Riding the Shadow

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Introduction

While mountain biking one clear December evening there was a beautiful sunset. Distracted by the setting Sun a friend lost focus and rode off the trail. Asked later about the cause of the accident, the culprit was a distracting question. How fast would a person have to climb (a ladder) vertically from the surface of the Earth to always see the Sun just about to set on the horizon while the Earth is rotating?

Assumptions

To simplify the problem, a few assumptions will be made.

- The point of reference (person) is positioned at the equator
- The Earth is rotating at a constant rate
- The Sun is treated like a point light source

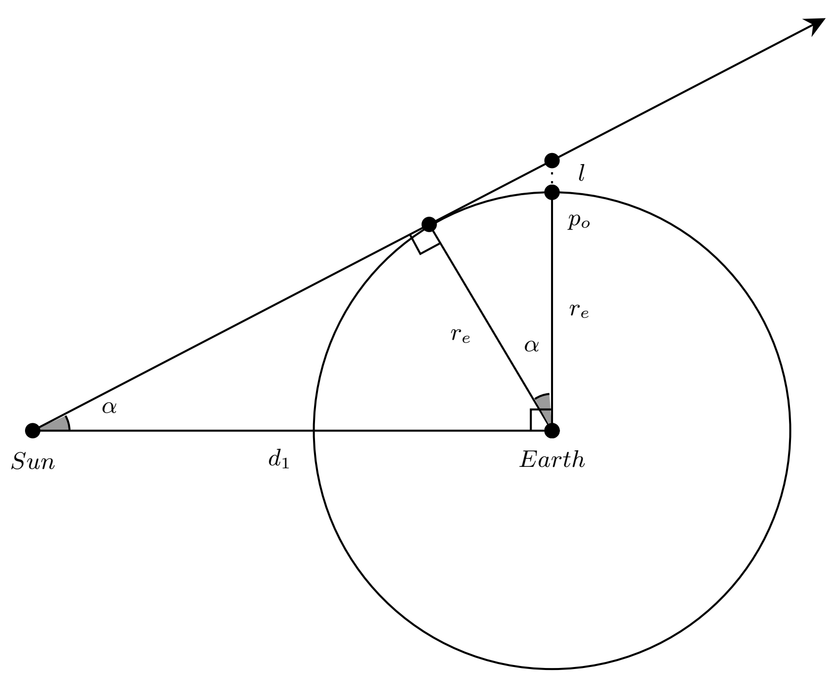


Figure 1: Diagram of the line of the shadow cast by the Sun.

Setup

If the Sun is just setting on the horizon then a position on the equator p_o the observation point will require an increase in height to stay just outside the approaching shadow as the Earth rotates, as shown in Figure 1. The average distance between the center of the Earth and the center of the Sun is denoted as d_1, the radius of the Earth is denoted as r_e, the height of the theoretical ladder (point of reference altitude) is denoted by l and the shadow angle as α, which can be determined using r_e and d_1.

$$\begin{aligned}\sin(\alpha) &= \frac{r_e}{d_1} \\ \alpha &= \sin^{-1}(\frac{r_e}{d_1}) \\ \alpha &\approx 0\end{aligned}$$

The Earth's radius relative to the distance between the Sun and the Earth is quite small. As a result α is very small, approximately 0. This simplifies the setup diagram. The updated setup is shown in Figure 2.

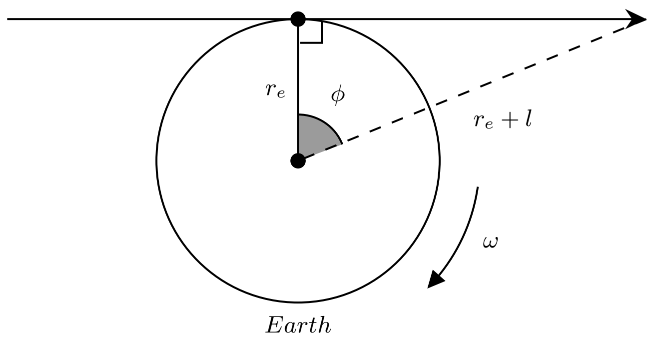


Figure 2: Setup

The equation for the height of the ladder based on the angle ϕ can be derived using the simplified diagram shown in Figure 2. where the the height l is in m meters and the angle ϕ is measured in *radians*.

$$\begin{aligned}\cos(\phi) &= \frac{r_e}{r_e + l} \\ r_e + l &= \frac{r_e}{cos(\phi)} \\ l &= r_e\left(\frac{1}{cos(\phi)} - 1\right) \\ l(\phi) &= r_e\left(\frac{1}{cos(\phi)} - 1\right) \qquad (1)\end{aligned}$$

