Maths pre-requisites for data analysis in neuroscience

1/ Linear Algebra

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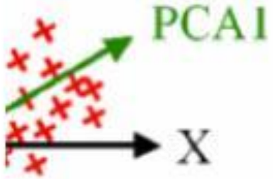
05/04/2019 - Workshop "computational neuroscience", pre-requisite course

Ultimate goal: understand and develop tools for the analysis of neural data

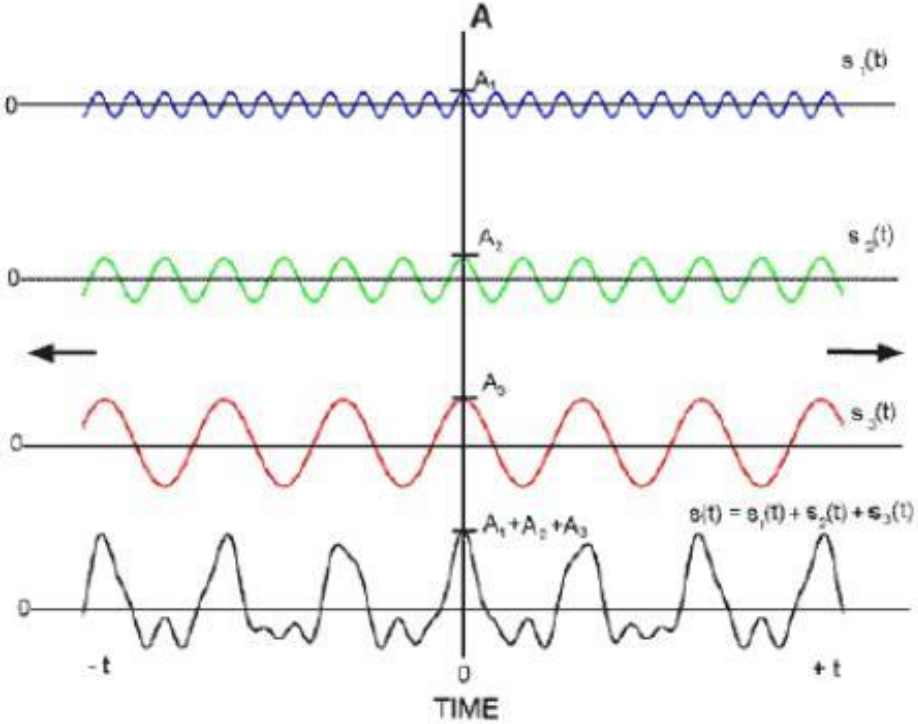
$$C_m \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta_T e^{\frac{V - V_T}{\Delta_T}} - u + I$$

$$\tau_w \frac{du}{dt} = a(V - E_L) - u$$

differential equations and modeling



dimensionality reduction



Fourier transforms, convolutions, and filtering out noise

"What is an eigenvector?"

"What exactly is PCA doing?"

"What *really* is a Fourier transform?"

Many slides from Lane McIntosh & Kiah Hardcastle

(NBIO course, Stanford Univ, https://web.stanford.edu/class/nbio228-01/info.html)

1/ Linear Algebra

- When and why is linear algebra useful?
- Vectors and their operations
- Matrices and their operations, special matrices
- Equation systems, matrices and determinants

Why linear algebra?

Why linear algebra?

1.63	5.20	7.66	8.12	3.22
4.98	5.90	8.21	9.29	20.10
10.10	8.57	5.73	8.17	2.22
0.02	0.21	0.14	0.93	1.40
9.27	10.27	13.12	8.90	9.01
7.44	6.98	5.62	8.20	7.21
100.10	8.22	7.54	60.10	1.69
40.20	29.21	12.45	10.41	8.90
32.33	21.59	10.21	4.99	2.62
2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

neuron 1	1.63	5.20	7.66	8.12	3.22
neuron 2	4.98	5.90	8.21	9.29	20.10
neuron 3	10.10	8.57	5.73	8.17	2.22
neuron 4	0.02	0.21	0.14	0.93	1.40
neuron 5	9.27	10.27	13.12	8.90	9.01
neuron 6	7.44	6.98	5.62	8.20	7.21
neuron 7	100.10	8.22	7.54	60.10	1.69
neuron 8	40.20	29.21	12.45	10.41	8.90
neuron 9	32.33	21.59	10.21	4.99	2.62
neuron 10	2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

voxel 1	1.63	5.20	7.66	8.12	3.22
voxel 2	4.98	5.90	8.21	9.29	20.10
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Datasets are matrices

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21
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A/ Vectors

- Definition and representation
- Basic operations
- Norm and angle
- Products

B/ Matrices

- Definition and basic operations
- Linear transformation M·V
- Matrix rank
- Multiplication
- Special matrices

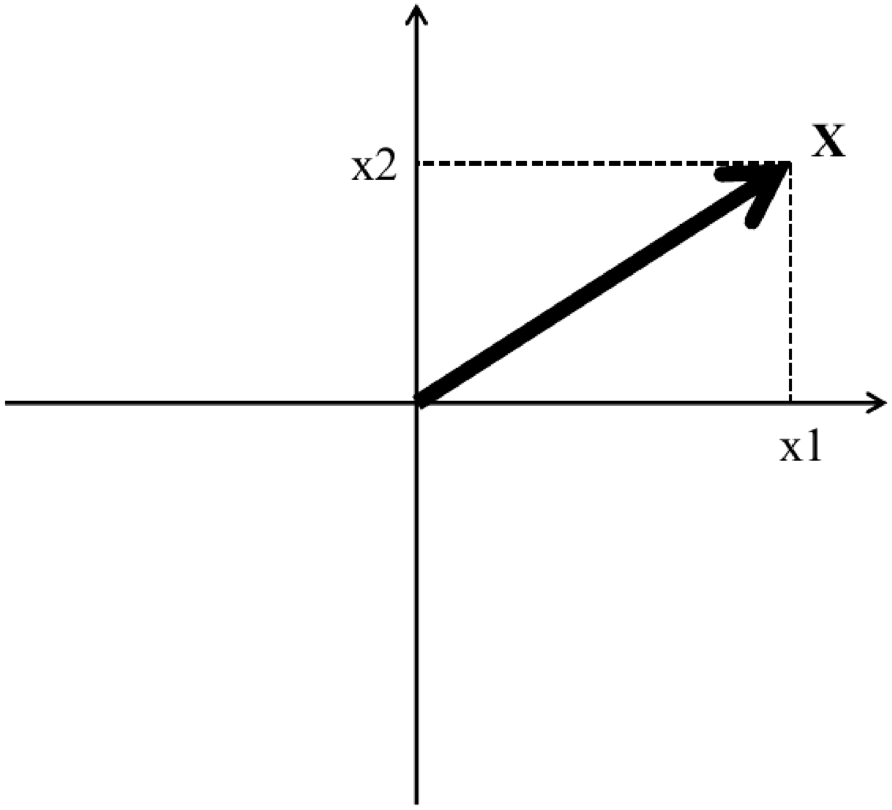
C/ Linear systems, matrices and determinant

