10/04 -

method 2 - variation of parameters

DE : $\boxed{y' = ay + f}$ (☆)

DE$_{(homo)}$: $y_h' = ay \to y_h = ce^{\int a(t)dt} \to y_h = ce^{A(t)}$ $\quad A(t) = \int a(t)dt$

pick a homogeneous solution ($\neq 0$) by choosing a value for c

for convenience, we choose $y_{h1} = e^{A(t)}$; $c = 1$

we consider $v(t) = \dfrac{y(t)}{y_{h1}(t)} \to \boxed{y(t) = v(t)y_{h1}(t)}$ (1) → aka change constant c to variable function $v(t)$

$y'(t) = (v(t)y_{h1}(t))'$

$= \boxed{v'(t)y_{h1}(t) + v(t)y_{h1}'(t)}$ (2)

Take (1) and (2) and plug into (☆)

$v' \quad + v(t)y_{h1}'(t) = av(t)y_{h1}(t) + f(t)$

$v'(t)y_{h1}(t) + (vy_{h1}' - avy_{h1}) = f(t)$

$v'(t)y_{h1}(t) + v(\underbrace{y_{h1}' - ay_{h1}}) = f(t)$

→ $y_{h1}' = ay_{h1}$ from DE(homo) so this is 0

Note: reason why we need y_h is to make that thing 0
the reason we chose y_{h1} is to make our lives easier

$v'(t)y_{h1}(t) = f(t)$

$v'(t) = \dfrac{f(t)}{y_{h1}(t)}$

$v(t) = \displaystyle\int \frac{f(t)}{y_{h1}(t)}\,dt + C$

plug back into (1)

$y(t) = \left(\displaystyle\int \frac{f(t)}{y_{h1}(t)}\,dt + C\right) y_{h1}(t)$

$= \boxed{e^{A(t)} \displaystyle\int \frac{f(t)}{y_{h1}(t)}\,dt + Ce^{A(t)}}$

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variation of parameters

ex: $y' = y + e^{-t}$ $\quad (a(t) = 1,\ f(t) = e^{-t})$

- $y_h' = y \to y_h = ce^t \to y_{h1} = e^t$
- let $v(t) = \dfrac{y(t)}{e^t} \to v(t)e^t = y(t)$ (1)

  $y'(t) = v'(t)e^t + v(t)e^t$ (2)
- $v'(t)e^t + v(t)e^t = v(t)e^t + e^{-t}$

  $v'(t)e^t + \underbrace{v(t)e^t - v(t)e^t} = e^{-t}$

  $v'(t) = e^{-2t}$

  $v(t) = -\frac{1}{2}e^{-2t} + C$
- $y(t) = (-\frac{1}{2}e^{-2t} + C)e^t = -\frac{1}{2}e^{-t} + ce^t$

ex: $y' = y + e^{-t}$ $\quad a(t) = 1$

- $u(t) = e^{-t}$

$e^{-t}(y' - y) = e^{-t}(e^{-t})$

$e^{-t}y' - e^{-t}y = e^{-2t}$

$(e^{-t}y)' = e^{-2t}$

$e^{-t}y = -\frac{1}{2}e^{-2t} + C$

$y = -\frac{1}{2}e^{-t} + ce^t$

back to method 1

ex: $y'-y=e^{-t}$ $\quad (y'-ay=f)$

find $u(t)$ s.t $\quad u(y'-y)=(uy)'$

$uy'-uy=u'y+uy'$

$-uy=u'y$

$y(u+u')=0$ for all $y(t)$

$\hookrightarrow (u+u')=0$

$u'=-u$

$u=e^{-t}$

blergh

$u(y'-ay)=uf$

$(uy)'=uf$

$uy=\int uf\,dt+c$

$y=\frac{1}{u}\int uf\,dt+c$

$=\frac{1}{e^{-t}}\left(\int e^{-t}\cdot e^{-t}\,dt+c\right)$

$=e^{t}\left(\int e^{-2t}\,dt+c\right)$

$=e^{t}\left(-\frac{1}{2}e^{-2t}\right)+ce^{t}$

$=-\frac{1}{2}e^{-t}+ce^{t}$

<u>Theorem</u> - the solutions of DE $y'=ay+f$ are of the form $y=y_h+y_p$

1) y_h is a nonzero solution of the homogenous part of DE $(y=y_h)$

2) y_p is a particular solution of $y'=ay+f$

<u>Remark</u> - if we know y_p is a solution of $y'=ay+f$, the <u>general solution</u> should be $y=y_p+y_h$ where $y_h'=ay_h$

$y=cy_h+y_p$ is a solution