Fields

field F is a set w/ 2 funcs and 10 axioms

$+: F \times F \to F$ $\qquad$ $\cdot: F \times F \to F$

$(x,y) \to x+y$ $\qquad$ $(x,y) \to x \cdot y$

addition $\qquad$ multiplication

A1: $(x+y)+z = x+(y+z)$	assoc
A2: $\exists 0 \in F$ s.t. $x+0 = 0 = 0+x$	add id
A3: $\exists (-x) \in F$ s.t. $x+(-x) = 0$	add inv
A4: $x+y = y+x$	comm
M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	assoc
M2: $\exists 1 \neq 0$ s.t. $x \cdot 1 = x = 1 \cdot x$	mult id
M3: $\forall x \neq 0, \exists (x^{-1}) \in F$ s.t. $x \cdot (x^{-1}) = 1$	mult inv
M4: $x \cdot y = y \cdot x$	comm
D1: $x \cdot (y+z) = xy + xz$	distr
D2: $(x+y) \cdot z = xz + yz$	distr

$\exists$: there exists

$\forall$: for all

Theorems

1. 0 is unique
2. 1 is unique
3. for any x, $(-x)$ is unique
4. for any $x \neq 0$, (x^{-1}) is unique
5. if $a+b = a+c$, then $b=c$
6. $0 \cdot a = 0$
7. if $ab = ac$ and $a \neq 0$, then $b=c$
8. if $ab=0$, then $a=0$ or $b=0$
9. $-(-a) = a$
10. $(-a)b = -(ab) = a(-b)$
11. $(-a)(-b) = ab$
12. if $a \neq 0$, then $a^{-1} \neq 0$ and $(a^{-1})^{-1} = a$

Vector Spaces

suppose F is a field, vector space V is a set w/ 2 functions and 8 axioms

$+: V \times V \to V$ $\qquad$ $\cdot: F \times V \to V$

$(u,v) \to u+v$ $\qquad$ $(c,v) \to cv$

A1: $(u+v)+w = u+(v+w)$	assoc
A2: $\exists 0_v \in V$ s.t. $0_v + v = v = v + 0_v$	add id
A3: $\exists (-v) \in V$ s.t. $v+(-v) = 0_v$	add inv
A4: $u+v = v+u$	comm
S1: $\exists 1 \in F$ s.t. $1v = v$	mult id
S2: $(cd)v = c(dv)$	assoc
D1: $(c+d)v = cv + dv$	distr
D2: $c(u+v) = cu + cv$	distr

Theorems:

1. $0_F v = 0_v$
2. $c 0_v = 0_v$
3. if $cv = 0_v$, then $c = 0_F$ or $v = 0_v$
4. $(-c)v = -(cv) = c(-v)$
5. $-v = -1 \cdot v$
6. $-(u+v) = -u-v$
7. if $cu = cv$ and $c \neq 0_F$, then $u=v$
8. if $cv = dv$ and $v \neq 0_v$, then $c=d$

Subspaces

suppose V is a vector space over field F. a subset $W \subset V$ is a subspace iff

1. closed under vector addition - for any $u,v \in W$, $u+v \in W$
2. closed under scalar multiplication - for any $v \in W$ and $c \in F$, $cv \in W$
3. $0_v \in W$
4. closed under negation - for any $v \in W$, $-v \in W$

Subspace Theorem: let V be a vector space over field F

suppose $W \subset V$ is a nonempty subset of V. then W is a subspace of V iff for any $u,v \in W$ and $c \in F$, $cu+v \in W$

$\{v_1, \dots, v_n\}$ are linearly dependent iff there exists $c_1, \dots, c_n$, not all 0, s.t. $c_1v_1 + \dots + c_nv_n = 0$
equivalent to there is a v_i that is a linear combination of the other vectors

$\{v_1, \dots, v_n\}$ are linearly independent if they aren't linearly dependent
if $S \subset T$ and S is lin dep, then T is lin dep
if $T \subset S$ and S is lin ind, then T is lin ind
if $0 \in S$, S is lin dep

$\text{span}(v_1, \dots, v_n) = \{ \sum_{i=1}^{n} c_i v_i \mid c_i \in F \text{ for } i = 1, \dots, n \}$

let V be a v.s./F. a subset $\mathcal{B} \subset V$ is a basis ~~if~~ of V if it is lin ind and spans V