- Linear equation systems
- Inverse matrix
- Determinant: geometry

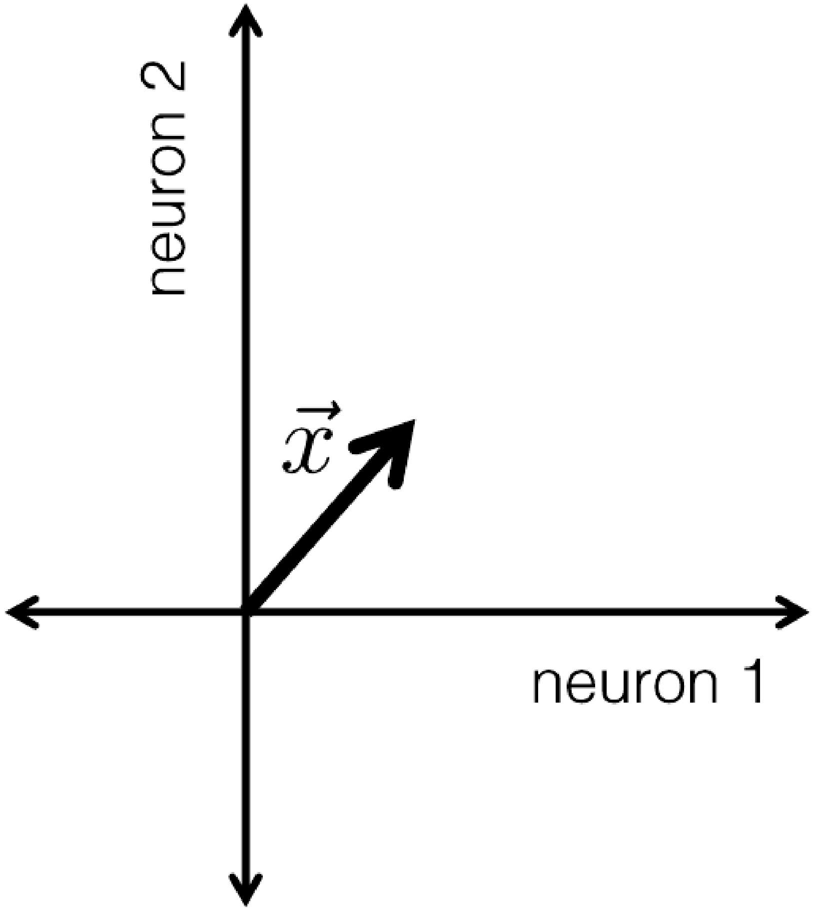
A/ Vectors

$$\mathrm{x} = (x_1, x_2, \dots, x_n)$$

$$\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

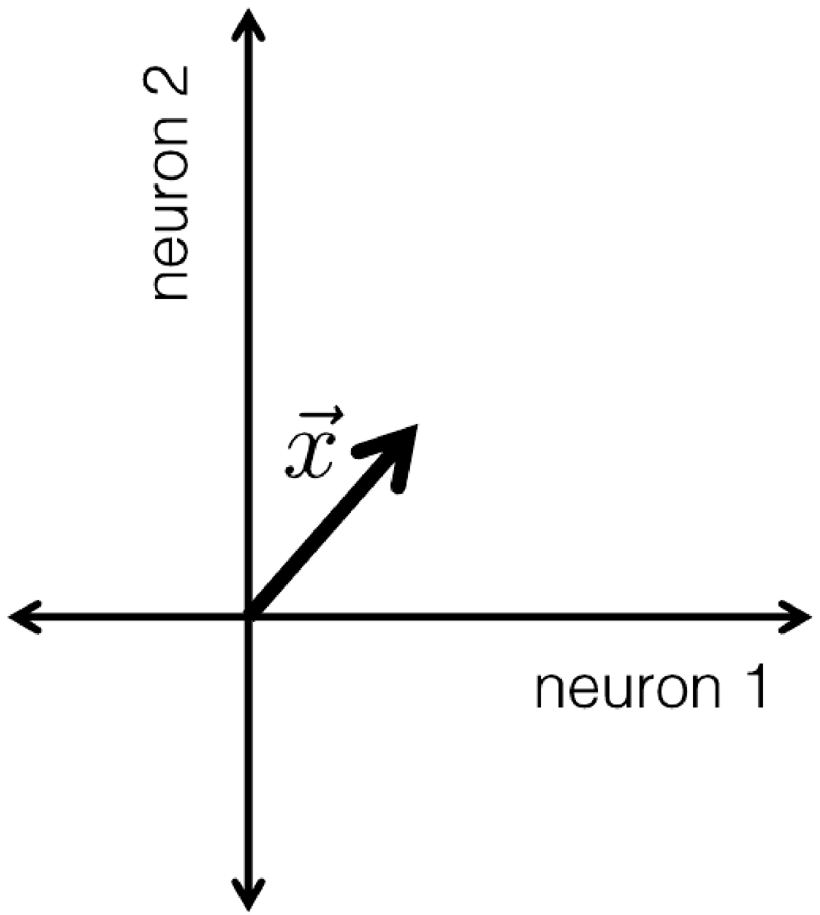


Scalar times vector



Measure 10 neurons at 100 HZ

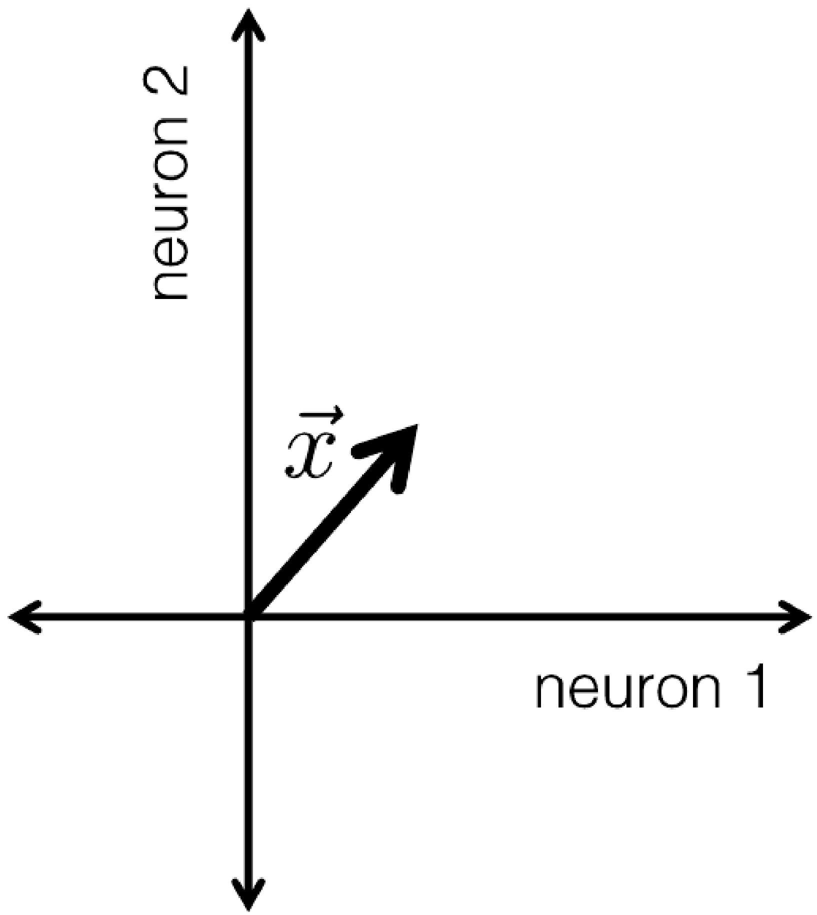
Scalar times vector



Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

Scalar times vector

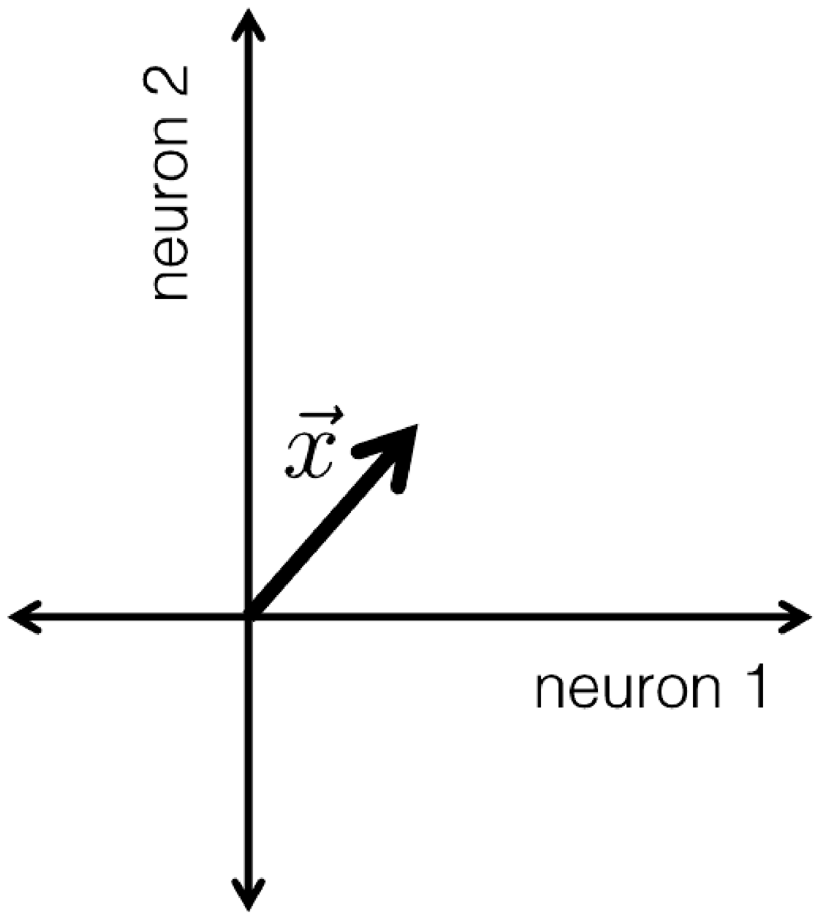


Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

Scalar times vector



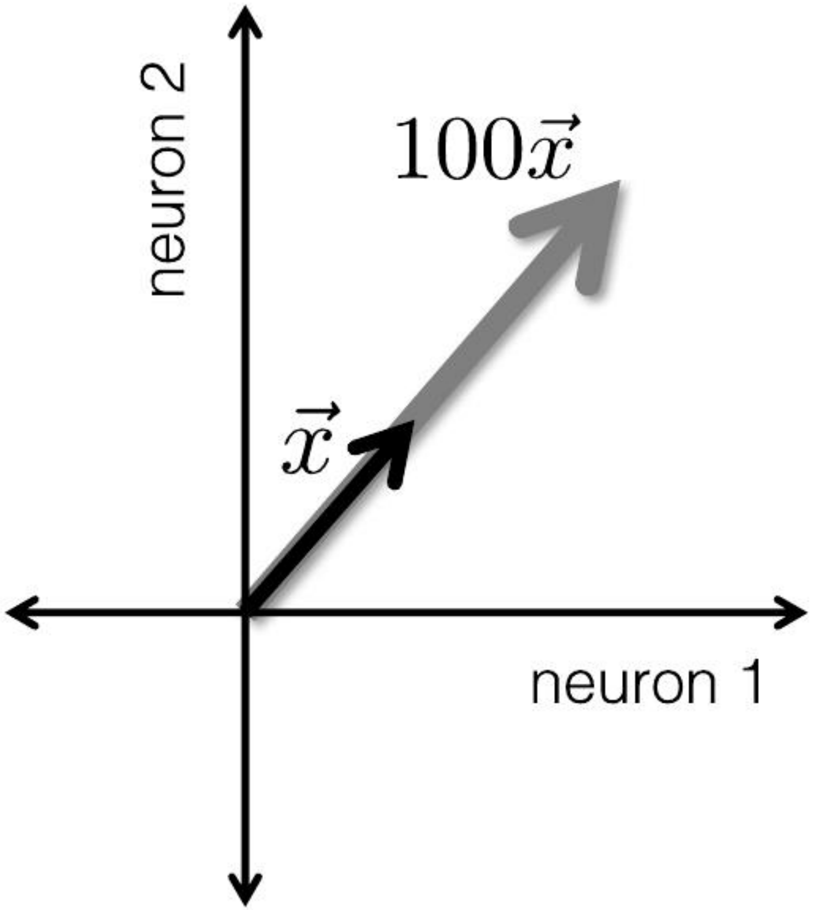
Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

➡ Multiply each measurement by 100

Scalar times vector



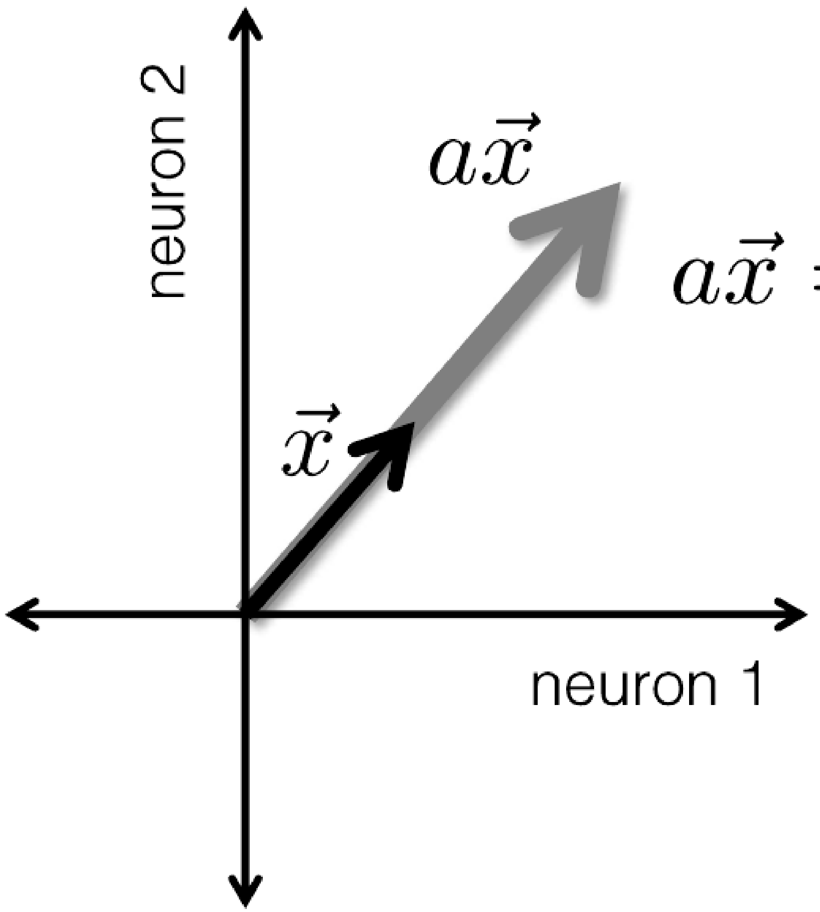
Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

→ Multiply each measurement by 100

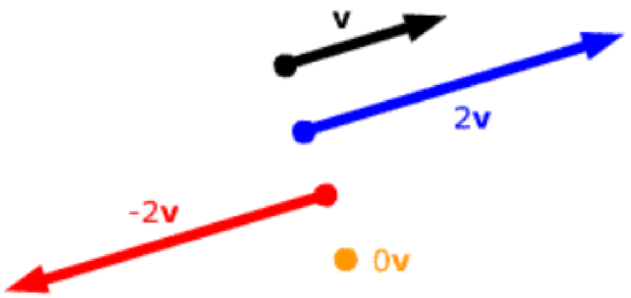
Scalar times vector



$$a\vec{x} = a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

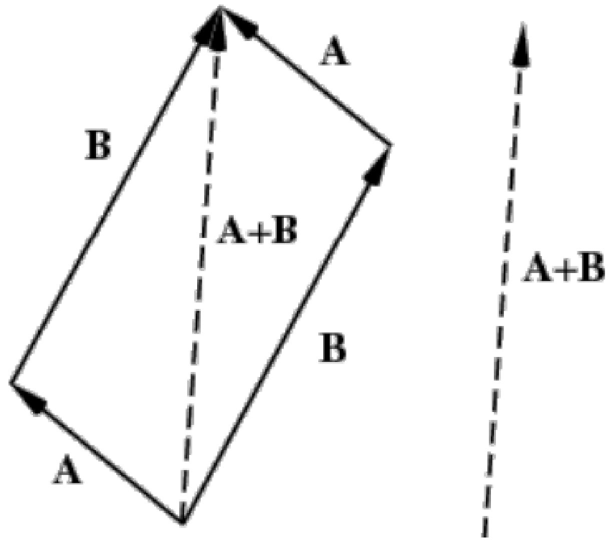
A/ Vectors

Scalar multiplication



$$\lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \equiv \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

