EEE 473 Medical Imaging
Homework #2

Boran Kılıç
22103444

Q1)

a) $f(r) = \dfrac{\delta(ar)}{r}$ find $\mathcal{H}\{f(r)\} = F(q)$

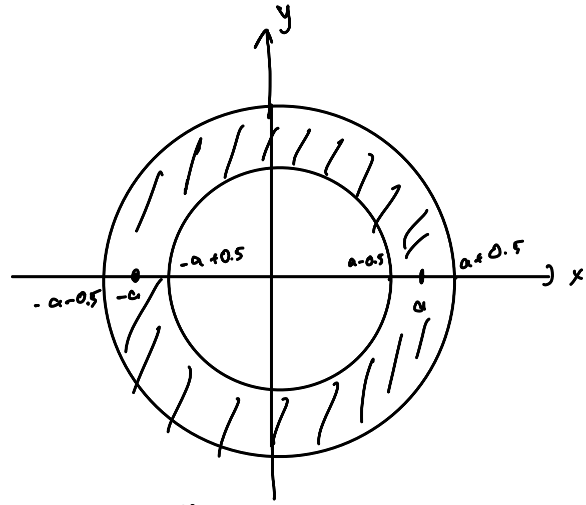
$$\delta(ar) = \delta(ax, ay)\, \pi a\sqrt{x^2+y^2} = \frac{1}{a^{\cancel{2}}}\, \delta(x,y)\, \pi \cancel{a} \sqrt{x^2+y^2} = \frac{1}{a}\, \underbrace{\delta(x,y)\, \pi\sqrt{x^2+y^2}}_{\delta(r)}$$

$$\Rightarrow \delta(ar) = \frac{1}{a}\,\delta(r)$$

$$f(r) = \frac{1}{a}\,\frac{\delta(r)}{r} = \frac{\pi}{a}\,\underbrace{\frac{\delta(r)}{\pi r}}_{\delta(x,y)} \Rightarrow \mathcal{H}\left\{\frac{\pi}{a}\,\frac{\delta(r)}{\pi r}\right\} = \mathcal{F}_{2D}\left\{\frac{\pi}{a}\,\delta(x,y)\right\} = \frac{\pi}{a} \,//$$

b) $f(r) = \text{rect}(r-a)$, where $a > 1$ find $\mathcal{H}\{f(r)\} = F(q)$

$$f(r) = \text{rect}(r-a) = \begin{cases} 1 & \text{if } |r-a| < 0.5 \\ 0 & \text{ow.} \end{cases}$$



"Annulus"

we have:

$$f(r) = \text{rect}(r-a) = \text{rect}\left(\frac{r}{2(a+0.5)}\right) - \text{rect}\left(\frac{r}{2(a-0.5)}\right)$$

$$= \text{rect}\left(\frac{r}{2a+1}\right) - \text{rect}\left(\frac{r}{2a-1}\right)$$

Using the linearity of Hankel transform:

$$F(q) = \mathcal{H}\{f(r)\} = \mathcal{H}\left\{\text{rect}\left(\frac{r}{2a+1}\right)\right\} - \mathcal{H}\left\{\text{rect}\left(\frac{r}{2a-1}\right)\right\}$$

$$F(q) = (2a+1)^2\, \text{jinc}\left((2a+1)q\right) - (2a-1)^2\, \text{jinc}\left((2a-1)q\right) \,//$$

where $\text{jinc}(q) = \dfrac{J_1(\pi q)}{2q}$

c) $f(r) = \delta(r-3)$

$$f(x,y) = \overbrace{\delta\left(\sqrt{(x-1)^2+(y-2)^2}-3\right)}^{\text{not circularly symmetric}} \xrightarrow{\mathcal{F}_{2D}} \mathcal{F}_{2D}\left\{\underbrace{\delta\left(\sqrt{x^2+y^2}-3\right)}_{\text{circularly symmetric}}\right\}(u,v) \; \underbrace{e^{-j2\pi(u+2v)}}_{\text{shifting property}}$$

So,

$$\mathcal{F}_{2D}\left\{\delta\left(\sqrt{x^2+y^2}-3\right)\right\} = \mathcal{H}\left\{\delta(r-3)\right\}(q) = 6\pi \, J_0(6\pi q) \quad \& \quad q=\sqrt{u^2+v^2}$$

Hence the answer becomes

$$\mathcal{F}_{2D}\left\{\delta(r-3)\right\}(u,v) = 6\pi \, J_0\left(6\pi\sqrt{u^2+v^2}\right) e^{-2\pi(u+2v)} \quad //$$

Q2)

a) $$\mathrm{MTF}(u,v) = \frac{|H(u,v)|}{H(0,0)} \quad \text{where} \quad H(u,v) = \mathcal{F}_{2D}\{h(x,y)\}(u,v)$$

