N-Coupled oscillations to Fourier Analysis

COURSE NAME: Mechanics, Oscillations and Waves (MOW)

PHY F111

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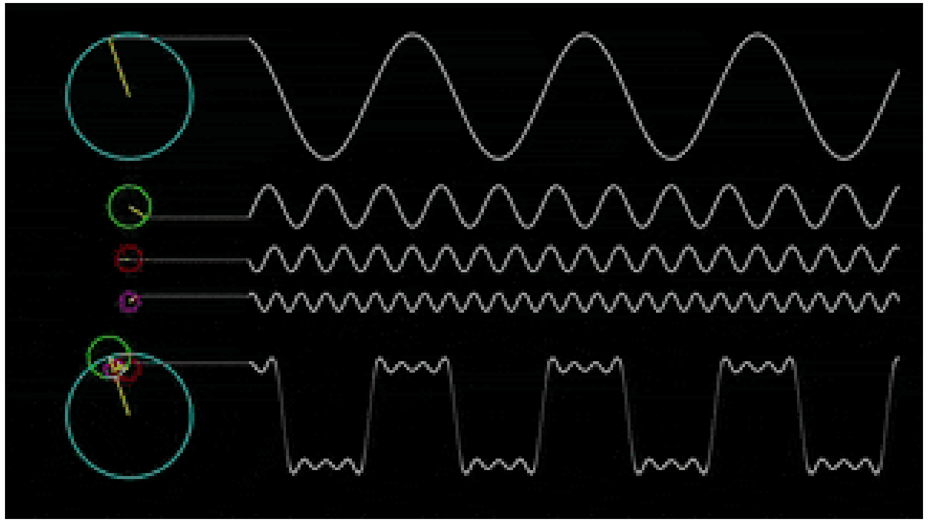
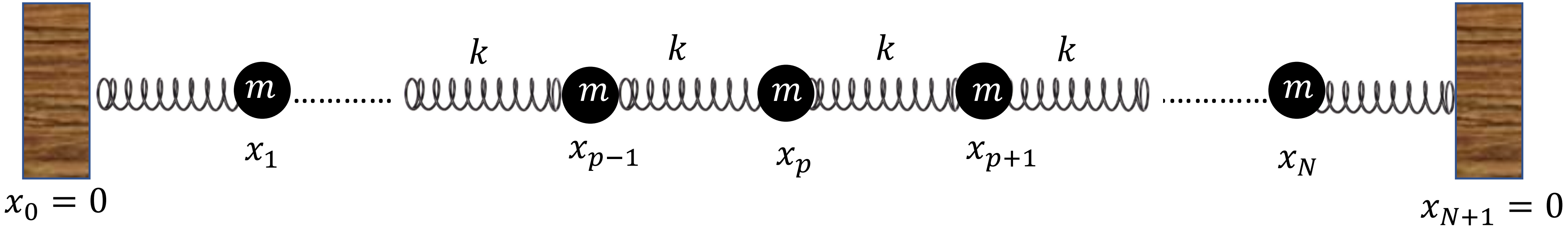


N coupled oscillators: longitudinal vibrations

- Consider a system of N coupled masses (each of mass m) with $N+1$ springs each having a spring constant k.
- The last two springs are connected to walls, where the displacement is zero as before.
- Displacements of the N coupled masses are $x_1, x_2, x_3, \dots\dots\dots, x_N$



- Force on the pth mass :

$$m\ddot{x}_p = -k(x_p - x_{p-1}) + k(x_{p+1} - x_p) = -2kx_p + kx_{p-1} + kx_{p+1} \qquad (25)$$

Remember these are N equations! Using the matrix method therefore we can write:

N coupled oscillators

Let the trial solution for the p^{th} mass be $x_p = A_p\, cos\omega t$ for a given normal mode. (26)

Then putting this in equation (25), we get:

$$-\omega^2 A_p = \omega_0^2 (A_{p-1} - 2A_p + A_{p+1})$$

Therefore, $\frac{A_{p-1}+A_{p+1}}{A_p} = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$

Note that these are basically N equations!

$$\omega_0^2 = \frac{k}{m}$$

For any particular value of ω, the right hand side is a constant. So left hand side should also be a constant and independent of p.

N coupled oscillators

- Let us write $x_p = A_p\, cos\omega t = C \sin p\theta\, cos\omega t$ where θ is some angle.
- Then we can write: $A_{p-1} + A_{p+1} = C\sin(p-1)\theta + C\sin(p+1)\theta = 2C\sin p\theta cos\theta$
- So, $\frac{A_{p-1}+A_{p+1}}{A_p} = 2cos\theta$ (Since $A_p = C\sin p\theta$)

  Constant and independent of p, but need to find θ

- Let's now impose the boundary conditions at the 'walls'. $A_0 = 0$ and $A_{N+1} = 0$
- So, $A_{N+1} = C\sin(N+1)\theta = 0,$
- so $(N+1)\theta = m\pi$ where $(m = 1,2,3, \ldots \ldots)$

$$\theta = \frac{m\pi}{N+1}$$

Therefore, $A_p = C\sin\frac{pm\pi}{N+1}$

N coupled oscillators

So,

$$\frac{A_{p-1}+A_{p+1}}{A_p} = \frac{2\omega_0^2-\omega^2}{\omega_0^2} = 2cos\frac{m\pi}{N+1}$$

$$1 - cos\varphi = 1 - \cos\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right)$$
$$= 1 - cos^2\frac{\varphi}{2} + sin^2\frac{\varphi}{2}$$
$$= 1 - 1 + sin^2\frac{\varphi}{2} + sin^2\frac{\varphi}{2} = 2sin^2\frac{\varphi}{2}$$

$$\omega^2 = 2\omega_0^2\left[1 - cos\frac{m\pi}{N+1}\right]$$

$$= 4\omega_0^2\, sin^2\left[\frac{m\pi}{2(N+1)}\right]$$

Frequency for the mth normal mode:

$$\boxed{\omega_m = 2\omega_0\, sin\left[\frac{m\pi}{2(N+1)}\right]}$$

$(m = 1,2,3, \dots\dots)$

Coordinate for ***p*th** particle in ***m*th** normal mode:

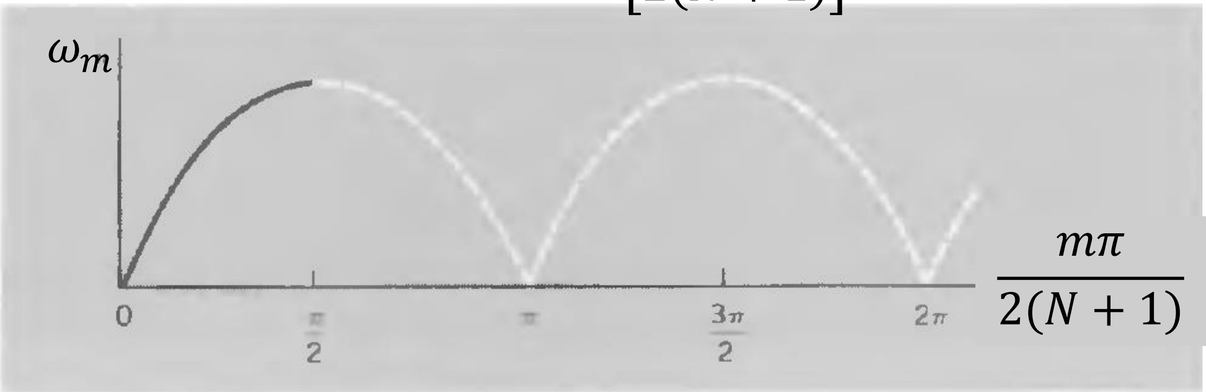
$$\boxed{x_{pm} = C_m \sin\frac{pm\pi}{N+1} cos\omega_m t}$$

- Question: How many possible values of frequency can we have here for the N oscillators? SO how many actual values can m have?