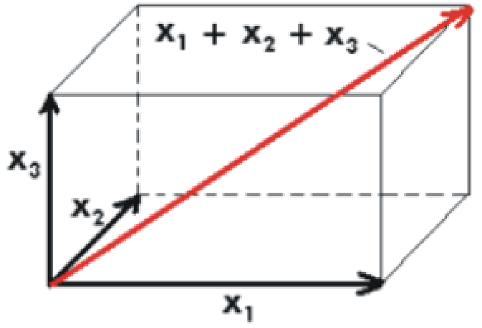
Addition



$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \equiv \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

Substraction

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

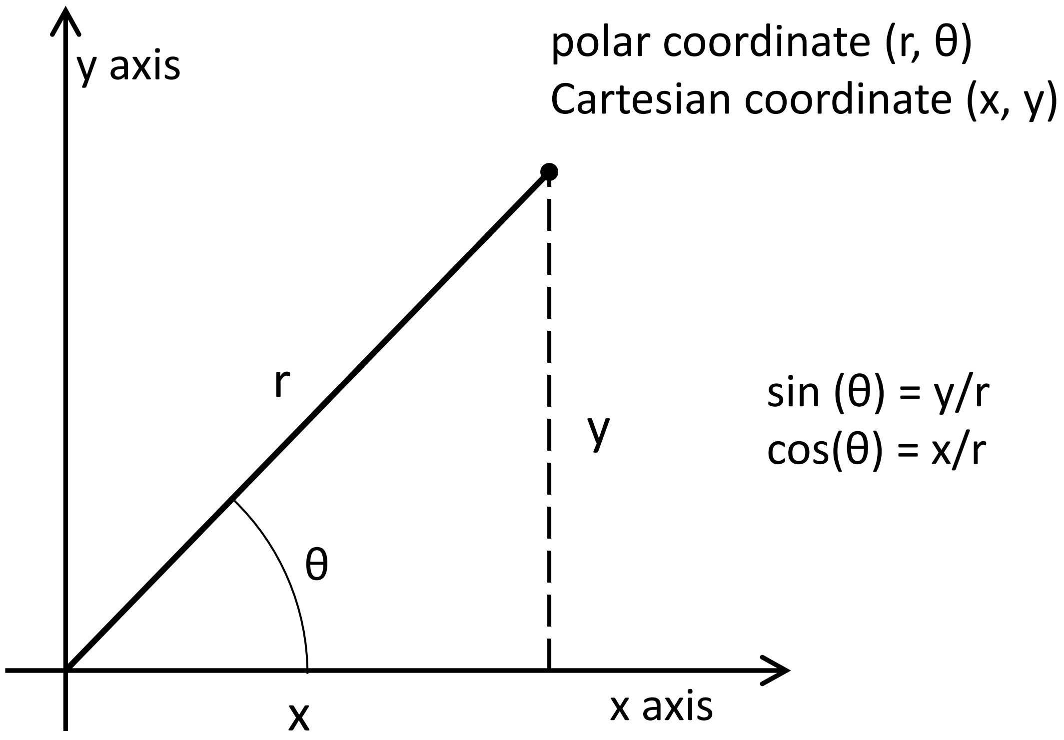


A/ Vectors

Norm (or magnitude): |x| or ||x||

$$\mathrm{x} = (x_1, x_2, \dots, x_n)$$

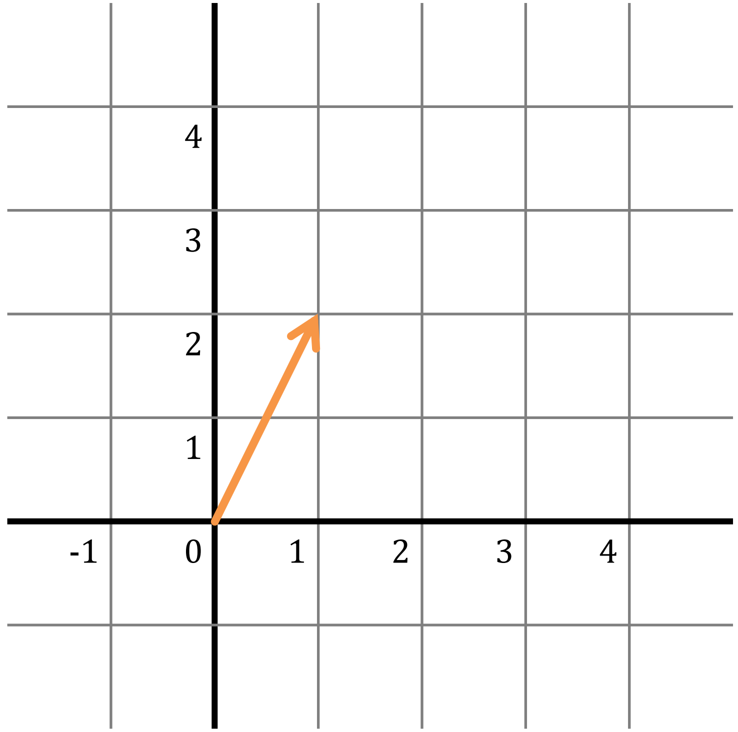
$$|\mathrm{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$



sin (θ) = y/r
cos(θ) = x/r

Exercises

1. Represent the following vectors of $\mathbb{R}^2$ on a plane with two orthogonal axes:
   a) $[1 \quad 2]$
   b) $[3 \quad -1]$
   c) $2[1 \quad -3]$
   d) $[2 \quad 4] - [0 \quad -2]$
   e) $[1 \quad 4] + 3[2 \quad -1]$
   f) $5[1 \quad 1] + 2[-2 \quad 1]$



2. Calculate the norm of the following vectors:
   a) $[3 \quad -1]$
   b) $[4 \quad 3 \quad 2]$
   c) $[1 \quad 2 \quad -1 \quad 3 \quad 1]$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

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Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

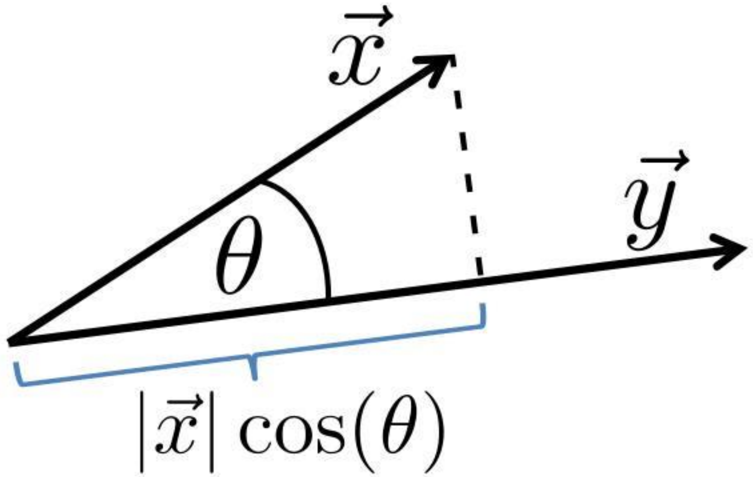
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

Condition: same inner dimension

$$\underbrace{1 \times \overbrace{N \qquad N}^{} \times 1}_{} \qquad\qquad \underbrace{1 \times 1}_{}$$

Result: outer dimensions combined

Dot product geometric intuition: "Overlap" of 2 vectors

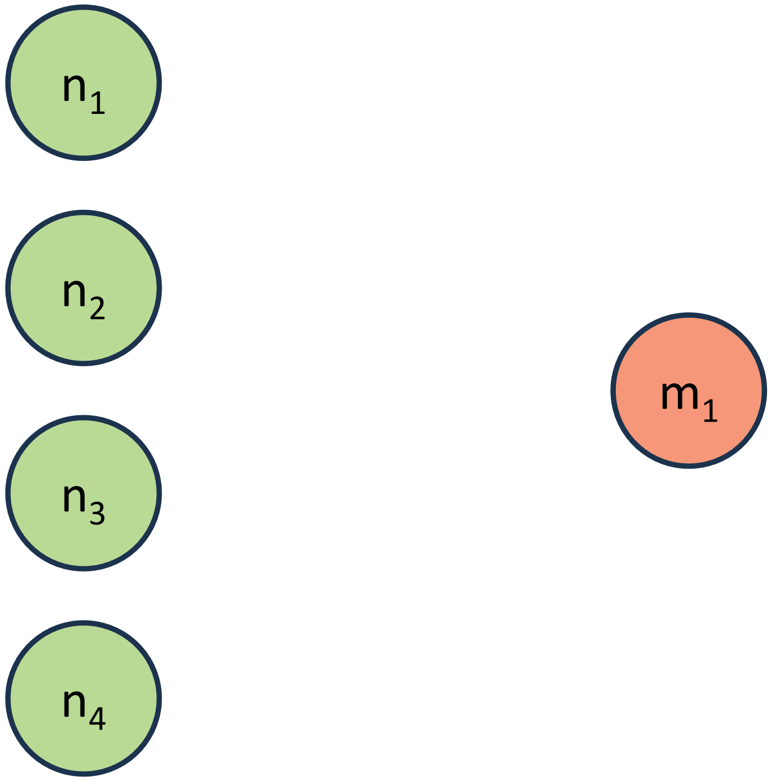


$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos(\theta)$$

Multiplication:

Dot product (inner product) Example 1

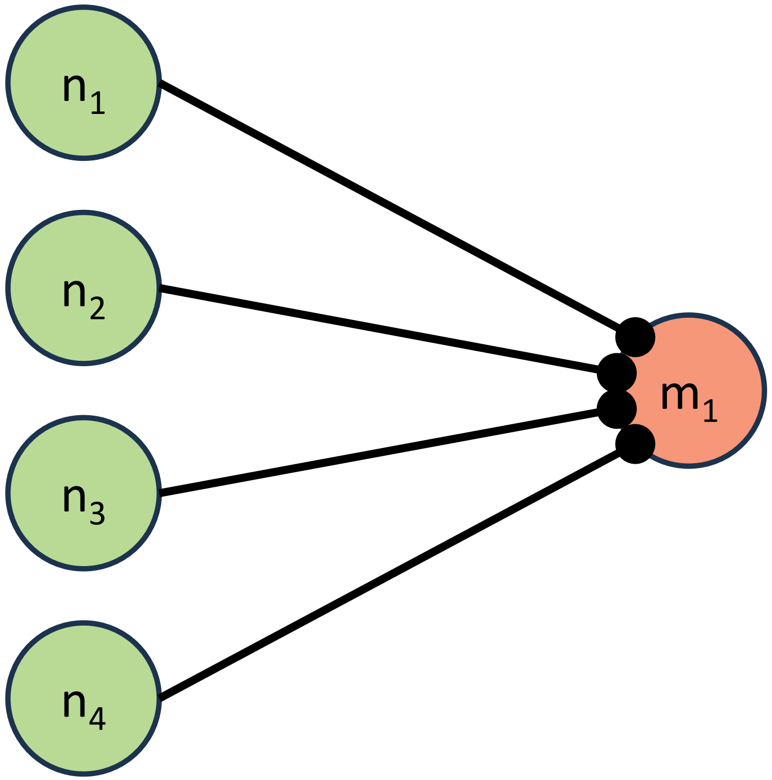
weighted average



Multiplication:

Dot product (inner product) Example 1

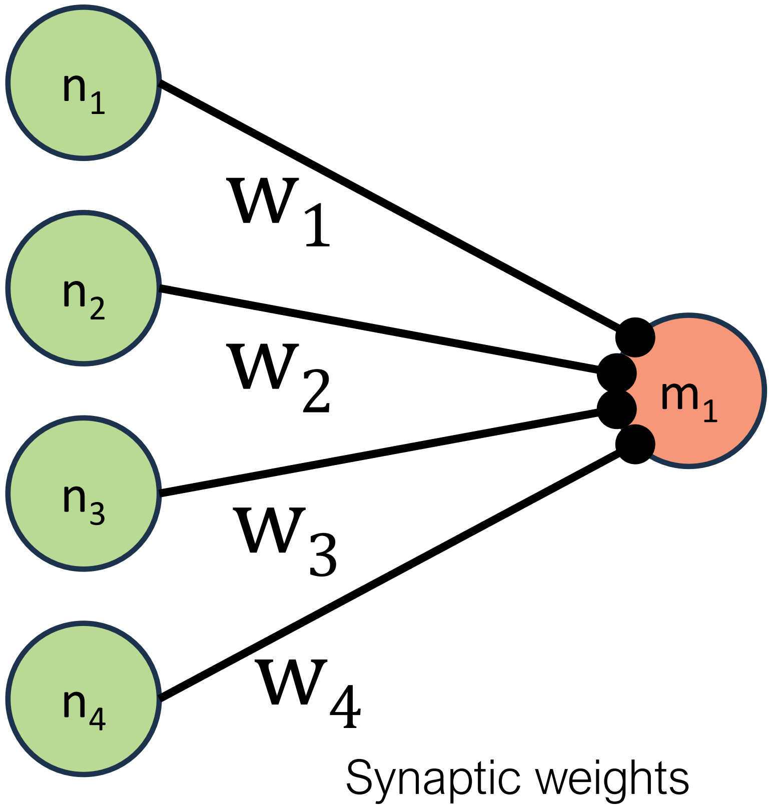
weighted average



Multiplication:

Dot product (inner product) Example 1

weighted average

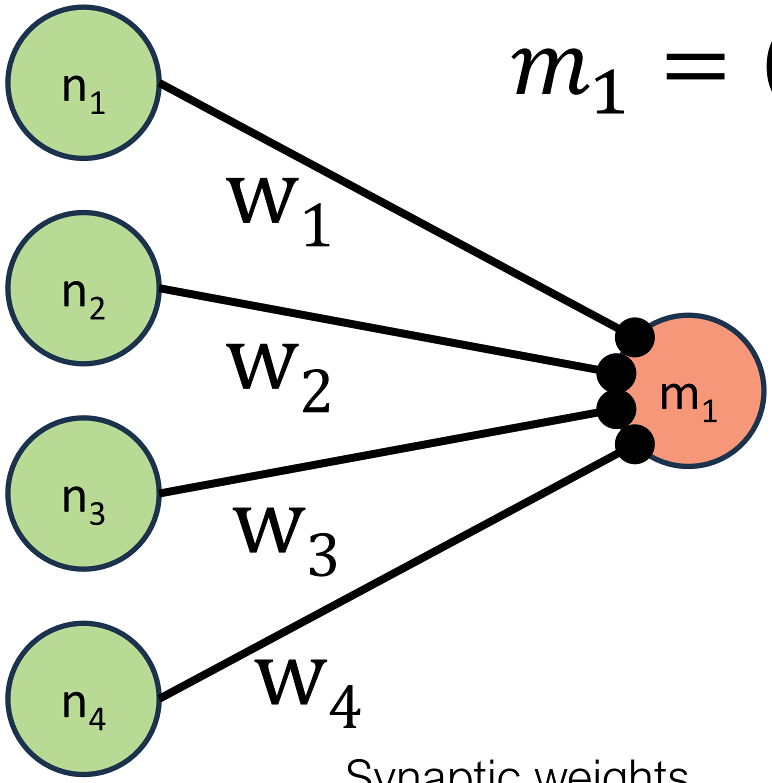


Multiplication:

Dot product (inner product) Example 1

weighted average

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$

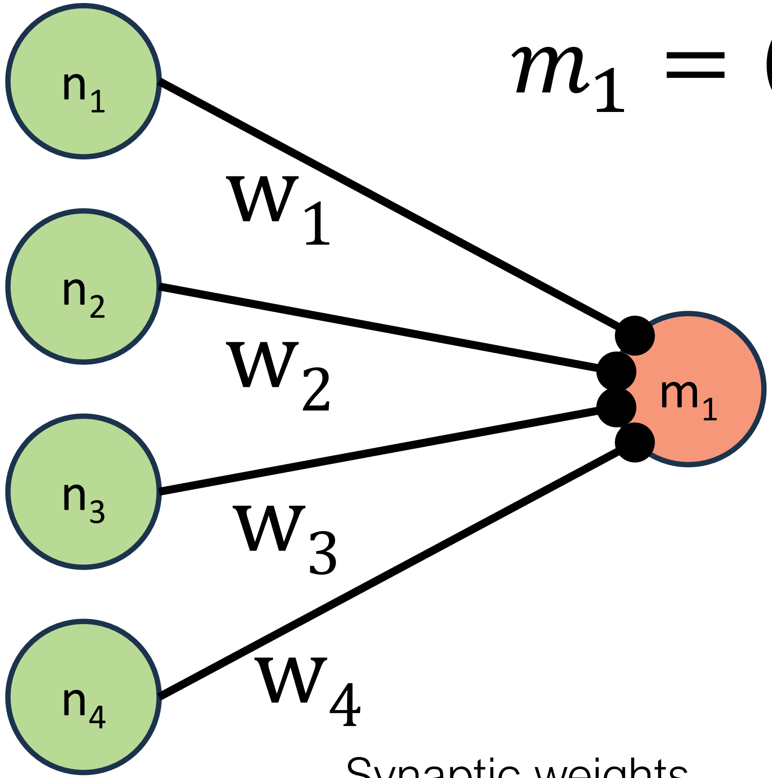


Synaptic weights

Multiplication:

Dot product (inner product) Example 1

weighted average



Synaptic weights

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$

$$= w_1 n_1 + w_2 n_2 + w_3 n_3 + w_4 n_4$$

$$= w^T n$$

Exercise

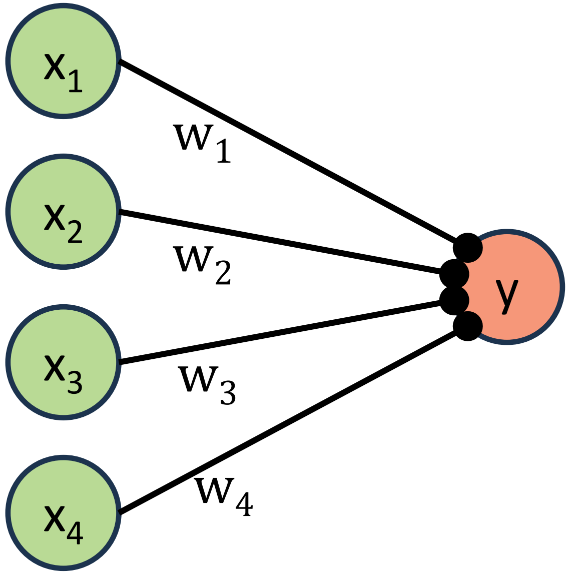
3. Calculate the product of these two vectors:

$$\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}$$

From vectors to matrices

Let's imagine a linear neuron receiving input from n presynaptic neurons which firing rates $(x_j)_{j=1:n}$ are scaled by synaptic weights $(w_j)_{j=1:n}$ and together determine the firing rate y of the output neuron:

$$y = w_1 x_1 + w_2 x_2 + \cdots w_n x_n = \sum w_j x_j$$

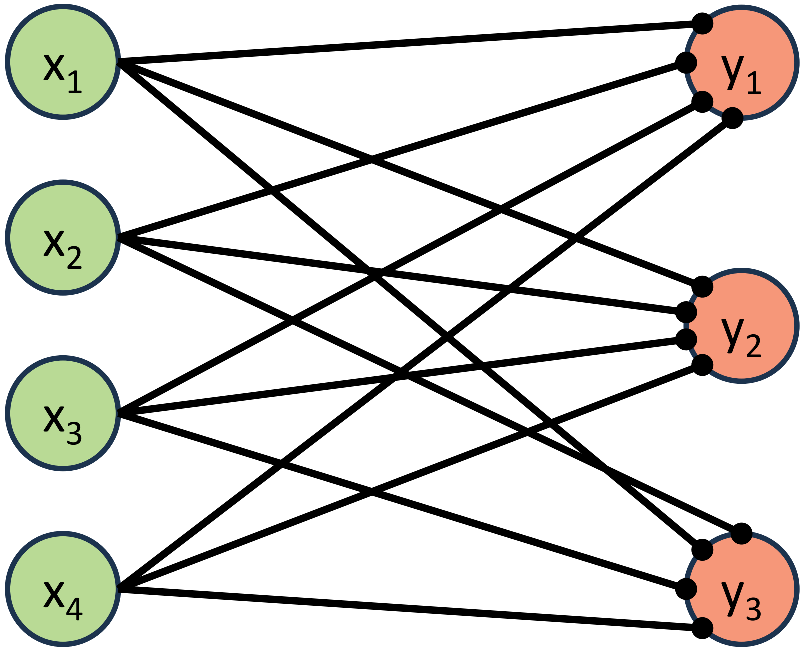


The output can be written as dot product of the weight vector and input vector:

$$y = \boldsymbol{W} \cdot \boldsymbol{X}$$

From vectors to matrices

Now let's suppose we have *m* output neurons with activities $(y_i)_{i=1:m}$, still given by a linear combination of the firing rates of their inputs neurons $(x_{ij})_{i=1:m,\, j=1:n}$ scaled by synaptic weights $(w_{ij})_{i=1:m,\, j=1:n}$



$$y_i = w_{i1}x_{i1} + w_{i2}x_{i2} + \cdots w_{in}x_{in} = \sum w_{ij}x_{ij}$$

From vectors to matrices

Now let's suppose we have m output neurons with activities $(y_i)_{i=1:m}$, still given by a linear combination of the firing rates of their inputs neurons $(x_{ij})_{i=1:m,\ j=1:n}$ scaled by synaptic weights $(w_{ij})_{i=1:m,\ j=1:n}$

n input neurons x m output neurons

We can summarize the weights w_{ij} into a matrix W with $m \times n$ values.

$$W = (w_{ij}) = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix}$$

From vectors to matrices

We already know that for each output neuron y_i :

$$y_i = \sum w_{ij} x_{ij}$$

We can generalize this for all the output neurons in the vector y with a more compact equation:

$$y = Wx$$

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C/ Linear systems, matrices and determinant

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Size of a Matrix

- The size of a matrix is always represented as **(# of rows) × (# of columns)**
- Example
  - $A = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1 & 5 \\ 0 & -2 & -1 \end{bmatrix}$ is a **3 × 3 matrix**
  - $B = \begin{bmatrix} 2 & \sqrt{2} \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$ is a **3 × 2 matrix**
- $A_{i,j}$ is the component of A in the *i*-th row and *j*-th column.
  - In the above example, $A_{3,2}$ = -2

Matrix representation of vectors

- Vectors can also be considered as matrices.
- Row vectors are **1 × *n* matrices**, where *n* is the number of components.
- Column vectors are ***n* × 1 matrice**s, where *n* is the number of component.
- Examples:
  - Column vector $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ is a 3 × 1 matrix.
  - Row vector $\begin{bmatrix} 2 & 1 \end{bmatrix}$ is a 1 × 2 matrix.

