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EEE 431- Digital Communications
Computational Assignment 3

Q1- Frequency Shift Keying

This part the assignment includes implementation of orthogonal signaling with frequency shift keying algorithm in MATLAB.

a)
The signal x(t) consisting of 5 random symbols and total duration 5T = 0.5s is generated and plotted in Figure 1.

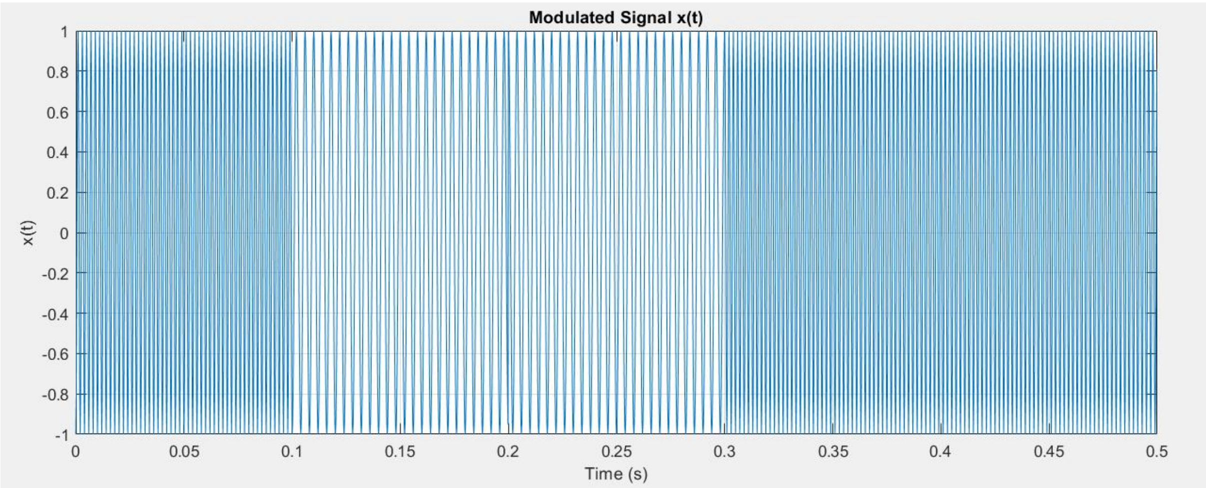


Figure 1: Modulated Signal x(t) 4FSK

b)

By inspection: $\psi_1(t) = \dfrac{s_1(t)}{\sqrt{E_{s_1}}}$, $\psi_2(t) = \dfrac{s_2(t)}{\sqrt{E_{s_2}}}$ will work.

- Orthogonality

$$\langle s_1(t), s_2(t)\rangle = \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt = \int_0^T \cos(2\pi 250t)\cos(2\pi 500t)\,dt$$

$$= \frac{1}{2}\int_0^{0.1} \cos(2\pi 250t - 2\pi 500t) + \cos(2\pi 250t + 2\pi 500t)\,dt$$

$$= \frac{1}{2}\int_0^{0.1} \cos(2\pi(-250)t) + \cos(2\pi 750t)\,dt$$

$$= \frac{1}{2}\left[\frac{\sin(-500\pi t)}{-500} + \frac{\sin(1500\pi t)}{1500\pi}\right]\Bigg|_0^{0.1} = 0$$

$$\boxed{\langle s_1(t), s_2(t)\rangle = 0}$$

– Normalization

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_{s_1}}}$$

$$E_1 = \int_0^T s_1(t)^2 dt = \int_0^T \cos^2(2\pi 250 t)\, h(t)\, dt$$

$$= \int_0^{0.1} \cos^2(2\pi 250 t)\, dt$$

$$= \int_0^{0.1} \frac{1 + \cos(2\pi 500 t)}{2} dt$$

$$= \int_0^{0.1} \frac{1}{2} dt + \int_0^{0.1} \frac{\cos(1000\pi t)}{2} dt$$

$$= 0.05 + \left[\frac{\sin(1000\pi t)}{1000\pi}\right]\Bigg|_0^{0.1}$$

$$= 0.05$$

$$\boxed{\psi_1(t) = 2\sqrt{5}\cos(250\pi t)\, h(t)}$$

$$\psi_2(t) = \frac{s_2(t)}{\sqrt{E_{s_2}}}$$

$$E_2 = \int_0^T s_2(t)^2 dt = \int_0^T \cos^2(2\pi 500 t)\, h(t)\, dt$$

$$= \int_0^{0.1} \cos^2(2\pi 500 t)\, dt$$

$$= \int_0^{0.1} \frac{1 + \cos(2\pi 1000 t)}{2} dt$$

$$= \int_0^{0.1} \frac{1}{2} dt + \int_0^{0.1} \frac{\cos(2000\pi t)}{2} dt$$

$$= 0.05 + \left[\frac{\sin(2000\pi t)}{2000\pi}\right]\Bigg|_0^{0.1}$$

$$= 0.05$$

$$\boxed{\psi_2(t) = 2\sqrt{5}\cos(500\pi t)\, h(t)}$$

2 orthonormal basis functions ⇒ $\boxed{\text{dimension of the signal space} = 2}$

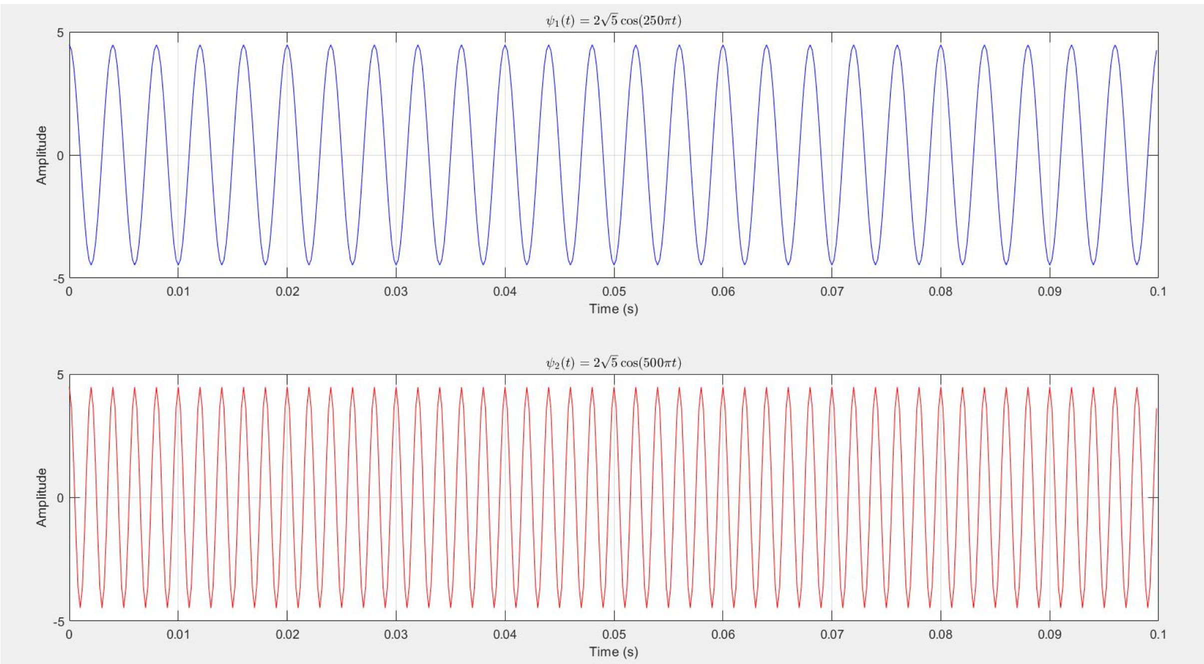


Figure 2: Basis functions of 4FSK

By taking the inner product of each representation with the basis functions signal constellation becomes:

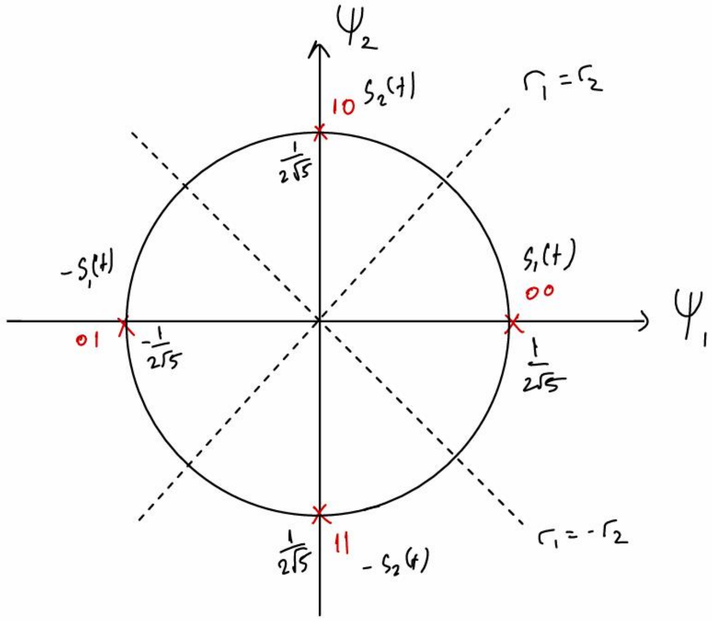


Figure 3: Signal constellation of 4FSK

c)
AWGN with variances $N_0/2 = 10^{-4}$, 10^{-2}, 10^{0} are applied to the signal x(t).

