Damped Harmonic Oscillator

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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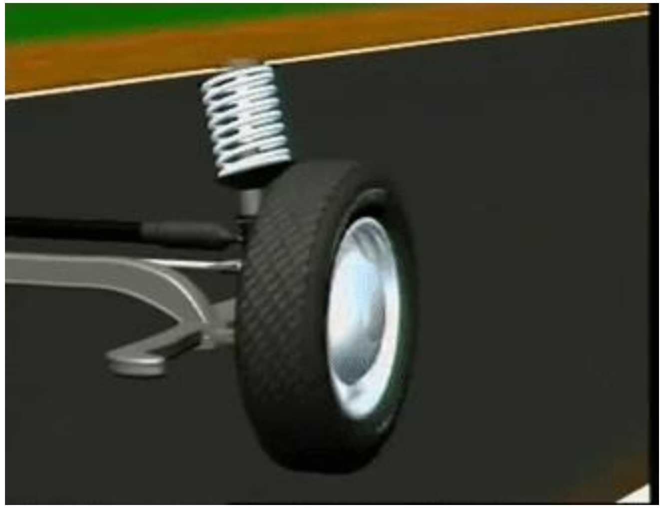
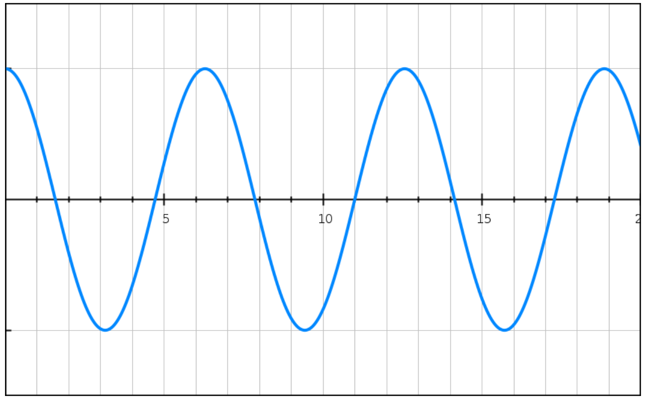
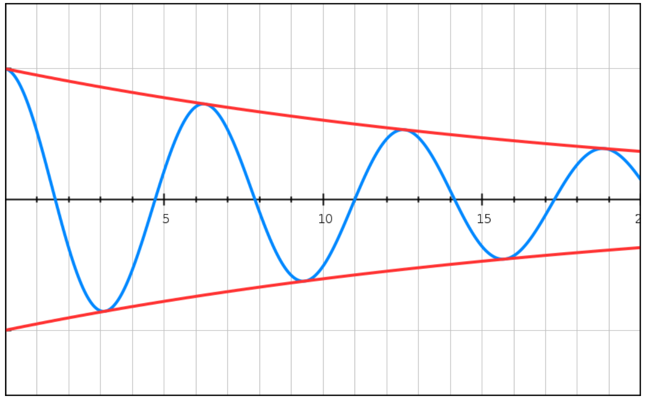


Damped oscillations



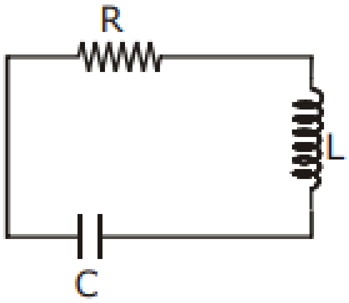
Undamped, free vibration



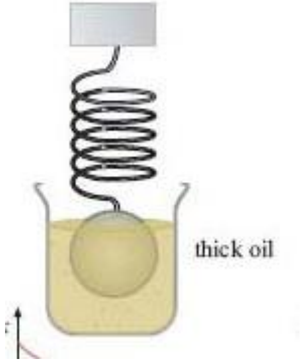
Damped, free vibration

- All free vibrations die away with time due to the presence of dissipative forces

Some examples of dissipative forces causing damped oscillations:



Resistance in an electrical oscillator



Viscous drag of the liquid

Air resistance and friction at the swing suspension point

Modelling damped motion

- The damping force on an object is a function of its velocity.

- For linear damping, we can write the damping force f as:

$$f = -b\dot{x} \qquad (1)$$

- So the equation for a damped harmonic oscillator is:

$$m\frac{d^2x}{dt^2} = -kx - b\dot{x} \qquad (2)$$

$$\boxed{\boldsymbol{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + {\omega_0}^2 x = 0} \textbf{ where } \boldsymbol{{\omega_0}^2 = k/m} \textbf{ and } \boldsymbol{\gamma = \frac{b}{m}}} \qquad (3)$$

Here damping constant γ has a dimension of frequency, while ω_0 is the undamped angular frequency.

Let's solve this! We will use complex exponentials.

Solving the equation

- We will solve equation 3 by using complex exponentials!

So let's write equation 3 as $$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + {\omega_0}^2 z = 0 \qquad (4)$$

We assume a solution of the form $$z = Ae^{i(pt+\alpha)} \qquad (5)$$

Putting equation 5 in 4, we obtain,

$$(-p^2 + ip\gamma + {\omega_0}^2)Ae^{i(pt+\alpha)} = 0$$

If this is to be satisfied for all values of t, we must have:

$$-p^2 + ip\gamma + {\omega_0}^2 = 0 \qquad (6)$$

This implies two separately satisfied conditions involving real and imaginary parts!

Solving the equation

Therefore, the quantity p **cannot be purely real** and we put $p = n + is$ where n and s are both real.