Coordinate Theorem - suppose V is a v.s./F and $v_1, \dots, v_n$ are dinstinct
then $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis for V iff for any $v \in V$,
there are unique scalars $c_1, \dots, c_n$ s.t. $v = \sum_{i=1}^{n} c_i v_i$
in such a case, we call $c_1, \dots, c_n$ the coordinates of v with respect to $\mathcal{B}$

Exchange Lemma - let V is a v.s./F
suppose $L \subset V$ is a set of m lin ind vectors and $G \subset V$ is an n-elmt spanning set for V
then 1) $m \leq n$
2) there is an $(n-m)$-element set $H \subset G$ s.t. $L \cup H$ spans V

$\dim(V)$ = dimension of v.s. V = unique int n s.t. every basis V has n elmts

suppose V is an n-dimensional v.s./F and $S \subset V$
1) if $|S| > n$ then S is dependent
2) if $|S| < n$ then S does not span V

Toss Out Theorem - let V be a v.s./F
if $S \subset V$ is a finite subset that spans V, then S contains a basis for V

Toss In Theorem - let V be a v.s./F and $S \subset V$ is a linearly independent subset
if $v \in V \setminus \text{span}(S)$, then $S \cup \{v\}$ is independent

Extension Theorem - let V be a f.d.v.s./F (finite dimensional vector space over F)
and let $W \subset V$ be a subspace
then every linearly independent subset of W is finite and part of a basis of W

suppose V and W are v.s./F

a <u>linear transformation</u> from V to W is a function $T: V \to W$ s.t.

1) for any $u, v \in V$, $T(u+v) = T(u) + T(v)$
2) for any $v \in V$ and $c \in F$, $T(cv) = cT(v)$

a <u>linear operator</u> is a l.t. $T: V \to V$ from a v.s. to itself

Properties of l.t.

1) $T(0_v) = 0_w$
2) $T(-v) = -T(v)$
3) $T(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} (T(c_i v_i))$
4) T is a l.t. iff, for all $u, v \in V$ and $c \in F$,
   $T(cu+v) = cT(u) + T(v)$
5) if $S: U \to V$ and $T: V \to W$ are linear
   $T \circ S$ is also linear
6) $1_w \circ T(v) = T(v) = T(v) \circ 1_w$
7) if $R: U \to V$, $S: V \to W$, $T: W \to X$,
   then $T \circ (S \circ R) = (T \circ S) \circ R$
8) if $S: V \to W$, $T: V \to W$ are linear,
   then $(S+T)$, defined as $(S+T)(v) = S(v) + T(v)$, is linear
9) if $T: V \to W$ is linear, $c \in F$
   then (cT), defined as $(cT)(v) = c \cdot T(v)$ is linear
10) suppose $R: U \to V$, $S_1, S_2: V \rightrightarrows W$, and $T: W \to X$ are linear, $c \in F$
    then $T \circ (S_1 + S_2) = T \circ S_1 + T \circ S_2$
    $(S_1 + S_2) \circ R = S_1 \circ R + S_2 \circ R$
    $(cT) \circ S = c(T \circ S) = T \circ (cS)$

$\mathcal{L}(V, W) = \{T: V \to W \mid T \text{ is a l.t.}\}$ = set of all l.t. from V to W

is a v.s. w/ sum and product functions as defined in 8 and 9

a <u>two-sided inverse</u> to a lt $T: V \to W$ is a lt $S: W \to V$ s.t.

$S \circ T = 1_v$ and $T \circ S = 1_w$

T is an <u>isomorphism</u> if it has a two-sided inverse

T is an iso iff T is bijective

suppose V and W are v.s

V and W are <u>isomorphic</u> ($V \cong W$) if there is an iso b/w them

$V \cong W$ iff $\dim V = \dim W$

<u>Universal Property of Vector Spaces (UPVS)</u>

suppose V and W are v.s./F and $B = \{v_1, \ldots, v_n\}$ is a basis of V

then given any vectors $w_1, \ldots, w_n$, not necessarily distinct

there is a unique lt $T: V \to W$ s.t. $T(v_i) = w_i$ for $i = 1, \ldots, n$

intuition: a lin transform can be defined on the basis of a v.s.

