Coupled oscillations

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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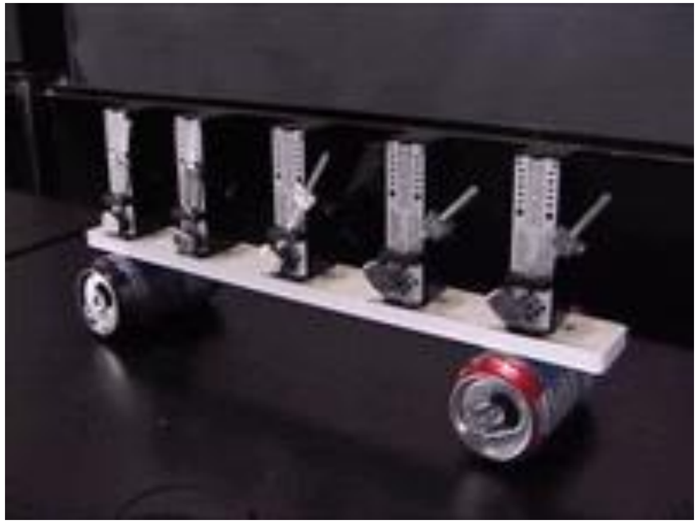
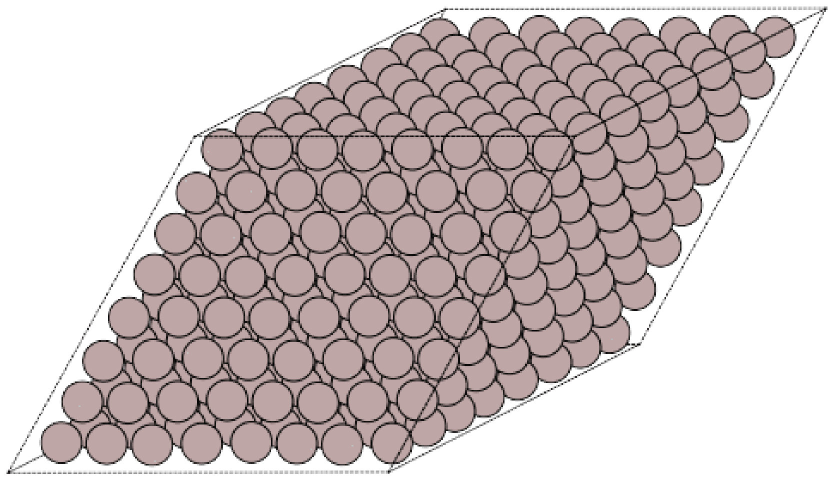


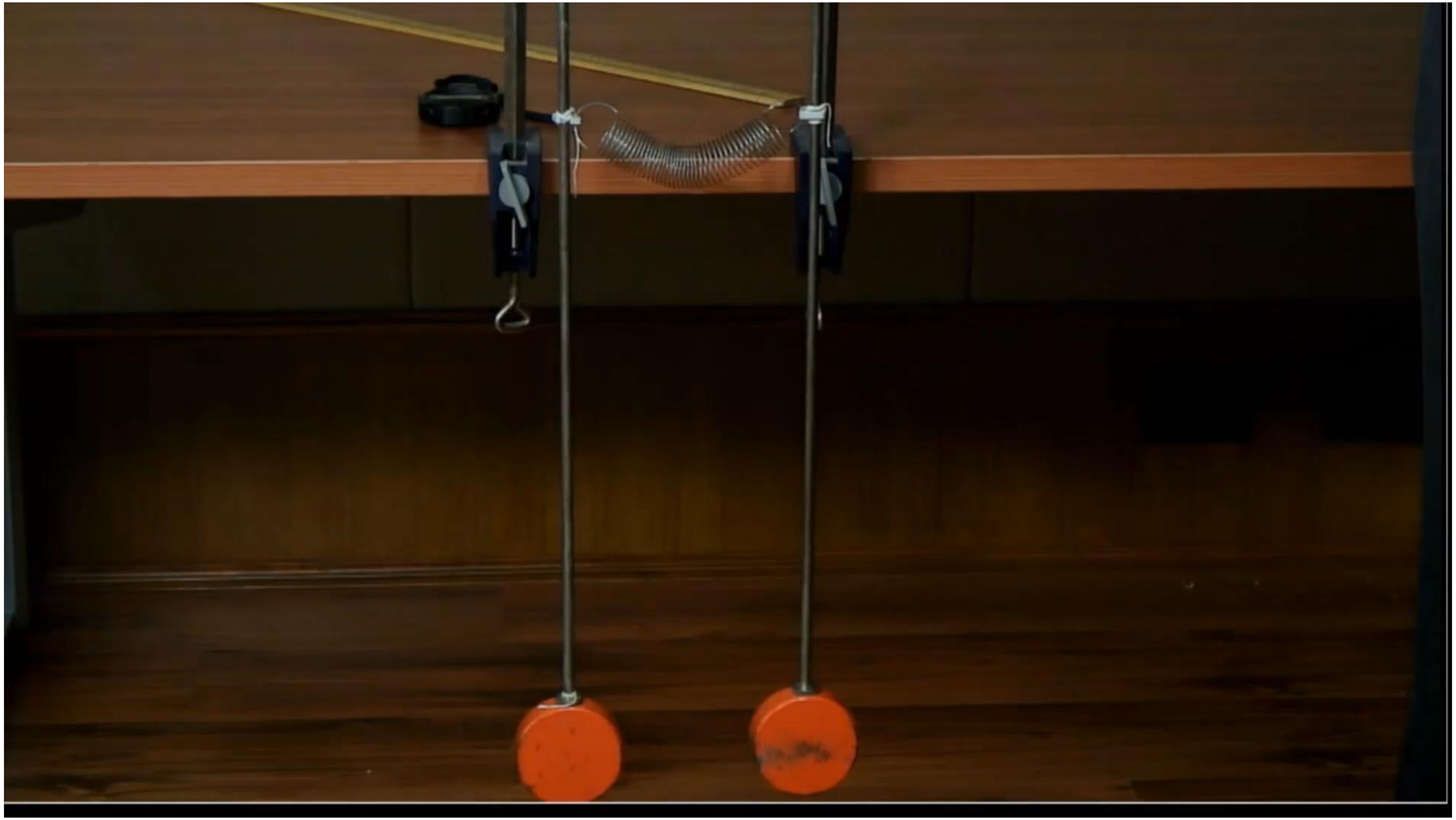
Coupled oscillations: why do we need to study?

- Earth and moon orbits
- Pacemakers for heart
- Radiofrequency and microwave applications (phase shifters and high frequency generation)
- Molecular spectroscopy (useful for pharma and chemical industries)
- To model signal transmission in neurons in our brains
- Artificial neural networks for machine learning and AI

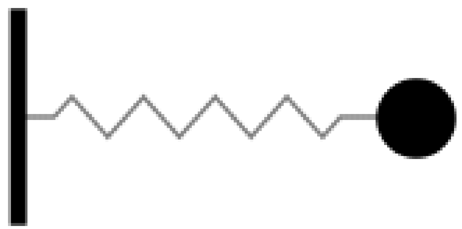


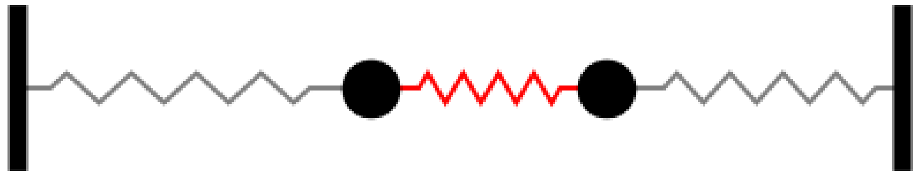
- Atoms as coupled oscillators!
- Gives us a measure of the specific heat and thermal expansion coefficients of solids!

