Riding the Shadow

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Introduction

While mountain biking one clear December evening there was a beautiful sunset. Entranced by the beauty a friend lost focus and rode off the trail. Asked later about the cause of the accident, the culprit was a distracting question. How fast would a person have to climb vertically from the surface of the Earth to always see the Sun just about to set on the horizon while the Earth is rotating?

Assumptions

To simplify the problem, a few assumptions will be made.

- The point of reference (person) is positioned at the equator
- The Earth is rotating at a constant rate
- The Sun is treated like a point light source

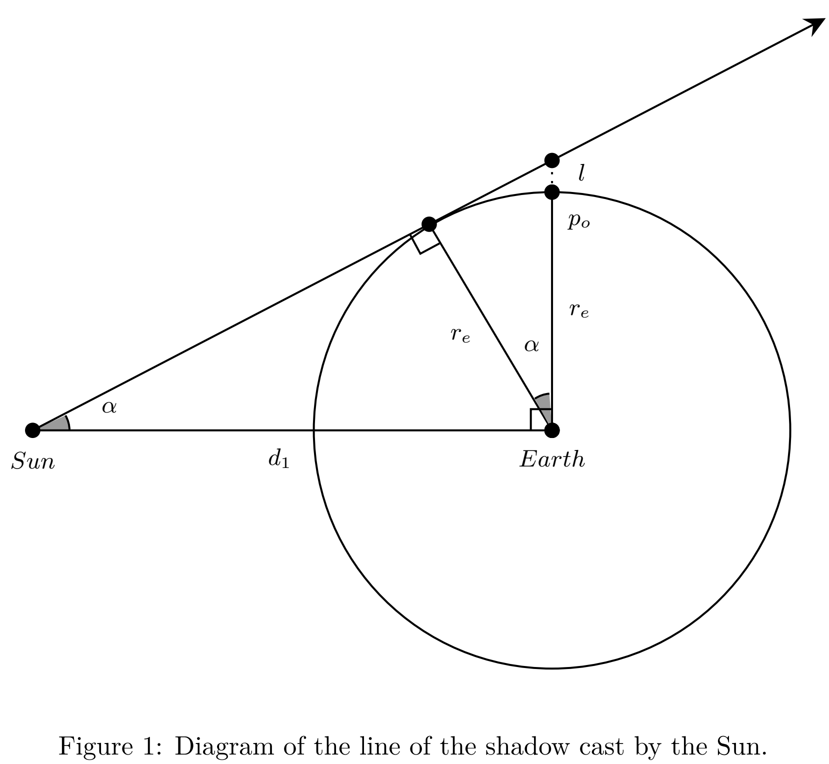


Figure 1: Diagram of the line of the shadow cast by the Sun.

Setup

If the Sun is just setting on the horizon then a position on the equator p_o the observation point will require an increase in height to stay just outside the approaching shadow as the Earth rotates, as shown in Figure 1. The average distance between the center of the Earth and the center of the Sun is denoted as d_1, the radius of the Earth is denoted as r_e, the height of the ladder is denoted by l and the shadow angle as α, which can be determined using r_e and d_1.

$$\sin(\alpha) = \frac{r_e}{d_1}$$
$$\alpha = \sin^{-1}(\frac{r_e}{d_1})$$
$$\alpha \approx 0$$

The Earth's radius relative to the distance between the Sun and the Earth is quite small. As a result α is very small, approximately 0. This simplifies the setup diagram. The updated setup is shown in Figure 2.

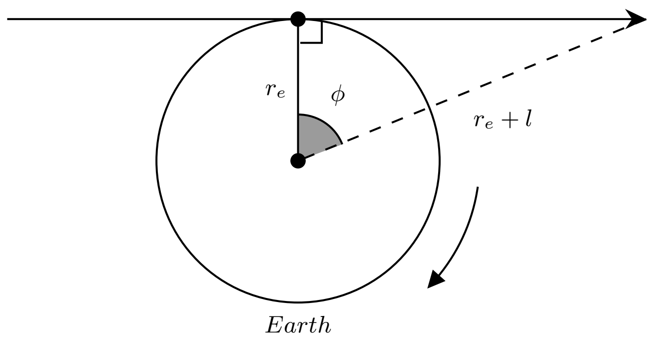


Figure 2: Setup

The equation for the height of the ladder based on the angle ϕ can be derived using the simplified diagram shown in Figure 2. where the the height l is in m meters and the angle ϕ is measured in rad.

