Waves

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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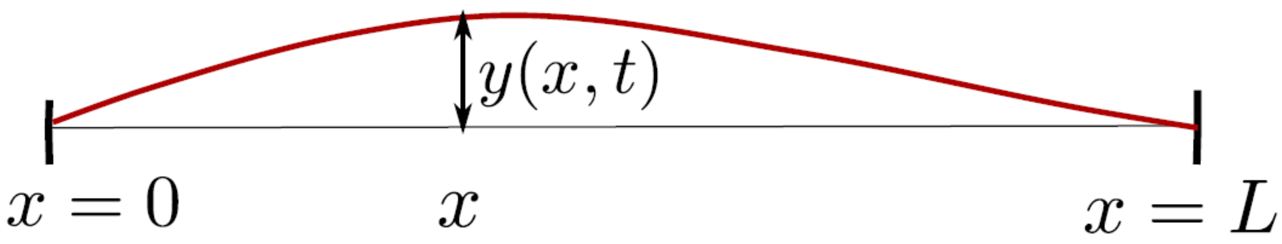
Waves

- Transport of vibrations in space and time
- Vibrations of continuous mediums: strings, membranes, solids
- Waves with no material medium: electromagnetic waves, gravitational waves
- Transverse or longitudinal
- A large number of branches of science deals with some kind of wave:
  Thermodynamics (infrared and microwaves-EM waves), Quantum mechanics (Schrodinger waves), Optics (EM waves), acoustics (sound waves), electronics and communication (EM waves), Astronomy (gravitational waves), geology (seismic waves), oceanography (tidal waves), medical diagnostics (ultrasonic waves) and so on..
- Waves transfer energy and momentum

Let’s begin with a simple string



- Normal mode: $y_n(x,t) = C_n \sin\frac{xn\pi}{L} cos\omega_n t \quad (n = 1,2,3, \dots \dots.)$
- Normal frequency: $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$
- We can write: $y_n(x,t) = C_n \sin\frac{xn\pi}{L} cos\omega_n t = \frac{1}{2} C_n \left(sin\left(\frac{n\pi x}{L} + \omega_n t\right) + sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right)$

Let's begin with a simple string

- We can write: $y_n(x,t) = \frac{1}{2}C_n\left(sin\left[\frac{n\pi}{L}\left(x + \sqrt{\frac{T}{\mu}}t\right)\right] + sin\left[\frac{n\pi}{L}\left(x - \sqrt{\frac{T}{\mu}}t\right)\right]\right)$

- Taking $\lambda = \frac{2L}{n}$ and $v = \sqrt{\frac{T}{\mu}}$

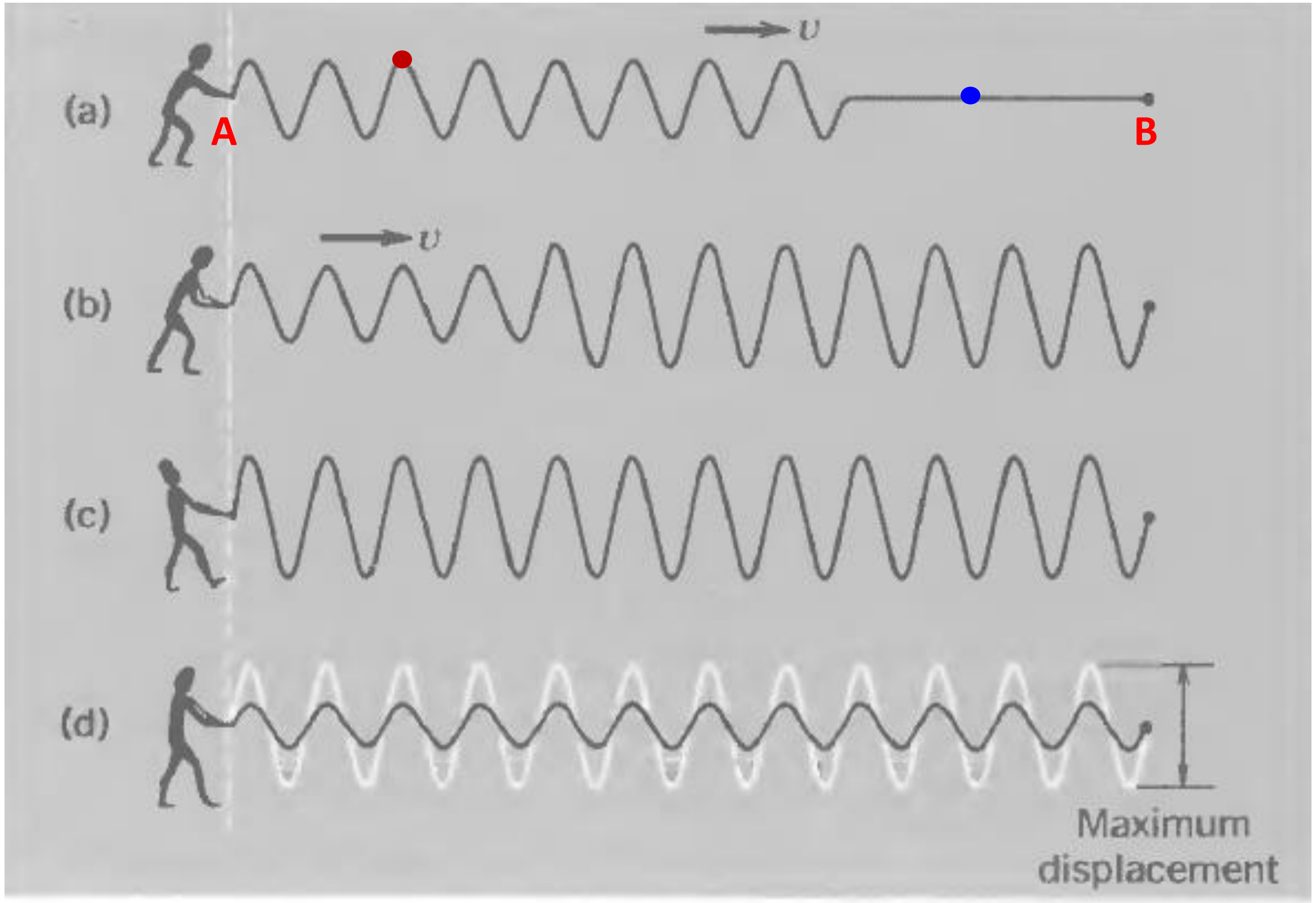
$$y_n(x,t) = \frac{1}{2}C_n\left(sin\left[\frac{2\pi}{\lambda}(x + vt)\right] + sin\left[\frac{2\pi}{\lambda}(x - vt)\right]\right) \qquad (1)$$