N coupled oscillators

Let's look at this plot:

$$\omega_m = 2\omega_0 \, sin\left[\frac{m\pi}{2(N+1)}\right]$$



For $m = 1$ to $m = N$ we have N characteristic frequencies.

For $m = N+1$, $\omega_{N+1} = 2\omega_0 \, sin\left[\frac{(N+1)\pi}{2(N+1)}\right] = 2\omega_0 \, sin\frac{\pi}{2}$ But this point corresponds to zero amplitude ($x_{p(N+1)} = 0$)

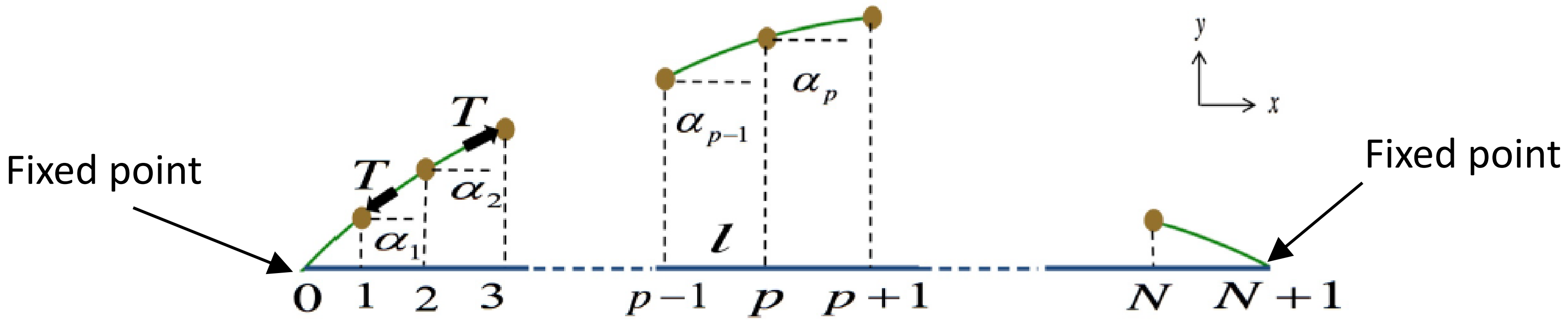
For $m = N+2$, $\omega_{N+2} = 2\omega_0 \, sin\left[\frac{(N+2)\pi}{2(N+1)}\right] = 2\omega_0 \, sin\left[\pi - \frac{N\pi}{2(N+1)}\right] = 2\omega_0 \, sin\left[\frac{N\pi}{2(N+1)}\right]$, so $\omega_{N+2} = \omega_N$

Similarly, $\omega_{N+3} = \omega_{N-1}$ and so on. So we would have only N normal frequencies for N coupled oscillators in 1D!

N-coupled oscillators: Transverse vibrations

- Let us consider a fixed elastic string to which N identical particles (each of mass m) are attached separated equidistantly by a distance l.

- Tension in the string is T. We consider small transverse displacements so that T remains the same.



Force acting on particle 1: $F_{1x} = -Tcos\alpha_0 + Tcos\alpha_1, \quad F_{1y} = -Tsin\alpha_0 + Tsin\alpha_1$

Force acting on particle 1: $F_{2x} = -Tcos\alpha_1 + Tcos\alpha_2, \quad F_{2y} = -Tsin\alpha_1 + Tsin\alpha_2$

Force acting on particle p: $F_{px} = -Tcos\alpha_{p-1} + Tcos\alpha_p, \quad F_{py} = -Tsin\alpha_{p-1} + Tsin\alpha_p \qquad (1)$

N-coupled oscillators: Transverse vibrations

When α is small, that is $y \ll l$, $\quad cos\alpha_p = 1 - \frac{\alpha_p^2}{2}$

So for the longitudinal vibrations, that is considering only forces along the X direction,

$$F_{px} = -Tcos\alpha_{p-1} + Tcos\alpha_p \;= \frac{1}{2}T(\alpha_{p-1}^2 - \alpha_p^2)$$

For small α, the difference between two second order terms is very small so we will ignore F_{px}

$$F_{py} = -Tsin\alpha_{p-1} + Tsin\alpha_p = -T\left(\frac{y_p - y_{p-1}}{l}\right) + T\left(\frac{y_{p+1} - y_p}{l}\right)$$

Therefore, $\quad \frac{d^2y_p}{dt^2} = \frac{F_{py}}{m} = \omega_0^2(y_{p+1} + y_{p-1} - 2y_p)$ where $\boxed{\omega_0^2 = \frac{T}{ml}}$

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0 \qquad (2)$$

(2) Represents N equations each giving a corresponding solution for ω_0.

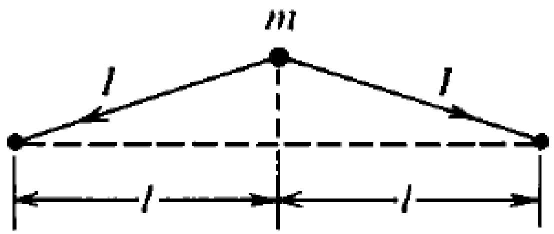
N-coupled oscillators: Transverse vibrations

Writing similar equation for each of the N particles we would have N differential equations.
Here, $y_0 = 0$ and $y_{N+1} = 0$ as these are fixed ends.

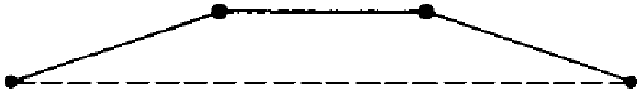
Let's consider a few special cases:

- $N = 1, \frac{d^2 y_1}{dt^2} + 2\omega_0^2 y_1 = 0 \; (as\ y_2 = 0\ and\ y_0 = 0)$
  Transverse harmonic motion with angular frequency

$\sqrt{2}\omega_0 = \sqrt{\frac{2T}{ml}}$

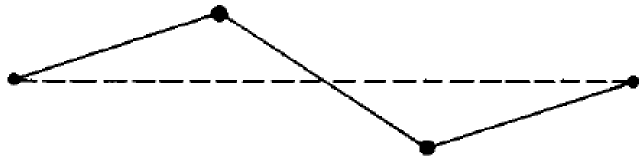


- $N = 2, \frac{d^2 y_1}{dt^2} + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0 (as\ y_0 = 0)$



