CNCC 20201 Exercise set #1

Exercises for Section 1: Linear Algebra

1. Represent the following vectors of $\mathbb{R}^2$on a plane with two orthogonal axes (canonical base). Vectors are written in *matrix* notation.

   (a) $\begin{bmatrix}0 & 2\end{bmatrix}$
   (b) $\begin{bmatrix}3 & -1\end{bmatrix}$
   (c) $2\begin{bmatrix}1 & -3\end{bmatrix}$
   (d) $\begin{bmatrix}2 & 4\end{bmatrix} - \begin{bmatrix}0 & -2\end{bmatrix}$
   (e) $\begin{bmatrix}1 & 4\end{bmatrix} + 3\begin{bmatrix}2 & -1\end{bmatrix}$
   (f) $5\begin{bmatrix}1 & 1\end{bmatrix} + 2\begin{bmatrix}-2 & 1\end{bmatrix}$
   (g) $\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$
   (h) $\begin{bmatrix}-2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$
   (i) $\begin{bmatrix}0 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$
   (j) $\left(\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix} + \begin{bmatrix}-2 & 1\\1 & 2\end{bmatrix}\right)\begin{bmatrix}1\\3\end{bmatrix}$

2. Find the scalar k for which the vector $k\begin{bmatrix}1 & 2 & 1 & 3 & 1\end{bmatrix}$ is a unit vector of $\mathbb{R}^5$.

3. Provide M and P to write the following equation system in the form $M\vec{v} = P$, where vector $\vec{v} = \begin{bmatrix}x & y & z\end{bmatrix}$ belongs to $\mathbb{R}^3$. Solve the following two equation systems.

$$\begin{aligned} x + y + 2z &= 3 \\ x + 2y + z &= 1 \\ 2x + y + z &= 0 \end{aligned} \qquad \begin{aligned} x + 2z &= 1 \\ -y + z &= 2 \\ x - 2y &= 1 \end{aligned}$$

4. Solve the following equation system for any scalar m and provide a geometric interpretation of the result.

$$\begin{aligned} x + my &= -3 \\ mx + 4y &= 6 \end{aligned}$$

5. First write out what is the dimensionality of the products shown here, then calculate the products.

$$x^T y = \begin{bmatrix}1 - 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} \quad Wx = \begin{bmatrix}w_{11} & w_{12}\\w_{21} & w_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \quad XX^T = \begin{bmatrix}x_{11} & x_{12} & x_{13}\\x_{21} & x_{22} & x_{23}\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\x_{21} & x_{22}\\x_{31} & x_{32}\end{bmatrix}$$

6. Find all possible products among the following matrices:

$$A = \begin{bmatrix}1 & 2 & 3\end{bmatrix}, \qquad B = \begin{bmatrix}1\\-2\end{bmatrix}, \qquad C = \begin{bmatrix}2 & 1\\-3 & 0\\1 & 2\end{bmatrix},$$

$$D = \begin{bmatrix}-2 & 5\\5 & 0\end{bmatrix} \qquad E = \begin{bmatrix}-1 & 1 & 3\\-1 & -4 & 0\\0 & 2 & 5\end{bmatrix}$$

7. Write the following linear function u from $\mathbb{R}^3$ to $\mathbb{R}^4$ as a matrix U in their canonical bases. Determine if $\vec{u_x} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \vec{u_y} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \vec{u_z} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$ are orthogonal.

$$u(x, y, z) = \begin{bmatrix}-x + y, & x - y, & -x + z, & -y + z\end{bmatrix}$$

8. Calculate, when they are defined, the products AB and BA for all the following cases:

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

9. Calculate AB and AC. May A be invertible? Find all 3×3 matrices M such that $AM = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

10. Given a and b in $\mathbb{R}^*$ (i.e. real but non-zero), find all matrices B in $\mathbb{R}^5$ that commute with A, i.e. such that AB=BA, for $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

11. Find all 2×2 matrices A and B such that $AB = 0$ and $BA \neq 0$.

12. Calculate A^n and B^n for all n (start with $2, 3, 4 \ldots$) for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

13. Show that $A^2 = 2I - A$. Rely on this result to show that A is invertible and find the inverse for $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

14. Show that $A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 1 \end{bmatrix}$ is invertible and calculate its inverse.

15. Let $(a_n), (b_n), (c_n)$ be three real sequences so that $a_0 = 1$, $b_0 = 2$, $c_0 = 7$ and verifying the recurrent relationships:

$$\begin{cases} a_{n+1} = & 3a_n + & b_n & \\ b_{n+1} = & & 3b_n + & c_n \\ c_{n+1} = & & & 3c_n \end{cases} \tag{1}$$

We want to express a_n, b_n, c_n only as a function of n.

(a) We consider the column vector $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$. Find a matrix A such that $X_{n+1} = AX_n$. Prove that $X_n = A^n X_0$

(b) Given $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Calculate N^2, N^3, then N^p for $p \geq 3$.

(c) Show that:

$$A^n = 3^n I + 3^{n-1} n N + 3^{n-2} \frac{n(n-1}{2} N^2$$

(d) Write a_n, b_n, and c_n as functions of n.

Hint. *If you are already familiar with Python or Matlab, you can evaluate the results of most of the exercises using code.*