Then, $$p^2 = n^2 + 2ins - s^2 \qquad (7)$$

Substituting this in equation (6) gives us:

$$-n^2 - 2ins + s^2 + in\gamma - s\gamma + {\omega_0}^2 = 0 \qquad (8)$$

From this we can write two separate equations taking the real and imaginary parts:

From the real part, $$-n^2 + s^2 - s\gamma + {\omega_0}^2 = 0 \qquad (9)$$

From the imaginary part, $$-2ns + n\gamma = 0 \qquad (10)$$

So from equation (10), we can write: $$s = \frac{\gamma}{2} \qquad (11)$$

Putting (11) in (9), we get: $$n^2 = {\omega_0}^2 - \frac{\gamma^2}{4} \qquad (12)$$

Solving the equation

Now we take equation (5), that is $z = Ae^{i(pt+\alpha)}$ and put equation $p = n + is$ in it.

$$z = Ae^{i(pt+\alpha)} = Ae^{i(nt+ist+\alpha)} = Ae^{-st}e^{i(nt+\alpha)}$$

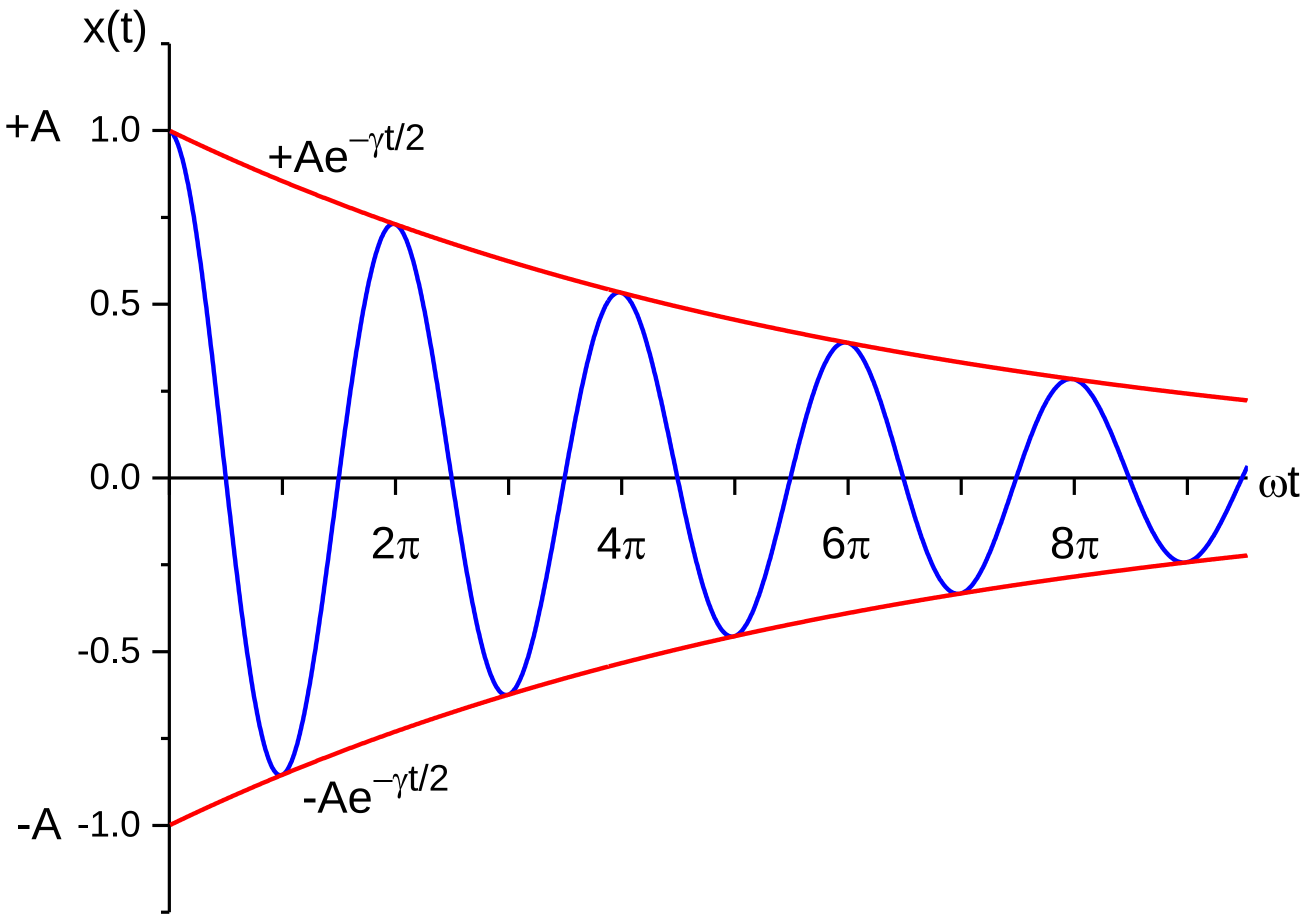
Taking the real part from this, $x = Ae^{-st}\cos(nt+\alpha)$ (13)

Putting equations (11) and (12) in this, that is putting in the values of n and s we get,

$$\boxed{\boldsymbol{x = Ae^{-\frac{\gamma t}{2}}\cos(\omega t + \alpha)}} \qquad (14)$$

