Week 3, Lecture 5 - Bayesian vs. frequentist approaches

Aaron Meyer

Outline

- Administrative Issues
- Bayesian Statistics
- A Couple Examples

Based on slides from Joyce Ho.

Frequentist vs Bayesian

Frequentist

- Data are a repeatable random sample (there is a frequency)
- Underlying parameters remain constant during repeatable process
- Parameters are fixed
- Prediction via the estimated parameter value

Bayesian

- Data are observed from the realized sample
- Parameters are unknown and described probabilistically (random variables)
- Data are fixed
- Prediction is expectation over unknown parameters

Freq v. Bayes can hugely influence how we interpret the world

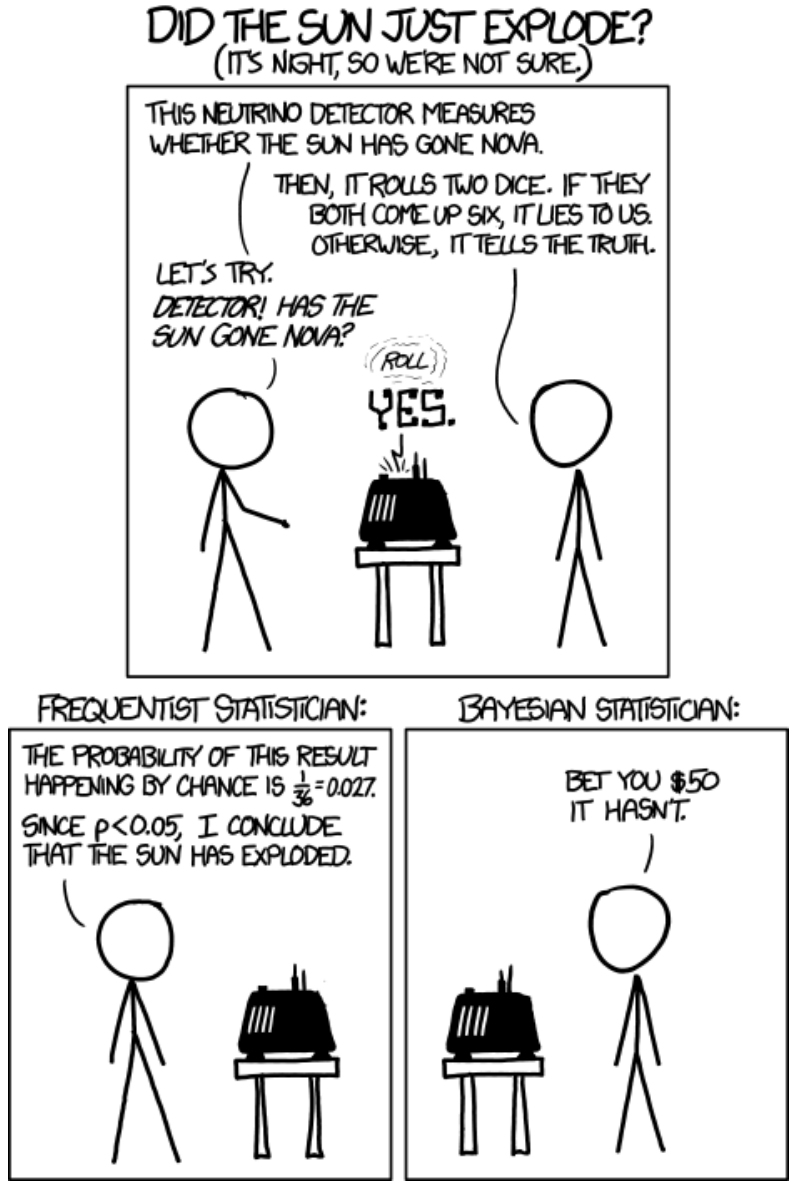


Figure: https://xkcd.com/1132/

Bayesian statistics

Bayesian - Derivation

Bayes' theorem may be derived from the definition of conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) \neq 0$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}, \text{ if } P(A) \neq 0$$

because

$$P(B \cap A) = P(A \cap B)$$

$$\Rightarrow P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$

$$\Rightarrow P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}, \text{ if } P(B) \neq 0$$

Classic Example: Binomial Experiment

- Given a sequence of coin tosses $x_1, x_2, \ldots, x_M$, we want to estimate the (unknown) probability of heads:

$$P(H) = \theta$$

- The instances are independent and identically distributed samples
- Note that x can take on many possible values potentially if we decide to use a multinomial distribution instead