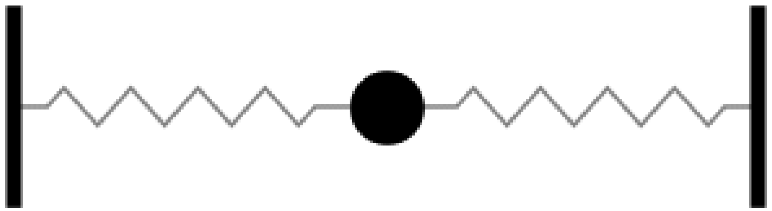
Two pendulums with a spring



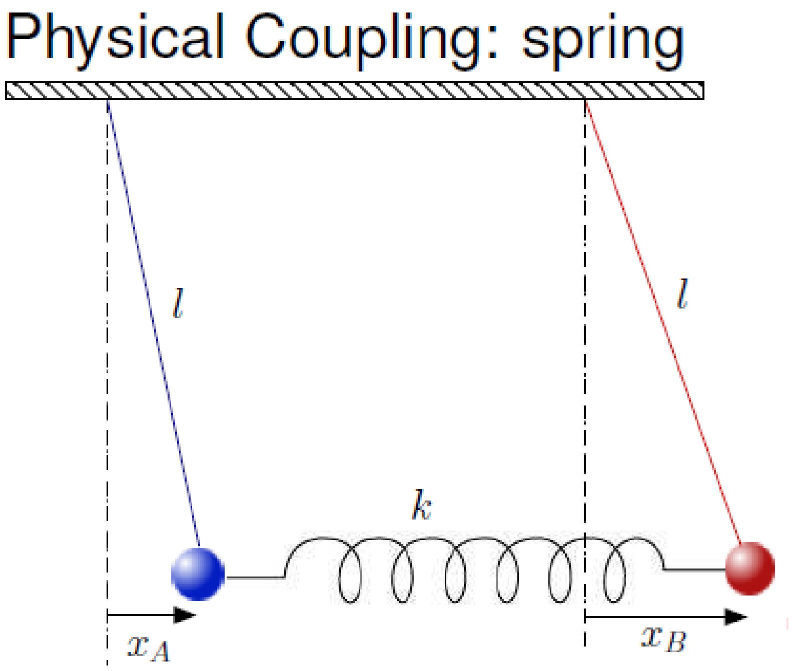
Block spring systems

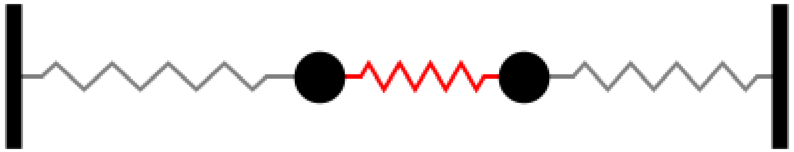






Two coupled oscillators





Analogous block-spring system

- Equation of motion:

$$m\ddot{x}_a = -\frac{mg}{l}x_a + k(x_b - x_a) \qquad (1)$$

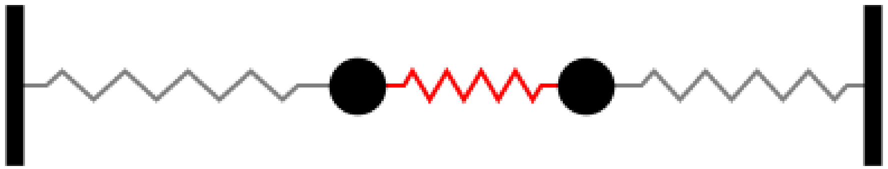
$$m\ddot{x}_b = -\frac{mg}{l}x_b - k(x_b - x_a) \qquad (2)$$

These are coupled equations: how do we uncouple them?

Solution: In terms of NORMAL MODES!

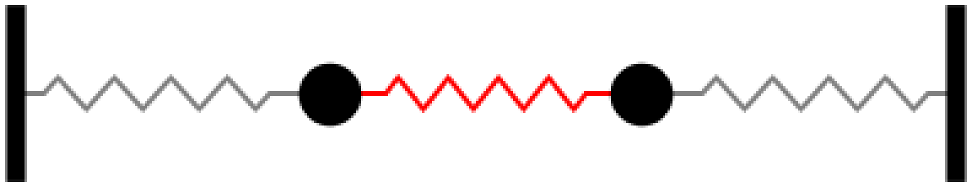
NORMAL MODES are solutions corresponding to SHM with common frequency for the whole system: in a normal mode all parts of the system oscillate with the same frequency and with a fixed phase relationship.

Two coupled oscillators and normal modes

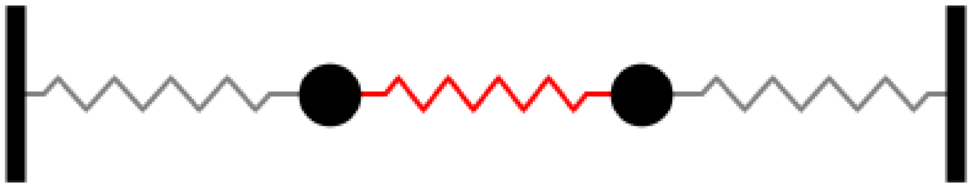


Any motion of the block spring system

It is so important to find the normal modes because any vibration of a system can be represented as a sum of its normal modes!



Normal mode 1



Normal mode 2

Back to the coupled pendulums

- Let us choose new coordinates: $\xi_1 = x_a + x_b$ (3)

$\xi_2 = x_a - x_b$ (4)

Normal coordinates

Adding (1) and (2)

$$m\ddot{\xi}_1 = -\frac{mg}{l}\xi_1 \quad \text{or} \quad \ddot{\xi}_1 + \omega_0^2\xi_1 = 0 \quad \text{where } \omega_0^2 = \frac{g}{l} \qquad (5)$$

Subtracting (2) from (1),

$$\ddot{\xi}_2 + \left(\omega_0^2 + \frac{2k}{m}\right)\xi_2 = 0 \qquad (6)$$

With these new coordinates ("normal coordinates"), equations are now uncoupled!

Solutions:

$$\xi_1 = A\,cos(\omega_1 t + \varphi_1), \quad \text{here} \quad \omega_1 = \omega_0 \qquad (7)$$

$$\xi_2 = B\,cos(\omega_2 t + \varphi_2), \quad \text{here} \quad \omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}} \qquad (8)$$

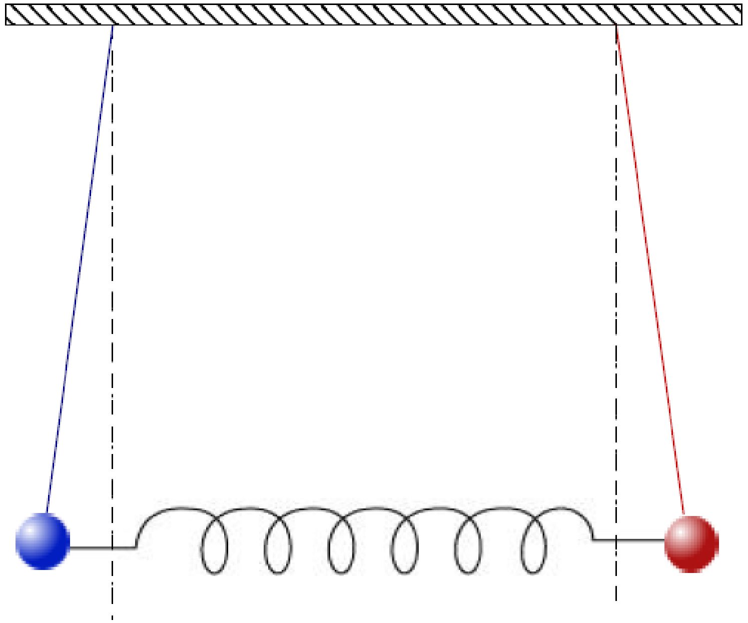
ω_1 and ω_2 are "normal frequencies"

The two normal modes

In the two normal modes, both pendula vibrate with a common frequency either ω_1 or ω_2

MODE 1:

- $\omega_1 = \omega_0$
- $\xi_2 = 0$
- $x_a = x_b$ at all times, so "*in-phase*" oscillation



MODE 2:

- $\omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}}$
- $\xi_1 = 0$
- $x_a = -x_b$ at all times, so "*out of phase*" oscillation

General motion of the coupled system

The general motion of the coupled system is a superposition of the two normal modes!