In order to find the velocity, time needs to be factored into the equation. This can be done using Earth's constant rate of rotation, denoted by ω measured in *radians* per second. The equation for angular velocity can be used to determine what to substitute in for the angle ϕ as shown.

$$\omega = \frac{v_e}{r_e} \qquad (2)$$

Now instead of using the angle ϕ, ω is changing over time and the length of the ladder l can be calculated based on time t where $t_0 \rightarrow \phi = 0$ and $t_f \rightarrow \phi = \frac{\pi}{2}$. The height equation can be rewritten relative to time as shown and the domain can be changed to t, measured in seconds, where $t_0 = 0$ and $t_f = 6$ hours.

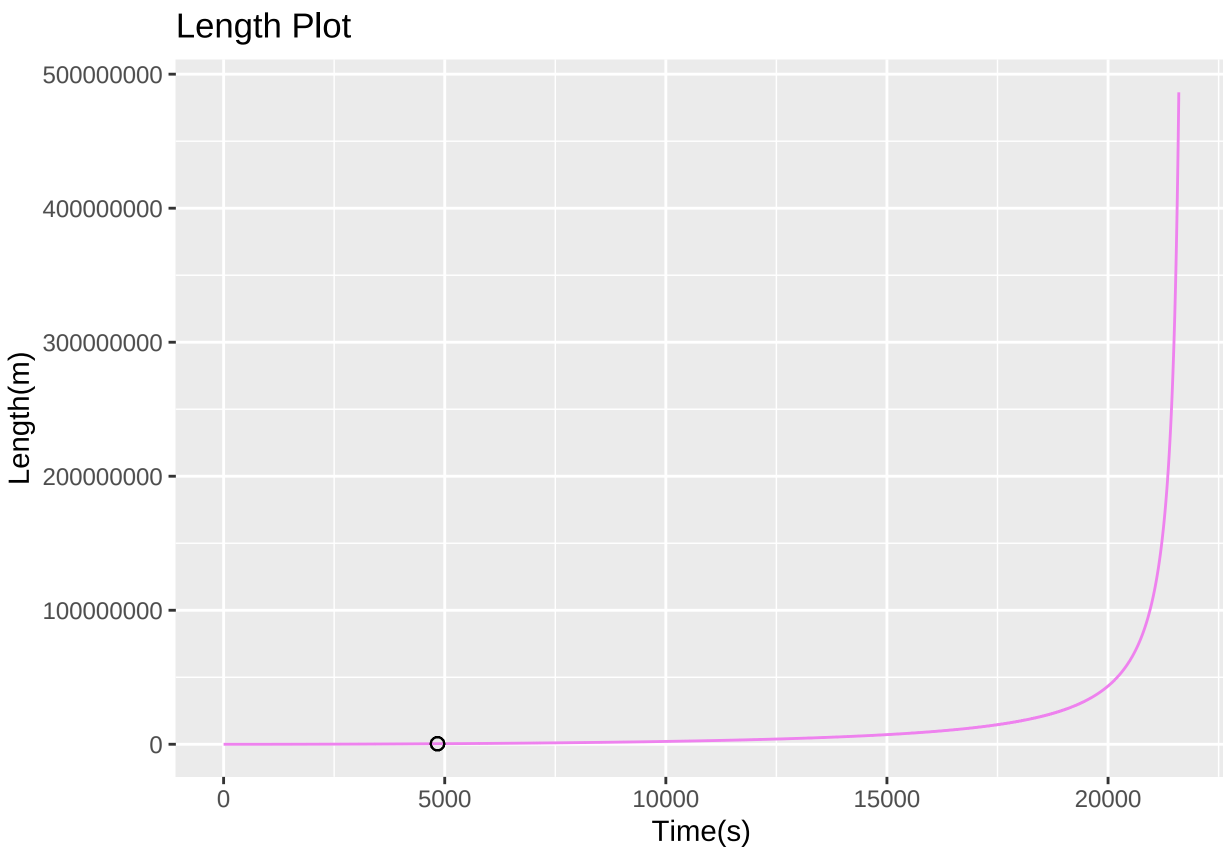
Analysis

Using Equation 1, Equation 2 can be substituted in for ϕ which must be replaced with a value that is unitless. A quick unit check on Equation 2 confirms that the value is unitless and can be substituted.