- Let's look at the first term: $y_{n1}(x,t) = \frac{1}{2}C_n sin\left[\frac{2\pi}{\lambda}(x + vt)\right]$

  At a later instant of time, the displacement will be

$$y_{n1}(x + \Delta x, t + \Delta t) = \frac{1}{2}C_n sin\left[\frac{2\pi}{\lambda}[(x + \Delta x) + v(t + \Delta t)]\right]$$

Let’s begin with a simple string



Introducing an oscillation at $t = 0$ which travels to end B with velocity v

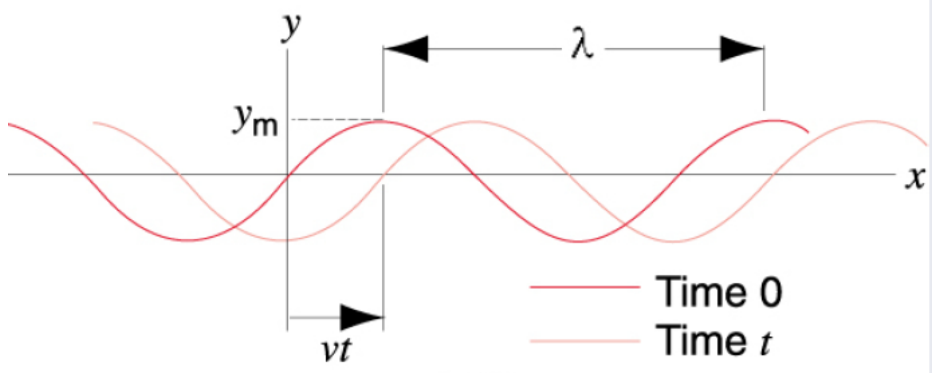
Reflected wave travelling back from end B to end A

Stationary/standing wave is produced at $t + \Delta t$. End A is held fixed now.

The stationary wave at a later time $t1$

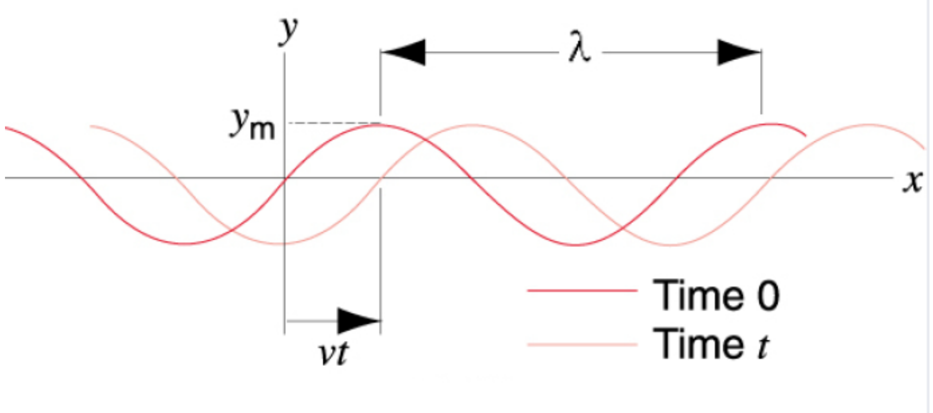
Let's begin with a simple string

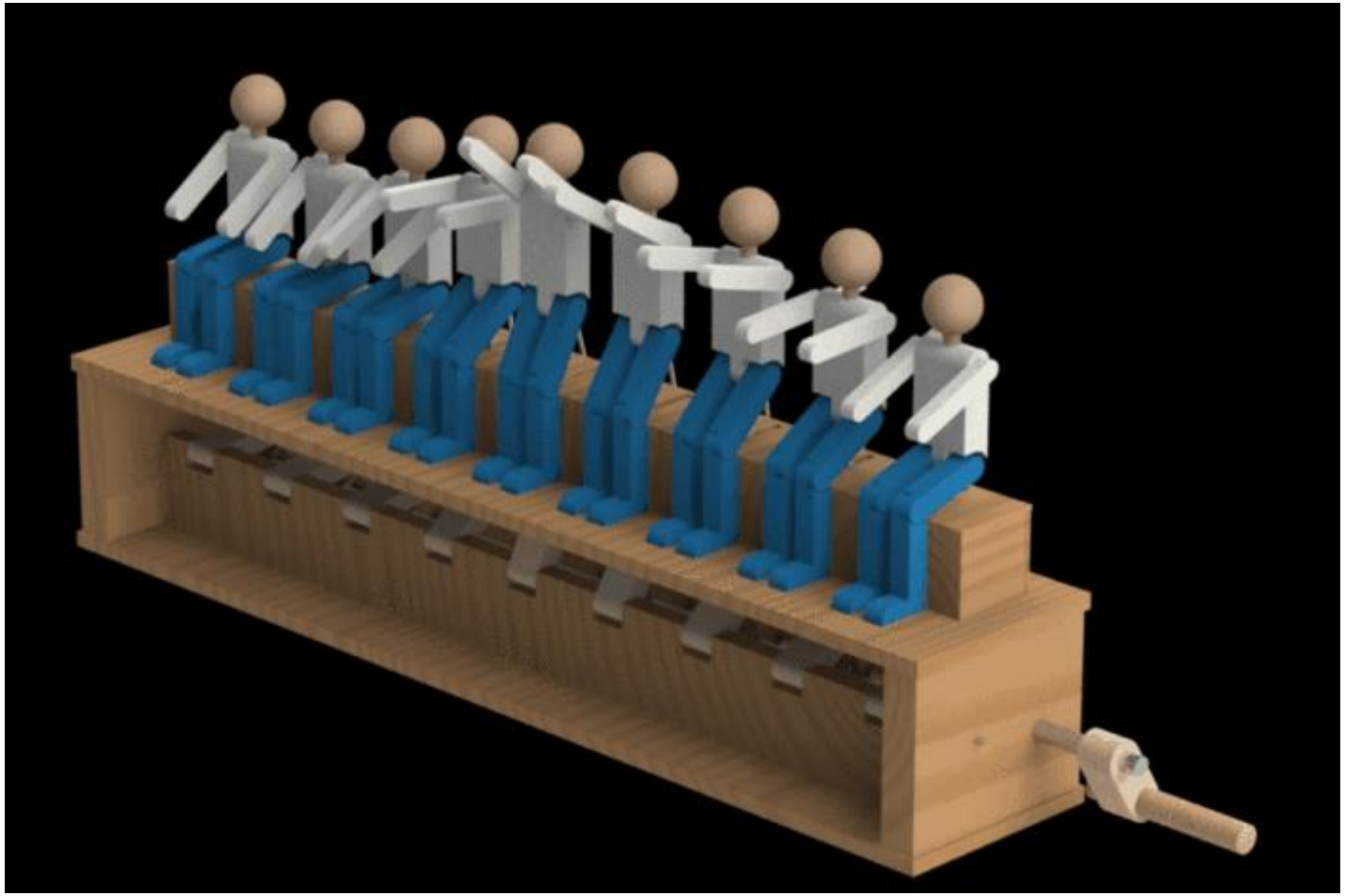
- Now can we find a situation such that $y_{n1}(x + \Delta x, t + \Delta t)$ and $y_{n1}(x, t)$ are equal?
- $y_{n1}(x + \Delta x, t + \Delta t) = y_{n1}(x, t)$
  $sin\left[\frac{2\pi}{\lambda}[(x + \Delta x) + v(t + \Delta t)]\right] = sin\left[\frac{2\pi}{\lambda}(x + vt)\right]$