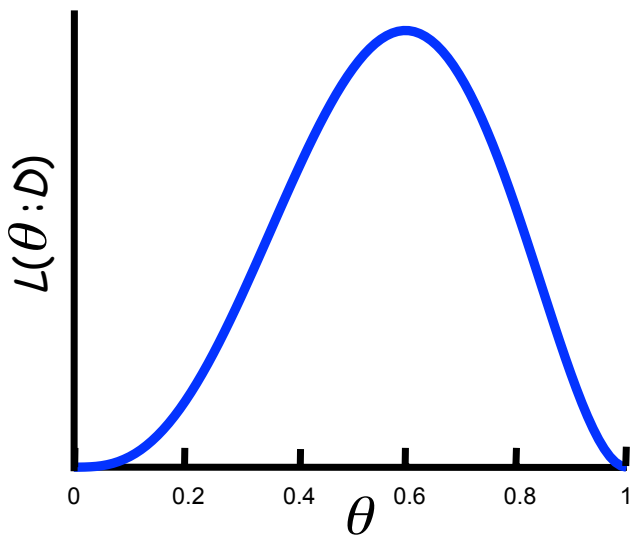
Likelihood Function

- How good is a particular parameter?
  - Answer: Depends on how likely it is to generate the data

$$L(\theta; D) = P(D \mid \theta) = \sum_m P(x_m \mid \theta)$$

- Example: Likelihood for the sequence: H, T, T, H, H

$$L(\theta; D) = \theta(1-\theta)(1-\theta)\theta\theta = \theta^3(1-\theta)^2$$



Maximum Likelihood Estimate (MLE)

- Choose parameters that maximize the likelihood function
  - Commonly used estimator in statistics
  - Intuitively appealing
- In the binomial experiment, MLE for probability of heads

$$\hat{\theta} = \frac{N_H}{N_H + N_T}$$

- Optimization problem approach

Is MLE the only option?

- Suppose that after 10 observations, MLE estimates the probability of a heads is 0.7, would you bet on heads for the next toss?
- How certain are you that the true parameter value is 0.7?
- Were there enough samples for you to be certain?

Bayesian Approach

- Formulate knowledge about situation probabilistically
  - Define a model that expresses qualitative aspects of our knowledge (e.g., forms of distributions, independence assumptions)
  - Specify a **prior** probability distribution for unknown parameters in the model that expresses our beliefs about which values are more or less likely
- Compute the **posterior** probability distribution for the parameters, given observed data
- Posterior distribution can be used for:
  - Reaching conclusions while accounting for uncertainty
  - Make predictions by averaging over posterior distribution

Posterior Distribution

- Posterior distribution for model parameters given the observed data combines the prior distribution with the likelihood function using Bayes' rule:

$$P(\theta \mid D) = \frac{P(\theta)P(D \mid \theta)}{P(D)}$$

- Denominator is just a normalizing constant so you can write it proportionally as:

$$\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$$

- Predictions can be made by integrating with respect to posterior:

$$P(\text{newdata} \mid D) = \int_{\theta} P(\text{newdata} \mid \theta)P(\theta \mid D)$$

Revisiting Binomial Experiment

- Prior distribution: uniform for θ in [0, 1]
- Posterior distribution:

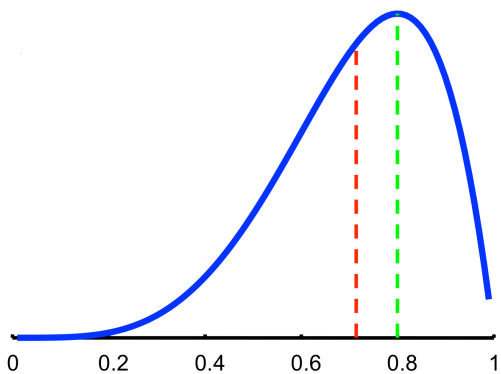
$$P(\theta \mid x_1, \ldots, x_M) \propto P(x_1, \ldots, x_M \mid \theta) \times 1$$

- Example: 5 coin tosses with 4 heads, 1 tail
  - MLE estimate:

$$P(\theta) = \frac{4}{5} = 0.8$$

  - Bayesian prediction:

$$P(x_{M+1} = H \mid D) = \int \theta P(\theta \mid D) d\theta = \frac{5}{7}$$



Bayesian Inference and MLE

- MLE and Bayesian prediction differ
- However...
  - IF prior is well-behaved (i.e., does not assign 0 density to any "feasible" parameter value)
  - THEN both MLE and Bayesian prediction converge to the same value as the training data becomes infinitely large

Features of the Bayesian Approach

- Probability is used to describe "physical" randomness and uncertainty regarding the true values of the parameters
  - Prior and posterior probabilities represent degrees of belief, before and after seeing the data
- Model and prior are chosen based on the knowledge of the problem and not, in theory, by the amount of data collected or the question we are interested in answering

Priors

- Objective priors: noninformative priors that attempt to capture ignorance and have good frequentist properties
- Subjective priors: priors should capture our beliefs as well as possible. They are subjective but not arbitrary.
- Hierarchical priors: multiple levels of priors
- Empirical priors: learn some of the parameters of the prior from the data ("Empirical Bayes")
  - Robust, able to overcome limitations of mis-specification of prior
  - Double counting of evidence / overfitting

Conjugate Prior

- If the posterior distribution are in the same family as prior probability distribution, the prior and posterior are called conjugate distributions
- All members of the exponential family of distributions have conjugate priors