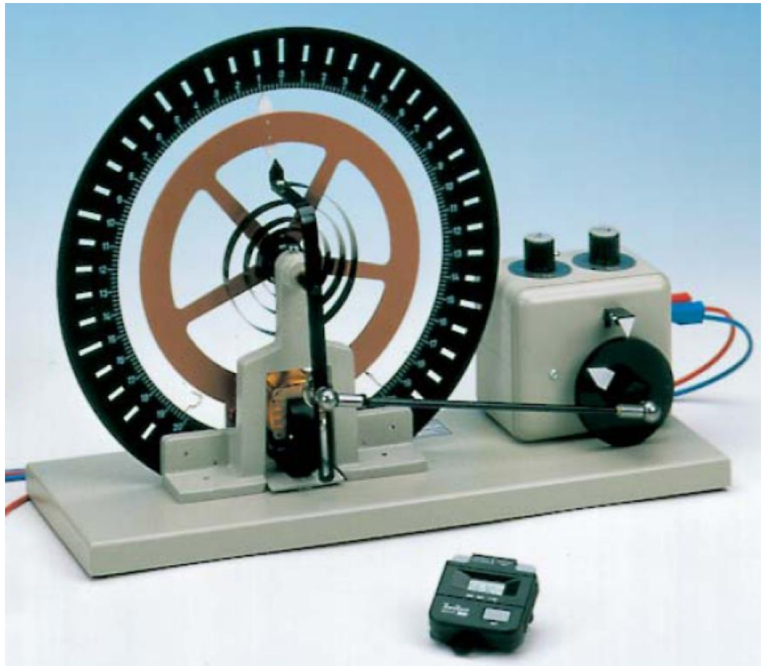
where, $\boldsymbol{\omega^2 = n^2 = {\omega_0}^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}}$

Solving the equation



$$x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega t + \alpha) \text{ for } \alpha = 0$$

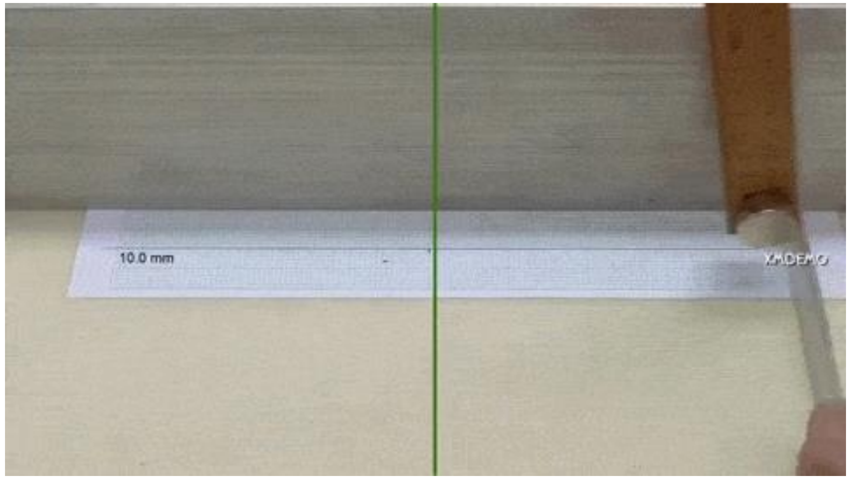
So how do we damp a system in a controlled way?

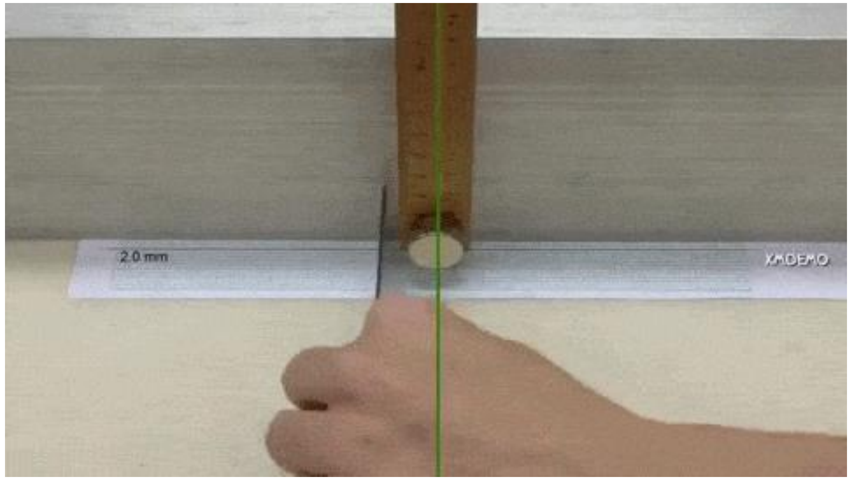


Mechanically:
Eg. A ‘Dashpot’: Cylinder with piston immersed in a viscous fluid

Electrically:
Eg. Use eddy current generated in an oscillating coil by an electromagnet

