11/07 - Review

Ch 9 $\vec{y}' = A\vec{y}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $\vec{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \dots & & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ \vdots \\ a_{n1}y_1 + \dots + a_{nn}y_n \end{bmatrix}$$

def - given an $n \times n$ matrix A, we say that $\lambda \in \mathbb{C}$ is an eigenvalue and $\vec{v} \in \mathbb{C}^n$ is an eigenvector associated to λ (or in the eigenspace of λ) if $A\vec{v} = \lambda\vec{v}$

thm - if $\lambda, \vec{v}$ are an eigenvalue/vector pair for A, then $y(x) = e^{\lambda x}\vec{v}$ is a solution to $\vec{y}' = A\vec{y}$

proof - $$(e^{\lambda x}\vec{v})' = \left(e^{\lambda x}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right)' = \begin{bmatrix} e^{\lambda x}v_1 \\ e^{\lambda x}v_2 \\ \vdots \\ e^{\lambda x}v_n \end{bmatrix}' = \begin{bmatrix} \lambda e^{\lambda x}v_1 \\ \lambda e^{\lambda x}v_2 \\ \vdots \\ \lambda e^{\lambda x}v_n \end{bmatrix} = \lambda e^{\lambda x}\vec{v}$$

$$A(e^{\lambda x}\vec{v}) = e^{\lambda x}A(\vec{v}) = e^{\lambda x}(\lambda\vec{v}) = \lambda e^{\lambda x}\vec{v}$$

how to find eigenvalues

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0$

$A\vec{v} - \lambda I\vec{v} = 0$

$(A - \lambda I)\vec{v} = 0$

$\Downarrow$

$\det(A - \lambda I) = 0$

eigenvalues of A are solutions to $\det(A - \lambda I) = 0$

ex: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 + 3\lambda + 2$

$(\lambda + 2)(\lambda + 1)$

$\lambda = -1, -2$

eigenvectors: eigenspace of λ is the nullspace/kernel of $A - \lambda I$

ex(cont): $A - \lambda I = A + 2I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\vec{v} = 0$ $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$2v_1 + v_2 = 0$

$v_2 = -2v_1$

$A - \lambda I = A + I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\vec{v} = 0$ $v_1 + v_2 = 0$, $v_2 = -v_1$ $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ex(cont): Solve $\vec{y}' = A\vec{y}$; $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\vec{y}_1 = e^{-2x}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{y}_2 = e^{-x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{y} = c_1 e^{-2x}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-x}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

multiplying matrices

$$\begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} ag+bi+ck & ah+bj+cl \\ dg+ei+fk & dh+ej+fl \end{bmatrix}$$

<u>laplace expansion</u> (expansion upon minors)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

1) choose a row or column
2) use the signs (on right)
3) for each element, multiply by the determinant of matrix after removing the row and col

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

ex: row 2

$$-d\cdot\det\left(\begin{bmatrix} b & c \\ h & i \end{bmatrix}\right) + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

ex: col 1

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

more facts about eigenvalues

- characteristic polynomial factors as $\det(A-\lambda I) = c(\lambda-\lambda_1)^{t_1}(\lambda-\lambda_2)^{t_2}\dots(\lambda-\lambda_k)^{t_k}$
- $t_1+t_2+\dots+t_k = n$
- <u>algebraic multiplicity</u> of λ_i is t_i
- <u>geometric multiplicity</u> of λ_i is dimension of its eigenspace

  $1 \le$ geometric multiplicity $\le$ algebraic multiplicity