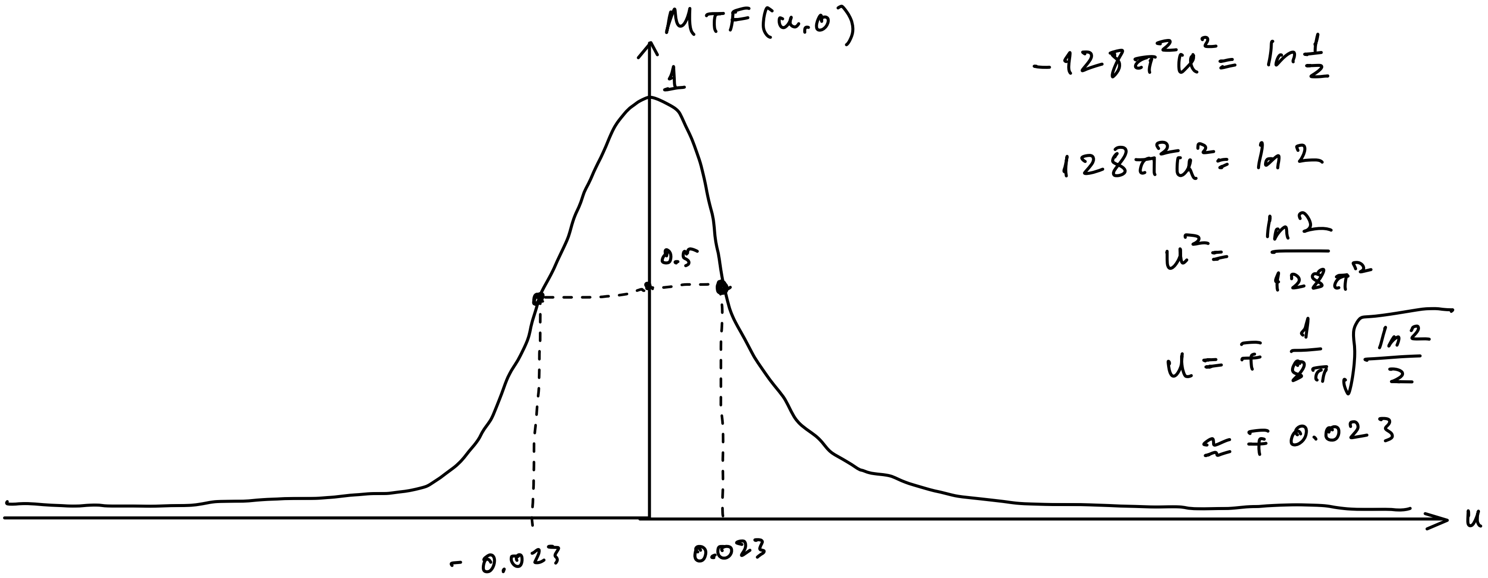
$$\mathcal{F}_{2D}\{h(x,y)\}(u,v) = \mathcal{F}_{2D}\left\{e^{-\frac{x^2+8y^2}{2\sigma^2}}\right\}(u,v) \quad \text{where} \quad \sigma=8$$

$$H(u,v) = \mathcal{F}_{2D}\left\{e^{-\pi\left(\frac{x^2}{2\pi\sigma^2}+\frac{y^2}{(2\pi\sigma^2/8)}\right)}\right\}(u,v)\Bigg|_{\sigma=8} = \frac{(2\pi\sigma^2)^2}{8} e^{-\pi\left(2\pi\sigma^2u^2+\frac{2\pi\sigma^2}{8}v^2\right)}\Bigg|_{\sigma=8}$$

$$H(0,0) = \frac{(2\pi\sigma^2)^2}{8}\Bigg|_{\sigma=8}$$

$$\mathrm{MTF}(u,v) = e^{-\pi\left(2\pi\sigma^2u^2+\frac{2\pi\sigma^2}{8}v^2\right)}\Bigg|_{\sigma=8} = e^{-128\pi^2u^2-16\pi^2v^2} \quad //$$

