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EEE 431- Digital Communications
Computational Assignment 1

This assignment includes implementations of uniform and non-uniform quantization.

a)

Given:

$$f_{X,Y}(x,y) = e^{-(x+y)} \quad , \quad x>0 \ , \ y>0 \qquad x,y \in \mathbb{R}^2$$

Let $Z = X - Y$

If $z > 0$, which also means $x = z+y > y > 0$, the marginal PDF of Z can be calculated as:

$$f_Z(z) = \int_0^\infty f_{X,Y}(y+z, y)\,dy = \int_0^\infty e^{-(z-2y)}\,dy = e^{-z}\int_0^\infty e^{-2y}\,dy = e^{-z}\left[-\frac{1}{2}e^{-2y}\right]\Bigg|_0^\infty = \frac{1}{2}e^{-z}$$

$$f_Z(z) = \frac{1}{2}e^{-z} \quad , \quad \text{if } z>0$$

if $z<0$, which also means $y = x - z > x > 0$, the marginal PDF of Z can be calculated as:

$$f_Z(z) = \int_0^\infty\int_0^\infty f_{X,Y}(x, x-z)\,dx\,dy = \int_0^\infty e^{-(2x-z)}\,dx = e^{z}\int_0^\infty e^{-2x}\,dx = e^{z}\left[-\frac{1}{2}e^{-2x}\right]\Bigg|_0^\infty = \frac{1}{2}e^{z}$$

$$f_Z(z) = \frac{1}{2}e^{z} \quad , \quad \text{if } z<0$$

$$\Rightarrow f_Z(z) = \frac{1}{2}e^{-|z|} \quad , \quad \forall z \in \mathbb{R}$$

This is the PDF of a Laplace distribution with mean 0 and scale b = 1.

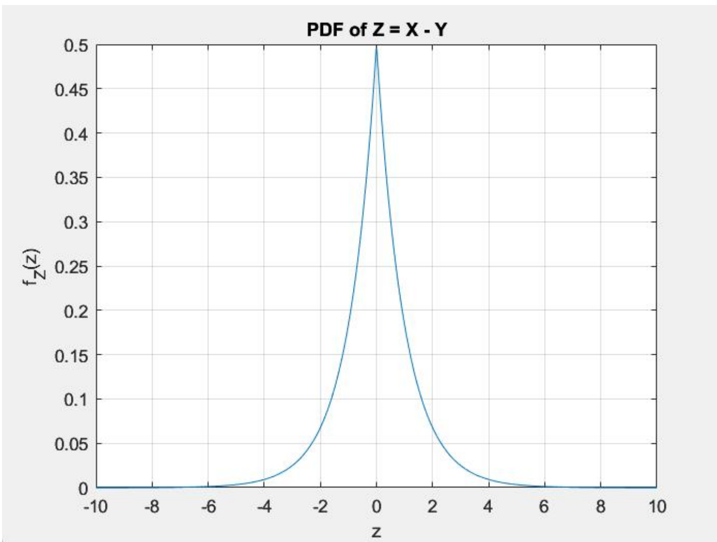

Figure 1: PDF of Z = X - Y

b)