$$\frac{v_e}{r_e}t = \frac{m}{s}\frac{1}{m}s \tag{3}$$

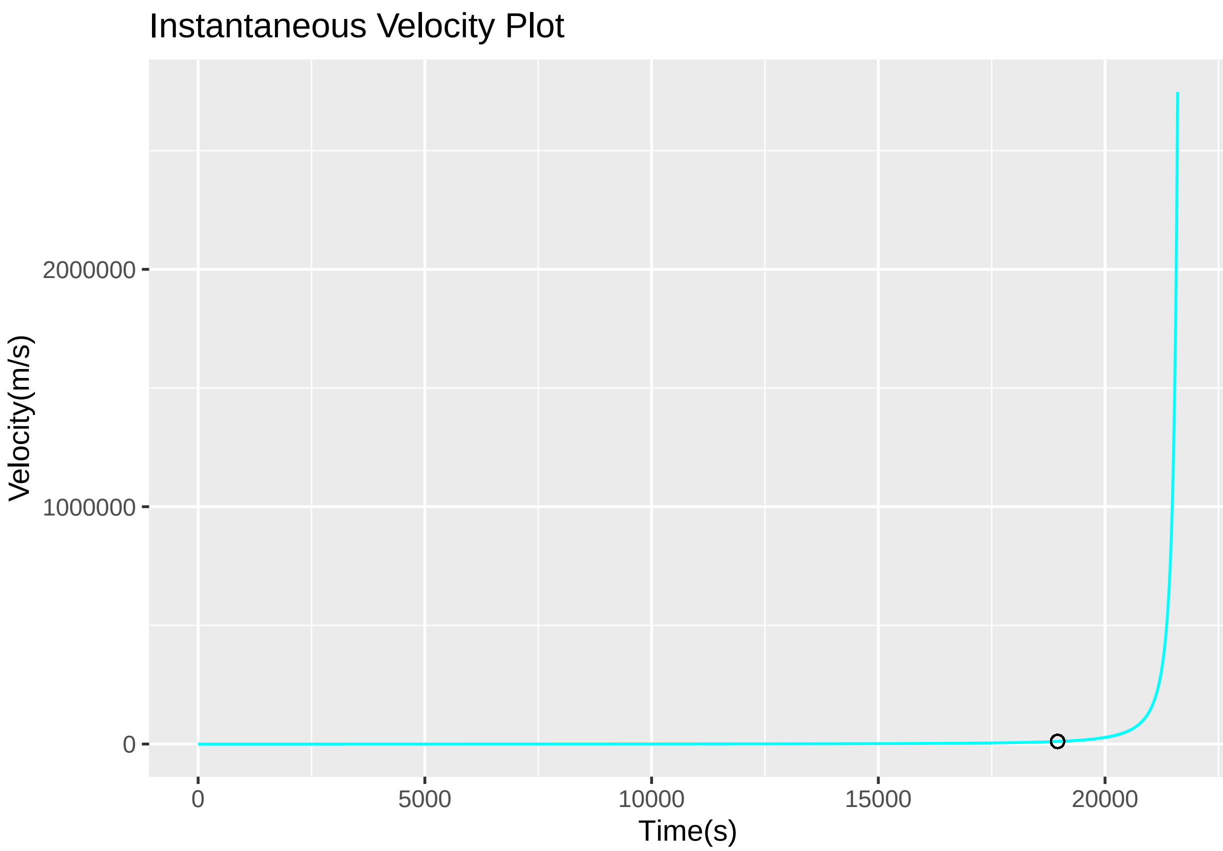
Given Equation 4. the length of the ladder at any point in time from 0 to 6 hours after Sunset can be plotted. A reference point is plotted to add context. The reference point shows the distance of the International Space Station from the surface of the Earth, which is approximately 254 *miles* (408,773 m).

$$l(t) = r_e\left(\frac{1}{cos\left(\frac{v_e}{r_e}t\right)} - 1\right) \tag{4}$$



Taking the derivative of the position function gives the velocity equation (Equation 5). Plotting the velocity function yields the graph below. A reference point is included to add context. The reference point signifies the escape velocity for a Falcon 9 rocket launch, roughly 11,000$\frac{m}{s}$.

$$l'(t) = v_e tan\left(\frac{v_e}{r_e}t\right) sec\left(\frac{v_e}{r_e}t\right) \tag{5}$$