Basic matrix operations

Multiplication by a scalar

$$-2\begin{bmatrix} 2 & 3 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 4 & -2 \\ 0 & 2 \end{bmatrix}$$

Transpose of a matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ has transpose } A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Addition of two matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Subtraction of two matrices

$$A - B = A + (-1)B$$

Matrix addition is commutative and associative

- A+B = B+A
- (A+B)+C = A+(B+C)

Vector and Matrix operations

- A vector can be multiplied with a matrix in two possible ways: Matrix-Vector multiplication or Vector-Matrix multiplication

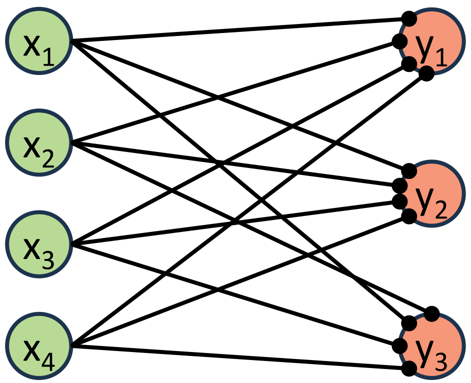
- **Matrix-Vector multiplication:** $Ab = c$
  - Matrix $A_{m \times n}$ can be multiplied with column vector $b_{n \times 1}$ to get a column vector $c_{m \times 1}$.

- **Vector-Matrix multiplication:** $aB = c$
  - Row vector $a_{1 \times n}$ can be multiplied with matrix $B_{n \times m}$ to get a row vector $c_{1 \times m}$.

Matrix-Vector multiplication

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

M × N　　　N × 1　　　M × 1



Matrix-Vector multiplication

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

M × N　　N × 1　　M × 1

Condition: same inner dimension　　Result: outer dimensions combined

Matrix-Vector multiplication

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

Rule: the i^{th} element of **y** is the dot product of the i^{th} row of **W** with **x**

Matrix-Vector multiplication

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

$$y_1 = w_{11}x_1 + w_{12}x_2 + \cdots + w_{1n}x_n$$

Matrix-Vector multiplication

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

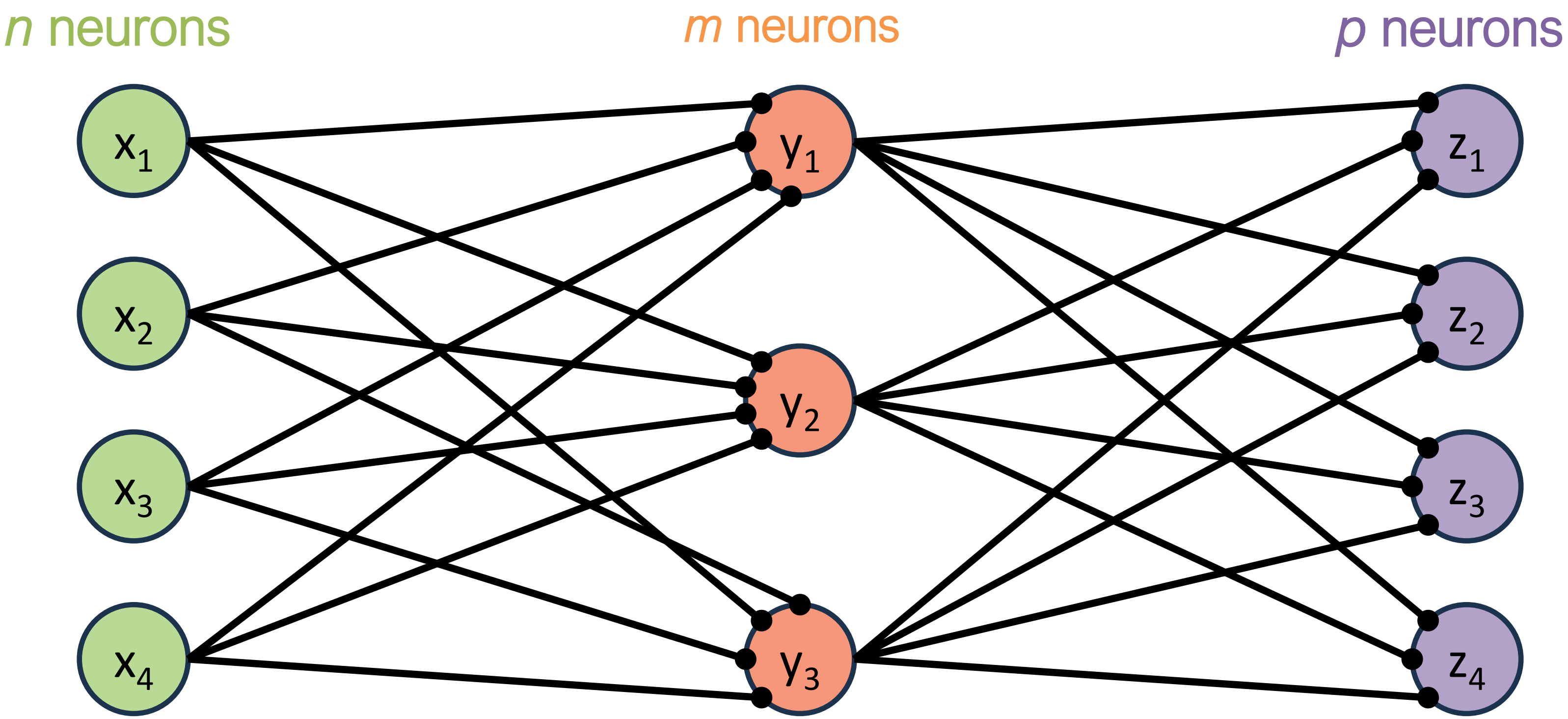
$$y_2 = w_{21}x_1 + w_{22}x_2 + \cdots + w_{2n}x_n$$

Matrix-Vector multiplication

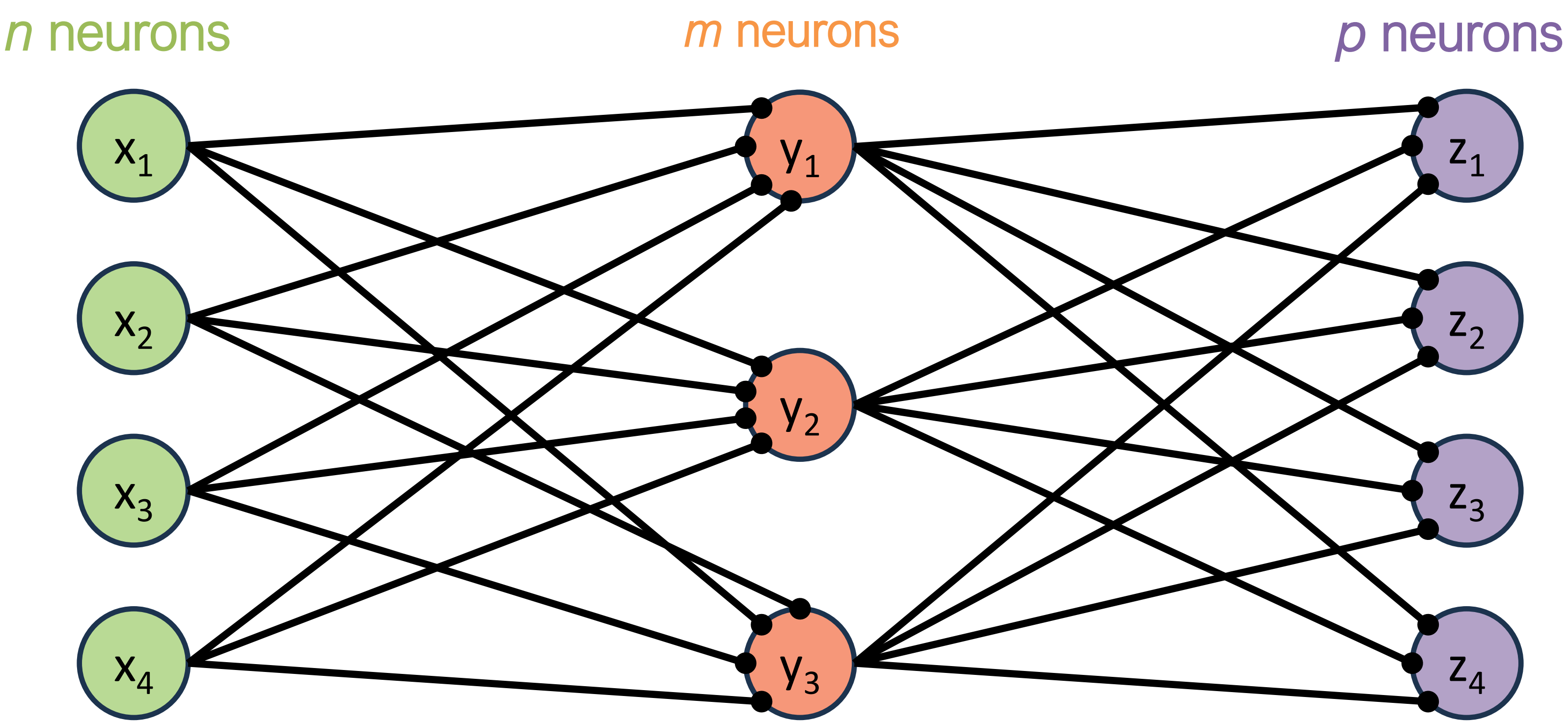
$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