So let's write $x_a$and x_b in terms of ξ_1 and ξ_2:

$$x_a = \frac{1}{2}(\xi_1 + \xi_2) = \frac{A}{2}\, cos(\omega_1 t + \varphi_1) + \frac{B}{2}\, cos(\omega_2 t + \varphi_2) \qquad (9)$$

$$x_b = \frac{1}{2}(\xi_1 - \xi_2) = \frac{A}{2}\, cos(\omega_1 t + \varphi_1) - \frac{B}{2}\, cos(\omega_2 t + \varphi_2) \qquad (10)$$

Example 1:

Boundary conditions:

$x_a(0) = M$ $\qquad\qquad$ $\dot{x}_a(0) = 0$

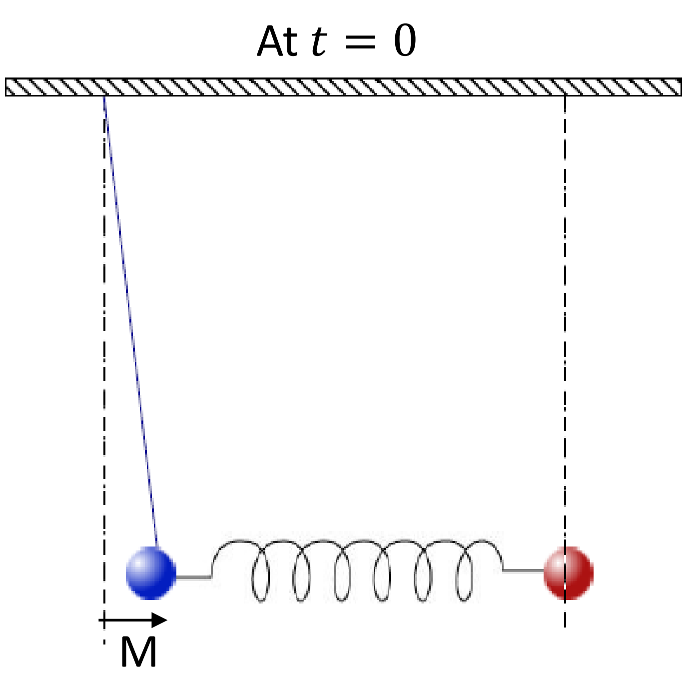
$x_b(0) = 0$ $\qquad\qquad$ $\dot{x}_b(0) = 0$

$Acos(\varphi_1) + Bcos(\varphi_2) = 2M$ (i)

$Acos(\varphi_1) - Bcos(\varphi_2) = 0$ (ii)

$\omega_1 Asin(\varphi_1) + \omega_2 Bsin(\varphi_2) = 0$ (iii)

$\omega_1 Asin(\varphi_1) - \omega_2 Bsin(\varphi_2) = 0$ (iv)



So, adding (iii) and (iv), $sin(\varphi_1) = 0$, so $\varphi_1 = 0, \pi$. Let's take for now $\varphi_1 = 0$. So, $\varphi_2 = 0$ also.

Subtracting (ii) from (i), $2Bcos(\varphi_2) = 2\mathrm{M}$, so since $\varphi_2 = 0$, $B = M$.

From (ii), $Acos(0) - Bcos(0) = 0$, so $A = B = M$

Note: It was the right pendulum that was oscillated at t =0 in the video!

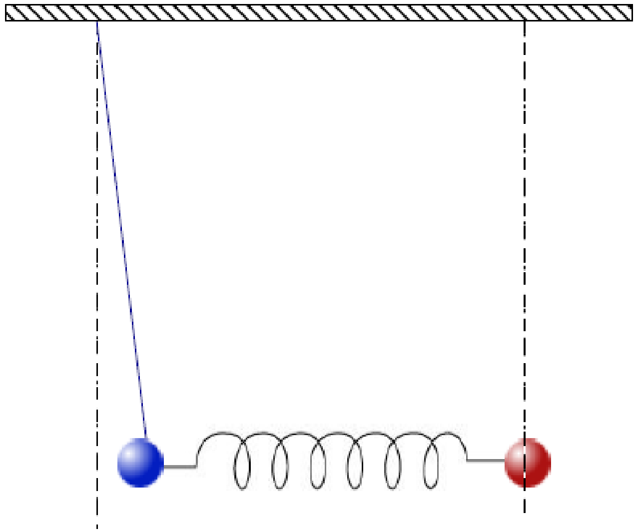
Example 1:

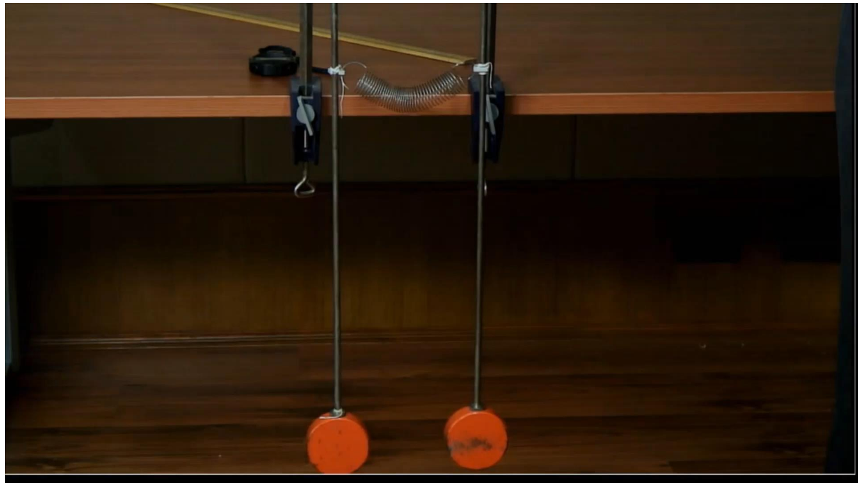
$$x_a(t) = \frac{A}{2}\left(cos(\omega_1 t) + cos(\omega_2 t)\right) = A\ cos\left(\frac{\omega_2 + \omega_1}{2}\right) t\ cos\left(\frac{\omega_2 - \omega_1}{2}\right) t$$

BEATS!

$$x_b(t) = \frac{A}{2}\left(cos(\omega_1 t) - cos(\omega_2 t)\right) = A\ sin\left(\frac{\omega_2 + \omega_1}{2}\right) t\ sin\left(\frac{\omega_2 - \omega_1}{2}\right) t$$

BEATS!

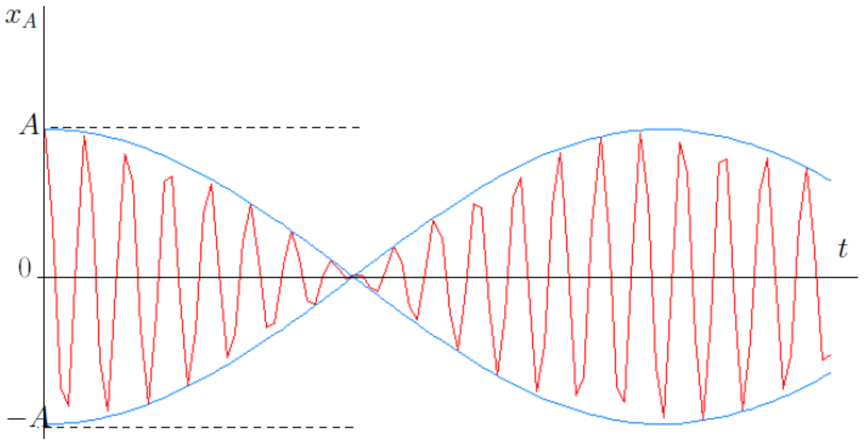




Example 1

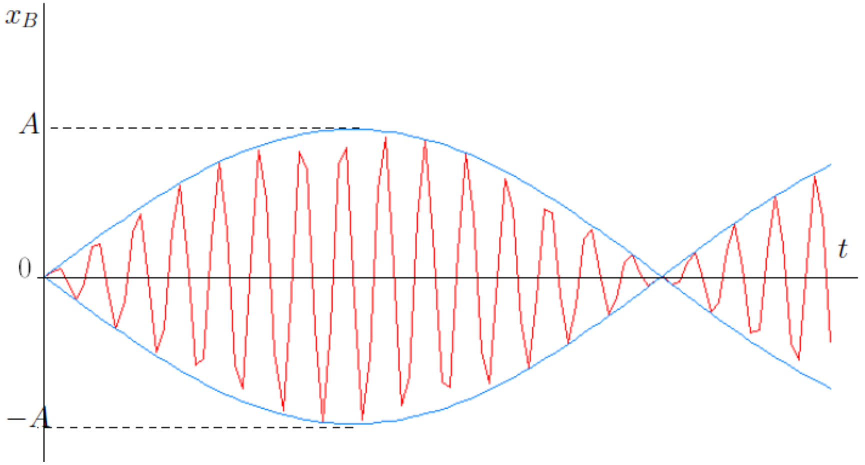
Pendulum A (blue one):

$$x_a(t) = A\,cos\left(\frac{\omega_2 + \omega_1}{2}\right) t\; cos\left(\frac{\omega_2 - \omega_1}{2}\right) t$$



Pendulum B (red one):

$$x_b(t) = A\,sin\left(\frac{\omega_2 + \omega_1}{2}\right) t\; sin\left(\frac{\omega_2 - \omega_1}{2}\right) t$$



How do we find the normal modes for a complex system?

