H_∞ Design with Pole Placement Constraints: An LMI Approach

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Abstract—This paper addresses the design of state- or output-feedback H_∞ controllers that satisfy additional constraints on the closed-loop pole location. Sufficient conditions for feasibility are derived for a general class of convex regions of the complex plane. These conditions are expressed in terms of linear matrix inequalities (LMI's), and our formulation is therefore numerically tractable via LMI optimization. In the state-feedback case, mixed H_2/H_∞ synthesis with regional pole placement is also discussed. Finally, the validity and applicability of this approach are illustrated by a benchmark example.

I. INTRODUCTION

MANY classical control objectives such as disturbance attenuation, robust stabilization of uncertain systems, or shaping of the open-loop response can be expressed in terms of H_∞ performance and tackled by H_∞-synthesis techniques [10], [12]. Since it only involves solving two Riccati equations, H_∞ synthesis has a low complexity comparable to that of linear-quadratic-Gaussian (LQG) synthesis [11]. However, H_∞ design deals mostly with frequency-domain aspects and provides little control over the transient behavior and closed-loop pole location. In contrast, satisfactory time response and closed-loop damping can often be achieved by forcing the closed-loop poles into a suitable subregion of the left-half plane (see, for instance, the discussion in [1]). In addition, fast controller dynamics can be prevented by prohibiting large closed-loop poles (often desirable for digital implementation). One way of simultaneously tuning the H_∞ performance and transient behavior is therefore to combine the H_∞ and pole-placement objectives.

While many authors deal with root clustering in the context of linear-quadratic regulation [13], [20], [25], [33], only a few papers consider the design of H_∞ controllers with additional specifications on the closed-loop poles. Moreover, most papers on this topic are restricted to the state-feedback case [4], [36]. This can be explained by the difficulty of incorporating pole-placement constraints in the standard state-space approach to H_∞ synthesis [11]. Except for very special regions [21], there seems to be no systematic way of modifying the two algebraic Riccati equations to enforce specific root-clustering objectives. Note that an attempt in this direction is made in [36]. There, the H_∞ and pole-placement constraints are turned into a system of coupled Lyapunov and Riccati equations, and a Lagrange multiplier formulation is used to solve this system

numerically. However, global convergence is not guaranteed, and the method is restricted to the state-feedback case.

It has been noticed recently that H_∞ synthesis can be formulated as a convex optimization problem involving linear matrix inequalities (LMI) [14], [22], [30]. These LMI's correspond to the inequality counterpart of the usual H_∞ Riccati equations. Because LMI's intrinsically reflect constraints rather than optimality, they tend to offer more flexibility for combining several constraints on the closed-loop system [7]. This important advantage of LMI techniques has been exploited in [8] to handle H_∞ design with pole clustering in the region $\mathrm{Re}\,(s) < -\alpha$ where α is a positive scalar. This paper extends these earlier results to a wide variety of stability regions.

Note that the LMI formulation of constrained H_∞ optimization is appealing from a practical standpoint. Indeed, LMI's can be solved by efficient interior-point optimization algorithms such as those described in [29], [28], [6], [34], and [27]. Moreover, software like MATLAB's *LMI Control Toolbox* [15] is now available to solve such LMI's in a fast and user-friendly manner.

This paper is organized as follows. Section II introduces the concept of an LMI region as a convenient LMI-based representation of general stability regions. By stability region, we mean any subregion of the open left half of the complex plane. Section III discusses state-feedback H_2/H_∞ synthesis with pole clustering in arbitrary LMI regions. In Section IV, these results are fully extended to the output-feedback case. Finally, the validity and practicality of this approach are demonstrated by an example drawn from the literature in Section V.

II. LMI FORMULATION OF POLE-PLACEMENT OBJECTIVES

This section discusses a new LMI-based characterization for a wide class of pole clustering regions as well as an extended Lyapunov Theorem for such regions. Prior to this presentation, we briefly recall the main motivations for seeking pole clustering in specific regions of the left-half plane. It is known that the transient response of a linear system is related to the location of its poles [24], [1]. For example, the step response of a second-order system with poles $\lambda = -\zeta\omega_n \pm j\omega_d$ is fully characterized in terms of the undamped natural frequency $\omega_n = |\lambda|$, the damping ratio ζ, and the damped natural frequency ω_d (see, e.g., [24, pp. 327–354]). By constraining λ to lie in a prescribed region, specific bounds can be put on these quantities to ensure a satisfactory transient response. Regions of interest include α-stability
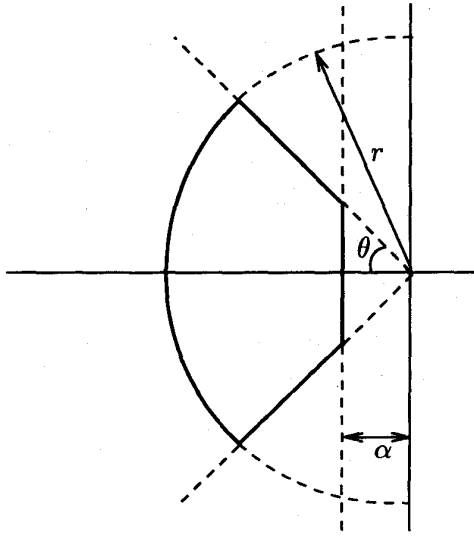
Fig. 1. Region $S(\alpha, r, \theta)$.

regions $\mathrm{Re}(s) \leq -\alpha$, vertical strips, disks, conic sectors, etc. Another interesting region for control purposes is the set $S(\alpha, r, \theta)$ of complex numbers $x + jy$ such that

$$x < -\alpha < 0, \quad |x + jy| < r, \quad \tan\theta\, x < -|y| \qquad (1)$$

as shown in Fig. 1. Confining the closed-loop poles to this region ensures a minimum decay rate α, a minimum damping ratio $\zeta = \cos\theta$, and a maximum undamped natural frequency $\omega_d = r\sin\theta$. This in turn bounds the maximum overshoot, the frequency of oscillatory modes, the delay time, the rise time, and the settling time [24].

After reviewing existing Lyapunov-based characterizations of pole clustering in stability subregions of the complex plane, the notion of the LMI region is introduced, and various illustrations of its usefulness are given.

