Approximate Answer to a Misunderstood Kinematics Problem

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Abstract

In 2023, a two-dimensional kinematics problem was presented as part of the Senior Section of the "Galileo" physics contest in Moldova. Interestingly, the students who proposed it incorrectly solved the problem. This article discusses the physical aspects of the problem, concluding that an exact mathematical solution cannot be found. Incorrect assumptions that led to a false solution are also analyzed. Using a computer simulation, an approximate answer is derived and interpreted.

INTRODUCTION

The "Galileo" national physics contest[1] is a competition in which pupils from 6th to 10th grade participate separately. It also contains the Senior category, dedicated to 11th and 12th graders. At the 11th edition of "Galileo", which took place on 16 December 2023, one of the authors proposed the problem "Wolf Around Lake"[2] in the Mechanics section of the Senior category. It was chosen to be the first problem for the Seniors.

The problem goes as follows: Let there be a circular lake, in the middle of which there is a man, and on the shore a wolf. The wolf can only move around the lake's circumference at constant speed u. The man, frightened by the wolf, will continuously swim in the opposite direction to the predator. What is the minimum swimming speed v of the man so that he can escape from the lake without being caught?

The university students, who were in charge of the problems, solved it, obtaining the answer that the minimum speed for the man in the lake is $v = u$.

The purposes of this article are:

- To solve the problem and discuss possible solutions;
- To check whether the students' solution is correct;
- To arrive at an exact or approximate answer;

- And to check the validity of the problem along with the decision to submit it to a contest.

Physical Aspect

As the problem states, the man moves at all times opposite to the wolf, which rotates on a circle of radius R. The best way to approach this problem is to define a $2D$ Cartesian coordinate system. We will attribute the coordinates x_1, y_1 for the wolf, and x_2, y_2 for the man.

The wolf will move on a circle with its center at $(0, 0)$, and $\theta = 0$ will be set on the y-axis. Because the man's initial position is in the center, we can consider that the wolf moves in only one direction along the circle, for now.

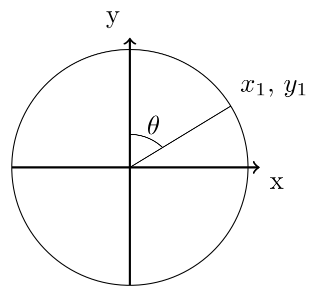


Figure 1: Position of the wolf

From the angular speed of the wolf, we can find that $\omega = \frac{d\theta}{dt} = \frac{u}{R}$. Because u and R are constants and independent of time, we can integrate both sides, $d\theta = \frac{u}{R}dt$, to obtain:

$$\theta = \frac{ut}{R} \tag{1}$$

$$\begin{aligned} x_1 &= R\sin\theta &&= R\sin\frac{ut}{R} \\ y_1 &= R\cos\theta &&= R\cos\frac{ut}{R} \end{aligned} \tag{2}$$

These are the coordinates of the wolf until the moment when we can determine whether the man has succeeded or has failed to reach the shore.

To find the direction of the man's velocity as a function of time, we will define a line passing through points (x_1, y_1) and (x_2, y_2).

We know that the definition of a line in a plane is $\frac{x-x_0}{m} = \frac{y-y_0}{n}$. Knowing that the direction vector of such a line is equal to $\hat{a} = (m\hat{x} + n\hat{y})/\sqrt{m^2 + n^2}$, we obtain the following:

$$\frac{x_2 - x_1}{m} = \frac{y_2 - y_1}{n} \tag{3}$$

$$\hat{v} = \frac{m\hat{x} + n\hat{y}}{\sqrt{m^2 + n^2}} \tag{4}$$

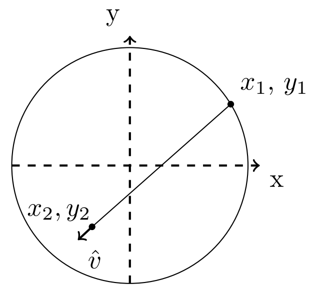


Figure 2: Movement of the man

$$\hat{v} = \frac{(x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{5}$$

Next, knowing the direction of the $\vec{v}$ vector, we can separate it into v_x and v_y, to find our differential equations.

