mathematical model - description of a system using mathematical concepts and language

How to create a mathematical model

1) problem formulation

   state the question

   identify factors, assumptions, and approximations

   represent the relationship between factors as equations

2) solve the problem

3) interpret and evaluate your solution in the context of the original problem

I. Spring Mass System

1) what formula can describe how the spring-mass system works?

factors:

- initial position: x_0
- spring constant: k
- ~~friction: F_f~~ assumption: friction doesn't matter
- mass of object: m
- position: x
- velocity: v
- acceleration: a
- force: F

relationships:

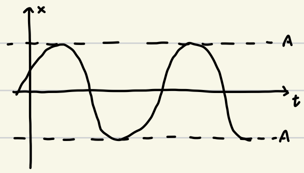
$F = -kx$ (Hooke's Law)

$F = ma = m\frac{d^2x}{dt^2}$ (Newton's 2nd Law)

2) solve $m\frac{d^2x}{dt^2} = -kx \Rightarrow mx'' + kx = 0$

$$m\lambda^2 + k = 0$$
$$\Rightarrow \lambda = \pm\sqrt{\frac{k}{m}}\,i$$
$$\Rightarrow x(t) = c_1 e^{t\sqrt{\frac{k}{m}}i} + c_2 e^{-t\sqrt{\frac{k}{m}}i}$$
$$= c_1\left(\cos\left(\sqrt{\tfrac{k}{m}}t\right) + i\sin\left(\sqrt{\tfrac{k}{m}}t\right)\right) + c_2\left(\cos\left(-\sqrt{\tfrac{k}{m}}t\right) + i\sin\left(-\sqrt{\tfrac{k}{m}}t\right)\right)$$
$$= c_3\cos\left(\sqrt{\tfrac{k}{m}}t\right) + c_4\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$
$$= \sqrt{c_3^2+c_4^2}\left(\frac{c_3}{\sqrt{c_3^2+c_4^2}}\cos\left(\sqrt{\tfrac{k}{m}}t\right) + \frac{c_4}{\sqrt{c_3^2+c_4^2}}\sin\left(\sqrt{\tfrac{k}{m}}t\right)\right)$$
$$= \sqrt{c_3^2+c_4^2}\left(\sin(\theta_2)\cos(\theta_1) + \cos(\theta_2)\sin(\theta_1)\right)$$
$$= \sqrt{c_3^2+c_4^2}\left(\sin(\theta_1+\theta_2)\right)$$
$$= \boxed{A\sin(\omega t + \theta)}$$

3) evaluate our solution: $x(t) = A\sin(\omega t + \theta)$



$x(t)$ is periodic with period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

$\max_t x(t) = A$, $\min_t x(t) = -A$, so A is the amplitude

$\omega t + \theta$ is the phase of the oscillation

issue: our model predicts that the spring will oscillate forever

solution: consider friction in our model

ex: $F_f = -c\frac{dx}{dt}$

solve $m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$

II. Population Dynamics

Case 1: Density Independent

1) can we predict the US population in 1, 2, 3, 4 years?

assumptions:

- birth/death rates are constant
- population is measured at fixed intervals
- population changes only from births/deaths

factors:

birth rate: b

death rate: d

initial population: N_0

distribution of population at different ages: $N^{(a)}$

~~migration rate~~

years from current time: t

population at time: $N(t)$

growth rate R

counting period: Δt

current time: t_0

relationships:

$$N(t+\Delta t) = N(t) + b \cdot N(t) \cdot \Delta t - d \cdot N(t) \cdot \Delta t$$
$$= N(t) + (b-d)\Delta t \cdot N(t)$$
$$= (1 + R\Delta t) N(t)$$