let $T: V \to W$ be a lt

kernal/null space of $T = \ker(T) = \{v \in V \mid Tv = 0\} \subset V$

nullity of $T = \dim(\ker(T))$

image/range of $T = \operatorname{im}(T) = \{w \in W \mid Tv = w \text{ for some } v \in V\} \subset W$

rank of $T = \dim(\operatorname{im}(T))$

Rank-Nullity Theorem - let $T: V \to W$ be a lin trans b/w v.s./F

suppose V is fin dim, then

1) $\operatorname{im}(T)$ and $\ker(T)$ are fdvs/F

2) $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$

suppose $T: V \to W$ is a lin trans b/w fdvs

the following are equivalent:

1) T is surjective

2) $\operatorname{im}(T) = W$

3) $\operatorname{rank}(T) = \dim W$

the following are equivalent

1) T is injective

2) $\ker(T) = \{0\}$

3) $\operatorname{nullity}(T) = 0$

Isomorphism Theorem - suppose $T: V \to W$ is a lin trans and $\dim V = \dim W < \infty$

the following are equivalent:

1) T is an isomorphism

2) T is injective

3) T is surjective

the coordinate matrix of a vector $v = \sum_{i=1}^{n} c_i v_i$ wrt a basis $\mathcal{B} = v_1, \dots, v_n$ is $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

the function $[\,]_{\mathcal{B}}$ is a linear iso from V to $F^{n \times 1}$

the matrix of $T: V \to W$ wrt bases $\mathcal{B} = v_1, \dots, v_m$ for V and $\mathcal{C} = w_1, \dots, w_n$ for W; $V \cong F^{n \times 1}$, $W \cong F^{m \times 1}$

is $[T]_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} | & & | \\ [Tv_1]_{\mathcal{C}} & \dots & [Tv_m]_{\mathcal{C}} \\ | & & | \end{bmatrix} \in F^{n \times m}$

think $[T]_{\mathcal{B}\mathcal{C}}$ as transform from basis $\mathcal{B}$ to basis $\mathcal{C}$

let $T: V \to W$ be a lin trans b/w fdvs V and W

let $\mathcal{B} = v_1, \dots, v_m$ be an ordered basis of V and $\mathcal{C} = w_1, \dots, w_n$ be an ordered basis for W

then there is a unique matrix $A \in F^{n \times m}$ s.t.

$A[v]_{\mathcal{B}} = [Tv]_{\mathcal{C}}$, namely $A = [T]_{\mathcal{B},\mathcal{C}}$

let $S: U \to V$ and $T: V \to W$ be fdvs, and let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be ordered bases for U, V, W

then $[T \circ S]_{\mathcal{B},\mathcal{D}} = [T]_{\mathcal{C},\mathcal{D}} \circ [S]_{\mathcal{B},\mathcal{C}}$

T is iso iff $[T]_{\mathcal{B},\mathcal{C}}$ is invertible

in this case, $[T]_{\mathcal{B},\mathcal{C}}^{-1} = [T^{-1}]_{\mathcal{C},\mathcal{B}}$

Matrix Theory Theorem - $\mathcal{L}(V,W) \cong F^{n \times m}$

i.e. the set of all lin trans from V to W is isomorphic to the set of $n \times m$ matrices, if $|V|=m$ and $|W|=n$

corollary: if V and W are fdvs, $|V|=m$ and $|W|=n$

then $\dim(\mathcal{L}(V,W)) = \dim V * \dim W$

def: let $\mathcal{B}, \mathcal{C}$ be two ordered bases for V

the change of basis matrix is $[1]_{\mathcal{B},\mathcal{C}}$

with key property $[1]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} = [1v]_{\mathcal{C}} = [v]_{\mathcal{C}}$

a matrix A is a CoB matrix iff A is invertible

Change of Bases Theorem - suppose $T: V \to W$ is a lt b/w fdvs/F

let $\mathcal{B}, \mathcal{B}'$ be ordered bases of V, let $\mathcal{C}, \mathcal{C}'$ be ordered bases of W

then $[T]_{\mathcal{B}',\mathcal{C}'} = [1_W]_{\mathcal{C},\mathcal{C}'}[T]_{\mathcal{B},\mathcal{C}}[1_V]_{\mathcal{B}',\mathcal{B}}$

let A and B be square matrices w/ entries in F

A is similar to B, or $A \sim B$, if there is an invertible matrix C

st $A = C^{-1}BC$

let $A, B \in F^{n \times n}$

$A \sim B$ iff there is an n-dim v.s. V/F and lin op. $T: V \to V$ s.t.

