12/12 - Review

1) 1st Order DEs - $y' = ay + f$ general solution: $y = cy_h + y_p$

a) separable equations

b) integrating factors - $e^{-\int a\,dt}$

c) variation of parameters - $y_p = vy_h \Rightarrow y_p' = v'y_h + vy_h'$

y_h is solution to homogeneous portion (we choose constant = 0)

d) exact equations - $P(x,y)dx + Q(x,y)dy = 0$ is exact iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

e) integrating factors for $P(x,y)dx + Q(x,y)dy = 0$

1-var case:

if $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of only x, $\mu(x) = e^{\int h(x)dx}$

if $h = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of only y, $\mu(y) = e^{-\int g(y)dy}$

or if you are given whether it's $\mu(x)$ or $\mu(y)$, just try solving

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \mu(x)P_y = \mu'(x)Q + \mu(x)Q_x$ (same process for $\mu(y)$)

f) homogeneous equations

a function f is homogeneous to degree n is $f(tx, ty) = t^n f(x,y)$

a DE in the form $Pdx + Qdy = 0$ is homogeneous if

P and Q are homogeneous to the same degree

assume there is a solution of the form $y = vx$

then $dy = vdx + xdv$

$P(x,y)dx + Q(x,y)dy = P(x,vx)dx + Q(x,vx)(vdx + xdv)$

$= x^n\left(P(1,v)dx + Q(1,v)(vdx + xdv)\right) = 0$

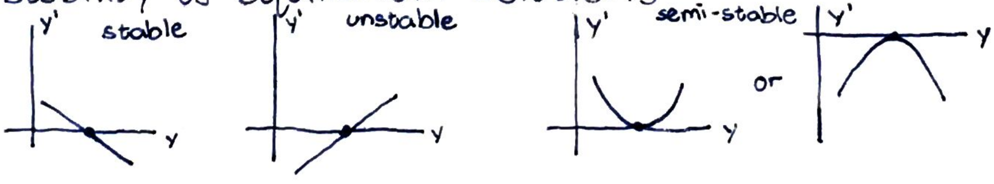
g) Existence and Uniqueness

i) $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous

ii) assume there exists a solution x above and below an equilibrium solution, then by IVT, they must intersect

iii) because they intersect, they must be the same solution this is a contradiction, so x cannot exist

h) stability of equilibrium solutions



2) 2nd Order DEs - $y'' + py' + qy = g$ general solution: $y = c_1y_1 + c_2y_2 + y_p$

a) Existence and Uniqueness

i) p, q, and g are continuous

ii) same process as before with initial condition (y_0, y_1), use IVT!

iii) by contradiction, your solution cannot cross equilibrium solutions

b) Wronskian

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{cases} 0 \text{ is } u \text{ and } v \text{ are not linearly independent} \\ \text{not } 0 \text{ if linearly independent} \end{cases}$$

use to check if two solutions are linearly independent

only when $y''+py'+qy=0$

c) constant coefficient method - $\lambda^2 + p\lambda + q = 0$

case 1: 2 distinct real roots λ_1, λ_2 $\Rightarrow$ $y_1 = e^{\lambda_1 t}$ $y_2 = e^{\lambda_2 t}$

case 2: 2 complex roots $\alpha \pm \beta i$ $\Rightarrow$ $y_1 = e^{\alpha t}\cos\beta t$ $y_2 = e^{\alpha t}\sin\beta t$

case 3: 1 repeated real root λ $\Rightarrow$ $y_1 = e^{\lambda t}$ $y_2 = te^{\lambda t}$

finds a particular solution

d) undetermined coefficient method - $y'' + py' + qy = g(t)$

case 1: $g(t) = ce^{at}$ $\Rightarrow$ $y_p = \underline{k}e^{at}$

case 2: $g(t) = A\cos wt + B\sin wt$ $\Rightarrow$ $y_p = \underline{a}\cos wt + \underline{b}\cos wt$

case 3: $g(t) = a_0t^n + a_1t^{n-1} + \ldots + a_{n-1}t + a_n$ $\Rightarrow$ $y_p = \underline{b_0}t^n + \underline{b_1}t^{n-1} + \ldots + \underline{b_n}$

solve for underlined variables

☆ note: if ke^{at} doesn't work, try tke^{at}

note: if $y'' + py' + qy = f + g$

if y_1 is a solution to $y'' + py' + qy = f$

if y_2 is a solution to $y'' + py' + qy = g$

then $y_p = y_1 + y_2$ is a solution to $y'' + py' + qy = f + g$

e) variation of parameters

$y_p = v_1y_1 + v_2y_2$

use const coefficents to get y_1 and y_2

$$v_1 = \int \frac{-y_2 g}{W}$$

$$v_2 = \int \frac{y_1 g}{W}$$

3) Linear Systems

a) find eigenvalues - solve $\det(A-\lambda I)=0$

b) for each eigenvalue, find eigenvectors - solve $(A-\lambda I)\vec{v}=0$

case 1: real root

just solve for $\vec{v}$

case 2: imaginary roots $\lambda=\alpha+\beta i$, $\bar{\lambda}=\alpha-\beta i$

solve for $\vec{v}$ with λ

separate into real and imaginary components (2x2)

