EEE-321: Signals and Systems

Section-1

LAB Assignment 1

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Part 1

MATLAB code of function SUMCS:

```
function [xs] = SUMCS(t,A,omega)
    xs = zeros(size(t));
    for i = 1 : length(A)
        xs = xs + A(i)*exp(1j*omega(i)*t);
    end
end
```

Figure1: MATLAB code for SUMCS

MATLAB code for calculating signal $x_S(t)$:

```
t = 0:0.001:1 ;
n = mod(22103444,41);
A = (3*rand(1,n)) + (3*rand(1,n)*1j);
omega = pi*rand(1,n);
xs = SUMCS(t,A,omega);
real_xs = real(xs);
imag_xs = imag(xs);
mgn_xs = abs(xs);
phase_xs = angle(xs);

%plot
figure
tiledlayout(2,2)

nexttile
plot(t,real_xs)
title("Real part of xs")
ylabel("Re[xs]")
xlabel("t")

nexttile
plot(t,imag_xs)
title("Imaginer part of xs")
ylabel("Im[xs]")
xlabel("t")

nexttile
plot(t,mgn_xs)
title("Magnitude of xs")
ylabel("Magnitude")
xlabel("t")

nexttile
plot(t,phase_xs)
title("Phase of xs")
ylabel("Phase")
xlabel("t")
```

Figure2: MATLAB code for $x_S(t)$

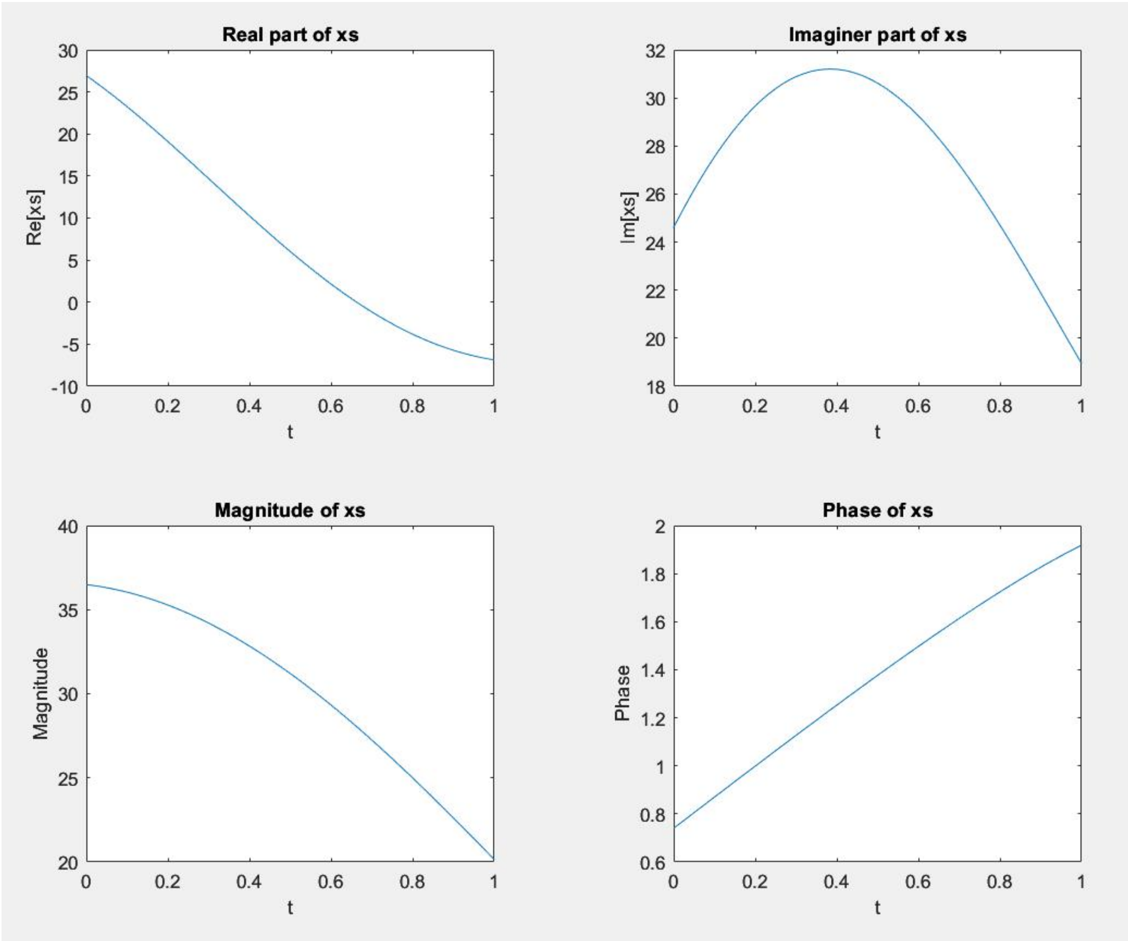
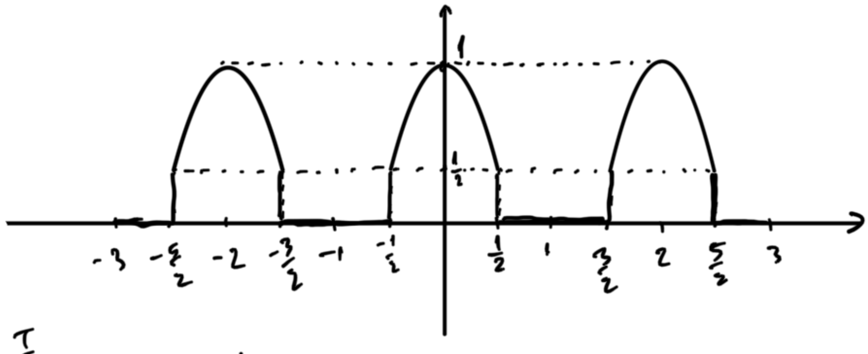