Transverse harmonic motion with angular frequency ω_0

$$\frac{d^2 y_2}{dt^2} + 2\omega_0^2 y_2 - \omega_0^2 y_1 = 0 \; (as\ y_3 = 0)$$



Transverse harmonic motion with angular frequency $\frac{\omega_0}{\sqrt{3}}$

N-coupled oscillators: Transverse vibrations

- $N = p$

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} + y_{p-1}\right) = 0 \quad \text{where } p = 1,2,3, \dots \dots \dots . . N \qquad (2)$$

Let's assume a trial solution of the form: $\boldsymbol{y_p = A_p\, cos\omega t}$ where $p = 1,2,3, \dots \dots \dots . . N$ (3)

Our aim is to find N values of A_p and N values of ω such that (3) satisfies N differential equations (2).

$$\frac{dy_p}{dt} = -\omega A_p sin\omega t \text{ where } p = 1,2,3, \dots \dots \dots . . N \qquad (4)$$

Putting (4) and (3) in (2) we get a set of N equations:

$$\begin{gathered}
(-\omega^2 + 2\omega_0^2)A_1 - \omega_0^2(A_2 + A_0) = 0 \\
(-\omega^2 + 2\omega_0^2)A_2 - \omega_0^2(A_3 + A_1) = 0 \\
. \qquad\qquad\qquad\qquad . \\
. \qquad\qquad\qquad\qquad . \\
(-\omega^2 + 2\omega_0^2)A_p - \omega_0^2\left(A_{p+1} + A_{p-1}\right) = 0 \\
. \qquad\qquad\qquad\qquad . \\
(-\omega^2 + 2\omega_0^2)A_N - \omega_0^2(A_{N+1} + A_{N-1}) = 0
\end{gathered}$$

N-coupled oscillators: Transverse vibrations

- We will follow a method similar to N coupled oscillators with longitudinal vibrations.
- Let's first write the set of N equations in a compact form:

$$(-\omega^2 + 2\omega_0^2)A_p - \omega_0^2\left(A_{p+1} + A_{p-1}\right) = 0 \text{ where } (p = 1,2,3, \dots \dots ..N)$$

Our boundary conditions are $A_0 = 0$ and $A_{N+1} = 0$

Therefore,
$$\frac{A_{p-1}+A_{p+1}}{A_p} = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$$

Just as **in the case of longitudinal vibrations**, we will write:

$$y_p = A_p\, cos\omega t = C \sin p\theta\, cos\omega t \quad \text{where } \theta \text{ is some angle}$$

Imposing the boundary conditions, we can show as before that:

$$\theta = \frac{n\pi}{N+1} \quad \text{where } (n = 1,2,3, \dots \dots \dots \dots .)$$

$$A_p = C \sin\frac{pn\pi}{N+1}$$

N-coupled oscillators: Transverse vibrations

Frequency for the n th normal mode: $$\boxed{\omega_n = 2\omega_0\, sin\left[\frac{n\pi}{2(N+1)}\right]}$$ $(n = 1,2,3, \dots \dots)$

Coordinate for pth particle in nth normal mode: $$\boxed{y_{pn} = C_n \sin\frac{pn\pi}{N+1} cos\omega_n t}$$

p: particle index
n: normal mode index

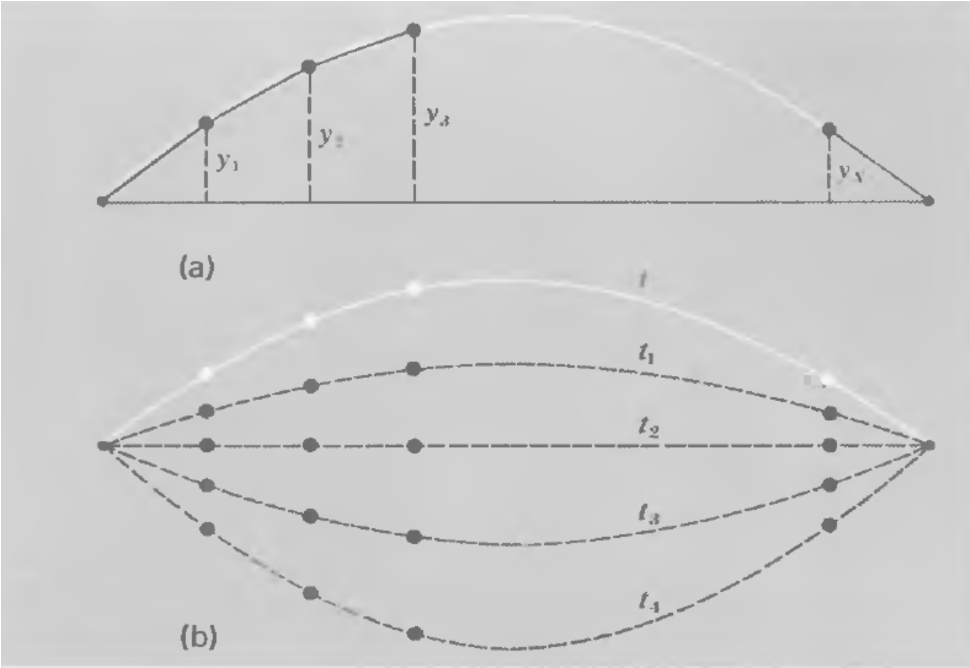
- Again as before we would consider only N values of n that is N normal modes, as values of n above N do not hold any physical meaning. (check N longitudinally coupled oscillators)