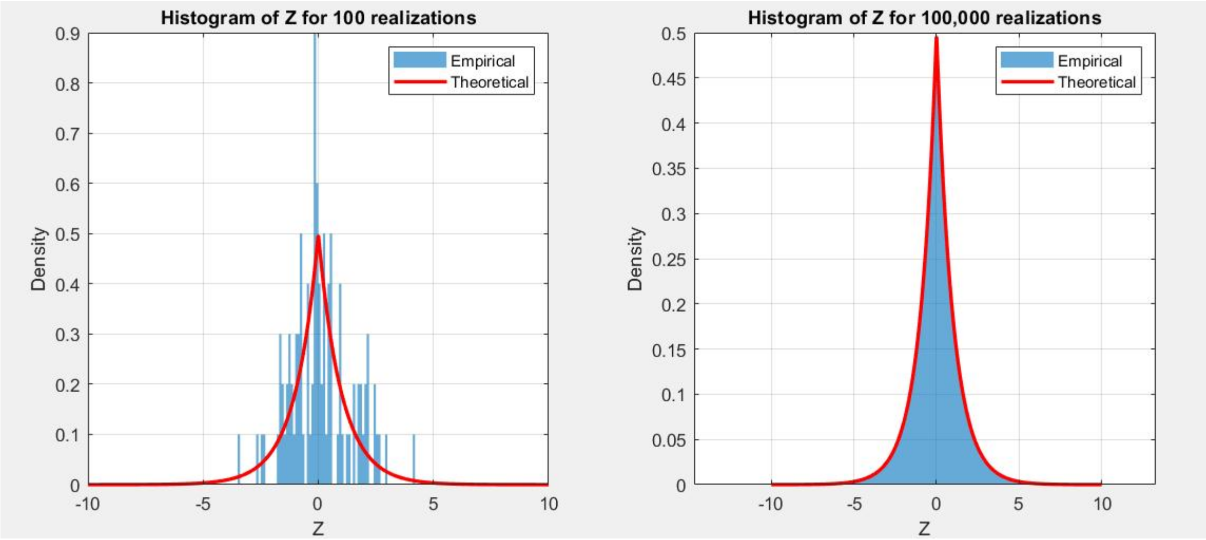

Figure 2: Histograms of Z for 100 and 100,000 realizations

As it can be seen from Figure 2, the histogram for 100 realizations (left) shows significant variation and does not closely match the theoretical Laplace distribution, mainly due to the small sample size. The empirical distribution appears more scattered, and the peak is not as sharp as the theoretical curve.

On the other hand, the histogram for 100,000 realizations (right) aligns closely with the theoretical PDF. As the number of realizations increases, the empirical distribution becomes smoother, resembling the shape of the Laplace distribution more accurately.

c)

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z)\,dz \qquad \text{where} \qquad f_Z(z) = \frac{1}{2} e^{-|z|}$$

$$E[Z^2] = \int_{0}^{\infty} z^2 \frac{1}{2} e^{-z}\,dz + \int_{-\infty}^{0} z^2 \frac{1}{2} e^{z}\,dz$$

Integration by parts
$\int u\,dv = uv - \int v\,du$

$$E[Z^2] = 2\int_{0}^{\infty} z^2 \frac{1}{2} e^{-z}\,dz = \int_{0}^{\infty} z^2 e^{-z}\,dz$$

$u = z^2 \quad dv = e^{-z}dz$
$du = 2z\,dz \quad v = -e^{-z}$

$$E[Z^2] = \left[\cancel{-z^2 e^{-z}}\right]_0^{\infty} - \int_0^{\infty} -e^{-z}\,2z\,dz = 2\int_0^{\infty} z e^{-z}\,dz = 2\left(\cancel{z e^{-z}} - \int_0^{\infty} -e^{-z}\,dz\right)$$

L'Hospital
$$\lim_{z\to\infty} -\frac{z^2}{e^z} = \lim_{z\to\infty} -\frac{2z}{e^z} = \lim_{z\to\infty} -\frac{2}{e^z} = 0$$

$u = z \quad v = e^{-z}$
$du = dz \quad dv = -e^{-z}dz$

$$E[Z^2] = 2\int_0^{\infty} e^{-z}\,dz = 2\left[-e^{-z}\right]_0^{\infty} = 2(0-(-1)) = 2$$

The average value of squares of these realizations are computed in MATLAB. The output on the command window:

```
Empirical average of Z^2: 1.9927
```

If we compare the results, the empirical result of 1.9927 is very close to the theoretical value of 2. The small difference is expected due to the variability in the random sampling process. With a larger sample size, the empirical average would converge even more closely to the theoretical value.

d)

A uniform quantizer divides the range of the signal Z into N = 8 intervals of equal width, and each interval is assigned to a quantization level. The quantized value of Z, $\tilde{Z}$ is the midpoint of the interval to which Z belongs.

The width of each interval:

$$\Delta = \frac{Z_{\max} - Z_{\min}}{N}$$

For each realization Z_i, the quantized value $\tilde{Z}_i$ can be found with:

$$\tilde{Z}_i = Z_{\min} + \Delta \cdot \left\lfloor \frac{Z_i - Z_{\min}}{\Delta} \right\rfloor + \frac{\Delta}{2}$$

Except for the $Z_i = Z_{max}$ case, for this case *floor()* function did not work as intended. Therefore the following adjustment added to the code to obtain accurate results:

$$\tilde{Z}_{\max} = Z_{\min} + \Delta \cdot \left\lfloor \frac{Z_{\max} - Z_{\min}}{\Delta} \right\rfloor - \frac{\Delta}{2}$$

