11/13 - 2×2 systems (cont)

case 3: $p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = (\lambda - \lambda_1)^2$

$\Leftrightarrow \Delta = (tr(A))^2 - 4det(A) = T^2 - 4D = 0$

subcases	dim. eigenspace	diagonalizable?		
case 3.1:	2	✓ $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$		$\vec{v}_1, \vec{v}_2$
case 3.2:	1	× $\begin{bmatrix} \lambda & a \\ 0 & \lambda_1 \end{bmatrix}$	$a \neq 0$	$\vec{v}_1$

case 3.1: $(A - \lambda_1 I)\vec{v} = 0$

$ker(A - \lambda_1 I)$ is eigenspace associated to λ_1, A

$dim(\mathbb{R}^2) = dim(ker(A - \lambda_1 I)) = 2 \Leftrightarrow (A - \lambda_1 \ )\vec{v} = 0$ for all $\vec{v} \in \mathbb{R}$

$\Leftrightarrow A - \lambda_1 I = 0 \Leftrightarrow A = \lambda_1 I \Leftrightarrow A$ is a diagonal matrix

$\vec{x}_1(t) = e^{\lambda_1 t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad \vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 = e^{\lambda_1 t}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\vec{x}_2(t) = e^{\lambda_1 t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

case 3.2: $dim(ker(A - \lambda_1 I)) = 1$, we have one eigenvector

thm: suppose $A_{2\times2}$ has an eigenvalue λ of multiplicity 2 and $dim(ker(A - \lambda_1 I)) = 1 \Leftrightarrow$ eigenspace has dim 1

1) let $\vec{v}_1$ be eigenvector

2) let $\vec{v}_2$ st $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1$

$A^2 - tr(A)A + det(A)I = 0$

$p(A) = 0$ by Cayley-Hamilton Thm

$(A - \lambda_1 I)^2 = 0$ matrix

$(A - \lambda_1 I)(A - \lambda_1 I)\vec{v} = 0$

$\Rightarrow (A - \lambda_1 I)\vec{v} \in ker(A - \lambda_1 I)$ for all $\vec{v} \in \mathbb{R}$

and $ker(A - \lambda_1 I) = span(\vec{v}_1)$ b/c $dim(ker(A - \lambda_1 I)) = 1$

$\Rightarrow (A - \lambda_1 I)\vec{v} = k\vec{v}_1$

hence, we can choose $\vec{v}$ lin. ind. to $\vec{v}_1$

$\Rightarrow (A - \lambda I)\vec{v} = k\vec{v}_1$; $k \neq 0$

choose $\vec{v}_2 = \frac{1}{k}\vec{v}$

proof is not required~~

then $\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$

$\vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$

fundamental set of solutions

ex: $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix}$

$p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$

$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

step 1 $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$ $\quad a - b = 0 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

step 2 pick $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(A+2I)\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1$

$\vec{v}_2 = \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_2(t) = e^{-2t}\begin{bmatrix} t \\ t-1 \end{bmatrix}$

↳ choose $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

or

$\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(A+2I)\vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \cdot \vec{v}_1$

$\vec{v}_2 = \frac{\vec{w}}{-1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

note: $\vec{v}_2^* = \vec{v}_2 + \alpha \vec{v}_1$ is still viable

ex: IVP

$\vec{y}' = A\vec{y} \quad A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \quad \vec{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$p(\lambda) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$

$A - \lambda I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$

$(A - 4I)\vec{v}_1 = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad a = 2b \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

let $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(A - 4I)\vec{w} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -1 \cdot \vec{v}_1$

$\vec{v}_2 = -\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\vec{y}_1(t) = e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{y}_2(t) = e^{4t}\begin{bmatrix} -1+2t \\ t \end{bmatrix}$