11/06 - 2x2 Systems

DE - $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Q - how to solve?

$A \to T^{-1}AT$

Classification of nxn matrix

1. two distinct eigenvalues (real) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1, \lambda_2 \in \mathbb{R}$
2. complex eigenvalues $(\lambda, \bar{\lambda})$ $\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} = Q^{-1}AQ$
3. one eigenvalues (real) $\begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$ $Q = [q_{ij}]_{n\times n}$; $q_{ij} \in \mathbb{C}$

Determine the Case

check $\det(\lambda I - A)$

trace(A) = sum of diagonal

$$= \det\left(\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}\right) = (\lambda - a)(\lambda - d) - bc$$
$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$
$$= \lambda^2 - (a-d)\lambda + (ad - bc)$$
$$= \lambda^2 - tr(A)\lambda + \det(A)$$

discriminant $\Delta = (tr(A))^2 - 4\det(A)$

1) $\Delta > 0 \Rightarrow$ 2 distinct real eigenvalues
2) $\Delta < 0 \Rightarrow$ 2 complex conjugate eigenvalues
3) $\Delta = 0 \Rightarrow$ 1 real eigenvalues

thm - $e^{\lambda_1 t}\vec{v}_1(t) = \vec{x}_1(t)$ and $e^{\lambda_2 t}\vec{v}_2(t) = \vec{x}_2(t)$ are lin. ind.

proof - by the prop, we only have to show that when $t=0$

$\vec{x}_1(0)$ and $\vec{x}_2(0)$ are lin ind

lemma - then $\vec{v}_1, \vec{v}_2$ are lin. ind.

proof by contradiction

suppose $\vec{v}_1, \vec{v}_2$ are lin dep $\Leftrightarrow$ $\vec{v}_2 = c_1\vec{v}_1$, $\vec{v}_1 = c_2\vec{v}_2$

without loss of generality (WLOG) assume $\vec{v}_1 = c\vec{v}_2$

1) $A\vec{v}_1 = \lambda\vec{v}_1$ $\lambda_1\vec{v}_1 = \lambda_2\vec{v}_1$

2) $A\vec{v}_1 = A(c\vec{v}_2) = c(A\vec{v}_2)$ so $\vec{v}_1 = 0$

$= c(\lambda_2\vec{v}_2)$ contradiction!

$= \lambda_2(c\vec{v}_2)$ $\vec{v}_1, \vec{v}_2$ are lin ind.

$= \lambda_2\vec{v}_1$

ex: $\vec{x}' = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}$

step 1: $\lambda I - A = \begin{bmatrix} \lambda+5 & -1 \\ 2 & \lambda+2 \end{bmatrix}$

$$\begin{aligned} d(\lambda I - A) &= (\lambda+5)(\lambda+2) + 2 \\ &= \lambda^2 + 7\lambda + 12 \\ &= (\lambda+4)(\lambda+3) \qquad \lambda_1 = -4,\ \lambda_2 = -3 \end{aligned}$$

step 2: find $\vec{v}_1, \vec{v}_2$

$\vec{v}_1$: $\begin{bmatrix} -4+5 & -1 \\ 2 & -4+2 \end{bmatrix} \vec{v}_1 = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} -3+5 & -1 \\ 2 & -3+2 \end{bmatrix} \vec{v}_2 = 0 \Rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \Rightarrow 2a_1 = b_1 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

step 3: $\vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1(t) = e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}$

$\vec{x}_2(t) = e^{\lambda_2 t} \vec{v}_2(t) = e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ 2e^{-3t} \end{bmatrix}$

$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$