Forced vibrations

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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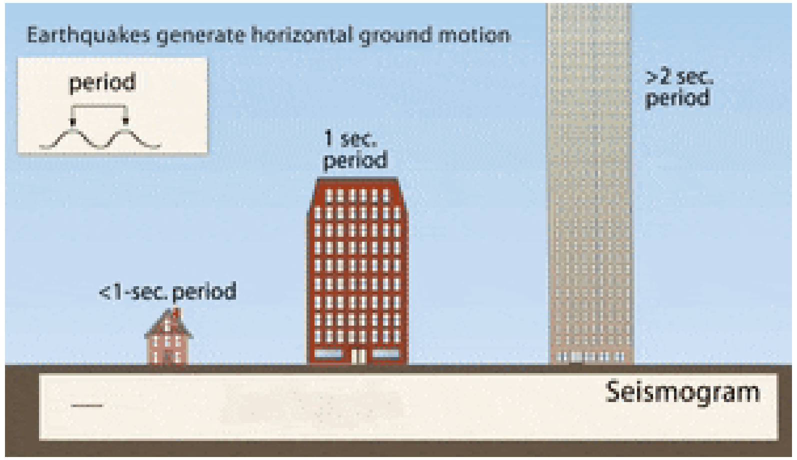
Forced oscillators

- An external driving force $F(t)$ acts on an oscillator.
- Let the force be harmonic, so $F(t) = F_0 e^{i\omega t}$
- Some examples:
  - Tides affected by periodic gravity of sun/moon
  - An electrical oscillator (LCR circuit)
  - Magnetic Resonance Imaging (MRI)
  - Musical instruments
  - Lasers



Forced undamped oscillators

- Let's write the equation of an oscillator that is harmonic and is forced, but undamped.

$$\boxed{m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t} \qquad (1)$$

Let's first look at what this equation implies qualitatively..

- Let the natural frequency of the oscillator without the force term be ω_0
- Then if $\omega < \omega_0$, the block moves 'in step' with the driving force. The restoring term kx dominates over the inertial term $m\frac{d^2x}{dt^2}$
  In phase with the driving force.
- if $\omega > \omega_0$, the block moves opposite to the driving force. The inertial term $m\frac{d^2x}{dt^2}$ dominates over the restoring term kx
  180° out of phase with the driving force.
- Check this yourself with a simple pendulum!

Forced undamped oscillators: solution

- Let's write the equation in terms of complex numbers to solve it.

$$m\frac{d^2z}{dt^2} + kz = F_0 e^{i\omega t} \qquad (2)$$

We will try the solution $z = z_0 e^{i(\omega t+\alpha)}$

Substituting in (1) we get, $(-m\omega^2 z_0 + kz_0)\mathrm{e}^{\mathrm{i}(\omega t+\alpha)} = F_0\mathrm{e}^{\mathrm{i}\omega t}$

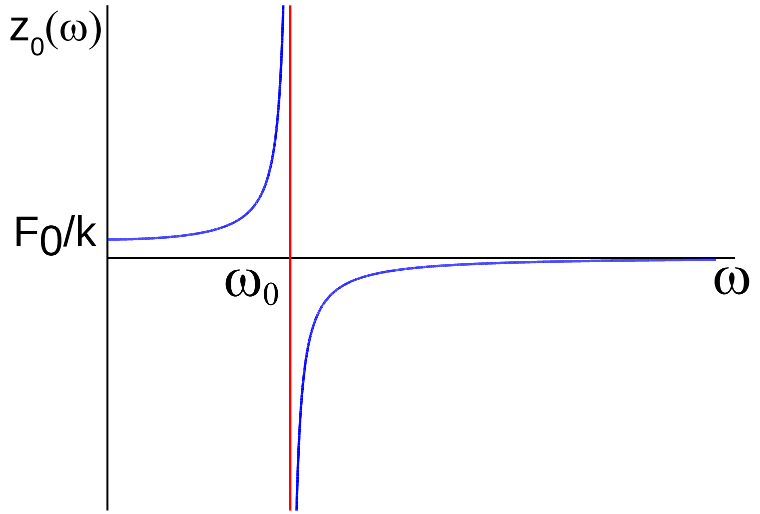
So, $$({\omega_0}^2 - \omega^2)z_0 = \frac{F_0}{m}\mathrm{e}^{-i\alpha} = \frac{F_0}{m}cos\alpha - i\frac{F_0}{m}sin\alpha \qquad (3)$$

This clearly has two solutions:

$$({\omega_0}^2 - \omega^2)z_0 = \frac{F_0}{m}cos\alpha \qquad (4)$$

$$0 = \frac{F_0}{m}sin\alpha \qquad (5)$$

$$\boxed{z_0 = \frac{F_0/m}{\omega_0^2-\omega^2}} \qquad (6)$$



Forced undamped oscillators: A and δ vs ω

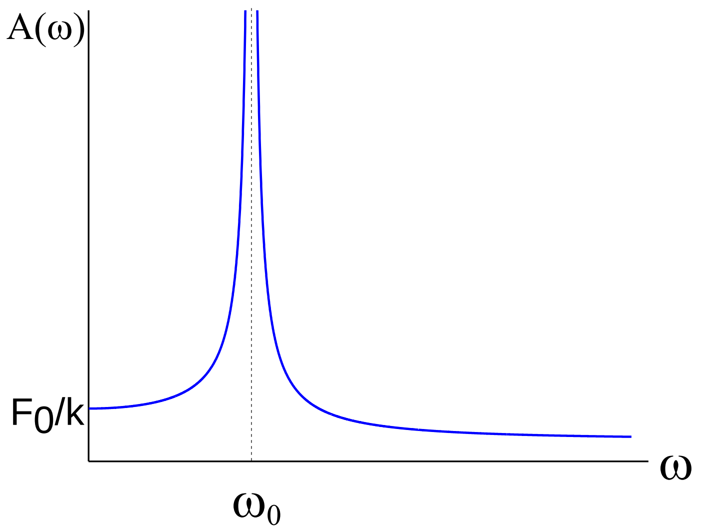
Now let's write $z = Ae^{(i\omega t+\alpha)}$ where we can see that:

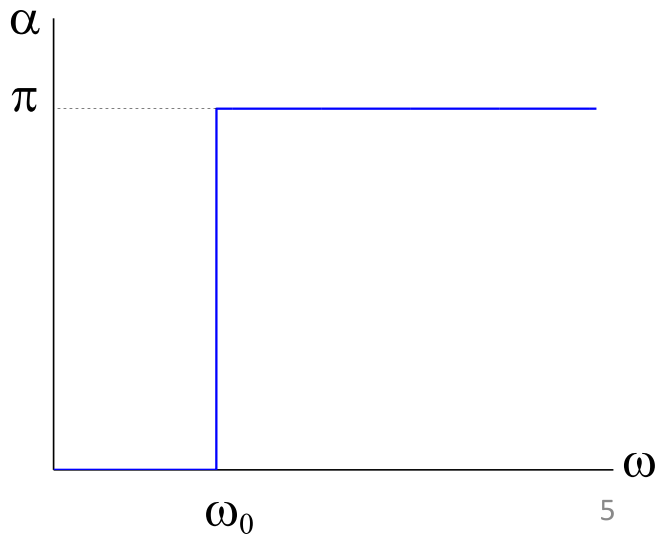
- $A = |z_0| = \frac{F_0/m}{|\omega_0^2 - \omega^2|}$ (7)
- $\alpha = 0$ if $\omega < \omega_0$ (8)