- This can only be possible if: $\Delta x + v\Delta t = 0,$ that is $\frac{\Delta x}{\Delta t} = -v$ *(2)*
- This implies : vibration is transferred in the **negative X-direction** with speed v
- Second term in equation (1) would have given us: the vibration transferred in the **positive X-direction** with speed $v.$ So, $(\Delta x - v\Delta t = 0,$ that is $\frac{\Delta x}{\Delta t} = v\,)$

The Mexican wave





Let’s continue with a simple string

- But these 2 travelling vibrations are two infinite sine waves. We have only a finite string of length L!

- The answer to this discrepancy lies again in the boundary conditions that is inherent to our vibrating string!

$$y(0, t) = 0 \;\; and \;\; y(L, t) = 0$$

That is the two ends of the string are fixed.

- Within these limits, the two waves in the positive and negative X direction will superpose to give a **“Stationary wave”** whose equation is the original equation for the normal mode of a continuous string.

$$y_n(x, t) = C_n \sin\frac{n\pi x}{L} cos\omega_n t \qquad (3)$$

Equation of a stationary wave   Equation of normal mode of a vibrating continuous string

The wave equation

- We know the two travelling wave conditions that are obtained for a vibrating string with both ends fixed.

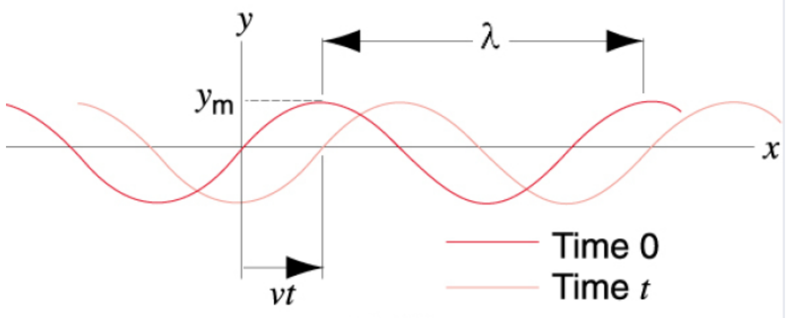
$$\Delta x - v\Delta t = 0$$
$$\Delta x + v\Delta t = 0$$

- Now we need to find an equation that can describe ***any wave with any boundary conditions.***

- For this, we will take only one of those two **infinite sine** waves.
  Let us choose the wave travelling in the **positive X direction**.

  For this, $$\Delta x - v\Delta t = 0$$

$$y(x,t) = A\,sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$



The wave equation

$$y(x,t) = A\,sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

'Partial derivate' of displacement y with respect to x and t

$$\frac{\partial y}{\partial x} = \frac{2\pi}{\lambda}Acos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{4\pi^2}{\lambda^2}Asin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{\partial y}{\partial t} = -\frac{2\pi v}{\lambda}Acos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

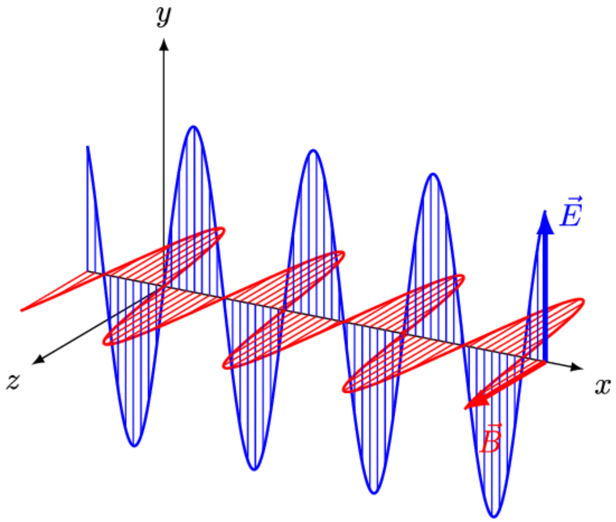
$$\frac{\partial^2 y}{\partial t^2} = -\frac{4\pi^2 v^2}{\lambda^2}Asin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

The Wave equation in 1 dimension

Example1: The equation of Light

In *one dimension*, *in vacuum*, the equation for any electromagnetic wave is:



$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

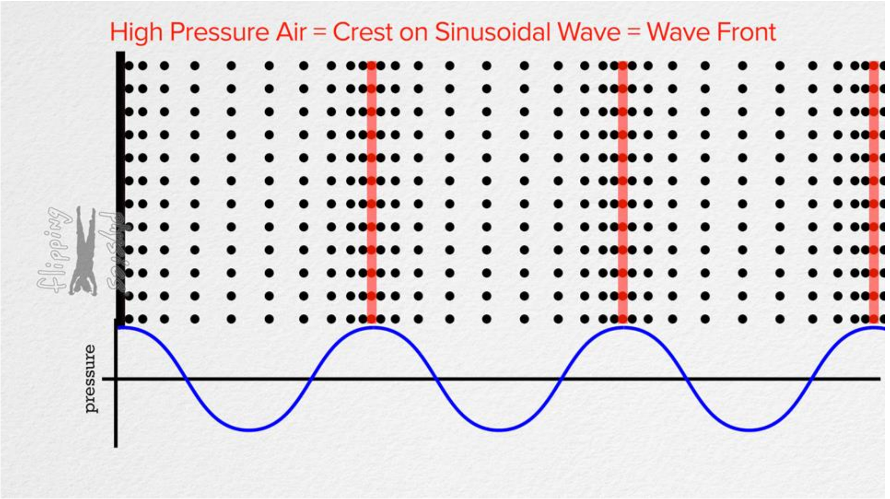
$$\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

- Light is an electromagnetic wave.
- So, E and B are the oscillating electric and magnetic fields propagating as ‘waves’ along X axis in vacuum.
- c is the velocity of light

Example2: The equation of Sound

Upto this, all was about transverse waves but the description of normal modes of longitudinal waves (N coupled oscillators with springs) was very similar.

