SHM and basics

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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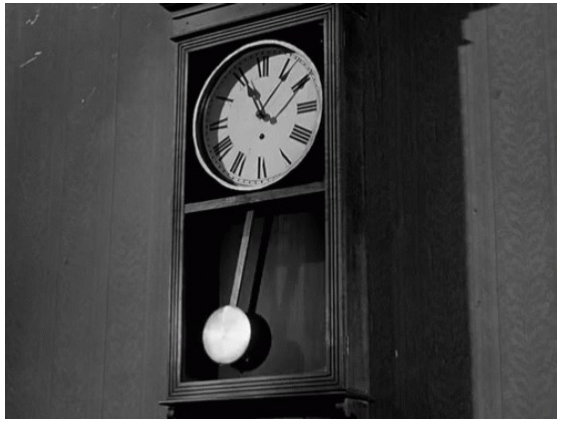
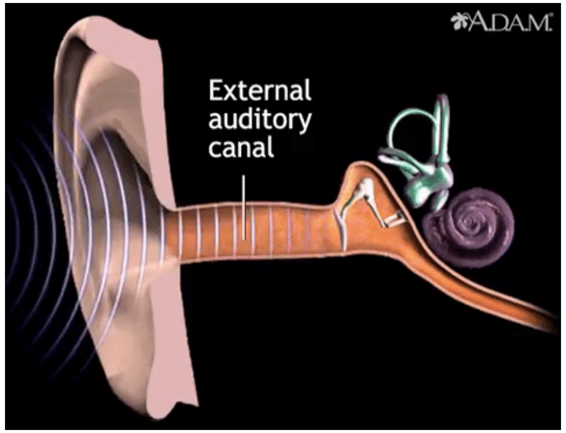
Simple harmonic motion all around us..





Before we begin: complex numbers

$z = x + iy$ where $i = \sqrt{-1}$

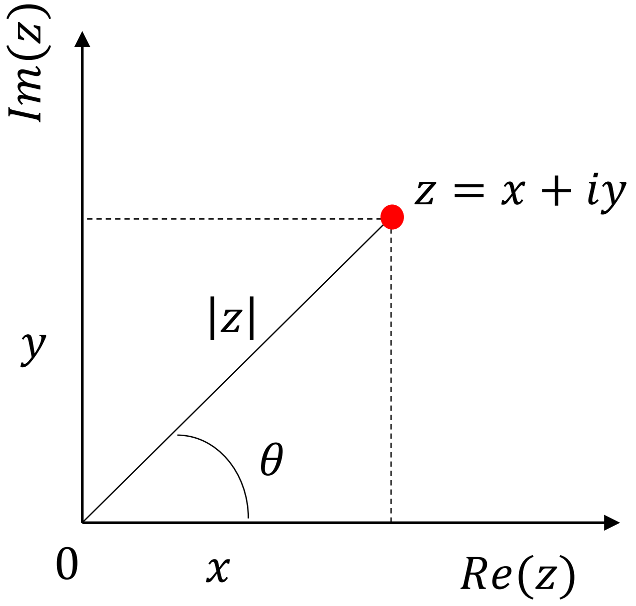
Amplitude : $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$

Argument: $\theta = tan^{-1}\left(\frac{y}{x}\right)$ where $0 \leq \theta < 2\pi$

So, $z = |z|(cos\theta + isin\theta)$

We would show later that using an equation called the Euler's formula, we could write this as:

$z = |z|e^{i\theta}$



Before we begin: differential equations

How to solve a homogeneous second order differential equation of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{where } a, b, c \text{ are constants}$$

We use a trial solution of the form: $y = Ae^{mx}$.

Putting this in the equation above we get,

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$$

$$am^2 + bm + c = 0 \qquad \text{‘auxiliary equation’}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{so this has two solutions } m_1 \text{ and } m_2$$

Case 1: $b^2 > 4ac$, so m has real roots.

$$y = Ae^{m_1x} + Be^{m_2x}$$

We get A and B using the boundary conditions, for eg, values of y and $\frac{dy}{dx}$ at $x = 0$.

Before we begin: differential equations

Case 2: $b^2 < 4ac$, so m has imaginary roots.

So we write $$m = \frac{-b \pm i\sqrt{4ac-b^2}}{2a}$$

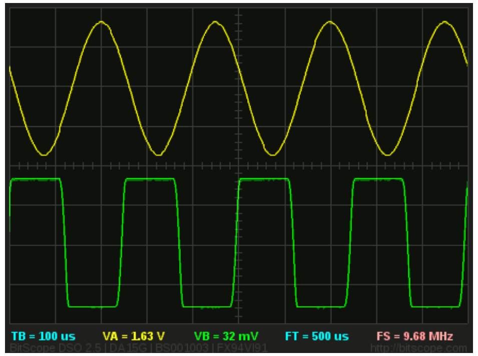
$$\begin{aligned} y &= Ae^{m_1 x} + Be^{m_2 x} \\ &= e^{-\frac{bx}{2a}}\left(Ae^{i\frac{\sqrt{4ac-b^2}}{2a}x} + Be^{-i\frac{\sqrt{4ac-b^2}}{2a}x}\right) \\ &= e^{-\frac{bx}{2a}}\left(Ae^{i\beta x} + Be^{-i\beta x}\right) \text{ where } \beta = \frac{\sqrt{4ac-b^2}}{2a} \end{aligned}$$

Here A and B are *complex constants.*

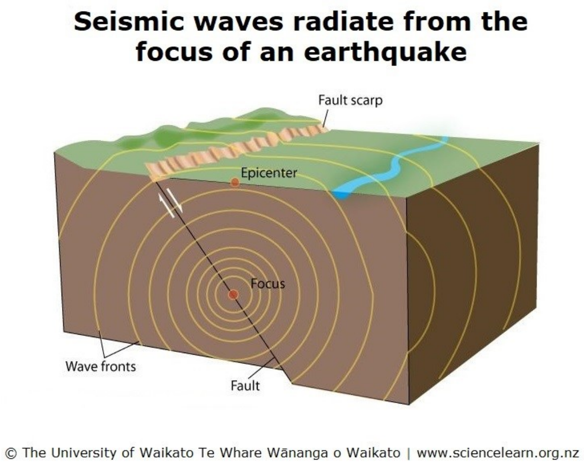
However we will show in the course that this can be written in terms of sine and cosine functions!