  $\alpha = \pi$ if $\omega > \omega_0$

Our observations:

- A depends on ω
- Resonance at $\omega = \omega_0 = \sqrt{\frac{k}{m}}$





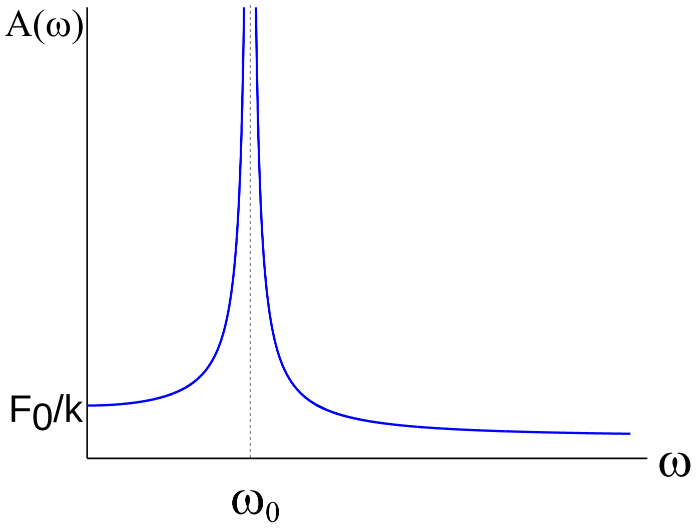
Forced undamped oscillators: problems

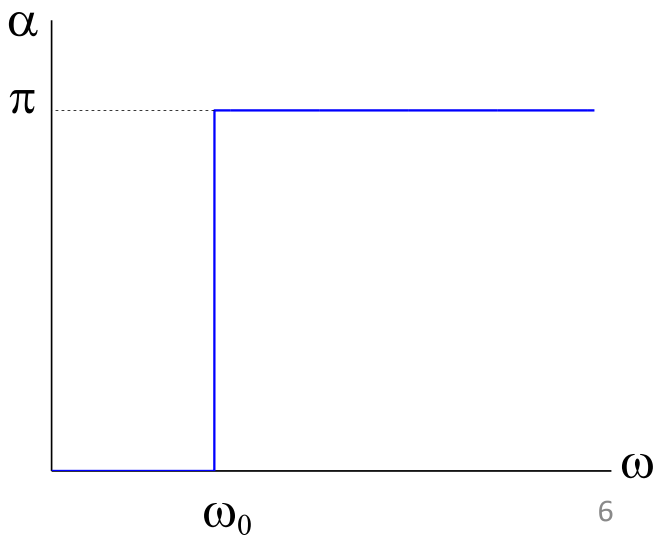
Problems in the model:

$$x = \frac{F_0/m}{|\omega_0^2 - \omega^2|} \cos(\omega t + \alpha) \qquad (9)$$

- Are the initial conditions satisfied?
  At $t = 0$, $x > 0$ or $x < 0$.
  So at the instant when the force starts to act, the displacement is already positive or negative!
  How can that be if the pendulum starts from rest?

- Amplitude becomes infinity at resonance!





Forced undamped oscillators: answers

Answers to the problems in the model:

- Solution is incomplete!

Complete solution is $x(t) = x_1(t) + x_2(t)$ (10)

$$x_1(t) = \frac{F_0/m}{|\omega_0^2 - \omega^2|}\cos(\omega t + \alpha)$$ ← "Particular solution"

$$x_2(t) = B\cos(\omega_0 t + \theta)$$ ← Homogeneous solution

(general solution of a free undriven oscillator $m\frac{d^2x}{dt^2} + kx = 0$)

This implies that the motion of a forced undamped oscillator is a 'superposition' of oscillations at two frequencies ω and ω_0.

- The situation is unphysical! Real systems always have damping so amplitude can never become infinity!

Forced undamped oscillators: the general solution

Let's use the solution (10) with $\theta = 0$:

$x(t) = C\cos(\omega t + \alpha) + B\cos(\omega_0 t)$ where $C = \frac{F_0/m}{|\omega_0^2 - \omega^2|}$

$\frac{dx}{dt} = -\omega C sin(\omega t + \alpha) - \omega_0 B sin(\omega_0 t)$

Putting the two boundary conditions:

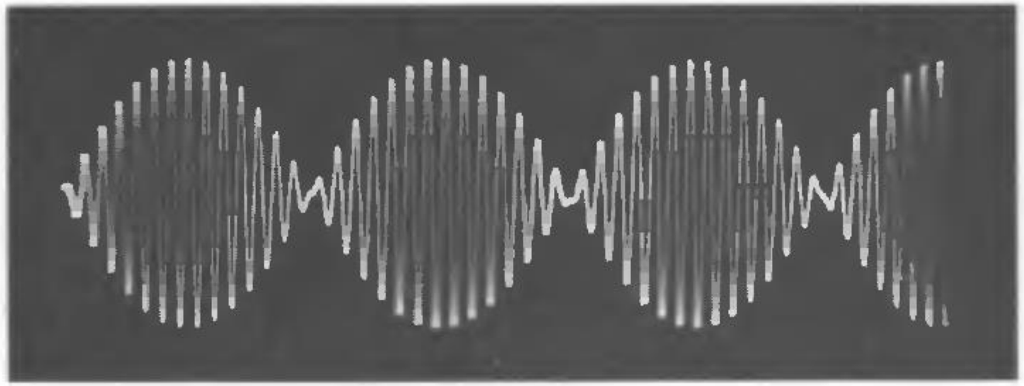
$x = 0 \; and \; \frac{dx}{dt} = 0 \; at \; t = 0$, we get:

$C\cos(\alpha) + B = 0$

$\sin(\alpha) = 0$, so $\alpha = 0 \; or \; \pi$

Taking $\alpha = 0$, $B = -C$

So, $x(t) = C\big(\cos(\omega t) - cos(\omega_0 t)\big) = \boxed{2C\, sin\left(\frac{\omega + \omega_0}{2}\right) t \; sin\left(\frac{\omega - \omega_0}{2}\right) t}$ (11)



Beats! This would continue indefinitely without damping!

For real systems, damping will kill the natural oscillations $Bcos(\omega_0 t)$ through a time interval known as the 'transient state'. Then the system will oscillate with the frequency of the driving force ω and reach the 'steady state'.

Forced Damped oscillators

$$\boxed{m\frac{d^2x}{dt^2}} + \boxed{b\frac{dx}{dt}} + \boxed{kx} = \boxed{F_0 \cos \omega t} \qquad (12)$$

Inertial term Damping term Restoring term Forcing term

This is equivalent to: : $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} cos\omega t$ where $\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$ (13)

Let's write this equation in terms of complex numbers: $\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$ (14)

$$z = Ae^{i(\omega t - \delta)} \qquad (15)$$

Putting (15) in (14), we get: $(-\omega^2 + i\omega\gamma + {\omega_0}^2)Ae^{i(\omega t-\delta)} = \frac{F_0}{m} e^{i\omega t}$

$$(-\omega^2 + {\omega_0}^2)A + i\omega\gamma A = \frac{F_0}{m} e^{i\delta}$$

Forced Damped oscillators: the steady state solution

Separating the real and imaginary parts:

$$(-\omega^2 + {\omega_0}^2)A = \frac{F_0}{m} cos\delta$$

$$\omega\gamma A = \frac{F_0}{m} sin\delta$$

Therefore, $$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2-\omega^2\right)^2+(\gamma\omega)^2\right]^{1/2}} \qquad (16)$$

$$\delta(\omega) = tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right) \qquad (17)$$