THE GENERAL METHOD:

$$m\ddot{x}_a = -\frac{mg}{l}x_a + k(x_b - x_a) \qquad (1)$$

$$m\ddot{x}_b = -\frac{mg}{l}x_b - k(x_b - x_a) \qquad (2)$$

A) When vibrating in the ith normal mode, with *zero initial velocity*, let the solutions be:

$$x_a = A_i\, cos\omega_i t \qquad\qquad \omega_0^2 = \frac{g}{l}$$

$$x_b = B_i\, cos\omega_i t$$

Hence,

$$\left[-m\omega_i^2 A_i + m\omega_0^2 A_i + k(A_i - B_i)\right]\cos\omega_i t = 0$$

$$\left[-m\omega_i^2 B_i + m\omega_0^2 B_i - k(A_i - B_i)\right]\cos\omega_i t = 0$$

Now we have to solve for ω_i, A_i and B_i. If the above equations are to be satisfied for all values of t then:

How do we find the normal modes for a complex system?

$$\left[-\omega_i^2 + \omega_0^2 + \frac{k}{m}\right] A_i - \frac{k}{m} B_i = 0 \qquad (11a)$$

$$\left[-\omega_i^2 + \omega_0^2 + \frac{k}{m}\right] B_i - \frac{k}{m} A_i = 0 \qquad (11b)$$

From the first equation we get, $\frac{A_i}{B_i} = \frac{\frac{k}{m}}{-\omega_i^2+\omega_0^2+\frac{k}{m}}$ (12a)

From the second equation we get, $\frac{A_i}{B_i} = \frac{-\omega_i^2+\omega_0^2+\frac{k}{m}}{\frac{k}{m}}$ (12b)

We do not know what are the values of A_i and B_i. A trivial solution would be $A_i = 0$ and $B_i = 0$ if equations 11a and 11b are independent.

However if 11a and 11b are dependent, then once A_i is chosen, B_i is fixed.

How do we find the normal modes for a complex system?

$$\frac{\frac{k}{m}}{-\omega_i^2 + \omega_0^2 + \frac{k}{m}} = \frac{-\omega_i^2 + \omega_0^2 + \frac{k}{m}}{\frac{k}{m}}$$

$$\left(\omega_0^2 + \frac{k}{m} - \omega_i^2\right)^2 - \frac{k^2}{m^2} = 0$$

$$\left(\omega_0^2 + \frac{k}{m} - \omega_i^2\right) = \pm\frac{k}{m}$$

So we have two solutions for ω_i. $\omega_1^2 = \omega_0^2$ and $\omega_2^2 = \omega_0^2 + \frac{2k}{m}$

Using $\omega_1^2 = \omega_0^2$ in 12a and 12b $\frac{A_i}{B_i} = 1$.

Using $\omega_2^2 = \omega_0^2 + \frac{2k}{m}$ in 12a and 12b $\frac{A_i}{B_i} = -1$

How do we find the normal modes for a complex system?

We have 2 sets of solutions:

$x_a = Ccos\omega_1 t$

$x_b = C\ cos\omega_1 t$ where $A_i = B_i = C$

and $x_a = Dcos\omega_2 t$

$x_b = -Dcos\omega_2 t$ where $A_i = -B_i = D$

Normal mode coordinates:

If we take $\xi_1 = x_a + x_b = 2Ccos\omega_1 t$

Then $\ddot{\xi}_1 + \omega_1^2 \xi_1 = 0$

If we take $\xi_2 = x_a - x_b = 2Dcos\omega_2 t$

Then $\ddot{\xi}_2 + \omega_1^2 \xi_2 = 0$

Since the differential equations are linear, sum of the solutions is also a solution. So we can write the general solution:

$$\boxed{\begin{aligned} x_a &= Ccos\omega_1 t + Dcos\omega_2 t \\ x_b &= C\ cos\omega_1 t - Dcos\omega_2 t \end{aligned}}$$

How do we find the normal modes for a complex system?

If we consider *non-zero initial velocity*, we write the phase terms in the general solution. Then we assume solutions of the form:

$$x_a = A_i \cos(\omega_i t + \varphi_i)$$

$$x_b = B_i \, cos(\omega_i t + \varphi_i)$$

Note: For the two coupled pendulums,
A) Either same phase for x_a and x_a in mode 1
B) Or phase difference of π in mode 2.

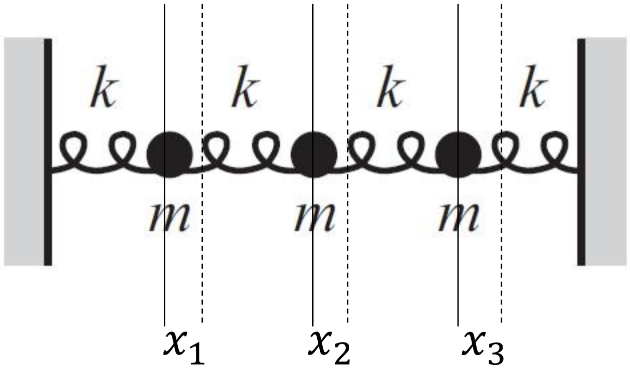
General solution: superposition of both normal modes.

$$x_a = C \cos(\omega_1 t + \varphi_1) + D \cos(\omega_2 t + \varphi_2) \quad (13)$$

$$x_b = C \cos(\omega_1 t + \varphi_1) - D \cos(\omega_2 t + \varphi_2) \quad (14)$$

C,D, $\varphi_1$and φ_2 are determined from initial positions and velocities

An example: 3 masses and 4 springs



$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$$
$$m\ddot{x}_3 = -k(x_3 - x_2) - kx_3$$

The normal modes are not that obvious now!

Let's take the trial solution as before for the i[th] normal mode:

$$x_1 = A_i\, cos\omega_i t$$
$$x_2 = B_i\, cos\omega_i t$$
$$x_3 = C_i\, cos\omega_i t$$

Using this trial solutions and taking $k = m\omega_0^2$ we can write,

$$\left[-m\omega_i^2 A_i + m\omega_0^2 A_i - m\omega_0^2 (B_i - A_i)\right] \cos \omega_i t = 0$$
$$\left[-m\omega_i^2 B_i + m\omega_0^2 (B_i - A_i) - m\omega_0^2 (C_i - B_i)\right] \cos \omega_i t = 0$$
$$\left[-m\omega_i^2 C_i + m\omega_0^2 (C_i - B_i) + m\omega_0^2 C_i\right] \cos \omega_i t = 0$$

An example: 3 masses and 4 springs

$$(-\omega_i^2 + 2\omega_0^2)A_i - \omega_0^2 B_i = 0$$
$$(-\omega_i^2 + 2\omega_0^2)B_i - \omega_0^2 A_i - \omega_0^2 C_i = 0$$
$$(-\omega_i^2 + 2\omega_0^2)C_i - \omega_0^2 B_i = 0$$

So, $\frac{A_i}{B_i} = \frac{\omega_0^2}{-\omega_i^2+2\omega_0^2}$ and $\frac{A_i+C_i}{B_i} = \frac{-\omega_i^2+2\omega_0^2}{\omega_0^2}$ and $\frac{B_i}{C_i} = \frac{-\omega_i^2+2\omega_0^2}{\omega_0^2}$

$$\frac{C_i}{B_i} = \frac{-\omega_i^2+2\omega_0^2}{\omega_0^2} - \frac{A_i}{B_i} = \frac{-\omega_i^2+2\omega_0^2}{\omega_0^2} - \frac{\omega_0^2}{-\omega_i^2+2\omega_0^2} = \frac{\omega_0^2}{-\omega_i^2+2\omega_0^2}$$

So, $\left((-\omega_i^2 + 2\omega_0^2)^2 - \omega_0^4\right)(-\omega_i^2 + 2\omega_0^2) = \omega_0^4(-\omega_i^2 + 2\omega_0^2)$

$$(-\omega_i^2 + 2\omega_0^2)(\omega_i^4 - 4\omega_i^2\omega_0^2 + 2\omega_0^4) = 0$$

So, $\boldsymbol{\omega_1^2 = 2\omega_0^2}$, $\boldsymbol{\omega_2^2 = (2-\sqrt{2})\omega_0^2}$, $\boldsymbol{\omega_3^2 = (2+\sqrt{2})\omega_0^2}$