Likelihood	Conjugate prior distribution	Prior hyperparameter	Posterior hyperparameters
Bernoulli	Beta	α, β	$\alpha + \sum x_i, \beta + n - \sum x_i$
Multinomial	Dirichlet	α	$\alpha + \sum x_i$
Poisson	Gamma	α, β	$\alpha + \sum x_i, \beta + n$

Linear Regression (Classic Approach)

$$y = w^\top x + \epsilon,\ \epsilon \sim N(0, \sigma^2)$$

$$P(y_i|w, x_i, \sigma^2) = N(w^\top x_i, \sigma^2)$$

$$P(y|w, X, \sigma^2) = \prod_i P(y_i|w, x_i, \sigma^2)$$

⇩ maximize log likelihood

$$\max\ \ln(P(y|w, x, \sigma^2) = \max \sum_i \ln(N(y_i|w, x_i, \sigma^2))$$

$$w_{\text{MLE}} = \text{argmin}_w \frac{1}{2} \sum_i (y_i - x_i^\top w)^2$$

$$w = (X^\top X)^{-1} X^\top y$$

Bayesian Linear Regression

- Prior is placed on either the weight, w, or the variance, σ
- Conjugate prior for w is normal distribution

$$P(w) \sim N(\mu_0, S_0)$$

$$P(w \mid y) \sim N(\mu, S)$$

$$S^{-1} = S_0^{-1} + \frac{1}{\sigma^2} X^T X$$

$$\mu = S\left(S_0^{-1}\mu_0 + \frac{1}{\sigma^2} X^T y\right)$$

- Mean is weighted average of OLS estimate and prior mean, where weights reflect relative strengths of prior and data information

Computing the Posterior Distribution

Analytical integration Works when "conjugate" prior distributions can be used, which combine nicely with the likelihood—usually not the case

Gaussian approximation Works well when there is sufficient data compared to model complexity—posterior distribution is close to Gaussian (Central Limit Theorem) and can be handled by finding its mode

Markov Chain Monte Carlo Simulate a Markov chain that eventually converges to the posterior distribution—currently the dominant approach

Variational approximation Cleverer way to approximate the posterior and maybe faster than MCMC but not as general and exact

