10/25 - undetermined coefficient (cont)

ex: $y'' - y' - 2y = 3e^{-t}$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda_1 = 2 \quad \lambda_2 = -1$

$y_1 = e^{2t} \quad y_2 = e^{-t}$

homogeneous y_h

$y_p = ke^{-t}$

$y_p' = -ke^{-t}$

$y_p'' = ke^{-t}$

$(k + k - 2k)e^{-t} = 3e^{-t}$

$0 = 3$

× nope!

note: $y_p = ke^{2t}$ doesn't work either

try - set $y_p = tke^{-t}$

$y_p' = -kte^{-t} + ke^{-t}$

$y_p'' = kte^{-t} - 2ke^{-t}$

$k(t-2)e^{-t} - k(1-t)e^{-t} - 2kte^{-t} = 3e^{-t}$

$k(t - 2 - 1 + t - 2t) = 3$

$-3k = 3$

$k = -1$

$y_p = -e^{-t}$

<u>theorem</u> - if ① $y_f'' + py_f' + qy_f = f$

② $y_g'' + py_g' + qy_g = g$

$(y_f + y_g)'' + p(y_f + y_g)' + q(y_f + y_g) = f + g$

$y_p'' + p\, y_p' + q\, y_p = f + g$

then $y_p = y_f + y_g$ is a solution to $y'' + py' + qy = f + g$

ex: $y'' - y' - 2y = e^{-2t} - 3e^{-t}$

① $y'' - y' - 2y = e^{-2t} \Rightarrow y_f = \frac{1}{4}e^{-2t}$

② $y'' - y' - 2y = 3e^{-t} \Rightarrow y_g = te^{-t}$

yea he just gave us these

$y_p = y_f + y_g = \frac{1}{4}e^{-2t} + te^{-t}$

variation of parameters

recall $y' + p(t)y = f(t)$ $\quad y_p = v(t)y_h$ $\quad y_h$ is solution for $y'+py=0$

2nd $\quad y_h'' + py_h' + qy_h = 0$ $\quad y''+py'+qy = g(t)$

$y_h = c_1 y_1(t) + c_2 y_2(t)$ $\quad y_p = v_1(t)y_1 + v_2(t)y_2$

result - how to solve $v_1 + v_2$ $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ = Wronskian

$$v_1 = \int \frac{-y_2(t)g(t)}{y_1y_2' - y_1'y_2}\,dt = \int \frac{-y_2(t)g(t)}{W(t)}\,dt$$

$$v_2 = \int \frac{y_1(t)g(t)}{y_1y_2' - y_1'y_2}\,dt = \int \frac{y_1(t)g(t)}{W(t)}\,dt$$

just memorize I guess

ex: $y'' + y = \tan t$ $\quad y_p = v_1y_1 + v_2y_2$

$y_h'' + y_h = 0$ $\quad = v_1\cos t + v_2 \sin t$

$\lambda^2 + \lambda = 0$ $\quad W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$\lambda = \pm i$

$\alpha = 0 \quad \beta = 1$

$y_1 = \cos t \quad y_2 = \sin t$

$$v_1 = \int -\sin t \tan t\,dt = \int -\frac{\sin^2 t \cos t}{\cos^2 t}\,dt$$

$u = \sin t$
$du = \cos t\, du$

$$= \int \frac{-u^2}{1-u^2}\,du = \int \frac{1-u^2-1}{1-u^2}\,du = \int 1 - \frac{1}{1-u^2}\,du$$

$$= \int 1 - \frac{1}{(1+u)(1-u)}\,du = \int 1 - \frac{1}{2(1+u)} - \frac{1}{2(1-u)}\,du$$

I skipped a lot of steps
$Au - Bu = 0$
$A + B = 1$

$$= u - \frac{1}{2}\ln|1+u| + \frac{1}{2}\ln|1-u|$$

$$= u + \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| = \sin t + \frac{1}{2}\ln\left|\frac{(1-\sin t)^2}{\cos^2 t}\right| = \sin t + \ln|\sec t - \tan t|$$

note - textbook says $\sin t - \ln|\sec t + \tan t|$

$$= \sin t + \ln\left|\frac{1}{\frac{1}{\cos t} + \frac{\sin t}{\cos t}}\right|$$

$$= \sin t + \ln\left|\frac{\cos t}{1+\sin t}\right|$$

$$= \sin t + \ln\left|\frac{\cos t - \cos t\sin t}{1-\sin^2 t}\right|$$

$$= \sin t + \ln\left|\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right|$$

$$= \sin t + \ln|\sec t - \tan t|$$

easier way to solve v_1

$$v_1 = -\int \sin t \tan t \, dt = -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{1-\cos^2 t}{\cos t} dt$$

$$= \int -\sec t + \cos t \, dt = -\underbrace{\ln|\tan t + \sec t|}_{\int \sec t \, dt} + \sin t$$

$$v_2 = \int \cos t \tan t \, dt = \int \sin t \, dt = -\cos t$$

$$y_p = v_1 y_1 + v_2 y_2 = (\sin t - \ln|\tan t + \sec t|)(\cos t) + (-\cos t)(\sin t)$$

$$= -\cos t \ln|\tan t + \sec t|$$

was it $y = c_1 y_1 + c_2 y_2 + y_p$?

general solution is $y = c_1 \cos t + c_2 \sin t - \cos t \ln|\tan t + \sec t|$

#orvhalloweek