Taking the real part, $$x = Acos(\omega t - \delta) \qquad (18)$$

Forced Damped oscillators: A and δ vs ω

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2\right]^{1/2}}$$

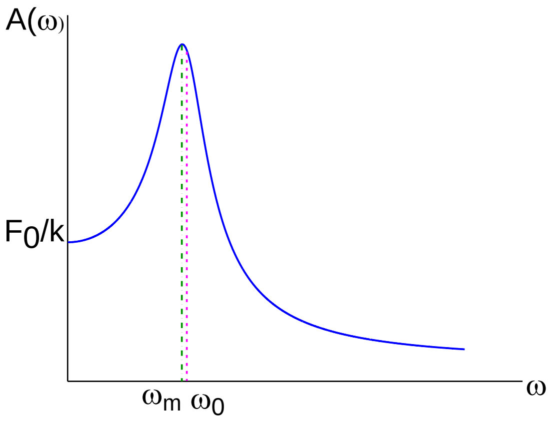
$$\delta(\omega) = tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

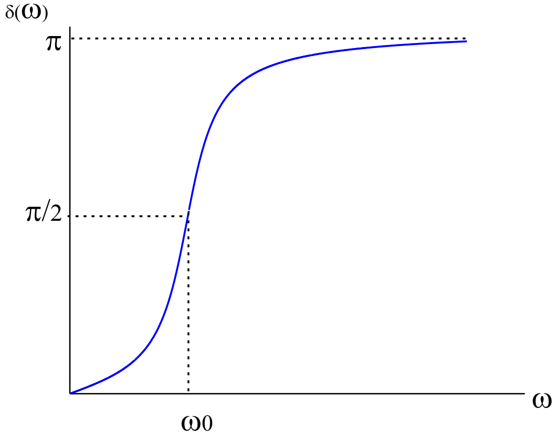
This is the "steady state" solution!

Observations:

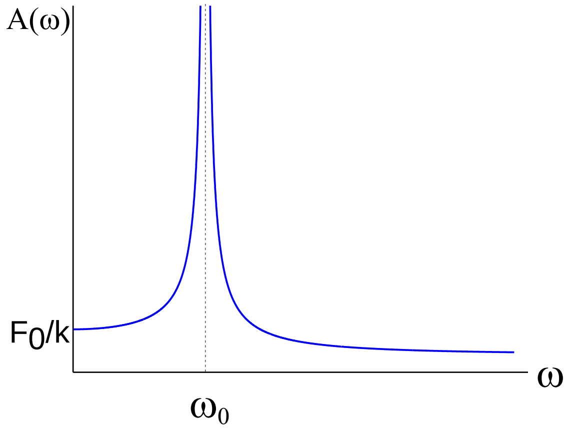
- A depends strongly on ω becoming max at resonance.
- For high damping, the max amplitude frequency becomes smaller than ω_0(negligible difference in most cases of practical interest)
- As ω is varied through ω_0 a phase change of π

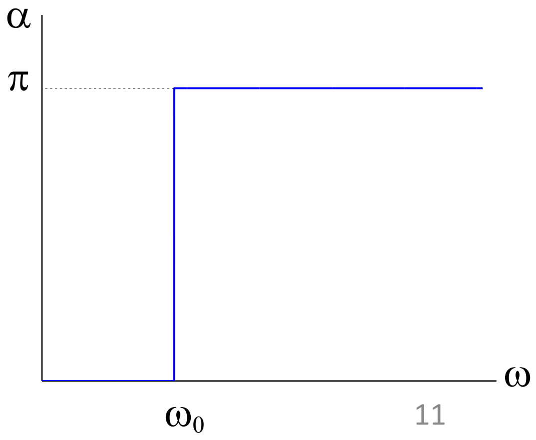
Forced damped





Forced undamped





Forced Damped oscillators: A and δ for different Q

What is the frequency ω_m at which the amplitude becomes maximum?

Putting, $\frac{dA}{d\omega} = 0$, we can find the maximum is at $\omega_m = \left(\omega_0^2 - \frac{\gamma^2}{2}\right)^{1/2}$ ← check this!

Remembering that Quality factor $Q = \frac{\omega_0}{\gamma}$,

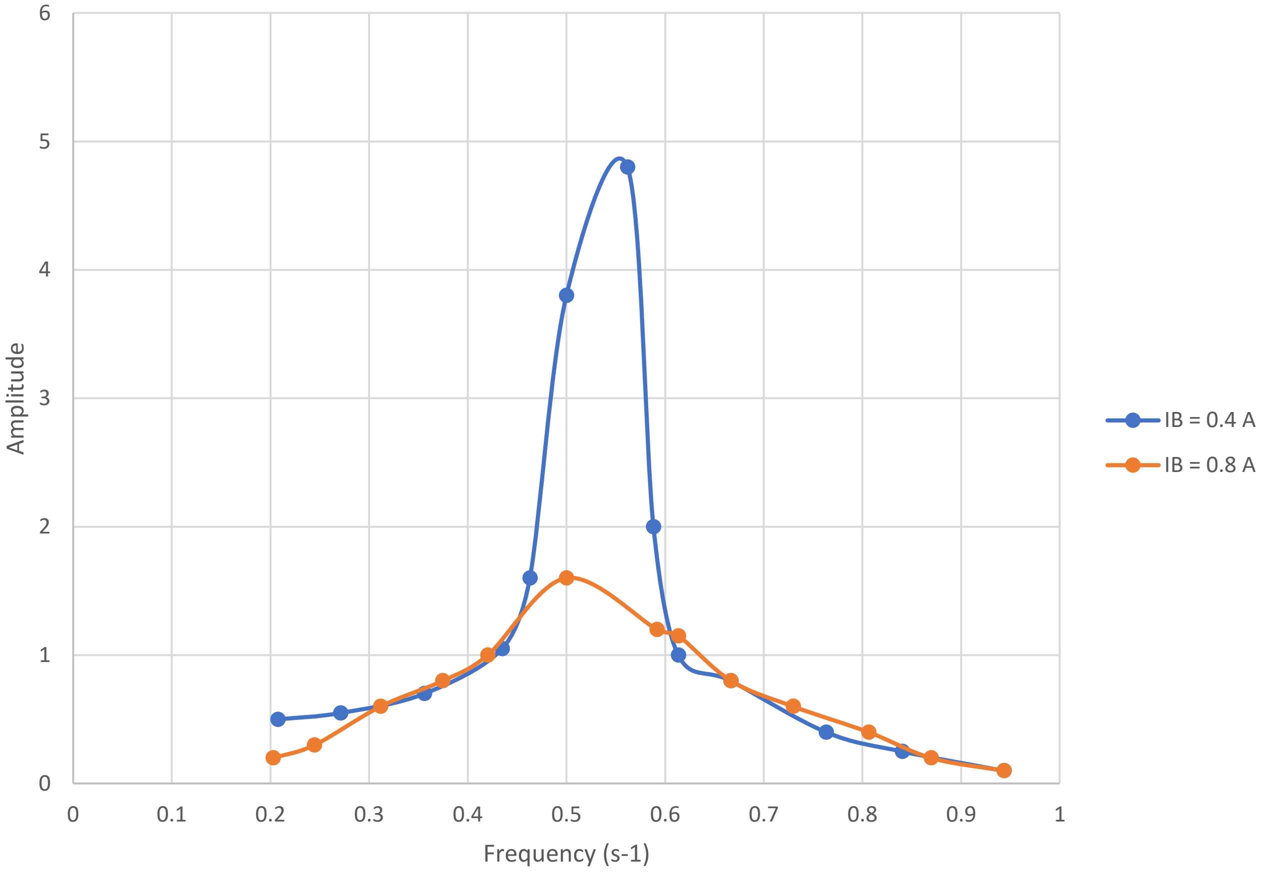
$$\omega_m = \omega_0 \left(1 - \frac{1}{2Q^2}\right)^{1/2} \quad (19)$$

$$A_m = A_0 \frac{Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}} \quad (20)$$