A. Lyapunov Conditions for Pole Clustering

Let $\mathcal{D}$ be a subregion of the complex left-half plane. A dynamical system $\dot{x} = Ax$ is called $\mathcal{D}$-stable if all its poles lie in $\mathcal{D}$ (that is, all eigenvalues of the matrix A lie in $\mathcal{D}$). By extension, A is then called $\mathcal{D}$-stable. When $\mathcal{D}$ is the entire left-half plane, this notion reduces to asymptotic stability, which is characterized in LMI terms by the Lyapunov theorem. Specifically, A is stable if and only if there exists a symmetric matrix X satisfying

$$AX + XA^T < 0, \qquad X > 0. \qquad (2)$$

This Lyapunov characterization of stability has been extended to a variety of regions by Gutman [19]. The regions considered there are polynomial regions of the form

$$\mathcal{D} = \left\{ z \in C: \sum_{0 \leq k, l \leq m} c_{kl} z^k \bar{z}^l < 0 \right\} \qquad (3)$$

where the coefficients c_{kl} are real and satisfy $c_{kl} = c_{lk}$. Note that polynomial regions are not fully general since, e.g., the

region $S(\alpha, r, \theta)$ in (1) cannot be represented in this form. For polynomial regions, Gutman's fundamental result states that a matrix A is $\mathcal{D}$-stable if and only if there exists a symmetric matrix X such that [19], [36]

$$\sum_{k,l} c_{kl} A^k X (A^T)^l < 0, \qquad X > 0. \qquad (4)$$

Note that (4) is derived from (3) by replacing $z^k \bar{z}^l$ with $A^k X (A^T)^l$. The Lyapunov theorem is clearly a particular case of this result.

Unfortunately, controller synthesis based on Gutman's characterization is hardly tractable due to the polynomial nature of (4). Recall that the closed-loop state-space matrices depend affinely on the controller state-space data [7], [14]. To keep the synthesis problem tractable in the LMI framework, it is necessary to use a characterization of pole clustering that preserves this affine dependence. In other words, it is necessary to use conditions that are affine in the state matrix A, such as the Lyapunov stability condition (2). Apart from a few special cases, there is no systematic way of turning (4) into an LMI in A. In fact, polynomial regions are not necessarily convex. With controller synthesis in mind, this motivates one to look for an alternative LMI-based representation of $\mathcal{D}$-stability regions.

B. LMI Regions

The class of LMI regions defined below turns out to be suitable for LMI-based synthesis. Hereafter, $\otimes$ denotes the Kronecker product of matrices (see, e.g., [18]), and the notation $M = [\mu_{kl}]_{1 \leq k, l \leq m}$ means that M is an $m \times m$ matrix (respectively, block matrix) with generic entry (respectively, block) μ_{kl}.

Definition 2.1 (LMI Regions): A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exist a symmetric matrix $\alpha = [\alpha_{kl}] \in R^{m \times m}$ and a matrix $\beta = [\beta_{kl}] \in R^{m \times m}$ such that

$$\mathcal{D} = \{ z \in C: \quad f_{\mathcal{D}}(z) < 0 \} \qquad (5)$$

with

$$f_{\mathcal{D}}(z) := \alpha + z\beta + \bar{z}\beta^T = [\alpha_{kl} + \beta_{kl} z + \beta_{lk} \bar{z}]_{1 \leq k, l \leq m}. \qquad (6)$$

Note that the characteristic function $f_{\mathcal{D}}$ takes values in the space of $m \times m$ Hermitian matrices and that "<0" stands for negative definite. ∎

In other words, an LMI region is a subset of the complex plane that is representable by an LMI in z and $\bar{z}$, or equivalently, an LMI in $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. As a result, LMI regions are convex. Moreover, LMI regions are symmetric with respect to the real axis since for any $z \in \mathcal{D}$, $f_{\mathcal{D}}(\bar{z}) = \overline{f_{\mathcal{D}}(z)} < 0$.

Interestingly, there is a complete counterpart of Gutman's theorem for LMI regions. Specifically, pole location in a given LMI region can be characterized in terms of the $m \times m$ block matrix

$$M_{\mathcal{D}}(A, X) := \alpha \otimes X + \beta \otimes (AX) + \beta^T \otimes (AX)^T$$
$$= [\alpha_{kl} X + \beta_{kl} AX + \beta_{lk} XA^T]_{1 \leq k, l \leq m} \qquad (7)$$

as follows.

Theorem 2.2: The matrix A is $\mathcal{D}$-stable if and only if there exists a symmetric matrix X such that

$$M_{\mathcal{D}}(A, X) < 0, \qquad X > 0. \qquad (8)$$

Proof: See the Appendix.

Note that $M_{\mathcal{D}}(A, X)$ in (7) and $f_{\mathcal{D}}(z)$ in (6) are related by the substitution $(X, AX, XA^T) \leftrightarrow (1, z, \bar{z})$. As an example, the disk of radius r and center $(-q, 0)$ is an LMI region with characteristic function

$$f_{\mathcal{D}}(z) = \begin{pmatrix} -r & q + z \\ q + \bar{z} & -r \end{pmatrix}. \qquad (9)$$

In this case, (8) reads

$$\begin{pmatrix} -rX & qX + AX \\ qX + XA^T & -rX \end{pmatrix} < 0, \qquad X > 0. \qquad (10)$$

Consider now the region $\mathcal{S}(\alpha, r, \theta)$ defined in (1) and take $\alpha = r = 0$. It was shown in [3] that the eigenvalues of A lie in the sector $\mathcal{S}(0, 0, \theta)$ if and only if there exists a positive definite matrix P such that

$$(W \otimes A)P + P(W \otimes A)^T < 0 \qquad (11)$$

where

$$W = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}.$$

On the other hand, $\mathcal{S}(0, 0, \theta)$ is an LMI region with characteristic function

$$f_\theta(z) = \begin{pmatrix} \sin\theta(z + \bar{z}) & \cos\theta(z - \bar{z}) \\ \cos\theta(\bar{z} - z) & \sin\theta(z + \bar{z}) \end{pmatrix}. \qquad (12)$$

From Theorem (2), A has its poles in $\mathcal{S}(0, 0, \theta)$ if and only if there exists $X > 0$ such that

$$\begin{pmatrix} \sin\theta(AX + XA^T) & \cos\theta(AX - XA^T) \\ \cos\theta(XA^T - AX) & \sin\theta(AX + XA^T) \end{pmatrix} < 0 \qquad (13)$$

or equivalently

$$(W \otimes A)\,\mathrm{Diag}\,(X, X) + \mathrm{Diag}\,(X, X)(W \otimes A)^T < 0. \qquad (14)$$

Compared to (11), this last condition gives additional information on the structure of P. It is also better from an LMI optimization perspective since the number of optimization variables is divided by four when replacing P by $\mathrm{Diag}\,(X, X)$.

