12/02 - exponential of matrix

general strategy: for a given matrix $A_{n\times n}$, find the fundamental set $\vec{y}' = A\vec{y}$

<u>step 1</u>: find char polynomial

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_q)^{k_q}$$ ← algebraic multiplicity

<u>step 2</u>: find the linearly independent solutions associated to λ_i

method 1: i) find basis of $\ker(A - \lambda_i I)$

$\vec{v}_{i1}, \vec{v}_{i2}, \dots, \vec{v}_{ir}$

$\vec{x}_{ij} = e^{tA}\vec{v}_{ij}$ $(j = 1, 2, \dots, r)$ is a solution

$= e^{\lambda_i t}\vec{v}_{ij}$ by truncation

ii) consider $\ker(A - \lambda_i I)^2$

note: $\dim(\ker(A - \lambda_i I)^2) = \dim(\ker(A - \lambda_i I)) + 1$

find $\vec{v}_{ir+1} \in \ker(A - \lambda_i I)^2 \setminus \ker(A - \lambda_i I)$

$\vec{x}_{ir+1} = e^{tA}\vec{v}_{ir+1} = e^{\lambda_i t}(\vec{v}_{ir+1} + t(A - \lambda_i I)\vec{v}_{ir+1})$

iii) if $r+1 < k_i$

$\ker(A - \lambda_i I)^3$ b/c $\dim(\ker(A - \lambda_i I)^3) = \dim(\ker(A - \lambda_i I)) + 1$

find $\vec{v}_{ir+2} \in \ker(A - \lambda_i I)^3 \setminus \ker(A - \lambda_i I)^2$

$\vec{x}_{ir+2} = e^{t\lambda_i}(\vec{v}_{ir+2} + t(A - \lambda_i I)\vec{v}_{ir+2} + \frac{1}{2}t^2(A - \lambda_i I)^2\vec{v}_{ir+2})$

⋮

until you're done

method 2: i) find the smallest p_i such that

$\dim(\ker(A - \lambda_i I)^{p_i}) = k_i$ $(p_i <= k_i)$

ii) find basis of $\ker(A - \lambda_i I)^{p_i}$

$\vec{v}_{i1}, \vec{v}_{i2}, \dots, \vec{v}_{ik}$

then $\vec{x}_{ij} = e^{tA}\vec{v}_{ij}$

$= e^{t\lambda_i}(\vec{v}_{ij} + t(A - \lambda_i I)\vec{v}_{ij} + \dots + \frac{t^{p-1}}{(p-1)!}(A - \lambda_i I)^{p-1}\vec{v}_{ij})$

ex: $A = \begin{bmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & 1 \end{bmatrix}$

$$\det(A) = -0 \begin{vmatrix} 5 & -3 & 2 \\ 10 & -5 & 4 \\ -4 & 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 7 & -3 & 2 \\ 12 & -5 & 4 \\ -4 & 2 & 1 \end{vmatrix} - 0 + 0$$

$$= 7 \begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 12 & 4 \\ -4 & 1 \end{vmatrix} + 2 \begin{vmatrix} 12 & -5 \\ -4 & 2 \end{vmatrix}$$

$$= 7(-5-8) + 3(12+16) + 2(24-20)$$

$$= -91 + 84 + 8 = 1$$

$$\det(A-\lambda I) = (1-\lambda) \begin{vmatrix} 7-\lambda & -3 & 2 \\ 12 & -5-\lambda & 4 \\ -4 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\Big((7-\lambda)((-5-\lambda)(1-\lambda)-8)$$
$$+ 3(12-\lambda+16)$$
$$+ 2(24-20-4\lambda)\Big)$$

char poly $= \lambda^4 - 2\lambda^3 + 2\lambda - 1 = (\lambda+1)(\lambda-1)^3$

$\lambda_1 = -1$ ker $(A+I)$ is 1 dim $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$

$\lambda_1 = 1$ $\begin{bmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 12 & 10 & -6 & 4 \\ -4 & -4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -4 & 2 & -2 \end{bmatrix}$ rank $(A-\lambda I) = 2$
so $\dim(A-\lambda I) = 2$

$6a + 5b - 3c + 2d = 0$
$-4a - 4b + 2c - 2d = 0$

1 0 2 0
1 -2 0 2