$$\vec{z}=e^{\lambda t}\vec{v}=e^{(\alpha+\beta i)t}(\vec{v}_1+i\vec{v}_2)=e^{\alpha t}(\cos\beta t+i\sin\beta t)(\vec{v}_1+i\vec{v}_2)$$

$$=\underbrace{e^{\alpha t}(\cos\beta t\,\vec{v}_1-\sin\beta t\,\vec{v}_2)}_{y_1}+i\underbrace{e^{\alpha t}(\sin\beta t\,\vec{v}_1+\cos\beta t\,\vec{v}_2)}_{y_2}$$

$$\vec{y}=c_1y_1+c_2y_2=c_1e^{\alpha t}(\cos\beta t\vec{v}_1-\sin\beta t\vec{v}_2)+c_2e^{\alpha t}(\sin\beta t\vec{v}_1+\cos\beta t\vec{v}_2)$$

if repeating root (algebraic multiplicity >1)

(2x2) for a $\vec{w}$ that is linearly independent to $\vec{v}_1$

$(A-\lambda_1 I)\vec{w}=k\vec{v}_1 \Rightarrow \vec{v}_2=\vec{w}/k$

$$\vec{y}=c_1e^{\lambda t}\vec{v}_1+c_2e^{\lambda t}(t\vec{v}_1+\vec{v}_2)$$

higher dimensions

i) find a basis of $\ker(A-\lambda I)$

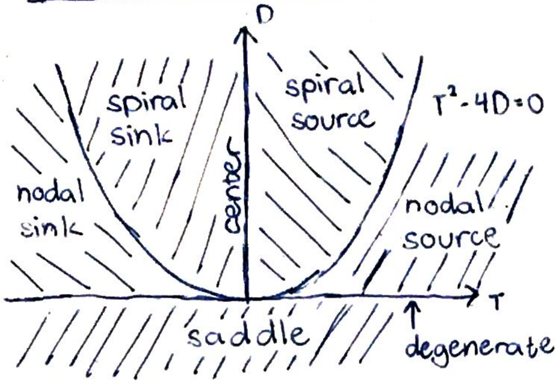
ii) if you still need more, find a vector in $\ker(A-\lambda I)^2$ not in $\ker(A-\lambda I)$

iii) repeat with higher powers until you have all the vectors you need

$$\vec{y}=e^{\lambda t}\left(\vec{v}+t(A-\lambda I)\vec{v}+\frac{1}{2!}t^2(A-\lambda I)^2\vec{v}+\ldots\right)$$

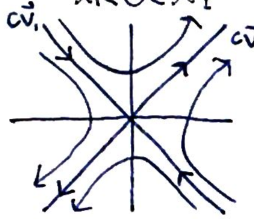
until it truncates

Phase Plane Portraits

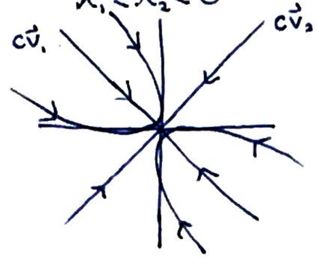


case 1: two real eigenvalues $T^2-4D>0$

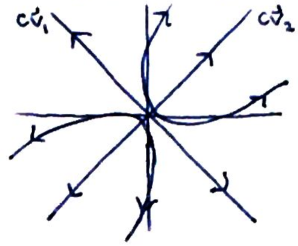
saddle point (G)
$\lambda_1<0<\lambda_2$



nodal sink (G)
$\lambda_1<\lambda_2<0$

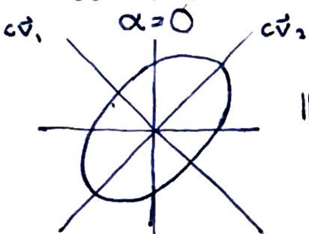


nodal source (G)
$0<\lambda_1<\lambda_2$



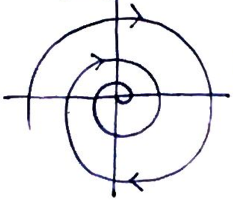
case 2: two complex eigenvalues $T^2-4D<0$

center (NG!)
$\alpha=0$



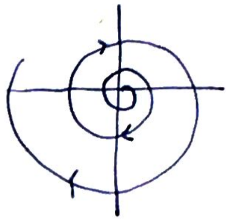
$\|\vec{v}_2\|>\|\vec{v}_1\|$

spiral sink (G)
$\alpha<0$



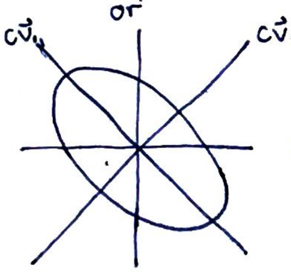
$a_{21}<0$

spiral source (G)
$\alpha>0$



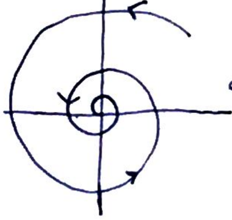
$a_{21}<0$

or



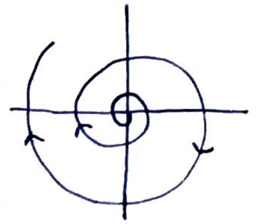
$\|\vec{v}_1\|>\|\vec{v}_2\|$

or



$a_{21}>0$

or



$a_{21}>0$

to check direction, plug $\begin{bmatrix}1\\0\end{bmatrix}$ into the equation