12/05 - Review

$\vec{y}' = A\vec{y}$

- $e^{A} = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$ 　　　- $e^{A+B} = e^{A}e^{B} = e^{B}e^{A} = e^{B+A}$ if A, B commute
- $e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots$
- $\frac{d}{dt}\left(e^{tA}\right) = Ae^{tA}$
- if $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, then $e^{A} = \begin{bmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{bmatrix}$

<u>theorem</u>: for $\vec{y}' = A\vec{y}$, if $\vec{v}$ is any vector then $\vec{y}(t) = e^{tA}\vec{v}$ solves $\vec{y}' = A\vec{y}$

proof: $\vec{y}'(t) = Ae^{tA}\vec{v} = A\vec{y}$

then $c_1e^{tA}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \, \end{bmatrix}, c_2e^{tA}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \dots, c_ne^{tA}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ forms a fundamental set of solutions

but hard to compute e^{tA}

if $A^k\vec{v} = 0$ for some k, then by truncation

$$e^{tA}\vec{v} = \vec{v} + tA\vec{v} + \frac{1}{2!}t^2A^2\vec{v} + \frac{1}{3!}t^3A^3\vec{v} + \dots + \frac{1}{(k-1)!}t^{k-1}A^{k-1}\vec{v}$$

also, we know

$$tA = \lambda tI + t(A-\lambda I) = \lambda tI + tA - \lambda tI = tA$$

$$e^{tA}\vec{v} = e^{\lambda tI + t(A-\lambda I)}\vec{v} = e^{\lambda tI}e^{t(A-\lambda I)}\vec{v} = e^{\lambda t}I\,e^{t(A-\lambda I)}\vec{v}$$

$$= e^{\lambda t}e^{t(A-\lambda I)}\vec{v}$$

$$= e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \frac{1}{2!}t^2(A-\lambda I)^2\vec{v} + \frac{1}{3!}t^3(A-\lambda I)^3\vec{v} + \dots\right)$$

we can compute $e^{tA}\vec{v}$ when $(A-\lambda I)^k\vec{v} = 0$ for some k

$\downarrow$ generalized eigenvectors

ex: $\vec{y}' = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{bmatrix}\vec{y}$

$\det(A-\lambda I) = (\lambda+1)(\lambda+2)$

$\lambda = -1$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\vec{y}_1 = e^{-t}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\lambda = -2$

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 3 & -5 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ 　$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$(A-\lambda I)^2 = \begin{bmatrix} -2 & 4 & -2 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 　$\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ ← just pick one that's lin ind to $\vec{v}_2$

$\vec{y}_2 = e^{-2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{y}_3 = e^{-2t}\left(\vec{v}_3 + t(A+2I)\vec{v}_3 + \underline{\frac{1}{2!}t^2(A+2I)^2\vec{v}_3} + \dots\right)$

note: $\vec{v}_3 \in \ker(A-\lambda I)^2$
so $(A-\lambda I)^2\vec{v} = 0$

$= e^{-2t}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = e^{-2t}\begin{bmatrix} 1+t \\ t \\ -1+t \end{bmatrix}$