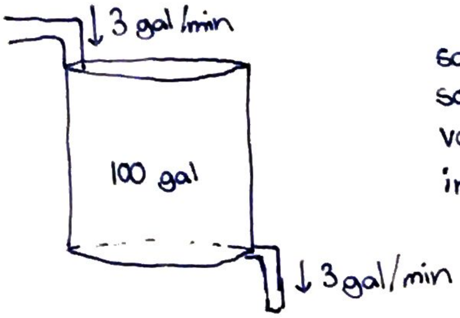
10/07 - Mixing Problem (Section 2.5)

ex:



salt solution (2 lb/gal) enter tank @ 3 gal/min

solution exits tank @ 3 gal/min

vol of liquid inside tank remains constant @ 100 gal

implicit assumption - perfect mixing

- the concentration of the solution in the drain = concentration of the solution in the tank

initial condition - at time $t=0$, tank contains 100 gal pure water (0 lb salt)

Question: How much salt is in the tank after 60 min

$x(t)$ = amount of salt in tank at time t

$x'(t) = \frac{dx}{dt}$ = amount of salt entering tank - amount of salt leaving tank

entering = concetration * rate = 2 lb/gal · 3 gal/min = 6 lb/min

leaving = concentration * rate = $x(t)$ lb/100 · 3 gal/min = $3x(t)$ lb/100 min

$$\frac{dx}{dt} = 6 - 3x(t)/100$$

↳ DE yay!

homogeneous solution - $x_h(t) = e^{-3/100\, t} + c$

↳ we choose c to be 0

solving this myself~~

$$x(t) = v(t)\, x_h(t)$$

$$x'(t) = v'(t)x_h(t) + v(t)\, x_h'(t)\, dt$$

$$v'(t)e^{-3/100\, t} + \cancel{v(t)\left(-\tfrac{3}{100}e^{-3/100\, t}\right)} = 6 - \cancel{v(t)\left(-\tfrac{3}{100}e^{-3/100\, t}\right)}$$

$$v'(t) = 6e^{3t/100}$$

$$v(t) = 200e^{3t/100} + c$$

$$x(t) = 200 + ce^{-3t/100}$$

$$x(0) = 0 = 200 + c \rightarrow c = -200$$

$$x(t) = 200 - 200e^{-3t/100}$$

Guess what? It's a lot easier to get the solution

$x_h = e^{-3t/100}$ homogeneous solution

x_p = any particular solution

assume at t_0 $x(t) = 200$, then $x(t)$ is always 200 so $x_p = 200$

general solution = $x(t) = cx_h + x_p$

$$x(t) = ce^{-3t/100} + 200$$

$$0 = c + 200 \rightarrow c = -200$$

$$\boxed{x(t) = -200e^{-3t/100} + 200}$$

$$-200e^{-180/100} + 200 = -200e^{-1.8} + 200$$

experiment

$$x(t) = cx_h + x_p$$

$$= ce^{-3t/100} + 200 - 200e^{-3t/100}$$

$$0 = c + 200 - 200$$

$$c = 0$$

Exact Equation

Review - $x(t)$, $y(t)$ are 1-var functions

$w(x,y)$ is 2-var function

$w(t) = w(x(t), y(t))$ is 1-var function

$w'(t) = \frac{d}{dt} w(t) = \frac{dw}{dt}$
$= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$
$\hookrightarrow dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$

ex: $w(x,y) = e^{x^2+y^2}$

$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy = 2xe^{x^2+y^2} dx + 2ye^{x^2+y^2} dy$

suppose $w(x,y) = c \rightarrow dw = 0 \rightarrow 2xe^{x^2+y^2} dx + 2ye^{x^2+y^2} dy = 0$

we consider a DE:

$P(x,y)\,dx + Q(x,y)\,dy = 0$ (E)

def - we say that the equation (E) is an exact DE if there exists ($\exists$)
a two variable function $w(x,y)$ such that
$\frac{\partial w}{\partial x} = P(x,y)$ and $\frac{\partial w}{\partial y} = Q(x,y)$

$\hookrightarrow w(x,y) = c$ is the general solution of the exact equation

Question - Given $P(x,y)dx + Q(x,y)dy = 0$, how do we check if it is exact?

Theorem - suppose $P(x,y)$, $Q(x,y)$ is continuously differentiable
aka $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, and $\frac{\partial Q}{\partial y}$ are differentiable
then the equation is exact if and only if (I think?) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

ex: $(x^2+y^2)\,dx + 2xy\,dy = 0$

$P(x,y) = x^2+y^2 \qquad \frac{\partial P}{\partial y} = 2y$

$Q(x,y) = 2xy \qquad \frac{\partial Q}{\partial x} = 2y$

$\Rightarrow$ Equation is exact