$$\cos(\phi) = \frac{r_e}{r_e + l}$$
$$r_e + l = \frac{r_e}{cos(\phi)}$$
$$l = r_e\left(\frac{1}{cos(\phi)} - 1\right)$$
$$l(\phi) = r_e\left(\frac{1}{cos(\phi)} - 1\right) \tag{1}$$

In order to find the velocity time needs to be factored into the equation. This can be done using Earth's constant rate of rotation, denoted by ω measured in radians per second. The equation for angular velocity can be used to determine what to substitute in for the angle ϕ as shown.

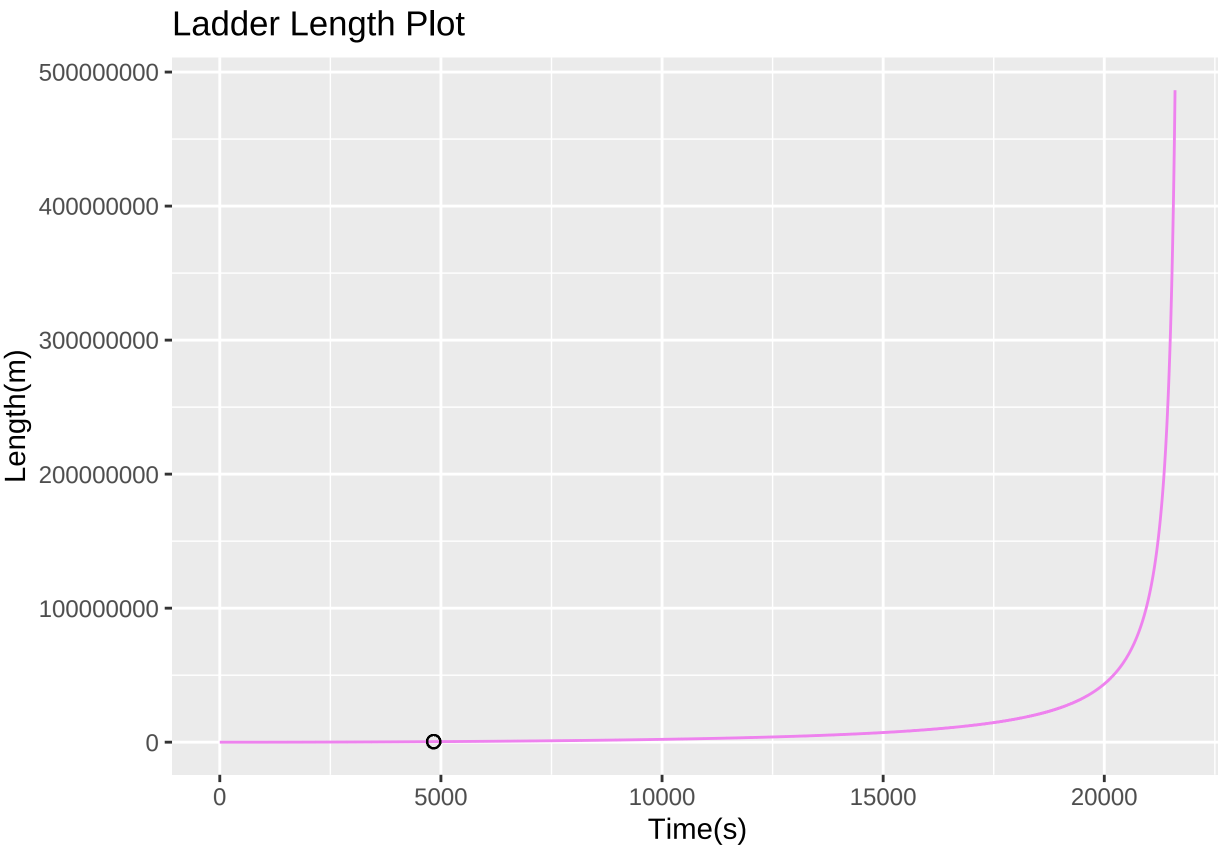
$$\omega = \frac{v_e}{r_e}$$

Now instead of using the constant ϕ, ω is changing over time and the length of the ladder l can be calculated based on time t where $t_0 \rightarrow \phi = 0$ and $t_f \rightarrow \phi = \frac{\pi}{2}$. The height equation can be rewritten relative to time as shown and the domain can be changed to t where $t_0 = 0$ and $t_f = 6$ hours.