C. Extent of the Class of LMI Regions

Theorem 2.2 answers our need for a characterization of stability regions that is affine in the A matrix. The convenience of LMI regions for synthesis purposes will be illustrated in the next two sections. Yet the case for LMI regions would be incomplete if this class of regions was insufficiently large. In the remainder of this section, we show that LMI regions not only include a wide variety of typical clustering regions, but also form a dense subset of the convex regions that are symmetric with respect to the real axis. In other words, LMI regions cover most practical needs for control purposes.
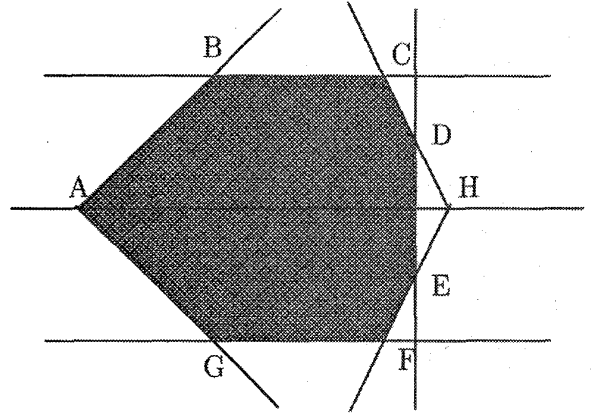

Fig. 2. Convex polygon as an intersection of LMI regions.

Let us first review some simple LMI regions. Considering functions of the form

$$f_{\mathcal{D}}(x + jy) = \begin{pmatrix} a_1 + b_1 x & a_{12} + b_{12}x + jy \\ a_{12} + b_{12}x - jy & a_2 + b_2 x \end{pmatrix} \qquad (15)$$

shows that conic sectors $ax + b < -|y|$, vertical half planes $x + \alpha < 0$ or $x + \alpha > 0$, vertical strips $h_1 < x < h_2$, horizontal strips $|y| < \omega$, as well as ellipses, parabolas, and hyperbolic sectors are all LMI regions.

To derive more complex regions, observe that the class of LMI regions is invariant under set intersection. Specifically, given two LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ and their associated characteristic functions $f_{\mathcal{D}_1}$ and $f_{\mathcal{D}_2}$, the intersection $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ is also an LMI region with characteristic function

$$f_{\mathcal{D}_1 \cap \mathcal{D}_2} = \mathrm{Diag}\,(f_{\mathcal{D}_1}, f_{\mathcal{D}_2}). \qquad (16)$$

As a result, the class of LMI regions includes all polygonal regions that are convex and symmetric with respect to the real axis. Indeed, any such region can be obtained as the intersection of conic sectors, vertical strips, and/or horizontal strips. An illustrative example is shown in Fig. 2. Here, the polygonal region (A, B, C, D, E, F, G) is the intersection of the half plane on the left of the line (DE), the horizontal strip delimited by the lines (BC) and (GF), and the two conic sectors $\widehat{CHF}$ and $\widehat{BAG}$. Since any convex region can be approximated by a convex polygon to any desired accuracy, we conclude that LMI regions are dense in the set of convex regions that are symmetric with respect to the real axis.

Another consequence of the intersection property (16) is that simultaneous clustering constraints can be expressed as a system of LMI's in the same variable X without introducing any conservatism.

Corollary 2.3: Given two LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$, a matrix A is both $\mathcal{D}_1$-stable and $\mathcal{D}_2$-stable if and only if there exists a positive definite matrix X such that $M_{\mathcal{D}_1}(A, X) < 0$ and $M_{\mathcal{D}_2}(A, X) < 0$.

Proof: Simply observe that $M_{\mathcal{D}_1 \cap \mathcal{D}_2}(A, X) = \mathrm{Diag}\,(M_{\mathcal{D}_1}(A, X), M_{\mathcal{D}_2}(A, X))$ since $\mathcal{D}_1 \cap \mathcal{D}_2$ has the characteristic function $\mathrm{Diag}\,(f_{\mathcal{D}_1}, f_{\mathcal{D}_2})$. Applying Theorem 2.2 to $\mathcal{D}_1 \cap \mathcal{D}_2$ completes the proof. ∎

To illustrate the implications of this property, consider the vertical strip defined by

$$\mathcal{D} = \{x + jy \in \mathbf{C}: \quad -h_1 < x < -h_2 < 0\}. \quad (17)$$

From Corollary 2.3, a matrix A has all its eigenvalues in $\mathcal{D}$ if and only if there exists a single $X > 0$ such that

$$AX + XA^T + 2h_2X < 0, \qquad AX + XA^T + 2h_1X > 0. \quad (18)$$

In comparison, (4) for the region $(x - h_1)(x - h_2) < 0$ reads

$$A^2X + X(A^2)^T + 2AXA^T + 2(h_1 + h_2)(AX + XA^T) + 4h_1h_2I < 0. \quad (19)$$

Another example is the region $\mathcal{S}(\alpha, r, \theta)$ defined by (1). This clustering region is difficult to express with classical representations. In contrast, Theorem 2.2 yields the following LMI characterization for $\mathcal{S}(\alpha, r, \theta)$:

$$AX + XA^T + 2\alpha X < 0 \quad (20)$$

$$\begin{pmatrix} -rX & AX \\ XA^T & -rX \end{pmatrix} < 0 \quad (21)$$

$$\begin{pmatrix} \sin\theta(AX + XA^T) & \cos\theta(AX - XA^T) \\ \cos\theta(XA^T - AX) & \sin\theta(AX + XA^T) \end{pmatrix} < 0. \quad (22)$$

This follows from observing that this region is the intersection of three elementary LMI regions: an α-stability region, a disk, and the conic sector $\mathcal{S}(0, 0, \theta)$ discussed in Section II-B.

III. STATE-FEEDBACK H_2/H_∞ DESIGN WITH POLE PLACEMENT

This section discusses state-feedback synthesis with a combination of H_2/H_∞ performance and pole assignment specifications. Here, the closed-loop poles are required to lie in some LMI region $\mathcal{D}$ contained in the left-half plane. Results are first derived in the nominal case and then extended to uncertain systems described by a polytope of models. Unconstrained mixed H_2/H_∞ synthesis is considered in [23], where an LMI-based synthesis procedure is proposed. These results are extended in [4] to include pole placement in a disk. The discussion below refines the results of [4] and extends them to the larger class of LMI stability regions.

