11/04 - Linear Systems (8.4, 8.5)

<u>def</u> - suppose $x_1(t), \dots, x_n(t)$ are unknown and differentiable

a <u>linear system</u> of DEs is a set of DEs

$$\left.\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \dots + a_{12}(t)x_n(t) + f_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \dots + a_{2n}(t)x_n(t) + f_2(t) \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \dots + a_{nn}(t)x_n(t) + f_n(t) \end{aligned}\right.$$

forcing terms

let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = [x_1(t), \dots, x_n(t)]^T$ $\quad \vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

n-dim vector-valued function

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}_{n\times n} = [a_{ij}(t)]_{n\times n}$$

$$\vec{x}'(t) = \frac{d}{dt}(\vec{x}(t)) = A(t)\vec{x}(t) + \vec{f}(t)$$

$n=1$ $\quad \vec{x}(t) = x_1(t) \quad \vec{f}(t) = f_1(t) \quad A(t) = a_{11}(t)$

$\vec{x}'(t) = a_{11}(t)x_1(t) + f_1(t)$ $\quad$ 1st order linear DE

<u>def</u> - if $\vec{f} \equiv 0$, then the linear system is homogeneous

<u>homogeneous system</u> $\quad \vec{x}'(t) = A\vec{x}(t)$

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \Rightarrow \underbrace{\left(\frac{d}{dt} - A\right)}_{\text{linear transformation}}\vec{x}(t) = 0$

$A, \frac{d}{dt}$ are linear transformations

$A(c_1\vec{v}_1 + c_2\vec{v}_2)$ $\qquad \frac{d}{dt}(c_1\vec{v}_1(t) + c_2\vec{v}_2(t))$

$= c_1A\vec{v}_1 + c_2A\vec{v}_2$ $\qquad = c_1\left(\frac{d}{dt}\vec{v}_1(t)\right) + c_2\left(\frac{d}{dt}\vec{v}_2(t)\right)$

we know $\vec{x} \in \ker\left(\frac{d}{dt} - A\right)$, kernal is a linear subspace

if $\vec{x}_1(t)\ \vec{x}_2(t) \in \ker\left(\frac{d}{dt} - A\right)$

then $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) \in \ker\left(\frac{d}{dt} - A\right)$

<u>thm</u> (E/U) - suppose $A(t) = [a_{ij}(t)]_{n\times n}$ $\vec{f}(t) = [f_i(t)]_{n\times 1}$ are continuous on interval $I = (\alpha, \beta)$ for $t_0 \in I$, $\vec{x}_0 \in \mathbb{R}^n$

$\vec{x}_1(t) = A\vec{x}(t) + \vec{f}(t)$, $\vec{x}(t_0) = \vec{x}_0$ has a unique solution defined for all $t \in (\alpha, \beta)$

def - a set of vector-valued functions (defined on $I=(\alpha,\beta)$)

$\{\vec{x}_1(t), \dots, \vec{x}_k(t)\}$ is

1) linearly dependent if there exists $c_1, c_2, \dots, c_k$ s.t. at least one $c_j \neq 0$ and $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = 0$

2) linearly independent if $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = 0$ if and only if $c_1 = c_2 = \dots = c_k = 0$

prop - let $\vec{x}_1(t), \dots, \vec{x}_k(t)$ be solutions to $\vec{x}'(t) = A\vec{x}(t)$

then if $\vec{x}_1(t), \dots, \vec{x}_k(t)$ is linearly dependent/independent

$\Leftrightarrow$ $\vec{x}_1(t_0), \dots, \vec{x}_k(t_0)$ is linearly dependent/independent

proof: U/E (similar to 2nd order linear DEs)

thm - suppose $\underbrace{\vec{x}_1(t), \dots, \vec{x}_n(t)}_{\text{fundamental set of solutions}}$ are linearly ind. solutions to $\vec{x}'(t) = A\vec{x}(t)$

if $\vec{x}(t)$ is a solution then $\vec{x}(t) = c_1\vec{x}_1(t) + \dots + c_k\vec{x}_k(t)$

prop - (n-dim) $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are n-dim vector-valued functions on $I=(\alpha,\beta)$ are lin. ind

$\Leftrightarrow$ $W(t) = \det(\vec{x}_1(t), \dots, \vec{x}_n(t)) \neq 0$