Analysis

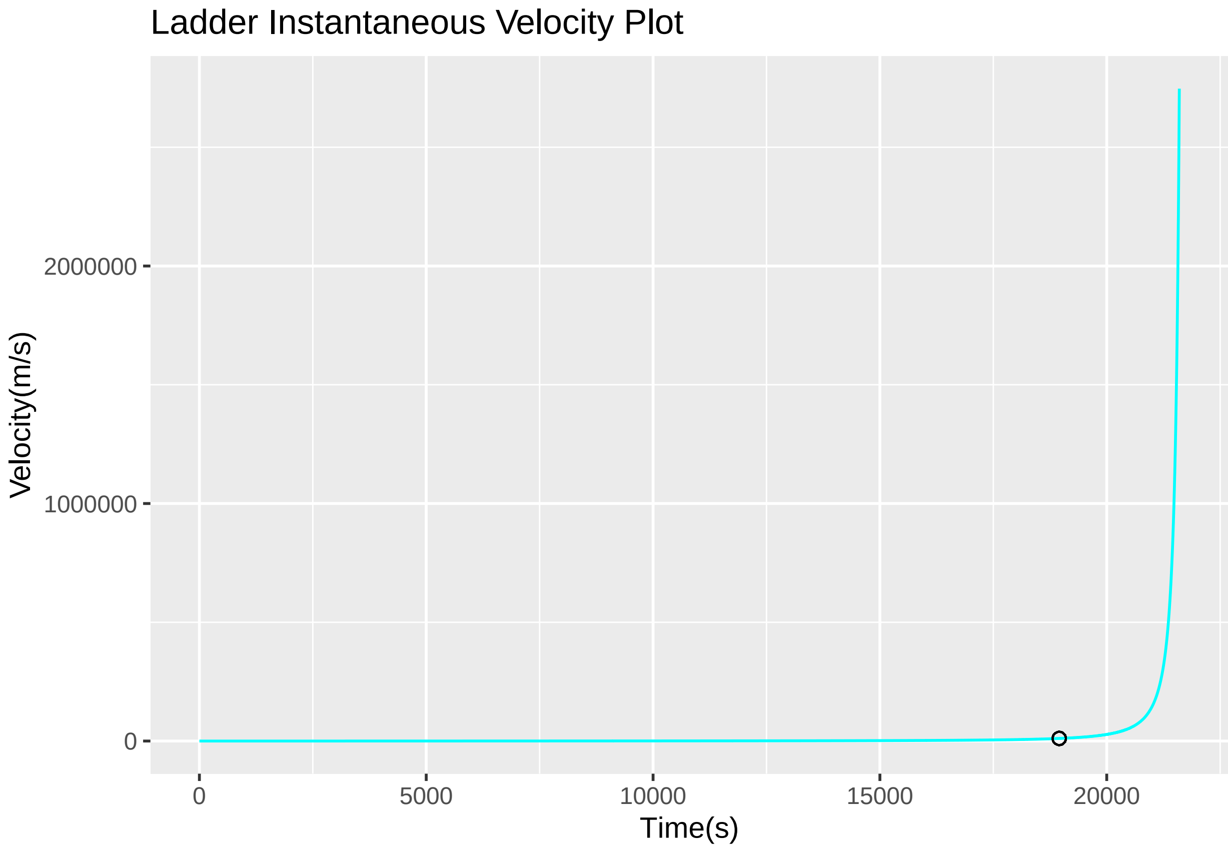
The length of the ladder at the point shown on the reference point of the Length plot is equivalent to the height of the International Space Stations orbit or 254 miles (408773 m).

$$l(t) = r_e \left(\frac{1}{cos\left(\frac{v_e}{r_e} t\right)} - 1 \right) \tag{2}$$