Consider a linear time-invariant (LTI) system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z_\infty(t) &= C_\infty x(t) + D_{\infty 1}w(t) + D_{\infty 2}u(t) \\ z_2(t) &= C_2x(t) + D_{22}u(t) \end{aligned} \quad (23)$$

and let $T_{wz_\infty}(s)$ [respectively, $T_{wz_2}(s)$] denote the closed-loop transfer function from w to z_∞ (respectively, from w to z_2) under state-feedback control $u = Kx$. For a prescribed closed-loop H_∞ performance $\gamma > 0$, our constrained mixed H_2/H_∞ problem consists of finding a state-feedback gain K that:

- places the closed-loop poles in some LMI stability region $\mathcal{D}$ with characteristic function (6);
- guarantees the H_∞ performance $\|T_{wz_\infty}\|_\infty < \gamma$;
- minimizes the H_2 performance $\|T_{wz_2}\|_2$ subject to these two constraints.

Let $(A_{cl}, B_{cl}, C_{cl\infty}, D_{cl\infty})$ and $(A_{cl}, B_{cl}, C_{cl2}, D_{cl2})$ denote realizations of T_{wz_∞} and T_{wz_2}. We first examine each specification separately. From Theorem 2.2, the pole-placement constraint is satisfied if and only if there exists $X_\mathcal{D} > 0$ such that

$$[\alpha_{kl}X_\mathcal{D} + \beta_{kl}A_{cl}X_\mathcal{D} + \beta_{lk}X_\mathcal{D}A_{cl}^T]_{1 \leq k,l \leq m} < 0. \quad (24)$$

Meanwhile, the H_∞ constraint is equivalent to the existence of a solution $X_\infty > 0$ to the LMI

$$\begin{pmatrix} A_{cl}X_\infty + X_\infty A_{cl}^T & B_{cl} & X_\infty C_{cl\infty}^T \\ B_{cl}^T & -I & D_{cl\infty}^T \\ C_{cl\infty}X_\infty & D_{cl\infty} & -\gamma^2 I \end{pmatrix} < 0. \quad (25)$$

This result is known as the Bounded Real Lemma (see, e.g., [2] and [31]). Consider finally the H_2 objective. Recall that $\|T_{wz_2}\|_2^2 = \text{Trace}(C_{cl2}PC_{cl2}^T)$, where P is the solution of the Lyapunov equation $A_{cl}P + PA_{cl}^T + B_{cl}B_{cl}^T = 0$. Hence $\|T_{wz_2}\|_2^2 < \text{Trace}(C_{cl2}X_2C_{cl2}^T)$ for any $X_2 > 0$ such that $A_{cl}X_2 + X_2A_{cl}^T + B_{cl}B_{cl}^T < 0$. Equivalently, $\|T_{wz_2}\|_2^2 <$ Trace (Y) whenever the symmetric matrices X_2 and Y satisfy

$$A_{cl}X_2 + X_2A_{cl}^T + B_{cl}B_{cl}^T < 0 \quad (26)$$

$$\begin{pmatrix} Y & C_{cl2}X_2 \\ X_2C_{cl2}^T & X_2 \end{pmatrix} > 0 \quad (27)$$

(note that (27) is equivalent to $Y > C_{cl2}X_2C_{cl2}^T$).

Our goal is to minimize the H_2 norm of T_{wz_2} over all state-feedback gains K that enforce the H_∞ and pole constraints. From the previous discussion, this is equivalent to minimizing Trace (Y) over all matrices $X_\mathcal{D}, X_\infty, X_2, Y$, and K satisfying (24)–(27). While this problem is not jointly convex in the variables $(X_\mathcal{D}, X_\infty, X_2, Y, K)$, convexity can be enforced by seeking a common solution

$$X = X_\mathcal{D} = X_\infty = X_2 > 0 \quad (28)$$

to (24)–(27). Admittedly with some conservatism, we are then left with computing

$$J(T_{wz_2}) := \inf\{\text{Trace}(Y): X, Y, K \text{ satisfy (24)–(27)} \\ \text{with } X_\mathcal{D} = X_\gamma = X_2 = X\} \quad (29)$$

[note that $X > 0$ is contained in the inequality (27)]. The auxiliary performance $J(T_{wz_2})$ is an upper estimate of the optimal H_2 performance subject to the H_∞ and pole-placement constraints.

Seeking a single Lyapunov matrix X that enforces multiple constraints has been used in other problems such as the quadratic stabilization of a polytope of plants (see [7] for a good survey). When $\mathcal{D}$ is the open left-half plane, the auxiliary performance $J(T_{wz_2})$ coincides with that introduced in [5] and successfully used in [23] for mixed H_2/H_∞ synthesis. Similarly, without the H_∞ constraint, the problem (29) reduces to the auxiliary cost minimization problem proposed in [20] and discussed in [33] in the case of pole placement in a disk. Finally, the same technique is used in [4] for H_2/H_∞ synthesis with root clustering in a disk. However, [4] combines the resulting conditions into a single LMI at the expense of additional conservatism. Note that the performance $J(T_{wz_2})$ cannot be improved by using dynamic state-feedback control.

This is proven in [23] when $\mathcal{D}$ is the open left-half plane, and this result extends to arbitrary LMI regions by essentially the same argument.

The optimization problem (29) is not yet convex because of the products KX arising in terms like $A_{cl}X$. However, convexity is readily restored by rewriting (24)–(27) in terms of X, Y, and the auxiliary variable $L := KX$ [7]. This simple change of variable leads to the following suboptimal LMI approach to H_2/H_∞ synthesis with pole assignment in arbitrary LMI regions.

Theorem 3.1: Let $\mathcal{D}$ be an LMI region contained in the open left-half plane and with (6). The H_2 performance index $J(T_{wz_2})$ defined by (29) can be computed as the global minimum of the following LMI optimization problem.

Minimize Trace (Y) over $X = X^T, Y = Y^T$, and L subject to the LMI constraints

$$[\alpha_{kl}X + \beta_{kl}U(X, L) + \beta_{lk}U(X, L)^T] < 0 \qquad (30)$$

$$\begin{pmatrix} U(X, L) + U(X, L)^T & B_1 & V(X, L)^T \\ B_1^T & -I & D_{\infty1}^T \\ V(X, L) & D_{\infty1} & -\gamma^2 I \end{pmatrix} < 0 \quad (31)$$

$$\begin{pmatrix} Y & C_2X + D_{22}L \\ (C_2X + D_{22}L)^T & X \end{pmatrix} > 0 \qquad (32)$$

where

$$U(X, L) := AX + B_2L, \qquad V(X, L) := C_\infty X + D_{\infty2}L. \qquad (33)$$

Assume that (30)–(32) are feasible and let (X^*, Y^*, L^*) be an optimal solution of this minimization problem. Then the state-feedback gain $K^* := L^*(X^*)^{-1}$ has guaranteed H_∞ performance γ, places the closed-loop poles in $\mathcal{D}$, and yields an H_2 performance that does not exceed $\sqrt{\text{Trace}\,(Y^*)}$.