TABLE 4-1: RESONANCE PARAMETERS OF DAMPED SYSTEMS

Q	ω_m/ω_0	A_m/A_0
$1/\sqrt{2}$	0	1
1	$1/\sqrt{2} = 0.707$	$2/\sqrt{3} = 1.15$
2	$\sqrt{\frac{7}{8}} = 0.935$	$8/\sqrt{14} = 2.06$
3	$\sqrt{\frac{17}{18}} = 0.973$	$18/\sqrt{35} = 3.04$
5	$\sqrt{\frac{49}{50}} = 0.990$	$50/\sqrt{99} = 5.03$
$\gg 1$	$1 - 1/4Q^2$	$Q[1 + 1/(8Q^2)]$

Extras: The experiment that you did in Phys lab I



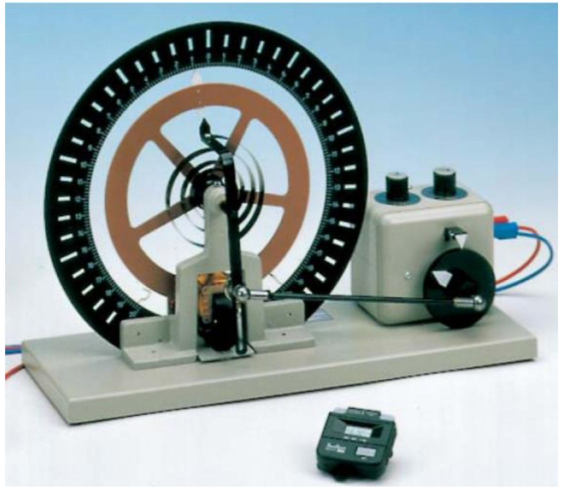
$$\omega_0 = 3.355 \text{ rad/s}$$

$$\gamma_{\text{blue}} = 0.163 \text{ /s}$$

$$\gamma_{\text{orange}} = 0.752 \text{ /s}$$

$$Q_{\text{blue}} = 20.6$$

$$Q_{\text{orange}} = 4.5$$



Extras: Some typical Q values

Car suspension	1
Tethered trolley	10
Simple pendulum	1,000
Guitar string	1,000
Quartz crystal of watch	105
Excited atom	107
Excited nucleus	1,012

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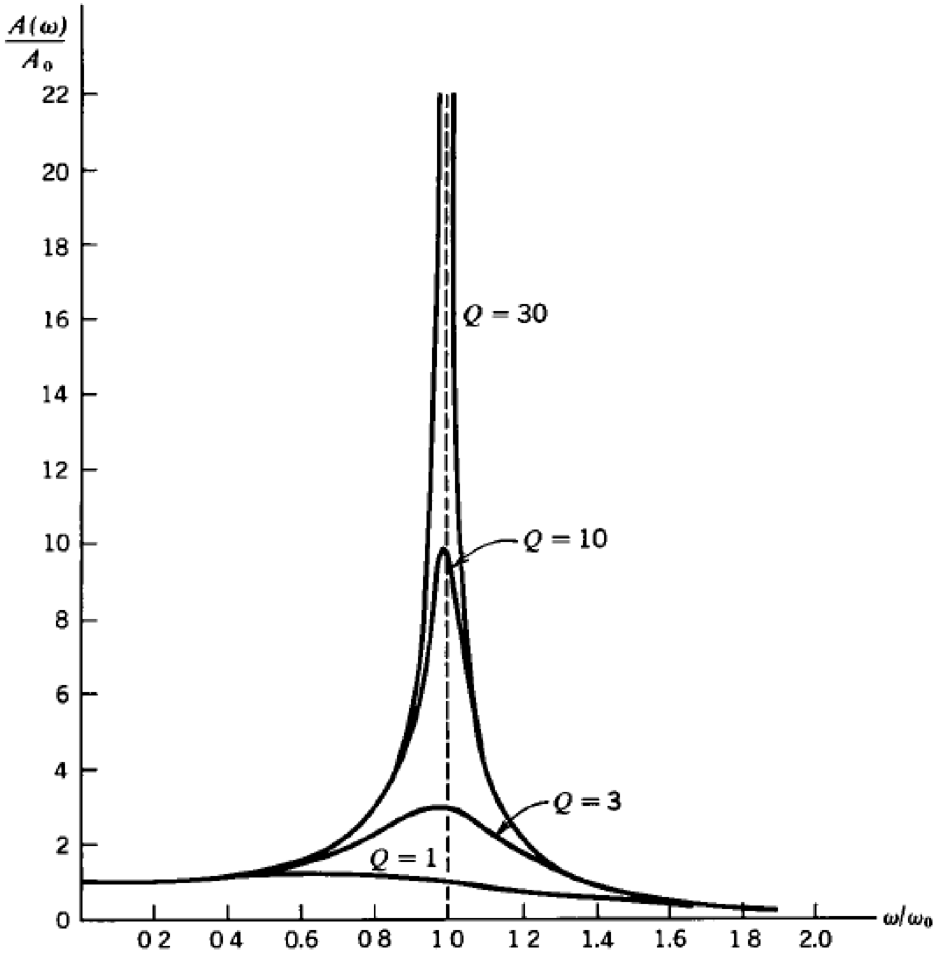
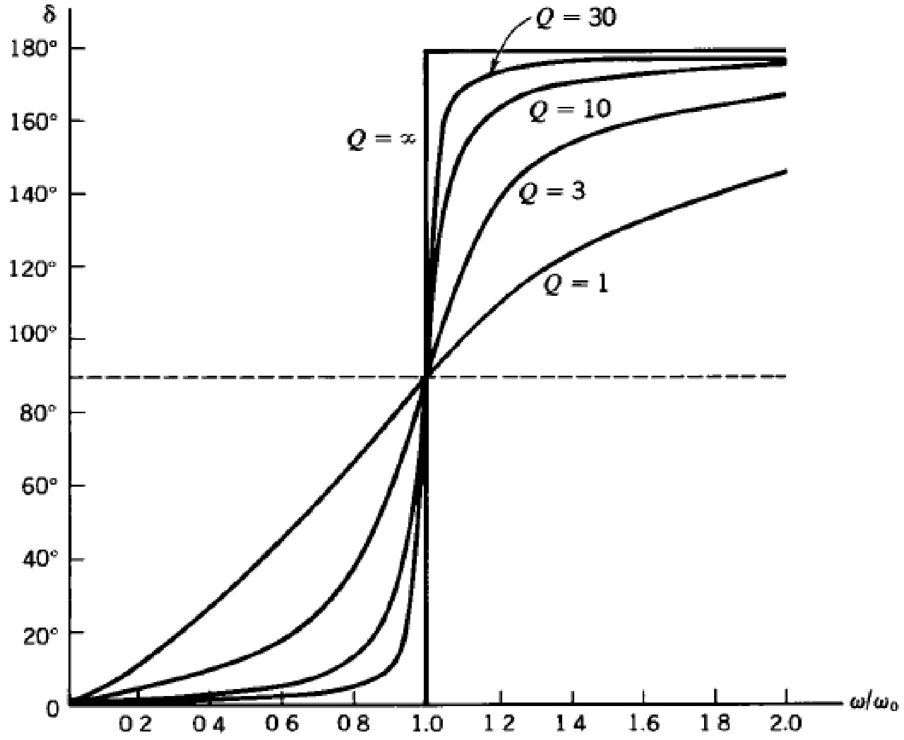
Forced Damped oscillators: A and δ for different Q