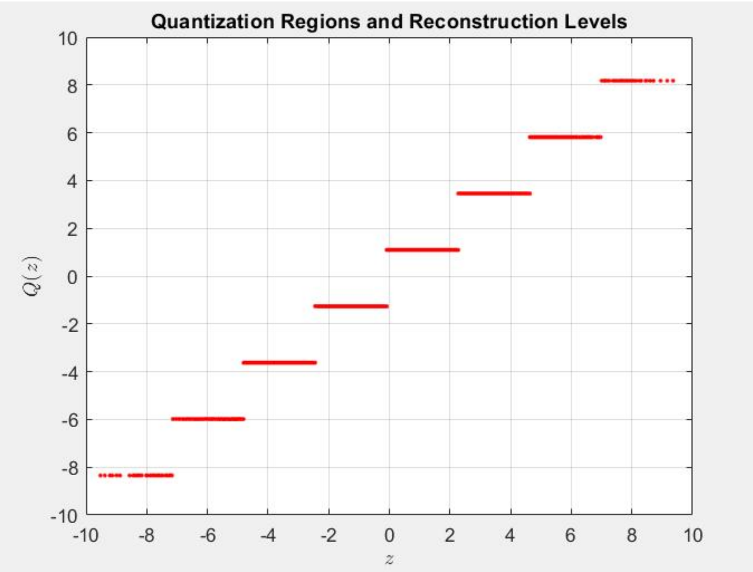

Figure 3: Quantization Regions and Reconstruction Levels (uniform)

```
Quantization Boundaries:
   -9.5277   -7.1672   -4.8067   -2.4462   -0.0856    2.2749    4.6354    6.9959    9.3564

Reconstruction Levels:
   -8.3474   -5.9869   -3.6264   -1.2659    1.0946    3.4551    5.8156    8.1762
```

Quantization error for each sample is the difference between the quantized version and the original value:

$$\text{Error}_i = Z_i - \tilde{Z}_i$$

The average power of the quantization error is the mean squared error:

$$P_e = \text{MSE} = E[(Z - \tilde{Z})^2]$$

The average source power is calculated in part (c).

$$P_{\text{signal}} = E[Z^2]$$

The Signal-to-Quantization Noise Ratio (SQNR) is defined as the ratio of the average signal power to the average quantization noise power:

$$\text{SQNR} = \frac{P_{\text{signal}}}{P_e}$$

In dB scale:

$$\text{SQNR (dB)} = 10 \log_{10} \left(\frac{P_{\text{signal}}}{P_e} \right)$$

After the quantization and SQNR calculations are done in MATLAB. The output on the command window:

```
Uniform quantizer
Average power of the quantization error: 0.5362
Signal power: 1.9927
SQNR (dB): 5.7012
```

Figure 4: Uniform quantizier results

e)

The probability mass function (PMF) is computed by dividing the frequency of each quantization level by the total number of realizations N = 100,000.

$$PMF = freq/N$$

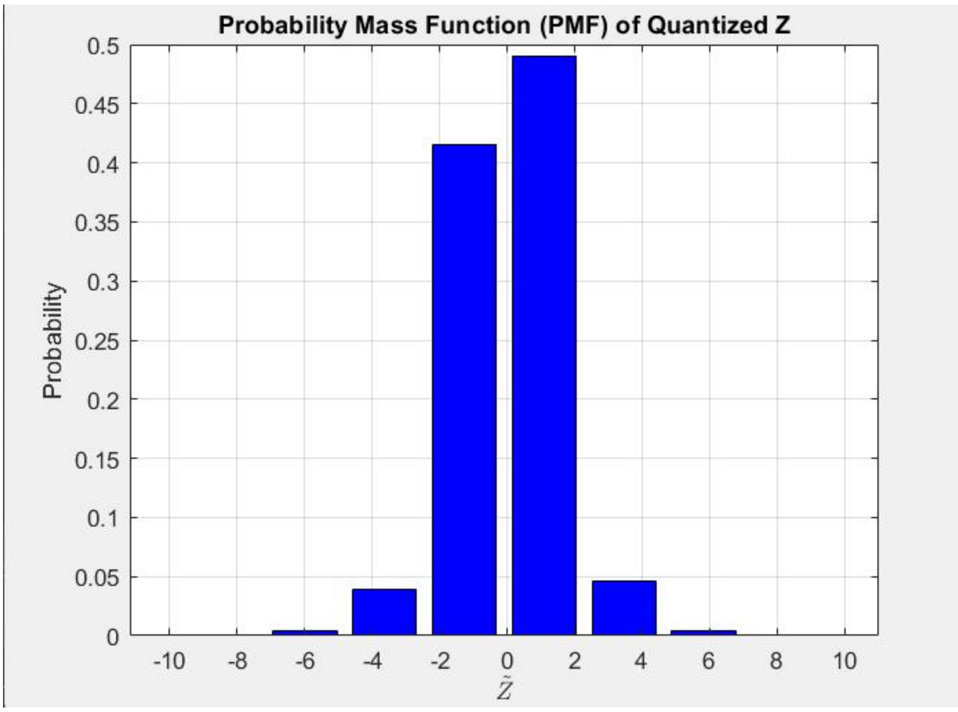

Figure 5: PMF of Quantized Z (uniform)

f)