Proof: Using the realization

$$A_{cl} = A + B_2K, \qquad B_{cl} = B_1, \qquad C_{cl\infty} = C_\infty + D_{\infty2}K,$$

$$D_{cl\infty} = D_{\infty1}, \qquad C_{cl2} = C_2 + D_{22}K, \qquad D_{cl2} = 0$$

for the closed-loop system, (30)–(32) readily follow from (24)–(27) together with the change of variable $L = KX$. ∎

Note that the constrained minimization of Theorem 3.1 is a standard LMI problem of the form

Minimize $c^T x$ subject to the LMI constraint $L(x) < 0$. (34)

In particular, it is readily solved with LMI optimization software such as the *LMI Control Toolbox* [15].

A. Extension to Uncertain Systems

The previous results are easily extended to uncertain systems described by a polytopic state-space model. For brevity, only pole placement is discussed here. See, e.g., [7] for similar generalizations to the H_∞ or H_2 performance.

Consider the uncertain LTI system

$$\dot{x} = Ax + Bu \qquad (35)$$

where the matrices A and B take values in the matrix polytope

$$(A, B) \in \left\{ \left(\sum_{i=1}^N p_iA_i, \sum_{i=1}^N p_iB_i \right) : \sum_{i=1}^N p_i = 1, \, p_i \geq 0 \right\}. \qquad (36)$$

Such polytopic models may result from convex interpolation of a set of models (A_i, B_i) identified in different operating points. They also arise in connection with affine parameter-dependent models

$$\dot{x} = A(p)x + B(p)u$$

where p is a vector of real uncertain parameters ranging in intervals and $A(p), B(p)$ are affine matrix-valued functions of p.

Consider the problem of computing a state-feedback gain K that forces the closed-loop eigenvalues into some LMI region $\mathcal{D}$ for all admissible values of A and B. Following [17], (35) is called quadratically $\mathcal{D}$-stabilizable if there exists a gain K and a single Lyapunov matrix $X > 0$ such that $M_\mathcal{D}(A + BK, X) < 0$ for all admissible values of A and B. Quadratic stabilizability in a sector $\mathcal{D}$ is considered in [3], where a sufficient condition is derived in terms of LMI's. It is shown below that this condition is also necessary, and the results of [3] are extended to arbitrary LMI regions. See also [17] for a treatment of the case where $\mathcal{D}$ is a disk.

Let $\mathcal{D}$ be any LMI region, suppose that (35) is quadratically $\mathcal{D}$-stabilizable with Lyapunov matrix X and state-feedback gain K, and let $L := KX$. Writing the condition $M_\mathcal{D}(A + BK, X) < 0$ at each vertex (A_i, B_i) of (36) yields the following necessary conditions on X, L:

$$[\alpha_{kl}X + \beta_{kl}(A_iX + B_iL) + \beta_{lk}(A_iX + B_iL)^T]_{k,l} < 0$$
$$\text{for } i = 1, \cdots, N \quad (37)$$

$$X > 0. \qquad (38)$$

Conversely, any solution (X, L) of this LMI system satisfies $M_\mathcal{D}(A + BK, X) < 0$ as readily seen when forming the weighted sum of the LMI's (37) with nonnegative coefficients $p_1, \cdots, p_N$. Hence (37) and (38) are necessary and sufficient for quadratic $\mathcal{D}$-stabilizability. When specialized to a sector, these conditions are exactly those derived in [3]. Note that LMI conditions for quadratic H_∞ and H_2 performance over (36) are obtained similarly by writing (30)–(32) at each vertex of the polytopic plant.

IV. OUTPUT-FEEDBACK H_∞ SYNTHESIS WITH POLE PLACEMENT

The state-feedback results of Section III are now generalized to the output-feedback case. For simplicity, we drop the H_2 objective and focus on H_∞ synthesis with regional constraints on the nominal closed-loop poles. See [15], [9], [26], and [32] for a treatment of the mixed H_2/H_∞ output-feedback problem.

The constrained H_∞ problem under consideration can be stated as follows. Given an LTI plant with

$$\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t)$$
$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t)$$
$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t) \qquad (39)$$

an LMI stability region $\mathcal{D}$, and some H_∞ performance $\gamma > 0$, find an LTI control law $u = K(s)y$ such that:

- The closed-loop poles lie in $\mathcal{D}$.
- $\|T_{wz}\|_\infty < \gamma$ where $T_{wz}(s)$ denotes the closed-loop transfer function from w to z.

Without loss of generality, it is assumed that $D_{22} = 0$. This amounts to a mere change of variable in the controller matrices and considerably simplifies the formulas.

The LTI controller $K(s)$ can be represented in state-space form by

$$\dot{x}_K(t) = A_Kx_K(t) + B_Ky(t)$$
$$u(t) = C_Kx_K(t) + D_Ky(t). \qquad (40)$$

Then $T_{wz}(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}$ with the notation

$$A_{cl} := \begin{pmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{pmatrix}$$
$$B_{cl} := \begin{pmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{pmatrix} \qquad (41)$$
$$C_{cl} := (C_1 + D_{12}D_KC_2, \ D_{12}C_K)$$
$$D_{cl} := D_{11} + D_{12}D_KD_{21}. \qquad (42)$$

Remark 4.1: Assume that $K(s)$ is strictly proper ($D_K = 0$) and that A does not have any fast mode, i.e., any eigenvalue with large real part. Then from the identity

$$\text{Trace}(A_{cl}) = \text{Trace}(A) + \text{Trace}(A_K)$$

we see that A_K has fast modes if and only if A_{cl} has fast modes. Consequently, fast controller dynamics can be prevented by constraining the closed-loop poles, e.g., in a disk centered at the origin and with appropriately small radius. This ability is valuable in the perspective of real-time digital implementation of the controller.