Vibrations and waves

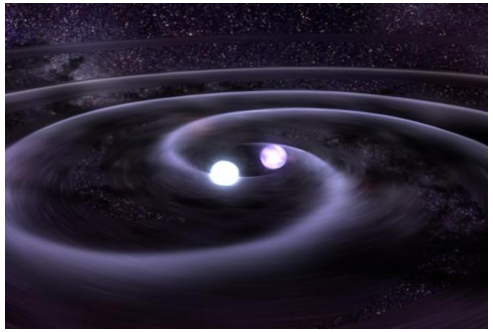
Complex wave pattern
from an ECG machine



Pure sinusoidal and square
waves generated by an
oscilloscope



Seismic waves that generate
an earthquake



Gravitational waves
from two merging
black holes

Vibrations are everywhere!

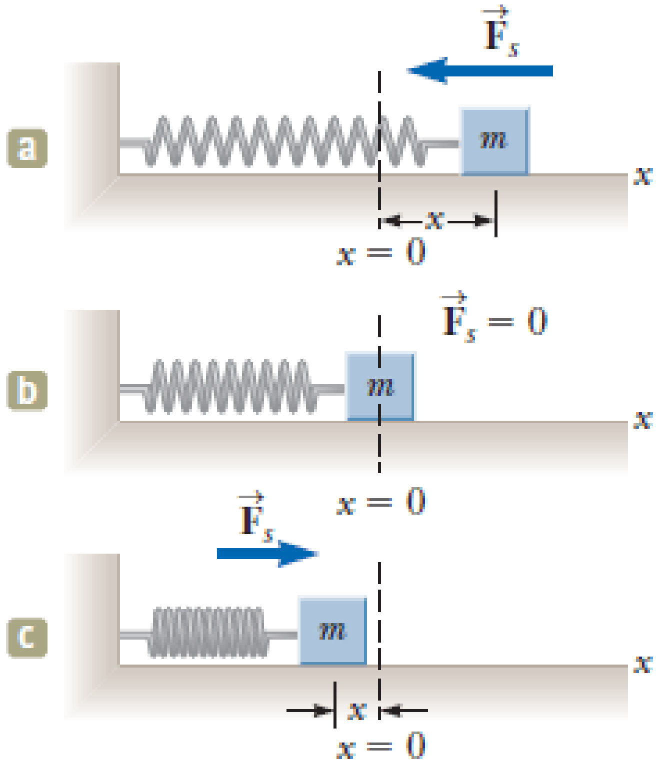
May be pure sinusoidal (eg. Vibration of a tuning fork)

Maybe have a complex pattern
(eg. An ECG signal from a human heart)

We will concern ourselves only with sinusoidal vibrations. Why so?

Why sine waves?

Reason 1



A block of mass m connected to the wall with a spring.

If we displace it by a distance x, then the restoring force F is

$$F(x) = -(\, k_1 x \; + k_2 x^2 \; + k_3 x^3 + \cdots)$$

Where k_1, k_2 and k_3 are constants

- When the oscillations are small (that means the higher order terms are negligible according to a set threshold, say 1 part in 1000 or 1 part in 10^6, we can only keep the first term)

So the equation of motion:

$$m\,\frac{d^2x}{dt^2} = -kx \qquad (1)$$

This has a solution of the form:

$$x = A\sin(\omega t + \varphi_o) \quad \text{where} \quad \omega = \frac{k}{m} \qquad (2)$$

- So finding a pure sine/cos vibration is always possible if the displacements are small enough!

Why sine waves?

Reason 2

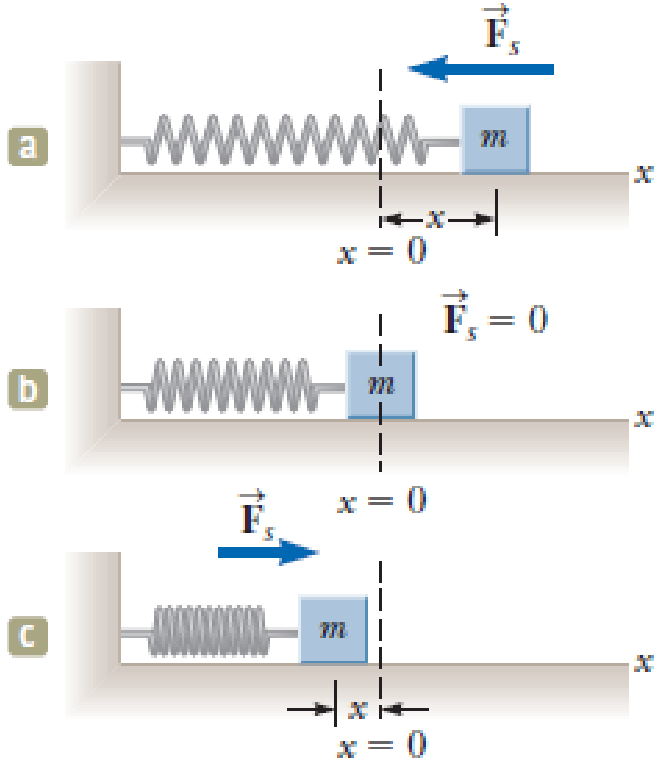
Fourier's theorem (J.B. Fourier 1807)

- Any periodic vibration with a time period T can be 'built up' from a set of pure sinusoidal vibrations of periods T, T/2, T/3,……. with appropriately chosen amplitudes!