b) $MTF(u,0) = e^{-128\pi^2 u^2}$



$$e^{-128\pi^2 u^2} = \frac{1}{2}$$

$$-128\pi^2 u^2 = \ln\frac{1}{2}$$

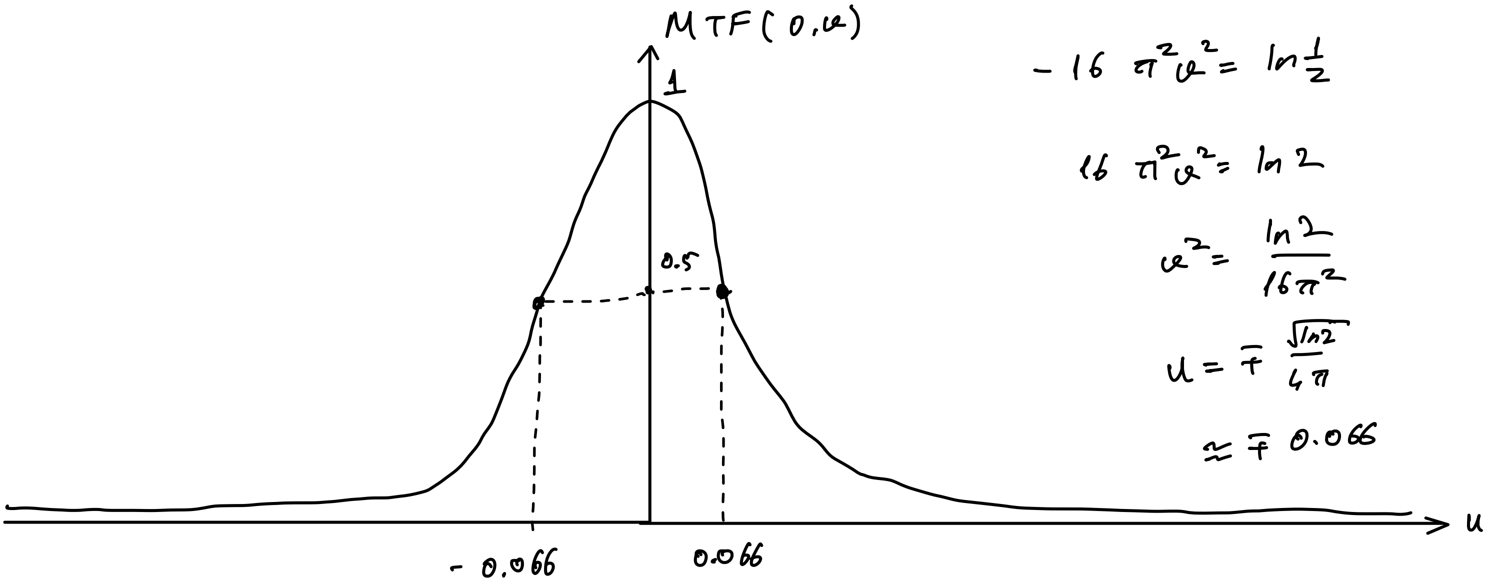
$$128\pi^2 u^2 = \ln 2$$

$$u^2 = \frac{\ln 2}{128\pi^2}$$

$$u = \mp \frac{1}{8\pi}\sqrt{\frac{\ln 2}{2}}$$

$$\simeq \mp 0.023$$

$MTF(0,v) = e^{-16\pi^2 v^2}$



$$e^{-16\pi^2 v^2} = \frac{1}{2}$$

$$-16\pi^2 v^2 = \ln\frac{1}{2}$$

$$16\pi^2 v^2 = \ln 2$$

$$v^2 = \frac{\ln 2}{16\pi^2}$$

$$u = \mp \frac{\sqrt{\ln 2}}{4\pi}$$

$$\simeq \mp 0.066$$

c) $f(x,y) = A + B\sin(2\pi u_0 x) = 4 + 3\sin\left(\frac{2\pi x}{20}\right) \Rightarrow$ $f_{max} = 4+3 = 7$

$f_{min} = 4-3 = 1$

$$m_f = \frac{f_{max} - f_{min}}{f_{max} + f_{min}} = \frac{7-1}{7+1} = \frac{6}{8} = \frac{3}{4} = 0.75 \quad //$$

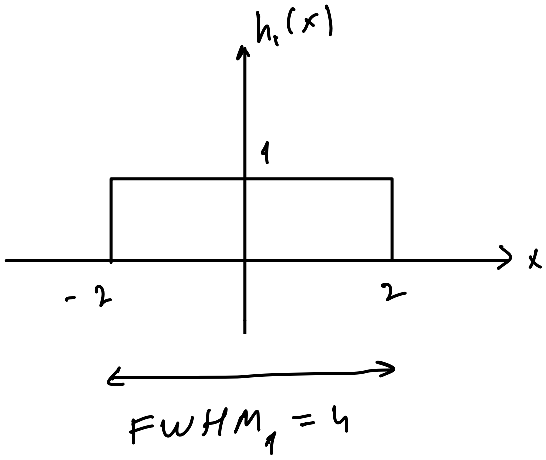
$$m_g = m_f \left.\frac{|H(u_0,0)|}{H(0,0)}\right|_{u_0=\frac{1}{20}} = m_f \, MTF(u_0, 0)\Big|_{u_0=\frac{1}{20}}$$