$$y_m = w_{m1}x_1 + w_{m2}x_2 + \cdots + w_{mn}x_n$$

Product of two matrices

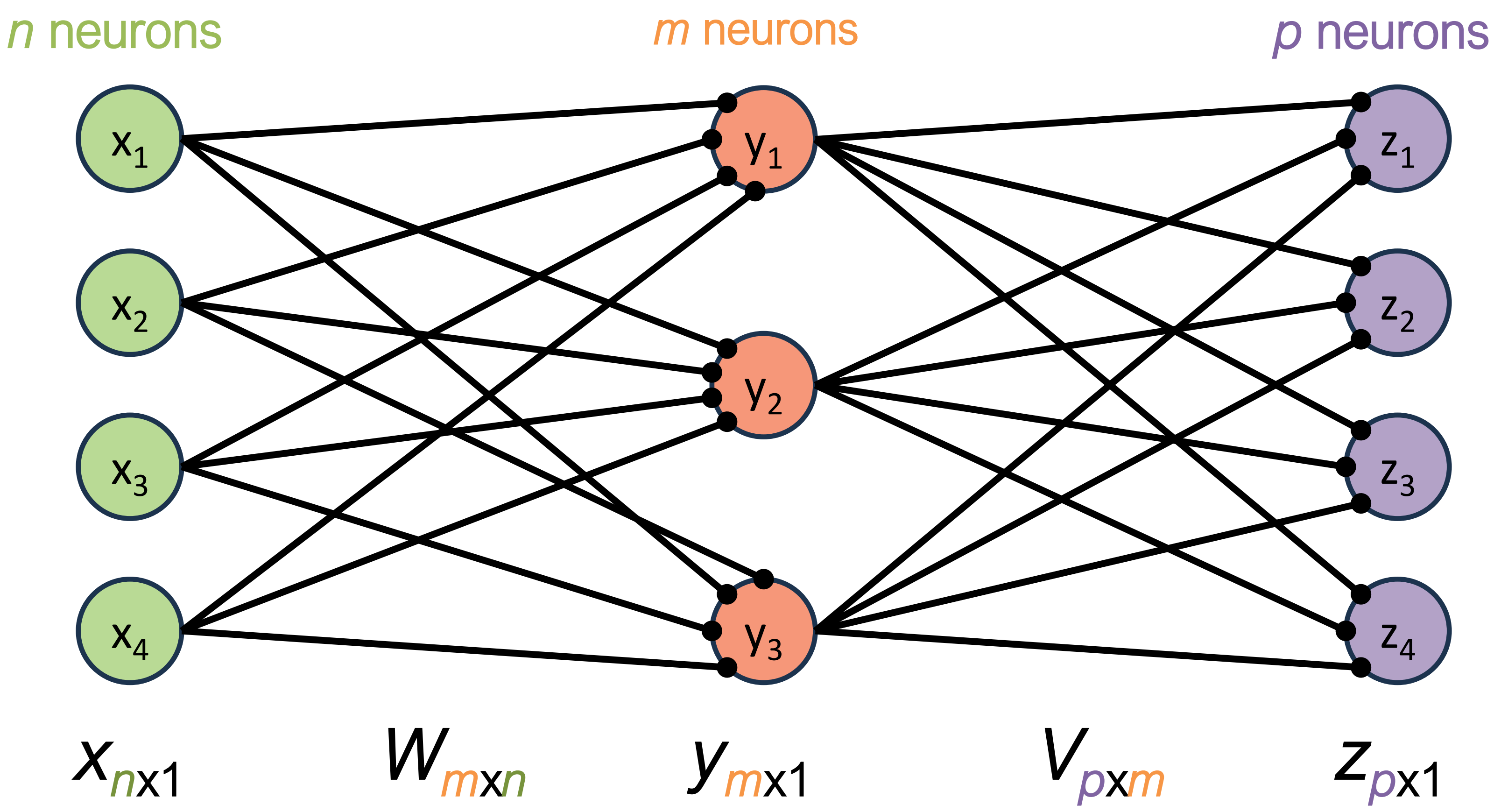


Product of two matrices



$x_{n\text{x}1}$ $W_{m\text{x}n}$ $y_{m\text{x}1}$ $V_{p\text{x}m}$ $z_{p\text{x}1}$

Product of two matrices



$$y = Wx \qquad z = Vy = V(Wx) = (VW)x$$

Product of two matrices

$$z = (VW)x \qquad \text{let's calculate } VW = U$$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = U \longrightarrow ? \times ?$$

P × M M × N

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

P × M M × N P × N

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{11} = v_{11}w_{11} + v_{12}w_{21} + \cdots + v_{1m}w_{m1}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{12} = v_{11}w_{12} + v_{12}w_{22} + \cdots + v_{1m}w_{m2}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{1n} = v_{11}w_{1n} + v_{12}w_{2n} + \cdots + v_{1m}w_{mn}$$

Product of two matrices

$$z = (VW)x \qquad \text{let's calculate } VW = U$$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{21} = v_{21}w_{11} + v_{22}w_{21} + \cdots + v_{2m}w_{m1}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{22} = v_{21}w_{12} + v_{22}w_{22} + \dots + v_{2m}w_{m2}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{2n} = v_{21}w_{1n} + v_{22}w_{2n} + \cdots + v_{2m}w_{mn}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{p1} = v_{p1}w_{11} + v_{p2}w_{21} + \cdots + v_{pm}w_{m1}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{p2} = v_{p1}w_{12} + v_{p2}w_{22} + \cdots + v_{pm}w_{m2}$$

Product of two matrices

$z = (VW)x$ let's calculate $VW = U$

$$\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & \dots & v_{2m} \\ \dots & \dots & \dots & \dots \\ v_{p1} & v_{p2} & \dots & v_{pm} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ u_{p1} & u_{p2} & \dots & u_{pn} \end{pmatrix}$$

$$u_{pn} = v_{p1}w_{1n} + v_{p2}w_{2n} + \cdots + v_{pm}w_{mn}$$

Special matrices: **diagonal matrix**

$$\overleftrightarrow{D} = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

$$\overleftrightarrow{D}\vec{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \dots \\ d_n x_n \end{pmatrix}$$

This acts like scalar multiplication

Exercises

4. Represent the following vectors of $\mathbb{R}^2$ on a plane with two orthogonal axes:

a) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

d) $\left(\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

5. Find all possible products among the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 5 \\ 5 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 1 & 3 \\ -1 & -4 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

Special matrices: **identity matrix**

$$\overleftrightarrow{1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\text{for all } \overleftrightarrow{A},\ \overleftrightarrow{1}\,\overleftrightarrow{A} = \overleftrightarrow{A}\,\overleftrightarrow{1} = \overleftrightarrow{A}$$

Special matrices: **inverse matrix**

$$\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{-1}\overleftrightarrow{A} = \overleftrightarrow{1}$$

Does the inverse always exist?

C/ Linear systems, matrices and determinants

Linear system

$$w_{11}x_1 + w_{12}x_2 + \cdots + w_{1n}x_n = y_1$$
$$w_{21}x_1 + w_{22}x_2 + \cdots + w_{2n}x_n = y_2$$
$$\ldots$$
$$w_{m1}x_1 + w_{m2}x_2 + \cdots + w_{mn}x_n = y_m$$

$$2x + 3y + z = 6$$
$$x - y + z = 1$$
$$x + y + z = 3$$

Matrix formulation

Given $\quad W: x \longrightarrow y$

Find $\quad V: y \longrightarrow x \quad$ such that $\quad x \xrightarrow{W} y \xrightarrow{V} x$

i.e. VWx=x, or VW=I

$$W^{-1}(W\mathbf{x}) = \mathbf{x}$$

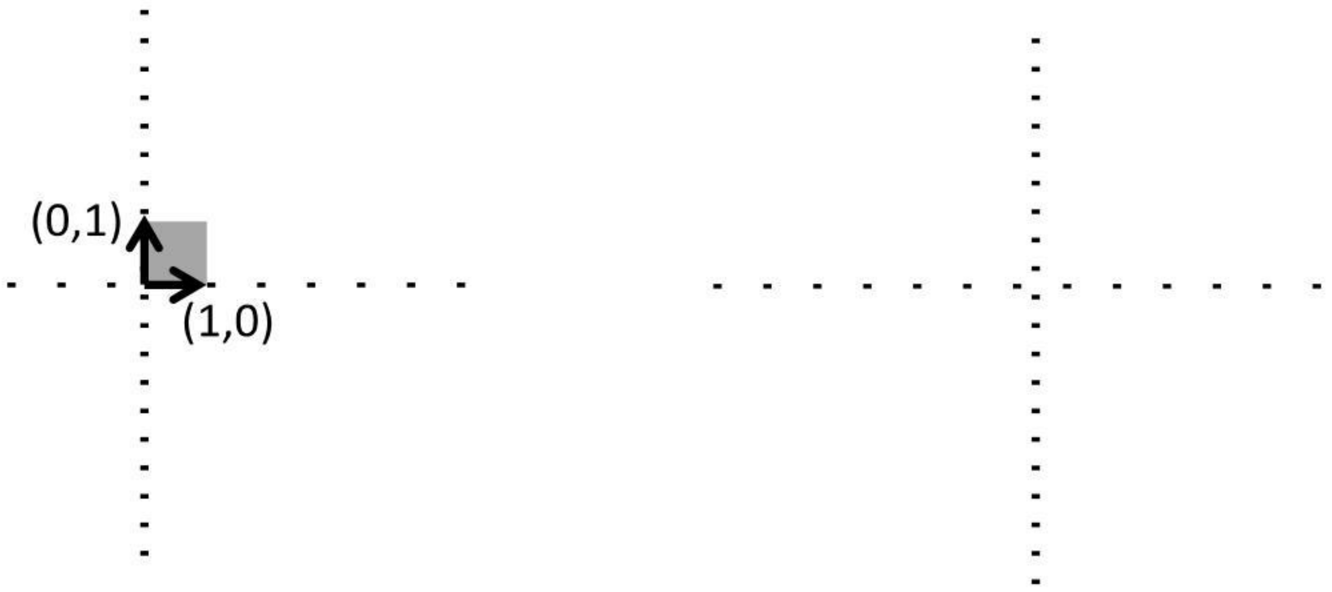
$$W^{-1}W = WW^{-1} = I$$

Some square matrices have no inverse and a non-invertible matrix is said to be *singular* (this is like the exception $k = 0$ for the scalar case). If an inverse does not exist it is for a very good reason. It happens when there is a non-zero vector $\mathbf{x}$ such
are two different vectors, 0 and $\mathbf{x}$, which map to 0 under W, hence W cannot have an inverse (in general a function must be *one-to-one* to have an inverse).

Another test for singularity of a matrix W is to calculate its *determinant*, a scalar quantity denoted by $\det(W)$ (sometimes $|W|$) which we will discuss later. A matrix is singular if

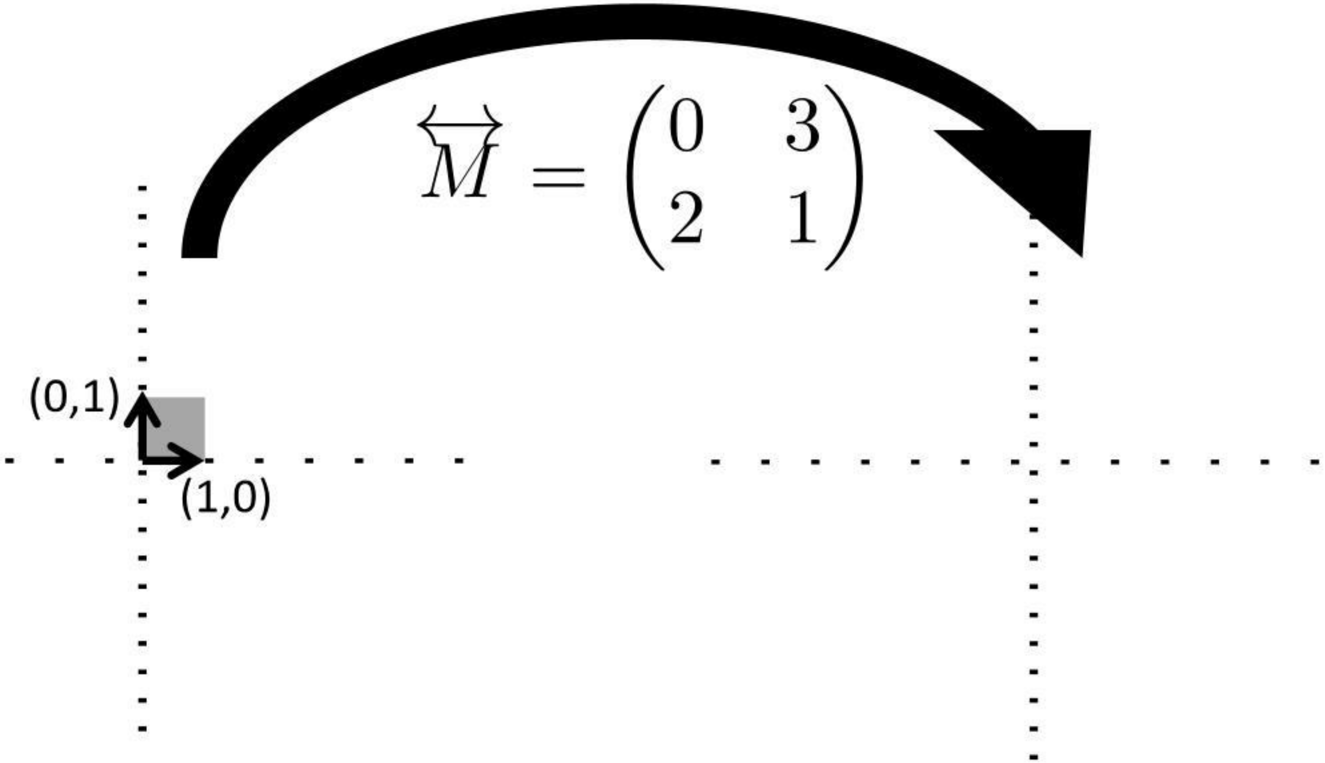
$$\det(W) = 0 \tag{76}$$

How does a matrix transform a square?