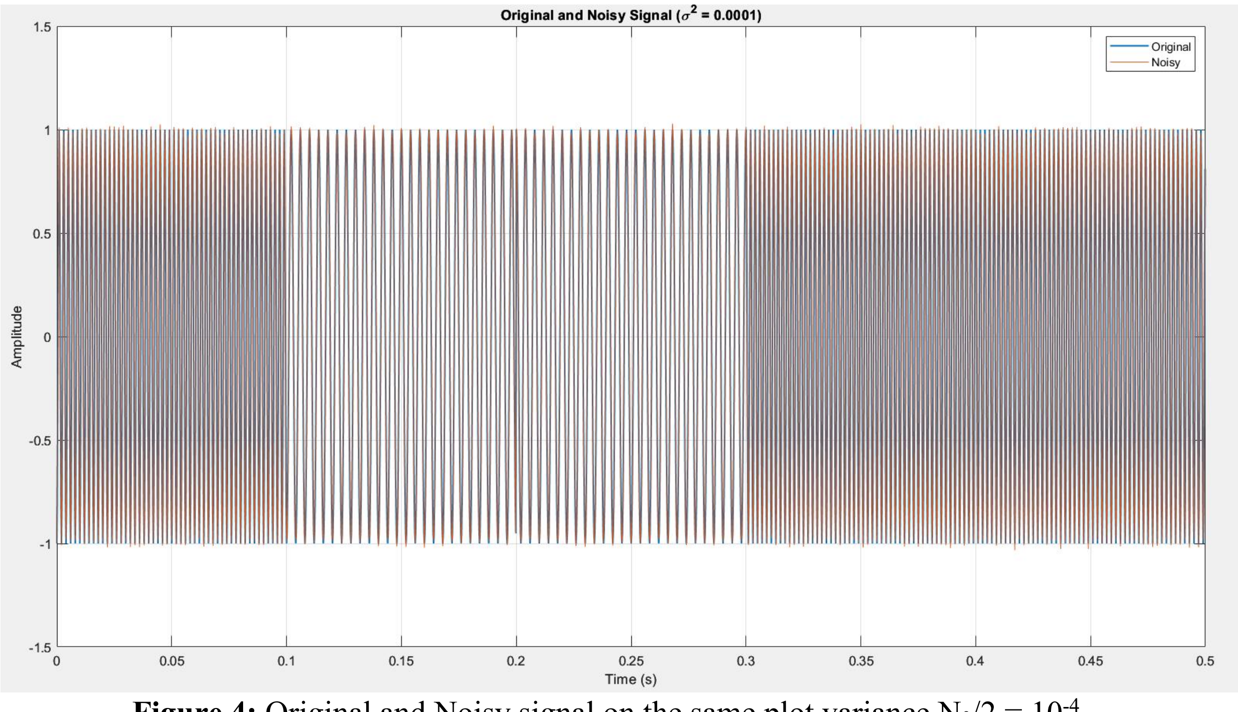


Figure 4: Original and Noisy signal on the same plot variance $N_0/2 = 10^{-4}$

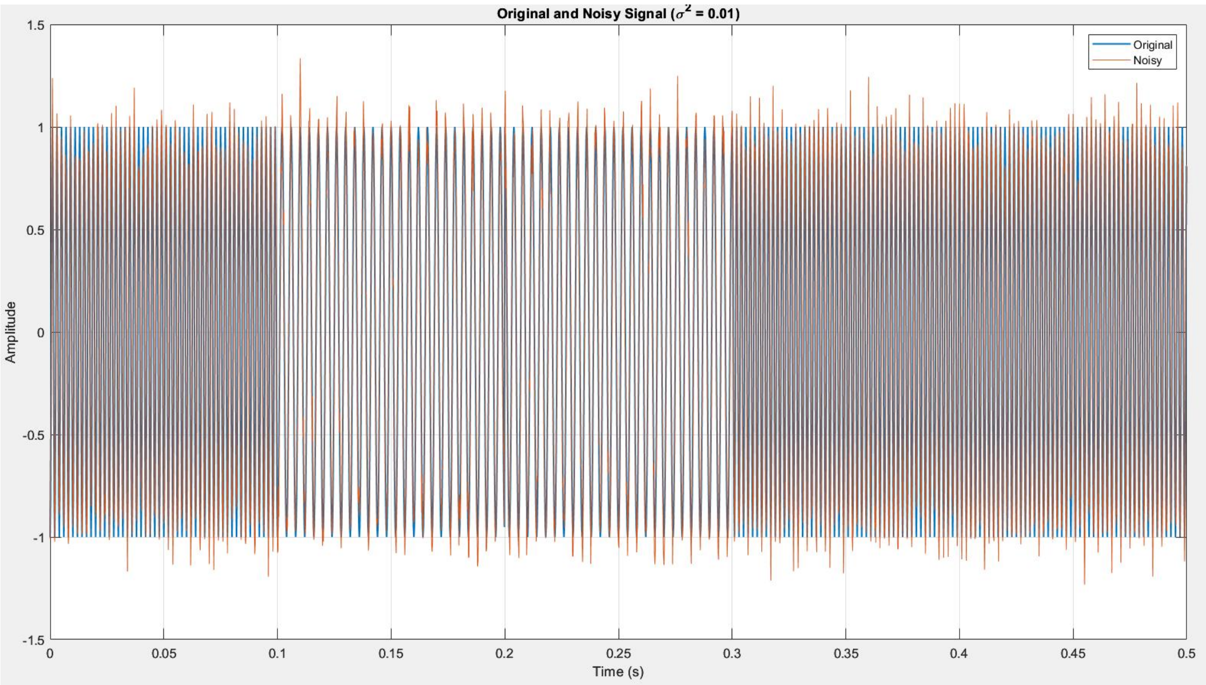


Figure 5: Original and Noisy signal on the same plot variance $N_0/2 = 10^{-2}$

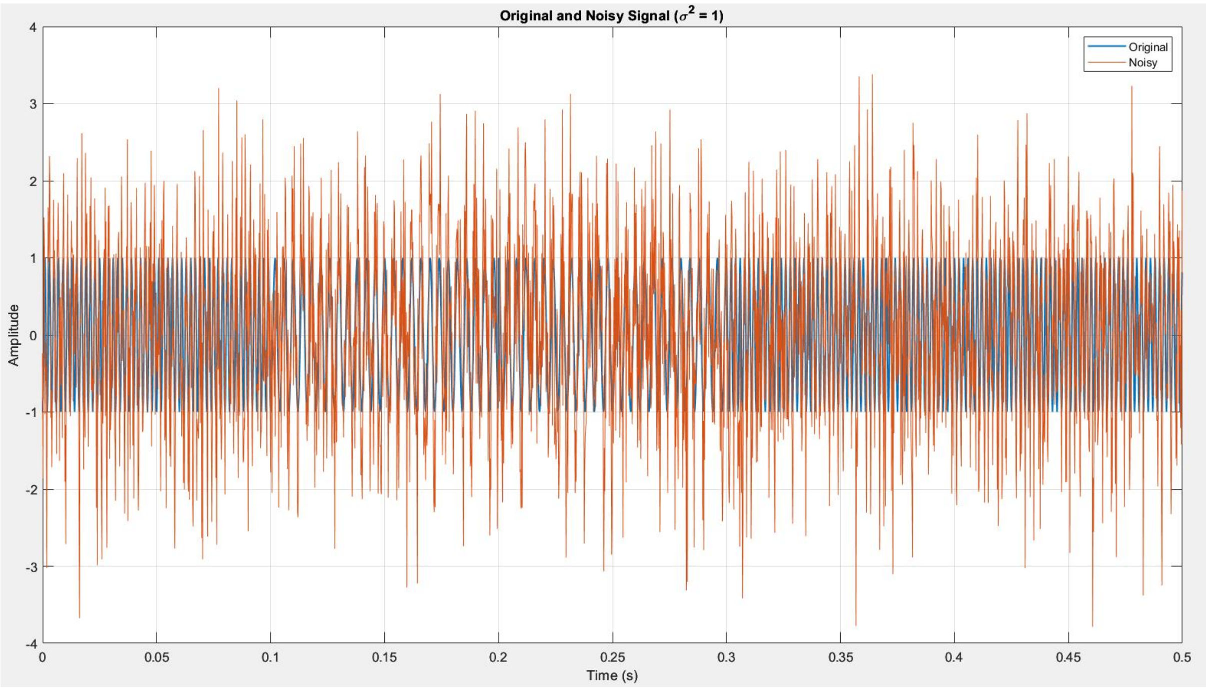


Figure 6: Original and Noisy signal on the same plot variance $N_0/2 = 10^{0}$

From Figures 4-6, it is visible that as the noise variance increases the distortion in the signal increases as expected.