$A(\omega) vs\ \omega$



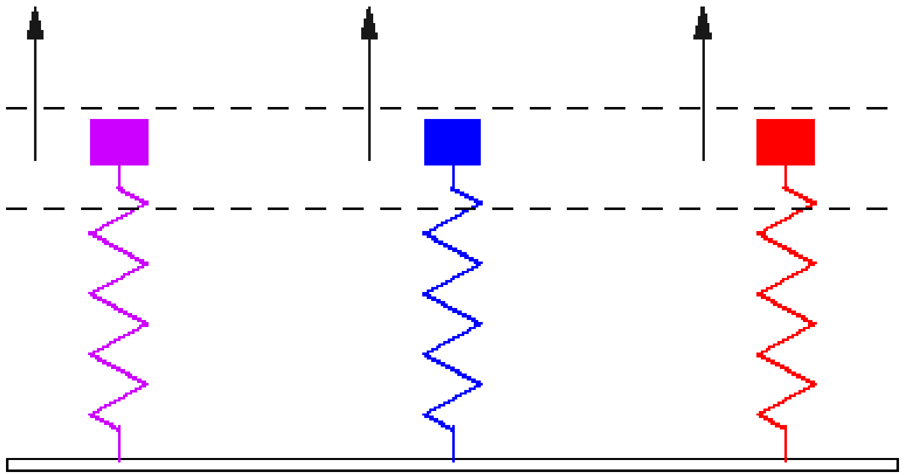
$\delta(\omega)\ vs\ \omega$

Observations and interpretations

$\omega \ll \omega_0$
$\delta \approx 0$

$\omega = \omega_0$
$\delta = \frac{\pi}{2}$

$\omega \gg \omega_0$
$\delta \approx \pi$

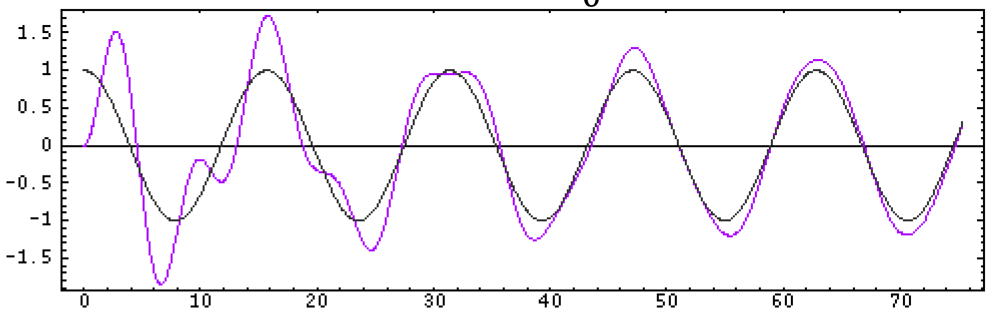


Restoring term dominates over inertial term

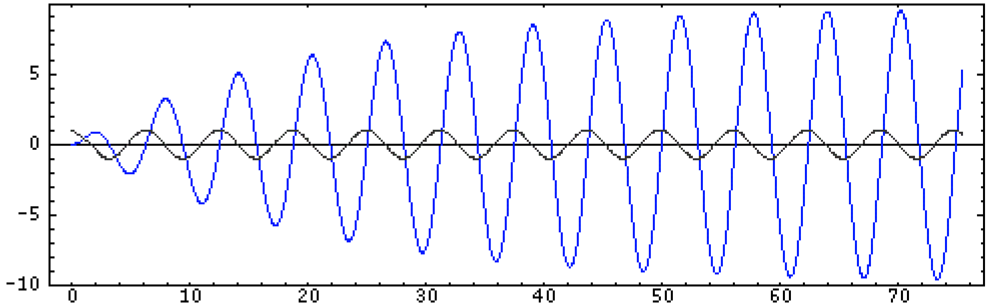
Inertial term equal to restoring term and entire energy is spent to overcome damping