In *one dimension, the LONGITUDINAL* vibration in an *elastic medium:*

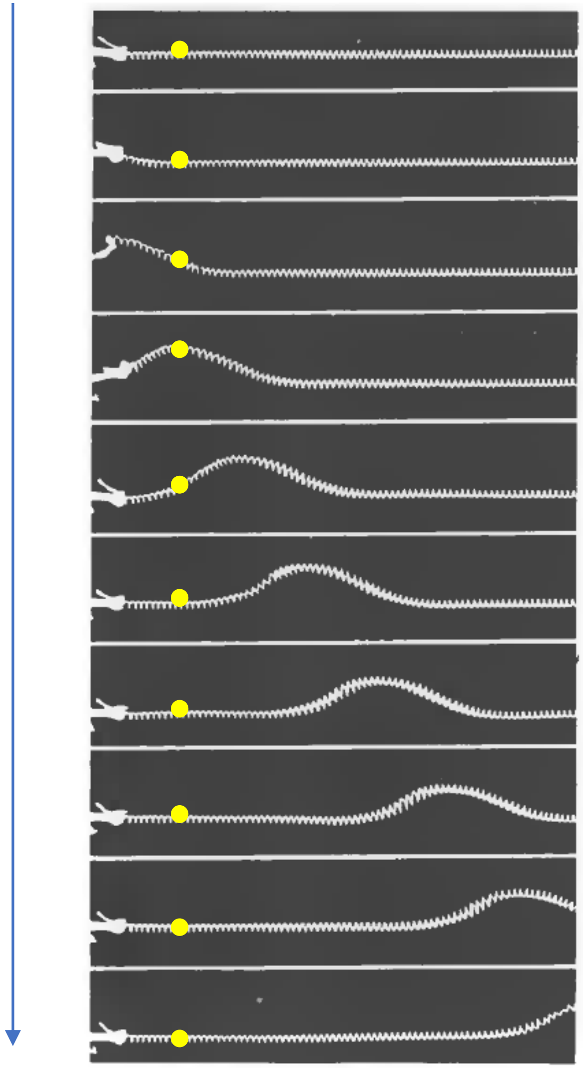


$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$$

- Sound is compressional waves travelling in an elastic medium.
- v is the velocity of sound in that medium.
- $\boldsymbol{v} = \sqrt{\frac{\boldsymbol{Y}}{\boldsymbol{\rho}}}$ in solids,
- $\boldsymbol{v} = \sqrt{\frac{\boldsymbol{K}}{\boldsymbol{\rho}}}$ in fluids,

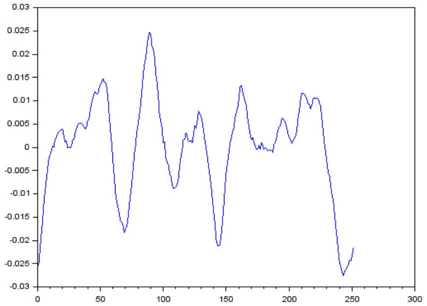
  where Y: Young's modulus, K: Bulk modulus, ρ: density

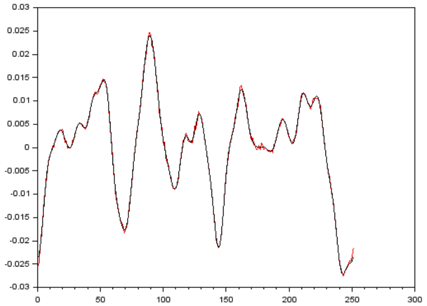
Wave pulses

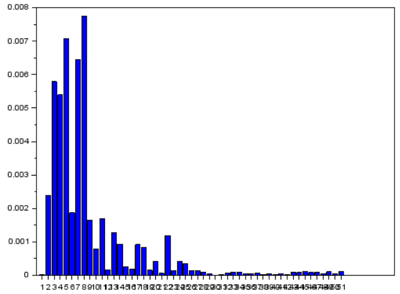


TIME

- Most waves are not infinite sine waves, but are rather ‘wave pulses’





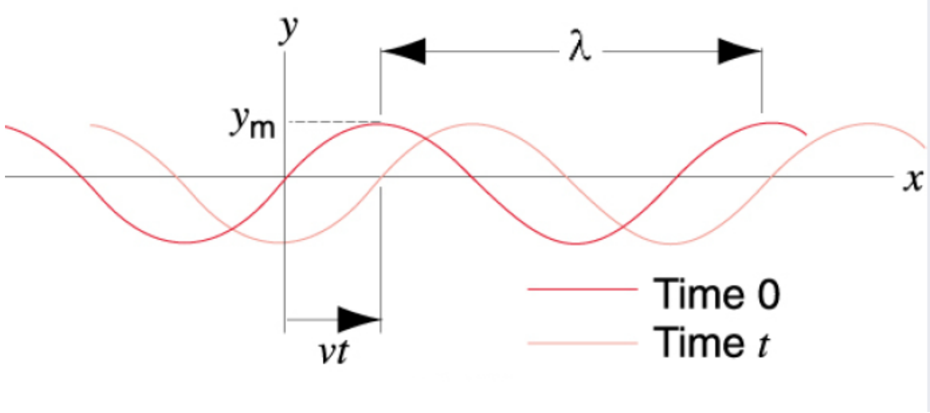


$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)$$

At a given point (fixed x)

$$y(t) = \sum_{n=1}^{\infty} B_n \, cos(\omega_n t - \delta_n)$$

Dispersion of waves



- **"Wave number"** :
  $\boldsymbol{\nu = \frac{1}{\lambda}}$ (wavenumber, more used in spectroscopy)
  $\boldsymbol{k = \frac{2\pi}{\lambda}}$ (angular wavenumber, more used in physics)
- Velocity : v
- Angular frequency: ω

