11/21 - systems of more than 2 equations

still true - if $\lambda_1, \vec{v}$ is an eigenvalue/eigenvector for A, then $\vec{y}(t) = e^{\lambda t}\vec{v}$ solves $\vec{y}' = A\vec{y}$

also true - for A un nxn matrix, the solutions to $\vec{y}' = A\vec{y}$ form an n-dimensional vector space

ex: $\vec{y}' = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}\vec{y}$

$\det(A-\lambda I) = (\lambda-3)(\lambda+5)(\lambda-6)$

$$\left.\begin{matrix} \lambda_1 = 3 & \lambda_2 = -5 & \lambda_3 = 6 \\ \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} & \vec{V}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} & \vec{V}_3 = \begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix} \end{matrix}\right] \text{given}$$

$$\boxed{\vec{y}(t) = c_1 e^{3t}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{6t}\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix}}$$

ex: $\vec{y}' = \begin{bmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}\vec{y}$

$$\det(A-\lambda I) = \begin{bmatrix} -4-\lambda & 8 & 8 \\ -4 & 4-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0 - 0 + (2-\lambda)\det\begin{bmatrix} -4-\lambda & 8 \\ -4 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)(\lambda^2+16)$$

$\lambda_1 = 2,\ \lambda_2 = 4i,\ \lambda_3 = -4i$

$$\begin{bmatrix} -4-2 & 8 & 8 \\ -4 & 4-2 & 2 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -6 & 8 & 8 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

do rref

$$\begin{bmatrix} -6 & 8 & 8 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_2} \begin{bmatrix} -6 & 8 & 8 \\ 6 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} -6 & 8 & 8 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} -6 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-8R_2} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_1 = 0,\ v_2 = -v_3 \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -4-4i & 8 & 8 \\ -4 & 4-4i & 2 \\ 0 & 0 & 2-4i \end{bmatrix} \xrightarrow[\frac{1}{2-4i}R_3]{\frac{1}{2}R_1,\ \frac{1}{2}R_2} \begin{bmatrix} -2-2i & 4 & 4 \\ -2 & 2-2i & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} -2i & 2+2i & 3 \\ -2 & 2-2i & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{iR_1} \begin{bmatrix} 2 & 2i-2 & 3i \\ -2 & 2-2i & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} 2 & 2i-2 & 3i \\ 0 & 0 & 3i+1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{stuff}} \begin{bmatrix} 2 & 2i-2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i-1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1-i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{y}(t) = c_1 e^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{bmatrix} + C_3\begin{bmatrix} \sin 4t - \cos 4t \\ \sin 4t \\ 0 \end{bmatrix}}$$

Preview

tricky part - if λ_0 has an algebraic mult of m

$$\det(A-\lambda I) = (\lambda-\lambda_0)^m$$

we should be able to find m linearly independent solutions with $e^{\lambda_0 t}$

if geo mult $< m$, how do we find the rest of the solutions?

given an eigenvalue λ with alg mult m, there are m solutions

$\vec{y}_1(t) = e^{\lambda t}\vec{w}_1$ where $(A-\lambda I)\vec{w}_1 = 0$

$\vec{y}_2(t) = e^{\lambda t}(t\vec{w}_1 + \vec{w}_2)$ where $(A-\lambda I)\vec{w}_2 = \vec{w}_1$

$\vec{y}_3(t) = e^{\lambda t}(\frac{1}{2}t^2\vec{w}_1 + t\vec{w}_2 + \vec{w}_3)$ where $(A-\lambda I)\vec{w}_3 = \vec{w}_2$

$\vec{y}_4(t) = e^{\lambda t}(\frac{1}{6}t^3\vec{w}_1 + \frac{1}{2}t^2\vec{w}_2 + t\vec{w}_3 + \vec{w}_4)$ where $(A-\lambda I)\vec{w}_4 = \vec{w}_3$

$\vdots$ $\vdots$

$\vec{y}_m(t) = e^{\lambda t}(\frac{1}{(m-1)!}t^{(m-1)}\vec{w}_1 + \dots + t\vec{w}_{m-1} + \vec{w}_m)$ where $(A-\lambda I)\vec{w}_m = \vec{w}_{m-1}$

2 questions:

- why?
- how to find $\vec{w}_i$?

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots$

$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$

$\vec{y}(t) = e^{tA}$

$\vec{y}'(t) = Ae^{tA}$