Inertial term dominates over restoring term

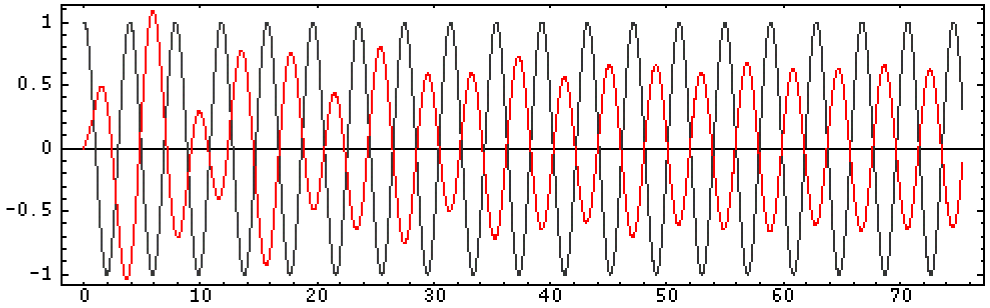
$\omega = 0.4\omega_0$



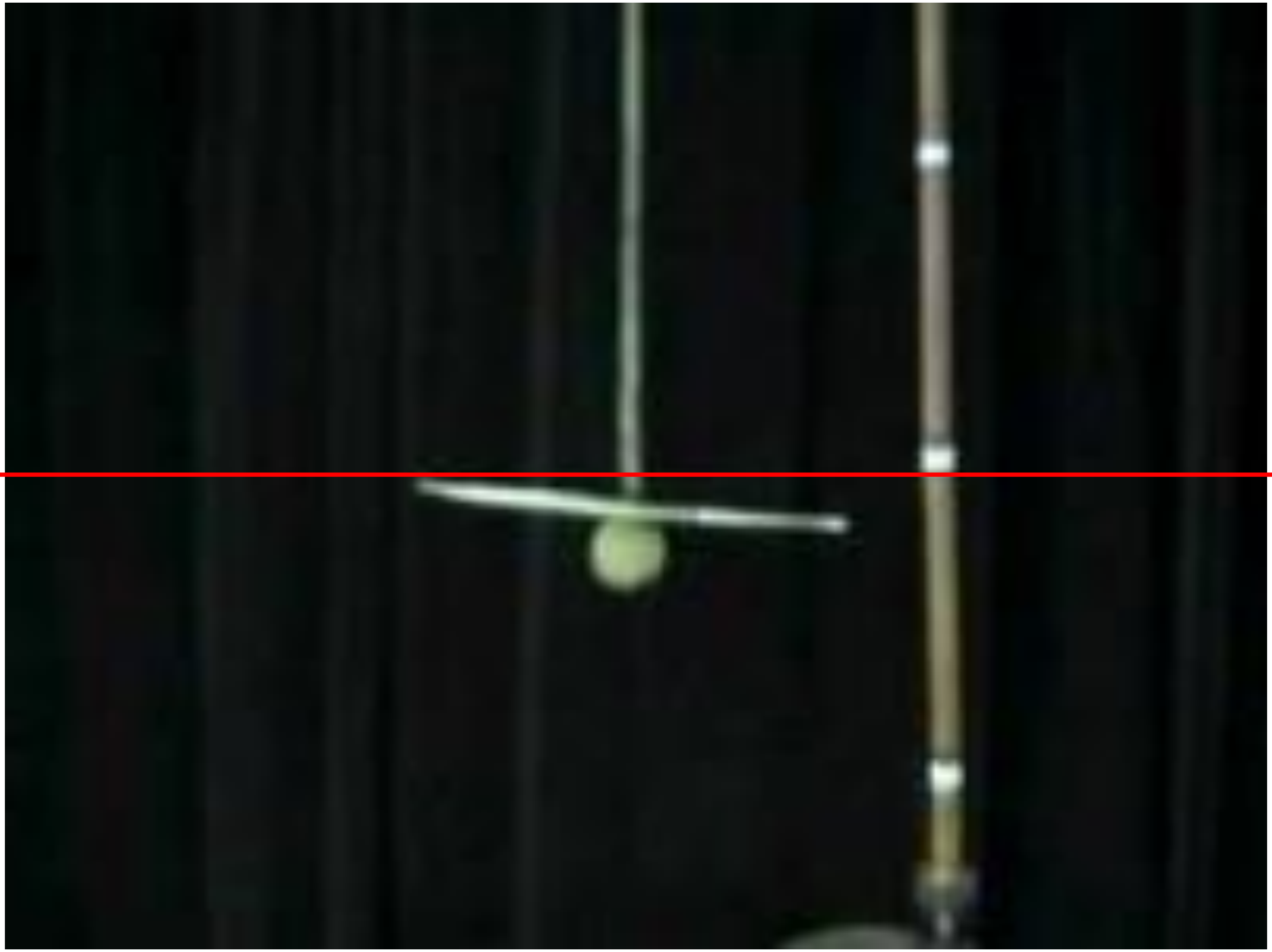
$\omega = 1.01\omega_0$



$\omega = 1.6\omega_0$



Observations and interpretations



The transient state

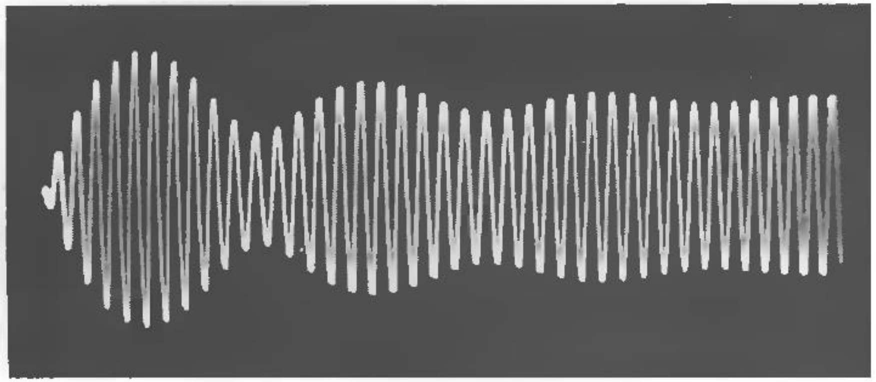
Let's look into the solution of the homogeneous equation: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \ kx \ \ = 0$ (21)

This has a solution of the form $x = e^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \alpha)$ where $\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}$ which is just the solution for an unforced damped oscillator.

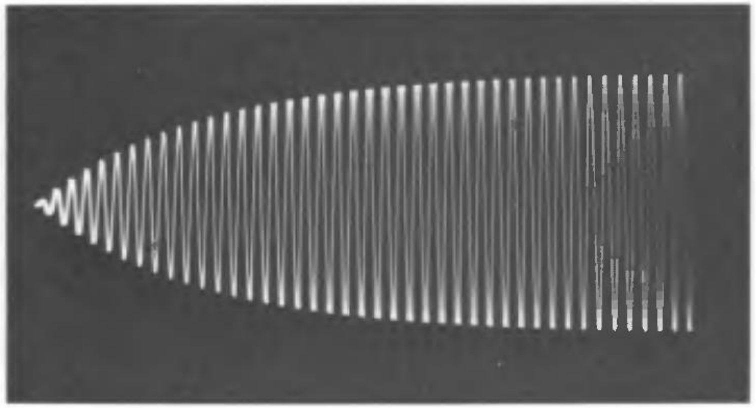
So complete solution of the forced damped oscillator:

$$x = A\cos(\omega t - \delta) + Be^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \alpha) \qquad (22)$$

Damping will kill the second term in the 'transient state'. Then the system will oscillate with the frequency of the driving force ω with a phase difference δ.



Transient behaviour off-resonance



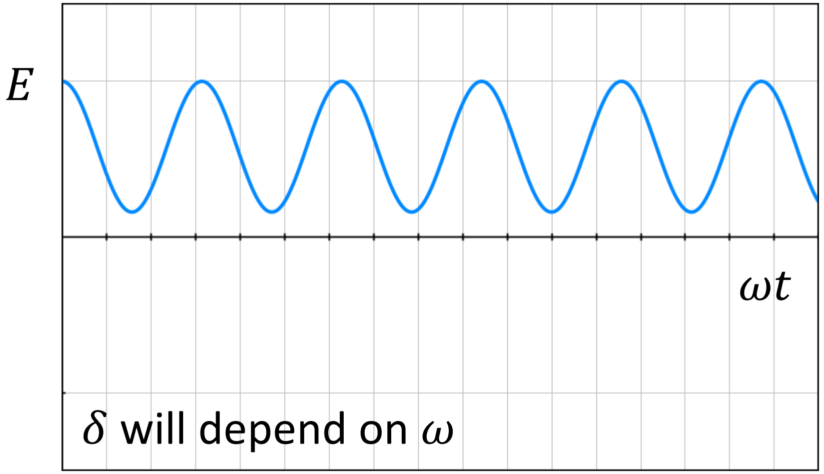
Transient behaviour at resonance

Energy considerations

- Energy of a forced damped oscillator is NOT constant.
- Let's consider the steady state solution.
- $x = A\cos(\omega t - \delta)$
- $\dot{x} = -\omega A\sin(\omega t - \delta)$
- $E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$$= \frac{1}{2}A^2(m\omega^2 \sin^2(\omega t - \delta) + k\cos^2(\omega t - \delta))$$ Here $k = m\omega_0^2$, so energy is a function of time!

$$= \frac{1}{2}A^2 m(\omega_0^2 + (\omega^2 - \omega_0^2)\sin^2(\omega t - \delta))$$



- The driving force feeds energy into the system. The damping dissipates energy from the system.