- We will discuss this in detail in later classes!

So we will not miss anything by concentrating on pure sine waves!

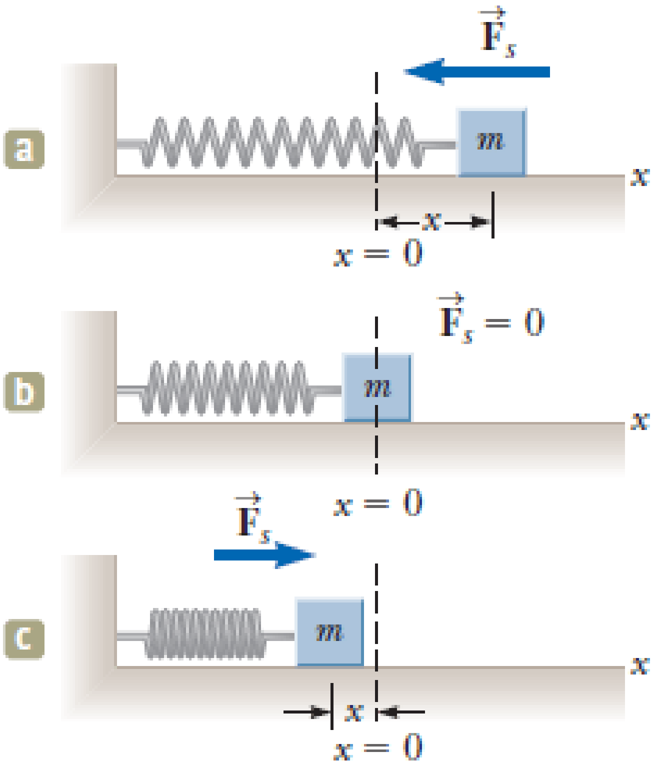
Simple harmonic motion (SHM)



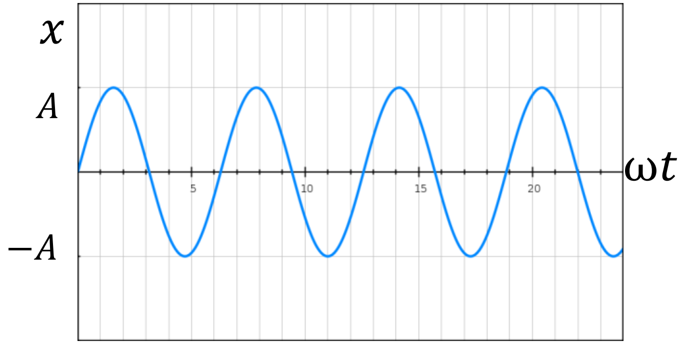
$$x = A\sin(\omega t + \varphi_o)$$

- What is the velocity and the acceleration?
- Is the velocity in the same phase with the displacement?
- Is the acceleration in the same phase with the displacement?
- How do we find out the amplitude and phase?
- What is the energy of the system?

Simple harmonic motion (SHM)

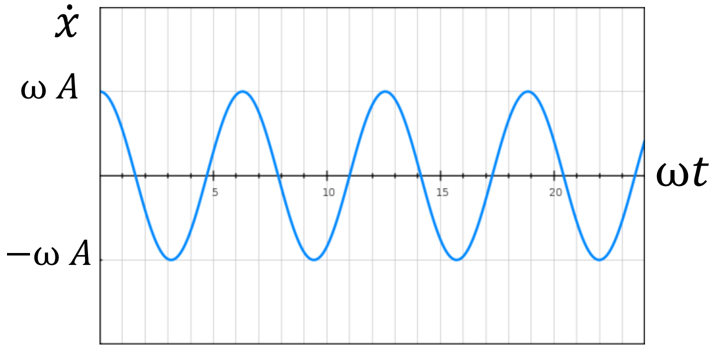


Displacement: $x = A\sin(\omega t + \varphi_o)$



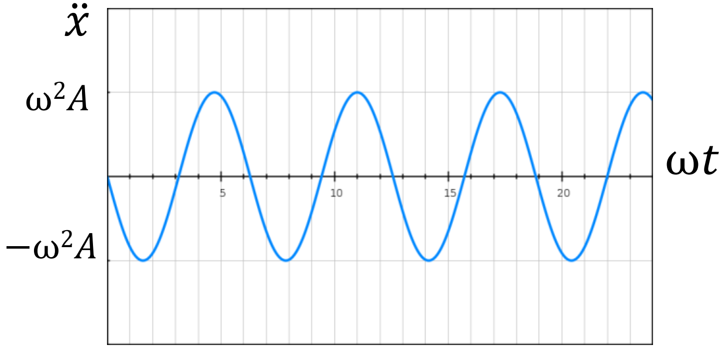
Velocity : $v = \dot{x} = \omega A\cos(\omega t + \varphi_o)$

