10/02 Class Summary - Integrating Factors

Given a first order differential equation in the form

$$y'(t) = a(t)\,y(t) + f(t)$$

How do you solve it?

$$y'(t) - a(t)y(t) = f(t)$$

Goal - find a $u(t)$ such that

$$u(t)\underbrace{\left(y'(t) - a(t)y(t)\right)}_{\text{left hand side}} = \left(u(t)y(t)\right)'$$

$$\boxed{u(t) = e^{-\int a(t)dt}}$$

Proof

$$e^{-\int a(t)dt}\left(y'(t) - a(t)y(t)\right) = \left(e^{-\int a(t)dt}y(t)\right)'$$
$$= \left(e^{-\int a(t)dt}\right)'y(t) + \left(e^{-\int a(t)dt}\right)y'(t)$$
$$= -a(t)e^{-\int a(t)dt}y(t) + e^{-\int a(t)dt}y'(t)$$
$$= e^{-\int a(t)dt}\left(y'(t) - a(t)y(t)\right)$$

---

ex from Snyder:

$$2y' = t - 4y \rightarrow y' + 2y = \tfrac{1}{2}t$$

$$a(t) = -2$$

$$u(t) = e^{-\int -2dt} = e^{2t} = \text{integrating factor}$$

$$(e^{2t})(y' + 2y) = \tfrac{1}{2}t(e^{2t})$$

$$e^{2t}y' + 2e^{2t}y = \tfrac{1}{2}t(e^{2t})$$

$$(e^{2t}y)' = \tfrac{1}{2}t(e^{2t})$$

reverse product rule

$$\int (e^{2t}y)'dt = \int \tfrac{1}{2}t(e^{2t})dt$$

$u = \frac{1}{2}t$ $\quad dv = e^{2t}dt$

$du = \frac{1}{2}dt$ $\quad v = \frac{1}{2}e^{2t}$

$$e^{2t}y = \tfrac{1}{2}t\cdot\tfrac{1}{2}e^{2t} - \int \tfrac{1}{2}e^{2t}\cdot\tfrac{1}{2}dt$$

$$e^{2t}y = \tfrac{1}{4}te^{2t} - \tfrac{1}{8}e^{2t} + c$$

$$\boxed{y = \tfrac{1}{4}t - \tfrac{1}{8} + ce^{-2t}}$$

Idk why you want to use this method but feel free $\longrightarrow$

same ex, diff method (Coca-Cola Method, Huynh)

1. solve homogeneous part $(y' = a(t)y)$

$$2y' = t - 4y \rightarrow y' = -2y + \tfrac{1}{2}t$$

$$z' = -2z$$

I'll use z to avoid confusion

$$\frac{dz}{z} = -2dt$$

$$\ln|z| = -2t$$

$$z = c'e^{-2t}$$

2. find a $u(t)$ such that $u(t)$ times your previously found solution will be a constant.

$u(t) = e^{2t}$ b/c $\underbrace{c'e^{-2t}(e^{2t}) = c'}_{\text{constant}}$

3. expand using the full original equation

$$(e^{2t}y(t))' = 2e^{2t}y(t) + e^{2t}y'(t)$$
$$= 2e^{2t}y(t) + e^{2t}(-2y + \tfrac{1}{2}t)$$
$$= \tfrac{1}{2}te^{2t}$$

$$\int (e^{2t}y(t))'dt = \int \tfrac{1}{2}te^{2t}dt$$

$$e^{2t}y(t) = \tfrac{1}{4}te^{2t} - \tfrac{1}{8}e^{2t} + c$$

$$\boxed{y = \tfrac{1}{4}t - \tfrac{1}{8} + ce^{-2t}}$$