10/14 - Existence and Uniqueness of Solutions of 1st Order DEs ( in Normal Form)

<u>Existence</u>

ex: $x' = f(t,x) = \begin{cases} 0 & x=0 \\ \frac{1}{x} & x\neq 0 \end{cases}$

if $x=0$, then initial value problem has no solutions

no solution can pass through $0$

<u>Thm</u> - Suppose $f(t,x)$ is defined and continuous on Rect $R$ in the tx-plane $([a,b],[c,d])$ then given a point $(t_0,x_0)\in R$, then the initial value problem $x'=f(t,x)$, $x(t_0)=x_0$ has a solution $x(t)$ defined in a interval containing $t_0$. Furthermore, the solution will be defined at least until $t\to(t,x(t))$ leaves Rectangle $R$

solution if continuous ~~~

<u>Uniqueness</u>

ex: $y' = y^{\frac{1}{3}}$ $y(0)=0$

1) $y_1(t)=0$ is a solution

2) $y_2(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2} & t>0 \\ 0 & t\leq 0 \end{cases}$

$\left|\lim\limits_{t\to0}\frac{y(t)-y(0)}{t0}\right| = \lim\limits_{t\to0}\left|\frac{y(t)}{t}\right|$

$\leq \lim\limits_{t\to0}\left(\frac{\frac{2}{3}|t|^{3/2}}{|t|}\right) = \lim\limits_{t\to0}\sqrt{|t|} = 0$

<u>Thm</u> - Suppose $f(t,x)$, $\frac{\partial f}{\partial x}(t,x)$ are both continuous on $R$ in the tx-plane

Suppose $(t_0,x_0)\in R$ and that the solution $x'=f(t,x)$, $y'=f(t,y)$ are two solutions satisfying $x(t_0)=y(t_0)=x$

Then as long as $((t,x(t)),(t,y(t)))$ stays in $R$, we have $x(t)=y(t)$

<u>Proof of Uniqueness</u> which makes absolutely no sense to me ^^

<u>Fact</u> - $\frac{\partial f}{\partial x}$ is continuous on $R$ means $\frac{\partial f}{\partial x}$ is bound on $\mathbb{R}$ ( $|\frac{\partial f}{\partial x}|\leq M$ on $R$)

we define $h(t)$ to be the difference betwen two solutions $x(t)$ and $y(t)$

$h(t)=x(t)-y(t)$

$h'(t)=x'(t)-y'(t) = f(t,x)-f(t,y)$ $\quad$ $\rightarrow \xi(t)$ is between $x(t)$ and $y(t)$ ??

by MVT $f(t,x)-f(t,y)=h'(t)=\frac{\partial f}{\partial x}(t,\xi(t))(x(t)-y(t)) = \frac{\partial f}{\partial x}(t,\xi(t))h$

$\left|\int_{t_0}^{t}\frac{h'}{h}\,dt\right| = \left|\int_{t_0}^{t}\frac{\partial f}{\partial t}(t,\xi(t))\,dt\right| \leq \left|\int_{t}^{t_0}M\,dt\right| \leq M|t-t_0|$

$\ln|h(t)|-\ln|h(t_0)|\leq M|t-t_0|$

$\left|\frac{h(t)}{h'(t)}\right|\leq e^{M(t-t_0)} \Rightarrow |h|\leq|h(t_0)|e^{M(t-t_0)} \Rightarrow |x(t)-y(t)|\leq|x(t_0)-y(t_0)|e^{M(t-t_0)}$

if $x(t_0)=y(t_0)$, aka they intersect at $t_0$, then $|x(t)-y(t)|\leq0$, so $x(t)=y(t)$

Suppose $y(t)$ satisfies $y' = (y-1)\cos(yt)$

$\frac{\partial f}{\partial y} = \cos(yt) + (-\sin(yt))(t)(y-1)$ is always continuous on R

Prove that 1) if $y(0)=1$ then $y(t)=1$

2) if $y(0)=2$, then $y(t)>1$

1) $y' = 0 = (1-1)\cos(1\cdot 0) = 0$ ✓ : $y(t)=1$

2) if $y(0)=2$, assume that the solution crosses $y(t)=1$

by IVT, $y(t)=1$ is unique

therefore the solution w/ IC $y(0)=2$ cannot intersect w/ line $y=1$, so $y>1$