Phase shift = π/2



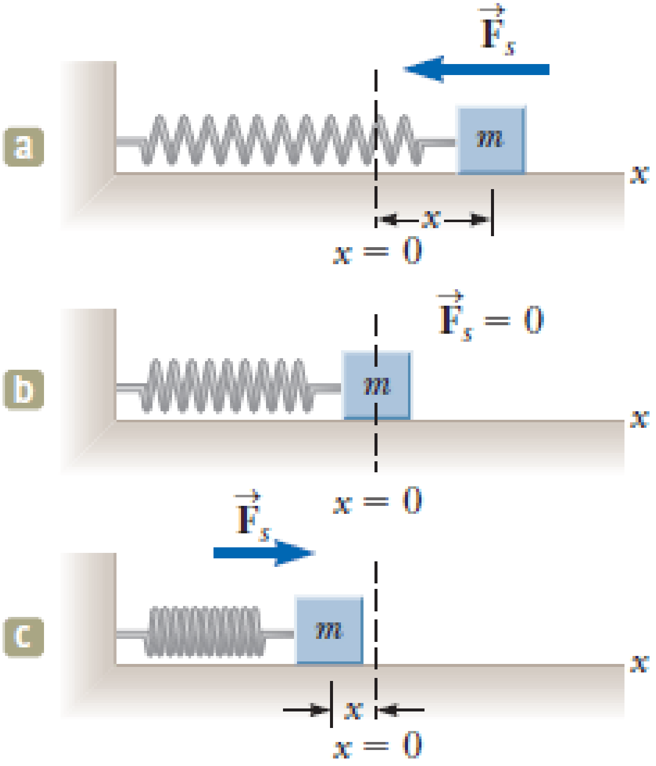
Acceleration : $f = \ddot{x} = -\omega^2 A\sin(\omega t + \varphi_o)$

Phase shift = π



All with $\varphi_o = 0$

Simple harmonic motion (SHM)



The equation for SHM:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

We solve this equation taking the boundary conditions:

$$At\ t = 0,\ \ x = x(0),\ \dot{x} = v(0),$$

The general solution of the SHM equation was $x = A\sin(\omega t + \varphi_o)$, so using the boundary conditions,

$$x(0) = Asin(\varphi_o) \text{ and } v(0) = \omega Acos(\varphi_o)$$

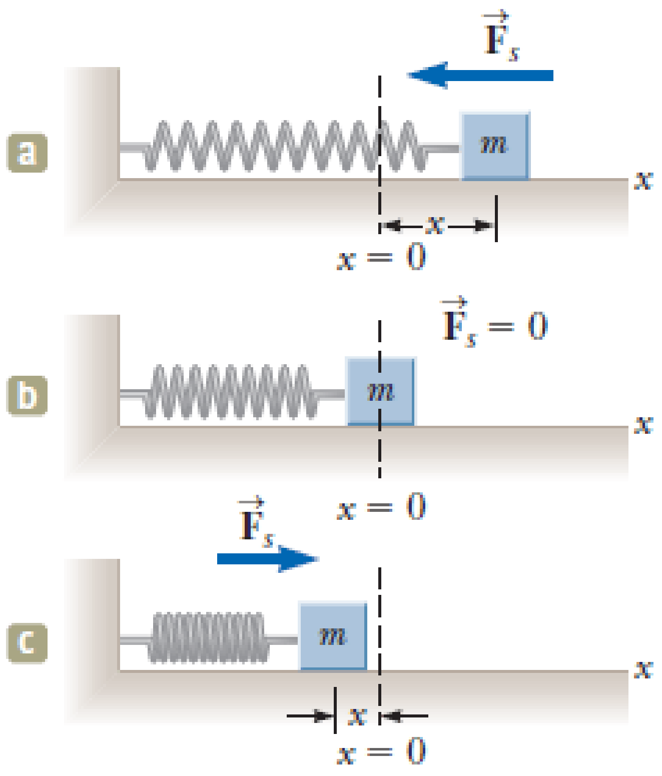
So,

$$A = \sqrt{(x(0))^2 + \left(\frac{v(0)}{\omega}\right)^2} \qquad (3)$$

$$\varphi_0 = tan^{-1}\left(\frac{x(0)\omega}{v(0)}\right) \qquad (4)$$

Now what about the energy of the system?

Simple harmonic motion (SHM): energy



Restoring force: $F(x) = -kx$

So, potential energy: $U(x) = -\int_0^x F dx = \frac{1}{2}kx^2$ (5)

Now kinetic energy: $K(x) = \frac{1}{2}m\dot{x}^2$ (6)

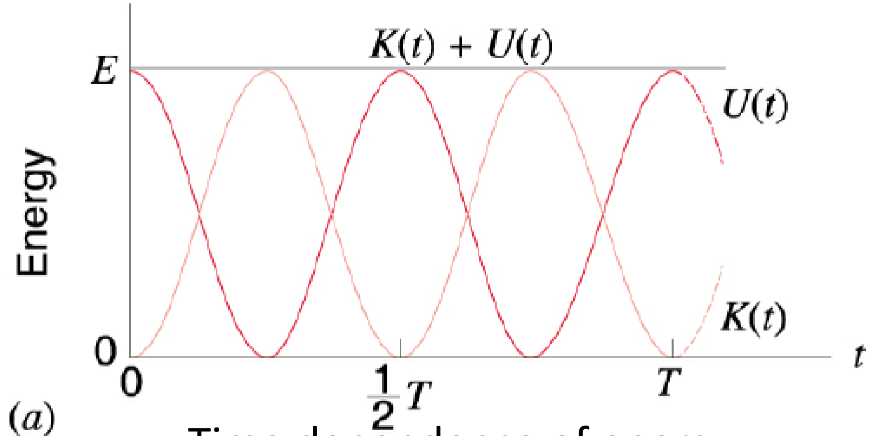
So total energy: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (7)

The potential energy is maximum at the ends, where the kinetic energy is zero. So we can write:

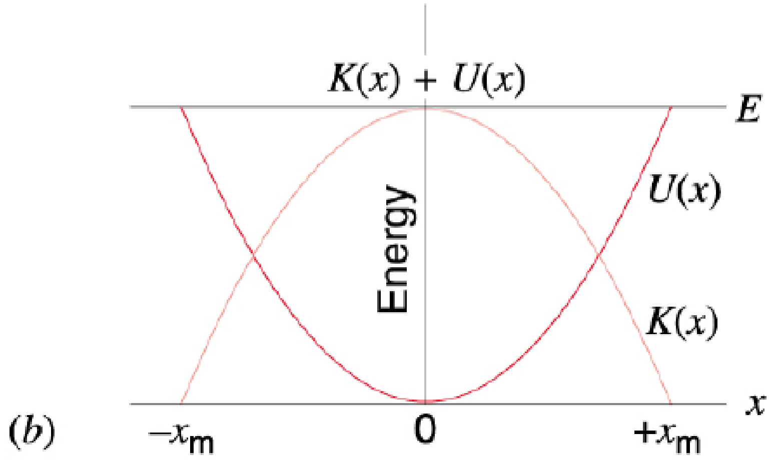
$$E = U_{max} = -\int_0^A F dx = \frac{1}{2}kA^2$$

Also at x = 0, kinetic energy is maximum and potential energy is zero. So, $E = U_{max} = K_{max} = \frac{1}{2}kA^2$ (8)

Simple harmonic motion (SHM):energy



Time dependence of energy



Space dependence of energy

$$E = \frac{1}{2}kA^2 \text{ where } k = m\omega^2$$

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}k\,A^2\sin^2(\omega t + \varphi_o)$$

$$K(x) = \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}k\,A^2cos^2(\omega t + \varphi_o)$$

Three examples: A physical pendulum

A simple pendulum

Limitations?

Physical/compound pendulum:
An arbitrarily shaped rigid object free to rotate in a vertical plane about a horizontal axis.

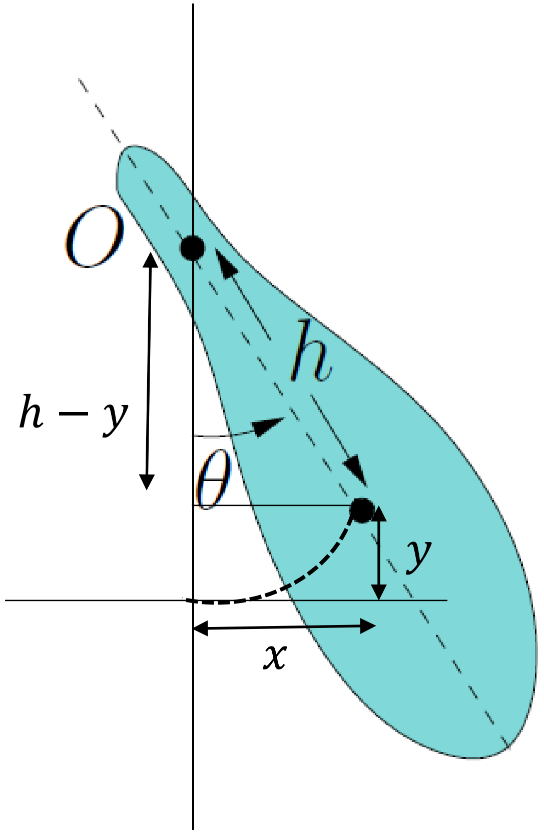
But how do we know the length?

Ketar’s pendulum was used to determine the value of g in Washington (1930) $9.80080 \pm 0.00003\ m/s^2$, accurate to within 1/1000th of a percent.

A bar pendulum

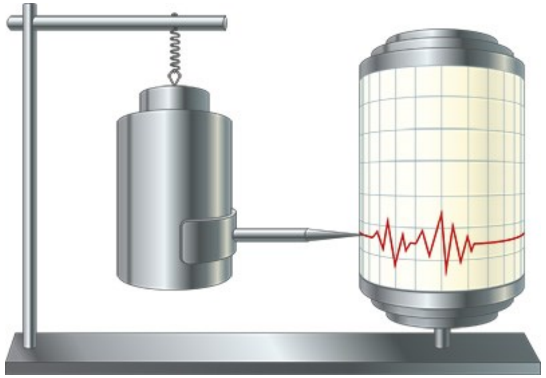
A Kater’s pendulum

Three examples: A physical pendulum



A physical pendulum

Kater's pendulum for accurate determinati on of g



Seismometer for measuring earthquakes

Kinetic energy = $\frac{1}{2} I \dot{\theta}^2$

where I is the moment of inertia about the point O.

Potential energy = $mgy \approx mgh \frac{\theta^2}{2}$

when θ is small.