Comparisons





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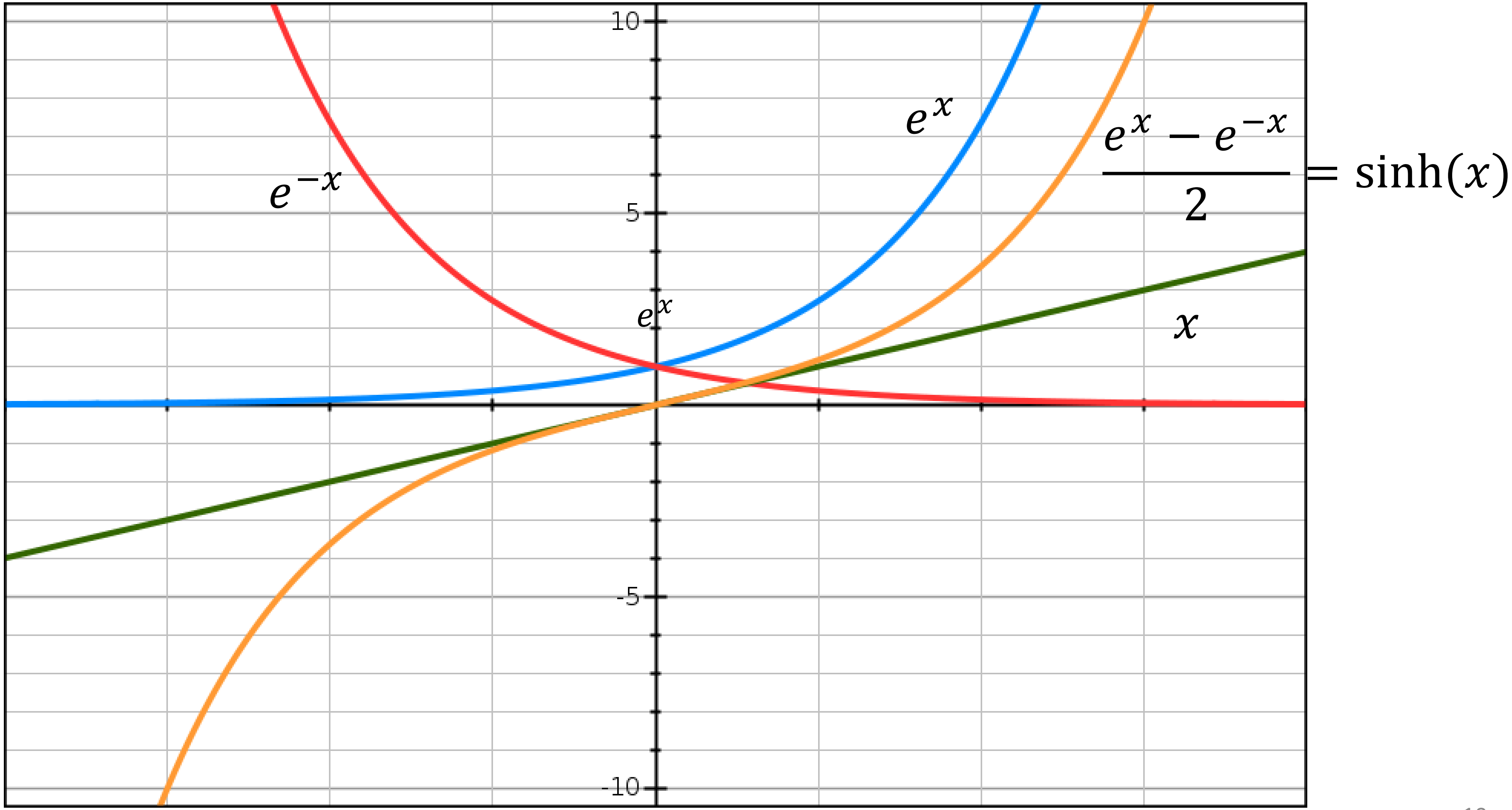
Plots of functions



Quality factor Q

A damped oscillator has two main quantities involved : $\boldsymbol{\omega_0}$ and $\boldsymbol{\gamma}$.

Both have dimensions of 1/time.

Let's define a quantity called the Quality Factor Q given by:

$$\boxed{\boldsymbol{Q = \frac{\omega_0}{\gamma}}} \qquad (15)$$

We have seen before that
$$\omega^2 = {\omega_0}^2 - \frac{\gamma^2}{4} \qquad (16)$$

This can be written as:
$$\omega^2 = {\omega_0}^2\left(1 - \frac{1}{4Q^2}\right) \qquad (17)$$

Now we will see how the damped vibrations look for different Q values.

Underdamped systems

$$\boldsymbol{\omega^2 = \omega_0{}^2 - \frac{\gamma^2}{4}}$$

Case A: **If the damping is low ($\boldsymbol{\gamma < 2\omega_0}$), $\boldsymbol{Q > \frac{1}{2}}$ and $\boldsymbol{\omega \approx \omega_0}$**

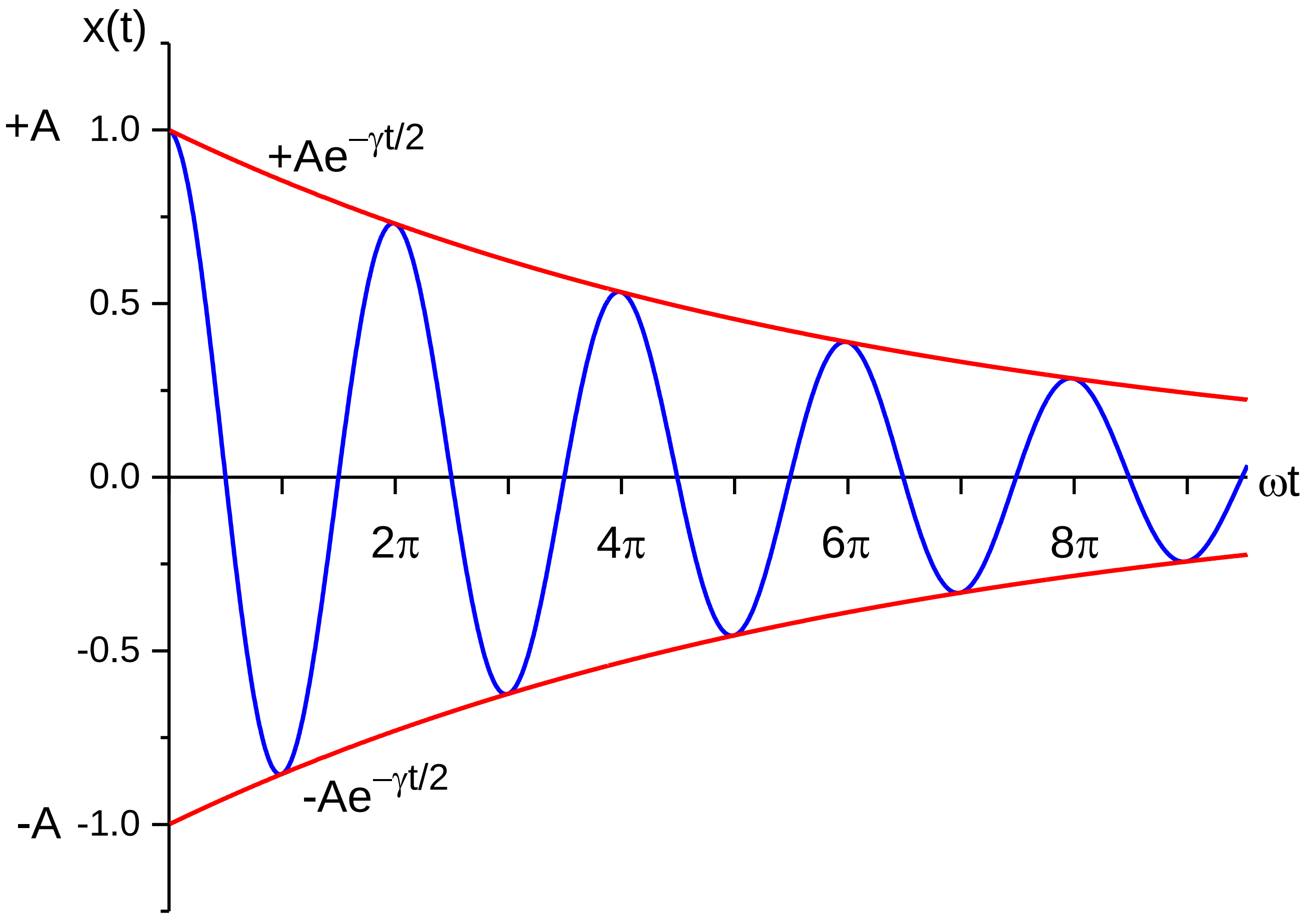
$$x = Ae^{-\frac{\omega_0 t}{2Q}}\cos(\omega_0 t + \alpha) \qquad (18)$$