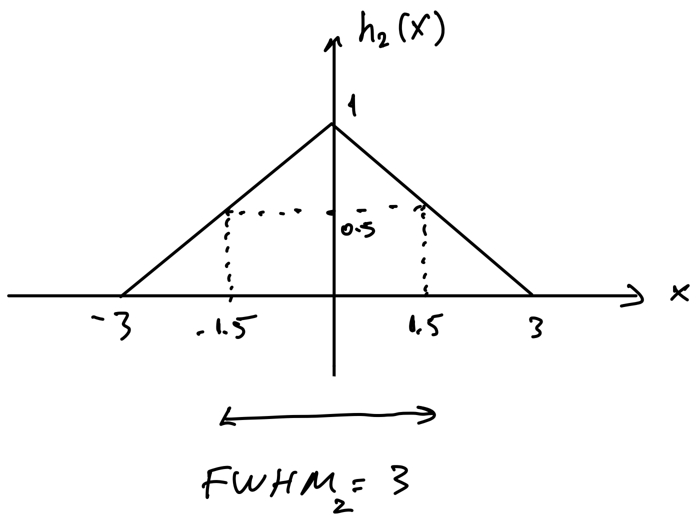
$$m_g = 0.75 \, e^{-128\pi^2\left(\frac{1}{20}\right)^2} \approx 0.032 \quad //$$

d) $f(x,y) = 4 + 3\sin\left(\frac{2\pi(x+y)}{20}\right) = A + B\sin(2\pi(u_0 x + v_0 y))$ $\quad m_f = \frac{B}{A} = \frac{3}{4} = 0.75$

$$m_g = m_f \, MTF(u_0, v_0) = 0.75 \, e^{-128\pi^2\left(\frac{1}{20}\right)^2 - 16\pi^2\left(\frac{1}{20}\right)^2} \approx 0.0215 \quad //$$

Q3) $h_1(x) = rect\left(\frac{x}{4}\right)$ $\quad h_2(x) = tri\left(\frac{x}{3}\right)$





$FWHM_2 < FWHM_1$ which means $h_2(x)$ have better spatial resolution.

$3 < 4$

FWHM describes the width of the system's PSF at half of its maximum value.

PSF represents how a single point source appears in the image.

Narrower this spread, sharper the image meaning high resolution.

b) $FWHM = \sqrt{FWHM_1^2 + FWHM_2^2} = \sqrt{4^2+3^2} = 5$ //

Q.4)

		Disease +	Disease −
test	+	$a=?$	$b=x$
test	−	$c=0$	$d=19x$

a)

the chance for person that found to have a positive result actually has the disease : Positive Predictive Value $= \frac{a}{a+b}$

b) "Assuming a perfectly sensitive test" stated in the article therefore,

$$\text{sensitivity} = \frac{a}{a+c} = \%100$$ which means $c=0$

$$\text{false positive rate}: \frac{b}{b+d} = 5\% = \frac{1}{20}$$

Let $b=x$

$$\frac{b}{b+d} = \frac{x}{x+d} = \frac{1}{20} \Rightarrow d = 19x$$

$$\text{prevalence} = \frac{a+c}{a+b+c+d} = \frac{1}{1000} = \frac{a}{a+20x} = \frac{1}{1000} \Rightarrow 999a = 20x$$

