frac{13}{12}\sin 3x - \frac{11}{8}\sin 4x - \frac{5}{4}\cos x + \frac{3}{4}\cos 2x + \frac{37}{20}\cos 5x$$

so that we can integrate to get

$$\langle \delta, \delta \rangle = (\frac{9}{16} + 2 + \frac{13^2}{48} + \frac{11^2}{16} + \frac{25}{16} + \frac{9}{8} + \frac{37^2}{80})\pi/(2\pi) = \frac{8027}{480}.$$

Lastly we compute

$$2\Lambda B_u = 2u_x J u_x + u J u_{xx}$$

$$= 2(\cos x - 3\sin 3x)(\sin x + 3\cos 3x) + (\sin x + \cos 3x)(\cos x - 9\sin 3x)$$

$$= 2\cos x \sin x - 6\sin 3x \sin x + 6\cos x \cos 3x - 18\cos 3x \sin 3x$$

$$+ \cos x \sin x + \cos x \cos 3x - 9\sin 3x \sin x - 9\cos 3x \sin 3x$$

$$= 3\cos x \sin x - 27\cos 3x \sin 3x - 15\sin 3x \sin x + 7\cos 3x \cos x$$

$$= \frac{3}{2}\sin 2x - \frac{27}{2}(\sin 6x) - \frac{15}{2}(\cos 2x - \cos 4x) + \frac{7}{2}(\cos 2x + \cos 4x) \quad (5.16)$$

so that

$$2\Lambda B_u = -\frac{27}{4}\sin 6x + \frac{3}{4}\sin 2x + \frac{11}{2}\cos 4x - 2\cos 2x$$

and now we have

$$B_u = -\frac{27}{24}\sin 6x + \frac{3}{8}\sin 2x + \frac{11}{8}\cos 4x - \cos 2x$$

and

$$\Lambda B_v = 2v_x J v_x + v J v_{xx}$$

$$= 2(2\cos 2x - \sin x)(2\sin 2x + \cos x) + (\sin 2x + \cos x)(4\cos 2x - \sin x)$$

$$= 8\cos 2x \sin 2x - 4\sin 2x \sin x + 4\cos 2x \cos x - 2\cos x \sin x$$

$$+ 4\cos 2x \sin 2x + 4\cos 2x \cos x - \sin 2x \sin x - \cos x \sin x$$

$$= -3\cos x \sin x + 12\cos 2x \sin 2x - 5\sin 2x \sin x + 8\cos 2x \cos x$$

$$= -\frac{3}{2}\sin 2x + 6\sin 4x - \frac{5}{2}(\cos x - \cos 3x) + 4(\cos x + \cos 3x) \quad (5.17)$$

So we simplify to get

$$\Lambda B_v = -\frac{3}{4}\sin 2x + 3\sin 4x + \frac{3}{4}\cos x + \frac{13}{4}\cos 3x$$

So that

$$-4\langle B_u, B_v\rangle = -4(-\frac{9}{32})\pi/(2\pi) = \frac{9}{16}.$$

Adding together the terms of the computation gives

$$\frac{8027}{480}+\frac{13}{4}-\frac{3(90)}{32}+\frac{9}{16}=\frac{5807}{480}=12.0979\bar{6}.$$

While the code gives

$$12.097966667.$$

5.4 Simulations

We can then use the code to compute the curvature of two given vector fields. We used this approach to try to find an example of negative curvatures. This attempt did not yield any results so we try a different approach. Next we use the code to compute the curvature of vector fields with random entries. As we can see in the code, we can control the length of these vector fields. Lastly we try to use random entries with some sort of decay condition on the coefficients. Here we bound the random entries by the Gaussian function, and here we can control the variance of that distribution. These attempts have yet to yield an example of negative curvature.

5.5 Outlook

One immediate result that would be interesting is either finding an example of negative curvature on $\mathcal{D}(S^1)/S^1$ with the $\dot{H}^{1/2}$ topology or proving non-negative curvature everywhere. We conjecture that the latter is true. It would then be interesting to use the positivity of the curvature in order to find an alternate proof of blow up of the Wunsch equation.

Another idea would be to find other solutions to the geodesic and consider the conjugate points along the geodesic flows. Even more interesting would be to find non-steady state solutions to the Euler-Arnold equation of the contactomorphism group and understand those implications. One possible direction to take would be to find another vector field X, which is not that Reeb vector field, such that $C(X,Y)\geq 0$ for all vector fields Y.

It would be interesting the infinte-dimensional manifold $\mathcal{D}(S^1)$ itself as a contact manifold given contact form

$$\theta(f) = \int_{S^1} f dx.$$

Recall that this makes sense since we can identify the vector fields on $\mathcal{D}(S^1)$ with functions on the circle. It is interesting since the Kirillov space $(\mathcal{D}(S^1)/S^1, \omega)$ with the is a Kähler manifold with the simplectic form

$$\omega(u, v) = \int_{S^1} u_x v dx$$

and we have that

$$\pi^* \omega = \theta.$$

In other words $(\mathcal{D}(S^1), \theta)$ is the Boothby-Wang fibration over the space $(\mathcal{D}(S^1), \omega)$.

Lastly it would be a good idea to prove strong Fredholmness in the case of $D_\theta(M)$ with the H^1 metric. In Chapter 3 we only sketch a proof. In the case of symplectomorphisms, Benn shows that strong Fredholmness of the exponential map does in fact need proving [6] and we suspect this is also the case here.