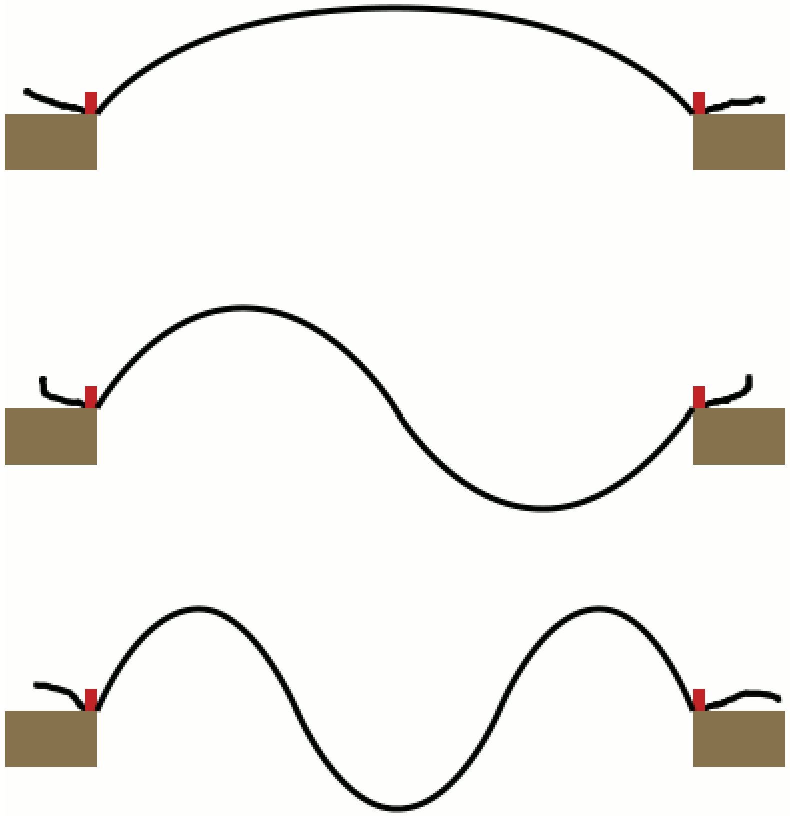
Let’s look at different normal modes

$n = 1$

$y_{p1} = C_1 \sin\frac{p\pi}{N+1} cos\omega_1 t$

$\omega_1 = 2\omega_0\, sin\left[\frac{\pi}{2(N+1)}\right]$

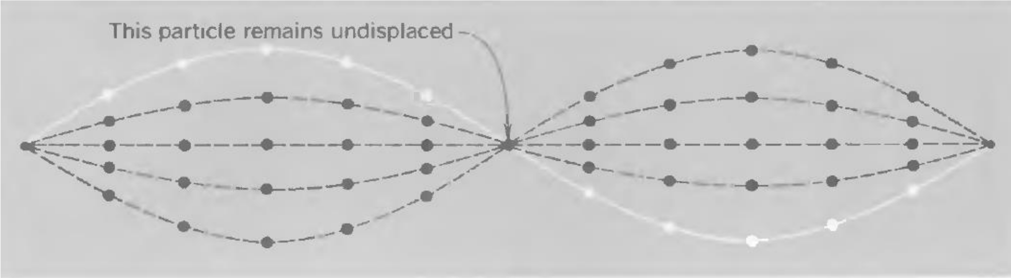




$n = 2$

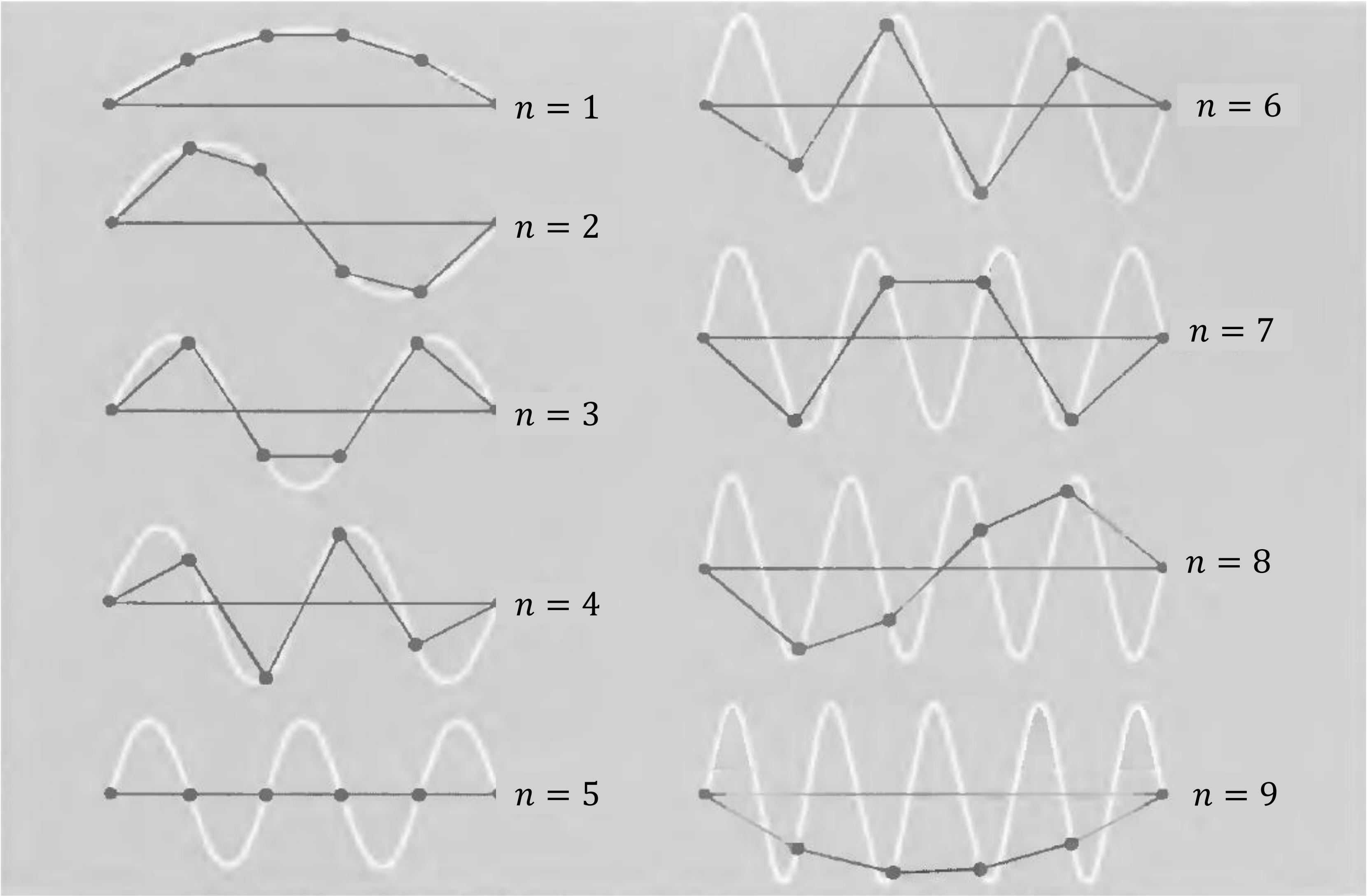
$y_{p2} = C_2 \sin\frac{2p\pi}{N+1} cos\omega_2 t$

$\omega_2 = 2\omega_0\, sin\left[\frac{\pi}{(N+1)}\right]$



Let's look at different normal modes: only 4 particles

4 normal modes



Repeat of 4, 3, 2,1 in terms of frequencies

All amplitudes are zero for $n = 5$

N oscillators: the highest frequency mode

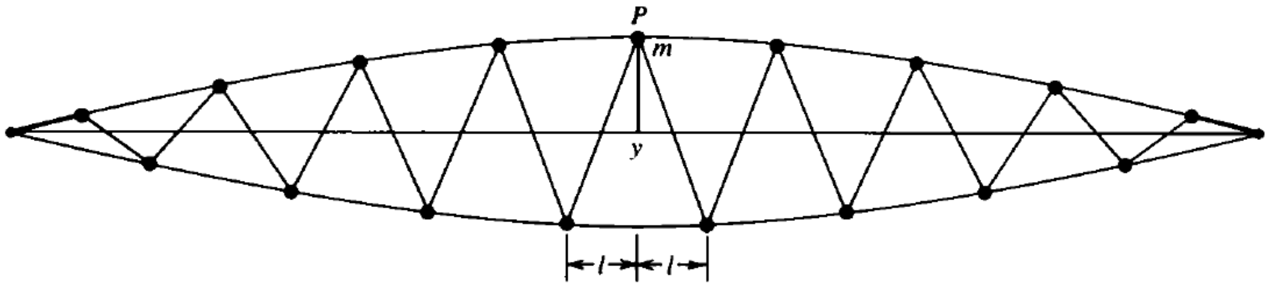
The maximum frequency for the highest possible mode with $n = N$ is given by:

$$\omega_{max} = 2\omega_0 \, sin\left[\frac{N\pi}{2(N+1)}\right] \approx 2\omega_0 sin\left(\frac{\pi}{2}\right) = 2\omega_0$$