$$A'(t) = Ae^{-\frac{\omega_0 t}{2Q}}$$ (A' is just a notation, not derivative)

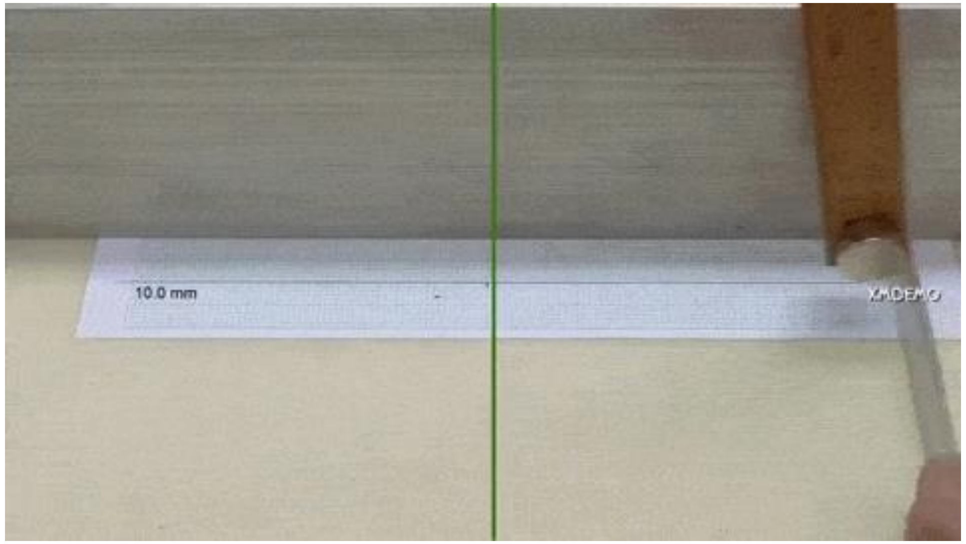
Let us measure the time t in number of complete oscillations n. Then we can write $t \approx \frac{2\pi n}{\omega_0}$

Therefore, $A'(n) \approx Ae^{-\frac{n\pi}{Q}} \rightarrow$ amplitude falls by a factor e in Q/π cycles of oscillation.

Underdamped system



$$x(t) = Ae^{-\frac{\omega_0 t}{2Q}} \cos(\omega_0 t + \alpha) \text{ for } \alpha = 0$$



Eg: Motion of a swing

Overdamped systems

$$\boldsymbol{\omega^2 = {\omega_0}^2 - \frac{\gamma^2}{4}}$$

Case B: If the damping is high ($\boldsymbol{\gamma > 2\omega_0}$), then $\boldsymbol{Q < \frac{1}{2}}$

$$x = Re(Ae^{-\frac{\gamma t}{2}}e^{i(\omega t+\alpha)})$$

If now $\omega_0^2 < \frac{\gamma^2}{4}$, then we can write: $\omega^2 = -\left(\frac{\gamma^2}{4} - \omega_0^2\right)$.

Therefore, $$\omega = \pm\, i \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{\frac{1}{2}} = \pm\, i\beta$$

SO, $e^{i\omega t} = e^{\mp\, \beta t}$ and this gives rise to two possible decay modes: $e^{-\left(\frac{\gamma}{2}+\beta\right)t}$ and $e^{-\left(\frac{\gamma}{2}-\beta\right)t}$ (take $\alpha = 0$)

So we can write a general solution of $x = A_1 \mathrm{e}^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 \mathrm{e}^{-\left(\frac{\gamma}{2}-\beta\right)t}$ (19)

Overdamped system

$$x = A_1 \mathrm{e}^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 \mathrm{e}^{-\left(\frac{\gamma}{2}-\beta\right)t}$$

$$x(0) = 0$$

$A_1 + A_2 = 0$, so $\quad A_2 = -A_1 = A$

$$x(t) = \frac{2A}{2} e^{-\frac{\gamma t}{2}} \left(-e^{-\beta t} + e^{+\beta t}\right)$$

$$= 2\mathrm{A} e^{-\frac{\gamma t}{2}} \sinh(\beta t)$$

where $\frac{e^{+\beta t} - e^{-\beta t}}{2} = \sinh(\beta t)$ is a hyperbolic function

