10/09 - Given $P(x,y)\,dx + Q(x,y)\,dy = 0$

How do you find $w(x,y)$ such that $\frac{\partial w}{\partial x} = P$ and $\frac{\partial w}{\partial y} = Q$

$$\frac{\partial w}{\partial x} = P(x,y) \qquad \frac{\partial w}{\partial y} = Q(x,y)$$

$$w(x,y) = \int P(x,y)\,dx + \Phi(y) \qquad w(x,y) = \int Q(x,y)\,dy + \Phi(x)$$

$$Q(x,y) = \frac{\partial}{\partial y}\left(\int P(x,y)\,dx + \Phi(y)\right)$$

$$= \frac{\partial}{\partial y}\left(\int P(x,y)\,dx\right) + \Phi'(y)$$

$$\int \Phi'(y)\,dy = \int\left(Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)\,dx\right)dy$$

$$w(x,y) = \int P(x,y)\,dx + \int Q(x,y)\,dy - \frac{\partial}{\partial y}\int P(x,y)\,dx$$ ask question!

ex: $(x^2+y^2)\,dx + 2xy\,dy = 0$

$$\frac{\partial}{\partial y} P(x,y) = 2y$$
$$\frac{\partial}{\partial x} Q(x,y) = 2y \quad \|$$
$$\int Q\,dx = 2xy$$

$$\frac{\partial w}{\partial x} = P(x,y) = x^2+y^2$$
$$\int \frac{\partial w}{\partial x}\,dx = \int x^2+y^2\,dx$$
$$= \frac{1}{3}x^3 + xy^2 + C(y)$$

$\frac{\partial}{\partial x}$

$$\frac{\partial w}{\partial y} = Q \Rightarrow \cancel{2xy} + C'(y) = \cancel{2xy}$$
$$C'(y) = 0$$
$$C(y) = \text{constant}$$

$$w(x,y) = \frac{1}{3}x^3 + xy^2 + c = 0$$

ex: $\sin(x+y)\,dx + (2y + \sin(x+y))\,dy = 0$

$$\frac{\partial}{\partial y} P = \cos(x+y)$$
$$\frac{\partial}{\partial x} Q = \cos(x+y) \quad \|$$

$$\frac{\partial w}{\partial x} = P(x,y) = \sin(x+y)$$
$$w = -\cos(x+y) + C(y)$$
$$\frac{\partial}{\partial y}\left(-\cos(x+y) + C(y)\right) = 2y + \sin(x+y)$$
$$\cancel{\sin(x+y)} + C'(y) = 2y + \cancel{\sin(x+y)}$$
$$C'(y) = 2y$$
$$C(y) = y^2 + c$$

$$w(x,y) = -\cos(x+y) + y^2 + C = 0$$

<u>Theorem</u> - Given a differential equation of the form

$P(x,y)\,dx + Q(x,y)\,dy = 0$ (not necessarily exact)

there always exists a non-zero function $\mu(x,y)$ such that

$\mu(x,y)P(x,y)\,dx + \mu(x,y)Q(x,y)\,dy = 0$ is exact equation

$\mu(x,y)$ is called the <u>integrating factor</u>

(No Proof, No General Way to Find $\mu(x,y)$)

Q - How to find integrating factor $\mu(x,y)$?

Special case 1 - separable

if $\frac{dy}{dx} = \frac{-P(x,y)}{Q(x,y)} = g(x) \cdot h(y)$ then we can find $\mu(x,y)$

but why bother?

Special case 2 - 1-var case, $\mu(x,y) = \mu(x)$ or $\mu(x,y) = \mu(y)$

1) if $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only

then $\mu(x) = e^{\int h(x)\,dx}$ is an IF

2) if $g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of y only

then $\mu(y) = e^{-\int g(y)\,dy}$ is an IF ex: $(xy-2)dx + (x^2-xy)dy = 0$

check exactness? $\frac{\partial}{\partial y} P(x,y) = x$

nope! $\frac{\partial}{\partial x} Q(x,y) = 2x - y$

check separable? $\frac{\partial y}{\partial x} = \frac{xy-2}{x^2-xy}$

nope!

check $\mu(x,y) = \mu(x)$ or $\mu(y)$?

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = x - 2x + y = y - x$

$h(x) = \frac{y-x}{x^2-xy} = \frac{y-x}{x(x-y)} = -\frac{1}{x}$ ✓

$g(x) = \frac{y-x}{xy-2} = ??$ ✗

$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{e^{\ln|x|}} = \frac{1}{|x|}$

$\frac{1}{x}$ and $-\frac{1}{x}$ are both IFs

$\left(\frac{1}{x}\right)\left((xy-2)dx + (x^2-xy)dy\right) = 0\left(\frac{1}{x}\right)$

$(y - \frac{2}{x})dx + (x-y)dy = 0$

$\frac{\partial}{\partial y} P(x,y) = 1$ $\quad ||$

$\frac{\partial}{\partial x} Q(x,y) = 1$

$w = \int P(x,y)dx = xy - 2\ln|x| + c(y)$

$\frac{\partial w}{\partial y} = Q \rightarrow x + c'(y) = x - y$

$c'(y) = -y$

$c(y) = -\frac{1}{2}y^2 + c$

$w = xy - 2\ln|x| - \frac{1}{2}y^2 + c = 0$