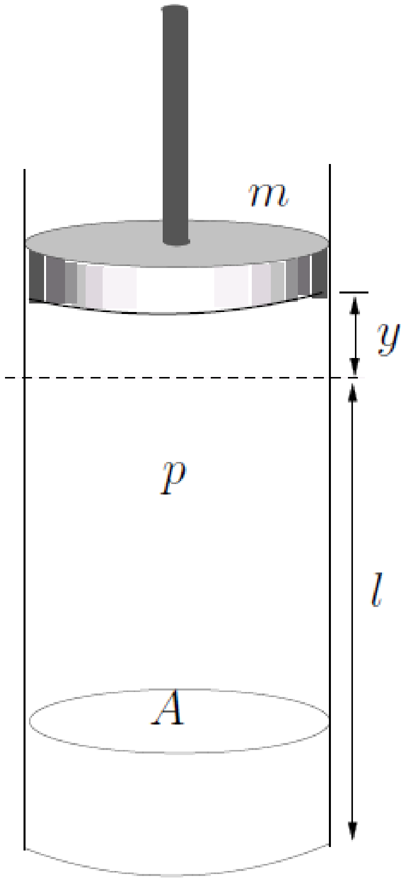
From conservation of energy,

$$\frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} mgh\theta^2 = E$$

$$\ddot{\theta} + \frac{mgh}{I} \theta = 0$$

$$\omega^2 = \frac{mgh}{I}$$

Three examples: An air spring



An air spring

Used in shock absorbers, vibration isolators, heavy duty vehicle suspensions

Piston of mass m oscillates about $y = 0$

Restoring force arises from the pressure change, as the volume of the air column changes.

$$F = \Delta Pressure \times Area$$

Under adiabatic conditions,

$$pV^{\gamma} = \text{constant}$$

$\gamma = 1.67$ for monoatomic gases, $\gamma = 1.40$ for diatomic gases

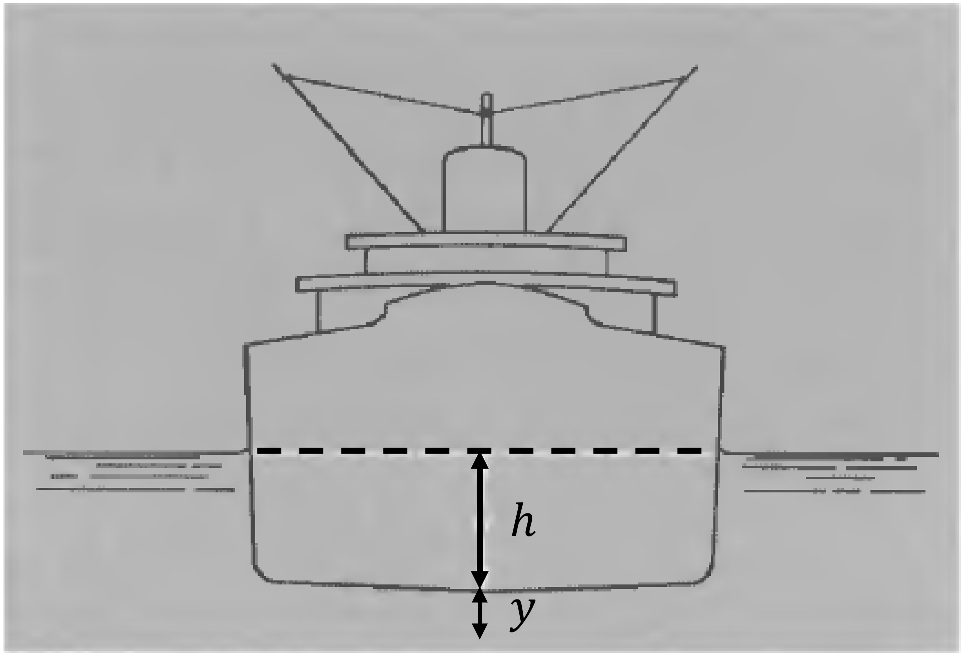
$$V^{\gamma}\frac{dp}{dV} + p\gamma V^{\gamma-1} = 0$$

$$dp = -\gamma p\frac{dV}{V} = -\gamma p\frac{Ay}{Al}$$

$$F = dp \times A = -\frac{\gamma p A y}{l}$$

$$\omega^2 = \frac{\gamma p A}{ml}$$

Three examples: A ship



A floating ship

- At equilibrium, ship of mass m floats with distance h submerged in water.
- Restoring force arises from the buoyancy arising from the displaced water when the ship is further submerged by a distance y.

$$F = -g\rho A y$$

ρ is density of water, A is the flat cross-sectional area of the bottom of the ship.

For a floating object, we know:

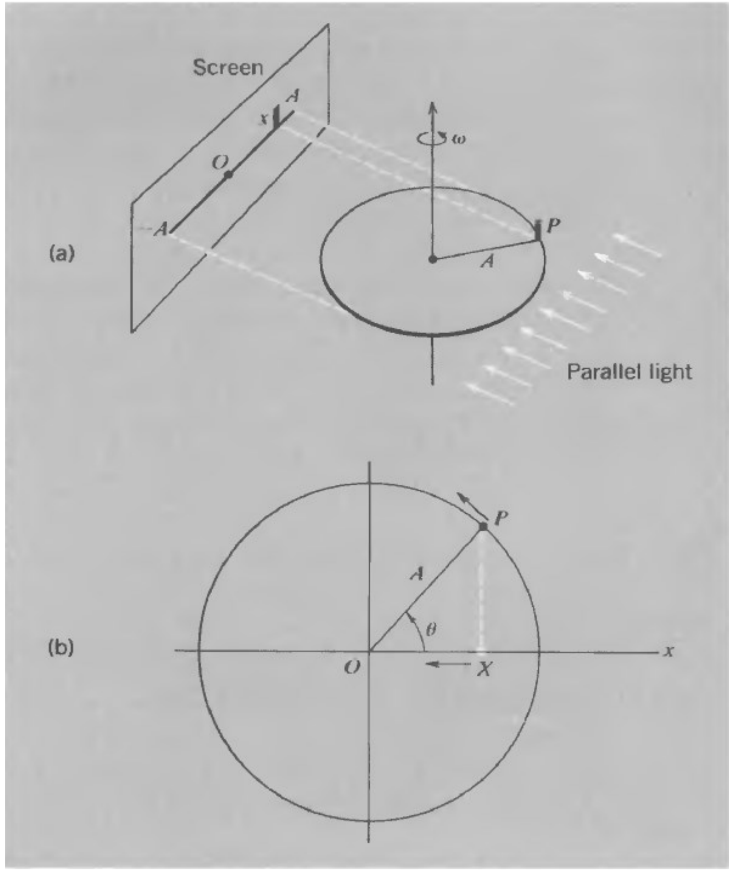
$$mg = Ah\rho g$$

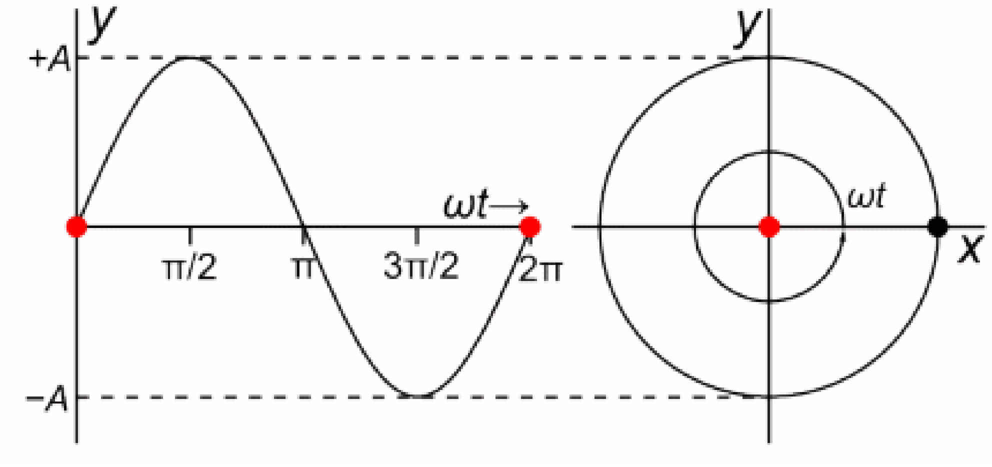
So, $$F = m\frac{d^2y}{dt^2} = -\frac{g\rho Ay.h}{h} = \frac{gmy}{h}$$

$$\omega^2 = \frac{k}{m} = \frac{g}{h}$$

If h = 10m, T= 6 s.

Rotational representation of SHM



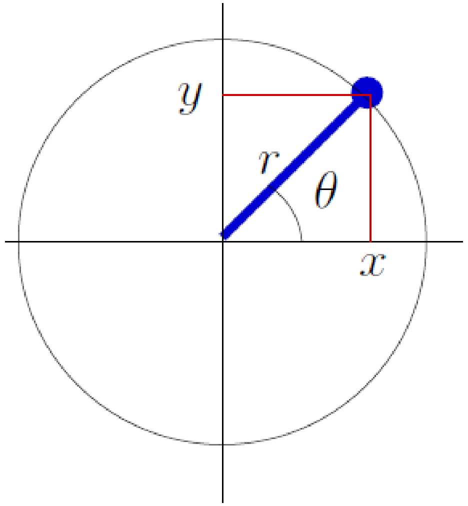


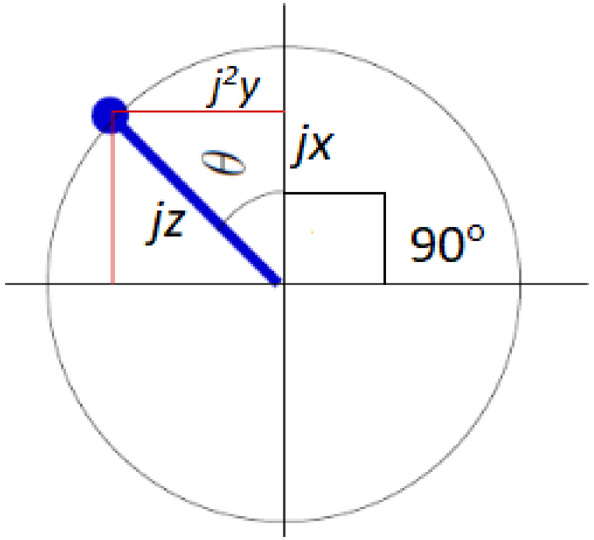
Particle moves with uniform angular speed ω along a circle of radius r

$$x = r\cos(\omega t + \varphi_0)$$
$$y = r\sin(\omega t + \varphi_0)$$

Therefore $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ is a rotating vector.

SHM in vector formalism





Let's represent the vector $\vec{r}$ in an X-Y coordinate system:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

$$x = r cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r sin\theta \qquad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Let's write this by dispensing the vector symbols and writing j as a general instruction to give a 90° counter-clockwise motion to account for the perpendicular axes.
So, $$r = x + jy$$

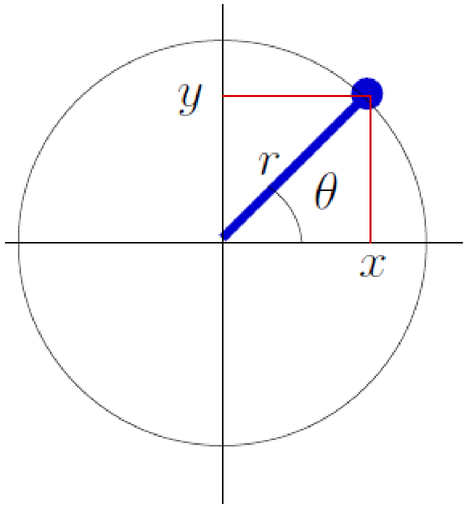
Multiplying both sides by j, that is rotating the entire vector by 90°, we get

$$jr = jx + j^2 y \quad \text{and} \quad r' = -x' + jy' = -y + jx$$

This indicates displacement along negative x direction. So we can write the identity:
$j^2 = -1$ or $j = \sqrt{-1}$

So we can write the vector $\vec{r}$ as a complex number in a complex coordinate system with real and imaginary axes!