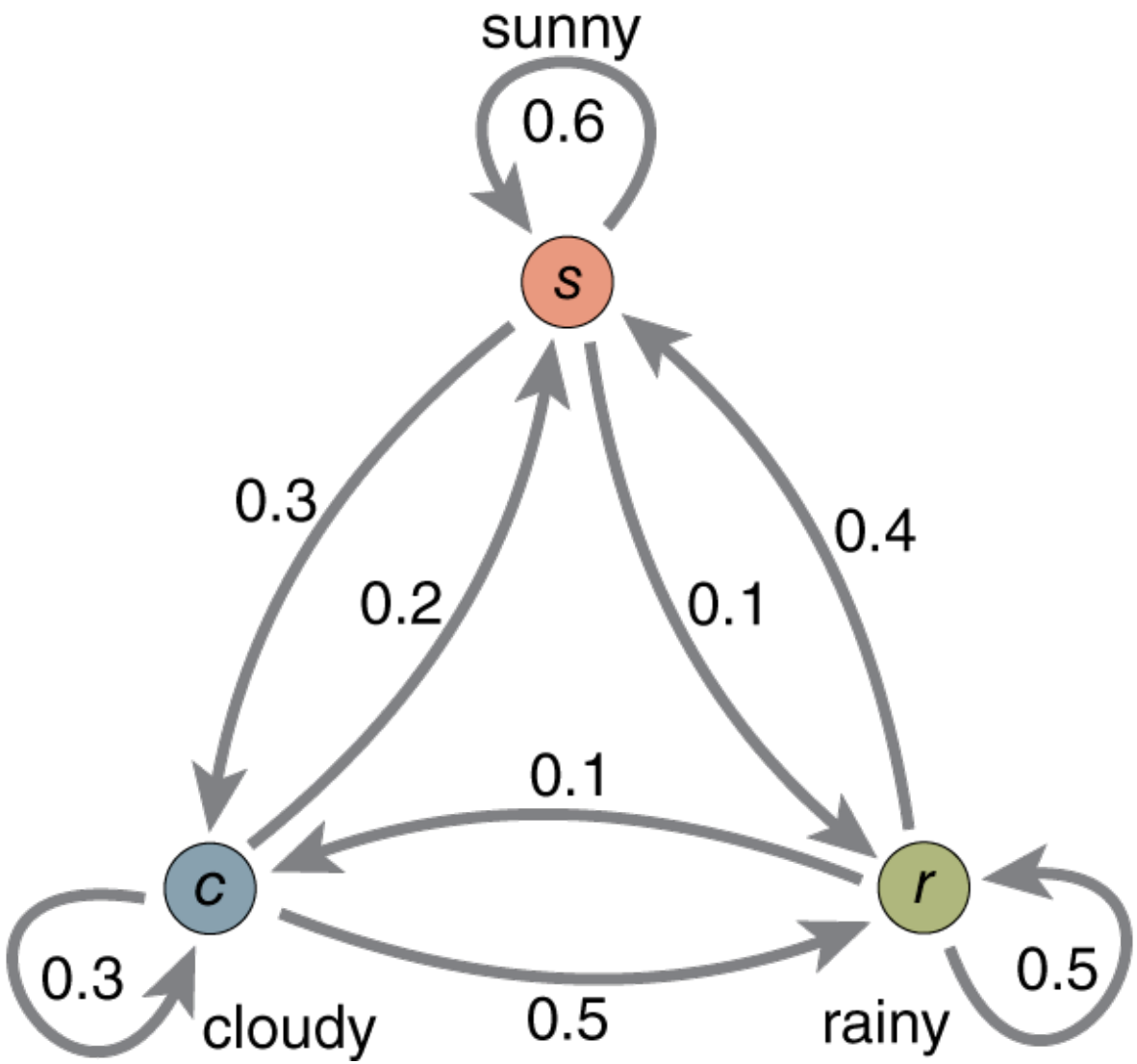
Approximate Bayesian Inference

- Stochastic approximate inference (MCMC)
  - Design an algorithm that draws sample from distribution
  - Inspect sample statistics
  - (Pros) Asymptotically exact
  - (Cons) Computationally expensive
  - (Cons) Tricky Engineering concerns
- Structural approximate inference (variational Bayes)
  - Use an analytical proxy that is similar to original distribution
  - Inspect distribution statistics of proxy
  - (Pros) Often insightful & fast
  - (Cons) Often hard work to derive
  - (Cons) Requires validation via sampling

Markov Chain Monte Carlo

Markov Chain Monte Carlo

A Simple Markov Chain



Markov Chains

- A random process has Markov property iff:

$$p(X_t \mid x_{t-1}, X_{t-2}, \ldots, X_1) = p(X_t \mid x_{t-1})$$

- Finite-state Discrete Time Markov Chains can be completely specified by the transition matrix P

$$P = [p_{ij}]; p_{ij} = P[X_t = j \mid X_{t-1} = i]$$

- Stationarity: As t approaches infinity, the Markov chain converges in distribution to its stationary distribution (independent of starting position)

Markov Chains

- Irreducible: any set of states can be reached from any other state in a finite number of moves
  - Assuming a stationary distribution exists, it is unique if the chain is irreducible
- Aperiodicity: greatest common divisor of return times to any particular state i is 1
- Ergodicity: if the Markov chain has a stationary distribution, aperiodic, irreducible then:

$$E_\pi[h(X)] = \frac{1}{N} \sum h(X^{(t)}) \text{ as } N \to \infty$$

MCMC Algorithms

- Posterior distribution is too complex to sample from directly, simulate a Markov chain that converge (asymptotically) to the posterior distribution
  - Generating samples while exploring the state space using a Markov chain mechanism
  - Constructed so the chain spends more time in the important regions
  - Irreducible and aperiodic Markov chains with target distribution as the stationary distribution
- Can be very slow in some circumstances but is often the only viable approach to Bayesian inference using complex models

The Monte Carlo Principle

- General Problem:

$$E_\pi[h(X)] = \int h(x)\pi(x)dx$$

- Instead, draw samples from the target density to estimate the function:
  - $X^{(1)}, X^{(2)}, \ldots, X^{(N)} \sim \pi(x)$
  - $E_\pi[h(X)] \approx \frac{1}{N} \sum h(X^{(t)})$

Metropolis-Hastings Algorithm

- Most popular MCMC (Metropolis, 1953; Hastings 1970)
- Main Idea:
  - Create a Markov chain whose transition matrix does not depend on the normalization term
  - Make sure the chain has a stationary distribution and is equal to the target distribution
  - After sufficient number of iterations, the chain converges to the stationary distribution