Following the argument of Section III, our specifications are feasible if and only if (24) and (25) hold for some positive definite matrices $X_\mathcal{D}$, X_∞ and some controller matrix $\Omega_K := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$. For convenience, we replace (25) by the equivalent condition

$$\begin{pmatrix} A_{cl}X_\infty + X_\infty A_{cl}^T & B_{cl} & X_\infty C_{cl}^T \\ B_{cl}^T & -\gamma I & D_{cl}^T \\ C_{cl}X_\infty & D_{cl} & -\gamma I \end{pmatrix} < 0 \qquad (43)$$

(this amounts to a mere rescaling of X_∞). As earlier, this problem is not tractable unless we require that the same Lyapunov matrix $X > 0$ satisfy both (24) and (43). We therefore restrict our attention to the following suboptimal formulation of H_∞ synthesis with pole placement:

Find $X > 0$ and a controller $K(s) \equiv \Omega_K$ that satisfy (24)
$$\text{and (43) with } X = X_\mathcal{D} = X_\infty. \qquad (44)$$

An additional difficulty in the output-feedback case is that (24) and (43) now involve nonlinear terms of the form $\mathcal{B}\Omega_K\mathcal{C}X$ that can no longer be removed by the change of variable $L := \Omega_KX$ used in Section III. This has been a long-standing obstacle to the derivation of LMI conditions for multi-objective output-feedback synthesis. Fortunately, these nonlinearities can also be eliminated by some appropriate change of controller variables, albeit a more sophisticated one than in the state-feedback case. This change of variable was introduced in [16] and is the core of our approach. Note that related ideas can be found in [26].

Let $A \in \mathbf{R}^{n \times n}$ and $D_{22} \in \mathbf{R}^{p_2 \times m_2}$, and let k be the controller order ($A_K \in \mathbf{R}^{k \times k}$). As in the state-feedback case, the change of controller variables is implicitly defined in terms of the (unknown) Lyapunov matrix X. Specifically, partition X and its inverse as

$$X = \begin{pmatrix} R & M \\ M^T & U \end{pmatrix}, \qquad X^{-1} = \begin{pmatrix} S & N \\ N^T & V \end{pmatrix},$$
$$R \in \mathbf{R}^{n \times n}, \ S \in \mathbf{R}^{n \times n}. \qquad (45)$$

We define the new controller variables as

$$\begin{cases} \mathcal{B}_K := NB_K + SB_2D_K \\ \mathcal{C}_K := C_KM^T + D_KC_2R \\ \mathcal{A}_K := NA_KM^T + NB_KC_2R + SB_2C_KM^T \\ \qquad \qquad + S(A + B_2D_KC_2)R. \end{cases} \qquad (46)$$

This change of variable has the following properties:

1) $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$ have dimensions $n \times n$, $n \times m_2$, and $p_2 \times n$, respectively.

2) If $M \in \mathbf{R}^{n \times k}$ and $N \in \mathbf{R}^{n \times k}$ have full row rank, then given $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$, and the matrices R, S, M, N, D_K, we can always compute controller matrices A_K, B_K, C_K that satisfy (46). Moreover, A_K, B_K, C_K are uniquely determined when $k = n$, that is, when M and N are square invertible.

The identity $XX^{-1} = I$ together with (45) gives

$$MN^T = I - RS. \qquad (47)$$

Thus, M and N have full row rank when $I - RS$ is invertible. The following technical lemma shows that invertibility of $I - RS$ can be assumed without loss of generality.

Lemma 4.2: Suppose that (44) has a solution $(X, K(s))$. Then the same problem has a solution $(\tilde{X}, \tilde{K}(s))$ with $I - RS$ invertible in the partitioning (45) of $\tilde{X}$ and $\tilde{X}^{-1}$.

Proof: Let k denote the order of the controller $K(s)$. First observe that there is no loss of generality in assuming $k \geq n$. Suppose indeed that $k < n$ and let $\lambda \in \mathcal{D}$. Then another solution of (44) is $(\tilde{X}, \tilde{K}(s))$ where $\tilde{X} := \text{Diag}(X, I_{n-k}) > 0$ and $\tilde{K}(s)$ is the full-order controller with realization

$$\tilde{A}_K := \text{Diag}(A_k, \lambda I_{n-k}), \qquad \tilde{B}_K := \begin{pmatrix} B_K \\ 0 \end{pmatrix}$$
$$\tilde{C}_K := (C_K, 0), \qquad \tilde{D}_K := D_K.$$

Assume $k \geq n$ and partition $X > 0$ as in (45). The Sherman–Morrison matrix inversion formula gives $S = (R -$

$MU^{-1}M^T)^{-1}$ whence

$$I - RS = -MU^{-1}M^T(R - MU^{-1}M^T)^{-1}.$$

Clearly, $I - RS$ is singular if and only if M is row-rank deficient. In such case, perturb M to make it full rank and denote by $\tilde{X}$ the resulting modification of X. Due to their strict nature, the inequalities (24), (43), and $X > 0$ remain satisfied by $\tilde{X}$ and $K(s)$ when the perturbation of M is small enough. Consequently, (44) always admits solutions with $I - RS$ invertible.   ∎

We are now ready to give tractable necessary and sufficient conditions for the solvability of (44). Note that these conditions constitute a complete output-feedback extension of the state-feedback results of Section III.

Theorem 4.3: Let $\mathcal{D}$ be an arbitrary LMI region contained in the open left-half plane and let (6) be its characteristic function. The modified problem (44) is solvable if and only if the following system of LMI's is feasible.

Find $R = R^T \in \mathbf{R}^{n \times n}$, $S = S^T \in \mathbf{R}^{n \times n}$, and matrices $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$, D_K such that

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} > 0 \tag{48}$$

$$\left[\alpha_{kl} \begin{pmatrix} R & I \\ I & S \end{pmatrix} + \beta_{kl}\Phi + \beta_{lk}\Phi^T \right]_{k,l} < 0 \tag{49}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{21}^T \\ \Psi_{21} & \Psi_{22} \end{bmatrix} < 0 \tag{50}$$

with the shorthand notation (and see (53) at the bottom of the page)

$$\Phi := \begin{bmatrix} AR + B_2\mathcal{C}_K & A + B_2 D_K C_2 \\ \mathcal{A}_K & SA + \mathcal{B}_K C_2 \end{bmatrix} \tag{51}$$

$$\Psi_{11} := \begin{bmatrix} AR + RA^T + B_2\mathcal{C}_K + \mathcal{C}_K^T B_2^T & B_1 + B_2 D_K D_{21} \\ (B_1 + B_2 D_K D_{21})^T & -\gamma I \end{bmatrix} \tag{52}$$

$$\Psi_{21} := \begin{bmatrix} \mathcal{A}_K + (A + B_2 D_K C_2)^T & SB_1 + \mathcal{B}_K D_{21} \\ C_1 R + D_{12}\mathcal{C}_K & D_{11} + D_{12} D_K D_{21} \end{bmatrix}. \tag{54}$$

Given any solution of this LMI system:
- Compute via SVD a full-rank factorization $MN^T = I - RS$ of the matrix $I - RS$ (M and N are then square invertible).
- Solve the system of linear equations (46) for $\mathcal{B}_K$, $\mathcal{C}_K$, and $\mathcal{A}_K$ (in this order).
- Set $K(s) := D_K + \mathcal{C}_K(sI - \mathcal{A}_K)^{-1}\mathcal{B}_K$.