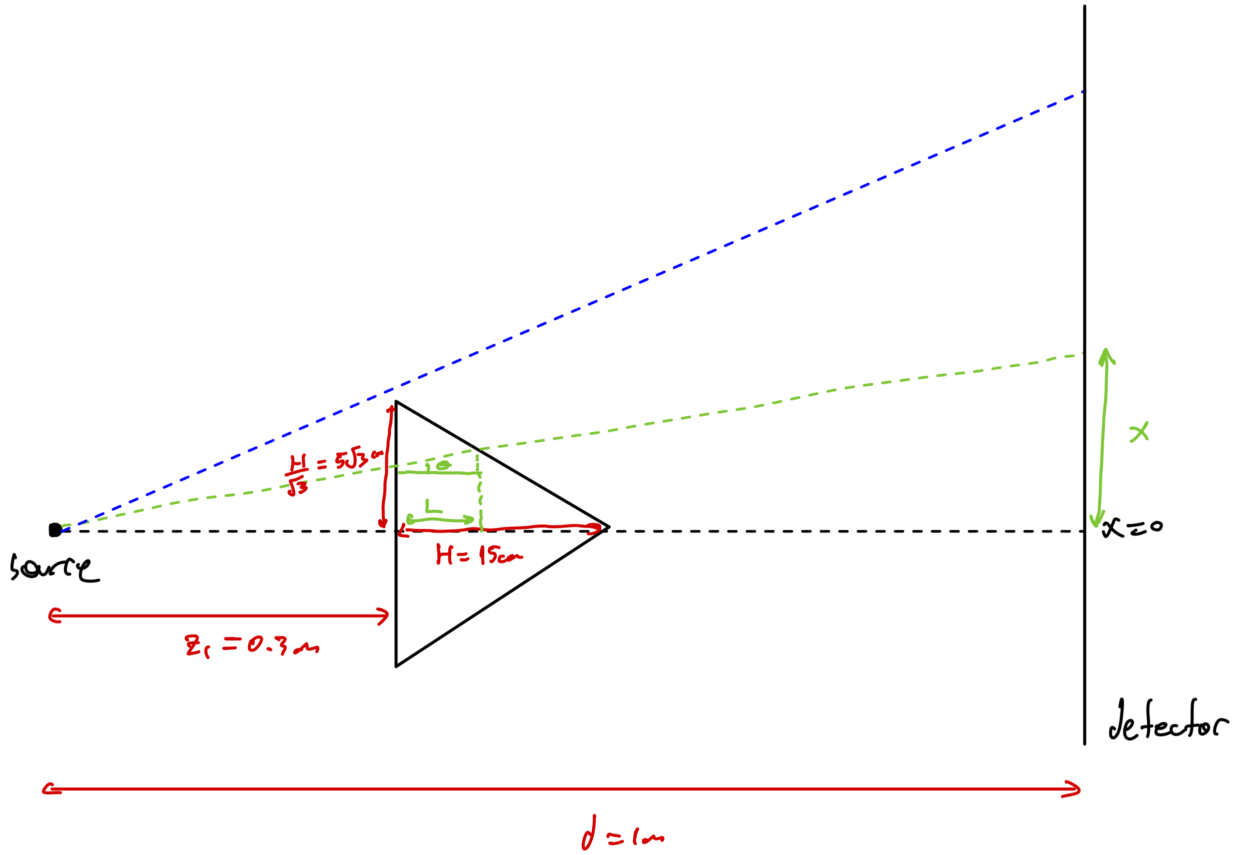
$$a = \frac{20x}{999}$$

$$PPV = \frac{a}{a+b} = \frac{\frac{20x}{999}}{\frac{20x}{999}+x} = \frac{20x}{1019x} = \frac{20}{1019} \approx 1.96\ \%$$ //

c) perfectly sensitive test assumption rules out the possible false negatives of the test. This means, by this assumption the performance of the test is assumed to be higher than actual performance. Therefore the assumption places an upper bound to the performance of the test.

Q 5)

a)



There are 2 possibilities, ray misses the object, ray enters from front, exits from the side

1) ray misses the object:

$$I_d(x) = I_0 \underbrace{\cos^3\theta}_{\text{inverse square law \& obliquity}} \qquad \text{where} \qquad \cos\theta = \frac{d}{\sqrt{d^2+x^2}}$$

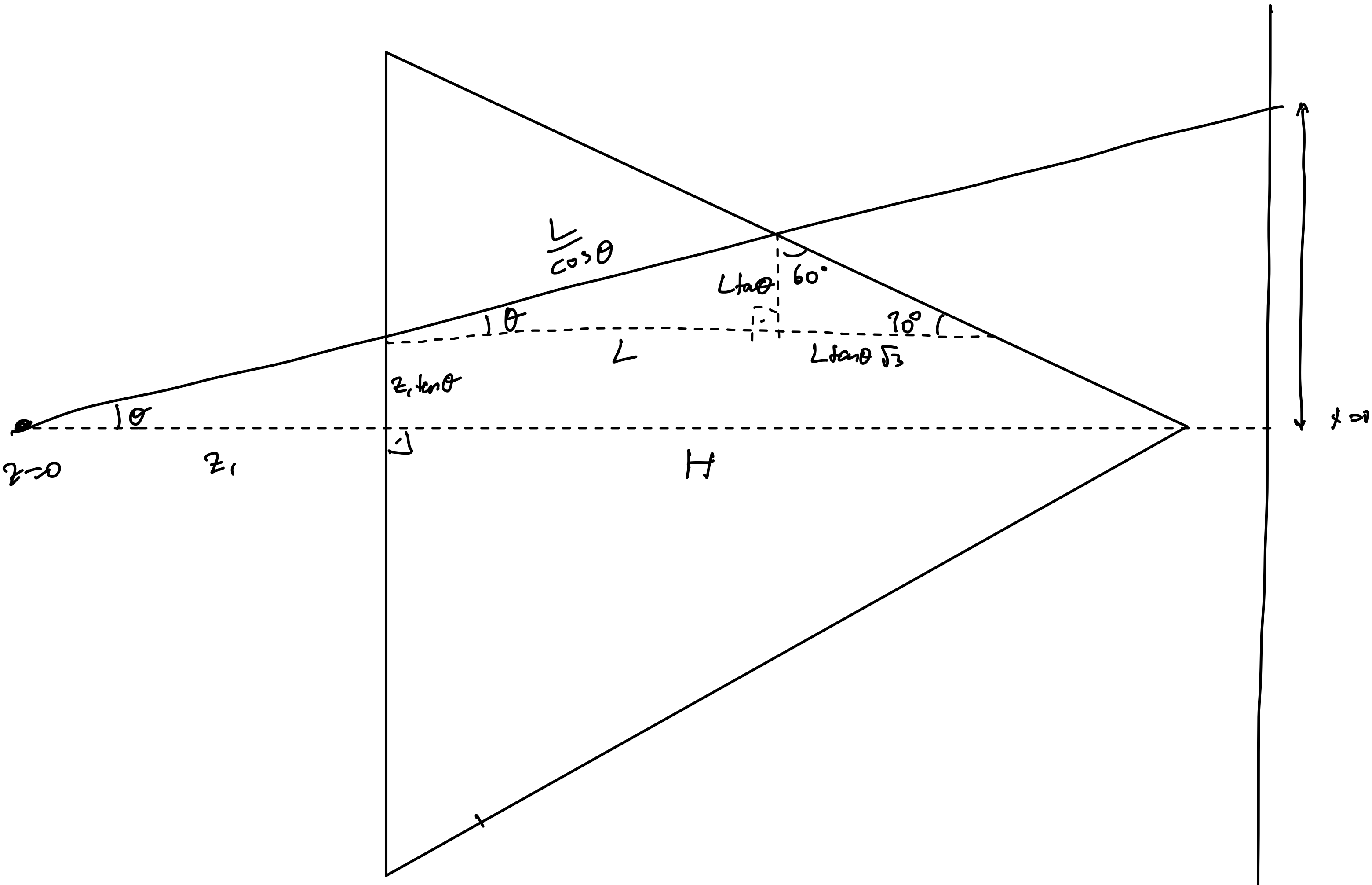
lower bound for $x = \dfrac{H/\sqrt{3}}{z_1} d$