Metropolis-Hasting Algorithm

At each iteration t

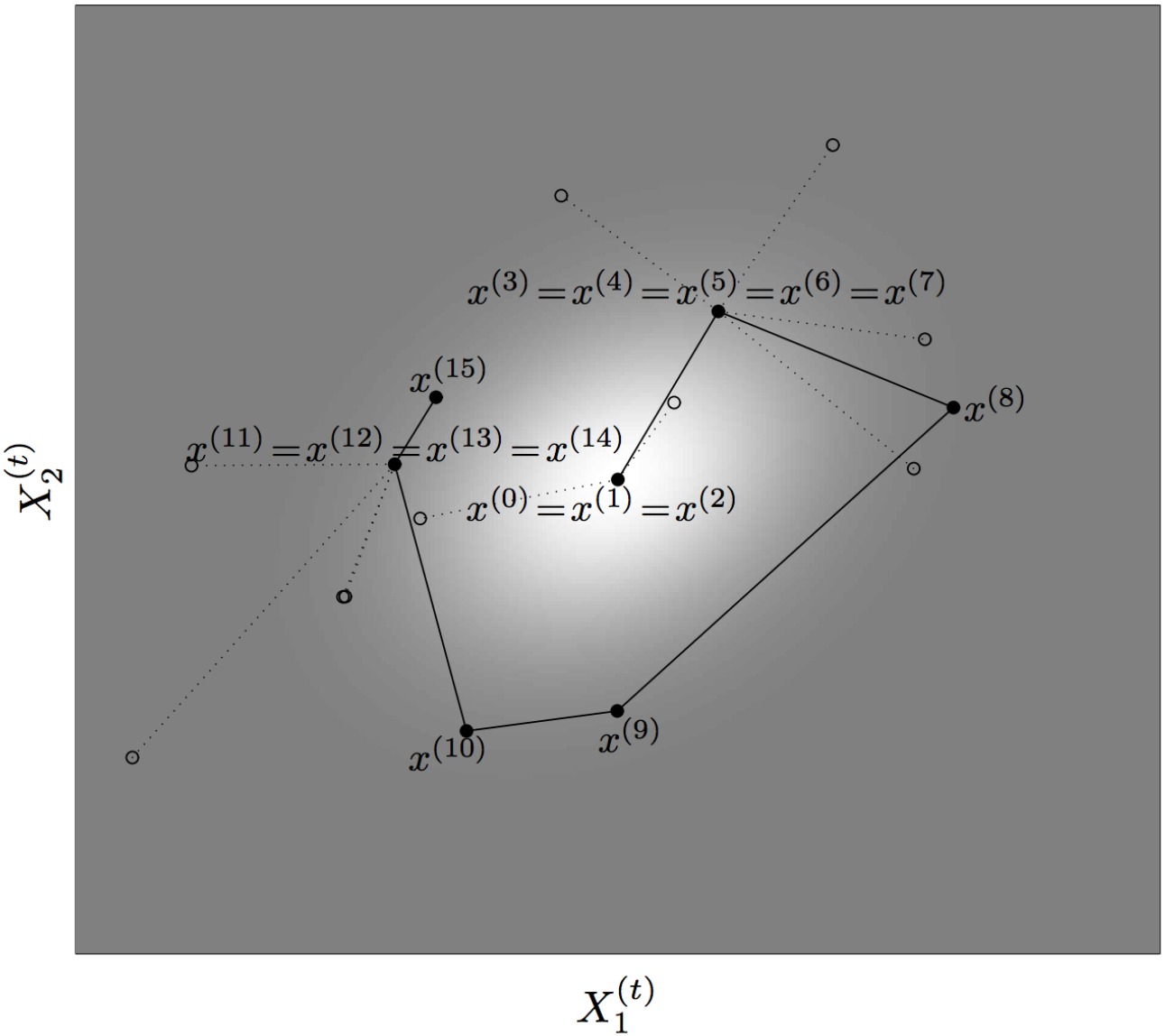
- Step 1: Sample a candidate point from proposal distribution

$$y \sim q\left(y \mid x^{(t)}\right)$$

- Step 2: Accept the next point with probability

$$\alpha\left(x^{(t)}, y\right) = \min\left[1, \frac{p(y)q(x^{(t)} \mid y)}{p(x^{(t)})q(y \mid x^{(t)})}\right]$$

Illustration of Metropolis-Hasting Algorithm



Variations of Proposal Distribution

- Random-walk is when proposal is dependent on previous state
  - $y \sim q(y \mid x^{(t)})$
- Symmetric proposal originally proposed by Metropolis (e.g., Gaussian distribution)
  - $q(x \mid y) \equiv q(y \mid x)$
- Independent sampler uses a proposal independent of x
  - $q(y \mid x) \equiv q(y)$

Metropolis-Hastings Notes

- Normalizing constant of the target distribution is not required
- Choice of proposal distribution is very important:
  - too narrow —> not enough mixing
  - too wide —> high correlations
- Usually q is chosen so the proposal distribution is easily to sample
- Easy to simulate several independent chains in parallel

Acceptance Rates

- Important to monitor the acceptance rate (fraction of candidate draws that are accepted)
- Too high means the chain is not mixing well and not moving around the parameter space quickly enough
- Too low means algorithm is too inefficient (too many candidate draws)
- General rules of thumb:
  - Random walk: Somewhere between 0.25 and 0.50
  - Independent: Closer to 1 is preferred

Gibbs Sampling (Geman & Geman, 1984)

- Popular in statistics and graphical models
- Special form of Metropolis-Hastings where we always accept a candidate point and we know the full conditional distributions
- Easy to understand, easy to implement
- Open-source, black-box implementations available

Gibbs Sampling

Sample or update in turn:

$$X_1^{(t+1)} \sim \pi\left(x_1 \mid x_2^{(t)}, x_3^{(t)}, \ldots, x_k^{(t)}\right)$$