Figure3: Real part, imaginary part, amplitude and phase versus time plots

Part 2

$$x(t) = \begin{cases} 1-2t^2 & \text{if } -\frac{w}{2} < t < \frac{w}{2} \\ 0 & \text{otherwise,} \end{cases} \qquad w < T$$

$T=2$, $w=1$, the graph of $x(t)$ over $-1.5T < t < 1.5T$:



$$X_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\frac{2\pi k t}{T}} dt$$

for $-\frac{T}{2} < t < -\frac{w}{2}$ and $\frac{w}{2} < t < \frac{T}{2}$ $x(t)=0$

so the integration becomes:

$$T X_k = \int_{-\frac{w}{2}}^{\frac{w}{2}} x(t) e^{-j\frac{2\pi k t}{T}} dt = \int_{-\frac{w}{2}}^{\frac{w}{2}} (1-2t^2) e^{-j\frac{2\pi k}{T}t} = \int_{-\frac{w}{2}}^{\frac{w}{2}} e^{-j\frac{2\pi k t}{T}} dt - 2 \underbrace{\int_{-\frac{w}{2}}^{\frac{w}{2}} t^2 e^{-j\frac{2\pi k t}{T}} dt}_{I}$$

to calculate the integral I, using Integration by Parts:

$$\boxed{\begin{aligned} &\int u\,dv = uv - \int v\,du \\ &u = t^2,\ dv = e^{-j\frac{2\pi k}{T}t} dt \\ &du = 2t\,dt,\ v = \frac{e^{-j\frac{2\pi k}{T}t}}{-j\frac{2\pi k}{T}} \end{aligned}}$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} t^2 e^{-j\frac{2\pi k t}{T}} dt = -t^2 \frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}} + 2 \underbrace{\int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} t\,dt}_{J}$$

to calculate the integral J we again use IBP where $dv = \frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} dt$, $u=t$, $du=dt$

$$v = \frac{e^{-j\frac{2\pi k}{T}t}}{\left(\frac{2\pi k}{T}\right)^2}$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} t\,dt = \frac{e^{-j\frac{2\pi k}{T}t}}{\left(\frac{2\pi k}{T}\right)^2} t \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}} - \int \frac{e^{-j\frac{2\pi k}{T}t}}{\left(\frac{2\pi k}{T}\right)^2} dt$$

$$J = \frac{e^{-j\frac{2\pi k}{T}t}}{\left(\frac{2\pi k}{T}\right)^2} t \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}} + \frac{e^{-j\frac{2\pi k}{T}t}}{j\left(\frac{2\pi k}{T}\right)^3} \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}}$$

$$TX_k = -\frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} - 2\left[t^2 \frac{e^{-j\frac{2\pi k}{T}t}}{j\frac{2\pi k}{T}} + 2\left(\frac{e^{-j\frac{2\pi k}{T}t}}{\left(\frac{2\pi k}{T}\right)^2} t + \frac{e^{-j\frac{2\pi k}{T}t}}{j\left(\frac{2\pi k}{T}\right)^3} \right)\right] \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}}$$

$w_0 = \frac{2\pi}{T} \rightarrow$ fundamental angular frequency

$$X_k = \frac{1}{T}\left[e^{-jw_0kt}\left[\frac{2t^2-1}{jw_0k} - \frac{4t}{(w_0k)^2} - \frac{4}{j(w_0k)^3} \right]\right] \Bigg|_{-\frac{w}{2}}^{\frac{w}{2}}$$

$$X_k = \frac{1}{T}\left[\frac{\frac{w^2}{2}-1}{jw_0k}\left(e^{-jw_0k\frac{w}{2}} - e^{jw_0k\frac{w}{2}} \right) + \frac{4\frac{w}{2}}{(w_0k)^2}\left(-e^{-jw_0k\frac{w}{2}} - e^{jw_0k\frac{w}{2}} \right) + \frac{4}{j(w_0k)^3}\left(e^{jw_0k\frac{w}{2}} - e^{-jw_0k\frac{w}{2}} \right)\right]$$

$$= \frac{1}{T}\left[\frac{w^2-2}{w_0k}\left(\frac{e^{-jw_0k\frac{w}{2}} - e^{jw_0k\frac{w}{2}}}{2j} \right) + \frac{4w}{(w_0k)^2}\left(\frac{e^{-jw_0k\frac{w}{2}} + e^{jw_0k\frac{w}{2}}}{2} \right) + \frac{8}{(w_0k)^3}\left(\frac{e^{jw_0k\frac{w}{2}} - e^{-jw_0k\frac{w}{2}}}{2j} \right)\right]$$

