11/22- Higher Dimensional Systems

A is $n\times n$ constant matrix

$\vec{y}\,' = A\vec{y}$

if λ_i is an eigenvalue of A and $\vec{v}_i$ is an eigenvector associated with λ_i
then $e^{\lambda_i t}\vec{v}_i$ is a solution to $\vec{y}\,' = A\vec{y}$

<u>thm</u> - suppose $\vec{v}_i$, $i=0,1,\dots,n$ is an eigenvector associated w/ eigenvalue λ_i

if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are lin ind, then $e^{\lambda_i t}\vec{v}_i$, $i=0,1,\dots,n$ form a fundamental set

<u>thm</u> - if $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct real eigenvalues of A

then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent

proof - $A\vec{v}_1 = \lambda_1\vec{v}_1$
$A\vec{v}_2 = \lambda_2\vec{v}_2$
$\vdots$
$A\vec{v}_n = \lambda_n\vec{v}_n$

if $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent

then $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0$

$A(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) = 0$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \dots + c_n\lambda_n\vec{v}_n = 0$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 + \dots + c_n\lambda_1\vec{v}_n = 0$

$c_2(\lambda_2-\lambda_1)\vec{v}_2 + \dots + c_n(\lambda_n-\lambda_1)\vec{v}_n = 0$

then $\vec{v}_2, \dots, \vec{v}_n$ are lin dep

then $\vec{v}_{n-1}, \vec{v}_n$ are lin dep

<u>rem</u> - in this case, A is (real) diagonalizable

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

ex: $\vec{y}\,' = A\vec{y}$ $\quad A = \begin{bmatrix} -9 & -3 & -7 \\ 3 & 1 & 3 \\ 11 & 3 & 9 \end{bmatrix}$

solution: char poly $p(\lambda) = \det(\lambda I - A) = -\det(A-\lambda I)$

$$= \begin{vmatrix} \lambda+9 & 3 & 7 \\ -3 & \lambda-1 & -3 \\ -11 & -3 & \lambda-9 \end{vmatrix} = (\lambda+9)(\lambda-1)(\lambda-9) + 3(-3)(-11) + 7(-3)(-3)$$
$$- (\lambda+9)(-3)(-3) - 3(-3)(\lambda-9) - 7(\lambda-1)(-11)$$

$$= \lambda^3 - \lambda^2 - 4\lambda + 4$$

rational zero theorem: b/c leading coefficient $=1 \Rightarrow$ rational roots are integers

$(\lambda^3 - 4\lambda) - (\lambda^2 - 4) = \lambda(\lambda^2-4) - 1(\lambda^2-4) = (\lambda-1)(\lambda-2)(\lambda+2)$

$\lambda = 1, 2, -2$

step 2: find eigenvectors

$\lambda = 1 \qquad (A - \lambda I)\vec{v} = \begin{bmatrix} -10 & -3 & -7 \\ 3 & 0 & 3 \\ -1 & 3 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

solve. $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = -2$

A Generalization A complex diagonizable

$$T^{-1}AT = \begin{bmatrix} \lambda_1 \to \vec{v}_1 & & & & \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & (\eta_1, \bar{\eta}_1) \begin{matrix} \to \vec{w}_1 \\ \to \bar{\vec{w}}_1 \end{matrix} & \\ & & & & \ddots \\ & & & & & \eta_m \bar{\eta}_m \end{bmatrix}$$

$\uparrow$
complex matrix

$\vec{z} = e^{\eta t} \vec{w}$

$= \underbrace{Re(\vec{z})}_{\text{solution 1}} + i \underbrace{(Im(\vec{z}))}_{\text{solution 2}}$

ex: $A = \begin{bmatrix} 5 & -2 & -2 \\ 7 & -4 & -2 \\ 3 & 1 & -1 \end{bmatrix}$

$p(\lambda) = \lambda^3 + \lambda + 10$ check $\pm 1, \pm 2, \pm 5, \pm 10$

$\lambda = -2$ is a root

$(\lambda + 2)\underbrace{(\lambda^2 - 2\lambda + 5)}_{\lambda = 1 \pm 2i}$

$(\lambda^3 + 2\lambda^2)(-2\lambda^2 - 4\lambda) + (4\lambda + \lambda - 10)$?? ?