An example: 3 masses and 4 springs

ω_i^2 has three solutions:

- $\omega_1^2 = 2\omega_0^2$ so, $B_i = 0$ and $A_i = -C_i$

  $\xi_1 = x_3 - x_1$

- $\omega_2^2 = (2 - \sqrt{2})\omega_0^2$ so, $B_i = \sqrt{2}A_i$ and $A_i = C_i$

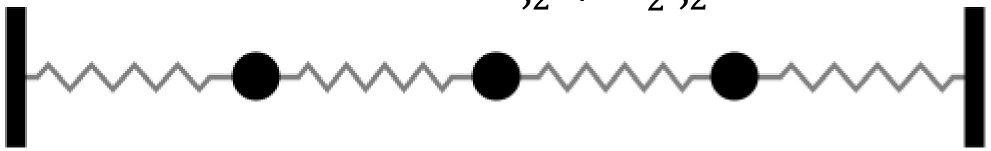
  $\xi_2 = x_3 + \sqrt{2}x_2 + x_1$

- $\omega_3^2 = (2 + \sqrt{2})\omega_0^2$ so, $B_i = -\sqrt{2}A_i$ and $A_i = C_i$

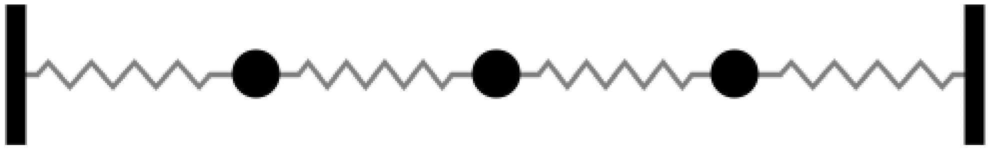
  $\xi_3 = x_3 - \sqrt{2}x_2 + x_1$

medium $\ddot{\xi}_1 + \omega_1^2\xi_1 = 0$

slow $\ddot{\xi}_2 + \omega_2^2\xi_2 = 0$



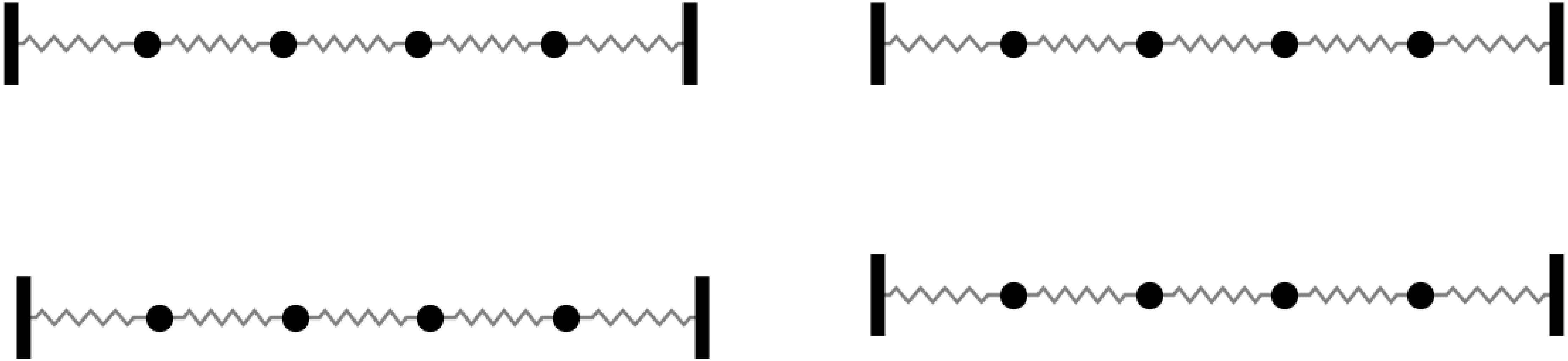
fast $\ddot{\xi}_3 + \omega_3^2\xi_3 = 0$



Animation from acs.psu.edu/drussell

More complex systems: 4 masses and 5 springs

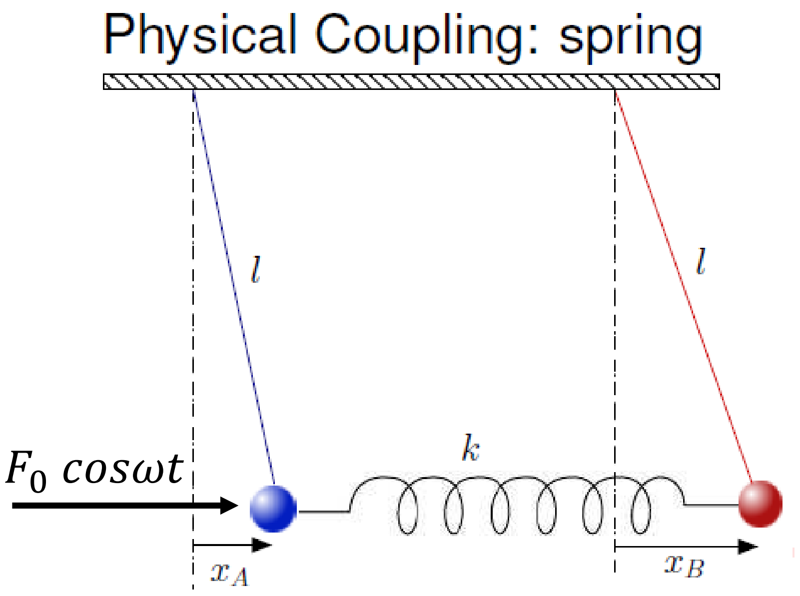
4 normal modes for 4 masses and 5 springs!



Animation from
acs.psu.edu/drussell

Forced vibrations of a coupled system

Pendulum A is driven by a periodic force $F = F_0\, cos\omega t$, no damping in the system.



$$\ddot{x}_A + \left(\omega_0^2 + \frac{k}{m}\right) x_A - \frac{k}{m} x_B = \frac{F_0}{m} cos\omega t \qquad (15)$$

$$\ddot{x}_B + \left(\omega_0^2 + \frac{k}{m}\right) x_B - \frac{k}{m} x_A = 0 \qquad (\text{here } \omega_0^2 = \frac{g}{l}) \qquad (16)$$

In terms of the normal coordinates, $\xi_1 = x_A + x_B$ and $\xi_2 = x_A - x_B$

Adding (15) and (16),

$$\ddot{\xi}_1 + \omega_1^2 \xi_1 = \frac{F_0}{m} cos\omega t \qquad (17)$$

Subtracting (15) and (16),

$$\ddot{\xi}_2 + \omega_2^2 \xi_2 = \frac{F_0}{m} cos\omega t \qquad (18)$$

Here, $\omega_1 = \omega_0 \;\; and \;\; \omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}}$

Forced vibrations of a coupled system

- Problem *translates into forced oscillations of two uncoupled oscillators with coordinates* ξ_1 *and* ξ_2

$$\xi_1 = \frac{F_0}{m}\frac{1}{\omega_1^2-\omega^2}\,cos\omega t \qquad (19)$$

$$\xi_2 = \frac{F_0}{m}\frac{1}{\omega_2^2-\omega^2}\,cos\omega t \qquad (20)$$

Steady state solutions

For the pendulum masses,

$$x_A = \frac{1}{2}(\xi_1 + \xi_2) = \frac{F_0}{2m}\frac{\omega_2^2+\omega_1^2-2\omega^2}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}cos\omega t = \frac{F_0}{m}\frac{\omega_0^2+\frac{k}{m}-\omega^2}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}cos\omega t = A\cos\omega t \quad (21)$$