How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



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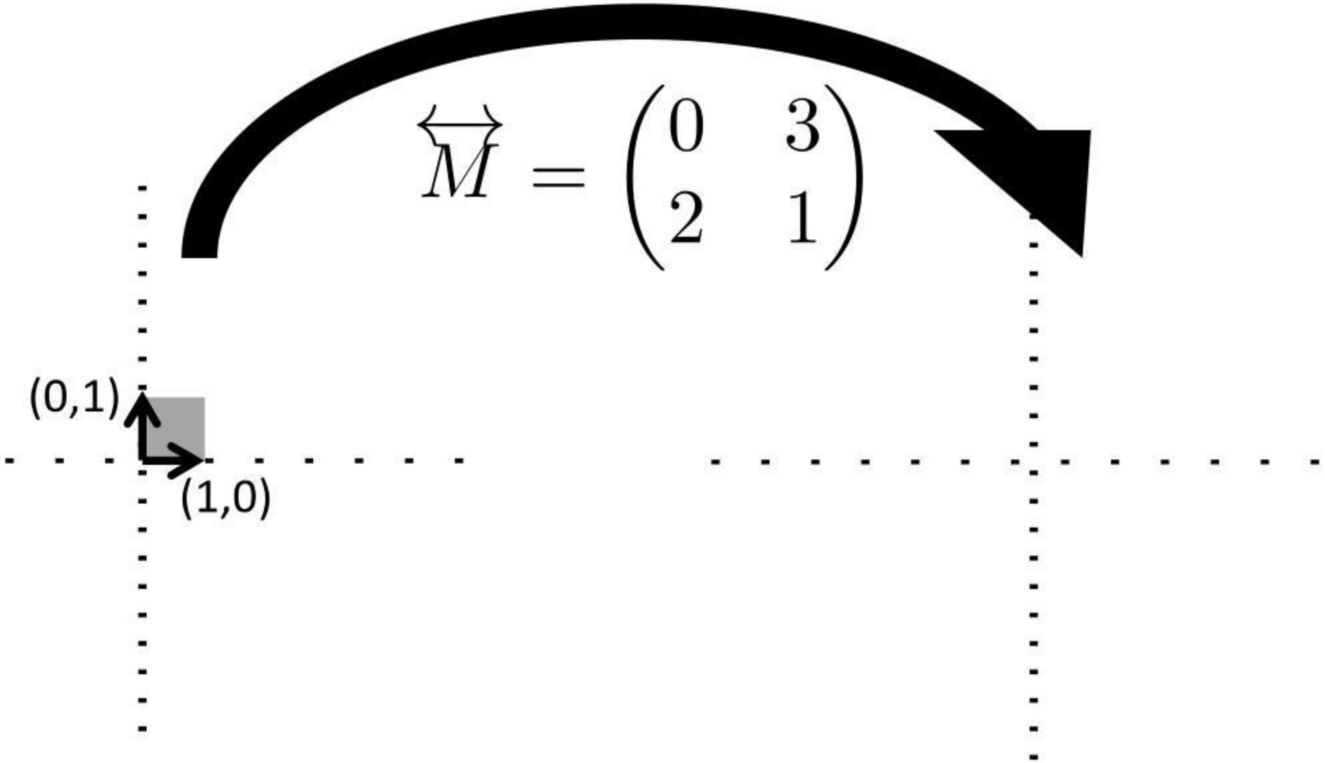
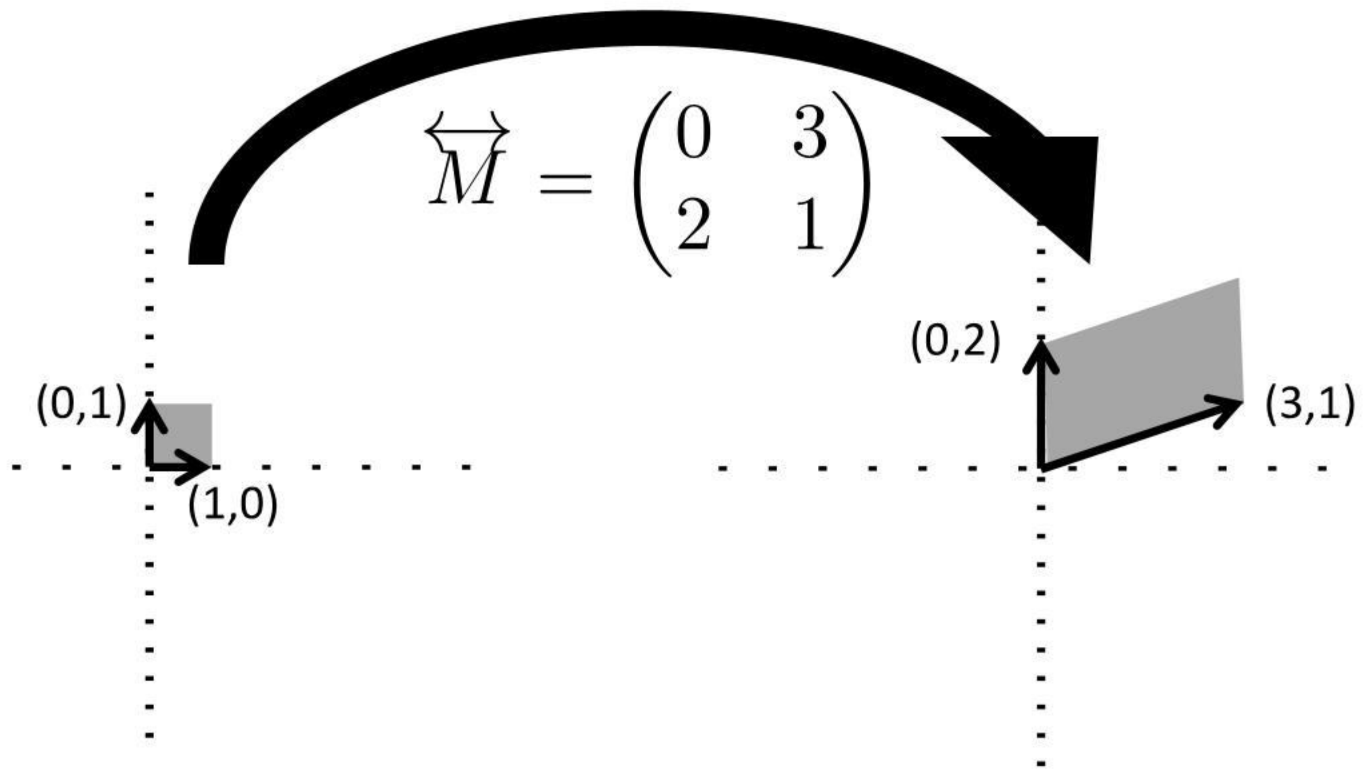


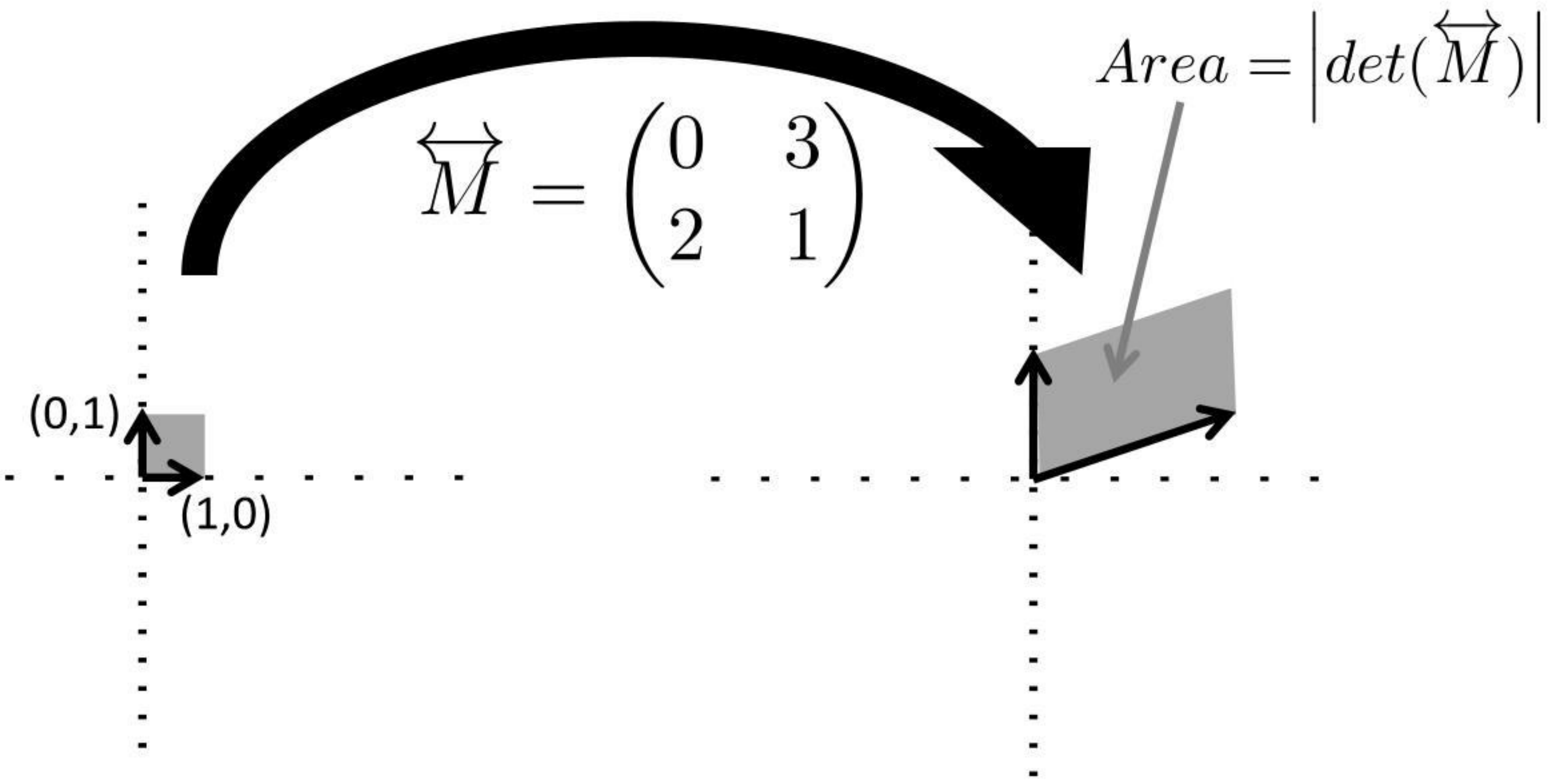
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

$$Mx?$$

How does a matrix transform a square?