Energy considerations

The average energy:

$$\langle E \rangle = \left\langle \frac{1}{2} A^2 (m\omega^2 sin^2(\omega t - \delta) + k cos^2(\omega t - \delta)) \right\rangle$$

$$= \frac{1}{4} m A^2 (\omega^2 + \omega_0^2)$$

$$\langle E \rangle \;= \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2} \qquad (23)$$

Power absorbed by a driven oscillator

Instantaneous power input : $P = F.v$

$$F = F_0 cos\omega t$$

$$x = Acos(\omega t - \delta)$$

$$P = -\omega A\, F_0\, cos\omega t\; sin(\omega t - \delta)$$

$$= -\omega A\, F_0\, cos\omega t\,(sin\omega t cos\delta - cos\omega t sin\delta)$$

$$= -\omega A\, F_0\,(cos\omega t\; sin\omega t cos\delta - cos^2\omega t sin\delta)$$

$$\int_0^T cos\omega t\; sin\omega t\; dt = 0$$

$$\int_0^T cos^2\omega t\; dt = \frac{T}{2}$$

$$\omega\gamma A = \frac{F_0}{m} sin\delta$$

Average power per cycle: $< P > = \frac{1}{T}\int_0^T P dt$

$$= \frac{1}{T}\int_0^T (-\omega A\, F_0\, cos\omega t\; sin\omega t cos\delta + \omega A\, F_0 cos^2\omega t sin\delta) dt$$

$$= \frac{1}{2}\,\omega A\, F_0 sin\delta = \frac{1}{2}\, m\gamma\omega^2 A^2 \quad (24)$$

Power dissipated by a driven oscillator

Power dissipated by the damping present in the system : $P = -bv.v = -m\gamma v^2$

$$x = A\cos(\omega t - \delta)$$

$$v = \dot{x} = -\omega A\sin(\omega t - \delta)$$

So, $P = -bv.v = -m\gamma v^2 = -m\gamma\omega^2 A^2 \sin^2(\omega t - \delta)$

Average power dissipated per cycle: $< P > = \frac{1}{T}\int_0^T P dt$

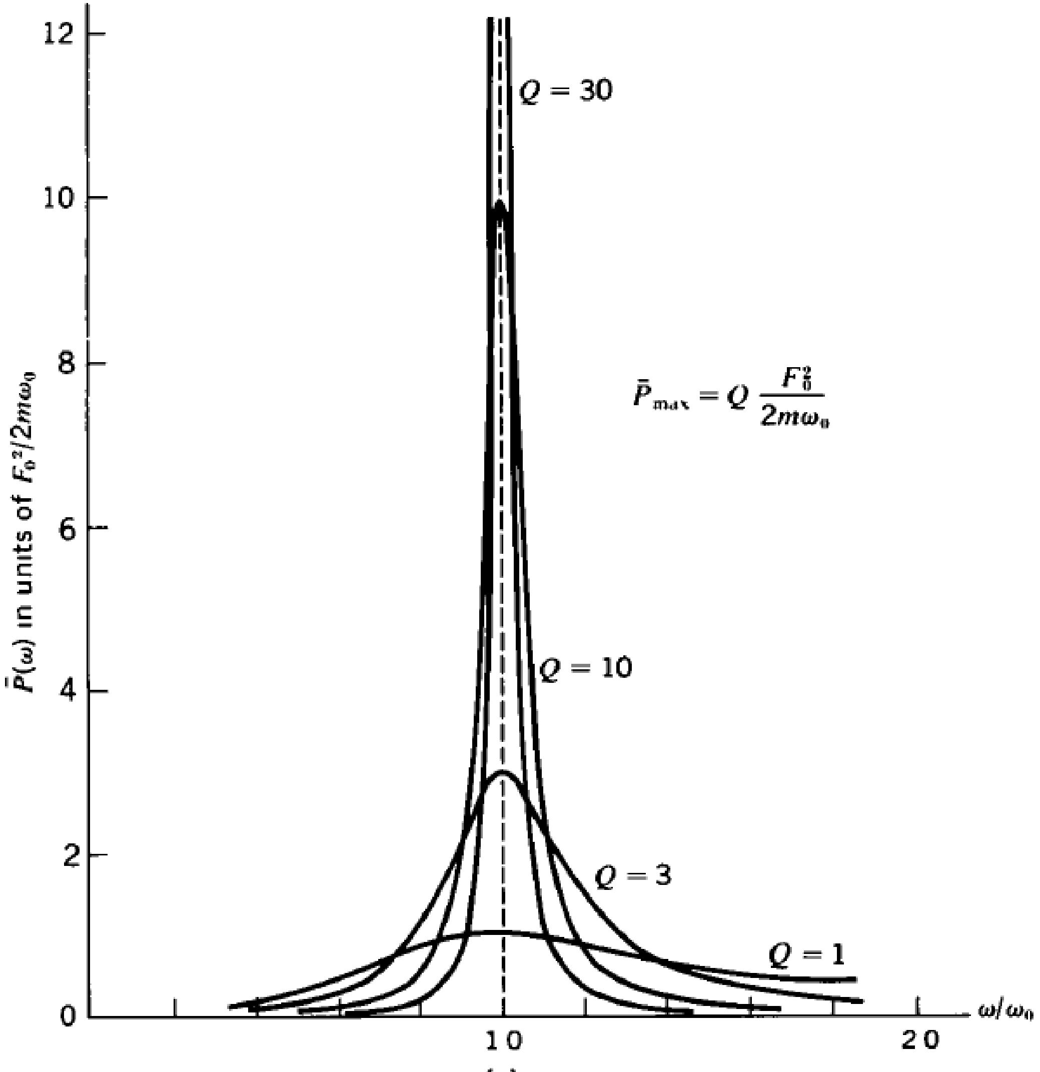
$$= -\frac{1}{T}\int_0^T m\gamma\omega^2 A^2 \sin^2(\omega t - \delta) dt$$

$$= -\frac{1}{2} m\gamma\omega^2 A^2$$

$$\int_0^T \sin^2(\omega t - \delta)\, dt = \frac{T}{2}$$

So at steady state, power absorbed per cycle = power dissipated per cycle!

Power absorbed by a driven oscillator



(a)

$$< P > \; = \frac{1}{2}\,\omega A\, F_0 sin\delta$$

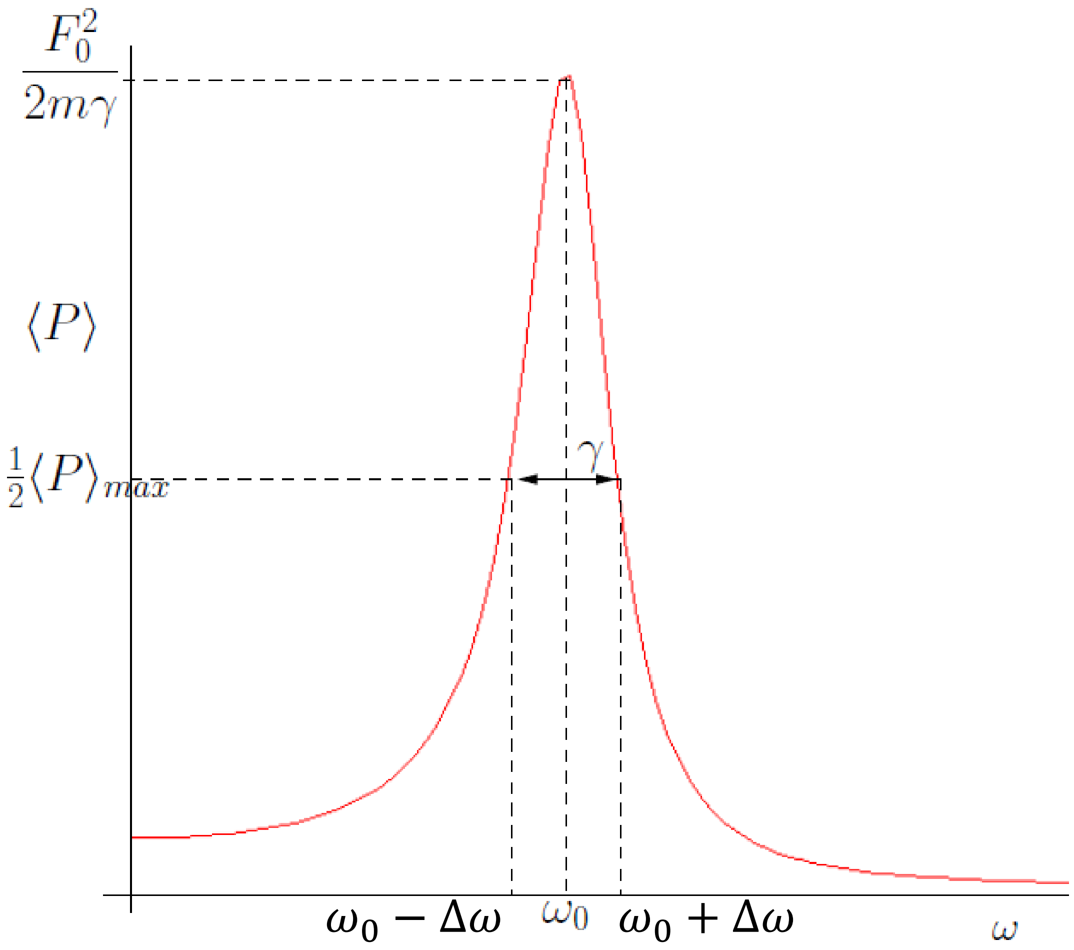
$$= \frac{F_0^2}{2m}\,\frac{\gamma\omega^2}{\omega^2\gamma^2+\left(\omega_0^2-\omega^2\right)^2}$$