Then $K(s)$ is an nth order controller that places the closed-loop poles in $\mathcal{D}$ and such that $\|T_{wz}\|_\infty < \gamma$.

Proof: *(Necessity):* Let $X > 0$ and $K(s)$ be solutions of (44) and partition X as in (45). It is readily verified that X satisfies the identity

$$X\Pi_2 = \Pi_1 \quad \text{with} \quad \Pi_1 := \begin{pmatrix} R & I \\ M^T & 0 \end{pmatrix}$$

$$\Pi_2 := \begin{pmatrix} I & S \\ 0 & N^T \end{pmatrix}. \tag{55}$$

From Lemma 4.2, we can always assume that M and N have full row rank which makes Π_2 a full-column-rank matrix. Pre- and postmultiply the inequality $X > 0$ by Π_2^T and Π_2, respectively. Using (55) and Π_2 full column rank, this yields $\Pi_2^T \Pi_1 = \begin{pmatrix} R & I \\ I & S \end{pmatrix} > 0$. Next, pre- and postmultiply (24) by the block-diagonal matrices $\text{Diag}(\Pi_2^T, \cdots, \Pi_2^T)$ and $\text{Diag}(\Pi_2, \cdots, \Pi_2)$, respectively. Carrying out the matrix products and performing the change of variable (46), this evaluates to the LMI (51). Similarly, the last LMI condition (50) is derived from (43) by pre- and postmultiplication by $\text{diag}(\Pi_2^T, I, I)$ and $\text{diag}(\Pi_2, I, I)$, respectively.

(Sufficiency): Suppose that R, S, $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$, D_K satisfy (48)–(50). Constraint (48) ensures that $I - RS$ is invertible and can be factorized as $I - RS = MN^T$ with M and N square invertible. The matrix Π_2 in (55) is then square invertible as well. Define $X := \Pi_1 \Pi_2^{-1}$, let $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$ be the unique solutions of (46), and consider the nth-order controller $K(s) := D_K + \mathcal{C}_K(sI - \mathcal{A}_K)^{-1}\mathcal{B}_K$. From the necessity part, (48)–(50) are derived from $X > 0$, (24), and (43) by congruence transformations involving Π_2. Since Π_2 is invertible, we can undo these congruences to conclude that X and $K(s)$ solve (44).   ∎

Remark 4.4: Theorem 4.3 provides a simple computational procedure to construct H_∞ controllers that assign the closed-loop poles in $\mathcal{D}$. Note that the variables R, S determine the Lyapunov matrix X by $X = \Pi_1 \Pi_2^{-1}$, while the variables $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$, D_K determine the controller $K(s)$.

Remark 4.5: The LMI conditions of Theorem 4.3 become necessary and sufficient when either the pole constraint or the H_∞ constraint is removed. In particular, Theorem 4.3 provides an exact and tractable solution to the problem of pole assignment in a specified convex region via output feedback.

Remark 4.6: The results of Theorem 4.3 are directly applicable to H_∞ optimization with pole-placement constraints. Specifically, solving the LMI problem

Minimize γ over R, S, $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$, D_K, γ

$$\text{satisfying (48)–(50)} \tag{56}$$

yields the smallest γ for which (44) has solutions. In turn, this gives an upper estimate of the optimal H_∞ performance subject to pole assignment in $\mathcal{D}$. Note that (56) is an LMI problem of type (34).

$$\Psi_{22} := \begin{bmatrix} A^T S + SA + \mathcal{B}_K C_2 + C_2^T \mathcal{B}_K^T & (C_1 + D_{12} D_K C_2)^T \\ C_1 + D_{12} D_K C_2 & -\gamma I \end{bmatrix} \tag{53}$$
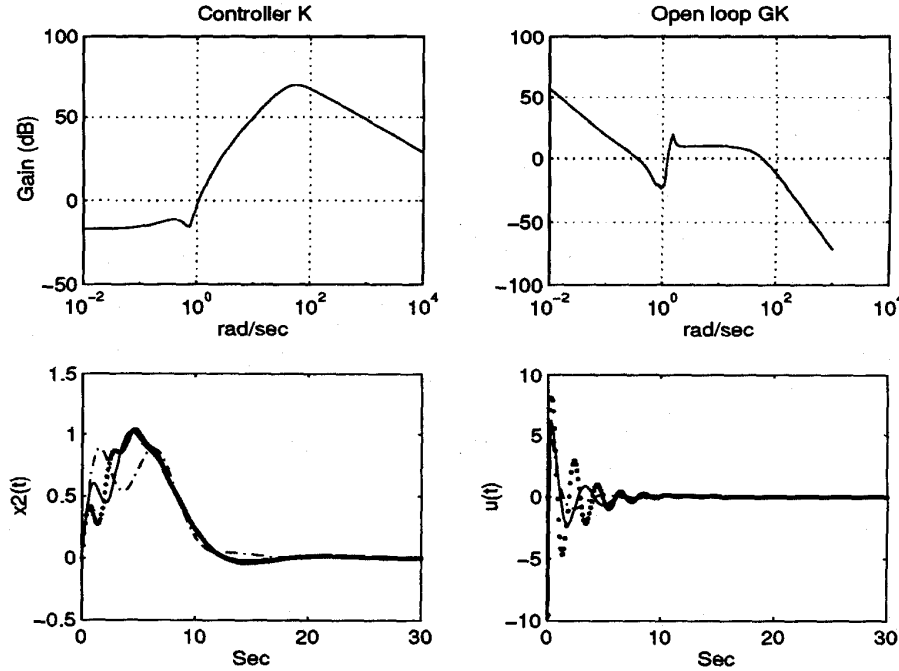
Fig. 3. Bode plots and impulse responses.

Remark 4.7: Using the change of variable (46) and the simplification (28), it is possible to derive suboptimal LMI solutions for a wide variety of multi-objective output-feedback problems. For instance, one can combine H_2 and H_∞ performance, passivity constraints, regional pole assignment, specifications on the settling time and overshoot, etc. [9], [32]. In addition, practical experience indicates that the restriction (28) is not overly conservative.

