EEE 431 - Digital Communications

Computational Assignment 3

Due: May 29, 2025, 23:59

In this assignment, you will implement various topics you learned in the class using MATLAB. Whenever a question requires plotting, numerical computation, or simulation, use MATLAB. Certain parts will also require analytical work. Please use a fixed random seed in your code so that your results are reproducible.

You can discuss the problems with your colleagues, but you need to write your code on your own without any collaboration with others/other tools (colleagues, Chegg, AI Chatbots, etc.) and you need to write your comments in your report with your own words and your own understanding.

1 Frequency Shift Keying

Consider the following two signals:

$$s_1(t) = \cos(2\pi 250t)h(t),$$
$$s_2(t) = \cos(2\pi 500t)h(t),$$

where $h(t)$ is 1 for $0 \leq t < T$, and $T = 0.1$ seconds. Message symbols are mapped as:

$$00 \rightarrow s_1(t),$$
$$01 \rightarrow -s_1(t),$$
$$10 \rightarrow s_2(t),$$
$$11 \rightarrow -s_2(t).$$

For example, if we generate a message that is 00011011, the corresponding modulated signal $x(t)$ is given by

$$x(t) = s_1(t) - s_1(t-T) + s_2(t-2T) - s_2(t-3T)\ .$$

Pick the sampling frequency as $F_s = 5\text{kHz}$.

(a) Generate 5 random symbols uniformly and construct the corresponding modulated signal $x(t)$. Plot $x(t)$.

(b) Obtain an orthonormal signal space for this modulation scheme in the time interval $[0, T)$. What is the dimension of the signal space? Plot the basis function(s). Represent all mappings in the signal space. Explain how you derived the signal space.

(c) Add zero-mean white Gaussian noise with variances $10^{-4}, 10^{-2}$, and 10^0 to the generated signal in Part (a). Plot the original signal and noisy version in the same figure for each noise level. Discuss the signal-to-noise ratio (SNR) definition. Does the number of samples per each symbol affect the definition of SNR?

(d) (No MATLAB) Based on the orthonormal signal space you found in Part (b), provide the optimal receiver, showing both the block diagram and specifying the ML rule. Calculate the probability of symbol error. Explain your methodology.

(e) Perform the optimal receiver for different noise values such that the probability of symbol error versus the SNR ranges from 0.5 to about 10^{-4}. For each noise level, generate 10^5 random symbols and use $F_s = 5$kHz. Plot also the theoretical probability of symbol error versus SNR. You can use the "semilogy" command in MATLAB. Comment on the results.

2 Binary Modulation and MAP vs. MLE

A signal $s(t)$ is defined as:

$$s(t) = \sin(2\pi 100 t)h(t) \ ,$$

where $h(t)$ is 1 for $0 \le t < T$, and $T = 0.01$ seconds. Message symbols are mapped as:

$$\begin{aligned} 0 &\rightarrow s(t) \\ 1 &\rightarrow -s(t) \ . \end{aligned}$$

Message symbols are generated uniformly. For example, if the message is 110, then the generated modulated signal, $x(t)$ is given by

$$x(t) = -s(t) - s(t-T) + s(t-2T) \ .$$

(a) Generate 5 random bits and construct the modulated signal $x(t)$. Use sampling frequency $F_s = 1$kHz. Plot $x(t)$.

(b) Obtain an orthonormal signal space for this modulation scheme in the time interval $[0, T)$. What is the dimension of the signal space? Plot the basis function(s). Represent both $s(t)$ and $-s(t)$ in the signal space.

(c) Add zero-mean white Gaussian noise with variances $10^{-4}, 10^{-2}$, and 10^0 to the generated signal in part (a). Plot the original signal and noisy version in the same figure for each noise level. Discuss the SNR definition and determine whether the sampling rate affects SNR or not.

(d) (No MATLAB) Based on the orthonormal signal space you found in part (b), provide the optimal receiver, showing both the block diagram and specifying the ML rule. Calculate the probability of error.

(e) Perform the optimal receiver for different noise values such that the probability of error versus the SNR ranges from 0.5 to about 10^{-4}. For each noise value, generate 10^5 random bits and use sampling frequency $F_s = 1$kHz. Plot also the theoretical probability of error versus SNR. Use the "semilogy" command in MATLAB. Comment on the results.

(f) (No MATLAB) Suppose that the generated bits are not equally likely. Assume that the probability of generating symbol 1 is 0.2. What is the optimal receiver structure? Is the receiver you built in part (d) still optimal?

(g) Suppose that the probability of generating symbol 1 is denoted by α. Perform the optimal receiver for various values of $\alpha \in (0, 0.5)$ by generating 10^5 random bits and use sampling frequency $F_s = 1$kHz. Use 10^{-2} variance for noise. Plot the probability of error versus α for both the α-optimal receiver and the optimal receiver found in Part (d) in the same figure. Comment on the results.

(h) Between the receivers in Part (d) and Part (g), which one is superior, and under what conditions? Is there any situation where you would choose one of the receivers even though it results in higher probability of error?

3 Asymmetric Modulation

(No MATLAB) Consider the following four signals:

$$
\begin{aligned}
s_1(t) &= 0, \\
s_2(t) &= -\cos(2\pi f_0 t), \\
s_3(t) &= \sin(2\pi f_0 t), \\
s_4(t) &= \cos(2\pi f_0 t),
\end{aligned}
$$

for $0 \leq t < T_s$, in which T_s is the symbol period, and $f_0 = N/T_s$, where $N \gg 1$ is an integer.

(a) Suppose that we picked the first basis function as $\phi_1(t) = c(\cos(2\pi f_0 t) + \sin(2\pi f_0 t))$, where c is a constant. Determine c to make sure that this basis function is normalized. Also, determine the second basis function $\phi_2(t)$ such that the two basis functions form an orthonormal basis. Explain your steps clearly. Represent each signal in the signal space.

(b) Based on the orthonormal signal space you found in Part (a), provide the optimal receiver, showing both the block diagram and specifying the ML rule. Describe the reasoning behind specifying the ML rule.

(c) The receiver receives the signal in the following form:

$$r(t) = s_i(t) + n(t) \ ,$$

where $s_i(t)$ is the transmitted signal and $n(t)$ is the noise process, which is zero-mean Gaussian with $N_0/2$ power spectral density. Calculate the probability of symbol error using the exact union bound and the looser version of the union bound. Comment on the difference in results.

(d) Calculate the exact conditional probability of symbol error given that $s_1(t)$ is transmitted. Make sure to include details in your calculations.