$$\begin{aligned} \frac{dx_2}{dt} &= \frac{(x_2 - x_1)v}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\ \frac{dy_2}{dt} &= \frac{(y_2 - y_1)v}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \end{aligned} \tag{6}$$

From Equations 2 and 6, we obtain the equations of motion dependent on time for the man in the lake:

$$\begin{aligned} \frac{dx_2}{dt} &= \frac{(x_2 - R\sin\frac{ut}{R})v}{\sqrt{x_2^2 + y_2^2 + R^2 - 2R(x_2\sin\frac{ut}{R} + y_2\cos\frac{ut}{R})}} \\ \frac{dy_2}{dt} &= \frac{(y_2 - R\cos\frac{ut}{R})v}{\sqrt{x_2^2 + y_2^2 + R^2 - 2R(x_2\sin\frac{ut}{R} + y_2\cos\frac{ut}{R})}} \end{aligned} \tag{7}$$

The system of equations 7 does not have an exact solution. Therefore, an accurate mathematical solution to this problem cannot be provided.

Limiting Case

In the attempt to solve this problem, it was often assumed that there exists a limiting case for the minimum speed of the man in the lake, that is, when the man and the wolf meet each other on the shore.

We will prove that this is not possible given the conditions. First, we will assume that they do meet at the same point at time τ, then we will analyze the motion at a very small period of time before the meeting takes place, $\tau - \epsilon$. The man must be within the circle at a small distance ϵv from a point on the circle, while the wolf is found on the circle at a distance ϵu from the same point.

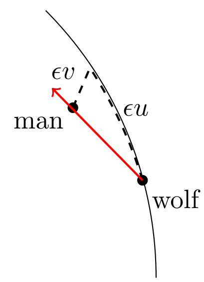


Figure 3: The wolf and the man ϵ time before meeting. The velocity vector of the man is pointed away from the shore.

As it can be seen in Figure 3, wherever the man is placed at this moment of time (ϵ is infinitesimal), his speed will be redirected to move away from the shore, as you can see the red line.

Therefore, we have proven that this is not a possible scenario. Take note that they may get very close when we seek the minimum speed, but it will never be at the same point.

METHODS

The conditions of the problem can be integrated into a computer simulation[3] using the formulas we obtained in the Introduction. The logic goes as follows: we define a set value for a small amount of time, let it be dt. At each moment of time t, the man moves a distance vdt in a straight line in the direction according to their coordinates, and then the process repeats itself at time $t + dt$.

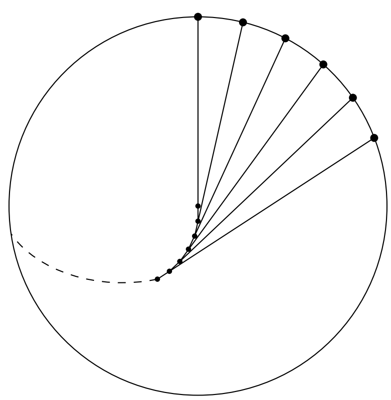


Figure 4: Simulation representation for large dt. The man moves in a small segment each moment of time, away from the wolf.

The closer dt is to 0, the closer we get to the real problem.

Ending Conditions

For the simulation program not to run forever, we need to specify the conditions that determine whether the man has managed to escape the circle or failed to do so.

- Success condition: The man manages to escape if, at any time, $x_2^2 + y_2^2 \geq R$.

- Fail condition: We know that the man cannot escape if at $t > 0$ the man is between the center of the circle and the wolf, being collinear with them.

The latter one is explained through the fact that, arrived in this position, the wolf can just remain stationary until the man gets back to the center, so we get back to the initial condition. We know that one of these will happen while the wolf keeps its $\hat{\theta}$ direction. This is how we explain that we need just the coordinates x_1 and y_1, which we got in the Physical Aspect section.

Conversion of the Formulas

We will set all the constants to 1, $u = R = 1$, and the value for v will represent the ratio $\frac{v}{u}$. Using these values, the equations obtained in the "Physical Aspect" section will be implemented in the program. Initially, we set the variables: $t = x_2 = y_2 = 0$ and $dt = 10^{-5}$.