2) solve the equation

$$N(t_0 + 1\Delta t) = (1 + R\Delta t) N(t_0) = (1 + R\Delta t) N_0$$
$$N(t_0 + 2\Delta t) = (1 + R\Delta t) N_1 = (1 + R\Delta t)^2 N_0$$
$$\vdots$$
$$N(t_0 + n\Delta t) = (1 + R\Delta t)^n N_0$$
$$= (1 + R\Delta t)^{\frac{1}{R\Delta t} n \cdot R\Delta t} N_0$$
$$= (1 + R\Delta t)^{\frac{1}{R\Delta t} Rt} N_0$$

let $t = n\Delta t$, $t_0 = 0$

$$\lim_{\Delta t \to 0} N(t) = \lim_{\Delta t \to 0} (1 + R\Delta t)^{\frac{1}{R\Delta t} Rt} N_0$$
$$= \lim_{\Delta t \to 0} e^{\frac{Rt}{R\Delta t} \ln(1 + R\Delta t)} N_0$$
$$= N_0 e^{Rt}$$

or

$$N(t + \Delta t) = N(t) + R \cdot \Delta t \cdot N(t)$$
$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = R \cdot N(t)$$
$$\frac{dN}{dt} = \lim_{\Delta t \to 0} RN(t) = RN(t)$$
$$\frac{dN}{N(t)} = R\,dt$$
$$\ln(N(t)) = Rt + c$$
$$N(t) = e^{Rt+c} = e^c e^{R0} = N_0 e^{Rt}$$

3) evaluate $N(n\Delta t) = (1 + R\Delta t)^n N_0$

if $(1 + R\Delta t) > 1$, then $\lim_{n \to +\infty} N(n\Delta t) = +\infty$, and the model should be modified

if $0 < (1 + R\Delta t) < 1$, then $\lim_{n \to +\infty} N(n\Delta t) = 0$, and the model is acceptable

evaluate $N(t) = N_0 e^{Rt}$

if $R > 0$, then $N(t)$ increases as t increases, $\lim_{t \to +\infty} N(t) = +\infty$, so the model should be modified

if $R < 0$, then $N(t)$ decreases as t increases, $\lim_{t \to +\infty} N(t) = 0$, so the model is acceptable

Case 2: Density Dependent

1) can we predict the US population in 1, 2, 3, 4 years?

assumptions:

- growth rate is density dependent, i.e. $R(t) = R \cdot N(t)$
- growth rate is almost constant when population is small
- growth rate is 0 when population is large

relationships:

$$\left.\begin{aligned} &R(N(t)) = a - bN(t) \\ &\boxed{\frac{dN}{dt} = R(N) \cdot N = (a - bN)N} \end{aligned}\right\} \text{logistic model, continuous case}$$

note: continuous models are in the form $\frac{dy}{dt}$

discrete models are in the form $y(t-1) = f(y(t))$

$$\left.\begin{aligned} &R(N(t)) = \frac{R_0}{1 + N(t-1)/M} \\ &N(t) = R(N(t))N(t-1) = \frac{R_0 N(t-1)}{1 + N(t-1)/M} \end{aligned}\right\} \text{Beverton-Holt model, discrete case}$$

2) solve the equation

$$\frac{dN}{dt} = N(a - bN) \quad \text{(continuous case)}$$

$$\frac{dN}{N(a-bN)} = dt$$

$$\left(\frac{1}{aN} + \frac{b}{a(a-bN)}\right) dN = dt$$

$$\frac{1}{a}\ln N - \frac{1}{a}\ln|a - bN| = t + c$$

$$\frac{1}{a}\ln\left|\frac{N}{a-bN}\right| = t + c$$

$$\ln\left|\frac{N}{a-bN}\right| = at + c$$

$$\frac{N}{a-bN} = e^{at+c} = ce^{at}$$

$$\frac{a-bN}{N} = ce^{-at}$$

$$\frac{a}{N} = ce^{-at} + b$$

$$N = \frac{a}{b + ce^{-at}}$$

$$\boxed{N = \frac{a}{b + (\frac{a}{N_0} - b)e^{-at}}}$$

since $N_0 = N(0) = \frac{a}{b+c}$

$b + c = \frac{a}{N_0}$

$c = \frac{a}{N_0} - b$

$$N(t) = \frac{R_0 N(t-1)}{1 + N(t-1)/M} \quad \text{(discrete case)}$$

$$\frac{1}{N(t)} = \frac{1}{R_0}\frac{1}{N(t-1)} + \frac{1}{R_0 \cdot M}$$

note that for the continuous case

$$\frac{1}{N(t)} = \frac{b}{a} + \left(\frac{a}{N_0} - b\right)\frac{e^{-at}}{a}$$