$$SNR = \frac{E_s}{N_0}$$

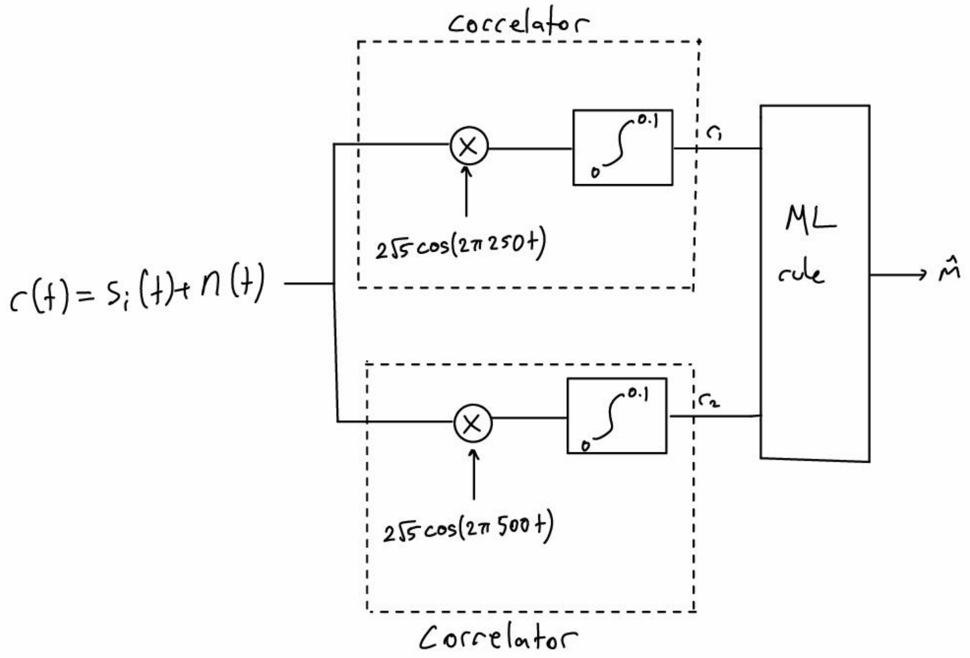
Signal to noise ratio (SNR) is defined as the ratio of the energy of the signal E_s to half the noise power spectral density N_0, representing how much signal power exists relative to noise, and indicating the quality of a digital communication system. The number of samples per each symbol does not directly affect the theoretical definition of SNR. However, it affects the practical estimation of SNR as, white noise includes all the frequencies that is impossible to achieve in practice. Very high frequency components of the AWGN are aliased to lower frequencies, therefore the PSD of the noise shrinks a little but not significant with large number of samples. Hence, SNR in practice changes.

Also to keep the total noise energy per symbol consistent with the theoretical $N_0/2$, the variance of the noise per sample must be scaled by F_s because in discrete-time, each sample represents a time interval 1/Fs so the per-sample noise variance must increase proportionally to maintain the correct total noise energy per symbol, matching continuous-time theory.

$$\sigma^2 = \frac{N_0}{2} \cdot F_s$$

d)

- Optimum coherent FSK receiver



$\underline{r} = \underline{s_i} + \underline{n}$

$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

ML Rule:

$$\hat{m} = \begin{cases} 00, & \text{if } r_1 \geq r_2 \ \& \ r_1 \geq -r_2 \\ 10, & \text{if } r_1 < r_2 \ \& \ r_1 \geq -r_2 \\ 01, & \text{if } r_1 < r_2 \ \& \ r_1 < -r_2 \\ 11, & \text{if } r_1 \geq r_2 \ \& \ r_1 < -r_2 \end{cases}$$

Probability of error:

Due to symetry: $P_{e,00} = P_{e,01} = P_{e,10} = P_{e,11} = P_e$

$$P_{e,00} = 1 - P_{c,00} = 1 - P(r_1 \geq r_2 \ \& \ r_1 \geq -r_2 \mid 00 \text{ sent})$$

$$= 1 - P(A + n_1 \geq 0 + n_2 \ \& \ A + n_1 \geq -(0 + n_2))$$

$$= 1 - P(\underbrace{n_1 - n_2}_{\mathcal{N}(0, N_0)} \geq -A \ \& \ \underbrace{n_1 + n_2}_{\mathcal{N}(0, N_0)} \geq -A)$$

$$= 1 - P(n_1 - n_2 \geq -A) P(n_1 + n_2 \geq -A)$$

$$= 1 - Q\left(\frac{-A}{\sqrt{N_0}}\right) Q\left(\frac{-A}{\sqrt{N_0}}\right)$$

$A = \sqrt{E_b}$

$$= 1 - \left(Q\left(-\sqrt{\frac{E_b}{N_0}}\right)\right)^2$$

$$P_e = 1 - \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

$$\boxed{P_e = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)}$$

e)

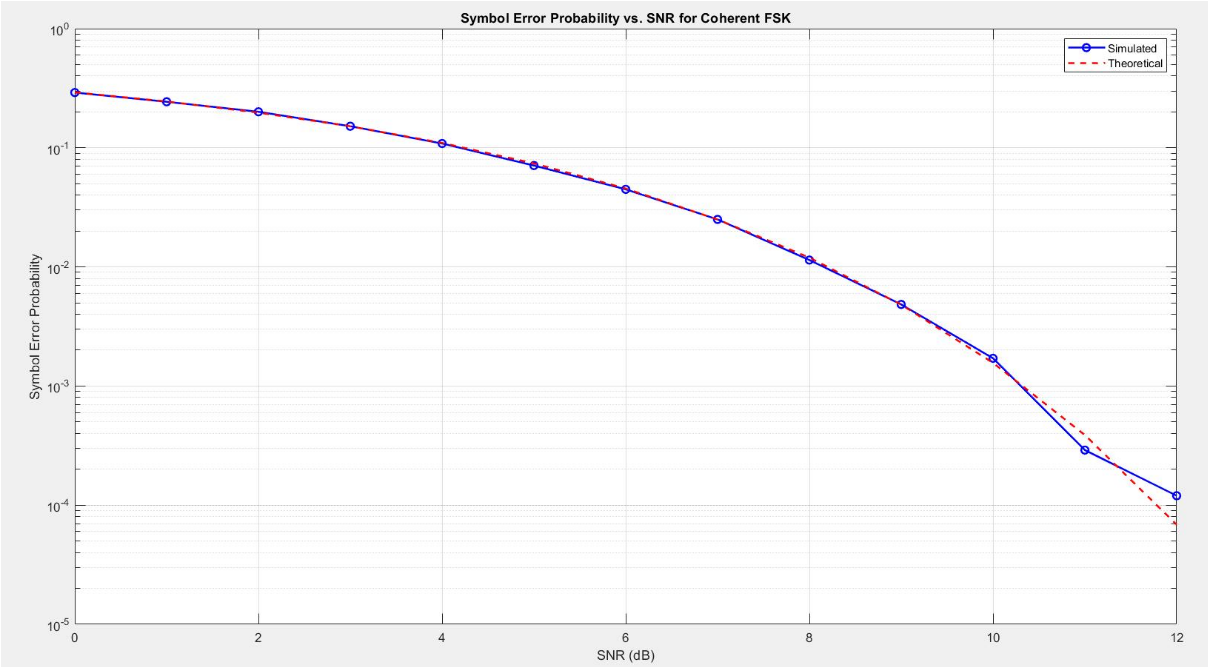


Figure 7: Simulated and Theoretical P_e vs SNR, 4FSK

In Figure 7, the plot shows the symbol error probability vs. SNR (dB) for coherent FSK, comparing simulated and theoretical results.

The simulated curve (blue) closely matches the theoretical curve (red dashed) over the entire SNR range. As expected, symbol error probability decreases exponentially with increasing SNR, indicating stronger noise immunity at higher signal power. At low SNR, the simulated error aligns well with theory but has more variability due to higher noise influence. At high SNR, the simulated error falls below 10^{-4}, showing great performance, and any small gap from theory is due to finite sample size and randomness in noise.

The simulation validates the theoretical model for coherent FSK detection. The close agreement confirms that the receiver implementation, noise scaling, and ML decision rule are correct. As SNR increases, the system becomes increasingly reliable, as predicted.

Q2- Binary Modulation and MAP vs. MLE

This part the assignment includes implementation of BPAM algorithm in MATLAB, probability of error calculations and simulation, and comparison of MAP and MLE.

a)
The signal x(t) consisting of 5 random symbols and total duration 5T = 0.05s is generated and plotted in Figure x.