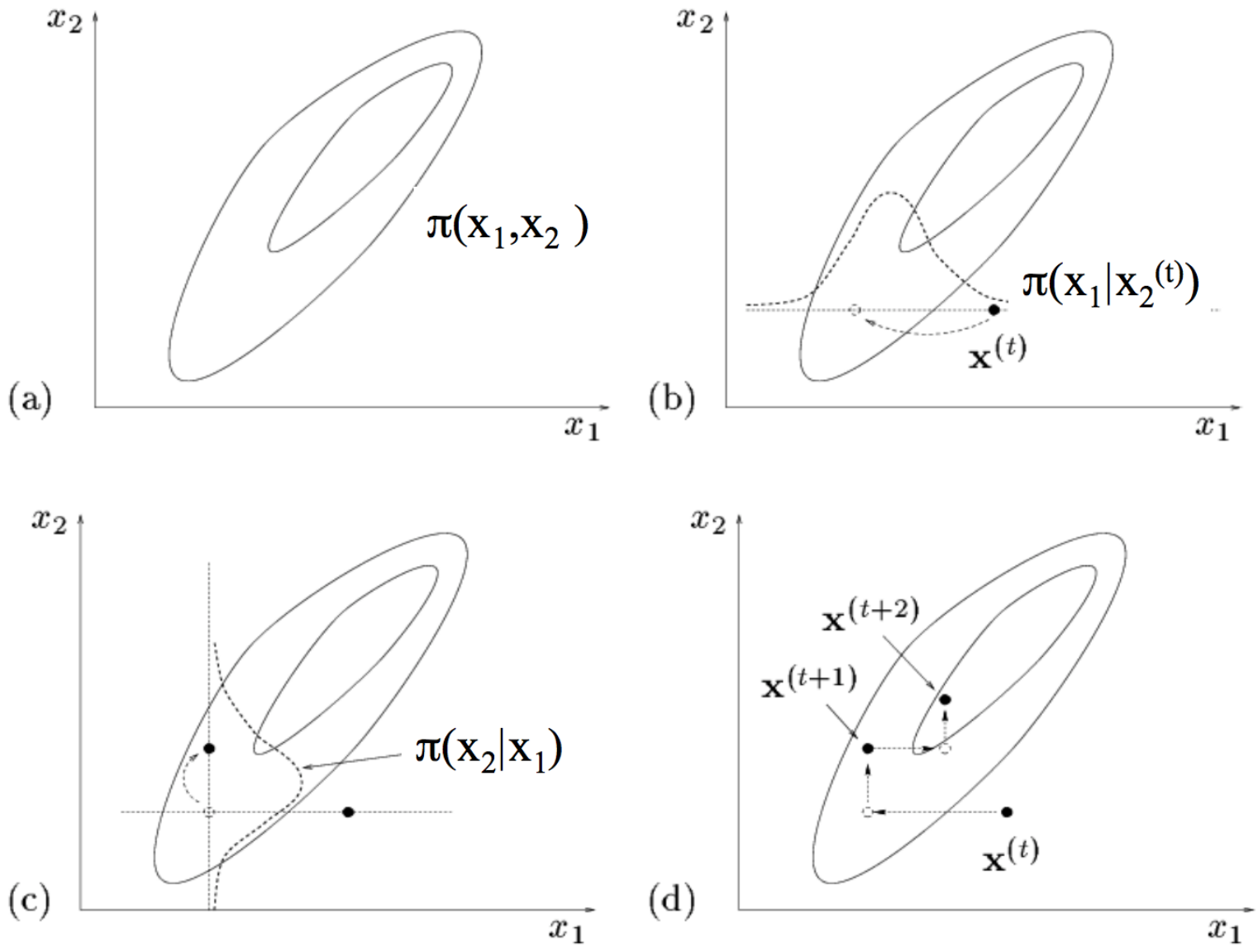
$$X_2^{(t+1)} \sim \pi\left(x_2 \mid x_1^{(t+1)}, x_3^{(t)}, \ldots, x_k^{(t)}\right)$$

$$X_3^{(t+1)} \sim \pi\left(x_3 \mid x_1^{(t+1)}, x_2^{(t+1)}, x_4^{(t)}, \ldots, x_k^{(t)}\right)$$

...

$$X_k^{(t+1)} \sim \pi\left(x_k \mid x_1^{(t+1)}, x_2^{(t+1)}, \ldots, x_{k-1}^{(t+1)}\right)$$

Illustration of Gibbs Sampler



Practicalities: Burn-In

- Convergence usually occurs regardless of our starting point, so can pick any feasible starting point
- Chain convergence varies depending on the starting point
- As a matter of practice, most people throw out a certain number of the first draws, known as the burn-in
- The remaining draws are closer to the stationary distribution and less dependent on the starting point
- Plot the time series for each quantity of interest and the auto-correlation functions to see if the chain has converged

Practicalities: Number of Chains

- Suggestion: Experiment with different number of chains
- Several long runs (Gelman & Rubin, 1992)
  - Gives indication of convergence
  - Sense of statistical security
- One very long run (Geyer, 1992)
  - Reaches parts other schemes cannot reach