$$\frac{1}{N(t-1)} = \frac{b}{a} + \left(\frac{a}{N_0} - b\right)\frac{e^{-at}e^{a}}{a}$$

$$\frac{1}{N(t)} - \frac{e^{-a}}{N(t-1)} = \frac{b}{a} - \frac{b}{a}e^{-a} = \frac{1}{e^{a}}\left(\frac{be^{a}}{a} - \frac{b}{a}\right)$$

$$\frac{1}{N(t)} = \frac{1}{e^{a}}\frac{1}{N(t-1)} + \frac{1}{e^{a}(a/(be^{a} + b))}$$

which is the same form as the discrete case

3) $\lim_{t\to+\infty} N(t) = \lim_{t\to+\infty} \frac{a}{b + (\frac{a}{N_0} - b)e^{-at}} = \frac{a}{b}$, so the population does not grow infinitely and the model is good

Case 3: Density Dependent with Age Distributions

1) assumptions:

- the birth/death rates will vary depending on the number of young/old citizens

factors:

- b_m is the birth rate for the population of age m
- d_m is the death rate for the population of age m

relationships:

$$N_0(t+1) = b_0 N_0(t) + b_1 N_1(t) + \dots + b_M N_M(t)$$
$$N_1(t+1) = N_0(t) - d_0 N_0(t) = (1-d_0)N_0(t)$$
$$N_2(t+1) = (1-d_1)N_1(t)$$
$$\vdots$$
$$N_M(t+1) = (1-d_{M-1})N_{M-1}(t)$$

rewrite the relationships:

$$\vec{N}(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_M(t) \end{bmatrix}$$

$$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \\ \vdots \\ N_M(t+1) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \cdots & b_M \\ 1-d_0 & 0 & \cdots & 0 \\ 0 & 1-d_1 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1-d_{M-1} \end{bmatrix} \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_M(t) \end{bmatrix}$$

note: the top row is birth rates for all age groups
ie. how many babies do they give birth to
the diagonal is the surival rate for the previous age group
ie. how many live to grow a year older

$\vec{N}(t+1) = L \cdot \vec{N}(t)$ where L is called the <u>Leslie matrix</u>

2) solve

a <u>stable age distribution</u> exists if a population approaches an age distribution that is independent of time as time increases (i.e. $\lim_{t\to+\infty} \frac{1}{\|\vec{N}(t)\|_1} \vec{N}(t) = \vec{v}$ where $\|\vec{N}(t)\|_1 = \sum_{i=0}^{M} |N_i(t)|$

to find the stable age distribution, find the maximum eigenvalue of the Leslie Matrix and its normalized eigenvector

proof: suppose L is diagonalizable, i.e. $L \in \mathbb{R}^{m\times m}$ has m eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ and m linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

and L can be written as $L = VDV^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \text{ and } V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

since $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent, we can write $\vec{N}(t=0) = \vec{N}_0 = \sum_{i=1}^{m} c_i \vec{v}_i$

since $\vec{N}(n\Delta t) = L \cdot \vec{N}((n-1)\Delta t) = \dots = L^n \vec{N}_0$

$$= L^n \sum_{i=1}^{m} c_i \vec{v}_i = \sum_{i=1}^{m} c_i L^n \vec{v}_i = \sum_{i=1}^{m} c_i \lambda^n \vec{v}_i$$

$$\Rightarrow \frac{1}{\lambda_1^n} \vec{N}(n\Delta t) = c_1 \vec{v}_1 + \sum_{i=2}^{m} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^n \vec{v}_i$$

$\Rightarrow$ if $|\lambda_1| > |\lambda_i|$ for $i = 2, \dots, n$

then $\lim_{n\to\infty} \frac{1}{\lambda_1^n} \vec{N}(n\Delta t) = \lim_{n\to\infty} c_1\vec{v}_1 + \sum_{i=2}^{m} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^n \vec{v}_i = c_1 \vec{v}_1$

$\Rightarrow$ therefore we can approximate $\vec{N}(n\cdot\Delta t)$ with $c_1 \lambda_1^n \vec{v}_1$ for large n

process:

1) Find the maximum eigenvalue of the Leslie Matrix L by solving $\det(L - \lambda I) = 0$
2) Confirm that $|\lambda_1| > |\lambda_i|$ for $i \geq 2$
3) Find $\vec{v}_1$ associated with λ_1
4) Normalize $\vec{v}_1$ by dividing by its L_1 norm, $\frac{\vec{v}_1}{\|\vec{v}_1\|_1}$ is the stable age distribution

III Phase Plane Solutions and Equilibrium Points Stability

a <u>phase plane</u> is a visual display of certain characteristics certain kinds of differential equations

an <u>autonomous differential equation</u> is an equation in the form $\frac{dy}{dt} = f(y)$

<u>Method 1</u>: Graphing to identify stability

1) Find the equilibrium points (constant solutions of the DE)

2) Plot $\frac{dN}{dt}$ as a function of N

Draw arrows to indicate direction (evaluate $\frac{dN}{dt}$ at N between and outside equilibrium points)

if $\frac{dN}{dt} > 0$, then N is increasing so the arrows point right

if $\frac{dN}{dt} < 0$, then N is decreasing so the arrows point left

If arrows on both sides point towards the equilibrium point, it is <u>stable</u>

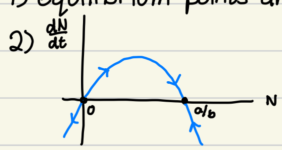
If arrows on both sides point away from the equilibrium point, it is <u>unstable</u>

note: one side pointing towards and one side pointing away is semistable

3) Based on the $\frac{dN}{dt}$ by N graph, draw an N by T graph

ex: logistic function $\frac{dN}{dt} = N(a-bN)$

1) equilibrium points are $N=0$, $N=\frac{a}{b}$

2) 

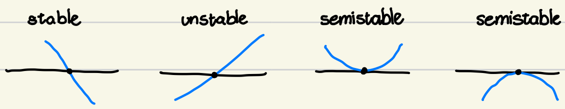
when $N<0$, $\frac{dN}{dt} < 0$

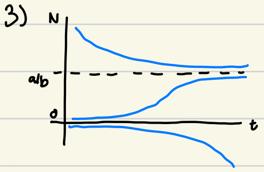
when $0<N<\frac{a}{b}$, $\frac{dN}{dt} > 0$

when $N > \frac{a}{b}$, $\frac{dN}{dt} < 0$

0 is unstable, $\frac{a}{b}$ is stable

note: alternatively:



3) 

converges because stable

diverges because unstable

<u>Method 2</u>: Analyze algebraically around equilibrium points to find stability

1) Find the equilibrium points (constant solutions of the DE)

2) Use Taylor's Theorem to analyze the neighborhoods of the equilibrium points

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \dots$$

ex: logistic function $\frac{dN}{dt} = N(a-bN)$

1) $N=0, \frac{a}{b}$

2) Taylor expansion of $f(N) = N(a-bN)$ at $N=\frac{a}{b}$

$$f(N) = f(\tfrac{a}{b}) + \frac{d}{dN}f(N)\Big|_{N=\frac{a}{b}} \cdot (N-\tfrac{a}{b}) + \frac{1}{2}\frac{d^2}{dN^2}f(N)\Big|_{N=\frac{a}{b}} \cdot (N-\tfrac{a}{b})^2$$

$$= 0 + \underbrace{(a-2b\cdot\tfrac{a}{b})(N-\tfrac{a}{b})}_{\text{use this as an approximation}} + \tfrac{1}{2}(-2b)(N-\tfrac{a}{b})^2$$

$$\approx -a(N-\tfrac{a}{b})$$

therefore $\frac{dN}{dt} \approx -a(N-\frac{a}{b})$ in the neighborhood of $N=\frac{a}{b}$

let $y = N-\frac{a}{b} \Rightarrow \frac{dy}{dt} = \frac{dN}{dt} = -a(N-\frac{a}{b}) = -ay$

$\frac{dy}{dt} = -ay \rightarrow y = ce^{-at} \Rightarrow N = ce^{-at} + \frac{a}{b}$ in the neighborhood of $N=\frac{a}{b}$

$\lim_{t\to+\infty} N(t) = \lim_{t\to+\infty} e^{-at} + \frac{a}{b} = \frac{a}{b}$, so $N=\frac{a}{b}$ is stable