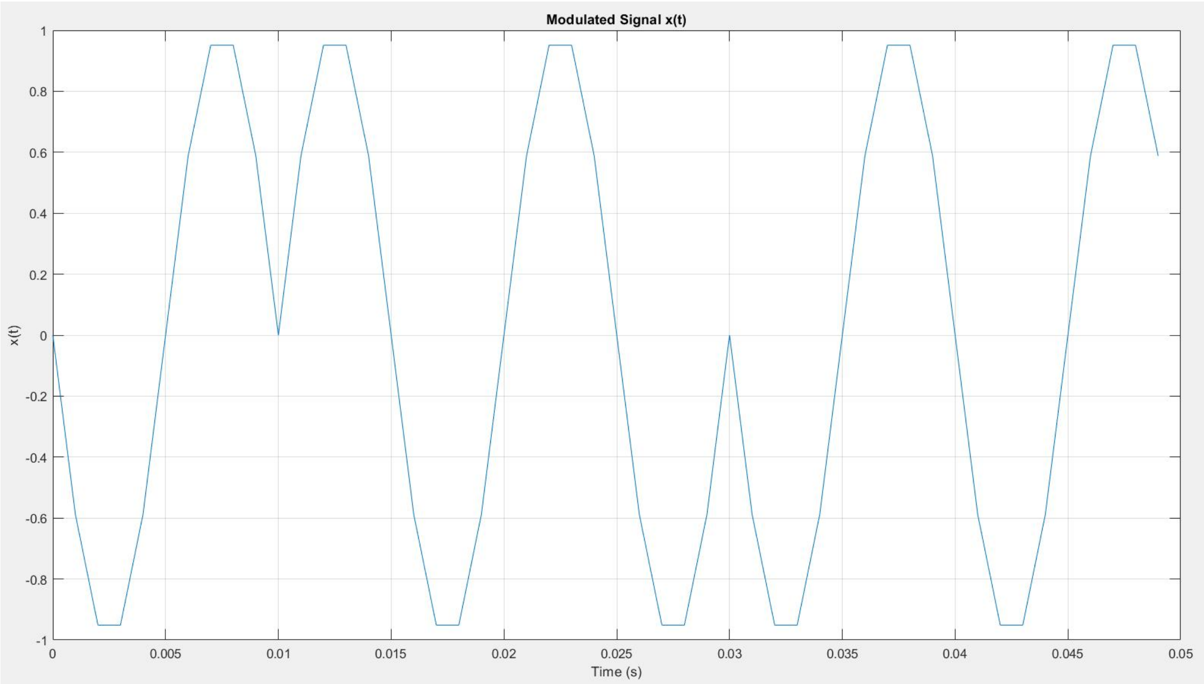


Figure 8: Modulated Signal x(t) BPAM

b)

By inspection:

$$\psi_1(t) = \frac{s(t)}{\sqrt{E_s}}$$ will work.

$$E_s = \int_0^T s^2(t)\,dt = \int_0^{0.01} \sin^2(2\pi 100t)\,dt = \int_0^{0.01} \frac{1-\cos(2\pi 200t)}{2}\,dt = \frac{1}{2}\int_0^{0.01} (1-\cos(400\pi t))\,dt$$

$$= \frac{1}{2}\left[0.01 - \left(\frac{\sin(400\pi t)}{400\pi}\right)\Bigg|_0^{0.01}\right] = 0.005 \qquad \sqrt{E_s} = \frac{\sqrt{5}}{10\sqrt{10}} = \frac{1}{10\sqrt{2}}$$

$$\boxed{\psi_1(t) = 10\sqrt{2}\,\sin(2\pi 100t)\,h(t)}$$

1 basis function ⇒ $\boxed{\text{signal space is 1 dimentional.}}$

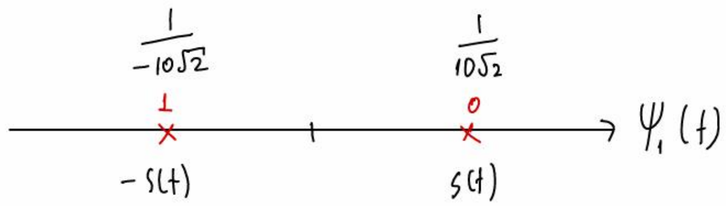


Figure x: Signal constellation of BPAM

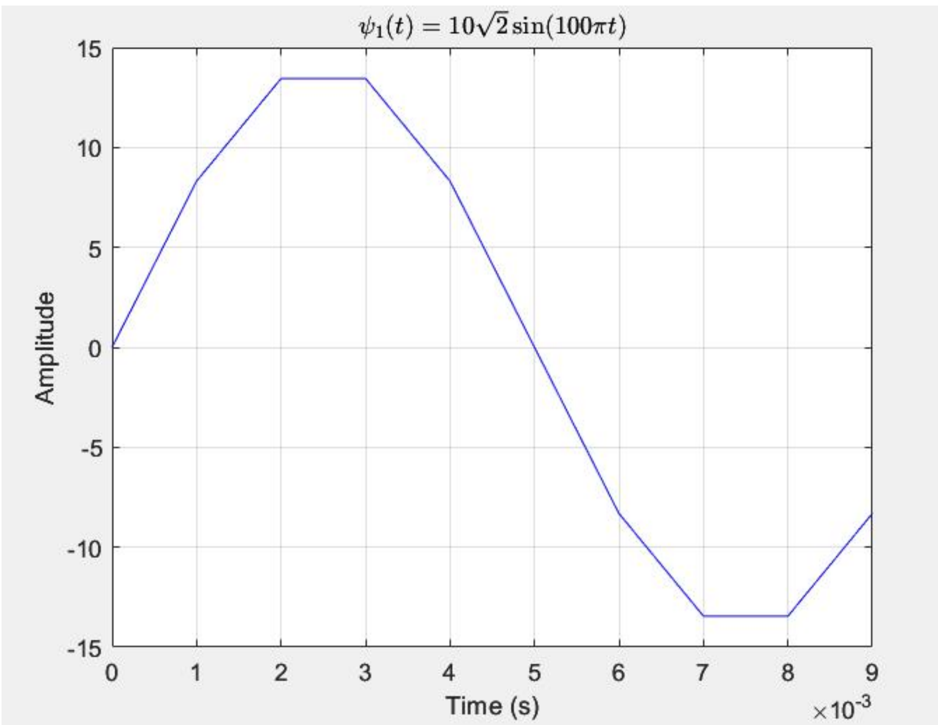


Figure 9: Basis function of BPAM

c)

AWGN with variances $N_0/2 = 10^{-4}$, 10^{-2}, 10^{0} are applied to the signal x(t).