Other Flavors of MC

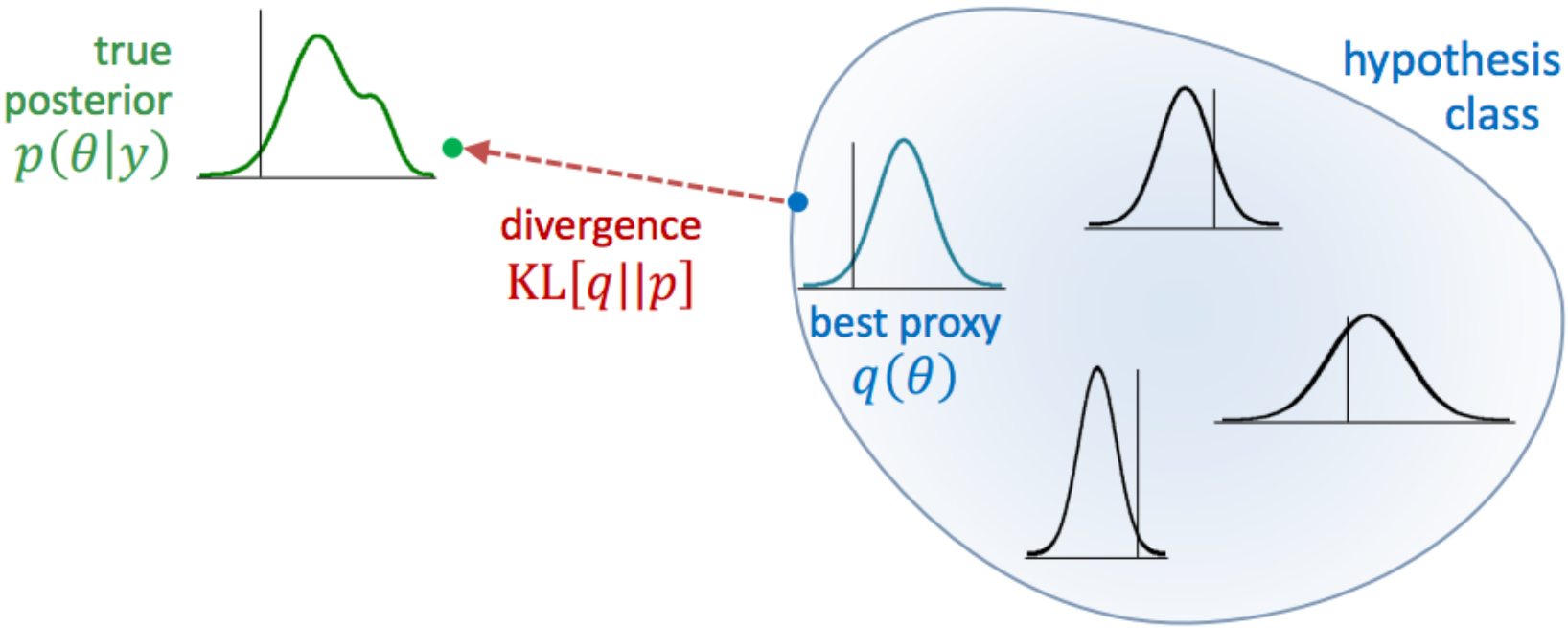
- Auxiliary Variable Methods for MCMC
  - Hybrid Monte Carlo (HMC)
  - Slice Sampler
- Reversible jump MCMC
- Adaptive MCMC
- Sequential Monte Carlo (SMC) and Particle Filters

Variational Approximation

Variational Approximation

Bayesian Inference via Variational Approximation

- Related to "mean field" and other approximation methods from physics
- Idea: Find an approximate density that is maximally similar to the true posterior



Kullback–Leibler divergence

$$D_{\mathrm{KL}}(P\|Q) = -\sum_i P(i)\,\log\frac{Q(i)}{P(i)}$$

or

$$D_{\mathrm{KL}}(P\|Q) = \int_{-\infty}^{\infty} p(x)\,\log\frac{p(x)}{q(x)}\,dx$$

The Mean-Field Form

- A common way of restricting the class of approximate posterior is to consider those posteriors that factorize into independent partitions

$$q(\theta) = \sum_i q_i(\theta_i)$$

- Each $q_i(\theta_i)$ is the approximate posterior for the ith subset of parameters
- This implies a straightforward algorithm for inference by cycling over each set of parameters given current sets of others

