11/19 - Review Thing

2x2 Systems $\vec{y}\,' = A_{2\times 2}\vec{y} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- eigenvalues - solve $\det(A - \lambda I) = 0$ or $\det(\lambda I - A) = 0$ (both work)
- eigenvectors - solve $(A - \lambda I)\vec{v} = 0$ for each eigenvalue λ
- trace - $tr(A) = a + d$ = sum of ↘ diagonal
- determinant - $\det(A) = ad - bc$
- solving for eigenvalues - $p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$

case 1: $tr(A)^2 - 4\det(A) > 0 \Rightarrow$ two distinct real roots λ_1, λ_2

- yay, solve for $\vec{v}_1, \vec{v}_2$

$$\boxed{y = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2}$$

case 2: $tr(A)^2 - 4\det(A) < 0 \Rightarrow$ two imaginary roots $\lambda, \bar{\lambda}$

- $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$
- Solve for $\vec{v}$ with λ (eigenvector for $\bar{\lambda}$ will be $\bar{\vec{v}}$ so that works too)
- separate into real and imaginary components
  $\vec{v} = \vec{v}_1 + i\vec{v}_2$
- $\vec{z} = e^{\lambda t}\vec{v} = e^{(\alpha + \beta i)t}(\vec{v}_1 + i\vec{v}_2) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{v}_1 + i\vec{v}_2)$
  $= e^{\alpha t}(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2 + i\sin\beta t\,\vec{v}_1 + i\cos\beta t\,\vec{v}_2)$
  $= \underbrace{e^{\alpha t}(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2)}_{y_1} + i\underbrace{e^{\alpha t}(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2)}_{y_2}$
- $$\boxed{\vec{y} = c_1 e^{\alpha t}(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2) + c_2 e^{\alpha t}(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2)}$$

case 3: $tr(A)^2 - 4\det(A) = 0 \Rightarrow$ one repeated real root λ

- case 3.1: A is a diagonal matrix $\left(A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right)$

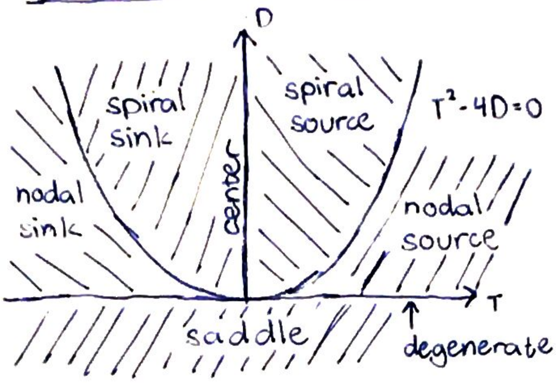
$$\boxed{\vec{y} = c_1 e^{\lambda t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

you don't have to use $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ but why make life difficult?

- case 3.2: A is not a diagonal matrix
  - solve for $\vec{v}_1$ with λ
  - pick a $\vec{w}$ that is lin. ind. with $\vec{v}_1$ ($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are nice)
  - solve for k: $(A - \lambda I)\vec{w} = k\vec{v}_1$
  - then $\boxed{\vec{v}_2 = \frac{\vec{w}}{k}}$

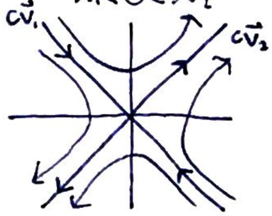
$$\boxed{\vec{y} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)}$$

Phase Plane Portraits

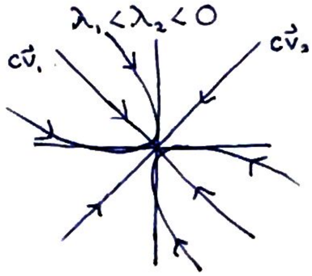


case 1: two real eigenvalues

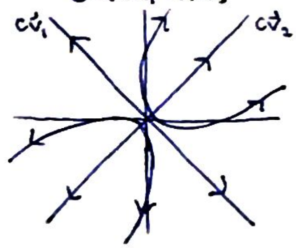
saddle point (G)
$\lambda_1 < 0 < \lambda_2$



nodal sink (G)
$\lambda_1 < \lambda_2 < 0$

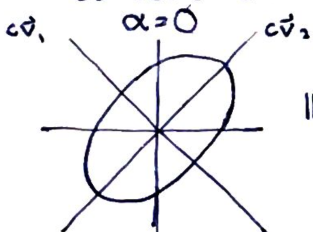


nodal source (G)
$0 < \lambda_1 < \lambda_2$



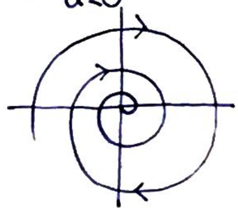
case 2: two complex eigenvalues

center (NG!)
$\alpha = 0$



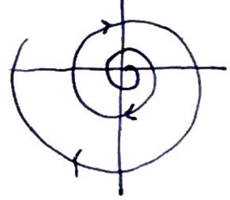
$\|\vec{v}_2\| > \|\vec{v}_1\|$

spiral sink (G)
$\alpha < 0$



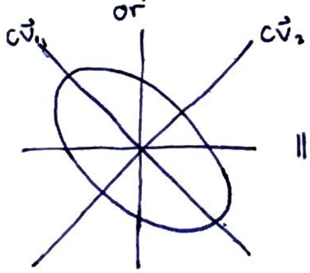
$a_{21} < 0$

spiral source (G)
$\alpha > 0$



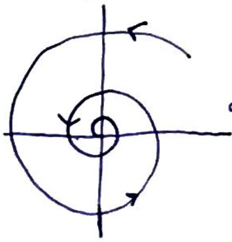
$a_{21} < 0$

or



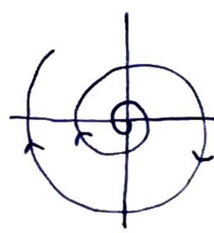
$\|\vec{v}_1\| > \|\vec{v}_2\|$

or



$a_{21} > 0$

or



$a_{21} > 0$