

Exercises for Section 1: Linear Algebra

1. Represent the following vectors of \mathbb{R}^2 on a plane with two orthogonal axes (canonical base). Vectors are written in *matrix* notation.

(a) $\begin{bmatrix} 0 & 2 \end{bmatrix}$

(f) $5 \begin{bmatrix} 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} -2 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(j) $\left(\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(c) $2 \begin{bmatrix} 1 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & -2 \end{bmatrix}$

(h) $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 4 \end{bmatrix} + 3 \begin{bmatrix} 2 & -1 \end{bmatrix}$

2. Find the scalar k for which the vector $k \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \end{bmatrix}$ is a unit vector of \mathbb{R}^5 .
3. Provide M and P to write the following equation system in the form $M\vec{v} = P$, where vector $\vec{v} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ belongs to \mathbb{R}^3 . Solve the following two equation systems.

$$\begin{array}{ll} x + y + 2z = 3 & x + 2z = 1 \\ x + 2y + z = 1 & -y + z = 2 \\ 2x + y + z = 0 & x - 2y = 1 \end{array}$$

4. Solve the following equation system for any scalar m and provide a geometric interpretation of the result.

$$\begin{array}{l} x + my = -3 \\ mx + 4y = 6 \end{array}$$

5. First write out what is the dimensionality of the products shown here, then calculate the products.

$$x^T y = [1 \ 1] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad Wx = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad XX^T = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$$

6. Find all possible products among the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 5 \\ 5 & 0 \end{bmatrix} \quad E = \begin{bmatrix} -1 & 1 & 3 \\ -1 & -4 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

7. Write the following linear function u from \mathbb{R}^3 to \mathbb{R}^4 as a matrix U in their canonical bases. Determine if $\vec{u}_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\vec{u}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\vec{u}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ are orthogonal.

$$u(x, y, z) = \begin{bmatrix} -x + y, & x - y, & -x + z, & -y + z \end{bmatrix}$$

8. Calculate, when they are defined, the products AB and BA for all the following cases:

$$\begin{aligned} \text{(a)} \quad A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \text{(b)} \quad A &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ \text{(c)} \quad A &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

9. Calculate AB and AC . May A be invertible? Find all 3×3 matrices M such that $AM = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

10. Given a and b in \mathbb{R}^* (i.e. real but non-zero), find all matrices B in \mathbb{R}^5 that commute with A , i.e. such that $AB=BA$, for $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

11. Find all 2×2 matrices A and B such that $AB = 0$ and $BA \neq 0$.

12. Calculate A^n and B^n for all n (start with $2, 3, 4, \dots$) for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

13. Show that $A^2 = 2I - A$. Rely on this result to show that A is invertible and find the inverse for $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

14. Show that $A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$ is invertible and calculate its inverse.

15. Let $(a_n), (b_n), (c_n)$ be three real sequences so that $a_0 = 1, b_0 = 2, c_0 = 7$ and verifying the recurrent relationships:

$$\begin{cases} a_{n+1} = 3a_n + b_n \\ b_{n+1} = 3b_n + c_n \\ c_{n+1} = 3c_n \end{cases} \quad (1)$$

We want to express a_n, b_n, c_n only as a function of n .

(a) We consider the column vector $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$. Find a matrix A such that $X_{n+1} = AX_n$. Prove

that $X_n = A^n X_0$

(b) Given $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Calculate N^2, N^3 , then N^p for $p \geq 3$.

(c) Show that:

$$A^n = 3^n I + 3^{n-1} n N + 3^{n-2} \frac{n(n-1)}{2} N^2$$

(d) Write a_n, b_n , and c_n as functions of n .

Hint. If you are already familiar with Python or Matlab, you can evaluate the results of most of the exercises using code.