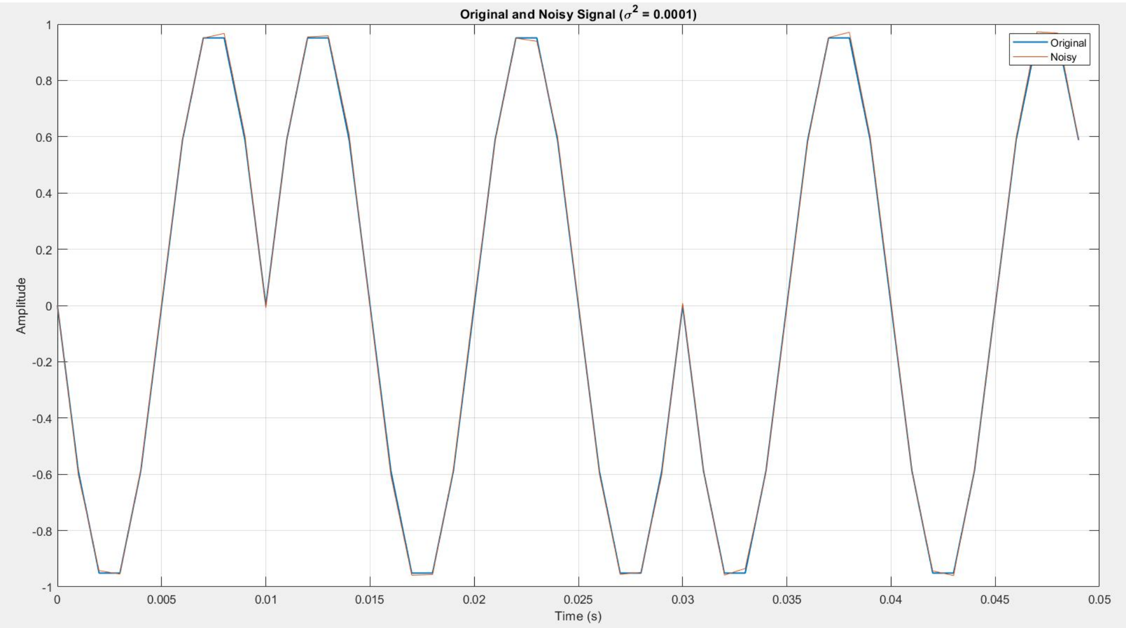


Figure 9: Original and Noisy signal on the same plot variance $N_0/2 = 10^{-4}$

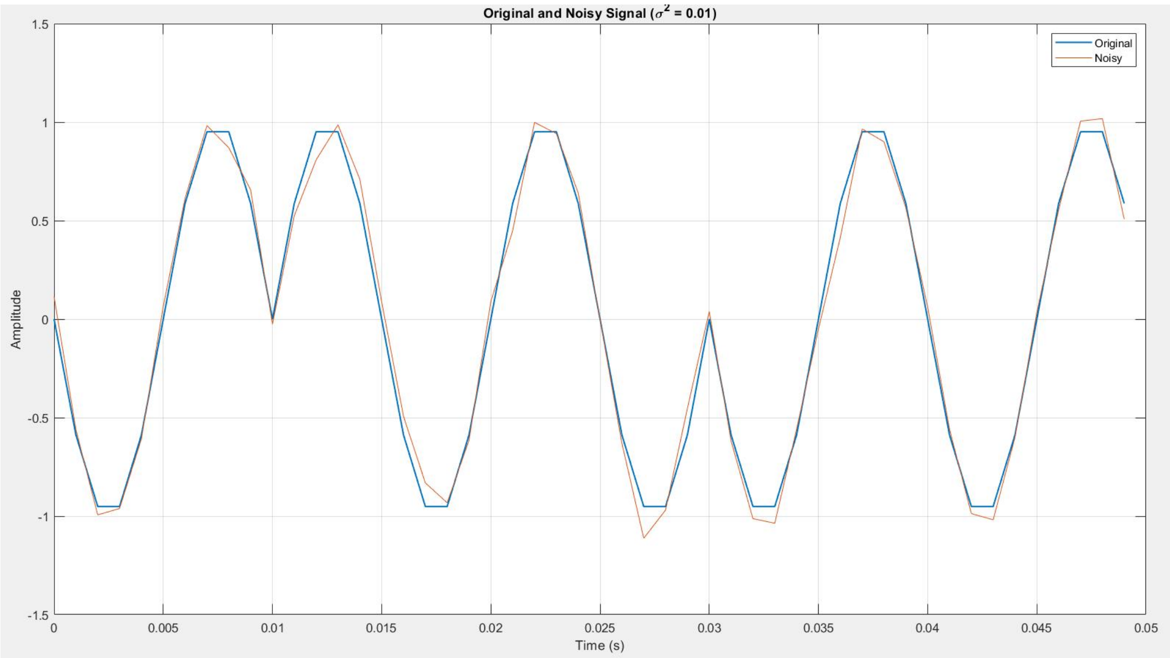


Figure 10: Original and Noisy signal on the same plot variance $N_0/2 = 10^{-2}$

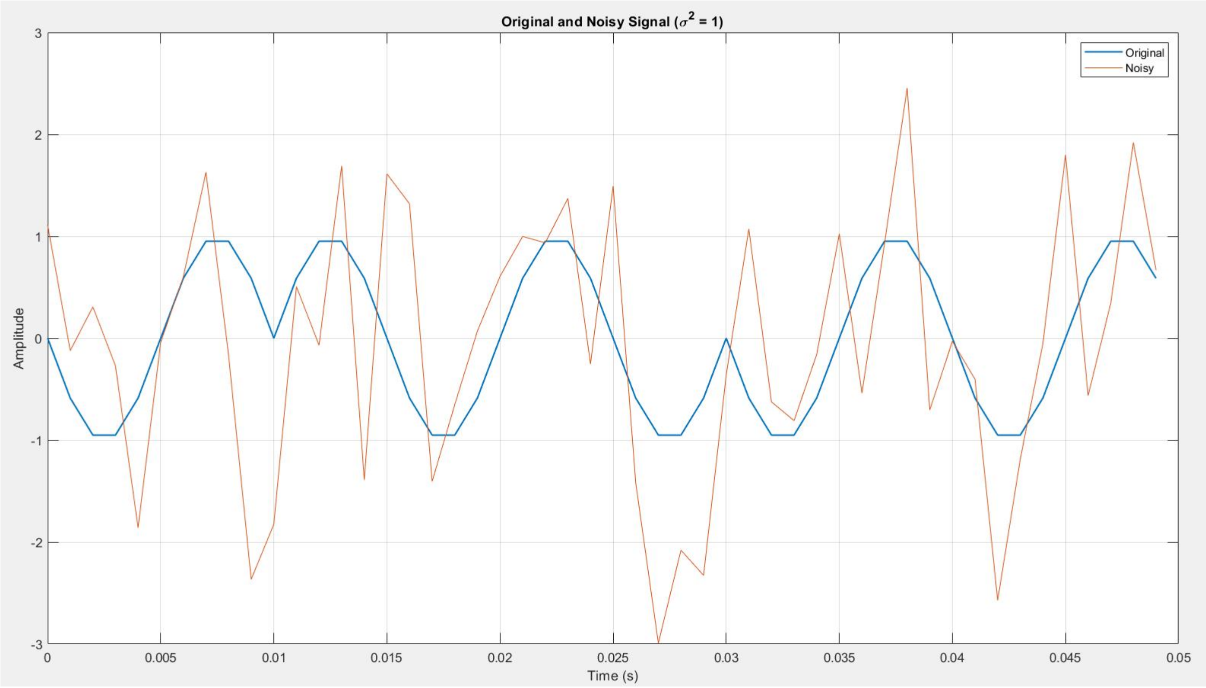


Figure 11: Original and Noisy signal on the same plot variance $N_0/2 = 10^0$

From Figures 9-11, it is visible that as the noise variance increases the distortion in the signal increases as expected.

$$SNR = \frac{E_s}{N_0}$$

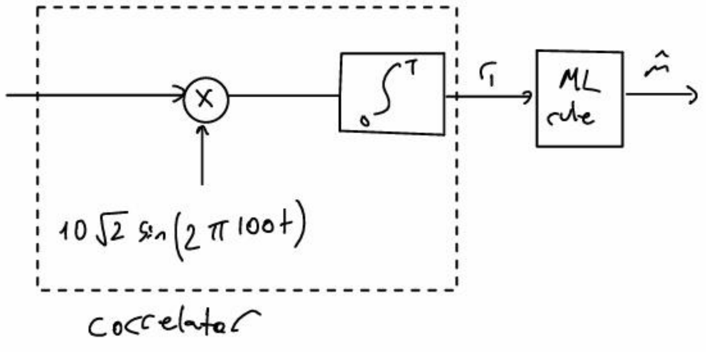
As discussed in the previous question, signal to noise ratio (SNR) is defined as the ratio of the energy of the signal E_s to half of the noise power spectral density N_0, representing how much signal power exists relative to noise, and indicating the quality of a digital communication system. The number of samples per each symbol does not directly affect the definition of SNR.

Also to keep the total noise energy per symbol consistent with the theoretical $N_0/2$, the variance of the noise per sample must be scaled by F_s to maintain the correct total noise energy per symbol.

$$\sigma^2 = \frac{N_0}{2} \cdot F_s$$

d)

Optimum Receiver:



$$\hat{m}\begin{cases} 0, & \text{if } r_1 > 0 \\ 1, & \text{if } r_1 < 0 \end{cases}$$

Probability of error:

$$P_e = \frac{1}{2}P_{e,1} + \frac{1}{2}P_{e,2} = \frac{1}{2}P(r_1 > 0 \mid 1 \text{ sent}) + \frac{1}{2}P(r_1 \leq 0 \mid 0 \text{ sent})$$

$$= \frac{1}{2}P(-A + n_1 < 0) + \frac{1}{2}P(A + n_1 \geq 0)$$

$$= \frac{1}{2}P\left(\frac{n_1}{\sqrt{N_0/2}} < \frac{-A}{\sqrt{N_0/2}}\right) + \frac{1}{2}P\left(\frac{n_1}{\sqrt{N_0/2}} \geq \frac{A}{\sqrt{N_0/2}}\right)$$

$$= \frac{1}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

$$E_b = \|s_i\|^2 = A^2 \rightarrow A = \sqrt{E_b} \qquad \sqrt{E} = \frac{1}{10\sqrt{2}}$$

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\frac{1}{10\sqrt{N_0}}\right)}$$

e)

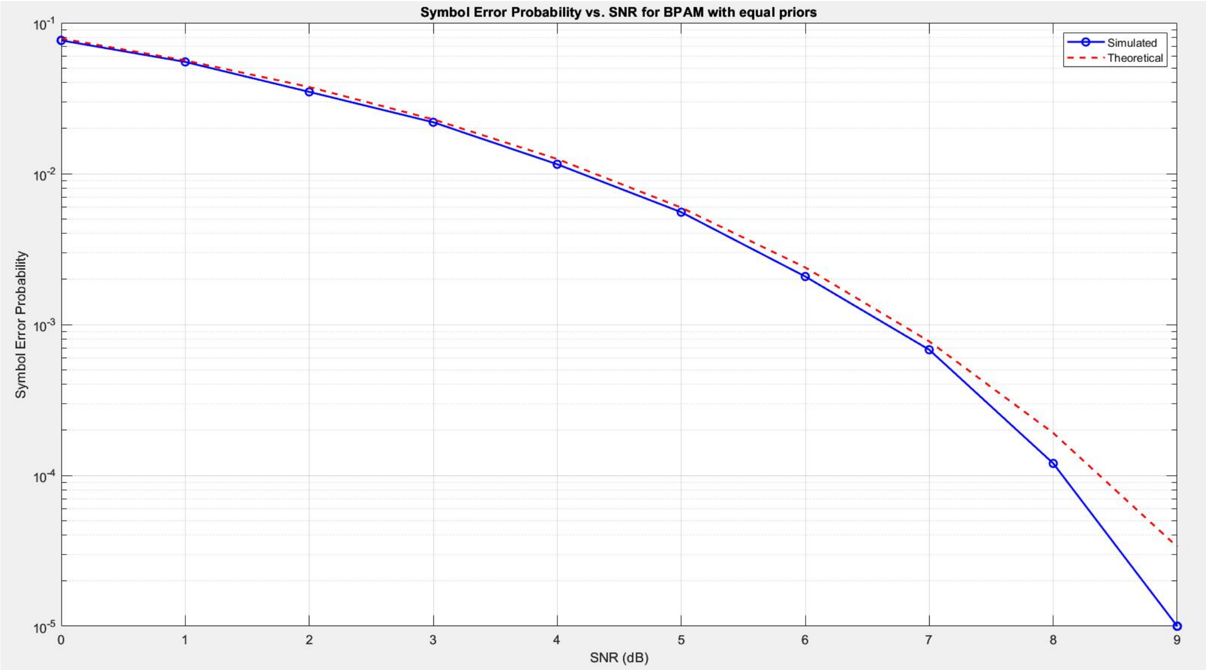


Figure 12: Simulated and Theoretical P_e vs SNR, BPAM with equal priors

In Figure 12, the plot shows the symbol error probability vs. SNR (dB) for BPAM with equal priors, comparing simulated and theoretical results.

