11/01 - Linear Algebra Review

I missed class today so I'm going off other people's stuff

def - matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n} = (a_{ij})_{m\times n}$; $a_{ij} \in \mathbb{R}$ ↳ real numbers

def - for an $n\times n$ matrix A, $\det(A) \in \mathbb{R}$

$$\det(A) = \sum_{\sigma \in S_n} \overset{\{-1,1\}}{\operatorname{sgn}(\sigma)} \cdot \prod_{i=1}^{n} a_{i\sigma(i)}$$ uh what?

$(1, 2, 3, \dots, n)$

$(2, 1, 3, \dots, n)$

for 2x2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

3x3: $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \det\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \det\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

or like $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ from each element in top row do product of ↘ minus product of ↙ (loop around)

$aei + bfg + cdh - afh - bdi - ceg$

ex: $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

if $A = A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \dots & a_{ij}(t) & \dots \\ a_{m1}(t) & \dots & a_{mn}(t) \end{bmatrix}$, then $\frac{dA}{dt} = \begin{bmatrix} a_{11}'(t) & \dots & a_{1n}'(t) \\ \dots & a_{ij}'(t) & \dots \\ a_{m1}'(t) & \dots & a_{mn}'(t) \end{bmatrix}$

def - an n-dimensional vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$; $v_i \in \mathbb{R}$

$A_{m\times n} \vec{v}_{n\times 1} = (A\vec{v})_{m\times 1}$ ↳ a vector

can think of it as canceling the middle $(m\times n)(n\times 1) = (m\times 1)$

$$A_{m\times n}\vec{v}_{n\times 1} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + \dots & + a_{2n}v_n \\ \vdots & \vdots \\ a_{m1}v_1 + \dots & + a_{mn}v_n \end{bmatrix}$$

for $\lambda \in \mathbb{R}$, $A(\lambda\vec{v}) = \lambda(A\vec{v})$

for $A = (a_{ij})_{m\times n}$, $B = (a_{ij})_{m\times n}$, $A + B = (a_{ij} + b_{ij})_{m\times n}$

NOTE: $AB \neq BA$ usually

for $A_{n\times n}, B_{n\times n}$, $A \cdot B = A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}_{n\times n} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \dots & A\vec{v}_n \\ | & | & & | \end{bmatrix}_{n\times n}$

$$\frac{d}{dt}(A(t) + B(t)) = A'(t) + B'(t)$$

$$\frac{d}{dt}(A(t) \cdot B(t)) = A'(t)B(t) + A(t)B'(t)$$

thm - given $A_{n\times n}$ matrix , $\det(A) \neq 0 \iff \operatorname{rank}(A) = n$

$\iff$ A is one-to-one

def - λ is an eigenvalue of $A_{n\times n}$ if $\exists$ a non-zero vector st $A\vec{v} = \lambda\vec{v}$

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{v}_1$ $\quad \lambda_1 = 2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad A\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}_1$ $\quad \lambda_2 = 1,\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

if $\vec{v}$ is an eigenvector, then $k\vec{v}$ is also an eigenvector

if $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v} = 0\vec{v}$

because A is a 2x2 matrix, choose two lin ind vectors

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\det(\vec{v}_1, \vec{v}_2) \neq 0$

Q: How to find eigenvalues and eigenvectors?

def - the characteristic polynomial of A

$$p(\lambda) = \det(\lambda I_n - A)$$

$$Id_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

prop - eigenvalues are roots of char. poly

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad (\lambda I - A) = \begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{bmatrix}$ $\Rightarrow \det(\lambda I - A) = (\lambda-2)(\lambda-1)$

$\lambda = 1, 2$

ex: $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda+4 & -6 \\ 3 & \lambda-5 \end{bmatrix}$ $\Rightarrow \det(\lambda I - A) = (\lambda+4)(\lambda-5) + 18$

$= \lambda^2 - \lambda\ 2$

$= (\lambda-2)(\lambda+1)$

$\lambda = -1, 2$

$\lambda_1 = 2$

$(\lambda_1 I)\vec{v}_1 = A\vec{v}_1$

$(\lambda_1 I - A)\vec{v}_1 = 0$

$\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}\vec{v}_1 = 0$

rref

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$a = b$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$

$\begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}\vec{v}_2 = 0$

$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$a - 2b = 0$

$a = 2b$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$