Functional Programming

Higher-order functions

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Higher-order functions

- Functions are first-class citizens in Haskell
- A function can be
  - stored in data
  - argument of a (**higher-order**) functions
  - returned from a function

Examples of higher-order functions

Most higher-order functions are polymorphic

```
map    :: (a -> b) -> [a] -> [b]
filter :: (a -> Bool) -> [a] -> [a]
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Example uses

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> map even [1..5]
[False,True,False,True,False]
> filter even [1..10]
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Haskell elides quantifiers in types

$$\begin{aligned} \mathsf{map} &:: \forall a.\forall b.(a \to b) \to [a] \to [b] \\ \mathsf{filter} &:: \forall a.(a \to \mathsf{Bool}) \to [a] \to [a] \end{aligned}$$

Instantiation

Instances of a polymorphic type

Consider

$$\mathrm{filter} \quad :: \quad \forall a.(a \rightarrow \mathrm{Bool}) \rightarrow [a] \rightarrow [a]$$

We may replace (instantiate) a by any type:

$$\begin{aligned} \mathrm{filter} \quad &:: \quad (\mathit{Int} \rightarrow \mathrm{Bool}) \rightarrow [\mathit{Int}] \rightarrow [\mathit{Int}] \\ \mathrm{filter} \quad &:: \quad ([\mathit{Char}] \rightarrow \mathrm{Bool}) \rightarrow [[\mathit{Char}]] \rightarrow [[\mathit{Char}]] \end{aligned}$$

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Instantiation rule

$$\textsc{Inst}\ \frac{e :: \forall a.t}{e :: t[a \mapsto t']}$$

Function types

What's the difference between these types?

```
Int -> Int -> Int
Int -> (Int -> Int)
(Int -> Int) -> Int
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How many arguments?

```
pick 1 = fst
pick 2 = snd
```

Curried functions

Compare these types

```
type T1 = Int -> Int -> Int
type T2 = (Int, Int) -> Int
```

- Both function types take two integers and return one
- T1 takes the arguments one at a time
- T2 takes both arguments as a pair

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Haskell prefers types like `T1`

- A **curried** type, after logician Haskell B. Curry
- Haskell's namesake
- Predefined functions `curry` and `uncurry` map between `T1` and `T2`
- (an isomorphism)

Typing Function Applications

Typing Rule

$$\textsc{App}\;\frac{e_1 :: t_2 \to t_1 \qquad e_2 :: t_2}{(e_1\ e_2) :: t_1}$$

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Designing a higher-order function

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Two functions on lists

```
sum []     = 0
sum (x:xs) = x + sum xs

product []     = 1
product (x:xs) = x * product xs
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The common pattern

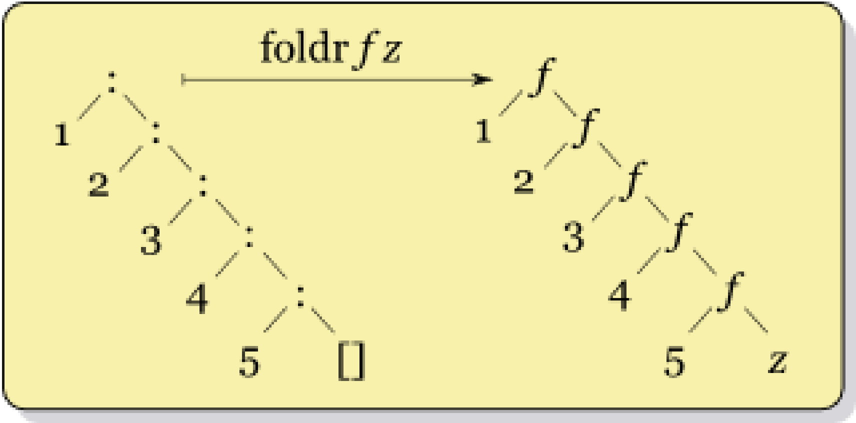
```
g []     = z
g (x:xs) = x `f` g xs
```

with parameters

- `z ::  b` is a value
- `f ::  a -> b -> b` is a combining function

⇒ Represent g by some `foldr' f z`

The foldr function: Intuition



The foldr function

Making the pattern into a higher-order function

Abstracting over value and combining function

```
foldr' f z []     = z
foldr' f z (x:xs) = x `f` foldr' f z xs
```

where

- `z ::` b is a value
- `f ::` `a -> b -> b` is a combining function

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What's the type of foldr'?