$$= \frac{1}{T}\left[\left(\frac{2-w^2}{w_0k} + \frac{8}{(w_0k)^3} \right)\left(\frac{e^{jw_0k\frac{w}{2}} - e^{-jw_0k\frac{w}{2}}}{2j} \right) + \frac{4w}{(w_0k)^2}\left(\frac{e^{-jw_0k\frac{w}{2}} + e^{jw_0k\frac{w}{2}}}{2} \right)\right]$$

$$= \frac{1}{T}\left[\left(\frac{2-w^2}{w_0k} + \frac{8}{(w_0k)^3} \right)\sin\left(w_0k\frac{w}{2}\right) + \frac{4w}{(w_0k)^2}\cos\left(w_0k\frac{w}{2}\right)\right]$$

substituting $w_0 = \frac{2\pi}{T}$

$$\boxed{X_k = \left(\frac{2-w^2}{2\pi k} + \frac{T^2}{\pi^3 k^3} \right)\sin\left(\frac{\pi w}{T}k\right) - \frac{wT}{\pi^2 k^2}\cos\left(\frac{\pi w}{T}k\right)}$$

for k=0 above equation is undefined. Therefore calculating X_0 from the original integral.

$$X_0 = \frac{1}{T}\int_{-\frac{W}{2}}^{\frac{W}{2}} (1-2t^2)\,dt = \frac{1}{T}\left(t-\frac{2}{3}t^3\Big|_{-\frac{W}{2}}^{\frac{W}{2}}\right) = \frac{1}{T}\left(\frac{W}{2}-\frac{2}{3}\frac{W^3}{8}-\left(-\frac{W}{2}+\frac{2}{3}\frac{W^3}{8}\right)\right)$$

$$= \frac{1}{T}\left(W-\frac{W^3}{6}\right) = \boxed{\frac{6W-W^3}{6T}}$$

Part 3

Using the results from Part 2, MATLAB code for FSWave is as follows:

```
function [xt] = FSWave(t,K,T,W)
    Xk = zeros(1,2*K+1);
    omega = zeros(1,2*K+1);
    for k = -K:K
        if k == 0
            Xk(K+1) =(1/T)*(W-W^3/6); %X0
        else
            Xk(k+K+1) = (((2-W^2)/(2*pi*k) + (T^2)/(pi^3*k^3))*sin((pi*W*k)/T) ...
                - ((W*T)/(pi^2*k^2))*cos((pi*W*k)/T));
        end
    omega(k+K+1) = 2*pi*k/T;
    end
    xt = SUMCS(t,Xk,omega);
end
```

Figure4: MATLAB code for FSWave

After the function code has written next step was to calculate the $\tilde{x}(t)$ where D_{11} = mod(22103444, 11), D_5 = mod(22103444, 5), T=2, W=1, K=20+D_{11} and t=[-5:0.001:5]. The MATLAB code is below:

```
D11 = mod(22103444,11);
D5 = mod(22103444,5);
K = 20 + D11;
T = 2;
W = 1;
t= -5:0.001:5;
xt = FSWave(t,K,T,W);
k = -20:20;
real_xt = real(xt);
imag_xt = imag(xt);

figure
tiledlayout(1,2)

nexttile
plot(t,real_xt)
title("Real part of xt")
ylabel("Re[xt]")
xlabel("t")

nexttile
plot(t,imag_xt)
title("Imaginary part of xt")
ylabel("Im[xt]")
xlabel("t")

max_real = max(real_xt);
min_real = min(real_xt);
max_imag = max(imag_xt);
min_imag = min(imag_xt);
```

Figure5: MATLAB code for $\tilde{x}(t)$

The plots of real and imaginary parts of $\tilde{x}(t)$ are in *Figure6* below:

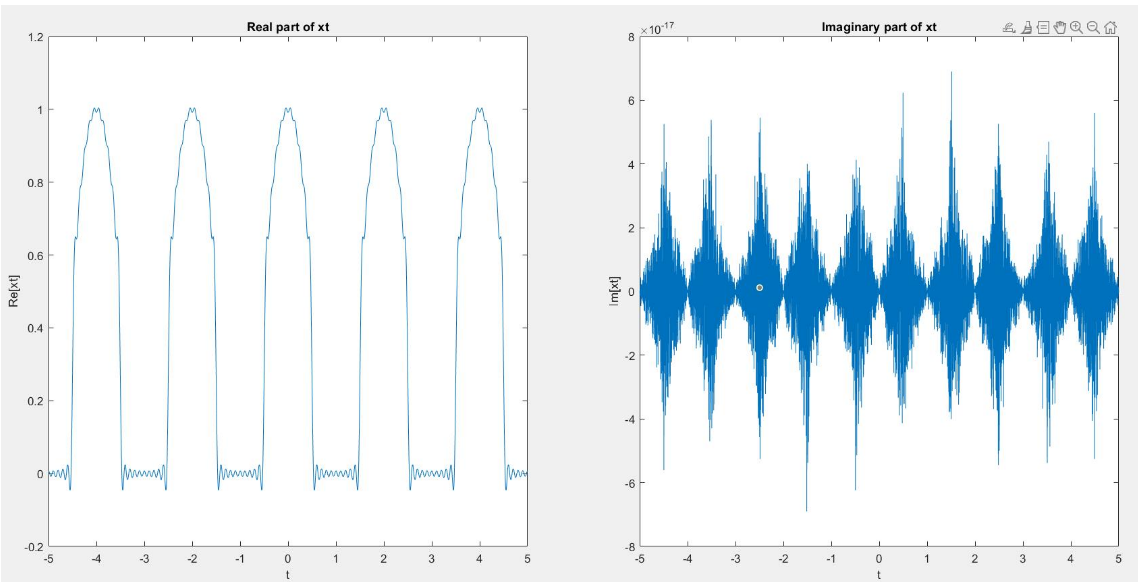


Figure6: Plots of real and imaginary parts of $\tilde{x}(t)$ *with* $K = 20+D_{11} = 20$