PDF: $f_Z(z) = \frac{1}{2} e^{-|z|}$

$$F_Z(z) = \frac{1}{2}\int_{-\infty}^{z} e^{-|z'|} dz'$$

If $z<0$:

$$F_Z(z) = \frac{1}{2}\int_{-\infty}^{z} e^{z'} dz' = \frac{1}{2} e^{z} \quad , \; z<0$$

If $z \geq 0$:

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{2} e^{-|z'|} dz' = \underbrace{\int_{-\infty}^{0} \frac{1}{2} e^{-|z'|} dz'}_{F(0)=\frac{1}{2}} + \int_{0}^{z} \frac{1}{2} e^{-|z'|} dz'$$

$$F_Z(z) = \frac{1}{2} + \int_{0}^{z} \frac{1}{2} e^{-z'} dz' = \frac{1}{2} - \frac{1}{2}\left(e^{-z} - 1\right) = 1 - \frac{1}{2} e^{-z}$$

$$F(z) = 1 - \frac{1}{2} e^{-z} \quad , \; z \geq 0$$

$$\text{CDF:} \quad F_Z(z) = \begin{cases} \frac{1}{2} e^{z} & , \; z<0 \\ 1 - \frac{1}{2} e^{-z} & , \; z \geq 0 \end{cases}$$

Inverse CDF, $F_Z^{-1}(p)$:

If $z<0$

$$F_Z(z) = \frac{1}{2} e^{z} = p \Rightarrow e^{z} = 2p \Rightarrow z = \ln(2p)$$

$$F_Z^{-1}(p) = \ln(2p) \quad , \; 0<p \leq 0.5$$

If $z \geq 0$

$$F_Z(z) = 1 - \frac{1}{2} e^{-z} = p \Rightarrow e^{-z} = 2(1-p) \Rightarrow -z = \ln(2(1-p)) \Rightarrow z = -\ln(2(1-p))$$

$$F_Z^{-1}(p) = -\ln(2(1-p)) \quad , \; 0.5 \leq p < 1$$

$$F_Z^{-1}(p) = \begin{cases} \ln(2p) & , \; 0<p \leq 0.5 \\ -\ln(2(1-p)) & , \; 0.5 \leq p < 1 \end{cases}$$

