Part 1.2 Fourier transform questions

a) $x(t) = e^{j2\pi f_0 t} \Rightarrow e^{j\omega_0 t}$ from the book $X(\omega) = 2\pi\delta(\omega - \omega_0)$

b) $\cos(2\pi f_0 t) \Rightarrow \cos(\omega_0 t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$ from the Euler's equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \cos(\omega_0 t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t} \cdot e^{-j\omega t}\, dt\right]$$

using the result from part (a):

$$\Rightarrow \frac{1}{2}\, 2\pi \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

$$= \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

c) $x(t) = \sin(2\pi f_0 t) \Rightarrow \sin(\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \frac{1}{2j}\int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t}\, e^{-j\omega t}\, dt\right]$$

Again, similar to part (b), using the result from part (a)

$$\Rightarrow \frac{2\pi}{2j}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \Rightarrow -\pi j\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

d) $x(t) = \text{rect}\left(\frac{t}{T_0}\right) = \begin{cases} 1, & \text{if } -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0, & \text{otherwise} \end{cases}$

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1 \cdot e^{-jwt}\, dt = -\frac{e^{-jwt}}{jw} \Bigg|_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \frac{2}{w}\left(\frac{-e^{-jw\frac{T_0}{2}} + e^{jw\frac{T_0}{2}}}{2j} \right) = \frac{2}{w} \sin\left(w \frac{T_0}{2}\right) = T_0 \frac{\sin\left(\frac{wT_0}{2}\right)}{\frac{wT_0}{2}} = T_0 \,\text{sinc}\left(\frac{wT_0}{2}\right)$$

e) $x(t) = e^{j2\pi f_0 t}\, \text{rect}\left(\frac{t}{T_0}\right)$

multiplying complex exponential in time domain leads to frequency shifting in frequency domain. Therefore shifting the result from part (d)

$w_0 = 2\pi f_0$

$$X(w) = T_0 \,\text{sinc}\left((w - w_0)\frac{T_0}{2}\right)$$

f) $x(t) = \cos(2\pi f_0 t)\, \text{rect}\left(\frac{t}{T_0}\right)$, $w_0 = 2\pi f_0$

$$\cos(2\pi f_0 t) = \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

therefore,

$$x(t) = \frac{1}{2}\left[e^{jw_0 t}\, \text{rect}\left(\frac{t}{T_0}\right) + e^{j(-w_0)t}\, \text{rect}\left(\frac{t}{T_0}\right) \right]$$

multiplying complex exponential in time domain leads to frequency shifting. Therefore the answer becomes:

$$X(w) = \frac{T_0}{2}\left[\text{sinc}\left((w - w_0)\frac{T_0}{2}\right) + \text{sinc}\left((w + w_0)\frac{T_0}{2}\right) \right]$$

g) $x(t) = \text{rect}\left(\frac{t - t_0}{T_0}\right)$ time shifting means multiplying with e^{-jwt_0} in the frequency domain.

$$X(w) = e^{-jwt_0} \cdot T_0 \,\text{sinc}\left(\frac{wT_0}{2}\right)$$

h) $x(t) = e^{j\omega_0 t} \, rect\left(\frac{t-t_0}{T_0}\right)$ again using time shifting and frequency shifting properties from previous question.

$\omega_0 = 2\pi f_0$

$$X(\omega) = e^{-j(\omega-\omega_0)t_0} \cdot T_0 \, Sinc\left(\frac{(\omega-\omega_0)T_0}{2}\right)$$

i) $x(t) = \cos(\omega_0 t) \, rect\left(\frac{t-t_0}{T_0}\right)$ $\quad \omega_0 = 2\pi f_0$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\Rightarrow x(t) = \frac{1}{2}\left[e^{j\omega_0 t} rect\left(\frac{t-t_0}{T_0}\right) + e^{j(-\omega_0)t} rect\left(\frac{t-t_0}{T_0}\right)\right]$$

Again, using time and frequency shifting properties:

$$X(\omega) = \frac{T_0}{2}\left[e^{-j(\omega-\omega_0)t} \, sinc\left(\frac{(\omega-\omega_0)T_0}{2}\right) + e^{-j(\omega+\omega_0)t} \, Sinc\left(\frac{(\omega+\omega_0)T_0}{2}\right)\right]$$