Discrete Time Signals

1 Introduction

Welcome to the CNCC 2021 workshop! This document serves as an introduction to signal processing. After reading through this document you should have a decent understanding of how discrete-time signals are represented using mathematical notation and what are their properties.

2 Basic Signals in D.T.

A discrete-time (D.T.) signal is any signal in the form:

$$x[n] : n \in \mathbb{Z} \to \mathbb{R} \textit{ or } \mathbb{C}$$

2.1 The unit impulse function $\delta[n]$

This signal is denoted by $\delta[n]$ and it's formula is given below:

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

As you can see, this signal is zero everywhere except at the point $n = 0$.

2.2 The unit step function $u[n]$

An extension of the unit impulse function is the unit step function, given by the formula:

$$u[n] = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The signal takes the value 1 for every $n \geq 0$. It can also be represented as an infinite sum of delayed unit impulse signals, which will be the topic of a later exercise.

2.3 The ramp function $r[n]$

The ramp function is given by the formula:

$$u[n] = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Its value keeps increasing linearly with n, if $n \geq 0$.

2.4 Sinusoidal signals

All continuous-time sinusoidal signals are periodic. However, in discrete-time, a sinusoidal sequence may or may not be periodic. Periodicity depends on the value of the signal's angular frequency Ω. For a discrete time signal to be periodic, its angular frequency must be a rational multiple of 2π. The general formula for a discrete-time sinusoidal signal is given as:

$$x[n] = sin(\Omega n + \phi)$$

Where the discrete-time angular frequency Ω is equal to $2\pi f_0$ and f_0 is the signal's frequency in Hz.

3 Even and Odd Signals

3.1 Even Signals in D.T.

A signal is said to be *even* or *symmetric* if it satisfies the following condition:

$$x[-n] = x[n]$$

3.2 Odd Signals in D.T.

A signal is said to be *odd* if it satisfies the following condition:

$$x[-n] = -x[n]$$

3.3 Periodic Signals in D.T.

A signal is said to be periodic **iff** it satisfies the following condition:

$$x[n + N] = x[n]$$

The condition above specifies that the signal is periodic if **and only if** the signal repeats itself after a period of N samples. For example, let's consider the following signal:

$$
\begin{aligned}
x[n] &= A\,cos(\Omega n + \phi) \\
x[n+N] &= A\,cos(\Omega(n+N) + \phi) \\
&\Rightarrow A\,cos(\Omega n + \Omega N + \phi)
\end{aligned}
$$

For the signal to be periodic, we know that the periodicity condition needs to be satisfied. From basic trigonometry, it follows that:

$$
\begin{aligned}
\Omega N &= 2\pi K \Rightarrow \\
N &= \frac{2\pi K}{\Omega} \Rightarrow \\
N &= \frac{K}{f_0}
\end{aligned}
$$

Where $K \in \mathbb{Z}$. The above states that the frequencies (in Hz) of discrete sinusoidal signals are separated by integral multiple of 2π.

Example #1

Let's consider the case where $x[n] = cos(\Omega\, n)$ with a frequency of $\Omega = \frac{\pi}{6}$;

$$
\begin{aligned}
x[n] &= cos(\frac{\pi\, n}{6}) \\
N &= \frac{2\pi K}{\Omega} = 12\,K
\end{aligned}
$$

If we choose $K = 1$, we get $N_0 = 12$; this is the *fundamental period* of our signal and it means that the signal $x[n]$ repeats itself every $N_0 = 12$ samples, and therefore is periodic.

Example #2

Now let's consider another case: $\Omega = \frac{1}{6}$;

$$
\begin{aligned}
x[n] &= cos(\frac{n}{6}) \\
N &= \frac{2\pi K}{\Omega} = 12\,\pi\,K
\end{aligned}
$$

As we can see, $\mathbb{N} \notin \mathbb{Z}^+$ for any $k \in \mathbb{Z}$ and thus the signal $x[n]$ is **not** periodic.

4 Signal Properties

Here we present a list of the basic properties of D.T. signals.

Time Shifting	$\{y[n]\} = \{x[n-k]\}$
Reflection	$\{y[n]\} = \{x[-n]\}$
Scaling	$\{y[n]\} = \alpha\{x[n]\}$
Addition	$\{y[n]\} = \{x_1[n]\} + \{x_1[n]\}$
Subtraction	$\{y[n]\} = \{x_1[n]\} + \{x_1[n]\}$
Linear Combination	$\{y[n]\} = \alpha_1\{x_1[n]\} + \alpha_2\{x_1[n]\}$

The properties listed above will be used in the exercises below. If you would like to read more, you can refer to this website:

TutorialsPoint: Digital Signal Processing - Operations on Signals; Shifting

5 Exercises

5.1 Draw the three basic signals $\delta[n]$, $u[n]$, $r[t]$ in the range $n \in [-5, 5]$

5.2 Draw the signal $y[n] = u[n] - u[n-3]$

5.3 Draw the signal $y[n] = 3(u[n] - u[n-3])$

5.4 Draw the signal $y[n] = \frac{2}{3}\,r[n-3]$

5.5 Draw the signal $y[n] = -\,\delta[n] + \delta[n-1] - \frac{1}{2}\,u[n-3] + \frac{1}{2}u[n-6]$

5.6 Draw the following signals in the specified range and classify them as even or odd

$\delta[n]$, $\forall n \in \mathbb{Z}$	even	odd
$u[n]$, $\forall n \in \{-5, 5\}$	even	odd
$r[n]$, $\forall n \in \{-5, 5\}$	even	odd
$u[n+3] - u[n-3]$, $\forall n \in \{-5, 5\}$	even	odd
$sin(\frac{\pi\, n}{5})$, $\forall n \in \{-20, 20\}$	even	odd

hint: use a computer to validate your results; signals may be neither even or odd.

5.7 Show that the signal $A\,sin(2\pi fn + \phi)$ is periodic.

5.8 Is the signal $y[n] = cos(\frac{3\pi n}{12}) + cos(\frac{\pi n}{13})$ periodic?

5.9 Fit a curve to the discrete periodic function in Example 1.

Interpret the results when the sample size is lower (10 samples) vs higher (50 samples).
Do the results change when you increase/decrease the number of samples? Explain your answer.

5.10 Fit a cosine curve to the function in Example 2.

Interpret the results change when the sample size is lower (5, 10 samples) vs higher (50 samples).
What are the differences you notice between the previous exercise's results and these?
Is it what you were expecting? Explain your answer.