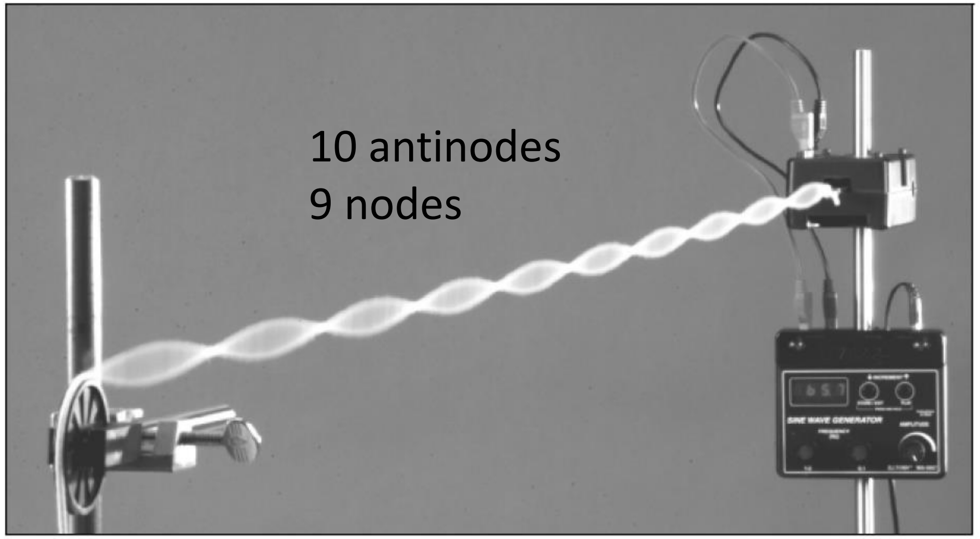
- $\boldsymbol{v = (2\pi) frequency \times \left(\frac{1}{2\pi}\right) wavelength = \frac{\omega}{k}}$

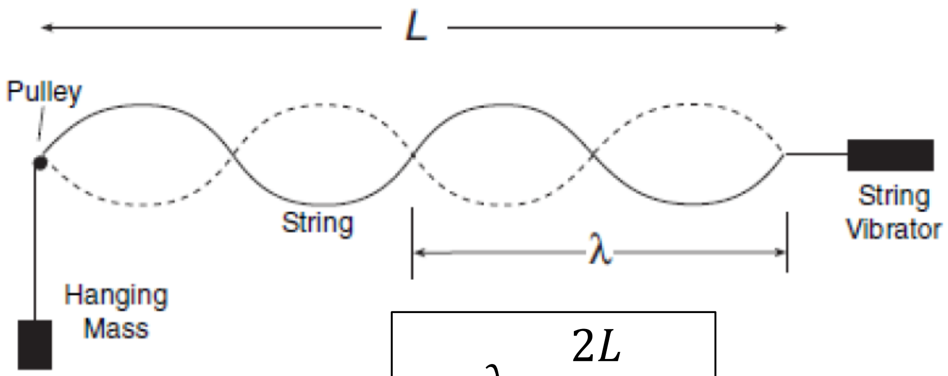
- The equation for a progressive wave in positive x direction: $y(x,t) = A\, sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$

- If it's a vibrating string, then $v = \sqrt{\frac{T}{\mu}}$. Now remember $\omega_n = \frac{n\pi}{L} v$ for a given normal mode.

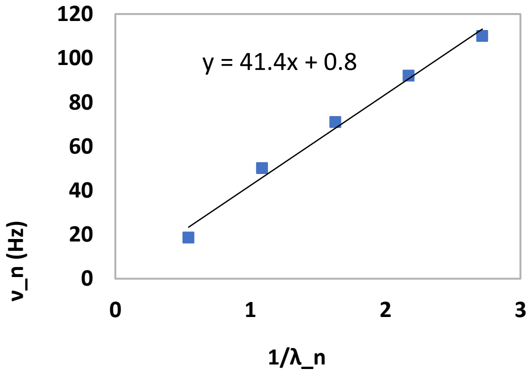
- So a string, under a given tension T , will carry sinusoidal waves of all frequencies with the same speed v!

Melde's experiment





$$\lambda = \frac{2L}{n}$$



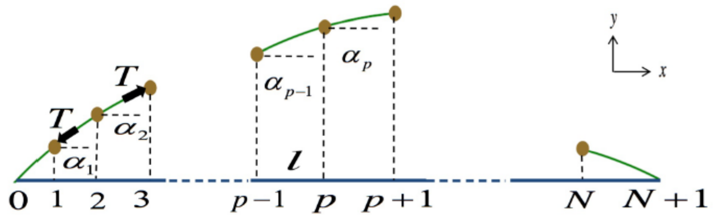
Total mass m (kg)	Number of antinodes n	λ_n=2L/n (m)	λ_n^{-1}(m^{-1})	ν_n(Hz)
0.055	5	0.368	2.717391304	110
	4	0.46	2.173913043	92
	3	0.613333333	1.630434783	71
	2	0.92	1.086956522	50
	1	1.84	0.543478261	18.5

Total mass m (kg)	Tension T=mg (N)	u=$(T/\mu)^{1/2}$ m/s	u (from the slope of the graph) m/s
0.0550	0.5390	41.6978	41.4

Dispersion of waves

- Let's go back to considering our string with discrete N-coupled oscillators for a moment.

$$\omega_n = 2\omega_0\, sin\left[\frac{n\pi}{2(N+1)}\right] \text{ where } \omega_0 = \sqrt{\frac{T}{ml}}$$



- When $n \ll N$, the equation for frequency becomes:

$$\omega_n = 2\omega_0 \frac{n\pi}{2(N+1)} = 2\sqrt{\frac{T}{ml}}.\frac{n\pi}{2(N+1)} = 2\sqrt{\frac{Tl^2}{ml.l^2}}.\frac{n\pi}{2(N+1)} = \sqrt{\frac{T}{m/l}}.\frac{n\pi}{(N+1)l} = \sqrt{\frac{T}{\mu}}.\frac{n\pi}{L} = \frac{n\pi}{L}v = \frac{2n\pi}{n\lambda}v = k_n\, v.$$

Here $(N+1)l = L$ and $v = \sqrt{\frac{T}{\mu}}$.

- **Same** velocity for **all** wavelengths! This is normally the case of a wave on a string. So **NON-DISPERSIVE.**

- These are waves with small frequency that is large wavelengths.
  Or now the wavelengths are much larger than the separation between the masses ($\lambda \gg l$)

Dispersion of waves

- Let's go back to considering our string with discrete N-coupled oscillators for a moment.