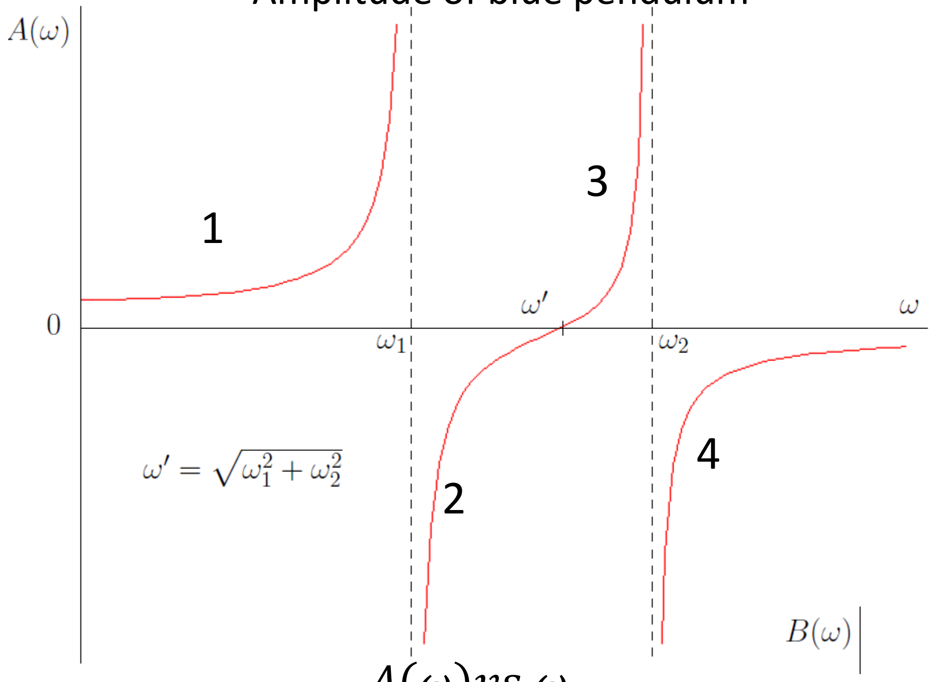
$$x_B = \frac{1}{2}(\xi_1 - \xi_2) = \frac{F_0}{2m}\frac{\omega_2^2-\omega_1^2}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}cos\omega t = \frac{F_0}{m}\frac{\frac{k}{m}}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)}cos\omega t = B\cos\omega t \quad (22)$$

$$\boxed{A(\omega) = \frac{F_0}{m}\frac{\left(\omega_0^2+\frac{k}{m}\right)-\omega^2}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)} \quad (23) \qquad B(\omega) = \frac{F_0}{m}\frac{\frac{k}{m}}{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)} \quad (24)}$$

Resonance at two frequencies: ω_1 and ω_2. So we can use this to experimentally determine the normal mode frequencies!