$A = [T]_{\mathcal{B}}$ and $B = [T]_{\mathcal{C}}$ for some bases $\mathcal{B}$ and $\mathcal{C}$ on V

the dual space of a v.s. V is $V^* = \mathcal{L}(V,F)$

$f \in \mathcal{L}(V,f)$ is a functional

Dual Basis Theorem - let V be a v.s. w/ basis $\mathcal{B} = v_1, \dots, v_n$

let $f_1, \dots, f_n$ be coordinate functionals s.t. $f_i(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

then $\mathcal{B}^* = f_1, \dots, f_n$ is an ordered basis of V^* and is called the dual basis to $\mathcal{B}$

there is a non-canonical (dependent on $\mathcal{B}$) iso $V \xrightarrow{\cong} V^*$

there is a non-canonical iso $V^* \xrightarrow{\cong} V^{**}$

there is a canonical iso $V \xrightarrow{\cong} V^{**}$

if v is nonzero, there exists a functional $f \in V^*$ s.t. $f(v) = 1$

Double Dual Theorem - let V be a fdvs/F

define $E_v \in V^{**}$ s.t. $E_v(f) = f(v)$ means "evaluate functional f at v"

then $E: V \to V^{**}$ s.t. $E(v) = E_v$ is an iso let $T: V \to W$ be a lt

the <u>transpose</u> of T is $T^t: W^* \to V^*$ given by the formula $T^t(g) = g \circ T \in V^*$

let V, W be fdvs, let $\mathcal{B}, \mathcal{C}$ be ordered bases for V, W respectively, let $\mathcal{B}^*, \mathcal{C}^*$ be ordered bases for V^*, W^*

let $T: V \to W$ be a lt

then $[T^t]_{\mathcal{C}^*, \mathcal{B}^*} = [T]_{\mathcal{B}, \mathcal{C}}^t$

let V be a vs/F and $S \subset V$ be a subset

then the <u>annihilator</u> of S is the set

$S^\circ = \{ f \in V^* \mid f(s) = 0 \text{ for all } s \in S \}$ = functionals that send S to 0

let $T: V \to W$ be a lt

then $\ker(T^t) = (\operatorname{im} T)^\circ$

$\operatorname{rank} T = \operatorname{rank} T^t$

let V be a fdvs, let $U \subset V$ be a subspase

then $\dim(V) = \dim(U) + \dim(U^\circ)$

let $A \in F^{m \times n}$ be an $m \times n$ matrix w/ entries in F

row rank A = dim(span(rows of A)) ; rows $\in F^{1 \times n}$

col rank A = dim(span(cols of A)) ; cols $\in F^{m \times 1}$

row rank A = col rank A = <u>rank A</u>

a <u>commutative ring</u> K is a set equipped w/ addition and multiplication

s.t. all the field axioms hold except possibly

1) multiplicative inverse

2) $1 \neq 0$

note: all fields are commutative rings

a function $D: K^{n \times n} \to K$ is <u>n-linear</u> if, for each $i = 1, \dots, n$

D is linear in the ith row of $A \in K^{n \times n}$ when the other $(n-1)$ rows are fixed

ie. $D \begin{bmatrix} — A_1 — \\ \vdots \\ — cA_i + A_i' — \\ \vdots \\ — A_n — \end{bmatrix} = cD \begin{bmatrix} — A_1 — \\ \vdots \\ — A_i — \\ \vdots \\ — A_n — \end{bmatrix} + D \begin{bmatrix} — A_1 — \\ \vdots \\ — A_i' — \\ \vdots \\ — A_n — \end{bmatrix}$

a lin comb of n-linear functions is n-linear

let $D: K^{n \times n} \to K$ be a n-linear func

D is <u>alternating</u> if $D(A) = 0$ whenever two rows of A are equal

i.e. if $i < j$ and $A_i = A_j$, then $D \begin{bmatrix} — A_1 — \\ \vdots \\ — A_i — \\ \vdots \\ — A_j — \\ \vdots \\ — A_n — \end{bmatrix}$

suppose $D: K^{n\times n} \to K$ is n-linear, and $D(A)=0$ when any two adjacent rows of A are equal

then 1) if $A \in K^{n\times n}$ and A' is obtained by interchanging any two rows of A, then $D(A) = -D(A')$