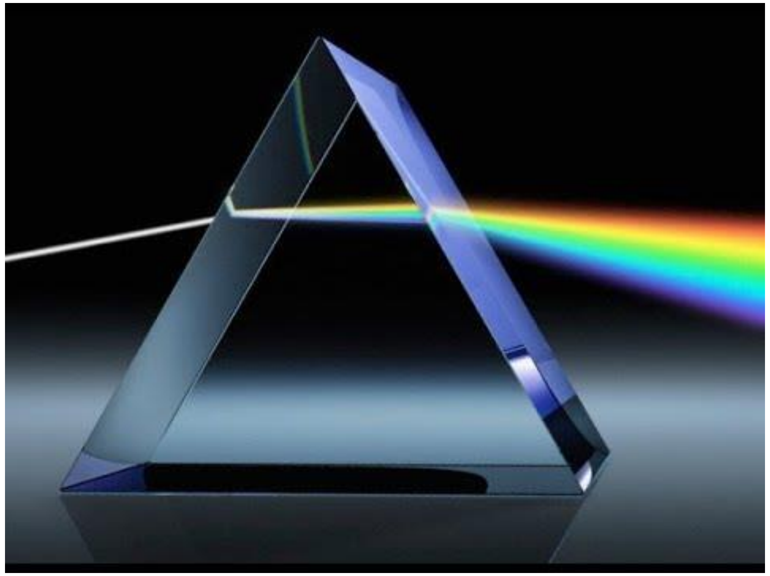
$$\omega_n = 2\omega_0\, sin\left[\frac{n\pi}{2(N+1)}\right] \text{ where } \omega_0 = \sqrt{\frac{T}{ml}}$$

- When $n \sim N$, the equation for frequency becomes $\omega_n = 2\omega_0\, sin\left[\frac{n\pi}{2(N+1)}\right]$, so $\omega_n < \frac{n\pi}{L}v$
- **Velocity** of the waves now depend on their **frequency**!
- Now the wavelengths are comparable to the separation between the masses ($\lambda \sim l$)

Dispersion: The frequency dependence of velocity

An example of dispersion

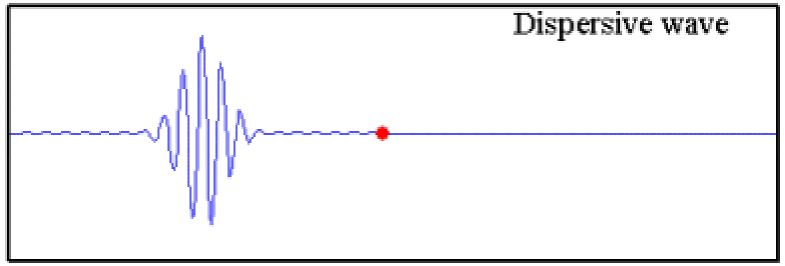
A prism

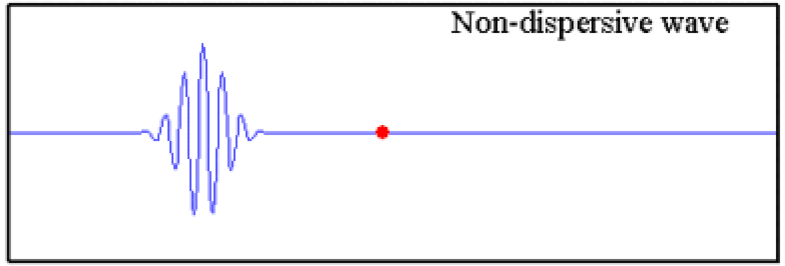


Snell's Law:

$$\mu = \frac{\sin i}{\sin r} = \frac{c}{v}$$

- Velocity is different for each wavelength, that is why we can split the spectrum.





Phase velocity and group velocity

- Let us consider two waves:

$$y_1 = A\,sin\left[\frac{2\pi}{\lambda_1}(x - v_1 t)\right]$$
$$y_2 = A\,sin\left[\frac{2\pi}{\lambda_2}(x - v_2 t)\right]$$

- Writing these in terms of wave number $k = \frac{2\pi}{\lambda}$ and angular frequency ω,

$$y_1 = A\,sin(k_1 x - \omega_1 t)$$
$$y_2 = A\,sin(k_2 x - \omega_2 t)$$

- The superposed wave:

$$y = A\,sin(k_1 x - \omega_1 t) + A\,sin(k_2 x - \omega_2 t)$$
$$= 2Asin\left(\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right) cos\left(\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right)$$

Phase velocity and group velocity

$$y = 2Asin\left(\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right) cos\left(\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right)$$

For comparison, the equation for BEATS: $\mathrm{x} = 2A \sin\frac{(\omega_1+\omega_2)t}{2} \cos\frac{(\omega_1-\omega_2)t}{2}$

- The **higher** frequency (average frequency $\frac{\omega_1+\omega_2}{2}$) travels with velocity $\frac{\omega_1+\omega_2}{k_1+k_2}$
- The **low**er frequency (frequency of the envelope $\frac{\omega_1-\omega_2}{2}$) travels with velocity $\frac{\omega_1-\omega_2}{k_1-k_2}$
- So weird to think of!
- Let's take both the wave numbers to be ***nearly*** equal $k_1 \approx k_2$ as an example. So,

$k_1 - k_2 = \Delta k$ and $\omega_1 - \omega_2 = \Delta\omega$

$\frac{k_1+k_2}{2} = k$ $\qquad$ $\frac{\omega_1+\omega_2}{2} = \omega$

Phase velocity and group velocity

So,

$$y = 2Asin(kx - \omega t)cos\left(\frac{\Delta kx - \Delta\omega t}{2}\right)$$

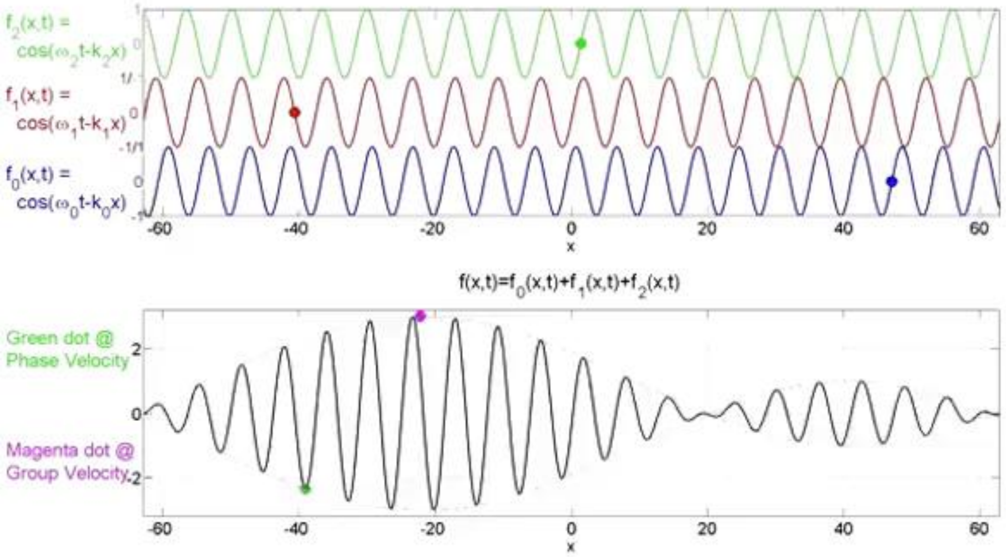
- These are two progressive waves with velocities $\dfrac{\omega}{k}$ and $\dfrac{\Delta\omega}{\Delta k}$