Our code will use a *while* loop with its first condition $t = t + dt$ with every run. For the coordinates of the wolf, from Equations 2 we have:

$$\begin{aligned} x_1 &= \sin t \\ y_1 &= \cos t \end{aligned} \tag{8}$$

Going back to Equations 6, the variables describing the position of the man will change as follows:

$$\begin{aligned} dx_2 &= \frac{v(x_2 - x_1)dt}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\ dy_2 &= \frac{v(y_2 - y_1)dt}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \end{aligned} \tag{9}$$

$$\begin{aligned} x_2 &= x_2 + dx_2 \\ y_2 &= y_2 + dy_2 \end{aligned} \tag{10}$$

We will also slightly modify the statement for the Fail condition. Because the ratio of variables will never be exactly equal in the program[4], our condition will be:

$$\left| \frac{x_2}{y_2} - \frac{x_1}{y_1} \right| \leq v dt \tag{11}$$

RESULTS AND DISCUSSION

Equal Speed

In this part, we will analyze the solution of the students, where they obtained that the minimum speed for the man to cross the shore is $v = u$. We will set $v = 1$ in our simulation.

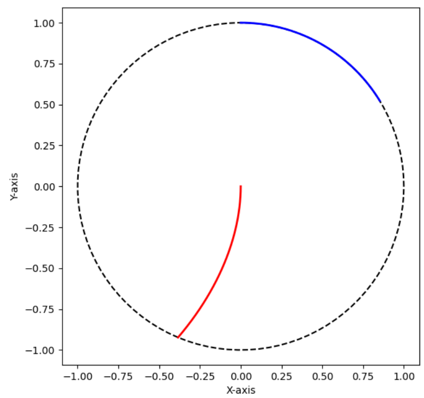
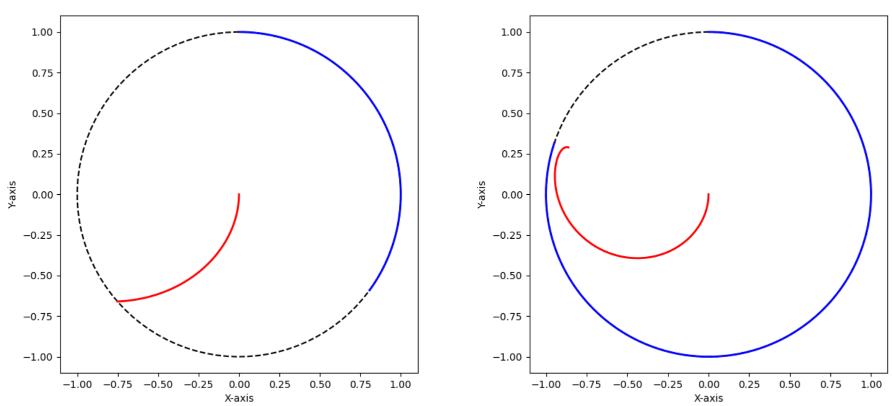


Figure 5: Graph for $v = u$. The blue arc is the path of the wolf, and the red line is the path of the man.

As it can be seen in Figure 5, the man has reached the shore, and the wolf wasn't close to catch him; therefore, we can try a smaller speed v.

Other Ratios

Next, to get an idea where to look for the minimum speed, we will simulate the cases for the ratios $\frac{1}{2}$, and $\frac{1}{3}$.



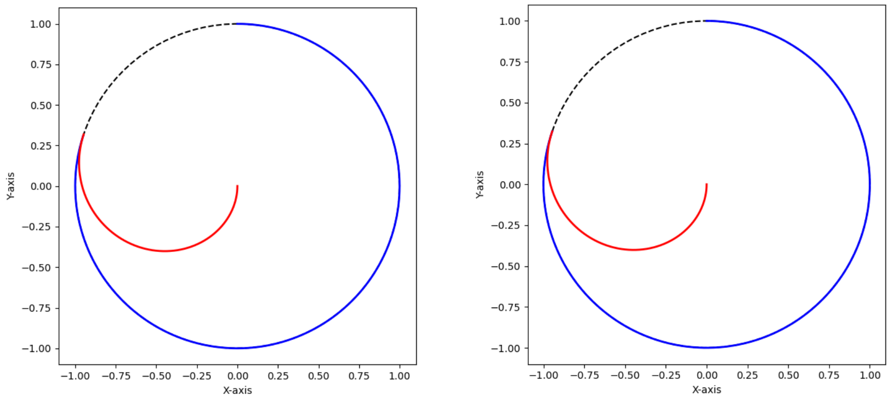
(a) (b)

Figure 6: Graphs for $v = 1/2u$ (a) and $v = 1/3u$ (b). In the first one, the man succeeds to escape, but in the second, the wolf catches him.