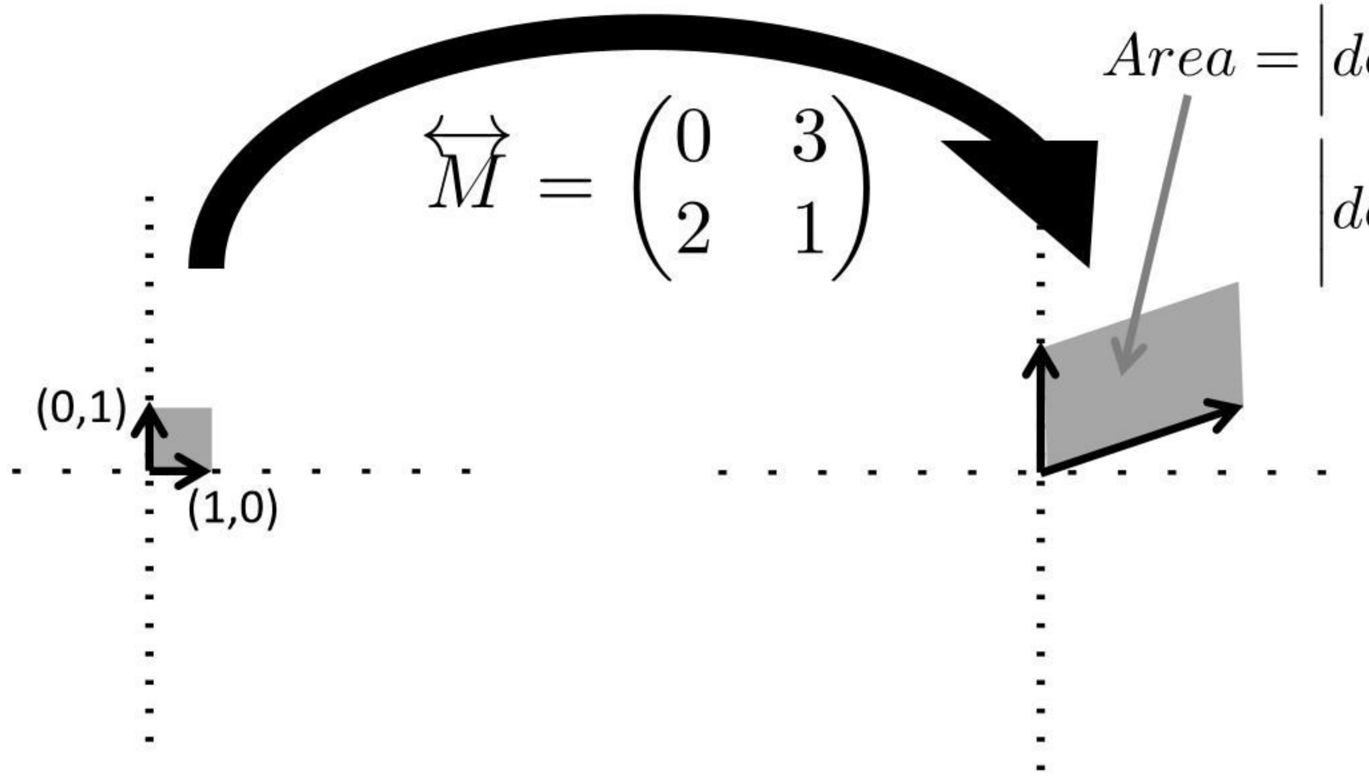


Geometric definition of the determinant: How does a matrix transform a square?



MATLAB: `det(M)`

Geometric definition of the determinant: How does a matrix transform a square?



$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

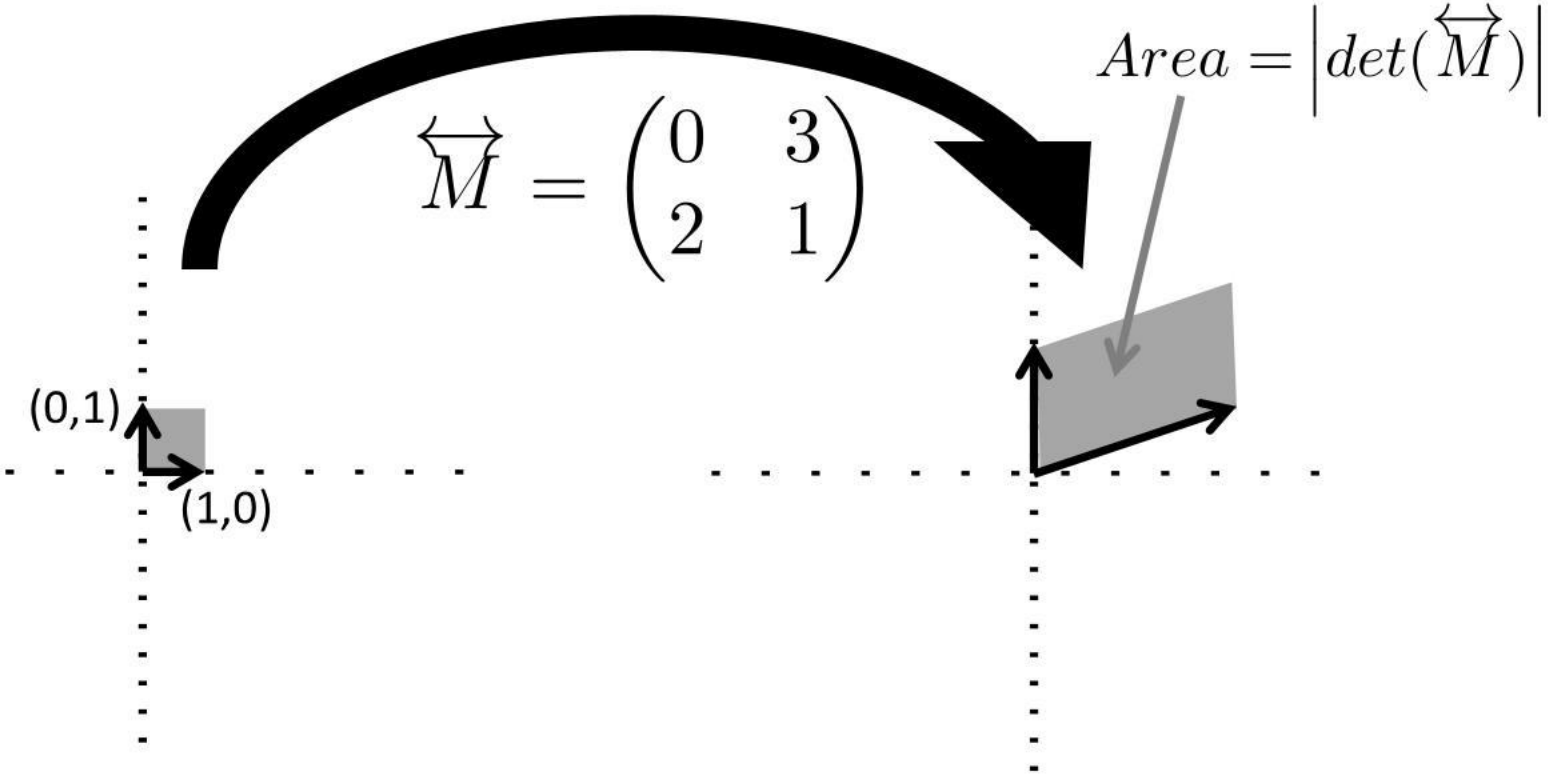
$$Area = \left| det(\overleftrightarrow{M}) \right|$$

$$\left| det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$$

MATLAB: `det(M)`

Geometric definition of the determinant: How does a matrix transform a square?

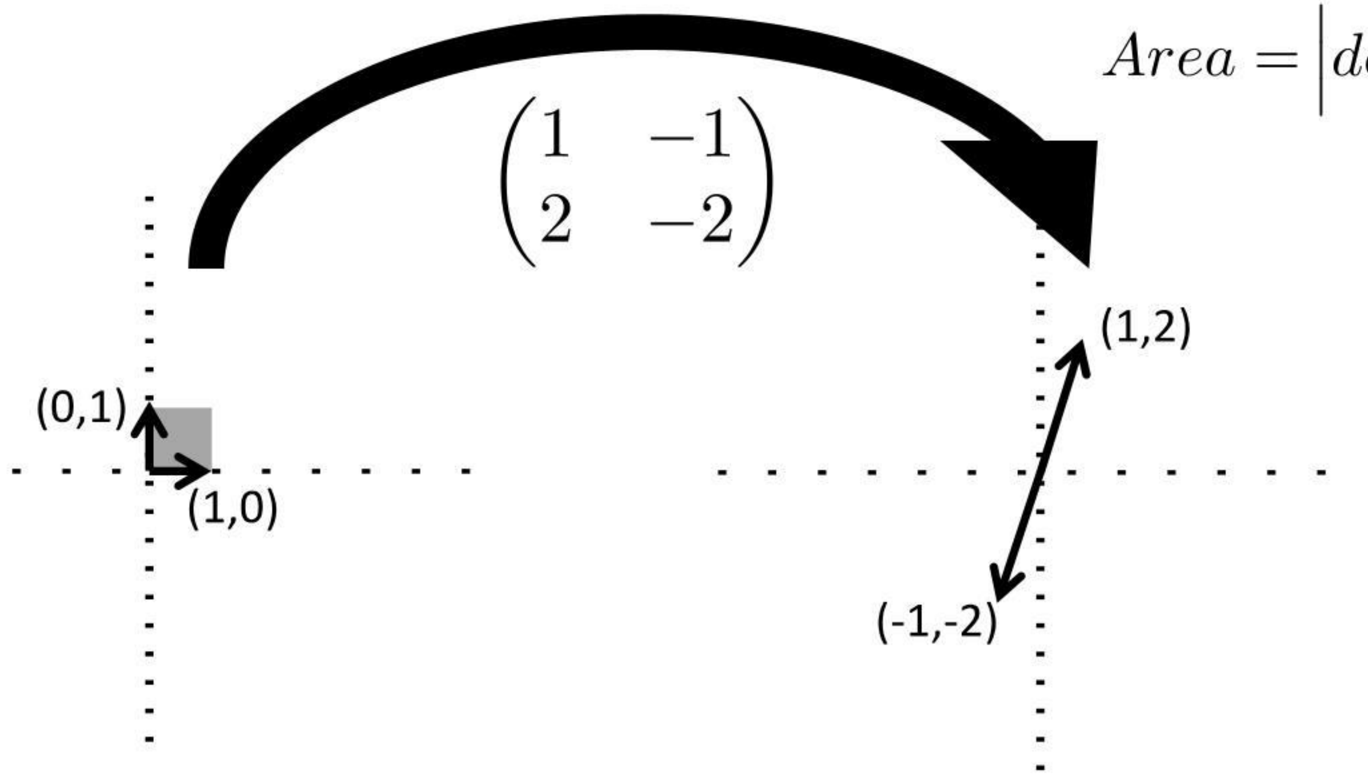
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Geometric definition of the determinant: How does a matrix transform a square?

$$Area = \left| det(\overleftrightarrow{M}) \right|$$



$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

no area!

this means the determinant is zero

what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?