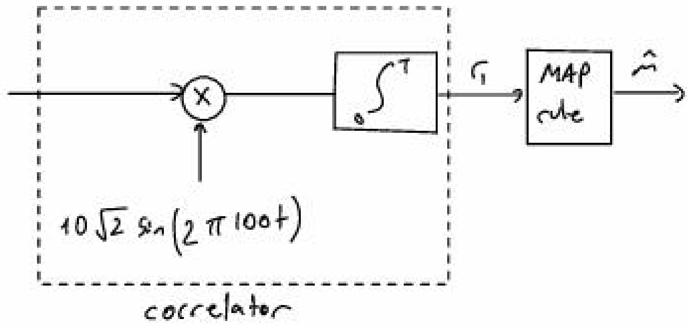
The simulated curve (blue) closely matches the theoretical curve (red dashed) over the entire SNR range. As expected, symbol error probability decreases exponentially with increasing SNR, indicating stronger noise immunity at higher signal power. At low SNR, the simulated error aligns well with theory but has more variability due to higher noise influence. At high SNR, the simulated error falls below 10^{-4}, showing great performance, and any small gap from theory is due to finite sample size and randomness in noise.

The simulation validates the theoretical model for BPAM with equal priors. The close agreement confirms that the receiver implementation, noise scaling, and ML decision rule are correct. As SNR increases, the system becomes increasingly reliable, as predicted.

f)

Since the prior probabilities are not equal, using MLE is not optimal. Therefore the estimator in part d is not the optimal receiver.

Instead MAP estimate should be utilized as follows:



$$\hat{m}\begin{cases} 0, & \text{if } r_1 > r_{th} \\ 1, & \text{if } r_1 < r_{th} \end{cases}$$

MAP rule:

$p = p(1) = 0.2$

$$p(1)\, f(r|1) \underset{0}{\overset{1}{\gtrless}} p(0)\, f(r|0)$$

$$p \cdot \cancel{\frac{1}{\sqrt{\pi N_0}}} \exp\left(-\frac{(r+\sqrt{E_b})^2}{N_0}\right) \underset{0}{\overset{1}{\gtrless}} (1-p)\cancel{\frac{1}{\sqrt{\pi N_0}}} \exp\left(-\frac{(r-\sqrt{E_b})^2}{N_0}\right)$$

$$\exp\left(-\frac{(r+\sqrt{E_b})^2}{N_0}\right) \underset{0}{\overset{1}{\gtrless}} \frac{1-p}{p} \exp\left(-\frac{(r-\sqrt{E_b})^2}{N_0}\right)$$

$$-\frac{(r+\sqrt{E_b})^2}{N_0} \underset{0}{\overset{1}{\gtrless}} \ln\left(\frac{1-p}{p}\right) - \frac{(r-\sqrt{E_b})^2}{N_0}$$

$$\frac{4r\sqrt{E_b}}{N_0} \underset{0}{\overset{1}{\gtrless}} \ln\left(\frac{1-p}{p}\right)$$

$$r \underset{0}{\overset{1}{\gtrless}} \ln\left(\frac{1-p}{p}\right)\frac{N_0}{4\sqrt{E_b}}$$

$$\boxed{r_{th} = \ln\left(\frac{1-p}{p}\right)\frac{N_0}{4\sqrt{E_b}}}$$

g)

Probability of error versus α for both the α-optimal receiver and the optimal receiver are plotted.

Probability of error of the MLE estimate BPAM is given by:

$$P_{e,MLE} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Probability of error of the MAP estimate of BPAM is given by:

$$P_{e,MAP} = \alpha \cdot Q\left(\frac{4E_b - N_0 \ln\left(\frac{1-\alpha}{\alpha}\right)}{4\sqrt{E_b N_0/2}}\right) + (1-\alpha) \cdot Q\left(\frac{4E_b + N_0 \ln\left(\frac{1-\alpha}{\alpha}\right)}{4\sqrt{E_b N_0/2}}\right)$$

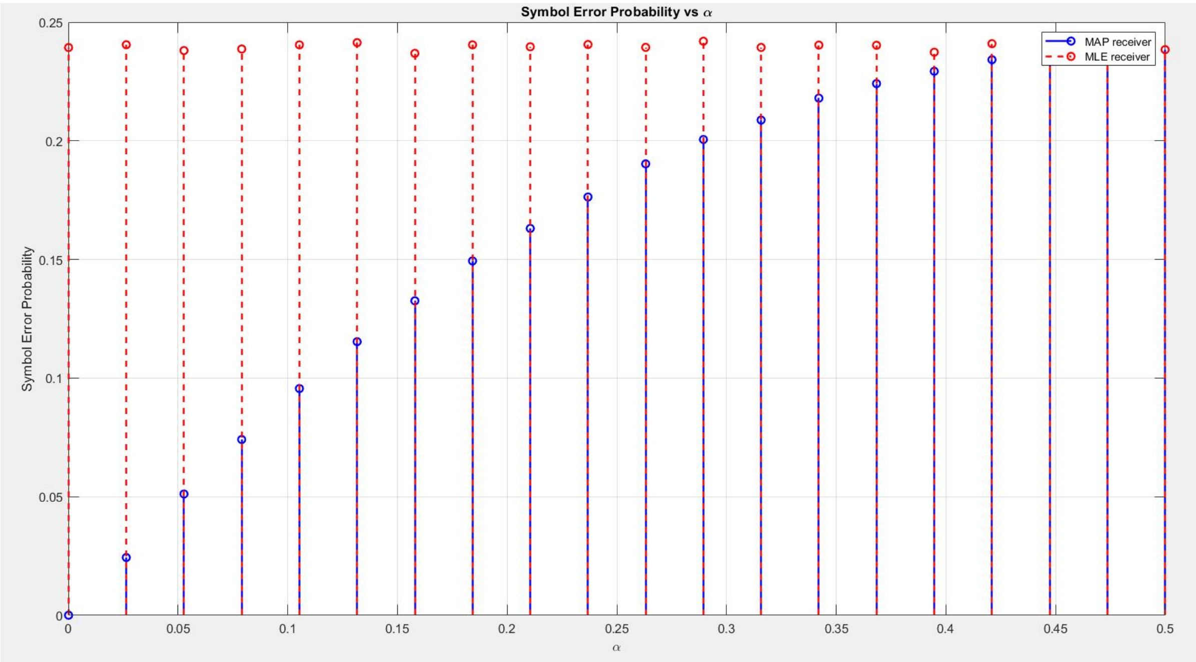


Figure 13: Probability of error versus α MAP and MLE

Figure 13 compares the symbol error probabilities of MAP and ML detectors for antipodal BPAM modulation as a function of the prior probability α. The ML detector, which assumes equal priors, results in a constant error probability regardless of α. In contrast, the MAP detector adapts to the prior information, resulting in a significantly lower error probability when α deviates from 0.5. This demonstrates the advantage of MAP detection in scenarios with unequal priors, as it effectively reduces the error by incorporating prior knowledge.

h)

The choice between the MLE and MAP receivers depends on the prior probabilities of the transmitted symbols. The MLE receiver, with a fixed decision threshold at zero, is optimal only when the priors are equal. However, when the symbol probabilities are unequal, the MAP receiver outperforms MLE by adjusting the decision threshold to account for prior information. This results in a lower overall probability of error, making MAP the preferred choice in practical scenarios where symbol transmission is biased or non-uniform.

However, while the MAP receiver offers optimal performance when symbol probabilities are known and unequal, there are practical scenarios where the simpler MLE receiver may be preferred. In cases where system simplicity, fairness, or robustness to unknown or varying priors is more critical than minimizing error, the MLE receiver can be a more suitable choice, even if it results in a slightly higher probability of error. Ultimately, the choice depends on the specific requirements and constraints of the communication system.

