11/27 - Application of exponential of matrices

<u>truncation</u>: $e^{tB}\vec{v} = I\vec{v} + tB\vec{v} + \frac{t^2}{2!}B^2\vec{v} + \dots + \frac{t^k}{k!}B^k\vec{v} + \dots$

suppose B is a "singular" matrix
then there exists a $\vec{v}$ s.t. $B\vec{v}=0$

1) if $B\vec{v}=0$, then $e^{tB}\vec{v} = \vec{v} + 0 + 0 + \dots$
2) if $B^2\vec{v}=0$, then $e^{tB}\vec{v} = \vec{v} + tB\vec{v}$

$\vdots$

k) if $B^k\vec{v}=0$, then $e^{tB}\vec{v} = \vec{v} + tB\vec{v} + \frac{t^2}{2!}B^2\vec{v} + \dots + \frac{t^{k-1}}{(k-1)!}B^{k-1}\vec{v}$

recall: $\vec{x}' = A\vec{x}$

if we let $x(t) = e^{tA}\vec{v}$ then

$\frac{d}{dt}(e^{tA}) = Ae^{tA}$

$\vec{x}' = \frac{d}{dt}(e^{tA}\vec{v}) = \frac{d}{dt}(e^{tA})\vec{v} = Ae^{tA}\vec{v} = A\vec{x}$

let $\vec{x}_1 = e^{tA}\vec{v}_1$, $\quad \vec{x}_1(0) = \vec{v}_1$

$\vdots$

$\vec{x}_n = e^{tA}\vec{v}_n \quad \vec{x}_n(0) = \vec{v}_n$

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are lin ind $\overset{E/U}{\Rightarrow} \vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$

fact: e^{tA} is generally hard to compute

instead we choose $\vec{v}_1, \dots, \vec{v}_n$ linearly independent calculate $\vec{x}_i = e^{tA}\vec{v}_i$

<u>motivation</u>: suppose λ is an eigenvalue

$tA = \lambda t I + t(A - \lambda I)$ $\quad$: $A - \lambda I$ is a singular matrix

$e^{tA} = e^{\lambda I t + t(A-\lambda I)}$

$= e^{\lambda I t} e^{t(A-\lambda I)} = e^{\lambda I t} e^{B}$ $\quad$ (b/c $\lambda t I$ commutes with $t(A-\lambda I)$)

if we choose $\vec{v}$ be the eigenvector associated w/ λ

$e^{tA}\vec{v} = e^{\lambda t I} e^{t(A-\lambda I)}\vec{v} = e^{\lambda t I}(e^{tB}\vec{v}) = e^{\lambda t I}\cdot\vec{v} \xrightarrow{\text{truncation}} (A-\lambda I)\vec{v} = 0$, $\; B\vec{v} = 0$

$= e^{\lambda t} I \vec{v} = e^{t\lambda}\vec{v}$

$\lambda t I = \begin{bmatrix} \lambda t & & \\ & \ddots & \\ & & \lambda t \end{bmatrix}$

$e^{\lambda t I} = \begin{bmatrix} e^{\lambda t} & & \\ & \ddots & \\ & & e^{\lambda t} \end{bmatrix}$

$= e^{\lambda t} I$

Similarly, if $(A - \lambda I)^2\vec{w} = 0$

then $e^{tA}\vec{w} = e^{\lambda t} I e^{t(A-\lambda I)} = e^{\lambda t}(\vec{w} + t(A-\lambda I)\vec{w})$

we choose $\vec{w} \in \ker((A-\lambda I)^2)$ and $\vec{w} \notin \ker(A - \lambda I)$