2) D is alternating

suppose K is a comm ring and $n>0$ is an int

let $D: K^{n\times n} \to K$

we say D is a determinant function if D is n-linear and alternating, and $D(I_n)=1$

for every comm ring K and $n>0$, there is a determinant function $D: K^{n\times n} \to K$

let $A \in K^{n\times n}$

$$\det(A) = \sum_{\sigma} (\operatorname{sgn}\sigma) A_{1,\sigma_1} \cdot A_{2,\sigma_2} \cdot \ldots \cdot A_{n,\sigma_n}$$
$$= \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det A(i|j) \text{ for any } 1 \le j \le n$$

let K be a comm ring and $n>0$ be an int

for any $A, B \in K^{n\times n}$, $\det(AB) = \det(A)\det(B)$

let $\sigma, \tau: \{1,\ldots,n\} \rightrightarrows \{1,\ldots,n\}$ be permutations of degree n

then $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

the scalar $c_{ij} = (-1)^{i+j} \det A(i|j)$ is called the (i,j)-cofactor of A

the classical adjoint to A is the transpose of the cofactor matrix

i.e. $(\operatorname{adj} A)_{ij} = c_{ji} = (-1)^{i+j} \det A(j|i)$

$$\operatorname{adj} A = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \in K^{n\times n}$$

let K be a comm ring, $A \in K^{n\times n}$

then $(\operatorname{adj} A) \cdot A = \det(A) I_n$

let K be a comm ring, $A \in K^{n\times n}$

A being an invertible matrix $\Leftrightarrow$ $\det A$ has a multiplicative inverse

if K is a field, then A being an invertible matrix $\Leftrightarrow$ $\det A \neq 0$

suppose $T: V \to V$ is a lin op on a vs/F

an <u>eigenvalue</u> of T is a scalar $\lambda \in F$ st there exists a nonzero $v \in V$ where $\lambda v = Tv$

a nonzero vector $v \in V$ is called an <u>eigenvector</u> of T associated to λ, or a λ-eigenvector

the collection of all $v \in V$ s.t $Tv = \lambda v$, including $v = 0$, is called the <u>eigenspace</u> associated to λ

or the λ-eigenspace

denote as $E(\lambda, T)$ or $E(\lambda)$ if T is clear

$E(\lambda, T) = \ker(T - \lambda I)$ is a subspace

a square matrix M is <u>diagonal</u> if every entry not on the diagonal is 0

i.e. $M_{ij} = 0$ if $i \neq j$

a lin op $T: V \to V$ is <u>diagonalizable</u> iff there is an ordered basis $\mathcal{B}$ for V s.t. $[T]_\beta$ is diagonal

a lin op $T: V \to V$ is diagonalizable iff V has an ordered basis consisting of eigenvectors for T (an <u>eigenbasis</u>)

let $T: V \to V$ be a lin op, let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues for T, and let v_i be a λ_i-eigenvector for $i = 1, \dots, n$

then $v_1, \dots, v_n$ are lin ind

let V be an n-dim vs, let $T: V \to V$ be a lin op

then T has at most n distinct eigenvalues

if T has n distinct eigenvalues, then T is diagonalizable

suppose $U_1, \dots, U_n \subset V$ are subspaces of V

the sum $U_1 + \dots + U_n$ is called a <u>direct sum</u> if

each element $u \in U_1 + \dots + U_n$ can be uniquely written as a sum $u = u_1 + \dots + u_n$; $u_i = U_i$ for $i = 1, \dots, n$

in such case, we write $U_1 \oplus \dots \oplus U_n$

$U_1 + \dots + U_n$ is a direct sum iff

the only way $0 = u_1 + \dots + u_n$; $u_i \in U_i$, is when $u_i = 0$ for $i = 1, \dots, n$

let V be a vs, let $T: V \to V$ be a lin op, let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues for T

