10/24 - discussion

solutions form a 2-D vector space

$y'' + p(x)y' + q(x)y = 0$

ex: $y'' + y = 0$

$f_1(x) = \sin(x)$
$f_2(x) = \cos(x)$
need to be linearly independent

general solution - $y = A\sin x + B\cos x$

note: $y_1 = \sin x$ and $y_2 = 5\sin x$ are both solutions but not linearly independent, so it does not cover the entirity of the solution

use Wronskian to check linear independence

$$W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$ so lin. ind.!

$$\begin{vmatrix} \sin x & 5\sin x \\ \cos x & 5\cos x \end{vmatrix} = 5\sin x\cos x - 5\sin x\cos x = 0,$$ not lin. ind!

$ay'' + by' + c = 0$

suppose $y = e^{\lambda x}$ is a solution

$a(e^{\lambda x})'' + b(e^{\lambda x})' + ce^{\lambda x} = 0$

$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$

$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0$ $\quad e^{\lambda x} \neq 0$

$a\lambda^2 + b\lambda + c = 0$

ex: $y'' + 5y' + 6y = 0$

solve $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+3)(\lambda+2) = 0$

$\lambda = -2, -3$

$y_1 = e^{-2x}$ $\quad y_2 = e^{-3x}$

if $\lambda = p \pm qi$ (from quadratic formula)

$y_1 = e^{(p+qi)x}$
$y_2 = e^{(p-qi)x}$
are solutions, but not real

1) since solutions form vector space, other lin ind solutions can also span
2) euler's formula: $e^{ix} = \cos x\, i\sin x$

proof: $y' = iy$ $\quad y(0) = 1$

$f_1(x) = e^{ix}$ $\quad f_1'(x) = ie^{ix} = if_1(x)$

$f_2(x) = \cos x + i\sin x$ $\quad f_2'(x) = -\sin x + i\cos x = i(i\sin x + \cos x) = if_2(x)$

$f_1(0) = 1$ $\quad f_2(0) = 1$

by E/U $f_1(x) = f_2(x)$ so $e^{ix} = \cos x + i\sin x$

$ay'' + by' + cy = 0 \rightarrow \lambda = p \pm qi$

$f_1(x) = e^{px} e^{qix}$ $\quad f_2(x) = e^{px} e^{-qix}$

$f_1(x) = e^{px}(\cos(qx) + i\sin(qx))$

$f_2(x) = e^{px}(\cos(qx) - i\sin(qx))$

$f_1(x) + f_2(x) = e^{px}(2\cos(qx)) \xrightarrow{\text{scale by } 1/2} f_1(x) = e^{px}\cos(qx)$

$f_1(x) - f_2(x) = e^{px}(2i\sin(qx)) \xrightarrow{\text{scale by } 1/2i} f_2(x) = e^{px}\sin(qx)$

are lin ind, you can prove yoursels ^.^

general solution - $y = c_1 e^{px}\cos(qx) + c_2 e^{px}\sin(qx)$

---

if $(\lambda - d)^2 = 0$, so $y = e^{dx}$ is one solution, then $y = xe^{dx}$ is also a solution

assume all other solutions are of the form $y = v(x)e^{dx}$

$a(v(x)e^{dx})'' + b(v(x)e^{dx})' + c(v(x)e^{dx}) = 0$

$\Downarrow$

$v(x) = Ax + B$

general solution - $y = (Ax+B)e^{dx} = c_1 xe^{dx} + c_2 e^{dx}$

---

ex: $y'' - 8y + 17y = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\lambda = \dfrac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$

$y_1(x) = e^{4x}\cos(x)$ $\quad y_2(x) = e^{4x}\sin(x)$

$y = c_1 e^{4x}\cos(x) + c_2 e^{4x}\sin(x)$ $\qquad y' = 4c_1 e^{4x}\cos x - c_1 e^{4x}\sin x + 4c_2 e^{4x}\sin x + c_2 e^{4x}\cos x$

$-4 = c_1$

$y = -4e^{4x}\cos(x) + 15e^{4x}\sin(x)$ $\qquad -1 = 4c_1 + c_2$

$-1 = -16 + c_2$

---

ex: $y'' - 4y' - 4y = 0$

$(\lambda - 2)^2 = 0$

$y_1 = e^{2x}$ $\quad y_2 = xe^{2x}$

$y = c_1 e^{2x} + c_2 xe^{2x}$ $\qquad y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 xe^{2x}$

$12 = c_1$ $\qquad -3 = 2c_1 + c_2$

$-27 = c_2$

$y = 12e^{2x} - 27xe^{2x}$