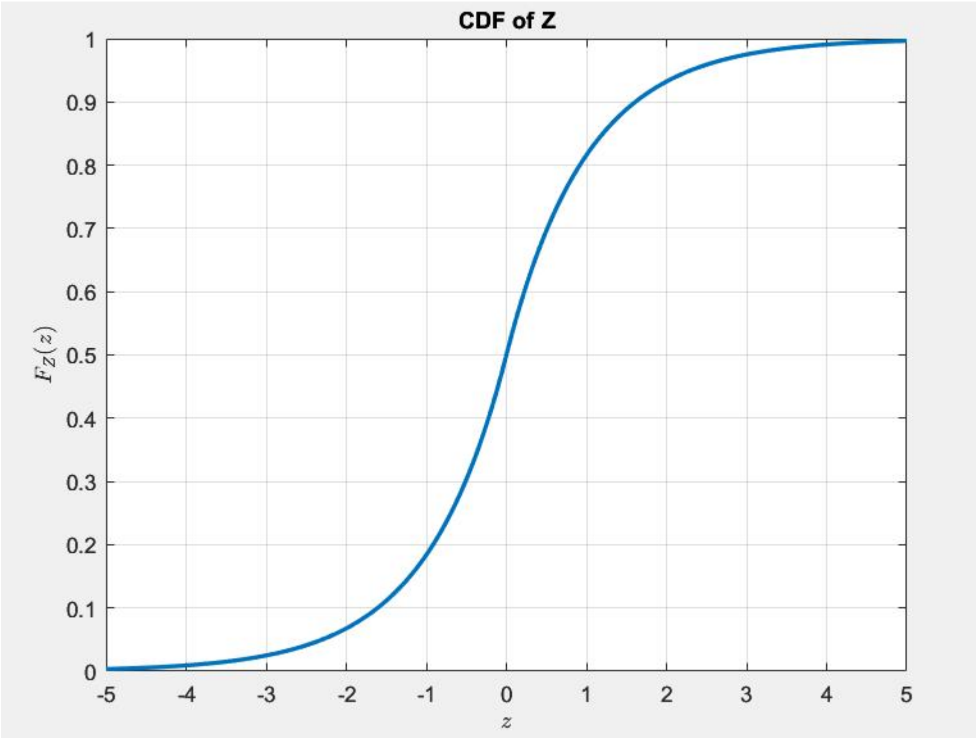

Figure 6: CDF of Z

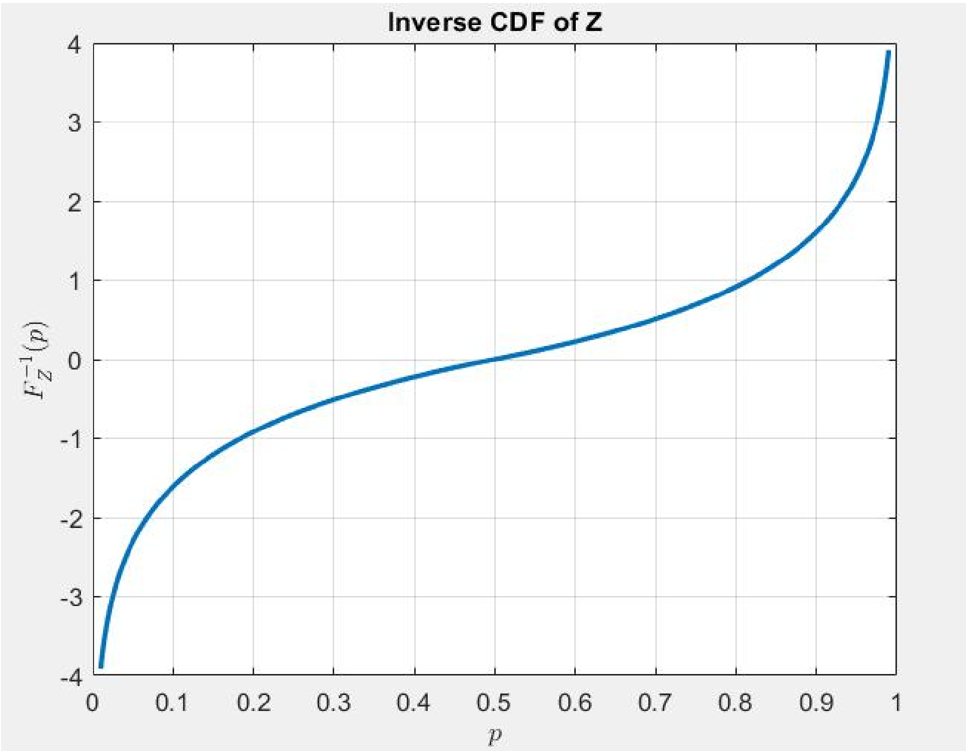

Figure 7: Inverse CDF of Z

g)

Figure 8 depcits that non-uniform quantizier includes compression and expansion steps before and after uniform PCM repectively.

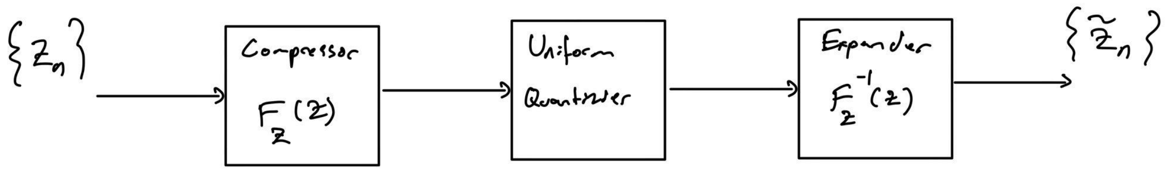

Figure 8: Non-uniform Quantizier

Quantization regions and corresponding reconstruction points are plotted using MATLAB.