$$I_d(x) = I_0 \cos^3\left(\frac{d}{\sqrt{d^2+x^2}}\right) \qquad \text{when} \qquad x > \frac{H}{\sqrt{3}\, z_1} d$$

2) ray goes through the object

$$I_d(x) = I_0 \cos^3\theta \ \exp\left\{-\mu_0 \frac{L}{\cos\theta}\right\} \qquad \text{if} \qquad |x| < \frac{H d}{\sqrt{3}\, z_1}$$

$$\cos\theta = \frac{d}{\sqrt{d^2+x^2}}$$



$$\frac{\frac{H}{\sqrt{3}} - z_1\tan\theta}{\frac{H}{\sqrt{3}}} = \frac{L + L\tan\theta\sqrt{3}}{H} \qquad \tan\theta = \frac{x}{d}$$

$$\frac{H - z_1\sqrt{3}\tan\theta}{\cancel{H}} = \frac{L + L\tan\theta\sqrt{3}}{\cancel{H}} \implies L = \frac{H - z_1\sqrt{3}\tan\theta}{1+\tan\theta\sqrt{3}}$$

$$I_d(x) = \begin{cases} \dfrac{I_s}{4\pi d^2}\dfrac{d^3}{(x^2+d^2)^{3/2}} & \text{for } |x| \geq \dfrac{Hd}{z_1\sqrt{3}} \\[2ex] \dfrac{I_s}{4\pi d^2}\dfrac{d^3}{(x^2+d^2)^{3/2}} \exp\left\{-\mu \dfrac{H - z_1\sqrt{3}\frac{|x|}{d}}{1+\sqrt{3}\frac{|x|}{d}} \cdot \dfrac{\sqrt{x^2+d^2}}{d}\right\} & \text{for } |x| < \dfrac{Hd}{z_1\sqrt{3}} \end{cases}$$

substituting in the values:

$$I_d(x) = \begin{cases} \dfrac{I_s}{4\pi} \dfrac{1}{(x^2+1)^{3/2}} \exp\left\{-20 \dfrac{0.15 - 0.3 \cdot \sqrt{3}\,|x|}{1 + \sqrt{3}\,|x|} \cdot \sqrt{x^2+1}\right\} & \text{for } |x| < \dfrac{1}{2\sqrt{3}} \\ \dfrac{I_s}{4\pi} \dfrac{1}{(x^2+1)^{3/2}} & \text{for } |x| \geq \dfrac{1}{2\sqrt{3}} \end{cases}$$

b)
Plot $I_d(x)$:

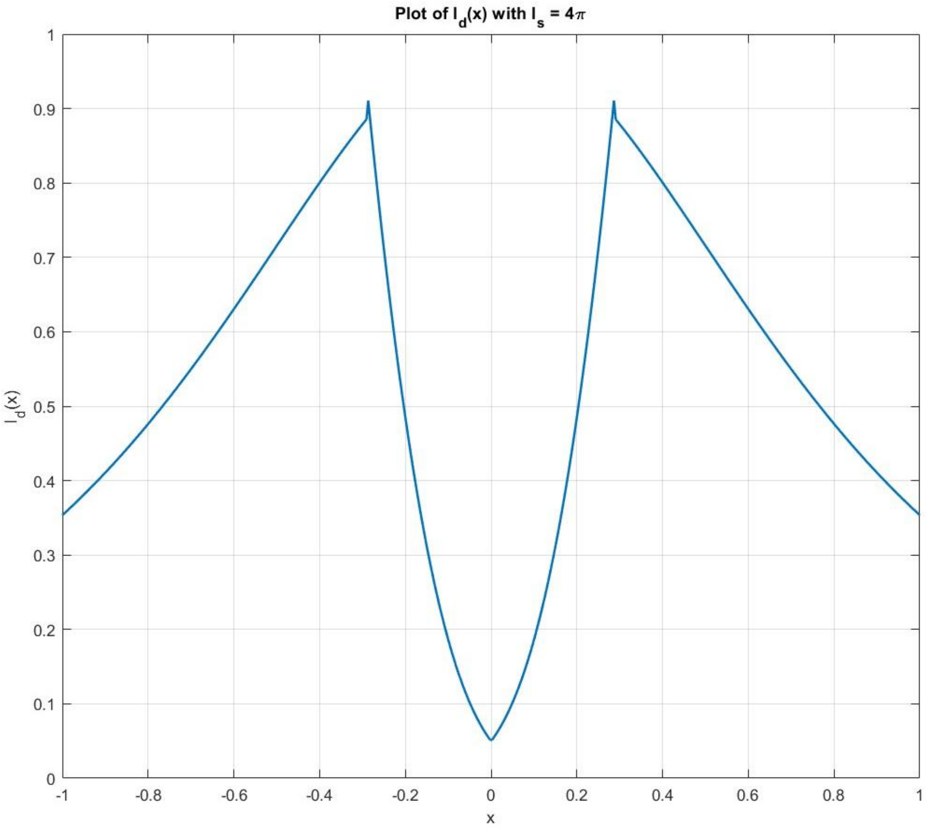


Fig.1 $I_d(x)$ plot with I_s assumed to be 4π

Q6)
a)

Fig.2 MPI image

b)&c)

```
Command Window
FWHM of Tracer A :    6.48 mm
SNR of Tracer A :     45.14

FWHM of Tracer X :    4.32 mm
SNR of Tracer X :     37.73

FWHM of Tracer B :    12.60 mm
SNR of Tracer B :     24.06
```

Fig.3 FWHM and SNR values

d)

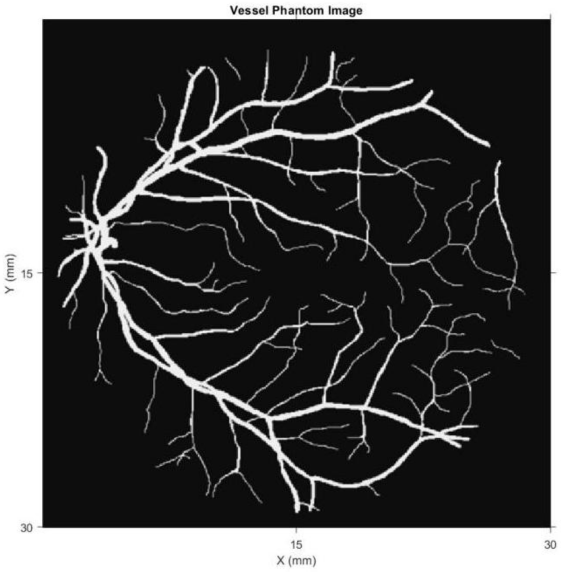


Fig.4 Vessel Phantom Image

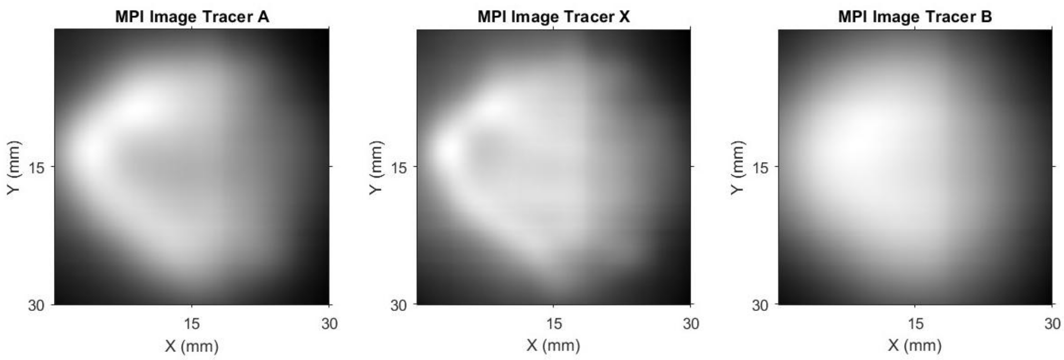


Fig.5 Resulting Images

A tracer with a smaller FWHM will produce a sharper image (less spread in the PSF), leading to better spatial resolution. Conversely, a larger FWHM will result in a more blurred image, decreasing spatial resolution. Indeed, this fact was observed in this question. It can be seen from *Fig.4* that resulting image of Tracer B is the most blurry image and it is consistent with the fact in *Fig.2* that Tracer B has the highest FWHM. On the other hand, Tracer X has the smallest FWHM value therefore the least blurry image.