Let's try it with Complex numbers!



Let's map the motion of the rotating vector $\vec{r}$ on to the complex plane.

$$z = x + iy,\ i = \sqrt{-1}$$

$$x = r cos\theta = Re(z)$$

$$y = r sin\theta = Im(z)$$

The complex exponential (Euler's formula):

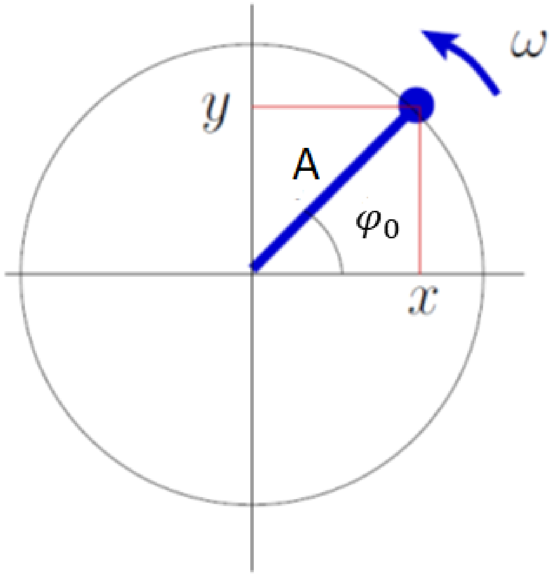
$$\boxed{cos\theta + i sin\theta = e^{i\theta}}$$

This equation, providing a connection between plane geometry (trigonometric functions) and algebra(exponential) was referred to by Feynman as "this amazing jewel…the most remarkable formula in mathematics"!

Some examples: $e^{i\pi} = -1, e^{i\frac{\pi}{2}} = i, e^{i2\pi} = 1$

So we can z in the polar form: $z = r(cos\theta + i sin\theta) = re^{i\theta}$. We will use this.

Rotating vectors in complex plane



$$z = Ae^{i\theta} = Ae^{i(\omega t+\varphi_0)}$$

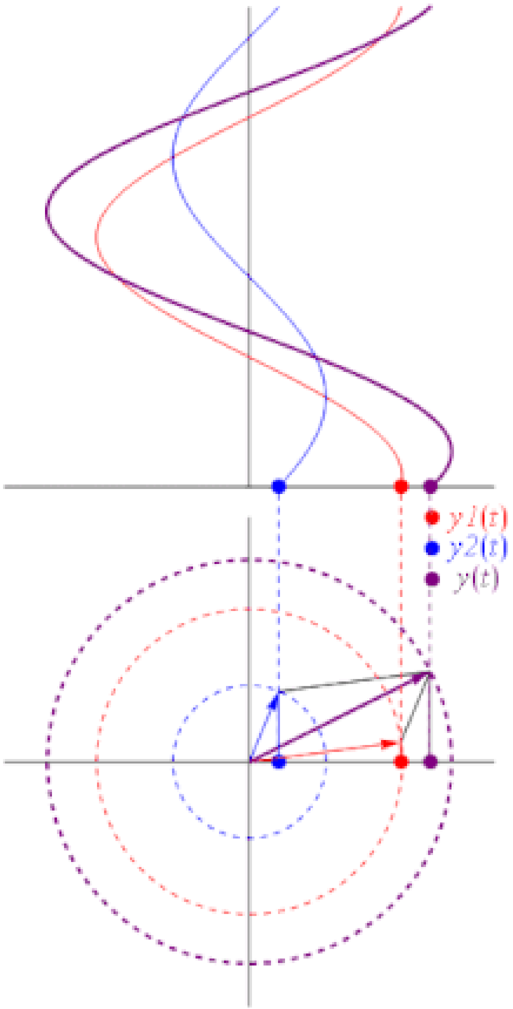
$$Re(z) = A\cos(\omega t + \varphi_0)$$

$$\mathrm{Im}(z) = A\sin(\omega t + \varphi_0)$$

$Ae^{i\varphi_0}$ with a given (static) value of theta is termed a ‘Phasor’, although people often use it to describe the whole function $Ae^{i\theta}$.

Represents SHM
Use: Superposition of SHMs can be calculated geometrically using phasors, also solution to SHM differential equations

Rotating vectors in complex plane



Represents SHM
Use: Superposition of SHMs can
be calculated geometrically using
phasors, also solution to differential equations

Applications:

1. Vibration of a microphone diaphragm
2. Analysis of ac power systems
3. Telecommunications: amplitude and frequency modulation

Solving the SHM equation

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$

Assume a solution of the form $z = e^{\lambda t}$

Using this I will get, $\lambda^2 + \omega^2 = 0$, so $\lambda = \pm i\omega$

The general solution is $z = Ae^{i\omega t} + Be^{-i\omega t}$ where A and B are complex constants

$$= R_1\, e^{i\theta_1} e^{i\omega t} + R_2 e^{i\theta_2} e^{-i\omega t}$$

Let's take only one solution to illustrate: $z = R_1\, e^{i(\omega t + \theta_1)}$

To get a real solution, we use the Euler's formula and write:

$$z = R_1 \cos(\omega t + \theta_1) + iR_1 \sin(\omega t + \theta_1)$$ where R_1 *is* real.

$$x = Re(z) = R_1 \cos(\omega t + \theta_1)$$
$$y = Im(z) = R_1 \sin(\omega t + \theta_1)$$