11/25 - Exponential of Matrix

Motivation: 1st Order DE: $x' = ax \Rightarrow x = e^{at} x_0$

$\hookrightarrow$ 1x1 matrix $\qquad \hookrightarrow x_0 = x(0)$

Nth Order DE: $\vec{x}' = A\vec{x} \Rightarrow \vec{x} = e^{At}\vec{v}$?

$\hookrightarrow$ nxn matrix

can you take exponential of matrix?

short answer: yes

Review: linear algebra

exponential function: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$

$e^A = 1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$ (A is nxn matrix)

to show e^A is a <u>well-defined matrix</u>

we have to check $(e^A)_{ij}$ is finite number

$A = (a_{ij})_{n \times n}$

norm of matrix $= \|A\| = \max_{1 \le i \le n,\ 1 \le j \le n} |a_{ij}|$

let $M = \|A\| \Rightarrow \|a_{ij}\| \le M$

$(e^A)_{ij} = I_{ij} + A_{ij} + \frac{1}{2!}(A^2)_{ij} + \frac{1}{3!}(A^3)_{ij} + \ldots$

$|A_{ij}| = |a_{ij}| \le M$

$(A^2)_{ij} = (a_{i1} \cdot a_{1j} + a_{i2} \cdot a_{2j} + \ldots + a_{in} \cdot a_{nj})$

$\le |a_{i1} \cdot a_{1j}| + |a_{i2} \cdot a_{2j}| + \ldots + |a_{in} \cdot a_{nj}|$

$\le M^2 + \ldots + M^2 = nM^2$

in general: $(A^k)_{ij} \le n^{k-1} \cdot M^k = \frac{(nM)^k}{n}$

$|(e^{+A})_{ij}| = |\sum \frac{1}{k!} \cdot (A^k)_{ij}| \le \sum_{k=1}^{+\infty} \frac{1}{k!} |(A^k)_{ij}| + |I_{ij}| \le \sum_{k=1}^{+\infty} \frac{1}{k!} \frac{(nM)^k}{n} + |I_{ij}|$

$= \frac{1}{n} \sum_{k=1}^{+\infty} \frac{1}{k!} (nM)^k + |I_{ij}|$

$= \frac{1}{n}(e^{nM} - 1) + |I_{ij}|$

so $(e^{+A})_{ij}$ is finite

so e^A is a well-defined matrix

how to calculate e^A?

<u>case 1</u>: A is diagonalizable

ex: $A = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{bmatrix} \Rightarrow A^k \begin{bmatrix} r_1^k & 0 \\ 0 & r_2^k \end{bmatrix}$

$$e^A = I + A + \dots + \frac{A^k}{k!} = \sum_{k=0}^{+\infty} \frac{1}{k!} \begin{bmatrix} r_1^k & 0 \\ 0 & r_2^k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{+\infty} \frac{r_1^k}{k!} & 0 \\ 0 & \sum_{k=0}^{+\infty} \frac{r_2^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{bmatrix}$$

if A is diagonalizable

$A = T^{-1} D T$; D is diagonal matrix $\begin{bmatrix} \lambda^1 & & \\ & \ddots & \\ & & \lambda^n \end{bmatrix}$

$A^2 = T^{-1} D T \cdot T^{-1} D T = T^{-1} D \cdot D \cdot T = T^{-1} D^2 T$

$$\sum_{k=0}^{+\infty} \frac{1}{k!} A^k = \sum_{k=0}^{+\infty} \frac{1}{k!} T^{-1} D^k T = T^{-1} \left(\sum_{k=0}^{+\infty} \frac{1}{k!} D^k \right) T = T^{-1} e^D T$$

<u>case 2</u>: A is not diagonalizable

no easy formula

prop 1: if $AB = BA$, then $e^A \cdot e^B = e^B \cdot e^A = e^{(A+B)}$ (no proof)

prop 2: let $B = -A$

$$e^A e^{-A} = e^{-A} e^A = e^{(A-A)} = e^0 = I$$

e^{-A} is is the inverse of e^A ($\Rightarrow e^A$ is nonsingular $\Rightarrow \det(e^A) \neq 0$)

let $e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \dots = \sum_{k=0}^{+\infty} \frac{1}{k!} t^k A^k$

fact: $\boxed{\frac{d}{dt}(e^{tA}) = Ae^{tA}}$ $\left(\frac{d}{dt} e^{ta} = ae^{ta} \right)$

$$\frac{d}{dt}(e^{tA}) = \frac{d}{dt}\left(\sum_{k=0}^{+\infty} \frac{1}{k!} t^k A^k \right)$$

b/c e^B converges absolutely for $\forall B_{n \times n}$ matrix

$$\frac{d}{dt}(e^{tA}) = \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \frac{d}{dt}\left(\frac{t^2}{2!} A^2 \right) + \dots$$

$$= 0 + A + tA^2 + \frac{t}{2} A^3 + \dots + \frac{t^k}{k!} A^{k+1}$$

$$= A\left(I + tA + \frac{t}{2} A^2 + \dots + \frac{t^k}{k!} A^k \right)$$

$$= Ae^{tA}$$