From these simulations, since the man managed to escape with $v = 0.5$, but didn't with speed $v = \frac{1}{3}$ (Simulation from Figure 6 (b) ends with the Fail condition), we therefore know that the minimum speed sits between $0.33 < v \leq 0.5$, and next we will search for it.

Minimum Speed

Deciding to stop at 6 significant digits, we obtained Success and Fail for the following simulations:



(a) (b)

Figure 7: Graphs for $v = 0.335683u$ (a) and $v = 0.335682u$ (b). From a distance, they look the same, but the first one ends with the success condition and the second with the fail condition.

To further dive into these simulations, let us analyze the last moments, when the ending condition takes place.

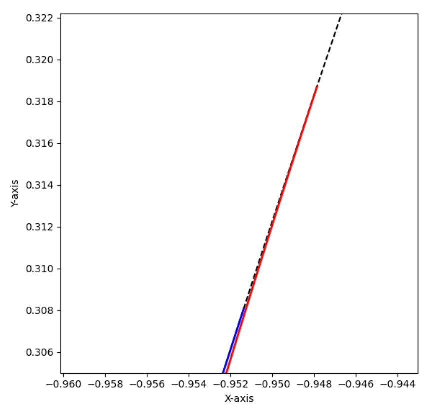


Figure 8: Close-up view of the last moments for $v = 0.335683u$. The man reaches the shore.

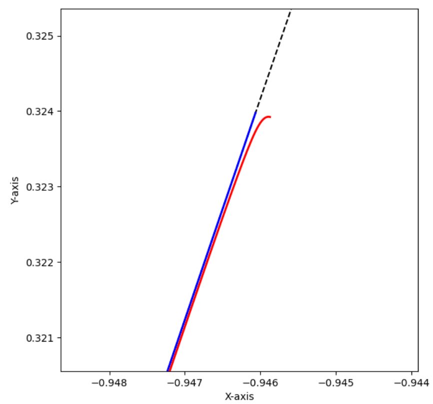


Figure 9: Close-up view of the last moments for $v = 0.335682u$. The Fail condition takes place.

As it can be seen in Figures 8 and 9, the corresponding ending conditions are satisfied such that $v = 0.335683u$ is the minimum speed for 6 significant digits. Having worked with $dt = 10^{-5}$ only, we will check if the value for the minimum speed holds for lower set values for dt.

Table 1: Outcome of crossing the shore

$dt =$	10^{-5}	10^{-6}	10^{-7}
$v = 0.335683$	+	+	+
$v = 0.335682$	-	-	-

From Table 1, we can conclude that $v = 0.335683u$ is a good approximate value for the minimum speed not just for $dt = 10^{-5}$, but for even smaller values.

CONCLUSION

This article has discussed a problem submitted at the "Galileo" physics contest. The problem consists of a man at the center of a lake and a wolf on the shore. The man always swims in the opposite direction from the wolf. The task is to find the minimum speed at which the man can escape. Attempting to solve the problem, we concluded that it leads to an unsolvable system of differential equations, which therefore lacks an exact mathematical solution. Thus, we put the conditions of the problem into a simulation to find the answer numerically. We found that the speed we were seeking is $v \approx 0.335683u$.

Contest Submission

As a mathematical solution that does not include computer simulations was not found, this problem would not be fit for a contest. However, such mistakes often happen, and a full score should have been given to at least the students who obtained the differential equations.

References

1. National Physics Contest held every year, the website of the contest: Hyperlink.
2. Problem from the Mechanics Section of the Senior Category, available at Hyperlink in Romanian. The original name was “Prădător”, meaning “Predator”, which was changed in this article to “Wolf around Lake”.
3. The program used to create the simulation was Python. To obtain the graphs of the simulations as images, the library matplotlib was used.
4. Limitations and issues of calculation of decimal numbers in Python: Hyperlink.