then $E(\lambda_1) + \dots + E(\lambda_n)$ is a direct sum

<u>diagonizability theorem</u> - suppose V is a fdvs/F and $T: V \to V$ is a lin op

let $\lambda_1, \dots, \lambda_n$ be all the distinct eigenvalues of T

then T is diagonalizable iff $\dim V = \dim E(\lambda_1) + \dots + \dim E(\lambda_n)$

let V be a fdvs/F, $T: V \to V$ a lin op, $\mathcal{B}$ an ordered basis for V, and $\lambda \in F$

then λ is an eigenvalue of T iff $\det(\lambda I - [T]_\beta) = 0$

let F be a field, $A \in F^{n \times n}$
the <u>characteristic polynomial</u> of A is $f_A = \det(tI - A)$

if $A \sim B$ (A and B are similar), then $f_A = f_B$

let V be a vs/F, let $\mathcal{B}, \mathcal{C}$ be ordered bases of V, let $T: V \to V$ be a lin op
then $f_{[T]_\mathcal{B}} = f_{[T]_\mathcal{C}}$

let F be a field, V a fdvs/F, $T: V \to V$ a lin op
then the <u>characteristic polynomial</u> of T is $f_T = f_{[T]_\mathcal{B}} = \det(tI - [T]_\mathcal{B})$ for any ordered basis $\mathcal{B}$

<u>characteristic polynomial thm</u> - let F be a field, V a fdvs/F, $T: V \to V$ a lin op
then the eigenvalues of T are the roots of the characteristic polynomial f_T

if V is a nonzero fdvs/$\mathbb{C}$ and $T: V \to V$ is a lin op
then by the fundamental theorem of algebra, f_T has a root, so T has an eigenvector

let F be $\mathbb{R}$ or $\mathbb{C}$, let V be a vs/F
an <u>inner product</u> is a function $\langle -, - \rangle : V \times V \to F$ with the following properties
1) (positive definite) for $v \in V$, $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geq 0$, with $\langle v, v \rangle = 0$ iff $v = 0$
2) (additivity in the first slot) for all $u, v, w \in V$, $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3) (homogeneity in the first slot) for all $u, v \in V$ and $\lambda \in F$, $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
4) (conjugate symmetry) for all $u, v \in V$, $\langle u, v \rangle = \overline{\langle v, u \rangle}$

an <u>inner product space</u> is a vs over $\mathbb{R}$ or $\mathbb{C}$ equipped with an inner product
first properties
1) for any $v \in V$, $\langle -, v \rangle . V \to F$ is a linear functional
2) for any $u, v, w \in V$, $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3) for any $u, v \in V$ and $\lambda \in F$, $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$
4) suppose $v \in V$ and $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$

let V be an ips (inner product space)
the <u>norm</u> of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$
propositions
1) $\|v\| = 0 \Leftrightarrow v = 0$
2) $\|\lambda v\| = |\lambda| \cdot \|v\|$

<u>pythagorean theorem</u> - let V be an ips, let $u,v \in V$, suppose $\langle u,v\rangle = 0$
then $\|u\|^2 + \|v\|^2 = \|u+v\|^2$

<u>orthogonal projections</u> - let V be an ips, let $u,v \in V$, $v \neq 0$
then we can split u into components parallel and perpendicular to v

$u_{\parallel} = \frac{\langle u,v\rangle}{\|v\|^2} \cdot v$

$u_{\perp} = u - u_{\parallel}$

<u>Cauchy-Schwarz Inequality</u> - suppose V is an ips and $u,v \in V$
then $|\langle u,v\rangle| \leq \|u\| \|v\|$

<u>triangle inequality</u> - suppose V is an ips and $u,v \in V$
then $\|u+v\| \leq \|u\| + \|v\|$

suppose V is an ips and $S \subset V$ is a subset
then S is an <u>orthonormal set</u> if
1) all pairs of distinct vectors are orthogonal
2) each vector in S has length 1

ie. S is an orthonormal set if
for all $u,v \in S$

$$\langle u,v\rangle = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v \end{cases}$$

an <u>orthonormal basis</u> (ONB) is a basis that is an orthonormal set
an <u>ordered orthonormal basis</u> is an ordered basis s.t. the set of vectors in the basis is an orthonormal set