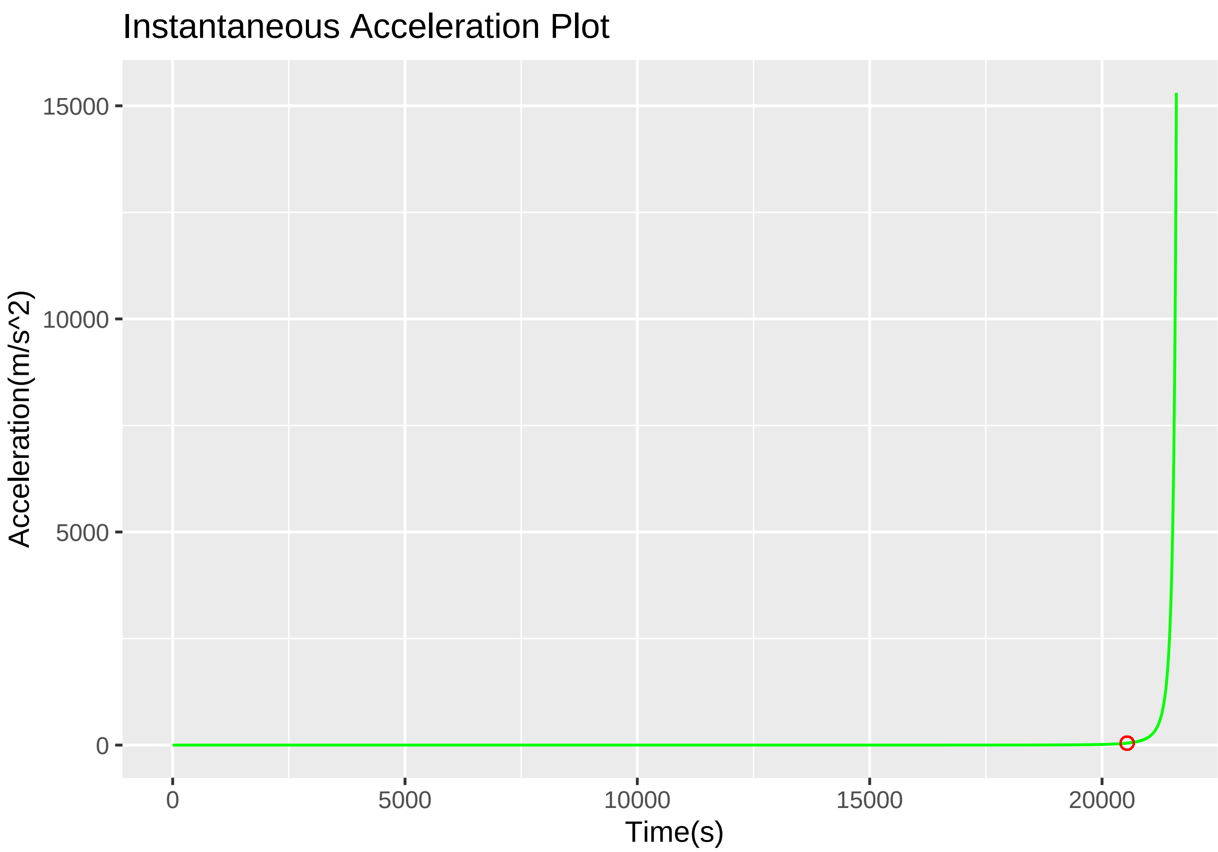



```
# Code for plotting
options(scipen=999) # Setup
t = seq(0, 60*60*6, 1) # Domain
vel = 460*tan(460/6378000*t)*1/cos(460/6378000*t)
values = c(vel)
# Create a variable that indicates what is being plotted
function.type = rep(c("Velocity(t)"), each = length(t))
xval = c(t)
plot.dat = data.frame(values, xval, function.type)
ggplot(plot.dat, aes(x = xval, y = values))+
  geom_line(color="cyan")+
  ggtitle("Instantaneous Velocity Plot")+
  xlab("Time(s)")+ ylab("Velocity(m/s)")+
  geom_point(shape=1, aes(x=18954.04, y=11000), colour="black", size=2)
```

```
# Code for solving equation
library(Ryacas)
# Find reference point example time for plot (finding root of equation)
t <- sym("t")
vel <- function(t) {460*tan(460/6378000*t)*1/cos(460/6378000*t)-11000}
vel0 <- vel(20500)
try(uniroot(vel, c(10000, 20500)))
```

Taking the derivative of the velocity equation gives the acceleration equation. Plotting yields the graph below. A reference point is included to add context. The red reference point signifies the most acceleration a person has ever been subjected to, or about $9g$'s ($46.2\frac{m}{s^2}$).

$$l''(t) = \frac{v_e^2}{r_e}\left(sin^2\left(\frac{v_e}{r_e}t\right)+1\right)sec^3\left(\frac{v_e}{r_e}t\right) \tag{6}$$



Conclusion

Ultimately, the question "How fast would a person have to climb a ladder to constantly see the Sun just about to set" can be answered using a given time (t) in seconds after which the Sun has set and Equation 5. At an hour after sunset a person would have to be on top of a ladder 221,193 m's high and climbing at 126.4 $\frac{m}{s}$. Which is about 137 *miles* and 280 *mph*.