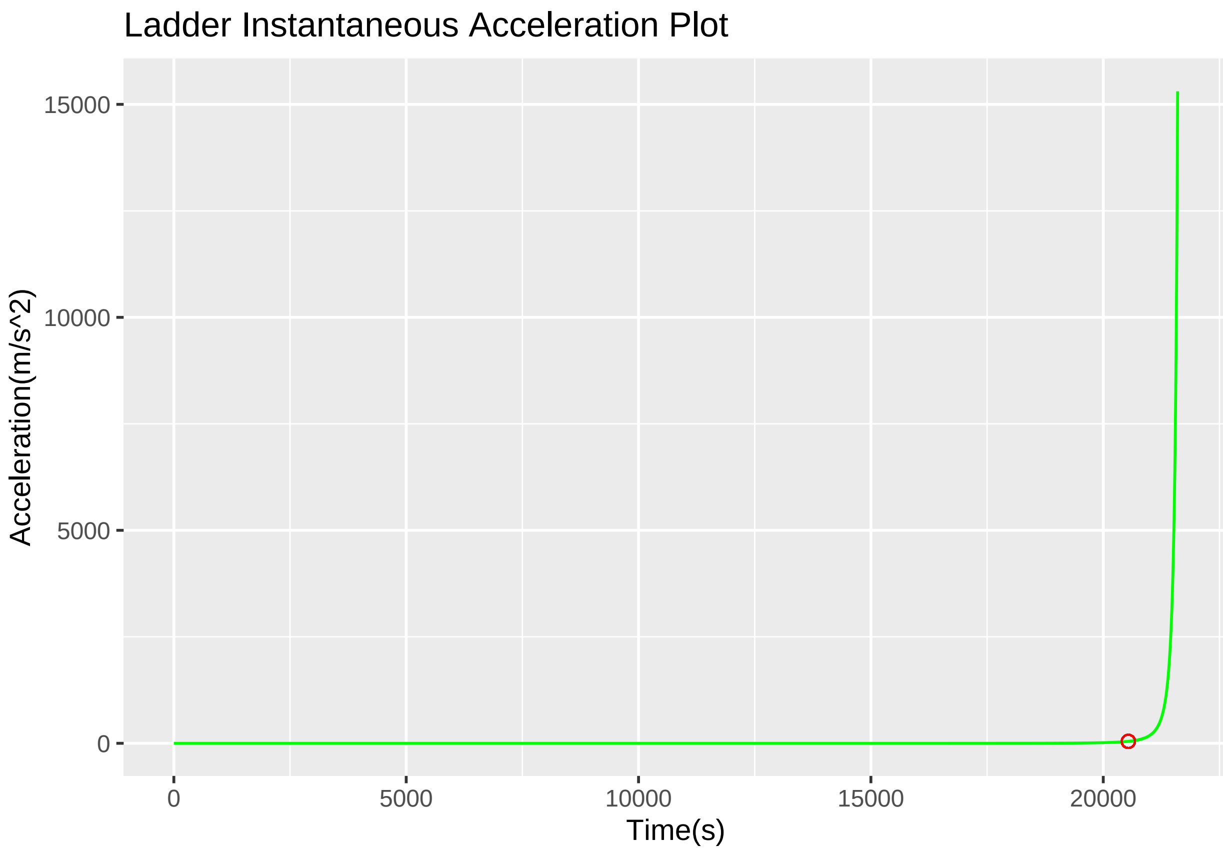
Taking the derivative of the position equation gives the velocity equation. The circle on this plot signifies the escape velocity for a Falcon 9 rocket launch, roughly 11,000$\frac{m}{s^2}$.

$$l'(t) = v_e tan\left(\frac{v_e}{r_e}t\right) sec\left(\frac{v_e}{r_e}t\right) \tag{3}$$



Taking the derivative of the velocity equation gives the acceleration equation.

$$l''(t) = \frac{v_e^2}{r_e}\left(sin^2\left(\frac{v_e}{r_e}t\right)+1\right)sec^3\left(\frac{v_e}{r_e}t\right) \tag{4}$$



For reference, the most acceleration a person has ever been subjected to is about $9g$'s or $46.2\frac{m}{s^2}$ shown on the acceleration plot as the red circle.

Conclusion