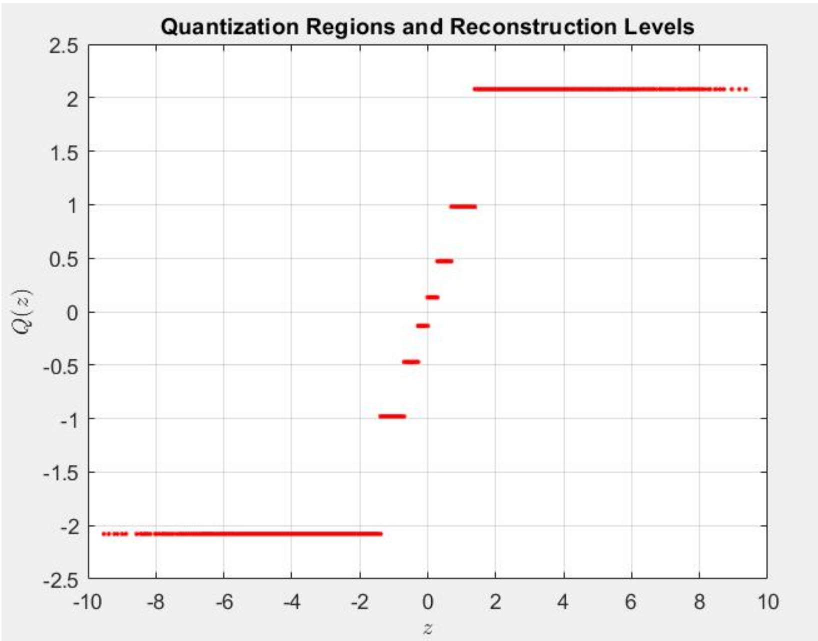

Figure 9: Quantization Regions and Reconstruction Levels (non-uniform)

```
Quantization Boundaries:
    0.0000    0.1250    0.2500    0.3750    0.5000    0.6250    0.7500    0.8750    1.0000

Reconstruction Levels:
    0.0625    0.1875    0.3125    0.4375    0.5625    0.6875    0.8125    0.9375
```

With the help of non-uniform PCM, the regions with higher probability are divided into smaller steps compared to uniform case, this can be observed from the graphs in *Figure 3 & 9*.

h)

The quantized version of Z is calculated using the non-uniform PCM quantizier proposed in part (g). The uniform quantization part in the algorithm is done in the same manner with part (d).

After the quantization and SQNR calculations are done in MATLAB following the same steps with part (d). The output on the command window is as follows:

```
Non-uniform PCM quantizer
Average power of the quantization error: 0.2848
Signal power: 1.9927
SQNR (dB): 8.4484
```

Figure 10: Non-uniform PCM quantizier results

Comparing the results in part (d) (*Figure 4*) with the results in part (h) (*Figure 10*), it is visible that average power of the quantization error calculated as 0.5362 for the uniform case whereas 0.2848 for the non-uniform case. This means, average power of the quantization error is almost halved when the non-uniform steps are added to the quantization process.

Similarly SQNR ratio is increased from 5.7012 to 8.4484 for the non-uniform case compared to uniform case. The non-uniform case results in a higher SQNR, indicating that the quantization noise is lower and the signal quality is improved compared to the uniform case.

i)

Probability of each possible quantization output is calculated by dividing the frequency of each quantization level by the total number of realizations N = 100,000.

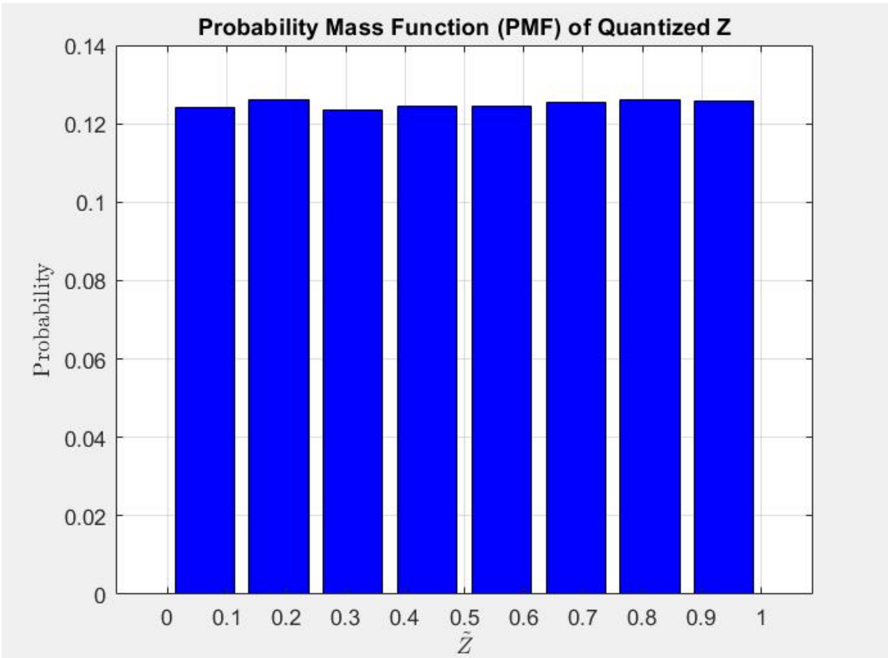

Figure 11: PMF of Quantized Z (non-uniform)