max_imag	6.9009e-17
max_real	1.0041
min_imag	-6.9009e-17
min_real	-0.0458

Figure7: Maximum and minimum values of the plots of $\tilde{x}(t)$

It can be seen from *Figure6* the graph of real part of $\tilde{x}(t)$ is very similar to the plot of x(t) drawn by hand in the Part 2. Although the plots are similar it can be seen that there are oscillations on the graph of real part, the reason of that oscillations is this graph is the graph of Fourier Synthesis which means the signal is summation of complex exponential functions. Maximum and minimum values are close to the analytical solution in Part 2 however it can be seen that there is more error in discontinuities. This is because approximating a discontinuity with Fourier Synthesis is harder.

Even though the original signal is purely real it can be seen that in Figure6 imaginary part is not zero. The reason for that is the round-off error of MATLAB. Sometimes MATLAB cannot store the exact values of the numbers. The reason why we get a really small number instead of zero when we computed $\sin(\frac{\pi}{6})$-0.5 is the same.

However the values in the imaginary part are really small compered to real part therefore they can be ignored.

Producing five different plots with replacing K in the code in *Figure5* five times with **K=2+***D*5, **K=7+***D*5, **K=15+***D*5, **K=50+***D*5, **K=100+***D*5 respectively where D_5 = mod(22103444, 5) = 4 and plotting only the real part. The plots are below:

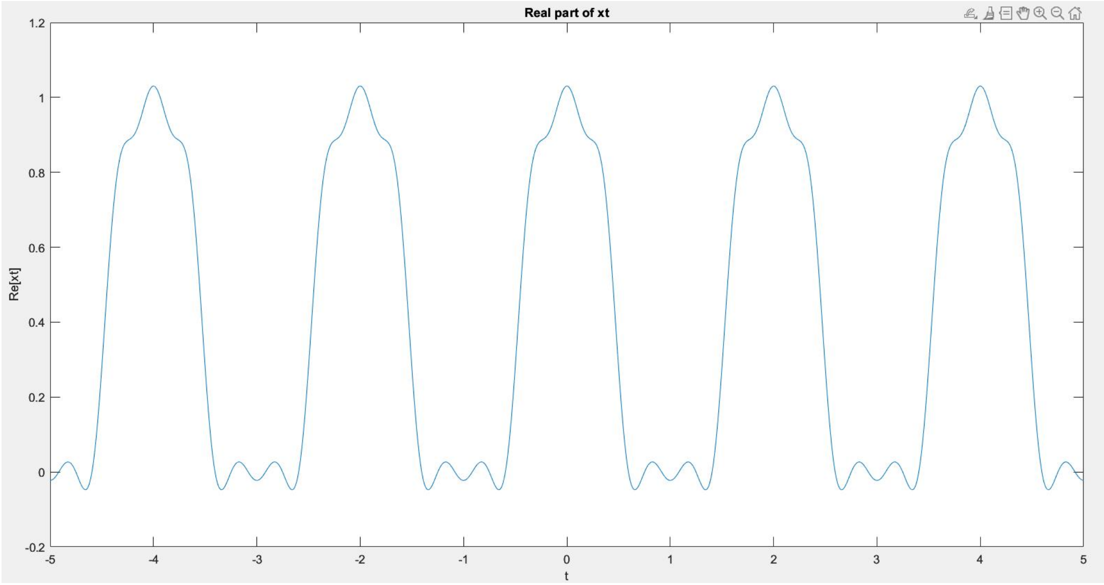


Figure8: Plot of real part of $\tilde{x}(t)$ with K = 2+D5 = 6

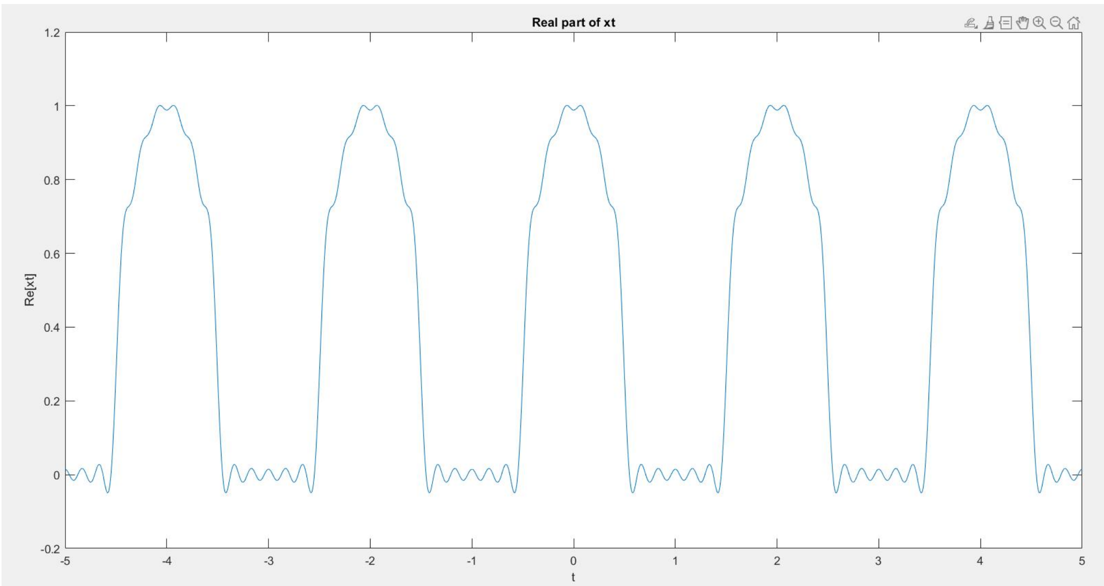


Figure9: Plot of real part of $\tilde{x}(t)$ with K = 7+D5 = 11

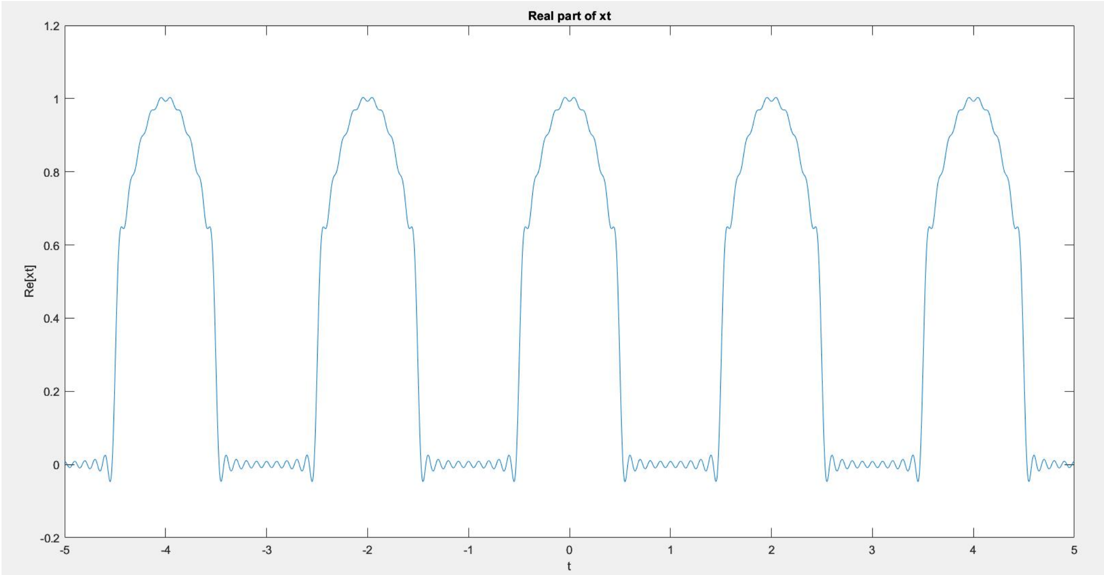


Figure10: Plot of real part of $\tilde{x}(t)$ with K = 15+D5 = 19

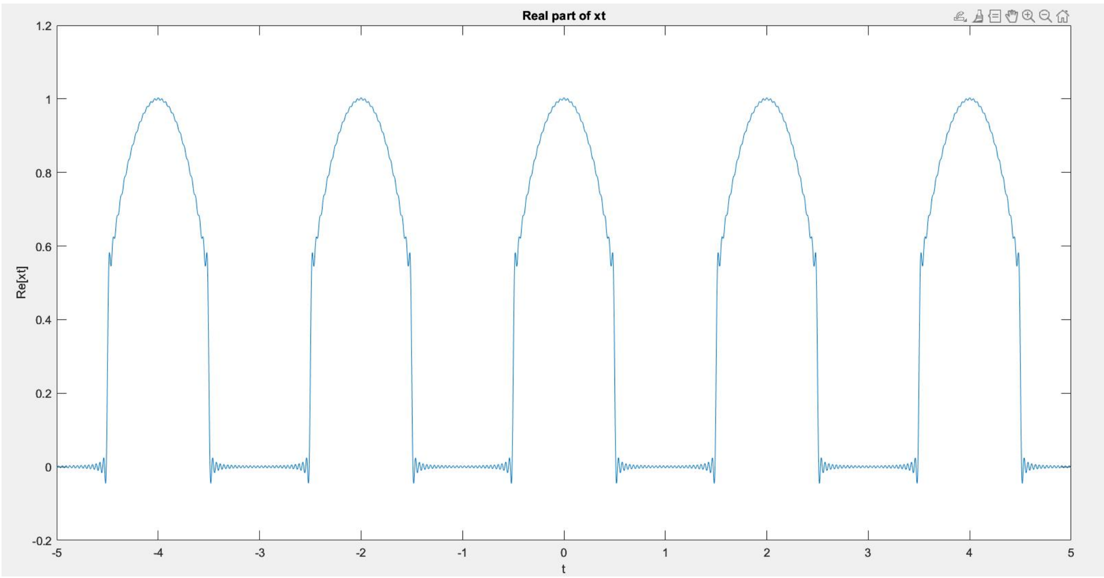


Figure11: Plot of real part of $\tilde{x}(t)$ with K = 50+D5 = 54

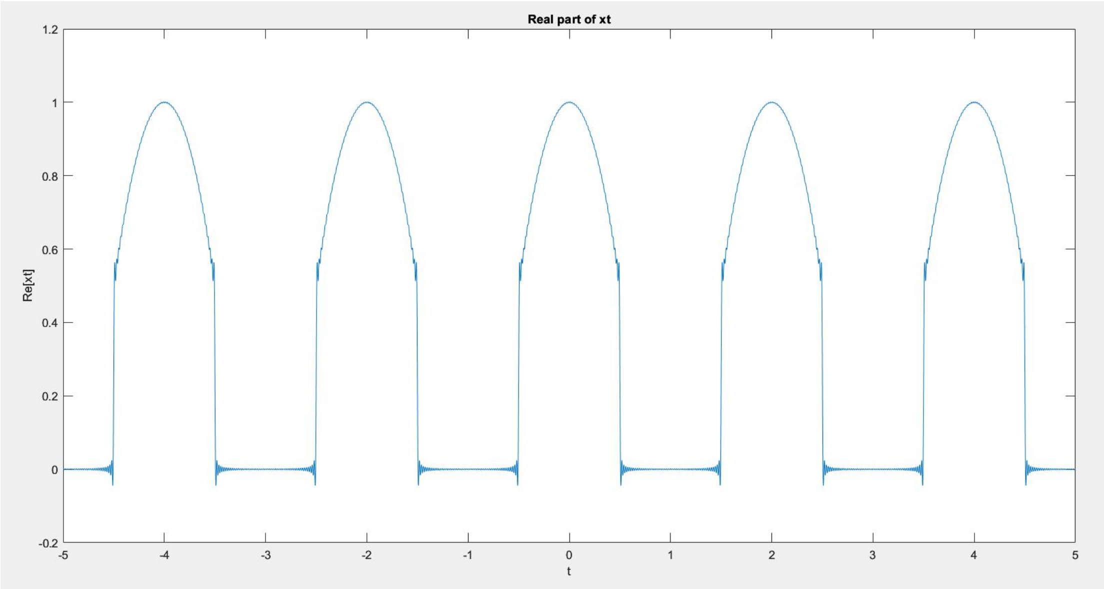


Figure12: Plot of real part of $\tilde{x}(t)$ with K = 100+D5 = 104

From *Figures8-12* it can be seen that as K increases the *c* looks more like x(t) this is because, as K increases the number of sinusoidals in the summation increases which increases the precision of the transformation. If K goes to infinity $\tilde{x}(t)$ will be equal to x(t).

Part 4

Part a) Equations for representing y(t):

$$y(t) = \sum_{k=-K}^{K} Y_k e^{j\frac{2\pi k}{T}t} = \sum_{k=-K}^{K} X_{-k} e^{j\frac{2\pi k t}{T}} = \sum_{k=-K}^{K} X_k e^{-j\frac{2\pi k}{T}}$$

$$\tilde{x}(t) = \tilde{x}(t)^* = \left(\sum_{k=-K}^{K} X_k e^{j\frac{2\pi k}{T}t} \right)^* = \sum_{k=-K}^{K} X_k^* e^{-j\frac{2\pi k}{T}t}$$