3 Asymmetric Modulation

a)

Normalize $\phi_1(t)$

$$\langle \phi_1(t), \phi_1(t) \rangle = 1$$

$$\langle \phi_1(t), \phi_1(t) \rangle = \int_0^{T_s} \phi_1^2(t)\, dt$$

$$= c^2 \int_0^{T_s} \left(\cos(2\pi f_0 t) + \sin(2\pi f_0 t)\right)^2 dt$$

$$= c^2 \int_0^{T_s} \underbrace{\cos^2(2\pi f_0 t) + \sin^2(2\pi f_0 t)}_{1} + 2\cos(2\pi f_0 t)\sin(2\pi f_0 t)\, dt$$

$$= c^2 \int_0^{T_s} 1 + 2\cos(2\pi f_0 t)\sin(2\pi f_0 t)\, dt$$

$$= c^2 \int_0^{T_s} 1 + \sin(4\pi f_0 t)\, dt$$

$$= c^2 \left[T_s + \frac{\cos(4\pi f_0 t)}{4\pi f_0} \Big|_0^{T_s} \right] = c^2 \left[T_s + \frac{\cos(4\pi f_0 T_s)}{4\pi f_0} \right]$$

$$= c^2 \left[T_s + \frac{\sin(4\pi N)}{8\pi f_0} \right] = c^2 T_s \Rightarrow c^2 T_s = 1 \Rightarrow \boxed{c = \frac{1}{\sqrt{T_s}}}$$

By inspection:

$$\phi_2(t) = \frac{1}{\sqrt{T_s}} \left(\cos(2\pi f_0 t) - \sin(2\pi f_0 t) \right) \quad \text{will work.}$$

Validate by showing:

$$\langle \phi_2(t), \phi_2(t) \rangle = 1 \quad \rightarrow \text{normalization}$$

$$\langle \phi_1(t), \phi_2(t) \rangle = 0 \quad \rightarrow \text{orthogonality}$$

Normalization

$$\langle \phi_2(t), \phi_2(t) \rangle = \frac{1}{T_s} \int_0^{T_s} \phi_2^2(t)\, dt$$

$$= \frac{1}{T_s} \int_0^{T_s} \left(\cos(2\pi f_0 t) - \sin(2\pi f_0 t) \right)^2 dt$$

$$= \frac{1}{T_s} \int_0^{T_s} \underbrace{\cos^2(2\pi f_0 t) + \sin^2(2\pi f_0 t)}_{1} - 2\cos(2\pi f_0 t)\sin(2\pi f_0 t)\, dt$$

$$= \frac{1}{T_s} \int_0^{T_s} 1 - 2\cos(2\pi f_0 t)\sin(2\pi f_0 t)\, dt$$

$$= \frac{1}{T_s} \int_0^{T_s} 1 - \sin(4\pi f_0 t)\, dt$$

$$= \frac{1}{T_s} \left[T_s - \frac{\cos(4\pi f_0 t)}{4\pi f_0} \Big|_0^{T_s} \right] = \frac{1}{T_s} \left[T_s - \frac{\cos(4\pi f_0 T_s)}{4\pi f_0} \right]$$

$$= \frac{1}{T_s} \left[T_s - \frac{\sin(4\pi N)}{4\pi f_0} \right] = \frac{1}{T_s} T_s = 1$$

$$\boxed{\langle \phi_2(t), \phi_2(t) \rangle = 1}$$

Ortogonalitiy:

$$\langle \phi_1(t), \phi_2(t) \rangle = \int_0^{T_s} \phi_1(t)\phi_2(t)\,dt$$

$$= \frac{1}{T_s} \int_0^{T_s} \cos^2(2\pi f_0 t) - \sin^2(2\pi f_0 t)\,dt$$

$$= \frac{1}{T_s} \int_0^{T_s} \cos(4\pi f_0 t)\,dt$$

$$= \frac{1}{T_s} \sin(4\pi f_0 T_s)$$

$$= \frac{1}{T_s} \sin(4\pi N) = 0$$

$$\boxed{\langle \phi_1(t), \phi_2(t) \rangle = 0}$$

Represent each signal with the basis functions:

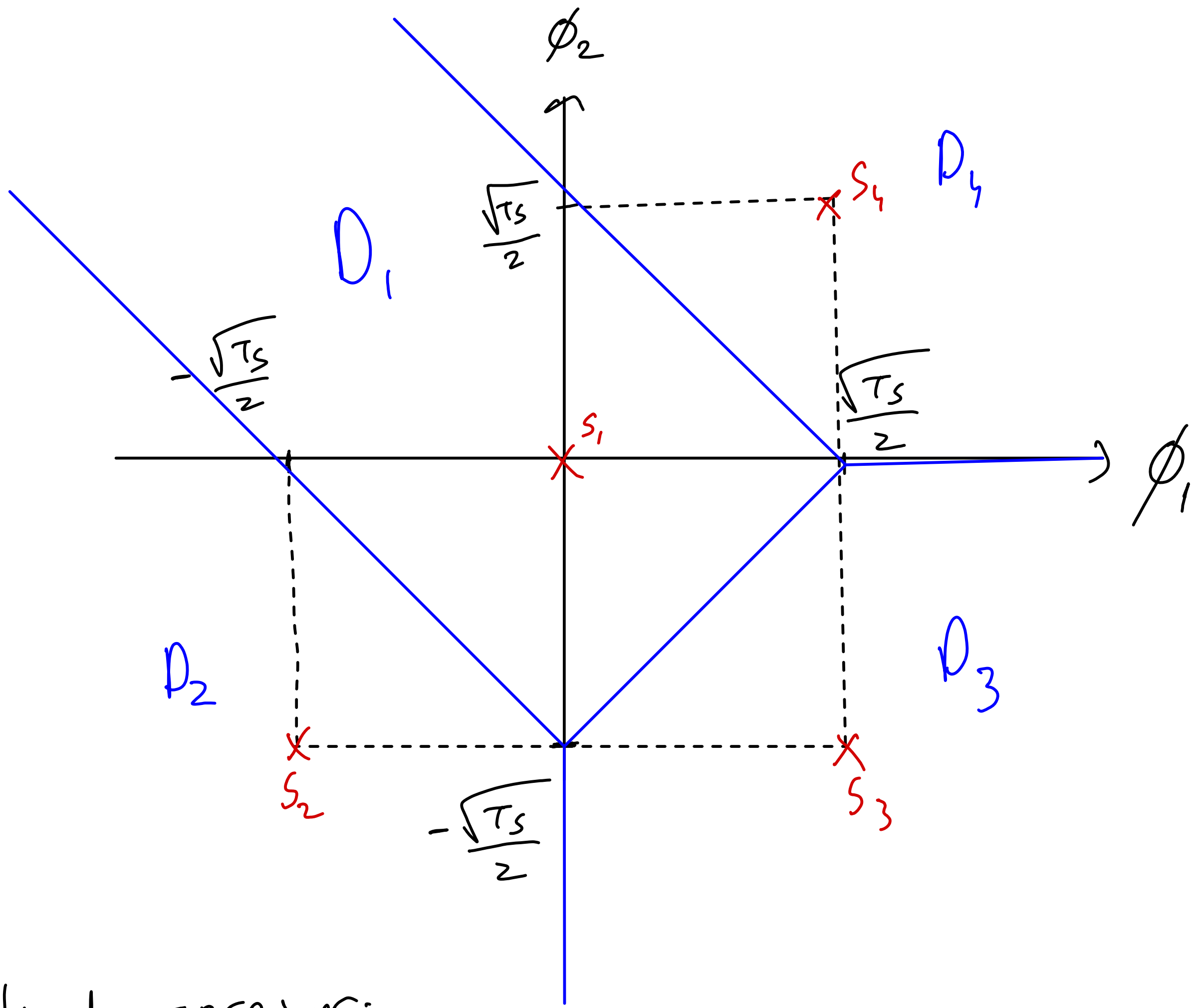
$$s_1(t) = 0\,\phi_1(t) + 0\,\phi_2(t)$$

$$s_2(t) = -\frac{\sqrt{T_s}}{2}\phi_1(t) - \frac{\sqrt{T_s}}{2}\phi_2(t)$$

$$s_3(t) = \frac{\sqrt{T_s}}{2}\phi_1(t) - \frac{\sqrt{T_s}}{2}\phi_2(t)$$

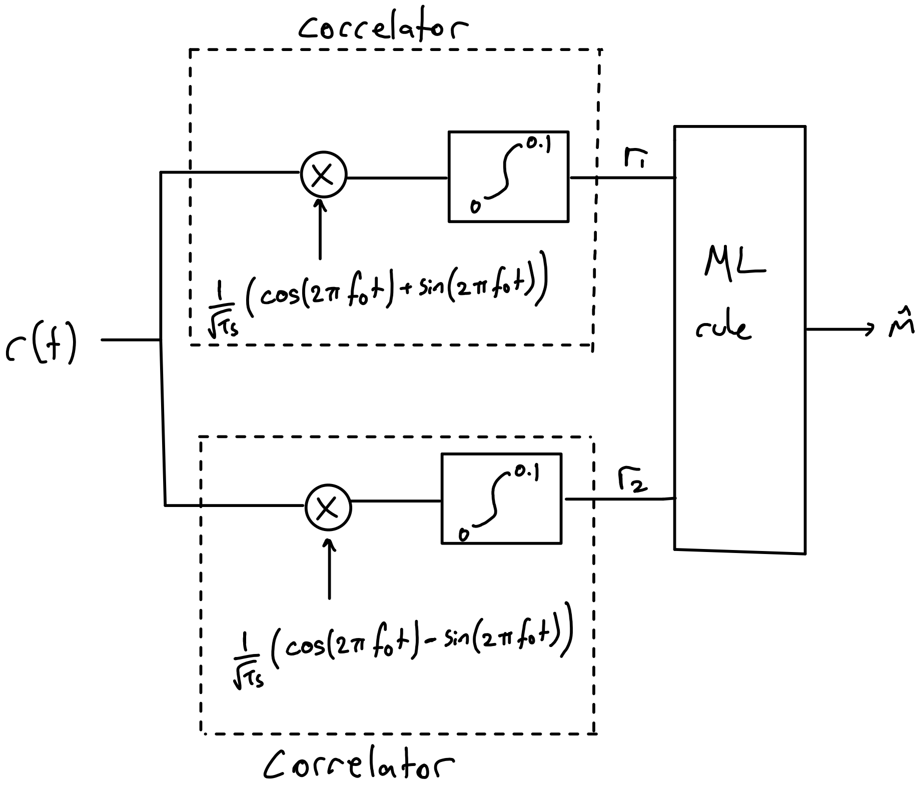
$$s_4(t) = \frac{\sqrt{T_s}}{2}\phi_1(t) + \frac{\sqrt{T_s}}{2}\phi_2(t)$$

Signal constellation diagram:



b)

Optimal receiver:



Since, signals have equal priors, ML rule is simply a minimum distance problem.