Comparing the PMF of the non-uniform quantized Z (*Figure 11*) with the uniform quantized Z (*Figure 5*). In the non-uniform case frequencies of occurrence for each possible quantization output are more or less equal to each other, indicating a more balanced distribution across the quantization levels. On the other hand, in the uniform quantized case, the PMF shows peaks, suggesting that certain values of quantization occur much more frequently than others. This difference in distribution suggests that the non-uniform quantization provides a more even representation of the signal, reducing the bias towards specific quantization levels seen in the uniform case.

j)

Lloyd-Max Algorithm is a quantization method that aims to minimize the distortion between a set of data values and their quantized representation. The algorithm follows these steps:

1. Initialization:

The algorithm starts with initial guess for the boundaries which determined as uniform quantization boundaries. These boundaries will define the intervals for the quantization levels. The quantization levels are determined as centers of each interval.

2. Expectation Step (Quantization)

In the Quantization, each value in the data Z is assigned to the closest reconstruction level.

3. Maximization Step (Recalculate Reconstruction Levels)

In the M-step, the new reconstruction levels are updated. For each quantization level, the data points assigned to that level are used to compute the mean. This new mean becomes the updated reconstruction level. After updating the reconstruction levels, the boundaries are also updated. The new boundaries are computed as the midpoints between adjacent reconstruction levels.

4. Convergence Check

The algorithm checks for convergence by comparing the updated reconstruction levels with the previous ones. If the maximum change between them is less than a small threshold (epsilon = 10^{-6}), the algorithm stops. Otherwise the algorithm goes back to step 2. If the algorithm did not converge after some number of iterations (max_iter = 100) algorithm also stops.

Quantization regions and corresponding reconstruction points are plotted using MATLAB.

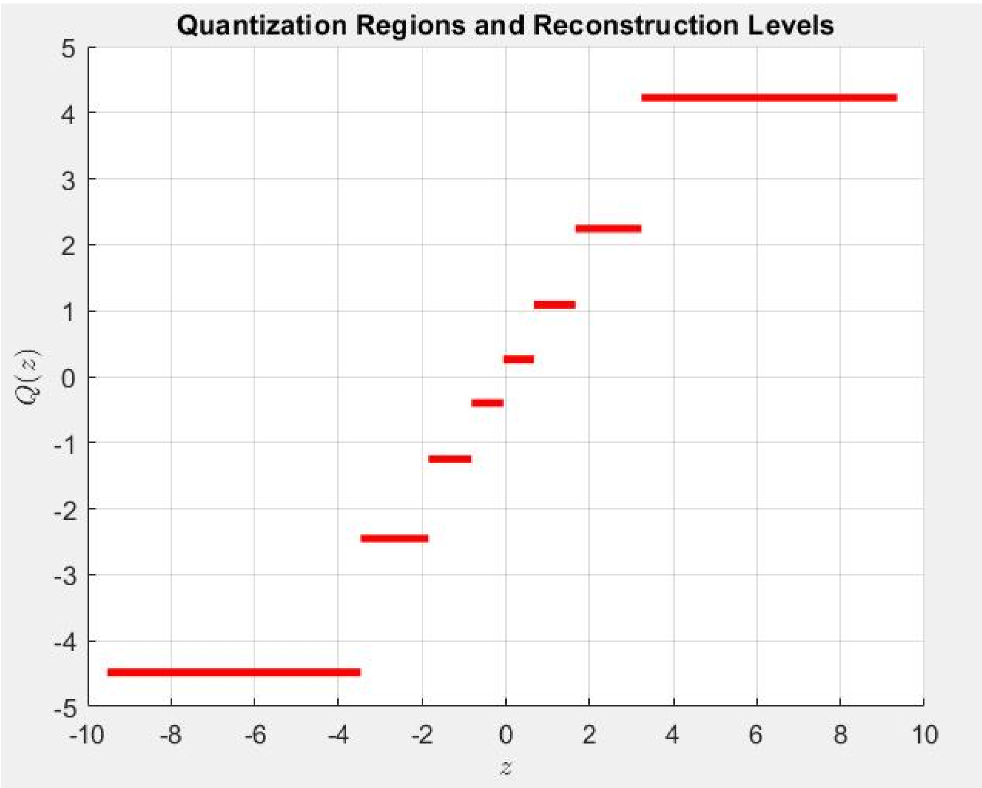

Figure 12: Quantization Regions and Reconstruction Levels (Lloyd-Max)

```
Quantization Boundaries:
   -9.5277   -3.4671   -1.8496   -0.8212   -0.0664    0.6760    1.6664    3.2353    9.3564

Reconstruction Levels:
   -4.4812   -2.4530   -1.2462   -0.3962    0.2634    1.0887    2.2441    4.2265
```

k)

The quantized version of Z is calculated using the non-uniform Loyd-Max quantizier proposed in part (j).