suppose $S \subset V$ is an orthonormal subset of ips V
then S is lin ind

suppose $\mathcal{B} = e_1, \dots, e_n$ is an OONB for ips V
then for $j = 1, \dots, n$ the linear map $\langle -, e_j\rangle : V \to F$ sends a vector $v \in V$ to the jth coord wrt $\mathcal{B}$
hence $v = \langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n$

if $\mathcal{B} = e_1, \dots, e_n$ is an OONB
then $\mathcal{B}^* = \langle -, e_1\rangle, \dots, \langle -, e_n\rangle$ is the dual basis

suppose V is an ips and $\mathcal{B} = e_1, \dots, e_n$ is an OONB of V
then for any $u,v \in V$, $\langle u,v\rangle = \sum_{i=1}^{n} \langle u, e_i\rangle \overline{\langle v, e_i\rangle}$
consequently, $\|v\|^2 = \langle v,v\rangle = \sum_{i=1}^{n} \langle v, e_i\rangle \overline{\langle v, e_i\rangle} = \sum_{i=1}^{n} |\langle v, e_i\rangle|^2$ (pythagorean theorem)

suppose $\mathcal{S} = \{e_1, \ldots, e_n\}$ is an orthonormal set in the ips V, and $e_1, \ldots, e_n$ are distinct

given any v,

let $v_{\parallel} = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n$

let $v_{\perp} = v - v_{\parallel}$

then

1) $v = v_{\parallel} + v_{\perp}$
2) $v_{\parallel} \in \operatorname{span} \mathcal{S}$
3) $v_{\perp}$ is orthogonal to every $w \in \operatorname{span} \mathcal{S}$

<u>Gram-Schmidt Process</u> - let V be an ips

suppose $\{v_1, \ldots, v_m\}$ is a lin ind set in V w/ v_i distinct

let $e_1 = \frac{v_1}{\|v_1\|}$

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}\|} = \frac{v_{j \perp \operatorname{span}(e_1, \ldots, e_{j-1})}}{\|v_{j \perp \operatorname{span}(e_1, \ldots, e_{j-1})}\|} \quad \text{for } j = 2, \ldots, m$$

then $\{e_1, \ldots, e_m\}$ is an orthonormal set w/ e_j distinct

and $\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$ for all $j = 1, \ldots, m$

<u>orthogonality theorem</u> - let V be a fdips

then V has an ONB (orthonormal basis)

<u>Approximation Theorem</u> - let V be an ips/$F = \mathbb{R}$ or $\mathbb{C}$, let $W \subset V$ be a finite subspace

choose an ordered ONB $\mathcal{B} = e_1, \ldots, e_n$ for W

given any $v \in V$, let $v_{\parallel} = \sum_{i=1}^{n} \langle v, e_i \rangle e_i$ and $v_{\perp} = v - v_{\parallel}$

then $v_{\parallel}$ is the *closest* vector to v in W

i.e. $\|v - v_{\parallel}\| \leq \|v - w\|$ for all $w \in W$

moreover, $\|v - v_{\parallel}\| = \|v - w\|$ iff $v_{\parallel} = w$

in particular, the choice of ONB does not matter

$v_{\parallel}$ is basically $\operatorname{proj}_W v$

let V be an ips/F, let $S \subset V$ be a subspace, let $v \in V$

$v \perp S$, i.e. v is orthogonal to S if $\langle v, s \rangle = 0$ for all $s \in S$

suppose V is an ips and $S \subset V$ is a subspace

then $S^{\perp} \subset V$ is a subspace

<u>Orthogonal Decomposition Theorem</u> - let V be an ips/F, let $W \subset V$ be a fd subspace

then $V = W \oplus W^{\perp}$

if V is an ips/F and $W \subset V$ is a subspace

1) $\dim V = \dim W + \dim W^{\perp}$
2) $(W^{\perp})^{\perp} = W$

Riest Representation Theorem - let V be a fdips$/F$
define the function $R: V \to \mathcal{L}(V,F)$
$R(v) = \langle -, v \rangle : V \to F$
then 1) R is bijective
2) $R(v+w) = R(v) + R(w)$ for all $v,w \in V$
3) $R(\lambda v) = \bar{\lambda} R(v)$ for all $v \in V$, $\lambda \in F$