$$= \frac{F_0^2\,\omega_0}{2kQ}\,\frac{1}{\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\frac{1}{Q^2}}$$

The power is maximum when $\omega = \omega_0$

$$\langle P\rangle_{\max} = \frac{F_0^2}{2m\gamma} \qquad (25)$$

Width of the resonance



$$< P > = \frac{F_0^2}{2m} \frac{\gamma \omega^2}{\omega^2 \gamma^2 + (\omega_0^2 - \omega^2)^2}$$

Since at resonance $\omega \approx \omega_0$

$$(\omega_0^2 - \omega^2)^2 = (\omega + \omega_0)^2 (\omega - \omega_0)^2 \cong 4\omega^2 (\omega - \omega_0)^2$$

$$< P > = \frac{F_0^2 \gamma}{2m} \frac{1}{4(\omega_0 - \omega)^2 + \gamma^2}$$

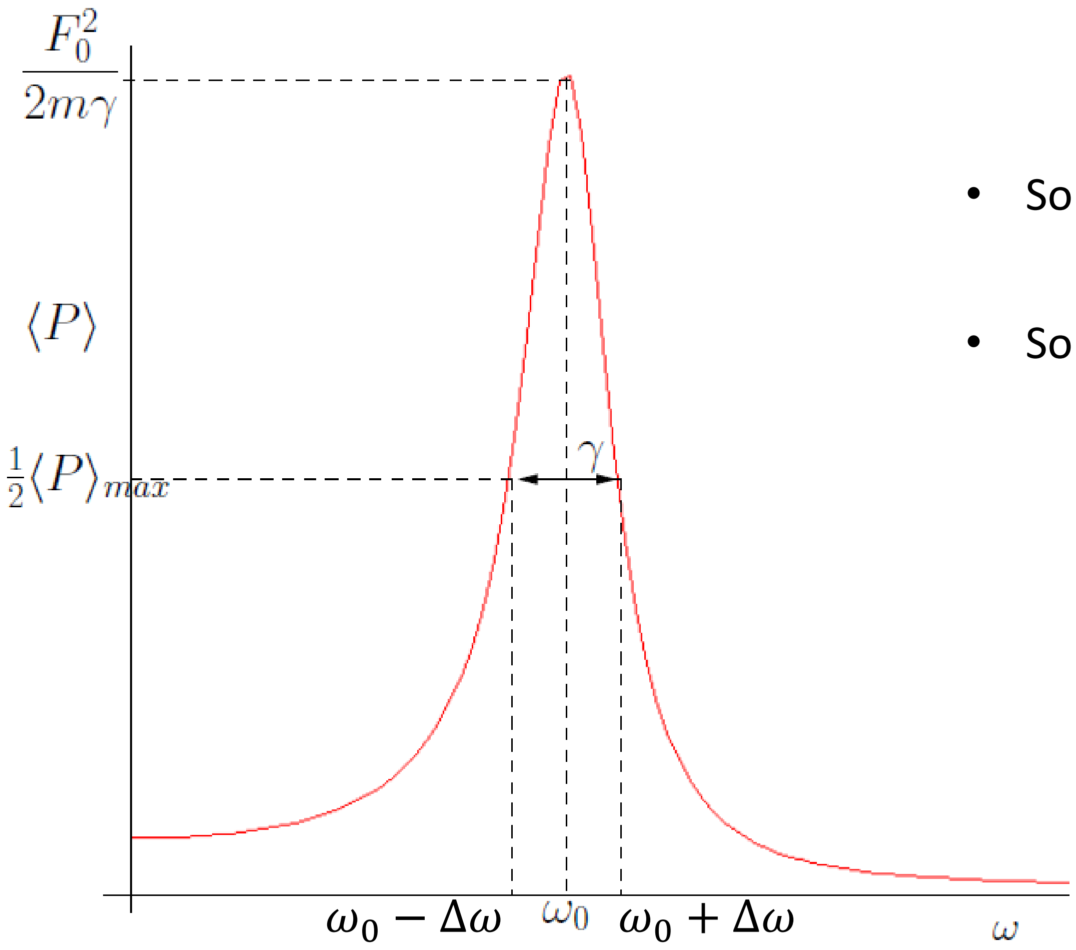
Let at frequencies $\omega = \omega_0 \pm \Delta\omega$ the power fall to half of $\langle P \rangle_{\max}$

$$\frac{F_0^2}{4m\gamma} = \frac{F_0^2 \gamma}{2m} \frac{1}{4(\Delta\omega)^2 + \gamma^2} \quad \text{so, } 4(\Delta\omega)^2 = \gamma^2$$

Full width of resonance at half maxima (FWHM):

$$2\Delta\omega = \gamma \qquad (26)$$

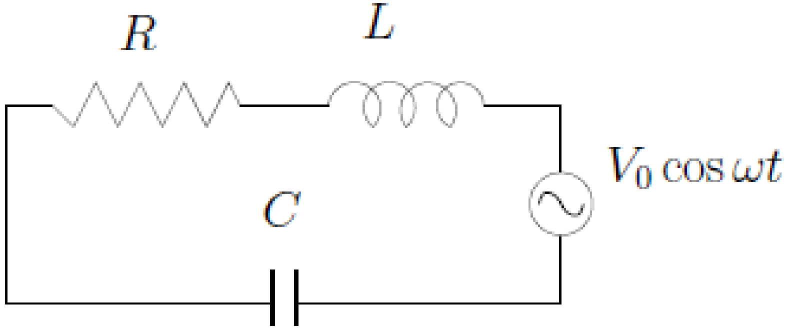
Width of the resonance



$$2\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$

- So if the resonance is "sharp", that is $2\frac{\Delta\omega}{\omega_0} \ll 1, Q$ is large.
- So if the resonance is "broad", that is $2\frac{\Delta\omega}{\omega_0} \gg 1, Q$ is small.

An example: LCR circuit in a radio



Mechanical analogy:

Mass $m \sim L$(inductance)
Friction $b \sim R$ (resistance)
Spring constant $k \sim 1/C$ (C is capacitance)

Natural frequency of charge oscillation:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Add voltage drop across each element:

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + IR + \frac{q}{C} = V_0 \cos \omega t$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = V_0 \cos \omega t$$

Damping constant : $\gamma = \frac{R}{L}$

Quality factor: $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

Use: AM/FM radios with analog tuners to tune to different frequencies