Example: Variational Inference

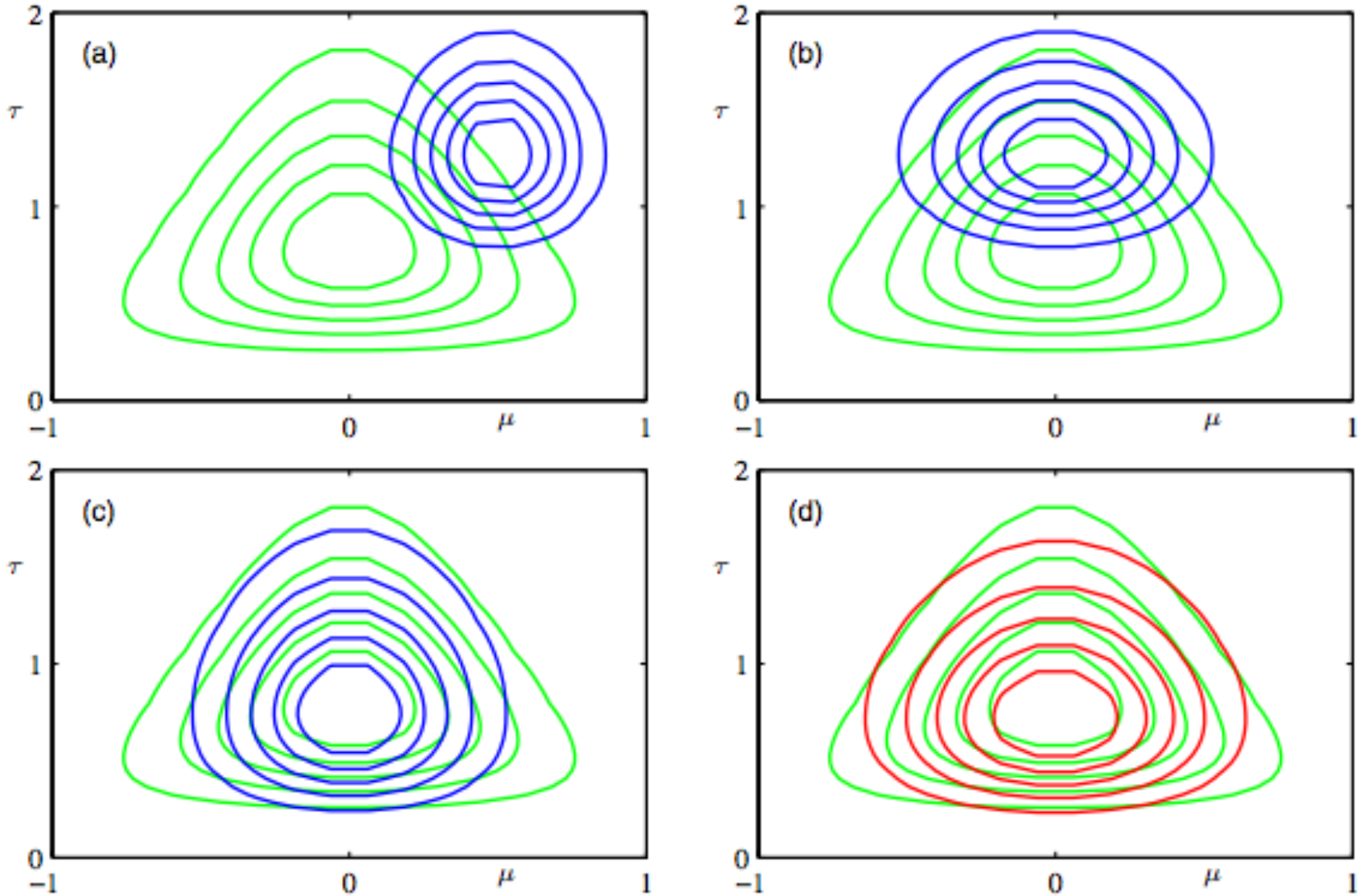


Figure: Figure 10.4 from Bishop PRML

Limitations and Criticisms of Bayesian Methods

- It is hard to come up with a prior (subjective) and the assumptions may be wrong
- Closed world assumption: need to consider all possible hypotheses for the data before observing the data
- Computationally demanding (compared to frequentist approach)
- Use of approximations weakens coherence argument

Bayesian statistics

Example problem - HIV test

- Rapid home tests will pick up an infection 97.7% of the time at 28 days after exposure (sensitivity).
- These same tests have a specificity of ~95%.
- 0.34% of the US population is estimated to be infected.

Given a positive test, what is the chance of the average person in the US being infected?
How would this change if ~10% of the population were infected?

Implementation - PyMC3

```
import pymc3 as pm

n = 100
heads = 61

with pm.Model() as coin_context:
    p = pm.Beta('p', alpha=2, beta=2)
    y = pm.Binomial('y', n=n, p=p, observed=heads)
    trace = pm.sample()

pm.summary(t, varnames=['p'])
```

Implementation - PyMC3

Output:

p:

```
  Mean             SD               MC Error         95% HP
  ------------------------------------------------------------

  0.615            0.050            0.002            [0.517

  Posterior quantiles:
  2.5              25               50               75
  |--------------|==============|==============|-----------

  0.517            0.581            0.616            0.654
```

Implementation - emcee

```
import emcee

def lnprob(p):
    return lnprior(p) + lnobs(p, heads, n)

sampler = emcee.EnsembleSampler(nwalkers=3, ndim=1, lnprob=lnpro

sampler.run_mcmc(pos, 500)

samples = sampler.chain
```

Further Reading

- PyMC3 (python)
- emcee (python)
- Stan (C++, python, R)
- Bayesian Data Analysis
- Probabilistic Programming & Bayesian Methods for Hackers