Amplitude at ω_{max}:

$$A_{pN} = C_N \sin\frac{pN\pi}{N+1} = C_N \sin\left(1 - \frac{1}{N+1}\right)p\pi = C_N \sin\left(p\pi - \alpha_p\right) = C_N(-1)^{p+1}\frac{p\pi}{N+1}$$

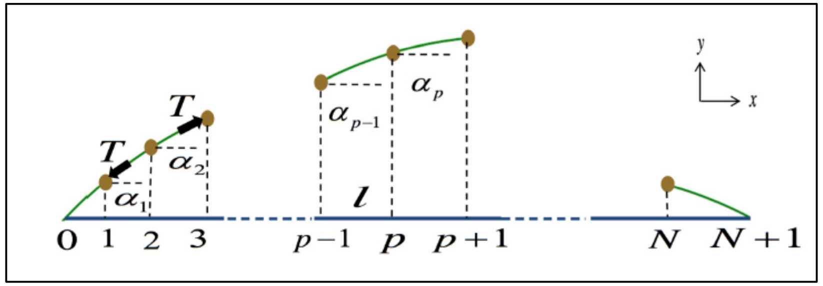
$$\frac{A_p}{A_{p+1}} = -\frac{\sin\dfrac{p\pi}{N+1}}{\sin\dfrac{(p+1)\pi}{N+1}}$$



For the highest mode, over most of the central region, displacements of adjacent masses are opposite and nearly equal!

We will revisit the normal modes of the string when we will do Waves!

When N becomes very large



$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \quad \text{where } \omega_0 = \sqrt{\frac{T}{ml}}$$

- Now, we will let N increase and l (spacing between neighbouring particles) decrease such that, the length of the string = $L = (N+1)l$ is a constant.

- We shall also decrease the mass m of each particle such that, The total mass of the system = $M = Nm$ also remains constant.

In that case,

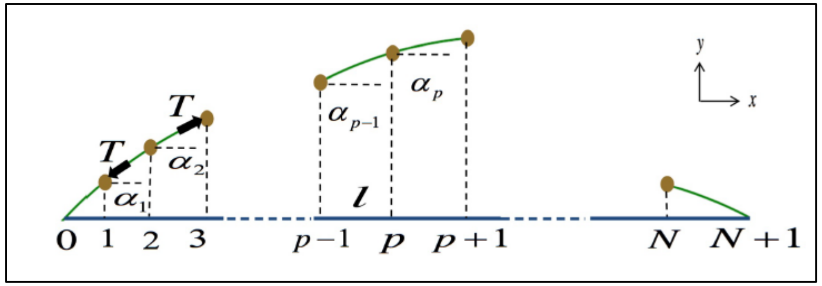
$$\sin\left[\frac{n\pi}{2(N+1)}\right] \approx \frac{n\pi}{2(N+1)}$$

$$\omega_n \approx 2\sqrt{\frac{T}{ml}}\,\frac{n\pi}{2(N+1)} = \sqrt{\frac{T}{m/l}}\,\frac{n\pi}{(N+1)l} = \sqrt{\frac{T}{\mu}}\,\frac{n\pi}{L} \quad \text{where } \mu = m/l$$

So,

$$\boxed{\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \quad \text{where } n = 1,2,3, \ldots \ldots \ldots}$$

When N becomes very large: the continuum limit



$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \quad \text{where } n = 1,2,3, \dots \dots \dots$$

Normal frequencies are integral multiples of the lowest frequency $\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\mu}}$

$$y_{pn} = C_n \sin\frac{pn\pi}{N+1} cos\omega_n t$$

Instead of using particle index p, let's specify position of each particle by a distance x from the fixed end of the string.

$$x = pl$$

$$\frac{pn\pi}{N+1} = \frac{pln\pi}{(N+1)l} = \frac{xn\pi}{L}$$

$$\boxed{y_n(x,t) = C_n \sin\frac{xn\pi}{L} cos\omega_n t \quad (n = 1,2,3, \dots \dots .)}$$

In the continuum limit, $N \to \infty$, x is a continuous variable ranging from 0 to L. Distance between adjacent particles $\to 0$

When N becomes very large: the continuum limit

- The equation $y_n(x,t) = C_n \sin\frac{xn\pi}{L} cos\omega_n t \quad (n = 1,2,3, \dots \dots.)$ has **no phase term** which means that the **initial velocity of all components on the string was zero (trial solution was $\boldsymbol{y_p = A_p\, cos\omega t}$).**

- But we can also write a solution without this condition by ***including phase (initial velocity: ≠ 0)***

$$y_n(x,t) = C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)$$

NORMAL MODES OF A REAL VIBRATING STRING WITH FIXED ENDS

$$\boxed{y_n(x,t) = C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)}$$ where $$\boxed{\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \quad \text{where } n = 1,2,3, \dots \dots \dots}$$

GENERAL SOLUTION IS SUM OF THE NORMAL MODES

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)}$$

When N becomes very large: the continuum limit

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)$$

If I take a snapshot of the vibrating string, then $cos(\omega_n t - \delta_n)$ is just a number. So now I can consider only the space dependence of the displacements.

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \text{where, } B_n = C_n\, cos(\omega_n t - \delta_n)$$

Just a constant number for a given mode

- If the initial velocity of the string is zero, then $\delta_n = 0$ and $B_n = C_n$ at $t = 0$
- If the initial velocity of the string is NOT zero, then $\delta_n \neq 0$ and $B_n = C_n cos\delta_n$ at $t = 0$

The Fourier series

Any well behaved function $f(x)$ which vanishes at $x = 0$ and $x = L$
(i.e. $f(0) = f(L) = 0$)

- $f(x)$ can be expanded as the series:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

Proof: Just showed

- $$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Proof: Will find

Note that this is **NOT** the most general statement of Fourier's theorem!

The Fourier series

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

How do we find B_n values for all the modes?

Prescription: Multiply both sides by $\sin\frac{n_1\pi x}{L}$ and integrate $\int_0^L dx$

$$\int_0^L y(x)\sin\frac{n_1\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L sin\frac{n_1\pi x}{L}\, sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \int_0^L \left[cos\frac{(n-n_1)\pi x}{L} - cos\frac{(n+n_1)\pi x}{L}\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\frac{L}{(n-n_1)\pi} sin\frac{(n-n_1)\pi x}{L} - \frac{L}{(n+n_1)\pi} sin\frac{(n+n_1)\pi x}{L}\right]\Bigg|_0^L$$

The Fourier series

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\frac{L}{(n-n_1)\pi} sin\frac{(n-n_1)\pi x}{L} - \frac{L}{(n+n_1)\pi} sin\frac{(n+n_1)\pi x}{L}\right] \Bigg|_0^L$$

- When $n \neq n_1$, everything vanishes!
- When $n = n_1$, we have a $\frac{0}{0}$ form!

