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MS 3474 10-3 Mon-Thurs

No Late Hw!!

09/27 - Introduction

1. motivations Physics - gravity, acceleration, velocity

Biology - S-curves, isometric system, population/logistic equations

$$\frac{dP(t)}{dt} = (1 - P/k)P$$

↳ k = "max population

2. definition of differential equations

	derivative	partial derivative
first order	$f'(t) = \frac{df}{dt}$	$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}$
second order	$f''(t) = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d^2 f}{dt^2}$	$\frac{\partial^2 w}{\partial x^2}$ $\frac{\partial^2 w}{\partial t^2}$ $\frac{\partial^2 w}{\partial x \partial t}$ $\frac{\partial^2 w}{\partial t \partial x}$

$\Phi(x_0, x_1, x_2, \ldots, x_n) = 0$

$\Phi(t, y, y', y'', y^{(3)}, \ldots, y^{(n)}) = 0$ <u>ordinary</u> diff eq b/c $y(t)$ has one var

$\Phi(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \ldots)$ <u>partial</u> diff eq

ex: $t + 4y \cdot y' = 0$

$4y \cdot y' = -t$

$y' = \frac{-t}{4y}$ $y' = f(t,y)$ ← normal form

$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n)})$

3. solutions to differential equations

<u>def</u> - a solution to the DEq $\Phi(t, y, y') = 0$ is a function $y(t)$ such that $\Phi(t, y(t), y'(t)) = 0$ for all t in the interval $y(t)$ is defined

↳ depends on only t

ex: $0t - y + y' = 0$ $(y = y')$ for ex, $y = e^t$ is a solution

$\Phi(x,y,z) = 0 \cdot x - y + z$ $y = 0$ is a solution

$y = ce^t$ is a solution

$y = ce^t$ is the <u>general solution</u> (contains all or almost all solutions)

- in some cases, general solution ≠ all solutions

note - in first order, $\Phi(t, y, y') = 0$, general solution will have one constant, $y = f(c, t)$

in nth order, $\Phi(t, y, y', \ldots, y^{(n)}) = 0$, general solution will have n constants, $y = f(c_1, c_2, \ldots, c_n, t)$

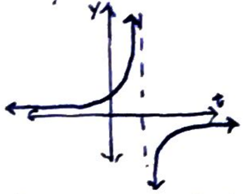
<u>def</u> - Initial Value Problem (IVP) is a DEq paired w/ an initial condition (IC)

ex: $y = y'$, $y(0) = 2$ then $y = ce^t \to 2 = ce^0 \to c = 2$

solution is $\boxed{y = 2e^t}$

def - Interval of Existence

ex: $y = \frac{1}{1-t}$



Given the interval $(-2,1)$

on $(-\infty, 1)$, $y(t)$ is continuous

$(-\infty, 1)$ is the interval of existence

$y = \frac{1}{1-t}$ is a solution to $y' = y^2$?

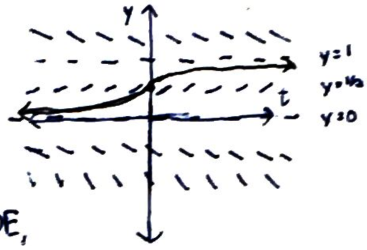
def - Directional fields

consider $y' = f(t,y) = 1 - y^2$, plug into Wolfram Alpha

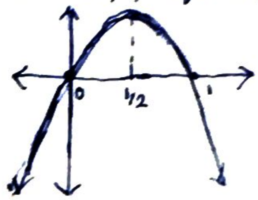
given $y' = f(t,y)$, we can assign a "unit-length" line segment to each point on the (t,y) plane

ex: $y' = (1-y)y = f(t,y)$

IC: $y = 1/2$



$f(y) = (1-y)y$ = quadratic



note $f(t,1) \neq 1$

if $f(t,1) = 1$, by E/U of DE,

then $f = 1$ always! but

$f(t, \frac{1}{2})$

Seperable Equations

$y' = g(t) \cdot f(y) \rightarrow y' = \frac{dy}{dt} = g(t) \cdot f(y)$

$\frac{dy}{f(y)} = g(t)\,dt \rightarrow \int \frac{dy}{f(y)} = \int g(t)\,dt$

let $F(y) = \int \frac{1}{f(y)} dy$ and $G(t) = \int g(t)\,dt$

then $\int \frac{dy}{f(y)} = \int g(t)\,dt \rightarrow F(y) = G(t) + C$ (implicit form)

$y = H(t, C)$ (explicit form)

$\hookrightarrow H = F^{-1}(G(t) + C)$

note: $H = F^{-1}$ if F^{-1} exists

Proof:

$y = y(t)$

$y'(t) = g(t) \cdot f(y(t))$

$y'(t) - g(t) \cdot f(y(t)) = 0$ for all t in domain of $f(t)$

$\hookrightarrow$ 1 var function of t

$\int_{t_0}^{t_1} \frac{y'(t)}{f(y(t))} - g(t)\,dt = 0$

$\int_{t_0}^{t_1} \frac{y'(t)}{f(y(t))}\,dt - \int_{t_0}^{t_1} g(t)\,dt = 0$

$dy = y'(t)dt$ $\int_{y(t_0)}^{y(t_1)} \frac{dy}{f(y)} - \int_{t_0}^{t_1} g(t)\,dt = 0$

by FTC $F(y(t_1)) - F(y(t_0)) - G(t_1) + G(t_0) = 0$

$F(y(t_1)) - G(t_1) = F(y(t_0)) - G(t_0)$ for all points $[t_0, t_1]$

fix t_0, $F(y(t_0)) - G(t_0) = C$ so $F(y(t)) - G(t) = C$

ex: Solve $y' = ty^2$

$\frac{dy}{dt} = ty^2 \rightarrow \frac{dy}{y^2} = t\,dt \rightarrow -\frac{1}{y} = \frac{1}{2}t^2 + C$ (implicit)

$y = -\frac{2}{t^2+2C}$ (explicit)

or $y = -\frac{2}{t^2+C_1}$; $C_1 = 2C$

↳ general solution

$\lim_{t\to\infty} \frac{2}{t^2+2C} = 0$ and $y=0$ is a solution

$y=0$ and $y = -\frac{2}{t^2+2c}$ are all the solutions

ex: $y' = \frac{e^t}{1+y}$; $y(0) = 1$

$\int (1+y)\,dy = \int e^t\,dt \rightarrow = y + \frac{1}{2}y^2 = e^t + c$

$y^2 + 2y - 2(e^t + C) = 0$

$y = \frac{-2 \pm \sqrt{4 + 8(e^t + C)}}{2} = -1 \pm \sqrt{1 + 2e^t + 2C}$

$-1 \pm \sqrt{1+2+2C} = 1$ so must be $+$

$\sqrt{3+2C} = 2$

$3 + 2C = 4$

$C = \frac{1}{2}$

solution: $y = -1 + \sqrt{2 + 2e^t}$