R is an "antilinear" iso of functionals

suppose V is an ips$/F$ and $v,w \in V$ are vectors s.t. $\forall u \in V$, $\langle u,v \rangle = \langle u,w \rangle$
then $v=w$

suppose V and W are fdips$/F$ and $T: V \to W$ is a lt
then there exists a lt $T^*: W \to V$ st $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$ for all $v \in V$, $w \in W$
the lt T^* is called the adjoint of T

Properties of adjoint
1) for any lt $S,T: V \rightrightarrows W$, $(S+T)^* = S^* + T^*$
2) for any lt $T: V \to W$, $\lambda \in F$, $(\lambda T)^* = \bar{\lambda} T^*$
3) for any lt $T: V \to W$, $(T^*)^* = T$
4) the adjoint of $1: V \to V$ is 1, i.e. $1^* = 1$
5) if U,V,W are fdips, $S: U \to V$, $T: V \to W$ are linear
then $(TS)^* = S^* T^*$

suppose $T: V \to W$ is a lt b/w fdips, $\mathcal{B} = v_1, \dots, v_m$ is an OONB of V, and $\mathcal{C} = f_1, \dots, f_n$ is an OONB of W
then $[T^*]_{\mathcal{C},\mathcal{B}} = \overline{[T]_{\mathcal{B},\mathcal{C}}}^t = [T]_{\mathcal{B},\mathcal{C}}^*$ (where $A^* = \bar{A}^t$)
i.e. the matrix of the adjoint of a lt is the conjugate transpose of the matrix of the lt (wrt OONB)

a lin op $T: V \to V$ is self-adjoint / hermitian if $T^* = T$

suppose V is a vs$/F$ and $T: V \to V$ is a lin op
a subspace $U \subset V$ is invariant under T or T-invariant if $Tu \in U$ for all $u \in U$
in this case, $T|_U : U \to U$ is T restricted to the subspace U

Real Spectral Theorem - suppose V is a nonzero fdips$/\mathbb{R}$ and $T: V \to V$ is a lin op
then V has an orthonormal eigenbasis wrt T iff T is hermitian
proof idea: ($\Rightarrow$) ONEB means $[T]_{\mathcal{B}}$ is diagonal, so $[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}$
($\Leftarrow$) induction on the dimension of V
T is hermitian $\Rightarrow$ it has an eigenvalue λ
let e_1 be a λ-eigenvector for T
then $U = \mathrm{span}(e_1)$ is T-invariant b/c $Te_1 = \lambda e_1 \in U$
so $U^\perp$ is T-invariant and T is hermitian on $U^\perp$

a lin op $T: V \to V$ is <u>normal</u> if $TT^* = T^*T$

<u>Complex Spectral Theorem</u> - suppose V is a nonzero fdvs/$\mathbb{C}$ and $T: V \to V$ is a lin op
then V has an orthonormal eigenbasis wrt T iff T is normal
proof idea: ($\Rightarrow$) ONEB means $[T]_B$ is diagonal
$[T^*T]_B = [T^*]_B [T]_B = [T]_B [T^*]_B = [TT^*]_B$ b/c diagonal matrices
($\Leftarrow$) induction of $\dim V$
T has an eigenvalue λ by the fundamental theorem of algebra
let e_1 be a λ-eigenvector for T
then e_1 is a $\bar{\lambda}$-eigenvector for T^*
then $U = \operatorname{span}(e_1)$ is T-invariant and T^*-invariant b/c $Te_1 = \lambda e_1 \in U$ and $T^* e_1 = \bar{\lambda} e_1 \in U$
so $U^\perp$ is also T-invariant and T^*-invariant, and T is normal on $U^\perp$

<u>Singular Value Decomposition</u> - suppose $T: V \to W$ is a lin trans b/w fdvs/$F = \mathbb{R}$ or $\mathbb{C}$
then there exists ordered orthonormal eigenbases
$\mathcal{B} = e_1, \dots, e_m$ for V
$\mathcal{C} = f_1, \dots, f_n$ for W
s.t. the matrix $[T]_{\mathcal{B},\mathcal{C}}$ is a rectangular diagonal matrix of the form

$$\begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix} \quad \text{where } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

the diagonal entries of $[T]_{\mathcal{B},\mathcal{C}}$ are called the <u>singular values</u> of T