In this case, the integral $\int_0^L sin^2\frac{n\pi x}{L}dx = \frac{1}{2}\int_0^L \left[1 - cos\frac{2n\pi x}{L}\right]dx = \frac{L}{2}$

$$\int_0^L y(x)\sin\frac{n\pi x}{L}dx = \frac{LB_n}{2}$$

$$\boxed{B_n = \frac{2}{L}\int_0^L y(x)\sin\frac{n\pi x}{L}dx}$$

How to find C_n and δ_n?

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)$$

We will use the boundary conditions..

Initial shape : $y_0(x) = y(x, t=0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\delta_n)$

Initial velocity: $\dot{y}_0(x) = \dot{y}(x, t=0) = \sum_{n=1}^{\infty} (-\omega_n C_n) \sin\frac{n\pi x}{L} sin(\delta_n)$

- Fourier series expansion of $y_0(x)$ gives $C_n cos(\delta_n)$
- Fourier series expansion of $\dot{y}_0(x)$ gives $-\omega_n C_n\, sin(\delta_n)$

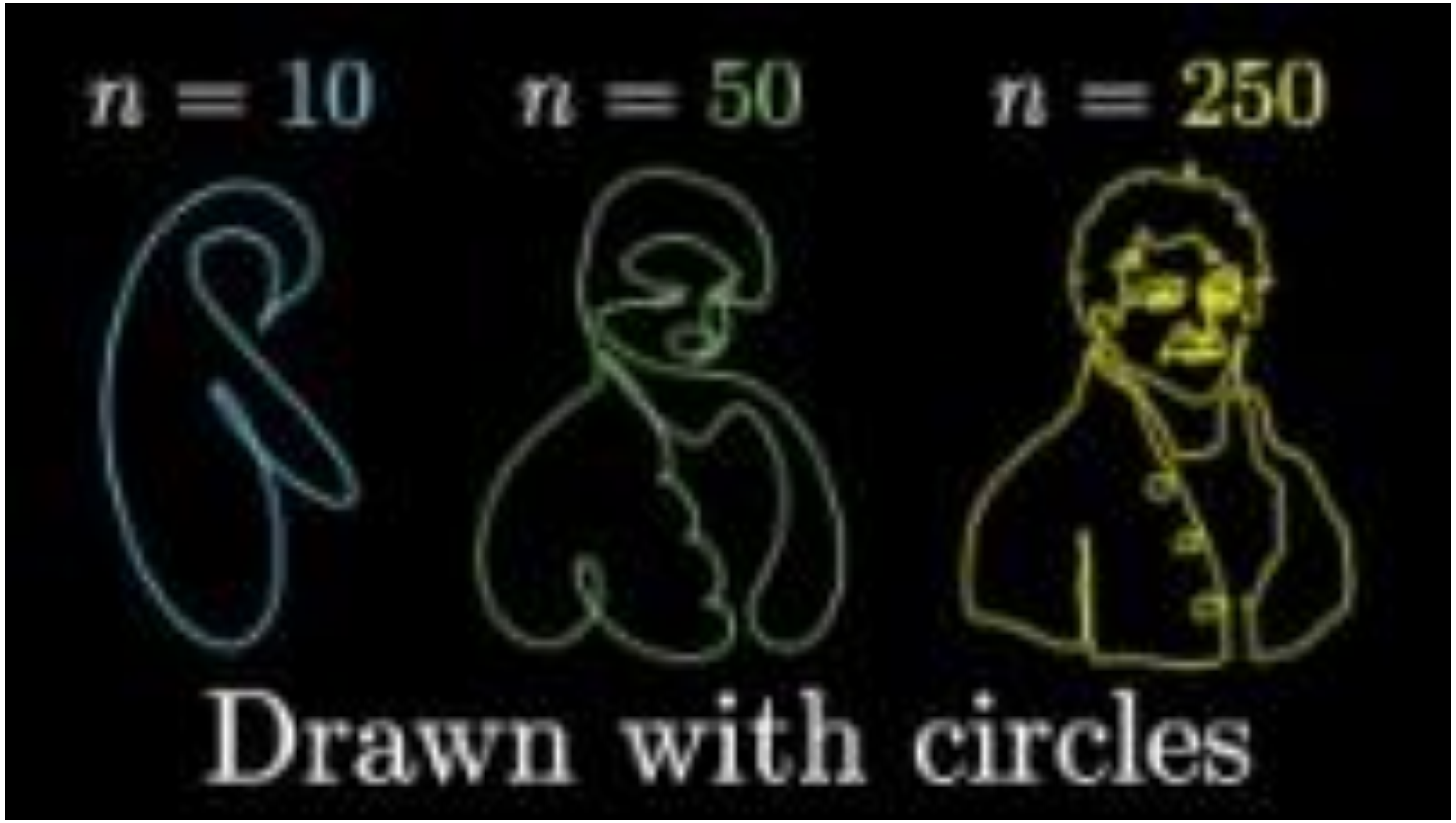
$$C_n cos(\delta_n) = \frac{2}{L}\int_0^L y_0(x) \sin\frac{n\pi x}{L} dx$$

$$-\omega_n C_n\, sin(\delta_n) = \frac{2}{L}\int_0^L \dot{y}_0(x) \sin\frac{n\pi x}{L} dx$$

If $\dot{y}_0(x) = 0$, then $\delta_n = 0$

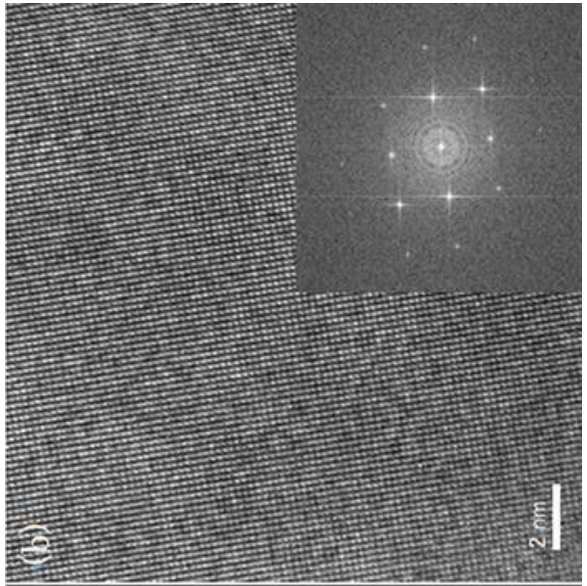
Some demos

Some demos

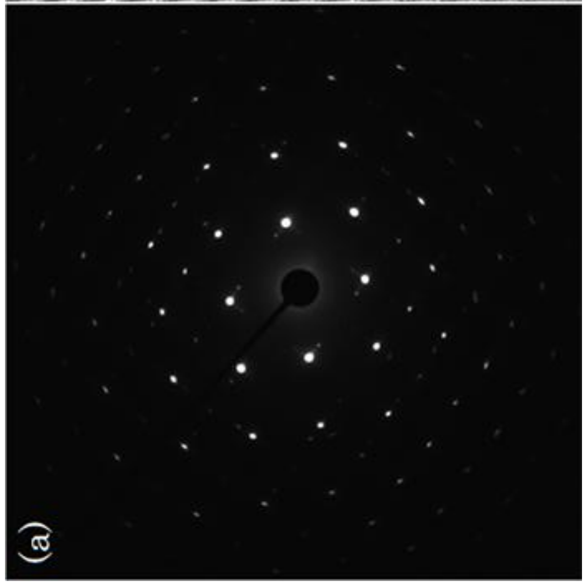


Applications

- Compress a jpg image
- Compress your .mp3 file and mail it!
- Solving important differential equation for example the heat diffusion equation and finding out the temperature distribution of an object, or solving the Laplace equation to find out the charge distribution.
- Image processing: Get noise-free pictures of atom columns in an electron microscope.
- In optics, quantum mechanics, crystallography, electrical engineering, telecommunication etc. etc.

