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integrating factor ver 2

$y' = ay + b$ $\qquad$ $y_h' = ay_h$ $\qquad$ $v = \frac{y}{y_h} \rightarrow vy_h = y$

$y' - ay = b$ $\qquad$ $y_h = e^{\int a(x)dx}$ $\qquad$ $(vy_h)' = ay + b$

$y_h' = ae^{\int a(x)dx}$

$vy_h' + v'y_h = avy_h + b$

$\cancel{vay_h} + v'y_h = \cancel{avy_h} + b$

$v' = \frac{b}{y_h}$

$P(x,y)dx + Q(x,y)dy = 0$ or $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

ex: $(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$

solution: $\underbrace{y^2 + (x^2+1)y - 3x^3}_{F(x,y)} = C$

exactness check:
$P = 2xy - 9x^2$
$P_y = 2x$
$Q = 2y + x^2 + 1$ ✓
$Q_x = 2x$

$\frac{\partial F}{\partial x} = F_x = 2xy - 9x^2 = P(x,y)$

$\frac{\partial F}{\partial y} = F_y = 2y + x^2 + 1 = Q(x,y)$ $\Longrightarrow$ $F_x + F_y\frac{dF}{dx} = 0$

$\frac{d}{dx}(F(x, y(x))) = 0$

So to solve $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$, we need to find an F such that $F_x = P(x,y)$ and $F_y = Q(x,y)$

Then the solution is $F(x,y) = C$

let's actually solve the example

1) $F_x = 2xy - 9x^2$ $\qquad$ $F = \int 2xy - 9x^2 dx = x^2y - 3x^3 + \Phi(y)$ $\quad \hookrightarrow y^2 + y$

$F_y = 2y + x^2 + 1$ $\qquad$ $F = \int 2y + x^2 + 1 dy = y^2 + x^2y + y + \Phi(x)$ $\quad \hookrightarrow -3x^3$

$F = x^2y - 3x^3 + y^2 + y = C$

2) $F_x = 2xy - 9x^2$ $\qquad$ $F = \int 2xy - 9x^2 dx = x^2y - 3x^3 + \Phi(y)$

$F_y = \cancel{x^2} + \Phi'(y) = 2y + \cancel{x^2} + 1$

$\Phi'(y) = 2y + 1$

$\Phi(y) = y^2 + y$

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def - if there is a function F with $F_x = P$ and $F_y = Q$ then $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$ is <u>exact</u>

<u>Clairaut's Theorem</u> - If F, F_x, F_y are all continuously differentiable then $F_{xy} = F_{yx}$

so if $\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q \Rightarrow P_y = Q_x \Rightarrow F_{xy} = F_{yx}$

ex: $2xy^2+4 = 2(3-x^2y)\frac{dy}{dx}$, $y(-1)=8$

$(2xy^2+4) - 2(3-x^2y)\frac{dy}{dx} = 0$

$P = 2xy^2+4 \qquad P_y = 4xy$

$Q = -2(3-x^2y) \qquad Q_x = 4xy$

so this eq. is exact!

$F = \int P\,dx = \int 2xy^2+4\,dx = x^2y^2+4x+\Phi(y)$

$F_y = 2x^2y + \Phi'(y) = -6+2x^2y = Q$

$\Phi'(y) = -6$

$\Phi(y) = -6y$

note - kind of confusing is you say $F(x,y)=C$

$x^2y^2+4x-6y = C$

$(-1)^2(8)^2+4(-1)-6(8) = C$

$64-4-48 = C$

$12 = C$

check initial condition!

$x^2y^2+4x-6y = 12$

$x^2y^2 - 6y + (4x-12) = 0$

$$y = \frac{6 \pm \sqrt{36-4x^2(4x-12)}}{2x^2} = \frac{3 \pm \sqrt{9-4x^3+12x^2}}{x^2}$$

$$\frac{3 \pm \sqrt{9+4+12}}{1} = 8 \qquad \frac{3+5}{1} = 8$$

$$\boxed{y = \frac{3+\sqrt{9-4x^3+12x^2}}{x^2}}$$

check interval of existence