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solve for y_1, y_2

$y_1' = y_1 + 2y_2$
$y_2' = 3y_1 + 2y_2$
$y_1(0) = 17$
$y_2(0) = 18$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y' = Ay \;;\; A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)(2-\lambda) - 6$$
$$= \lambda^2 - 3\lambda + 2 - 6$$
$$= \lambda^2 - 3\lambda - 4$$
$$= (\lambda - 4)(\lambda + 1)$$
$$\lambda = 4, \lambda = -1$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$17 = c_1 \cdot 2 + c_2 \cdot 1$
$18 = 3c_1 - c_2$
$5c_1 = 35$
$c_1 = 7$
$c_2 = 3$

$$\vec{y} = 7e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 14e^{4t} + 3e^{-t} \\ 21e^{4t} - 3e^{-t} \end{bmatrix}$$

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rules:

1) If $(\lambda, \vec{v})$ is an eigenvalue/vector for A, then $\vec{y}(t) = e^{\lambda t}\vec{v}$ solves $\vec{y}' = A\vec{y}$
2) solutions to $\vec{y}' = Ay$ form a vector space, thus linear combinations of the solutions found by rule 1 are also solutions

case 1: A has two real eigenvalues λ_1 and λ_2 with eigenvectors $\vec{v}_1, \vec{v}_2$

solution: $\vec{y}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

case 2: A has two complex eigenvalues $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ with eigenvectors $\vec{w} = \vec{w}_1 + i\vec{w}_2$ and $\bar{\vec{w}} = \vec{w}_1 - i\vec{w}_2$

Note: $\vec{v}_1 = \vec{w}$, $\vec{v}_2 = \bar{\vec{w}}$

solution: $\vec{y}(t) = c_1 e^{at}((\cos bt)\vec{w}_1 - (\sin bt)\vec{w}_2) + c_2 e^{at}((\sin bt)\vec{w}_1 + (\cos bt)\vec{w}_2)$

case 3: A has one eigenvalue λ with geometric multiplicity 2, with linearly independent $\vec{v}_1, \vec{v}_2$

then A must be of the form $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

solution: $\vec{y}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$

case 4: A has one eigenvalue λ with geometric multiplicity 1, with eigenvector $\vec{v}$

pick $\vec{v}_2$ such that $(A - \lambda I)\vec{v}_2 = \vec{v}$

pick a $\vec{w}$ lin ind from $\vec{v}$, $(A - \lambda I)\vec{w} = k\vec{v}$, so $\vec{v}_2 = \frac{\vec{w}}{k}$

solution: $\vec{y}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}[\vec{v}t + \vec{v}_2]$

ex: $\vec{y}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \vec{y}$ $\qquad \vec{y}(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 + 65$$

$$\lambda = \frac{4 \pm \sqrt{16-272}}{2} = \frac{4 \pm \sqrt{-256}}{2} = \boxed{2 \pm 8i}$$

since $1 \le$ geo mult $\le$ alg mult and alg mult $= 1$, then geo mult $= 1$ and dim of eigenspace $= 1$ so both eq give same sols

choose 1

$$\begin{bmatrix} 3-2-8i & -13 \\ 5 & 1-2-8i \end{bmatrix} \vec{v} = \begin{bmatrix} 1-8i & -13 \\ 5 & -1-8i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow 5v_1 = (1+8i)v_2 \Rightarrow v_1 = \frac{(1+8i)v_2}{5}$$

$$\Rightarrow v_1 = (1+8i), \quad v_2 = 5$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 1+8i \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix} i}$$

proof: do rref

$(1-8i)(1+8i)$	$-13(1+8i)$
$1+64$	$13(-1-8i)$
$\frac{65}{13}$	$\frac{13}{13}(-1-8i)$
5	$-1-8i$

some method I don't quite understand

$$\vec{z}(t) = e^{\lambda t}\vec{v} = e^{(2+8i)t}\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right)$$

$$= e^{2t}(\cos 8t \pm i \sin 8t)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right)$$

$$= e^{2t}\left(\cos 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right) + ie^{2t}\left(\sin 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right)$$

$$\bar{\vec{z}}(t) = e^{2t}\left(\cos 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right) - ie^{2t}\left(\sin 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right)$$

or just memorize the rule

$$\vec{y}(t) = c_1 e^{2t}\left(\cos 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \sin 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin 8t \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \cos 8t \begin{bmatrix} 8 \\ 0 \end{bmatrix}\right)$$