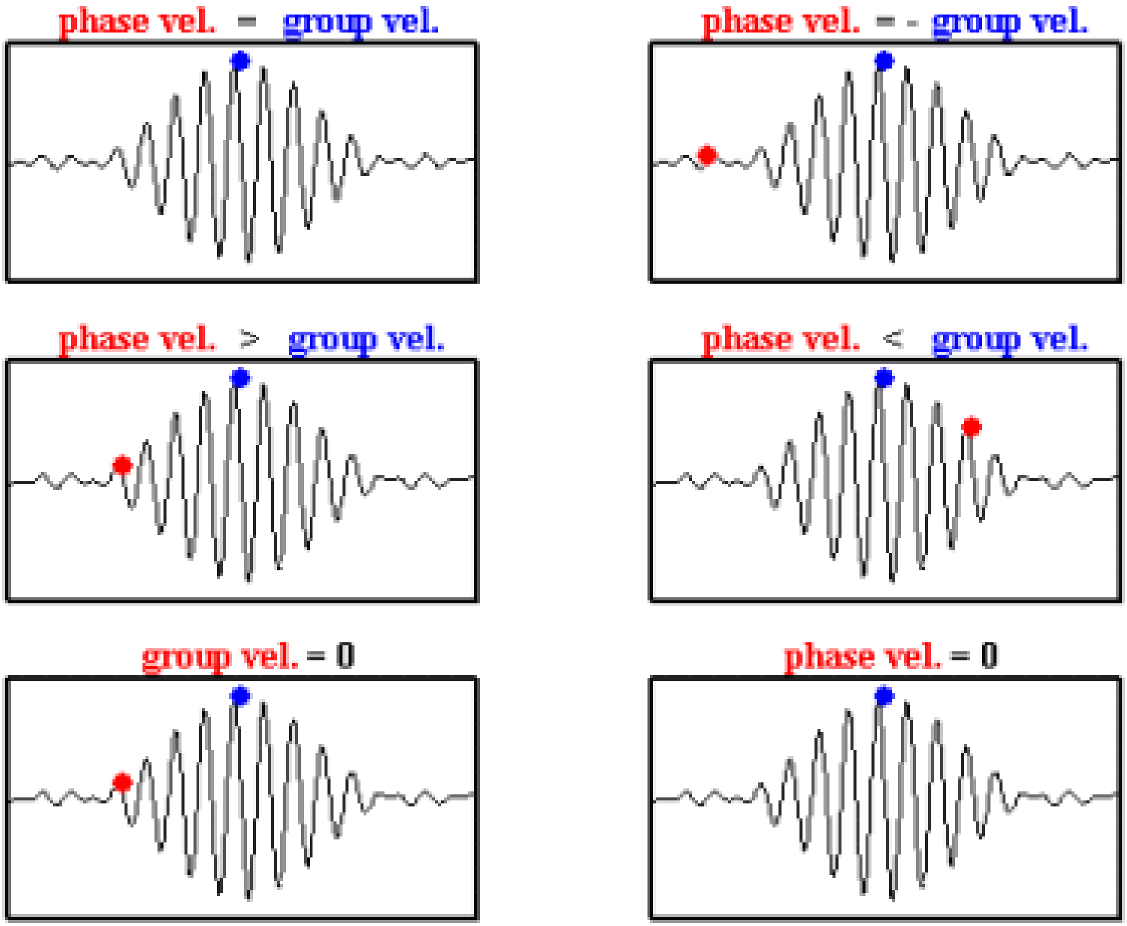
- PHASE VELOCITY: $\dfrac{\omega}{k}$ $\longleftarrow$ Velocity of a wave whose wave vector is k, the average wave vector.

- GROUP VELOCITY: $\dfrac{\Delta\omega}{\Delta k}$ $\longleftarrow$ Velocity of the envelope.

- The energy always travels with the group velocity!

Demo: phase velocity and group velocity





Phase velocity

Group velocity

Ripples on a pond: group and phase velocities

https://www.youtube.com/watch?v=dESm6VjfSNs

- ‘Gravity waves' on water, the longer wavelength waves travel faster and the shorter wavelength waves travel slower. This is normal dispersion of waves. This is what we get for our prism and for this particular waves. Here phase velocity > group velocity

- The second group of waves called the 'capillary waves' in which case the longer wavelengths travel slower than the shorter wavelengths. This is called Anomalous dispersion, and we will not discuss about that. Here group velocity > phase velocity

Example problems: Q1

7–5 A long uniform string of mass density 0.1 kg/m is stretched with a force of 50 N. One end of the string ($x = 0$) is oscillated transversely (sinusoidally) with an amplitude of 0.02 m and a period of 0.1 sec, so that traveling waves in the $+x$ direction are set up.

(a) What is the velocity of the waves?

(b) What is their wavelength?

(c) If at the driving end ($x = 0$) the displacement (y) at $t = 0$ is 0.01 m with dy/dt negative, what is the equation of the traveling waves?

Example problems

Example problems: Q2

7–18 The motion of ripples of short wavelength ($\lesssim 1$ cm) on water is controlled by surface tension. The phase velocity of such ripples is given by

$$v_p = \left(\frac{2\pi S}{\rho\lambda}\right)^{1/2}$$

where S is the surface tension and ρ the density of water.

(a) Show that the group velocity for a disturbance made up of wavelengths close to a given λ is equal to $3v_p/2$.

(b) What does this imply about the observed motion of a group of ripples traveling over a water surface?

(c) If the group consists of just two waves, of wavelengths 0.99 and 1.01 cm, what is the distance between crests of the group?

