10/21 - 2nd Order Linear DE w/ Const Coefficient

solve $y''+py'+qy=f(t)$ where p,q are const

recall the general solution is $y(t)=y_p+c_1y_1(t)+c_2y_2(t) = y_p + y_h(t)$

$y_h(t)$ is a solution to homogenous equation $y''+py'+qy=0$

Step 1 - Solve $y''+py'+qy=0$ (homogenous w/ const coefficient)] 4.3

Get $y_h(t)=c_1y_1(t)+c_2y_2(t)$

Step 2 - Find a particular solution y_p for $y''+py'+qy=f(t)$

Method 1 - Undetermined Coefficient] 4.5

Pros - fast, easy to calculate ; Cons - has some requirements on $f(t)$

Method 2 - Variation of Parameters] 4.6

Pros - no requirements for $f(t)$; Cons - calculation is more complicated

Q: How to solve $y''+py'+qy=0$ (Step 1)

A: $\lambda^2+p\lambda+q(\lambda^0)=\lambda^2+p\lambda+q=0$

$\Delta = p^2-4q$

$\Delta>0$	two distinct real roots λ_1,λ_2	$e^{\lambda_1 t}, e^{\lambda_2 t}$
$\Delta<0$	two complex roots $\alpha+\beta i$	$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t$
$\Delta=0$	one repeated root λ_0	$e^{\lambda_0 t}, te^{\lambda_0 t}$

Motivation (not required):

2 real roots - consider $y=e^{\lambda t} \rightarrow y'=\lambda e^{\lambda t}=\lambda y$, $y''=\lambda^2 e^{\lambda t}=\lambda^2 y$

$y''+py'+qy=0$; p and q are constant

$=\lambda^2 y+p\lambda y+qy$

$=(\lambda^2+p\lambda+q)y=0$, solve for λ

if λ_1 and λ_2 are two roots,

then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions

linearly independent $\Leftrightarrow$ Wronskian $=\det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{\lambda_1 t}\lambda_2 e^{\lambda_2 t} - e^{\lambda_2 t}\lambda_1 e^{\lambda_1 t}$

$=(\lambda_2-\lambda_1)(e^{(\lambda_1+\lambda_2)t}) \neq 0$

Complex root - consider $\lambda^{+} = \alpha + \beta i$, then $z(t) = e^{\lambda^{+} t}$

$z'' + pz' + qz = (\lambda^2 + p\lambda + q)z = 0 \rightarrow e^{(\alpha+\beta i)t}$ is a solution

$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + (\sin\beta t)i)$ ①

consider $\lambda^{-} = \alpha - \beta i$, then $z(t) = e^{\lambda^{-} t}$ is a solution

$e^{(\alpha-\beta i)t} = e^{\alpha t}(\cos\beta t - (\sin\beta t)i)$ ②

$\frac{1}{2}(① + ②) = e^{\alpha t}\cos\beta t$

$\frac{1}{2i}(① - ②) = e^{\alpha t}\sin\beta t$

Repeated Root - consider $y'' + py' + qy = \underbrace{\frac{\partial^2}{\partial t^2}}_{=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right)} y + p\frac{\partial}{\partial t}y + qy = \left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)y = 0$

$y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$ are solutions

$y_{\lambda_1\lambda_2} = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_2 t} - e^{\lambda_1 t})$ is a solution

$\lim\limits_{\lambda_2 \to \lambda_1} y_{\lambda_1\lambda_2} = \frac{\partial}{\partial\lambda}(e^{\lambda t})\Big|_{\lambda = \lambda_1} = te^{\lambda_1 t}$

$\lim\limits_{\lambda_2 \to \lambda_1}\left(\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)y_{\lambda_1\lambda_2}\right) = \left(\frac{\partial}{\partial t} - \lambda_1\right)^2(te^{\lambda_1 t}) = 0$

so $te^{\lambda_1 t}$ is a solution to $y'' + py' - qy = 0$

what I think he's trying to say is if $\lambda_1 = \lambda_2$, $e^{\lambda_1 t} = e^{\lambda_2 t}$ are solutions but so are $te^{\lambda_1 t} = te^{\lambda_2 t}$