10/23 - Undetermined Coefficient Method

Goal - Find a particular solution to $y'' + py' + qy = f(t)$

Restriction - $f(t)$ has to be in one of the following forms to use undetermined coefficient method

1) $ce^{at} \rightarrow y_p = \underline{k}e^{at}$

2) $A\cos\omega t + B\sin\omega t \rightarrow y_p = \underline{a}\cos\omega t + \underline{b}\sin\omega t$

3) $a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1}t + a_n \rightarrow y_p = \underline{b_0}t^n + \underline{b_1}t + \ldots + \underline{b_n}$

ex for 1) $y'' - y' - 2y = 2e^{-2t}$

$y_p = ke^{-2t}$

$y_p' = -2ke^{-2t}$

$y_p'' = 4ke^{-2t}$

$4ke^{-2t} + 2ke^{-2t} - 2ke^{-2t} = 2e^{-2t}$

$4ke^{-2t} = 2e^{-2t}$

$k = 1/2 \rightarrow y_p = \frac{1}{2}e^{-2t}$

solve for y_1 and y_2

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

$y_1 = e^{2t} \quad y_2 = e^{-t}$

general solution

$y = c_1 y_1 + c_2 y_2 + y_p$

$\boxed{y = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{2}e^{-2t}}$

ex for 2) $y'' + 2y' - 3y = 5\sin 3t \ (+ 0\cos 3t)$

$y_p = a\cos 3t + b\sin 3t$

$y_p' = -3a\sin 3t + 3b\cos 3t$

$y_p'' = -9a\cos 3t - 9b\sin 3t$

$\lambda^2 + 2\lambda - 3\lambda = 0$

$\lambda = -3, 1$

$y_1 = e^{-3t} \quad y_2 = e^{t}$

$\boxed{y = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$

$-9a\cos 3t - 9b\sin 3t - 6a\sin 3t + 6b\cos 3t$
$-3a\cos 3t - 3b\sin 3t = 5\sin 3t$

$-12a\cos 3t - 12b\sin 3t - 6a\sin 3t + 6b\cos 3t = 5\sin 3t$

$(-12a + 6b)\cos 3t + (-12b - 6a)\sin 3t = 5\sin 3t$

$-12a + 6b = 0 \qquad -12b - 6a = 5$

$a = \frac{1}{2}b \qquad -15b = 5$

$a = -\frac{1}{6} \qquad b = -\frac{1}{3}$

$y_p = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$

ex for 3) $y'' + 2y' - 3y = 3t + 4$

$y_p = at + b$

$y_p' = a$

$y_p'' = 0$

$\lambda^2 + 2\lambda - 3\lambda$

$\boxed{y = c_1 e^{-3t} + c_2 e^{t} - t - 2}$

$0 + 2a - 3at - 3b = 3t + 4$

$-3a = 3 \qquad 2a - 3b = 4$

$a = -1 \qquad b = -2$

$y = -t - 2$

ex for 2) but w/ complex numbers

$$y'' + 2y' - 3y = 5\sin 3t$$

① $iy'' + 2iy' - 3iy = 5i\sin 3t$

② $x'' + 2x' - 3x = 5\cos 3t$

① + ② $= (x+iy)'' + 2(x+iy)' - 3(x+iy) = 5(i\sin 3t + \cos 3t)$

$= 5e^{3it}$

let $z(t) = x(t) + iy(t)$

belongs to $\mathbb{C}$

$$z'' + 2z' - 3z = 5e^{3it}$$

$z_p = ke^{3it}$

$z_p' = 3kie^{3it}$

$z_p'' = -9ke^{3it}$

$(-9k + 6ki - 3k)e^{3it} = 5e^{3it}$

$-12k + 6ki = 5$

$k = \frac{5}{-12+6i} = -\frac{1}{6}\frac{5}{2-i} = -\frac{1}{6}\frac{\cancel{5}(2+i)}{\cancel{5}} = -\frac{1}{6}(2+i)$

$z_p = -\frac{1}{6}(2+i)e^{3it}$

$= -\frac{1}{6}(2+i)(\cos 3t + i\sin 3t)$

$= -\frac{1}{6}(2\cos 3t + 2i\sin 3t + i\cos 3t - \sin 3t)$

$= -\frac{1}{6}((2\cos 3t - \sin 3t) + \underbrace{(2\sin 3t + \cos 3t)}i)$

$y_p = -\frac{1}{6}(2\sin 3t + \cos 3t)$