Appendix - MATLAB codes

hw2q5.m

```
Is = 4*pi;
x = linspace(-1, 1, 500);
Id = zeros(size(x));

for i = 1:length(x)
if abs(x(i)) < 1 / (2 * sqrt(3))
Id(i) = (Is / (4 * pi)) * (1 / (x(i)^2 + 1)^(3/2)) * ...
exp(-20 * (0.15 - 0.3 * sqrt(3) * abs(x(i))) / ...
(1 + sqrt(3) * abs(x(i)))) * sqrt(x(i)^2 + 1);
else
Id(i) = (Is / (4 * pi)) * (1 / (x(i)^2 + 1)^(3/2));
end
end

plot(x, Id, 'LineWidth', 1.5);
xlabel('x');
ylabel('I_d(x)');
title('Plot of I_d(x) with I_s = 4\pi');
grid on;
```

hw2q6.m

```
clear
close all
clc
MPI_data = load("MPI_data.mat");
MPI_image = MPI_data.MPI_image;
figure;
imshow(MPI_image,[]);
axis on;

xTicks = linspace(0, size(MPI_image, 2), 7);
yTicks = linspace(0, size(MPI_image, 1), 3);

xLabels = linspace(0, 90, numel(xTicks));
yLabels = linspace(0, 30, numel(yTicks));

set(gca, 'XTick', xTicks, 'XTickLabel', xLabels);
set(gca, 'YTick', yTicks, 'YTickLabel', yLabels);

xlabel('X (mm)');
ylabel('Y (mm)');
title('MPI Images left to right: Tracer A, Trace X, Tracer B');

TracerA = MPI_image(:, 1:500);
TracerX = MPI_image(:, 501:1000);
TracerB = MPI_image(:, 1001:1500);

noise = MPI_image(1:50,1:50);
rho_N = mean(std(noise));
mu_A = max(TracerA(:));
mu_X = max(TracerX(:));
mu_B = max(TracerB(:));

FWHM_A = findFWHM(TracerA);
fprintf("FWHM of Tracer A :\t %.2f mm \n",FWHM_A);
```

```
SNR_A = mu_A/rho_N;
fprintf("SNR of Tracer A :\t %.2f \n",SNR_A);
fprintf("\n");

FWHM_X = findFWHM(TracerX);
fprintf("FWHM of Tracer X :\t %.2f mm \n",FWHM_X);
SNR_X = mu_X/rho_N;
fprintf("SNR of Tracer X :\t %.2f \n",SNR_X);
fprintf("\n");

FWHM_B = findFWHM(TracerB);
fprintf("FWHM of Tracer B :\t %.2f mm \n",FWHM_B);
SNR_B = mu_B/rho_N;
fprintf("SNR of Tracer B :\t %.2f \n",SNR_B);


vessel_phantom = load("vessel_phantom.mat");
vessel_phantom = vessel_phantom.vessel_phantom;

figure;imshow(vessel_phantom,[]);
axis on;
xTicks = linspace(0, size(vessel_phantom, 2), 3);
yTicks = linspace(0, size(vessel_phantom, 1), 3);
xLabels = linspace(0, 30, numel(xTicks));
yLabels = linspace(0, 30, numel(yTicks));
set(gca, 'XTick', xTicks, 'XTickLabel', xLabels);
set(gca, 'YTick', yTicks, 'YTickLabel', yLabels);
xlabel('X (mm)');
ylabel('Y (mm)');
title('Vessel Phantom Image');

resultA = conv2(vessel_phantom, TracerA, 'same');
resultX = conv2(vessel_phantom, TracerX, 'same');
resultB = conv2(vessel_phantom, TracerB, 'same');

figure;

subplot(1,3,1);imshow(resultA,[]);
title('MPI Image Tracer A');
axis on;
xTicks = linspace(0, size(resultA, 2), 3);
yTicks = linspace(0, size(resultA, 1), 3);
xLabels = linspace(0, 30, numel(xTicks));
yLabels = linspace(0, 30, numel(yTicks));
set(gca, 'XTick', xTicks, 'XTickLabel', xLabels);
set(gca, 'YTick', yTicks, 'YTickLabel', yLabels);
xlabel('X (mm)');
ylabel('Y (mm)');

subplot(1,3,2);imshow(resultX,[]);
title('MPI Image Tracer X');
axis on;
set(gca, 'XTick', xTicks, 'XTickLabel', xLabels);
set(gca, 'YTick', yTicks, 'YTickLabel', yLabels);
xlabel('X (mm)');ylabel('Y (mm)');

subplot(1,3,3);imshow(resultB,[]);
title('MPI Image Tracer B');
axis on;
```

```
set(gca, 'XTick', xTicks, 'XTickLabel', xLabels);
set(gca, 'YTick', yTicks, 'YTickLabel', yLabels);
xlabel('X (mm)');ylabel('Y (mm)');


function FWHM = findFWHM(A)
[max_value, linear_index] = max(A(:));
[~, col] = ind2sub(size(A), linear_index);

target_value = max_value/2;
tolerance = 0.015;

column_data = A(:, col);
indices = find(abs(column_data - target_value) <= tolerance);

if ~isempty(indices)
FWHM = abs((col - indices(1))*2*(30/500)) ;
else
fprintf('No values close to %.2f were found \n', target_value);
end
end
```