Eg: Automatic door-close

Critically damped systems

$$\boldsymbol{\omega^2 = \omega_0{}^2 - \frac{\gamma^2}{4}}$$

Case C: **If ($\boldsymbol{\gamma = 2\omega_0}$), then $\boldsymbol{Q = \frac{1}{2}}$ and $\boldsymbol{\omega = 0}$ and $\boldsymbol{\beta = 0}$**

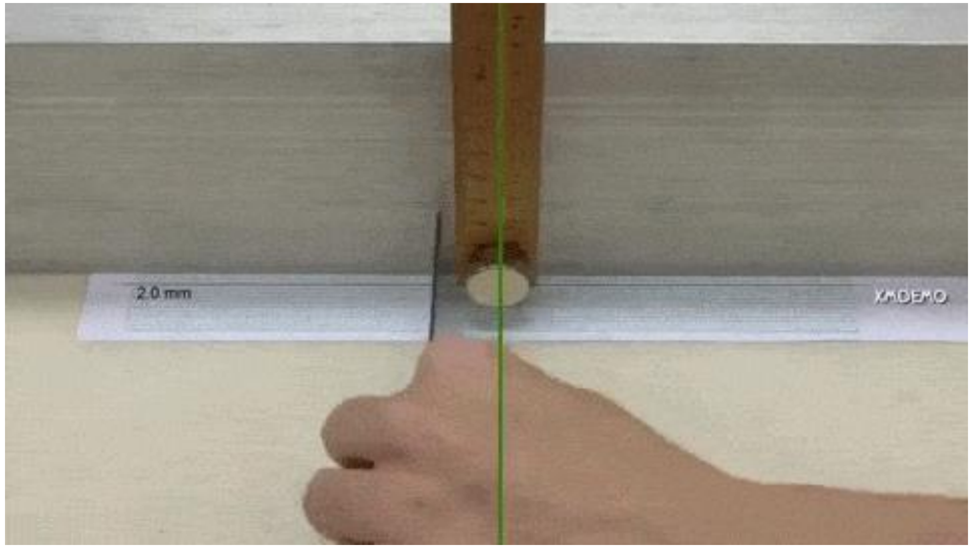
Equation (19) then has only one term and not two!

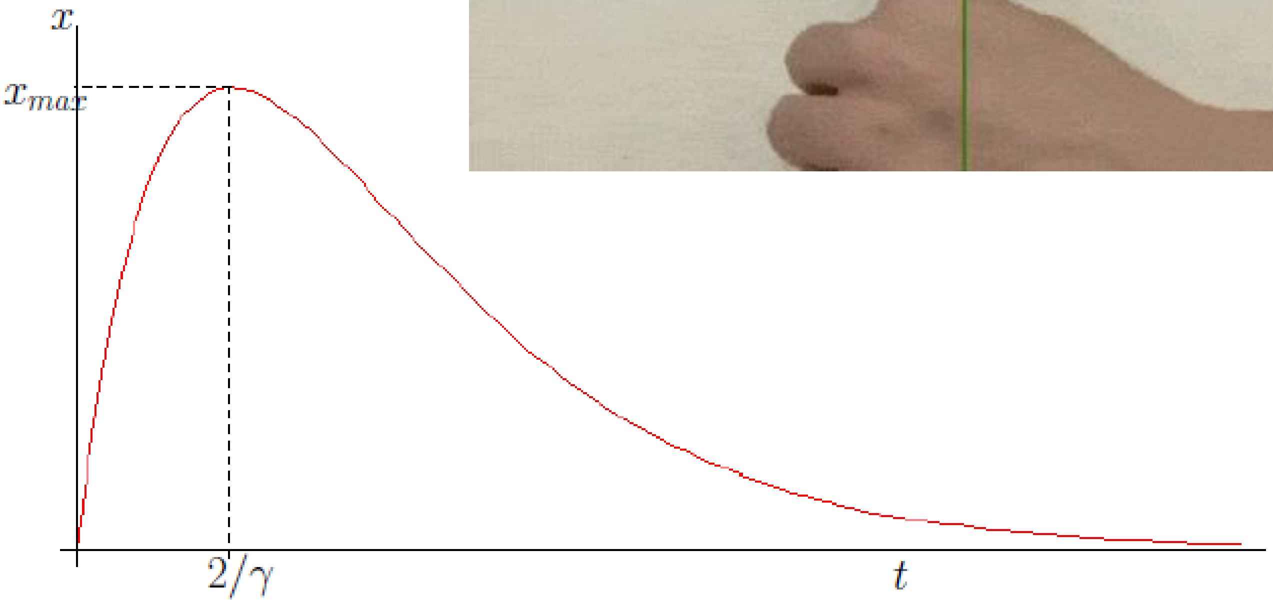
We still need two adjustable constants and so an appropriate solution is

$$x = (A + Bt)\mathrm{e}^{-\frac{\gamma t}{2}} \qquad (20)$$

You can check by substitution that (20) will satisfy the basic damped oscillator equation (equation(3)) when $\gamma = 2\omega_0$.