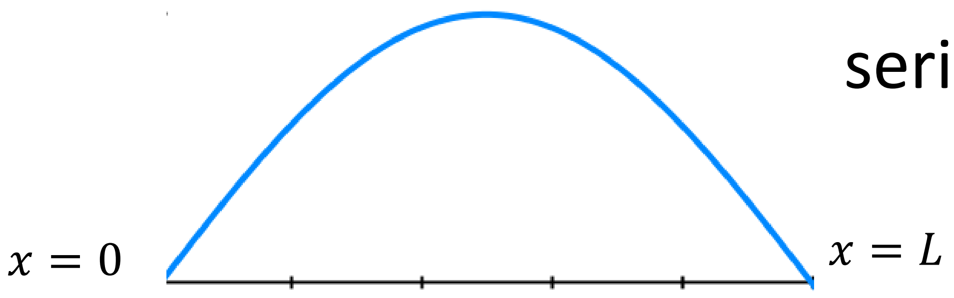


A gold crystal seen under an electron microscope.



Fourier transform (diffraction pattern) of a gold crystal

The Fourier series: Problem 1-A plucked string



Find the Fourier series expansion

Our list of formulas

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} cos(\omega_n t - \delta_n)$$

$y(x,0) = Asin\frac{\pi x}{L}$ for $0 < x < L$

$B_1 = C_1 cos\delta_1 = A$, $B_n = 0$ for $n = 2,3,4,\ldots..$

If string is released from rest, then $\delta_1 = 0$, so $C_1 = A$

$$\boldsymbol{y(x,t)} = C_1 \sin\frac{\pi x}{L} cos(\omega_1 t - 0) = Asin\frac{\pi x}{L} cos\omega_1 t$$

The Fourier series: Problem 2-A plucked string

$$y(x,0) = \frac{hx}{a} \quad \text{for } 0 < x < a$$

$$= \frac{h(L-x)}{L-a} \quad \text{for } a < x < L$$

Find the Fourier series expansion, then find the special case when $a = L/2$

The Fourier series: Problem 2

$y(x,0)$ has a Fourier series expansion:

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad \text{with} \qquad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{2}{L}\left[\int_0^a \frac{hx}{a}\sin\frac{n\pi x}{L}\,dx + \int_a^L \frac{h(L-x)}{L-a}\sin\frac{n\pi x}{L}\,dx\right]$$

$$\text{First term} = \frac{h}{a}\left[x\frac{-cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]\Bigg|_0^a - \frac{h}{a}\int_0^a\left(\frac{-cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)dx$$

$$= -\frac{hL}{n\pi a}\left[acos\frac{n\pi a}{L}\right] + \frac{L}{n\pi}\frac{Lh}{n\pi a}\left[sin\frac{n\pi x}{L}\right]\Bigg|_0^a$$

$$= -\frac{hL}{n\pi}cos\frac{n\pi a}{L} + \frac{L^2 h}{n^2\pi^2 a}sin\frac{n\pi a}{L}$$

The Fourier series: Problem 2

Second term $= \frac{h}{L-a}\left[(L-x)\frac{-cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]\Bigg|_a^L \quad - \frac{h}{L-a}\int_a^L(-1)\left(\frac{-cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)dx$

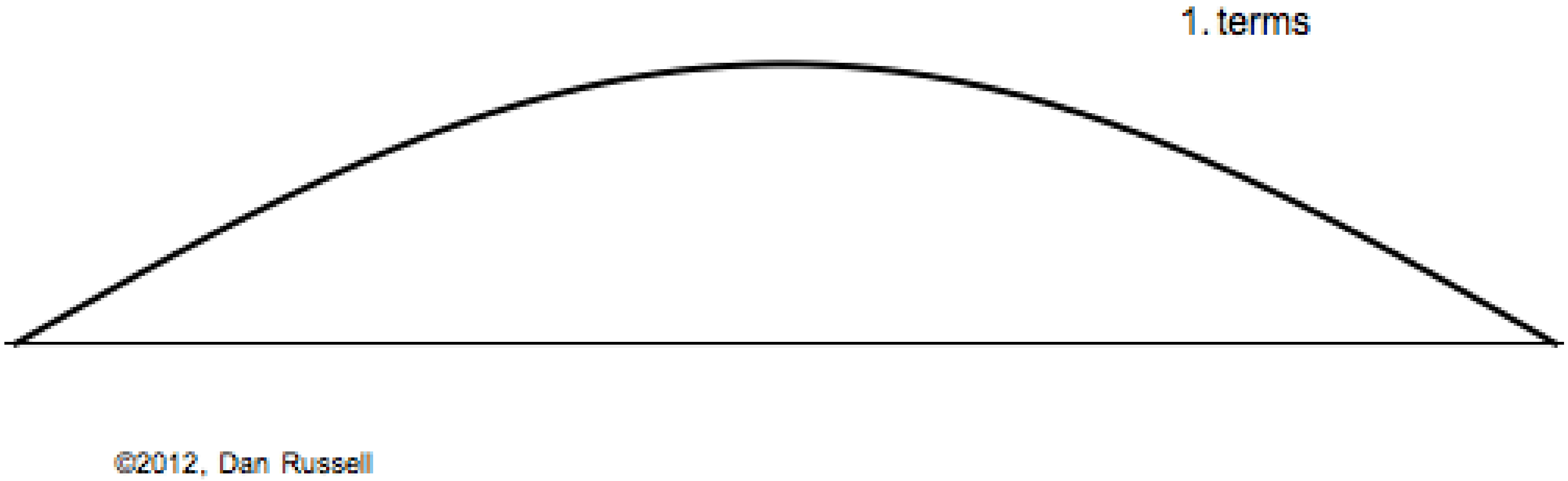
$$= -\frac{hL}{n\pi(L-a)}\left[-(L-a)cos\frac{n\pi a}{L}\right] - \frac{L}{n\pi}\,\frac{Lh}{n\pi(L-a)}\left[sin\frac{n\pi x}{L}\right]\Bigg|_a^L$$

$$= \frac{hL}{n\pi}cos\frac{n\pi a}{L} - \frac{L^2 h}{n^2\pi^2(L-a)}\left(-sin\frac{n\pi a}{L}\right)$$