$$= \sum_{k=-K}^{K} X_k e^{-j\frac{2\pi k t}{T}} = y(t)$$

As it can be seen from the equations this operation is equivalent of taking the complex conjugate of the $\tilde{x}(t)$. This operation can be applied in MATLAB by adding k=-k; in to the for loop as it can be seen from the *Figure13*.

```
function [xt] = FSWave(t,K,T,W)
    Xk = zeros(1,2*K+1);
    omega = zeros(1,2*K+1);
    for k = -K:K
        k = -k;
        if k == 0
            Xk(K+1) =(1/T)*(W-W^3/6); %X0
        else
            Xk(k+K+1) = (((2-W^2)/(2*pi*k) + (T^2)/(pi^3*k^3))*sin((pi*W*k)/T) ...
                - ((W*T)/(pi^2*k^2))*cos((pi*W*k)/T));
        end
    omega(k+K+1) = 2*pi*k/T;
    end
    xt = SUMCS(t,Xk,omega);
end
```

Figure13: Modified MATLAB code for achieving the specified change in part a

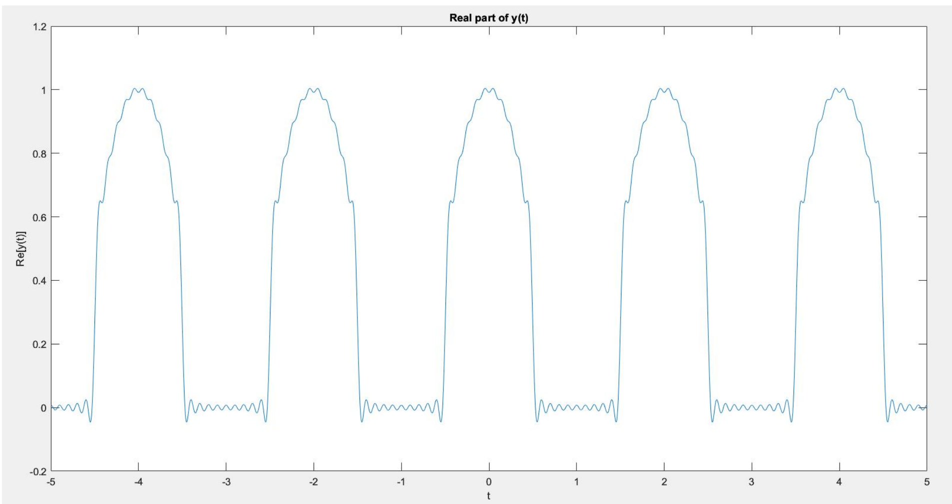


Figure14: Plot of real part of y(t) in part a

As it can be seen from Figure14, if the plot is compared to Figure6 specified change in our function actually did not change the plot. This is because x(t) is an real and even function.

