11/08 $z = r(\cos\beta + \sin\beta i)$

$|z| > 0 \quad |z| = r = e^{\alpha} \quad \alpha, \beta \in \mathbb{R}$

$z = e^{\alpha} e^{\beta i} = e^{\alpha + \beta i} = e^{w}$

Q: $Re(e^{\alpha+\beta i}) \quad Im(e^{\alpha+\beta i})$

$\quad = e^{\alpha}\cos\beta \quad\quad = e^{\alpha}\sin\beta$

my paper is dying up here

Conjugation: $z = a+bi \quad \bar{z} = a - bi$

$z \cdot \bar{z} = a^2 - (bi)^2 = a^2 + b^2 = r^2 = |z|^2$

$\frac{1}{a+bi} = \frac{1}{a+bi}\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$

$\frac{c+di}{a+bi} \cdot \frac{a-bi}{a-bi} = ?$

$e^{\bar{w}} = e^{\alpha - \beta i} = e^{\alpha}(\cos\beta - \sin\beta i) = \overline{e^{a}(\cos\beta + \sin\beta i)} = \overline{e^{w}}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \vec{x}' = A\vec{x} \quad$ char polynomial: $\lambda^2 - tr(a)\lambda + det(A)$

Case 2: $\Delta = tr(a) - 4\,det(A) < 0 \Rightarrow$ two complex roots

$\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}}{2} = \underbrace{\frac{1}{2}tr(A)}_{= a} \pm \underbrace{\frac{1}{2}\sqrt{-\Delta}}_{= b}\, i$

$\lambda = a + bi \quad \bar{\lambda} = a - bi$

complex conjugate roots of char poly

Q: How to solve $\underbrace{(\lambda I - A)}_{\text{complex matrix}}\vec{w} = 0$ (find eigenvectors for λ)

$(a+bi)I = \begin{bmatrix} a+bi & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & a+bi \end{bmatrix}$ to find a solution, set $\vec{w} = \begin{bmatrix} w_1 = \alpha_1 + \beta_1 i \\ w_2 = \alpha_2 + \beta_2 i \end{bmatrix}$

we will have a solution $\vec{w}$ associated w/ λ

$\overline{A\vec{w} = \lambda\vec{w}}$

$\overline{(A\vec{w})} = \overline{(\lambda\vec{w})}$

LHS $= \bar{A}(\overline{\vec{w}}) = A(\overline{\vec{w}}) = \bar{\lambda}(\overline{\vec{w}})$

real

$A(\overline{\vec{w}}) = \bar{\lambda}(\overline{\vec{w}})$

$\vec{w}_1 \quad \vec{w}_1$

consider $\vec{x}_1(t) = e^{\lambda t}\vec{w}$ are lin ind solutions

$\vec{x}_2(t) = e^{\bar{\lambda}t}\overline{\vec{w}} \leftarrow \overline{(e^{\lambda t})} = e^{\overline{\lambda t}} = e^{\bar{\lambda}t}$ b/c conjucation of real is itself

then $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$\vec{x}_1, \vec{x}_2$ are complex valued vectors

$c_1, c_2 \in \mathbb{C}$

Q: how to find two solutions $\vec{y}_1(t)$, $\vec{y}_2(t)$ real valued vectors

A: set $\vec{x}_1 = \vec{a}(t) + i\vec{b}(t)$

$\vec{x}_2 = \vec{a}(t) - i\vec{b}(t)$

$$\begin{bmatrix} e^t + \sin t \cdot i \\ 1 + \cos t \cdot i \end{bmatrix} = \begin{bmatrix} e^t \\ 1 \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$\vec{x}(t) = c_1(\vec{a}(t) + \vec{b}(t)i) + c_2(\vec{a}(t) - \vec{b}(t)i)$

$= (c_1 + c_2)\vec{a}(t) + (c_1 - c_2)(\vec{b}(t)i)$

1) $c_1 = c_2 = \frac{1}{2} \Rightarrow \vec{x}_1(t) = \vec{a}(t)$

2) $c_1 = -c_2 = -\frac{1}{2} \Rightarrow \vec{x}_2(t) = \vec{b}(t)i$

<u>thm</u> - general solution $x(t) = k_1 \cdot \underbrace{Re(e^{\lambda t}\vec{w})}_{\hookrightarrow \vec{a}(t)} + k_2 \underbrace{Im(e^{\lambda t}\vec{w})}_{\hookrightarrow \vec{b}(t)}$

$k_1, k_2 \in \mathbb{R}$

ex: $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda - 2 \end{bmatrix}$

$(\lambda)(\lambda - 2) + 2 = 0$

$\lambda^2 - 2\lambda + 2 = 0$ $\quad \Rightarrow \lambda = 1 + i, \bar{\lambda} = 1 - i$

$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad a, b \in \mathbb{C}$$

row 2 is a multiple of row 1 so we can ignore it

$(1+i)a - b = 0$

let $a = 1$, $b = 1+i$

$$\vec{x}_1 = e^{(1+i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \begin{bmatrix} e^t(\cos t + \sin t\, i) \\ e^t(\cos t + \sin t\, i)(1+i) \end{bmatrix}$$

$\vec{x}_2$ is conjugate of $\vec{x}_1$

$e^{(1+i)t} = e^t(\cos t + \sin t\, i)$

$$= \Bigg[\underbrace{\begin{pmatrix} e^t \cos t \\ e^t(\cos t + \sin t) \end{pmatrix}}_{\vec{y}_1(t)} + \underbrace{\begin{pmatrix} e^t \sin t\, i \\ e^t(\cos t + \sin t) i \end{pmatrix}}_{\vec{y}_2(t)} \Bigg]$$