Critically damped system





$$x = (A + Bt)e^{-\frac{\gamma t}{2}}$$

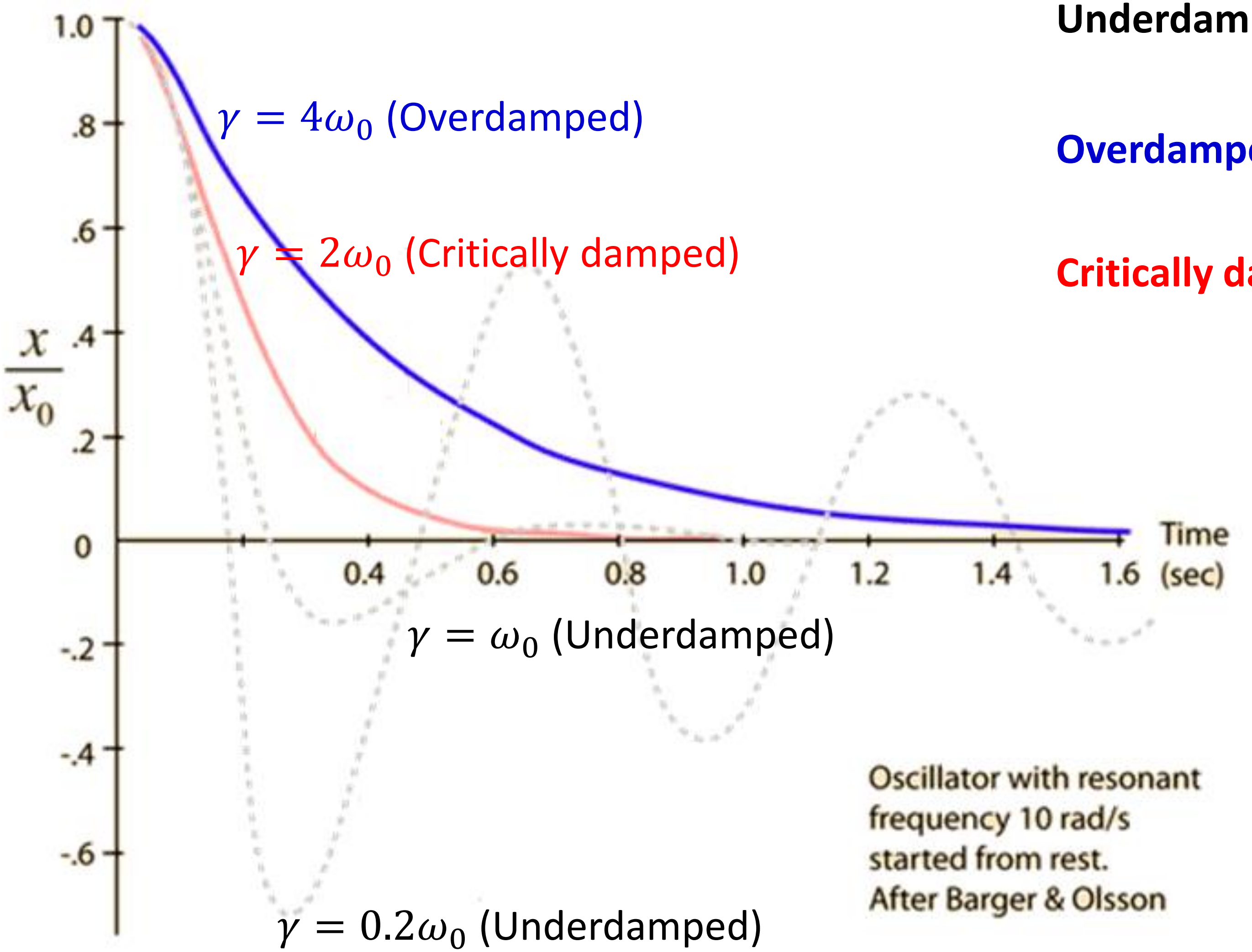
$$x(0) = 0$$

$$x(t) = Bte^{-\frac{\gamma t}{2}}$$

Decay of amplitude is fastest for critically damped!

Eg: Shock absorber in a car

Comparisons: Let's plot them together!



Underdamped: $x = Ae^{-\frac{\gamma t}{2}}\cos(\omega_0 t + \alpha)$

Overdamped: $x = A_1 e^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 e^{-\left(\frac{\gamma}{2}-\beta\right)t}$

Critically damped: $x = (A + Bt)e^{-\frac{\gamma t}{2}}$

Decay of amplitude is fastest for critically damped!

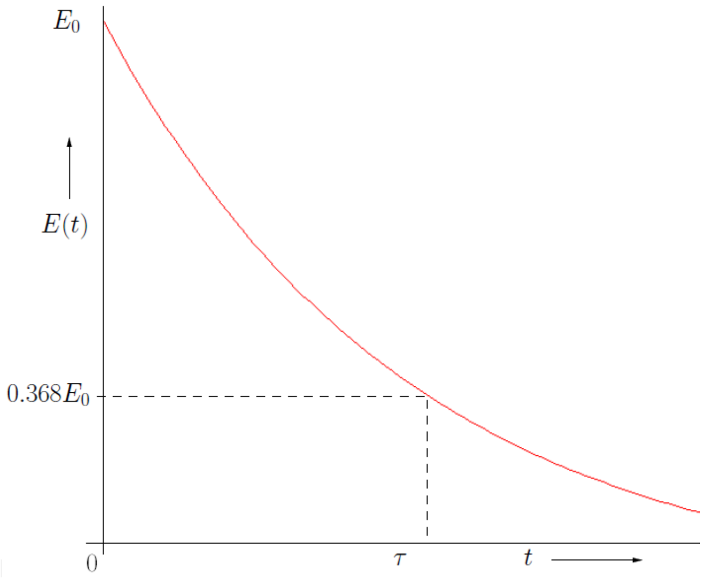
http://hyperphysics.phy-astr.gsu.edu/

Energy considerations

Under damping, amplitude falls off as: $A(t) = A_0 e^{-\frac{\gamma t}{2}}$

If $\gamma << \omega$, then we can approximate the oscillation as an SHM with a 'nearly' constant amplitude A

So we can write the total mechanical energy of the oscillator as:



$$E = \frac{1}{2}kA(t)^2$$

$$E(t) = \frac{1}{2}k\,A_0^2\,e^{-\gamma t} = E_0 e^{-\gamma t} \quad (21)$$

Also, $Q = \dfrac{Energy\ stored}{Energy\ lost\ per\ radian} = \dfrac{E}{-t_r\frac{dE}{dt}} = \dfrac{E\omega}{\gamma E}$

