

### Exercises for Section 1: Linear Algebra

1. Represent the following vectors of  $\mathbb{R}^2$  on a plane with two orthogonal axes (canonical base). Vectors are written in *matrix* notation.

(a)  $\begin{bmatrix} 0 & 2 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 & -1 \end{bmatrix}$

(c)  $2 \begin{bmatrix} 1 & -3 \end{bmatrix}$

(d)  $\begin{bmatrix} 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & -2 \end{bmatrix}$

(e)  $\begin{bmatrix} 1 & 4 \end{bmatrix} + 3 \begin{bmatrix} 2 & -1 \end{bmatrix}$

(f)  $5 \begin{bmatrix} 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} -2 & 1 \end{bmatrix}$

(g)  $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(h)  $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(i)  $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(j)  $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Find the scalar  $k$  for which the vector  $\begin{bmatrix} 1 & 2 & 1 & 3 & 1 \end{bmatrix}$  is a unit vector of  $\mathbb{R}^5$ .
3. Provide  $M$  and  $P$  to write the following equation system in the form  $MX = P$ , where vector  $X = \begin{bmatrix} x & y & z \end{bmatrix}$  belongs to  $\mathbb{R}^3$ . Solve the following two equation systems.

$$\begin{array}{rcl} x + y + 2z = 3 & & x + 2z = 1 \\ x + 2y + z = 1 & & -y + z = 2 \\ 2x + y + z = 0 & & x - 2y = 1 \end{array}$$

4. Solve the following equation system for any scalar  $m$  and provide a geometric interpretation of the result.

$$\begin{array}{l} x + my = -3 \\ mx + 4y = 6 \end{array}$$

5. First write out what is the dimensionality of the products shown here, then calculate the products.

$$x^T y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad Wx = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad XX^T = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$$

6. Find all possible products among the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 5 \\ 5 & 0 \end{bmatrix} \quad E = \begin{bmatrix} -1 & 1 & 3 \\ -1 & -4 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

7. Write the following linear function  $u$  from  $\mathbb{R}^3$  to  $\mathbb{R}^4$  as a matrix  $U$  in their canonical bases. Determine if  $\vec{u}_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ ,  $\vec{u}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ ,  $\vec{u}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$  are orthogonal.

$$u(x, y, z) = \begin{bmatrix} -x + y, & x - y, & -x + z, & -y + z \end{bmatrix}$$

8. Calculate, when they are defined, the products  $AB$  and  $BA$  for all the following cases:

$$\begin{aligned} \text{(a)} \quad A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \text{(b)} \quad A &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ \text{(c)} \quad A &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

9. Calculate  $AB$  and  $AC$ . May  $A$  be invertible? Find all  $3 \times 3$  matrices  $M$  such that  $AM = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

10. Given  $a$  and  $b$  in  $\mathbb{R}^*$  (i.e. real but non-zero), find all matrices  $B$  in  $\mathbb{R}^5$  that commute with  $A$ , i.e. such that  $AB=BA$ , for  $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ .

11. Find all  $2 \times 2$  matrices  $A$  and  $B$  such that  $AB = 0$  and  $BA \neq 0$ .

12. Calculate  $A^n$  and  $B^n$  for all  $n$  (start with  $2, 3, 4, \dots$ ) for  $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

13. Show that  $A^2 = 2I - A$ . Rely on this result to show that  $A$  is invertible and find the inverse for  $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ .

14. Show that  $A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$  is invertible and calculate its inverse.

15. Let  $(a_n), (b_n), (c_n)$  be three real sequences so that  $a_0 = 1, b_0 = 2, c_0 = 7$  and verifying the recurrent relationships:

$$a_{n+1} \dots$$

We want to express  $a_n, b_n, c_n$  only as a function of  $n$ .

(a) We consider the column vector  $X_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$ . Find a matrix  $A$  such that  $X_{n+1} = AX_n$ . Prove that  $X_n = A^n X_0$

(b) Given  $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  Calculate  $N^2, N^3$ , then  $N^p$  for  $p \geq 3$ .

(c) Show that:

$$A^n = 3^n I + 3^{n-1} n N + 3^{n-2} \frac{n(n-1)}{2} N^2$$

(d) Write  $a_n, b_n$ , and  $c_n$  as functions of  $n$ .

**Hint.** If you are already familiar with Python or Matlab, you can evaluate the results of most of the exercises using code.