V. EXAMPLE

This example is adapted from the benchmark problem in [35] and illustrates our LMI approach to output-feedback H_∞ synthesis with regional pole assignment. Consider a two-mass-spring system with

$$\begin{cases} \ddot{x}_1 + k(x_1 - x_2) & = & u \\ \ddot{x}_2 + k(x_2 - x_1) & = & w \\ y & = & x_2 \end{cases} \qquad (57)$$

where w is a disturbance and y is the controlled output. The stiffness k is uncertain and ranges in $[0.5, 2]$. Using output-feedback control $u = K(s)y$, our goal is to minimize the effect of the disturbance w on the output y.

To this end, we seek to minimize the H_∞ norm of the closed-loop transfer from w to $\begin{pmatrix} u \\ \rho y \end{pmatrix}$ while assigning the closed-loop poles in the intersection of the half-plane $x < -\alpha = -0.25$ with the disk of radius $r = 60$ centered at the origin. The parameter ρ is used to compromise between the control effort and the disturbance rejection performance. Meanwhile, the α-stability and disk constraints are used to enforce a settling time of about 15 s for the impulse response and to prevent fast controller dynamics, respectively.

The design was performed for $k = 0.5$. Applying the LMI conditions of Theorem 4.3, we obtained the following controller for $\rho = 0.2$:

$$K(s) = -\frac{(72s + 11.52)((s - 5.58)^2 + 43.8^2)}{(s + 2.1)(s + 28)((s + 36)^2 + 36^2)}.$$

Fig. 3 shows the Bode plots of the controller and open loop as well as the responses $y(t)$ and $u(t)$ for a unit impulse disturbance w. The dashed line corresponds to $k = 0.5$, the solid line to $k = 1$, and the dotted line to $k = 2$. Note that the 15 s settling time is robustly achieved and that the control magnitude $|u(t)|$ does not exceed 10.

VI. CONCLUSIONS

We have introduced a new LMI characterization for general convex subregions of the complex plane and demonstrated its usefulness for H_∞ synthesis with closed-loop pole clustering constraints. In the state-feedback case, a systematic LMI approach to mixed H_2/H_∞ synthesis with pole placement in LMI regions has been presented. These results have then been fully extended to output-feedback H_∞ synthesis with pole assignment in arbitrary LMI regions.

Admittedly with some degree of conservatism, these results offer numerically tractable means of performing multi-objective and/or constrained controller design. This LMI approach has been implemented in the *LMI Control Toolbox* and tested on a variety of problems with satisfactory results.

APPENDIX

Lemma A.1: Let $X \in C^{n \times n}$ be a Hermitian positive definite matrix. Then its real part $\text{Re}\,(X)$ is a real symmetric positive definite matrix.

Proof: From $X = \operatorname{Re}(X) + j \operatorname{Im}(X)$ and $X = X^H$, it follows that $\operatorname{Re}(X)$ is symmetric and $\operatorname{Im}(X)$ is skew-symmetric. As a result, $v^T X v = v^T \operatorname{Re}(X) v$ for any $v \in R^n$, whence $\operatorname{Re}(X) = \operatorname{Re}(X)^T > 0$. ∎

Proof: Proof of Theorem 2.2 (Sufficiency): Let λ be any eigenvalue of A, and let $v \in C^n$ be a nonzero vector such that $v^H A = \lambda v^H$. Using the identity

$$(I_m \otimes v^H) M_{\mathcal{D}}(A, X)(I_m \otimes v) = (v^H X v) f_{\mathcal{D}}(\lambda) \qquad (58)$$

it is immediate that $M_{\mathcal{D}}(A, X) < 0$ and $X > 0$ imply $f_{\mathcal{D}}(\lambda) < 0$, or equivalently $\lambda \in \mathcal{D}$. Hence, A is $\mathcal{D}$-stable when (8) holds.

(Necessity): Suppose that A is $\mathcal{D}$-stable. We must establish the existence of a real matrix $X > 0$ such that $M_{\mathcal{D}}(A, X) < 0$. To this end, it is useful to extend the function $M_{\mathcal{D}}$ to complex matrices $A \in C^{n \times n}$ and $X = X^H \in C^{n \times n}$ as follows:

$$M_{\mathcal{D}}(A, X) := \alpha \otimes X + \beta \otimes (AX) + \beta^T \otimes (AX)^H.$$

First consider the case where A is a diagonal matrix $\Delta = \operatorname{Diag}(\lambda_i)$ with $\lambda_i \in \mathcal{D}$. It is easily verified that

$$M_{\mathcal{D}}(\Delta, I) = U^T \operatorname{Diag}(f_{\mathcal{D}}(\lambda_i)) U \qquad (59)$$

where U is some permutation matrix. Consequently, $M_{\mathcal{D}}(\Delta, X) < 0$ holds for $X := I$.

In the general case, let Δ be the diagonal matrix of the eigenvalues of A (counting multiplicities in the characteristic polynomial). From the previous discussion, we have $M_{\mathcal{D}}(\Delta, I) < 0$. Working with the Jordan canonical form of A, we can construct a sequence of invertible matrices T_k such that $\lim_{k \to \infty} T_k^{-1} A T_k = \Delta$ [for instance, take $T_k = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ if $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$]. Since $M_{\mathcal{D}}(Y, I)$ is a continuous function of Y, it follows that

$$\lim_{k \to \infty} M_{\mathcal{D}}(T_k^{-1} A T_k, I) = M_{\mathcal{D}}(\Delta, I) < 0. \qquad (60)$$

Hence $M_{\mathcal{D}}(T_k^{-1} A T_k, I) < 0$ for k large enough. Pick such a k and let $T := T_k$. Using the identity $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ [18] and the definition (7), it is easily verified that

$$(I_m \otimes T) M_{\mathcal{D}}(T^{-1} A T, I)(I_m \otimes T^H) = M_{\mathcal{D}}(A, TT^H). \qquad (61)$$

Together with $M_{\mathcal{D}}(T^{-1} A T, I) < 0$, this shows that $M_{\mathcal{D}}(A, X) < 0$ for $X := TT^H > 0$.

To complete the proof, observe that even though $X = TT^H$ is not necessarily real, its real part also satisfies $\operatorname{Re}(X) > 0$ and $M_{\mathcal{D}}(A, \operatorname{Re}(X)) = \operatorname{Re}(M_{\mathcal{D}}(A, X)) < 0$ as a consequence of Lemma A.1 and A real. ∎

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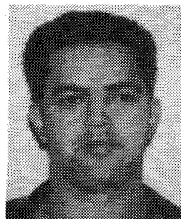
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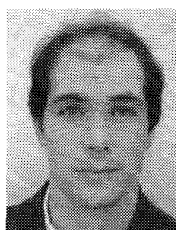
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