```
foldr' :: (a -> b -> b) -> b -> [a] -> b
```

Also known as reduce

map + reduce = MapReduce

Foldr in action

sum and product

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product xs = foldr (*) 1 xs
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more functions

```
or      xs = undefined
and     xs = undefined
concat  xs = undefined

maximum (x:xs) = undefined
```

Foldr puzzles

f1

```
f1 xs = foldr (:) [] xs
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```
f3 xs = foldr snoc [] xs
  where snoc x ys = ys++[x]
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```

f4

```
f4 f xs = foldr fc [] xs
  where fc x ys = f x:ys
```

Transforming functions

Useful operations on functions

- partial application
- operator sections
- function composition
- anonymous functions aka lambda expressions
- eta conversion

Partial application

```
take :: Int -> [a] -> [a]
take 5 :: [a] -> [a]

foldr :: (a -> b -> b) -> b -> [a] -> b
foldr (+) :: Int -> [Int] -> Int
foldr (+) 0 :: [Int] -> Int
```

- Partial application = function application with “too few” arguments
- Result is a function
- Can be used like any other function

Operator sections

```
-- subtraction
(-) :: Int -> Int -> Int
-- subtract one
(- 1) :: Int -> Int
-- subtract from one
(1 -) :: Int -> Int

-- less than 0
(< 0) :: Int -> Bool
-- greater than 0
(0 <) :: Int -> Bool
```

can be done with every infix function

Function composition

Example

- Remove spaces from string as in this example

```
removeSpaces "abc def \n ghi" == "abcdefghi"
```

Function composition

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- In module `Data.Char`

  `isSpace :: Char -> Bool`

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isSpace :: Char -> Bool
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```
removeSpaces xs = filter (not . isSpace) xs
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Function composition

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- Operator "." is function composition defined by

  `(f . g) x = f (g x)`

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- What's the type of `(.)`?

Function composition

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- In module `Data.Char`

  `isSpace :: Char -> Bool`

- yields definition

  `removeSpaces xs = filter (not . isSpace) xs`

- Operator “.” is function composition defined by

  `(f . g) x = f (g x)`

- What’s the type of `(.)`?
- `(.)  ::  (b -> c) -> (a -> b) -> (a -> c)`

Anonymous functions

- Usual function definition

```
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```

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- Often for function used in one place as in

  ```
  f3 xs = foldr snoc [] xs
    where snoc x ys = ys++[x]
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```
f3 xs = foldr snoc [] xs
  where snoc x ys = ys++[x]
```

- Equivalently replace `snoc` by its definition

```
f3' xs = foldr (\ x ys -> ys++[x]) [] xs
```

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f3 xs = foldr snoc [] xs
  where snoc x ys = ys++[x]
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- Equivalently replace `snoc` by its definition

```
f3' xs = foldr (\ x ys -> ys++[x]) [] xs
```

- Shorten more using partial application

```
f3'' xs = foldr (\ x -> (++[x])) [] xs
```

Eta conversion

1. A number of definitions have the form

   `f x = g x`

   where `x` does not occur in `g`
2. In such cases, the formal parameter `x` is redundant:

   `f = g`

   is an equivalent definition.
3. The transformation from (1) to (2) is called **eta reduction**.[1]
4. The typing of an eta-reduced definition is more restricted.

[1] Reverse transformation: **eta expansion**; both directions: **eta conversion**

Examples for eta-reduced definitions

```
sum = foldr (+) 0
product = foldr (*) 1
or = foldr (||) False
and = foldr (&&) True
concat = foldr (++) []
removeSpaces = filter (not . isSpace)
```

Exercises

```
takeLine :: String -> String
-- takeLine "abc\ndef\nghi\n" == "abc"

takeWhile' :: (a -> Bool) -> [a] -> [a]
dropWhile' :: (a -> Bool) -> [a] -> [a]
```

Exercises

```
lines :: String -> [String]
-- lines "abc\ndef\nghi\n" == ["abc", "def", "ghi"]

segments' :: (a -> Bool) -> [a] -> [[a]]

words :: String -> [String]
-- words "abc def  ghi" == ["abc","def","ghi"]
```

Exercises

Define a function that counts how many times words occur in a text and displays each word with its count.

```
wordCounts :: String -> [String]
```

Example use

```
*Main> putStr (wordCounts "hello clouds\nhello sky")
clouds: 1
hello: 2
sky: 1
```

Wrapup

Higher-order functions

- take functions as parameters,
- often have polymorphic types,
- abstract common patterns (map, filter, foldr),
- enable powerful programming techniques (partial application, operator sections, function composition, anonymous functions, eta conversion).