For low damping, $\omega \approx \omega_0$, so $Q = \dfrac{\omega_0}{\gamma}$

Q is large for less energy loss: "Quality" of the oscillation

Energy considerations: a little more detail

$$\boldsymbol{x = Ae^{-\frac{\gamma t}{2}}\cos(\omega t + \alpha)}$$

$$v = \dot{x} = -\frac{1}{2}Ae^{-\frac{\gamma t}{2}}[\gamma\cos(\omega t + \alpha) + 2\omega\sin(\omega t + \alpha)\ ]$$

Potential energy: $E_P = \frac{1}{2}kx^2$ where $k = m\omega_0^2$

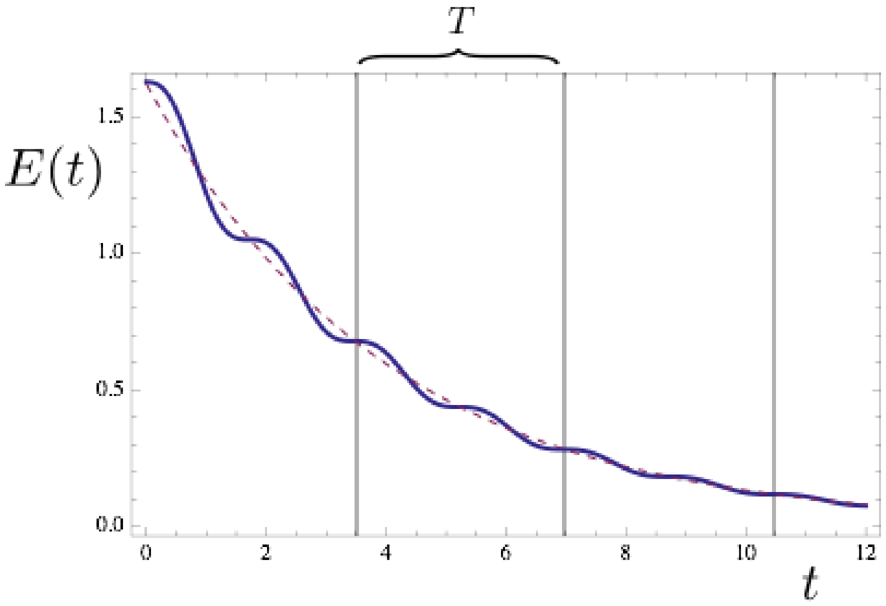
Kinetic energy: $E_k = \frac{1}{2}mv^2 = \frac{1}{8}mA^2e^{-\gamma t}[\gamma^2cos^2(\omega t + \alpha) + 4\gamma\omega\cos(\omega t + \alpha)\sin(\omega t + \alpha) + 4\omega^2sin^2(\omega t + \alpha)]$

Total energy: $E = E_P + E_k = \frac{1}{8}mA^2e^{-\gamma t}[\gamma^2cos^2(\omega t + \alpha) + 4\gamma\omega\cos(\omega t + \alpha)\sin(\omega t + \alpha) + 4\omega^2sin^2(\omega t + \alpha) +$

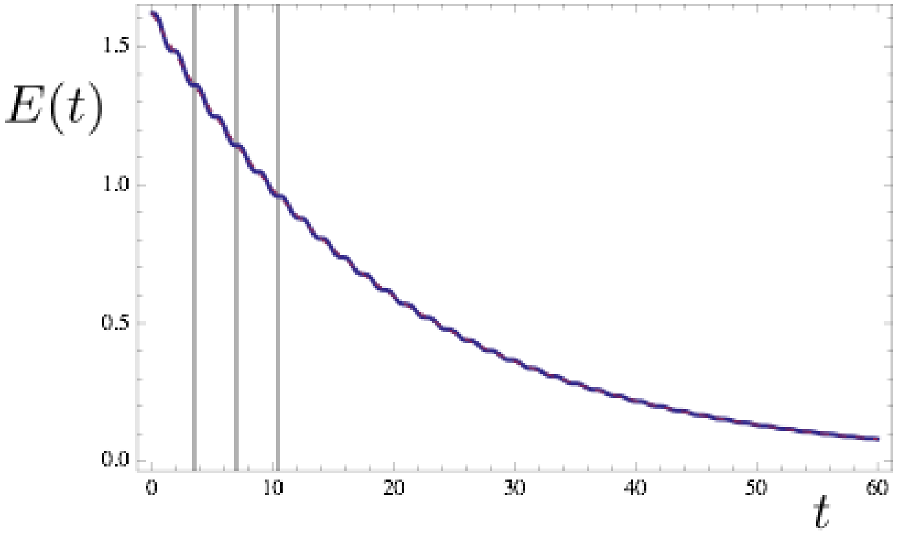
Using:
a) $2cos^2\theta = 1 + cos2\theta$

b) $\omega^2 = {\omega_0}^2 - \frac{\gamma^2}{4}$

Energy considerations: a little more detail



Energy of a damped oscillator. Force ∝ velocity, so energy is both oscillatory and decaying.



Energy of an underdamped oscillator with $\gamma \ll 2\omega_0$. Approximately exponentially decaying energy.

An example

A tuning fork having a natural frequency of 440 Hz is struck. How long will it take to reduce the energy by a factor of 5 if Q = 6912?

$$\omega_0 = 2\pi n = 2\pi \times 440 = 2765\, rad/s$$

Using $E(t) = E_0 \mathrm{e}^{-\gamma t}$, we can write $\frac{E_0}{E_0 \mathrm{e}^{-\gamma t}} = 5 \text{ or } t = \frac{1}{\gamma} \ln 5.$

Now $\gamma = \frac{\omega_0}{Q}$

$\gamma = \frac{\omega_0}{Q} = \frac{2765}{6912} = 0.4\, s^{-1}$, hence $t = 4s$