Part b) Equation for representing y(t):

$$y(t) = \sum_{k=-K}^{K} X_k \, e^{\frac{-j2\pi k t}{T}} \, e^{\frac{j2\pi k t}{T}} = \sum_{k=-K}^{K} X_k \, e^{\frac{j2\pi k (t-t_0)}{T}} = \tilde{x}(t-t_0)$$

As it can be seen from the equation this operation corresponds to time-shifting. This operation can be applied in MATLAB by adding

```
Xk(k+K+1) = (Xk(k+K+1))*(exp(-1i*2*pi*k*t0/T));
```

term in the for loop, it can be seen from Figure15.

```
function [xt] = FSWave(t,K,T,W)
    Xk = zeros(1,2*K+1);
    omega = zeros(1,2*K+1);
    t0 = 0.6;
    for k = -K:K

        if k == 0
            Xk(K+1) =(1/T)*(W-W^3/6); %X0
        else
            Xk(k+K+1) = (((2-W^2)/(2*pi*k) + (T^2)/(pi^3*k^3))*sin((pi*W*k)/T) ...
                - ((W*T)/(pi^2*k^2))*cos((pi*W*k)/T));
            Xk(k+K+1) = (Xk(k+K+1))*(exp(-1i*2*pi*k*t0/T));
        end
    omega(k+K+1) = 2*pi*k/T;
    end
    xt = SUMCS(t,Xk,omega);
end
```

Figure15: Modified MATLAB code for achieving the specified change in part b

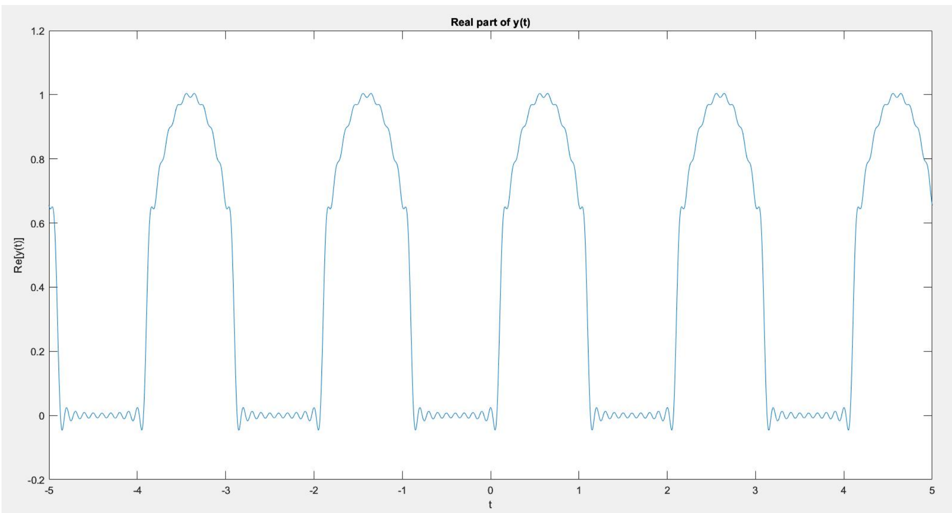


Figure16: Plot of real part of y(t) in part b

If *Figures 6 & 16* compared, it can be seen that the function is shifted by t_0=0.6 .

Part c) Equation for representing y(t):

$$\frac{d\hat{x}(t)}{dt} = \frac{d}{dt}\left(\sum_{k=-K}^{K} X_k e^{\frac{j2\pi k t}{T}}\right) = \sum_{k=-K}^{K} X_k j\left(\frac{2\pi k}{T}\right) e^{j\frac{2\pi k t}{T}} = y(t)$$

As it can be seen from the above equation this operation is equivalent of taking the derivative of the *x̃(t)*. This operation can be applied in MATLAB by adding

```
Xk(k+K+1) = (Xk(k+K+1))*(exp(-1i*2*pi*k*t0/T));
```

term in to the for loop as it can be seen from the *Figure17*.

```
function [xt] = FSWave(t,K,T,W)
    Xk = zeros(1,2*K+1);
    omega = zeros(1,2*K+1);
    for k = -K:K
        if k == 0
            Xk(K+1) =(1/T)*(W-W^3/6); %X0
        else
            Xk(k+K+1) = (((2-W^2)/(2*pi*k) + (T^2)/(pi^3*k^3))*sin((pi*W*k)/T) ...
                - ((W*T)/(pi^2*k^2))*cos((pi*W*k)/T));
            Xk(k+K+1) = (Xk(k+K+1))*(1i*k*2*pi/T);
        end
    omega(k+K+1) = 2*pi*k/T;
    end
    xt = SUMCS(t,Xk,omega);
end
```