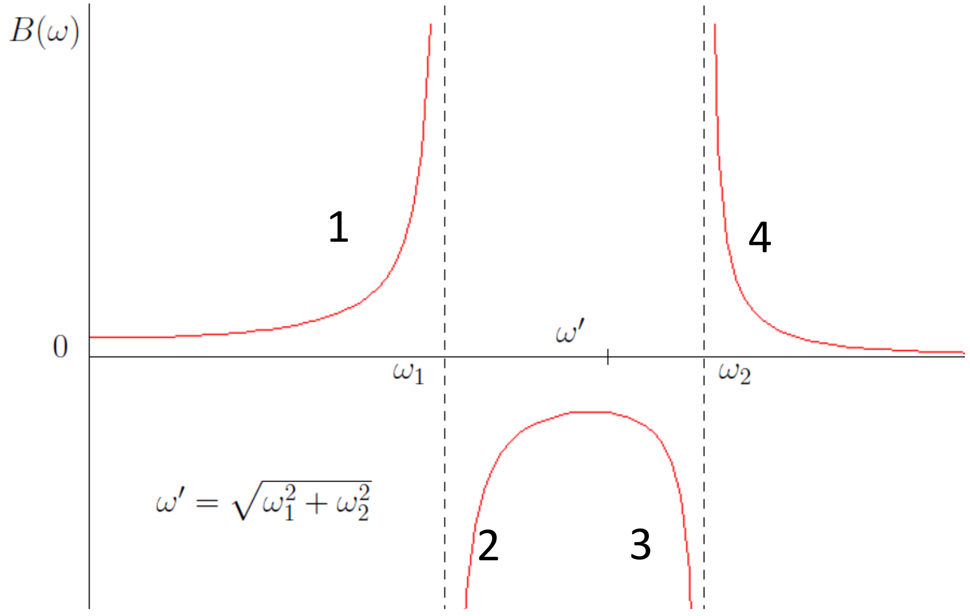
Forced vibrations of a coupled system

Amplitude of blue pendulum

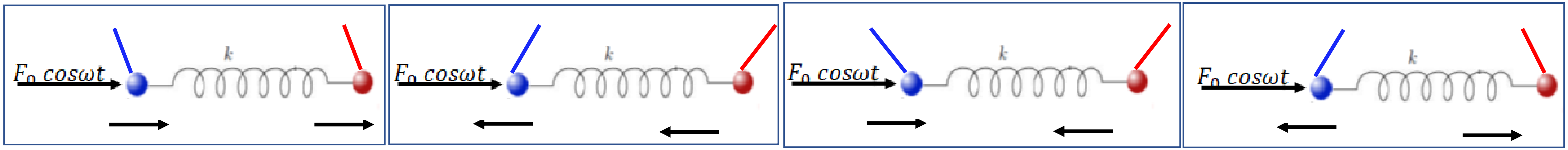


$A(\omega) vs\ \omega$

Amplitude of red pendulum



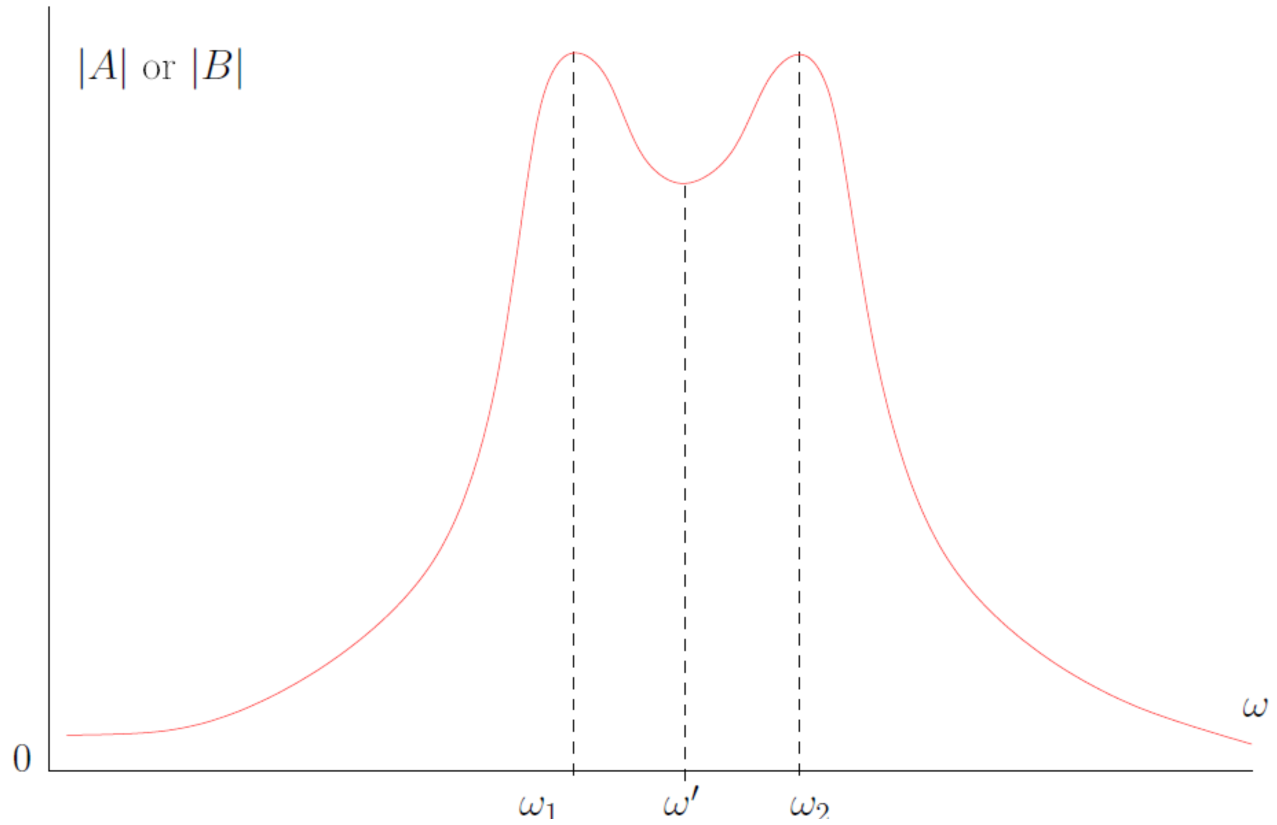
$B(\omega) vs\ \omega$



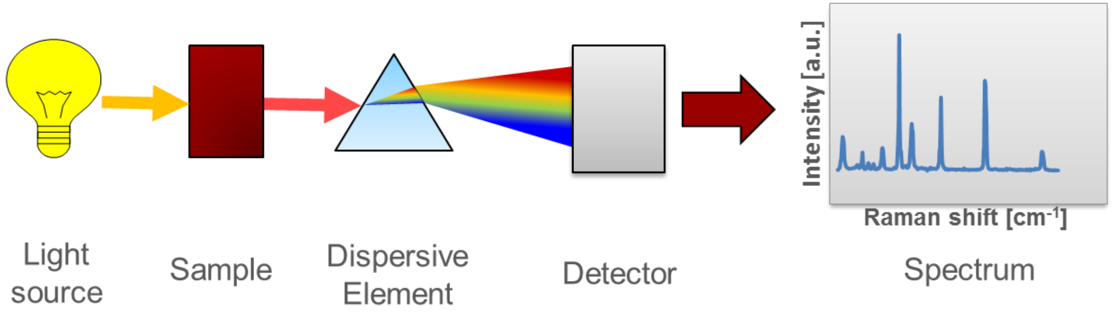
1 2 3 4

Forced vibrations of a coupled system

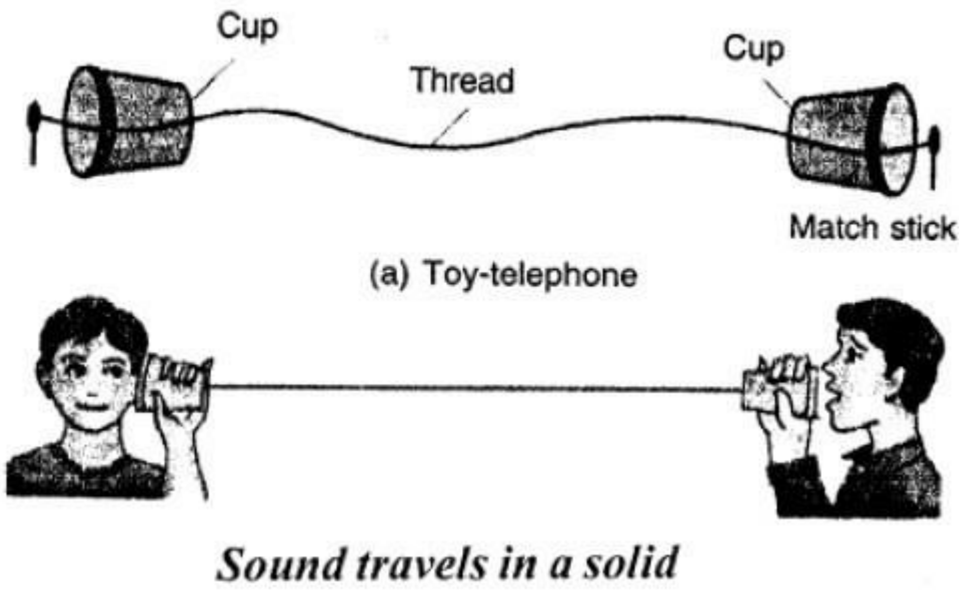
When damping is considered:



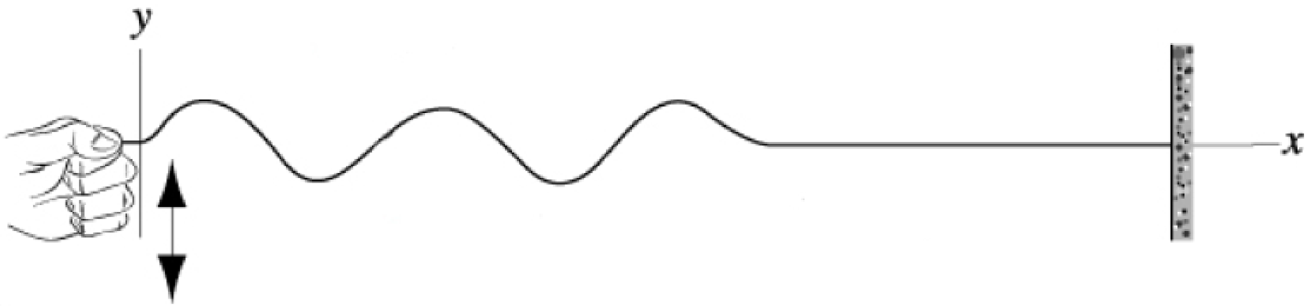
Forced coupled oscillators: Two examples



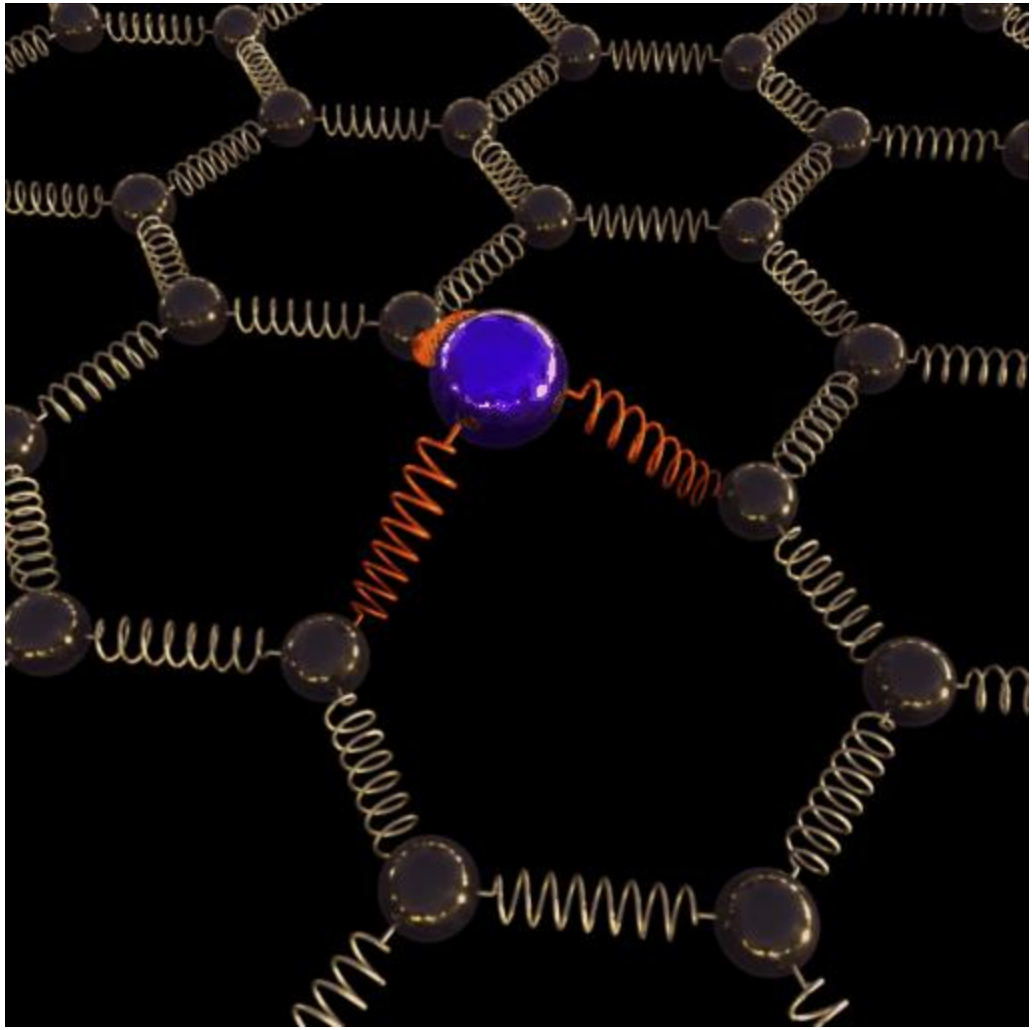
Light passing through a solid: Infrared and Raman spectroscopy



Sound passing through a solid: Acoustic waves



Shaking a string attached to a wall



A silicon atom in a graphene sheet excited by an electron beam

Hage *et al.*, Science 367 1124 (2020)

Extras: Matrix method

How do we find the normal modes for a complex system?

$$\left[-\omega_i^2 + \omega_0^2 + \frac{k}{m}\right] A_i - \frac{k}{m} B_i = 0$$

$$\left[-\omega_i^2 + \omega_0^2 + \frac{k}{m}\right] B_i - \frac{k}{m} A_i = 0$$