After the quantization and SQNR calculations are done in MATLAB following the same steps with part (d). The output on the command window is as follows:

```
LLyod-Max Quantizier
Average power of the quantization error: 0.1066
Signal power: 1.9927
SQNR (dB): 12.7169
```

Figure 13: Lloyd-Max quantizier results

When comparing the results with part (d) (uniform quantization) and part (h) (non-uniform quantization), it is evident that the Lloyd-Max quantizer offers improved performance. The quantization error in the Lloyd-Max case is 0.2867, compared to 0.5362 for the uniform quantizer in part (d) and 0.2848 for the non-uniform quantizer in part (h). Additionally, the SQNR for the Lloyd-Max quantizer is 12.7169, which is higher than both the uniform case (5.7012) and the non-uniform case (8.4484). This shows the Lloyd-Max algorithm's advantage in reducing noise and improving signal quality.

l)

Probability of each possible quantization output is calculated by dividing the frequency of each quantization level by the total number of realizations N = 100,000.

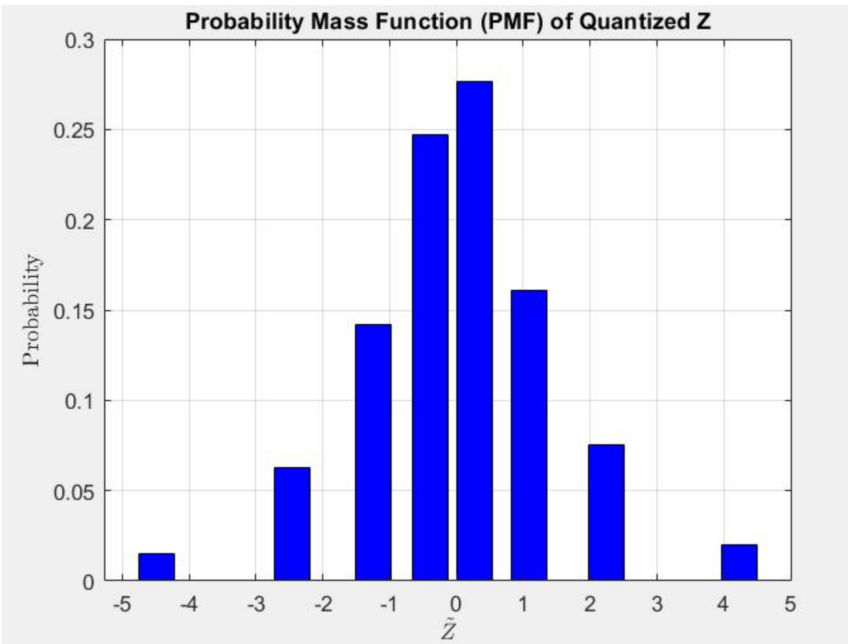

Figure 14: PMF of Quantized Z (Lloyd-Max)

In part *Figure 14*, the PMF of quantization levels of Lloyd-Max quantizer, does not exhibit a marked improvement over the non-uniform quantizer. The PMF for the Lloyd-Max quantized signal is still somewhat biased towards certain quantization levels, similar to the uniform case, but with slightly different frequency distributions. Therefore, the Lloyd-Max algorithm does not necessarily offer a more balanced distribution than the non-uniform quantizer, but it can still result in a lower quantization error and better signal quality due to its optimization of quantization regions based on the signal's probability density.

m)

The Lloyd-Max algorithm is not guaranteed to outperform other quantization techniques in all scenarios. While it provides superior performance in minimizing quantization error and maximizing SQNR in most cases, its effectiveness depends on the signal's characteristics and the complexity of the data. For signals with highly irregular or non-stationary distributions, the Lloyd-Max algorithm may struggle to provide optimal results without extensive adaptation. Additionally, the computational complexity of Lloyd-Max can be higher than simpler methods like uniform quantization, making it less practical for real-time systems or applications with limited computational resources.