Figure17: Modified MATLAB code for achieving the specified change in part c

The modified version of the signal *x̃(t)* can be seen below in *Figure18*:

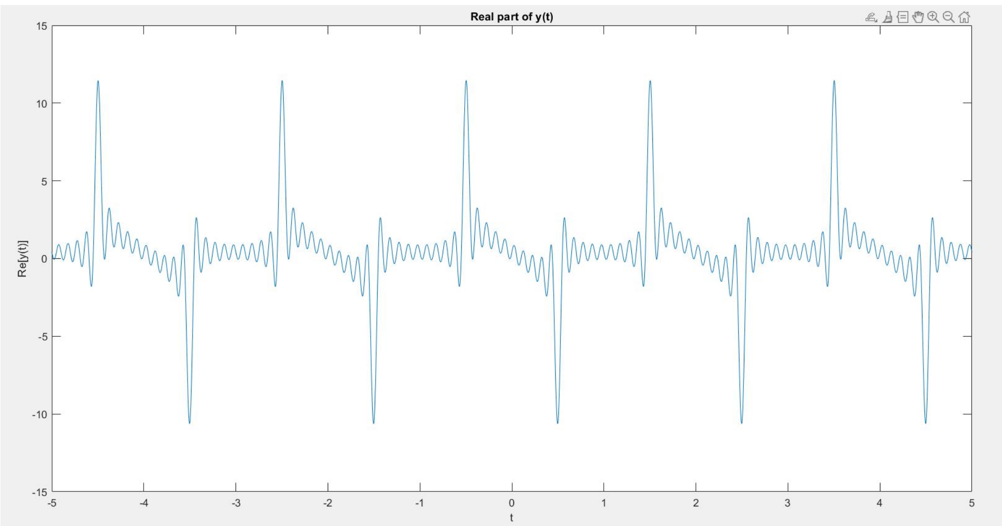


Figure18: Plot of real part of y(t) in part c

Part d) This operation can be applied in MATLAB by adding

```
negative=Xk(1:K);
```

```
negative=[flip(negative),zeros(1,K+1)];
positive=Xk(K+2:2*K+1);
positive=[zeros(1,K+1),flip(positive)];
Xk=negative+positive;
```

into the function as it can be seen from the *Figure19*.

```
function [xt] = FSWave(t,K,T,W)
    Xk = zeros(1,2*K+1);
    omega = zeros(1,2*K+1);

    for k = -K:K
        if k == 0
            Xk(K+1) =(1/T)*(W-W^3/6); %X0
        else
            Xk(k+K+1) = (((2-W^2)/(2*pi*k) + (T^2)/(pi^3*k^3))*sin((pi*W*k)/T) ...
                - ((W*T)/(pi^2*k^2))*cos((pi*W*k)/T));
        end
    omega(k+K+1) = 2*pi*k/T;
    end

    negative=Xk(1:K);
    negative=[flip(negative),zeros(1,K+1)];
    positive=Xk(K+2:2*K+1);
    positive=[zeros(1,K+1),flip(positive)];
    Xk=negative+positive;

    xt = SUMCS(t,Xk,omega);
end
```

Figure19: Modified MATLAB code for achieving the specified change in part d

The modified version of the signal $\tilde{x}(t)$ can be seen below in *Figure20*:

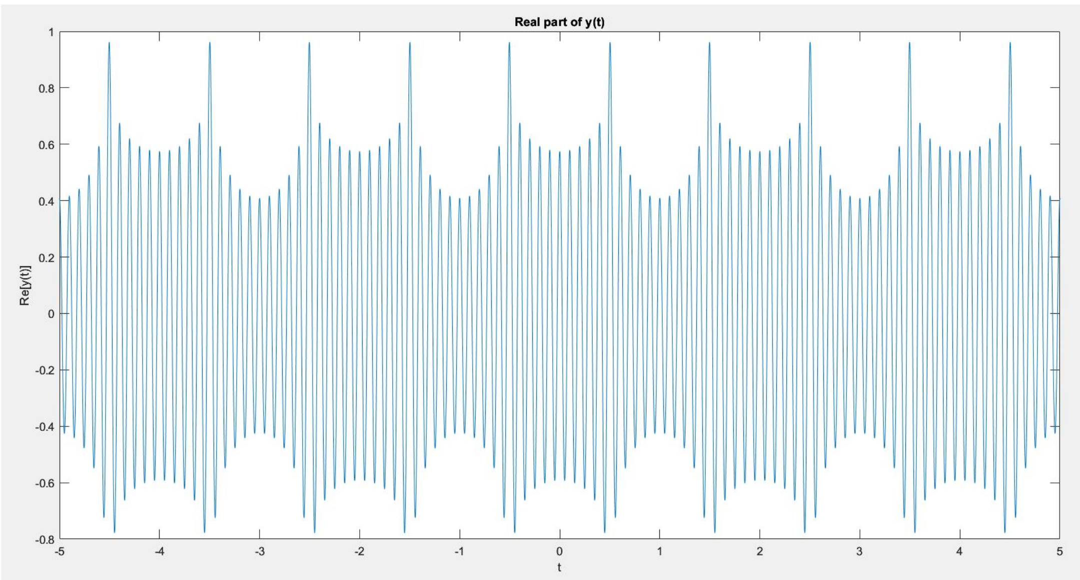


Figure20: Plot of real part of y(t) in part d