ML Rule Decision Regions

$$D_1 = \{(r_1, r_2) : \|r - s_1\|^2 < \|r - s_2\|^2, \; \|r - s_1\|^2 < \|r - s_3\|^2, \; \|r - s_1\|^2 < \|r - s_4\|^2\}$$

$$= \{(r_1, r_2) : r_1^2 + r_2^2 < \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2, \; r_1^2 + r_2^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2, \; r_1^2 + r_2^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2\}$$

$$\boxed{D_1 = \{(r_1, r_2) : r_1 + r_2 > -\frac{\sqrt{T_s}}{2}, \; r_1 - r_2 < \frac{\sqrt{T_s}}{2}, \; r_1 + r_2 < \frac{\sqrt{T_s}}{2}\}}$$

$$D_2 = \{(r_1, r_2) : \|r - s_2\|^2 < \|r - s_1\|^2, \; \|r - s_2\|^2 < \|r - s_3\|^2, \; \|r - s_2\|^2 < \|r - s_4\|^2\}$$

$$= \{(r_1, r_2) : \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < r_1^2 + r_2^2, \; \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2, \; \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2\}$$

$$\boxed{D_2 = \{(r_1, r_2) : r_1 + r_2 < -\frac{\sqrt{T_s}}{2}, \; r_1 < 0\}}$$

$$D_3 = \{(r_1, r_2) : \|r - s_3\|^2 < \|r - s_1\|^2, \; \|r - s_3\|^2 < \|r - s_2\|^2, \; \|r - s_3\|^2 < \|r - s_4\|^2\}$$

$$= \{(r_1, r_2) : \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < r_1^2 + r_2^2, \; \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2, \; \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2\}$$

$$\boxed{D_3 = \{(r_1, r_2) : r_1 > 0, \; r_2 < 0, \; r_1 - r_2 > \frac{\sqrt{T_s}}{2}\}}$$

$$D_4 = \{(r_1, r_2) : \|r - s_4\|^2 < \|r - s_1\|^2, \; \|r - s_4\|^2 < \|r - s_2\|^2, \; \|r - s_4\|^2 < \|r - s_3\|^2\}$$

$$= \{(r_1, r_2) : \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2 < r_1^2 + r_2^2, \; \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 + \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2, \; \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 - \frac{\sqrt{T_s}}{2}\right)^2 < \left(r_1 - \frac{\sqrt{T_s}}{2}\right)^2 + \left(r_2 + \frac{\sqrt{T_s}}{2}\right)^2\}$$

$$\boxed{D_4 = \{(r_1, r_2) : r_1 + r_2 > \frac{\sqrt{T_s}}{2}, \; r_2 > 0\}}$$

c)

Union Bound:

$$P_e \le \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{k=1 \\ k\neq m}}^{M} Q\left(\frac{\|\underline{s}_m - \underline{s}_k\|}{\sqrt{2N_0}}\right)$$

$M=4$

$$P_e \le \frac{1}{4} \sum_{m=1}^{4} \sum_{\substack{k=1 \\ k\neq m}}^{4} Q\left(\frac{\|\underline{s}_m - \underline{s}_k\|}{\sqrt{2N_0}}\right)$$

Calculate all the distances:

$$\|s_1 - s_2\| = \|s_1 - s_3\| = \|s_1 - s_4\| = \sqrt{\left(\frac{\sqrt{T_s}}{2}\right)^2 + \left(\frac{\sqrt{T_s}}{2}\right)^2} = \sqrt{\frac{T_s}{2}}$$

$$\|s_2 - s_3\| = \|s_3 - s_4\| = \sqrt{(0)^2 + \left(\frac{\sqrt{T_s}}{2} + \frac{\sqrt{T_2}}{2}\right)^2} = \sqrt{T_s}$$

$$\|s_2 - s_4\| = \sqrt{\left(-\frac{\sqrt{T_s}}{2} - \frac{\sqrt{T_s}}{2}\right)^2 + \left(-\frac{\sqrt{T_s}}{2} - \frac{\sqrt{T_s}}{2}\right)^2} = \sqrt{2T_s}$$

$$P_e \le \frac{1}{4}\Big[Q\left(\sqrt{\frac{T_s}{4N_0}}\right)3 + Q\left(\sqrt{\frac{T_s}{4N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right) + Q\left(\sqrt{\frac{T_s}{N_0}}\right)$$

$$+ Q\left(\sqrt{\frac{T_s}{4N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right)$$

$$+ Q\left(\sqrt{\frac{T_s}{4N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right)\Big]$$

$$\boxed{P_e \le \frac{3}{2} Q\left(\sqrt{\frac{T_s}{4N_0}}\right) + Q\left(\sqrt{\frac{T_s}{2N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{T_s}{N_0}}\right)}$$

→ Union Bound

Loose Union Bound:

$$d_{min} = \min_{\substack{m,k \\ m \neq k}} \| s_m - s_k \|$$

$$P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{k=1 \\ k \neq m}}^{M} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = (M-1)\, Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

$$P_e \leq (M-1)\, Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

$$d_{min} = \| s_1 - s_2 \| = \| s_1 - s_3 \| = \| s_1 - s_4 \| = \sqrt{\frac{T_s}{2}}$$

$$\boxed{P_e \leq 3Q\left(\sqrt{\frac{T_s}{4N_0}}\right)} \rightarrow$$

Loose Union Bound

d)

Exact conditional probability of symbol error:

$$P_{e,m} = P(A_{m,1} \cup A_{m,2} \cup \dots \cup A_{m,m-1} \cup A_{m,m+1} \dots \cup A_{m,M}\}$$

where $A_{m,k} = \{ \| \underline{r} - \underline{s_m} \| > \| \underline{r} - \underline{s_k} \| \}$, given $\underline{s_m}$ is sent.

Given s_1 is sent:

$$P_{e,s_1} = P(A_{s_1,s_2} \cup A_{s_1,s_3} \cup A_{s_1,s_4})$$

$$P_{e,s_1} = 1 - P_{c,s_1}$$

$$P_{c,s_1} = P(A_{s_1,s_1})$$

$$P_{e,s_1} = 1 - P\left(r_1 + r_2 > -\frac{\sqrt{T_s}}{2} \cup r_1 - r_2 < \frac{\sqrt{T_s}}{2} \cup r_1 + r_2 < \frac{\sqrt{T_s}}{2}\right)$$

$$= 1 - P\left(n_1 + n_2 > -\frac{\sqrt{T_s}}{2} \cup n_1 - n_2 < \frac{\sqrt{T_s}}{2} \cup n_1 + n_2 < \frac{\sqrt{T_s}}{2}\right)$$

$$= 1 - P\left(\frac{n_1 + n_2}{\sqrt{N_0}} > -\sqrt{\frac{T_s}{4N_0}}\right) P\left(\frac{n_1 - n_2}{\sqrt{N_0}} < \sqrt{\frac{T_s}{4N_0}}\right) P\left(\frac{n_1 + n_2}{\sqrt{N_0}} < \sqrt{\frac{T_s}{4N_0}}\right)$$

$$= 1 - Q\left(-\sqrt{\frac{T_s}{4N_0}}\right)\left(1 - Q\left(\sqrt{\frac{T_s}{4N_0}}\right)\right)\left(1 - Q\left(\sqrt{\frac{T_s}{4N_0}}\right)\right)$$

$$\boxed{P_{e,s_1} = 1 - \left(1 - Q\left(\sqrt{\frac{T_s}{4N_0}}\right)\right)^3}$$

The loose union bound provides a simple upper estimate of the error probability but tends to significantly overestimate it by assuming all error events are equally likely and independent. A more refined version of the union bound offers a tighter estimate by accounting for different distances and weights of error events, though it still remains an upper bound. In contrast, the exact expression captures the true probability of error for a given symbol by considering the actual structure of the constellation and the dependencies between error events, making it more accurate than either union bound.