Writing in terms of a 'matrix':

$$\underbrace{\begin{pmatrix} \omega_0^2 + \frac{k}{m} - \omega_i^2 & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} - \omega_i^2 \end{pmatrix}}_{\text{2 x 2 matrix}} \underbrace{\begin{pmatrix} A_i \\ B_i \end{pmatrix}}_{\text{2 x 1 matrix}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\text{Null matrix}} \qquad (11)$$

From theory of linear equations, non-trivial solution is possible only if the 'determinant' of the coefficients is zero (Cramer's rule):

$$\begin{vmatrix} \omega_0^2 + \frac{k}{m} - \omega_i^2 & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} - \omega_i^2 \end{vmatrix} = 0 \qquad (12)$$

How do we find the normal modes for a complex system?

$$\begin{vmatrix} \omega_0^2 + \frac{k}{m} - \omega_i^2 & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} - \omega_i^2 \end{vmatrix} = 0 \qquad (12)$$

So,

$$\left(\omega_0^2 + \frac{k}{m} - \omega_i^2\right)^2 - \frac{k^2}{m^2} = 0$$

$$\left(\omega_0^2 + \frac{k}{m} - \omega_i^2\right) = \pm\frac{k}{m}$$

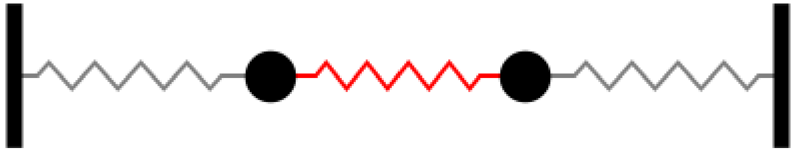
Solving this we get solutions for ω_i. They are:

$\omega_1^2 = \omega_0^2$ and $\omega_2^2 = \omega_0^2 + \frac{2k}{m}$ These are called "Eigenvalues"

$\begin{pmatrix} A_i \\ B_i \end{pmatrix}$ is called an "Eigenvector".

How do we find the normal modes for a complex system?

For $\boldsymbol{\omega_1^2 = \omega_0^2}$, let's put this in equation (11).



$$\begin{pmatrix} \omega_0^2 + \frac{k}{m} - \omega_0^2 & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} - \omega_0^2 \end{pmatrix} \begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this we can write two equations: $\frac{k}{m} A_i - \frac{k}{m} B_i = 0$ and $-\frac{k}{m} A_i + \frac{k}{m} B_i = 0$. Both gives $A_i = B_i = C$.

So, $x_a = A_i \, cos\omega_i t = Ccos\omega_1 t$
$x_b = B_i \, cos\omega_i t = Ccos\omega_1 t$

Since we do not know the constant C, we can represent the eigenvector $\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The normal coordinate $\xi_1 = x_a + x_b$

How do we find the normal modes for a complex system?

For $\boldsymbol{\omega_2^2 = \omega_0^2 + \frac{2k}{m}}$, let's put this in equation (11).

$$\begin{pmatrix} \omega_0^2 + \frac{k}{m} - \omega_0^2 - \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \omega_0^2 + \frac{k}{m} - \omega_0^2 - \frac{2k}{m} \end{pmatrix} \begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this we can write two equations: $-\frac{k}{m}A_i - \frac{k}{m}B_i = 0$ and $-\frac{k}{m}A_i - \frac{k}{m}B_i = 0$. Both gives $A_i = -B_i = D$.

So, $x_a = A_i\, cos\omega_i t = Dcos\omega_2 t$
$x_b = B_i\, cos\omega_i t = -Dcos\omega_2 t$

As before, we can write: $\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and

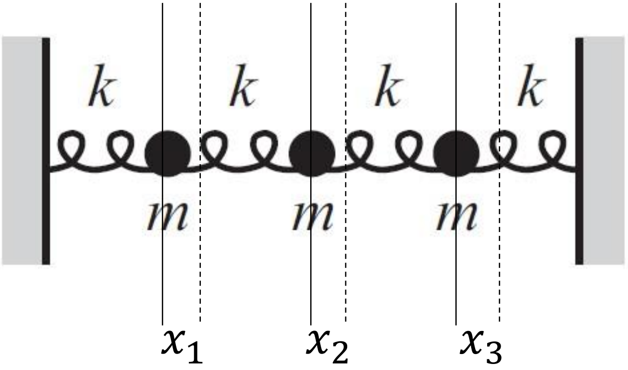
Normal coordinate $\xi_2 = x_a - x_b$

Since the differential equations are linear, sum of two solutions, is also a solution. So,

$$x_a = C\cos\omega_1 t + D\cos\omega_2 t$$
$$x_b = C\cos\omega_1 t - D\cos\omega_2 t$$

C and D determined from initial conditions

An example: 3 masses and 4 springs



$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$$
$$m\ddot{x}_3 = -k(x_3 - x_2) - kx_3$$

The normal modes are not that obvious now! So let's use the determinant method.

Let's take the trial solution as before for the ith normal mode:

$x_1 = A_i\, cos\omega_i t$
$x_2 = B_i\, cos\omega_i t$
$x_3 = C_i\, cos\omega_i t$

Using this trial solutions and taking $k = m\omega_0^2$ we can write,

$$\left[-m\omega_i^2 A_i + m\omega_0^2 A_i - m\omega_0^2(B_i - A_i)\right]\cos\omega_i t = 0$$
$$\left[-m\omega_i^2 B_i + m\omega_0^2(B_i - A_i) - m\omega_0^2(C_i - B_i)\right]\cos\omega_i t = 0$$
$$\left[-m\omega_i^2 C_i + m\omega_0^2(C_i - B_i) + m\omega_0^2 C_i\right]\cos\omega_i t = 0$$

An example: 3 masses and 4 springs

$$\begin{pmatrix} -\omega_i^2 + 2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega_i^2 + 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega_i^2 + 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_i \\ B_i \\ C_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As before, following Cramer's rule, A_i, B_i and C_i will have non trivial solution if:

$$\begin{vmatrix} -\omega_i^2 + 2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega_i^2 + 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega_i^2 + 2\omega_0^2 \end{vmatrix} = 0$$

Therefore,

$$(-\omega_i^2 + 2\omega_0^2)\left((-\omega_i^2 + 2\omega_0^2)^2 - \omega_0^4\right) + \omega_0^2\left(-\omega_0^2(-\omega_i^2 + 2\omega_0^2)\right) = 0$$

$$(-\omega_i^2 + 2\omega_0^2)\left(\omega_i^4 - 4\omega_i^2\omega_0^2 + 2\omega_0^4\right) = 0$$

An example: 3 masses and 4 springs

ω_i^2 has three solutions:

- $\omega_1^2 = 2\omega_0^2$ $\quad \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

  $\xi_1 = x_3 - x_1$

- $\omega_2^2 = (2 - \sqrt{2})\omega_0^2$ $\quad \begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

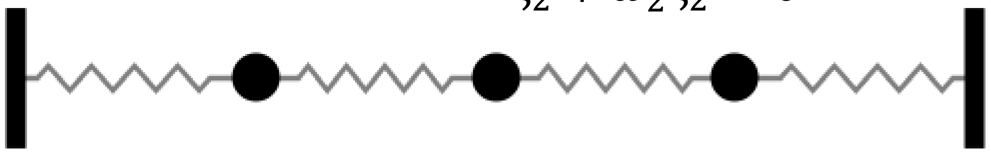
  $\xi_2 = x_3 + \sqrt{2}x_2 + x_1$

- $\omega_3^2 = (2 + \sqrt{2})\omega_0^2$ $\quad \begin{pmatrix} A_3 \\ B_3 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

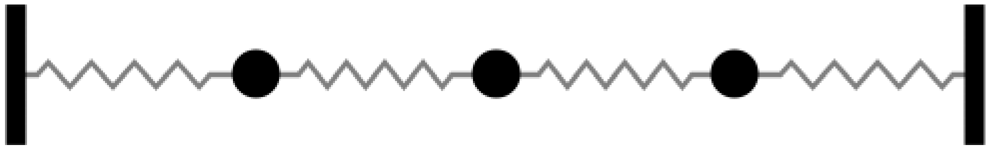
  $\xi_2 = x_3 - \sqrt{2}x_2 + x_1$

medium $\quad \ddot{\xi}_1 + \omega_1^2 \xi_1 = 0$

slow $\quad \ddot{\xi}_2 + \omega_2^2 \xi_2 = 0$



fast $\quad \ddot{\xi}_3 + \omega_3^2 \xi_3 = 0$



Animation from
acs.psu.edu/drussell