These are just a few ideas up for consideration but we give these just in order to provide future direction for the work done in this thesis.

Bibliography

[1] R. Abraham, J. E. Marsden, and T. Ratiu. Manifold, tensor analysis, and applications. Springer, 1988.

[2] V. I. Arnold and B. Khesin. Topological methods in hydrodynamics. Springer, 1998.

[3] V.I. Arnold. On the differential geometry of infinite-dimensional lie groups and its applications to the hydrodynamics of perfect fluids. Vladimir I. Arnold: collected works, 2:33–69, 2014.

[4] M. Bauer, M. Bruveris, M. Harms, and P. W. Michor. Geodesic distance for right invariant sobolev metrics of fractional order on the diffeomorphism group. Ann. Glob. Anal. Geom., 44:5–21, 2013.

[5] M. Bauer, B. Kolev, and S. C. Preston. Geometric investigations of a vorticity model equation. J. Diff. Eq., 260:478–516, 2016.

[6] J. Benn. The L^2 geometry of the symplectomorphism group. Ph.D. Thesis University of Notre Dame, 2015.

[7] D. Blair. Riemannian geometry of contact and symplectic manifolds. Springer, 2010.

[8] B. Chhay. Contactomorphism group with the L^2 metric on stream functions. Ann. Glob. Anal. Geom., 49:205–211, 2016.

[9] B. Chhay and S. C. Preston. Geometry of the contactomorphism group. Diff. Geo. Appl., 40:223–231, 2015.

[10] A. Constantin and B. Kolev. On the geometric approach to the motion of inertial mecahnical systems. J. Phys., 35:R51–R79, 2002.

[11] D. G. Ebin. On the space of riemannian metrics. Bull. Amer. Math. Soc., 74:1001–1003, 1968.

[12] D. G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math., 92:102–163, 1970.

[13] D. G. Ebin, G. Misiołek, and S. C. Preston. Singularities of the exponential map on the volume-preserving diffeomorphism group. GAFA, 16:157–164, 1980.

[14] D. G. Ebin and S. C. Preston. Riemannian geometry of the contactomorphism group. Arnold Math. J., 1:5–21, 2014.

[15] L. Evans. Partial differential equations. American Mathematical Society, 1998.

[16] G. Folland. Real analysis. John Wiley and Sons, 1999.

[17] B. Khesin, J. Lenells, G. Misiołek, and S. C. Preston. Curvatures of sobolev metrics on diffeomorphism groups. Pure Appl. Math. Q., 9:291–332, 2013.

[18] A. A. Kirillov. Kähler structure on the k-orbits of a group of diffeomorphisms of the circle. Funktsional. Anal. i Prilozhen, 21:122–125, 1987.

[19] A. A. Kirillov and D. V. Yur'ev. Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$. Funktsional. Anal. i Prilozhen, 21:35–46, 1987.

[20] S. Kouranbaeva. The camassa-holm equation as a geodesic flow on the diffeomorphism group. J. Math. Phy., 40, 1999.

[21] A. Kriegl and P. W. Michor. The convenient setting of global analysis. American Mathematical Society, 1997.

[22] S. Lang. Fundamentals of differential geometry. Springer, 2001.

[23] A. Majda and A. Bertozzi. Vorticity and incompressible flow. Cambridge University Press, 2002.

[24] P. W. Michor and D. Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphism groups. Doc. Math., 10:217–246, 2005.

[25] G. Misiołek. A shallow water equation as a geodesic flow on the bott-virasoro group. J. Geo. Phys., 24:203–208, 1998.

[26] G. Misiołek and S. C. Preston. Fredholm properties of riemannian exponential maps on diffeomorphism groups. Invent. Math., 179:191–246, 2009.

[27] H. Okamoto, T. Sakajo, and Marcus Wunsch. On a generalization of the connstantin-lax-majda equation. Nonlinearity, 21:2447–2461, 2008.

[28] S. C. Preston. On the volumorphism group, the first conjugate point is always the hardest. Comm. Math. Phys., 267:493–513, 2006.

[29] S. C. Preston and P. Washabaugh. Global existence and blowup for geodesics in universal teichmüller space. arXiv:1603.07061 [math.DG], 2016.

[30] T. Ratiu and R. Schmid. The differentiable structure of three remarkable diffeomorphism groups. Math. Z., 177:81–100, 1981.

[31] W. Rudin. Functional Analysis. McGraw-Hill, 1991.

[32] E. Shelukhin. The hofer norm of a contactomorphism. arXiv:1411.1457 [math.SG], 2015. version 3.

[33] N. K. Smolentsev. A biinvariant metric on the group of symplectic diffeomorphisms and equations $(\partial/\partial t)\Delta F = \{F, F\}$. Siberian Math. J., 27:120–126, 1986.

[34] N. K. Smolentsev. Curvature of the classical diffeomorphism groups. Siberian Math. J., 35:169–176, 1994.

[35] N. K. Smolentsev. Diffeomorphism groups of compact manifolds. J. Math. Sci., 146:6213–6312, 2007.

[36] E. Stein and R. Shakarchi. Real analysis. Princeton University Press, 2005.

[37] L.-P. Teo. The velling-kirillov metric on the universal teichmüller curve. J. d'Anal. Math., 93:271–307, 2004.

[38] M. Wunsch. On the geodesic flow on the group of diffeomorphisms of the circle with the fractional sobolev right-invariant metric. J. Nonlinear Math. Phys., 17:7–10, 2010.