So, $B_n = \frac{2}{L}\left[\frac{L^2 h}{n^2\pi^2 a}sin\frac{n\pi a}{L} + \frac{L^2 h}{n^2\pi^2(L-a)}\left(sin\frac{n\pi a}{L}\right)\right]$

$$= \frac{2Lh}{n^2\pi^2}\left(\frac{L}{a(L-a)}\right)sin\frac{n\pi a}{L} = \frac{2h}{n^2\pi^2}\,\frac{L^2}{a(L-a)}\,sin\frac{n\pi a}{L}$$

The plucked string: How the terms are added



$$y(x,0) = \sum_{n=1}^{\infty} \frac{2h}{n^2\pi^2} \frac{L^2}{a(L-a)} \, sin \frac{n\pi a}{L} \sin \frac{n\pi x}{L}$$

The Fourier series: Problem 2 special case

If $\boldsymbol{a = \frac{L}{2}}$,

$$B_n = \frac{2h}{n^2\pi^2}\,\frac{L^2}{a(L-a)}\,sin\,\frac{n\pi a}{L} \;=\frac{8h}{n^2\pi^2}\,sin\,\frac{n\pi}{2}$$

$= 0$ if n is even

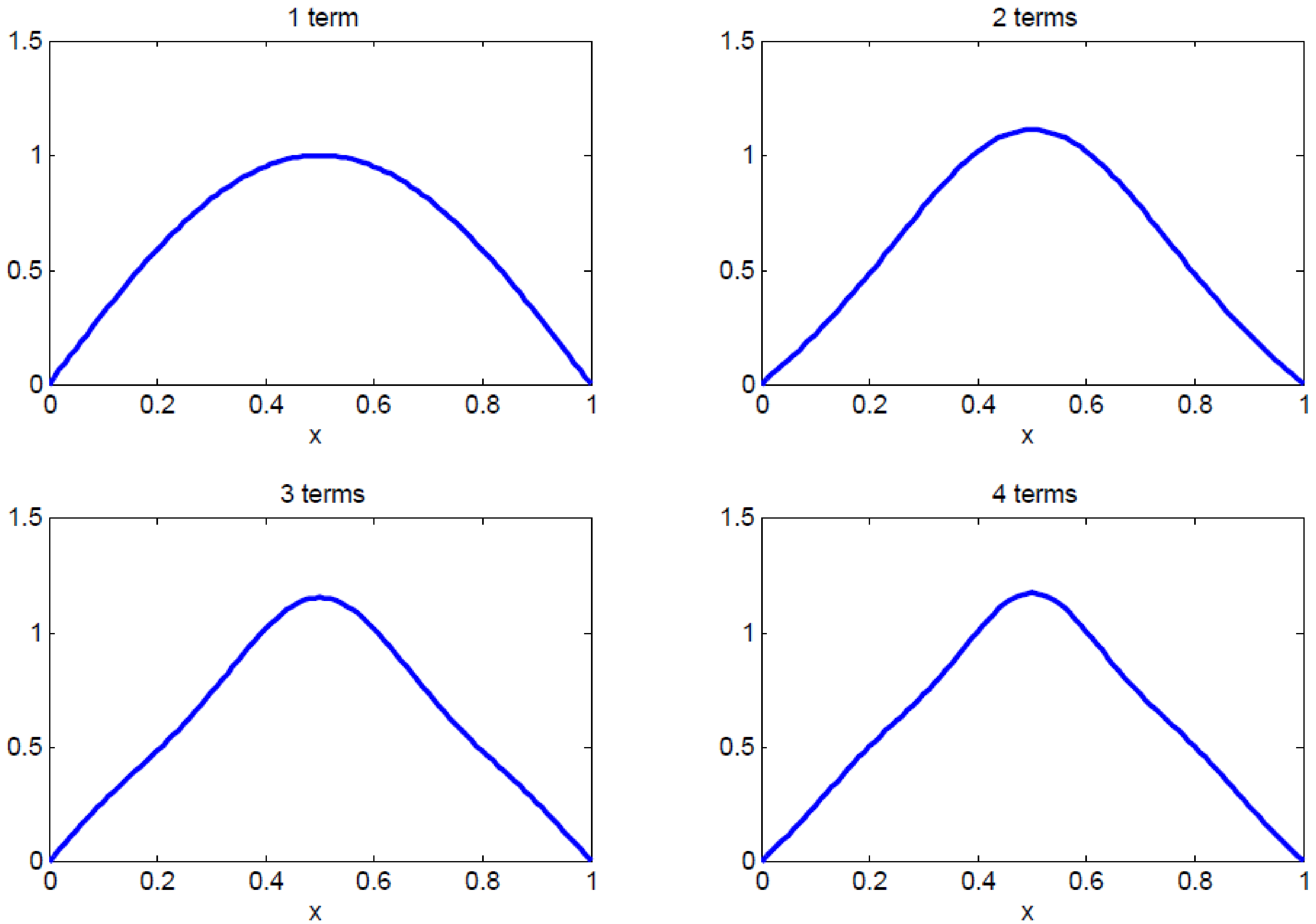
$= \frac{8h}{(2m+1)^2\pi^2}(-1)^m$ if n is odd, where $n \;=\; (2m+1)$

$m = 0,1,2,3, \ldots \ldots.$

So, $\quad y(x,0) = \sum_{m=1}^{\infty} \frac{8h}{(2m+1)^2\pi^2}(-1)^m \sin\frac{(2m+1)\pi x}{L}$

$$= \frac{8h}{\pi^2}\left(sin\,\frac{\pi x}{L} - \frac{1}{9}\,sin\,\frac{3\pi x}{L} + \frac{1}{25}\,sin\,\frac{5\pi x}{L} - \cdots\right)$$

The Fourier series: Problem 2 special case



The Fourier series: Problem 2

$$y(x,0) = \sum_{m=1}^{\infty} \frac{8h}{(2m+1)^2\pi^2} (-1)^m \sin\frac{(2m+1)\pi x}{L}$$

So at any time t**, if the initial velocity of the string is zero again as before,**

$$y(x,t) = \sum_{m=1}^{\infty} \frac{8h}{(2m+1)^2\pi^2} (-1)^m \sin\frac{(2m+1)\pi x}{L} cos(\omega_{2m+1}t)$$

Orthogonality

- When you have two vectors $\vec{A}$ and $\vec{B}$ perpendicular to each other, $\vec{A}.\vec{B} = 0$
  The vectors are 'orthogonal' to each other.
  So $A_x B_x + A_y B_y + A_z C_z = 0$ or $\sum_{p=1}^{3} A_p B_p = 0$.

- Consider the integral: $\int_0^\infty sin\left(\frac{n_1 \pi x}{l}\right) sin\left(\frac{n_2 \pi x}{l}\right) dx = 0$ for $n_1 \neq n_2$.

- Going back (!) to the discrete case of N oscillators from the continuum limit, we can write this as :

$$\frac{L}{N}\sum_{p=1}^{N} sin\left(\frac{n_1 \pi p}{N}\right) sin\left(\frac{n_2 \pi p}{N}\right) = 0 \text{ for } n_1 \neq n_2.$$

- Instead of three dimensions now we have orthogonality in N dimensions! All the orthogonal functions are linearly independent and the complete set can be summed together to generate any kind of function.

The Fourier series: more problems

Please do A P French problems 6-14 and 6-15 for practice!