Example problems

Example problems: Q3

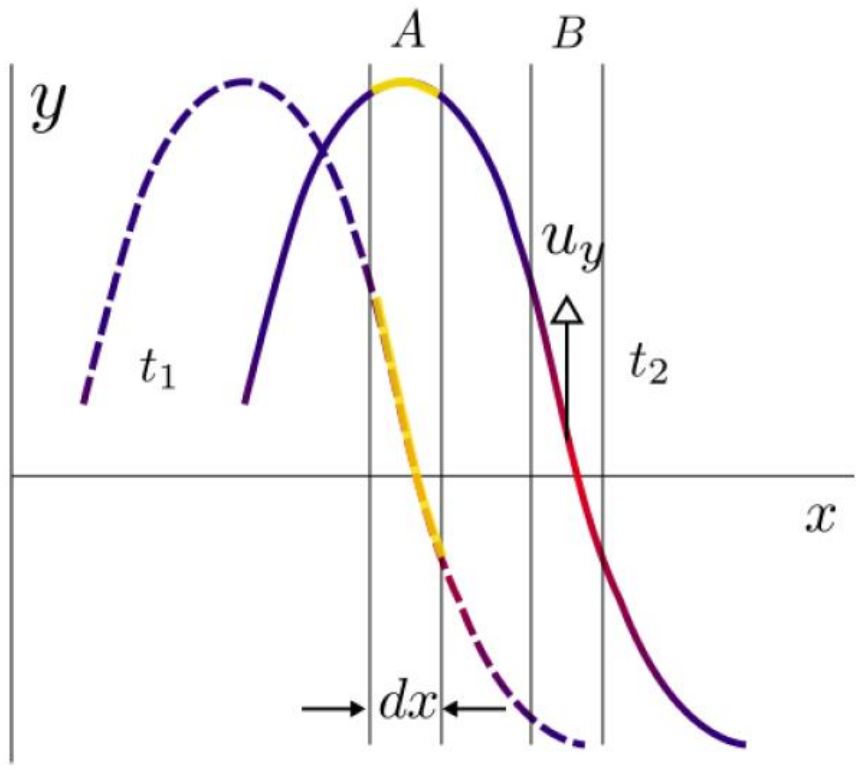
Q3. Consider a triangular pulse. The velocity of the pulse on a string is 25 m/s. To produce the pulse, the end is given a displacement y such that:

$$y(0,t) = 0.5\,t\ for\ 0 < t < 1$$
$$= 1 - 0.5\,t\ for\ 1 < t < 2.$$

What is the shape of the pulse at t = 4s?

Plot the transverse velocity of the string at t= 4s.

Energy of a vibrating string



As wave passes, energy passes from A to B

Kinetic energy:

KE = $\frac{1}{2}dm.{v_y}^2$

$$= \frac{1}{2}(\mu dx)\ {v_y}^2$$ where μ is the mass per unit length.

Consider the equation of a sine wave on a string propagating along the positive x direction:

$$y = Asin(kx - \omega t)$$

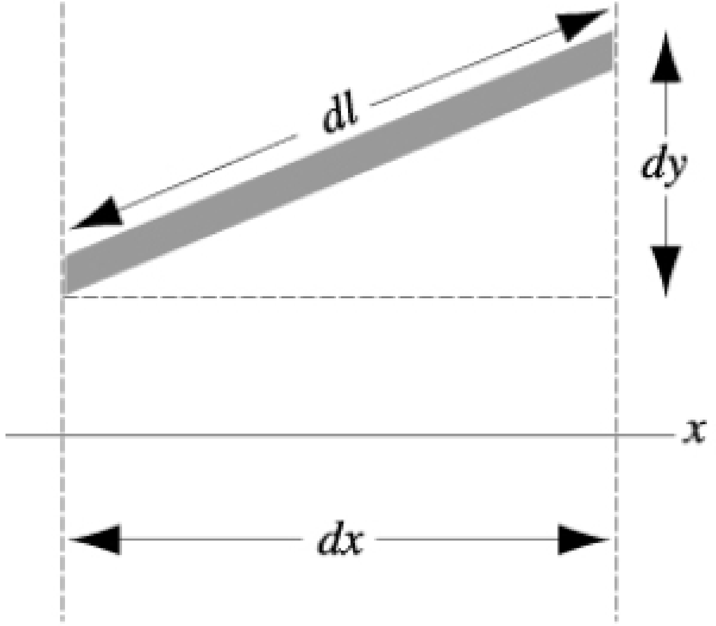
So $v_y = \frac{dy}{dt} = -\omega Acos(kx - \omega t)$

So KE = $\frac{1}{2}(\mu dx)\omega^2 A^2\ cos^2(kx - \omega t)$

Integrating, to get the total kinetic energy **over one wavelength**:

$\int_0^{\lambda}\frac{1}{2}(\mu dx)\omega^2 A^2\ cos^2(kx - \omega t) = \boldsymbol{\frac{1}{4}\mu\lambda\omega^2 A^2}$

Energy of a vibrating string



$$v_x = \sqrt{\frac{T}{\mu}}$$

$$\omega = k v_x$$

Potential energy:

Potential energy in length dx of the string = work done by tension T in stretching it.

PE $= T(dl - dx)$ where ds is the new length of the string after stretching.

$$\text{PE} = T(dl - dx) = T\left(\sqrt{(dx)^2 + (dy)^2} - dx\right)$$

$$= -Tdx\left(1 - \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\right)$$

If dy/dx <<1, we can do a binomial expansion, and keep only the first two terms such that: PE $= \frac{1}{2} T dx \left(\frac{\partial y}{\partial x}\right)^2$

$$= \frac{1}{2}\mu v_x^2 k^2 A^2 cos^2(kx - \omega t)dx = \frac{1}{2}\mu\omega^2 A^2 cos^2(kx - \omega t)dx$$

Integrating, to get the total potential energy **over one wavelength**:

$$\int_0^{\lambda} \frac{1}{2}\mu\omega^2 A^2 cos^2(kx - \omega t)dx = \boldsymbol{\frac{1}{4}\mu\lambda\omega^2 A^2}$$

Energy of a vibrating string

To sum up:

Kinetic energy = Potential energy

Total energy per wavelength: **E =** $\boldsymbol{\frac{1}{2}\mu\lambda\omega^2 A^2}$

Example problems: Q4

7–23 One end of a stretched string is moved transversely at constant velocity u_y for a time τ, and is moved back to its starting point with velocity $-u_y$ during the next interval τ. As a result, a triangular pulse is set up on the string and moves along it with speed v. Calculate the kinetic and potential energies associated with the pulse, and show that their sum is equal to the total work done by the transverse force that has to be applied at the end of the string.

Example problems