Method 3: Perturbation analysis

1) Find the equilibrium points (constant solutions of DE)

2) Substitute an equilibrium point + $\varepsilon N_1(t)$ into the DE and solve

ex: logistic equation $N(a-bN)$

to evaluate stability at $N(t)=\frac{a}{b}$, use $N(t)=\frac{a}{b}+\varepsilon N_1(t)$ where $\varepsilon \ll \frac{a}{b}$

$$\frac{d}{dt}\left(\frac{a}{b}+\varepsilon N_1(t)\right) = \varepsilon \frac{dN_1}{dt}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a}{b}+\varepsilon N_1(t)\right) &= \left(\frac{a}{b}+\varepsilon N_1(t)\right)\left(a-b\left(\frac{a}{b}+\varepsilon N_1(t)\right)\right) \\ &= \left(\frac{a}{b}+\varepsilon N_1(t)\right)\left(-b\varepsilon N_1(t)\right) \\ &= -a\varepsilon N_1(t) - b\varepsilon^2 N_1^2(t)\end{aligned}$$

$$\Rightarrow \frac{dN_1}{dt} = -aN_1(t) - \underbrace{\cancel{b\varepsilon N_1(t)}}_{\varepsilon \text{ is small}} \approx -aN_1(t)$$

$$\Rightarrow N_1(t) = ce^{-at}$$

$$\Rightarrow \lim_{t\to\infty} N(t) = \lim_{t\to\infty} N_1(t) + \frac{a}{b} = \frac{a}{b}$$ so $N(t)$ is stable at $\frac{a}{b}$

IV COVID modeling (Final Project)

SIR model

1) factors:

Susceptible

Infected

Recovered

β : contact rate of individuals

γ : recovery rate of infected

assumptions:

$S \to I \to R$

total population is fixed

S decreases proportionally to S, I, β

I increases by how much S decreases

I decreases by how much R increases

R increases proportional to I and γ

relationships:

$N = S + I + R$; $R = N - I - S$

$$\frac{dS}{dt} = -\frac{\beta \cdot S \cdot I}{N}$$

$$\frac{dI}{dt} = \frac{\beta \cdot S \cdot I}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

2) solve

normalize using $s = \frac{S}{N}$, $i = \frac{I}{N}$, $r = \frac{R}{N}$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{N}\frac{dS}{dt} = \frac{1}{N}\left(-\frac{\beta SI}{N}\right) = -\beta si$$

$$\frac{di}{dt} = \frac{1}{N}\frac{dI}{dt} = \frac{1}{N}\left(\frac{\beta SI}{N} - \gamma I\right) = \beta si - \gamma i$$

$r = 1 - i - s$

where $s, i, r \in [0, 1]$

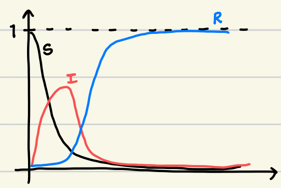
3) analyze

$\frac{ds}{dt} = -\beta si < 0 \Rightarrow s \downarrow$

$\frac{di}{dt} = (\beta s - \gamma) \cdot i = 0 \Rightarrow i = 0,\ s = \frac{\gamma}{\beta}$ are equilibrium points

$\frac{di}{dt} > 0 \Rightarrow \beta s - \gamma > 0 \Rightarrow s > \frac{\gamma}{\beta}$

$\frac{di}{dt} < 0 \Rightarrow \beta s - \gamma < 0 \Rightarrow s < \frac{\gamma}{\beta}$



SIR with vital dynamics (N is not constant)

1) b: birth rate

d: death rate

2)

$$\frac{dS}{dt} = -\frac{\beta SI}{N} + bN - dS$$

$$\frac{dI}{dt} = -\frac{\beta SI}{N} - \gamma I - dI$$

$$\frac{dR}{dt} = \gamma I - dR$$

SIRS without vital dynamics

1) α = return to susceptible rate

2)

$$\frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I - \alpha R$$

SIRS with vital dynamics

2)

$$\frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R + bN - dS$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - dI$$

$$\frac{dR}{dt} = \gamma I - \alpha R - dR$$

V Solving DEs

1) Find the solution of a linear 1st-order DE of the form

$$\begin{cases} \frac{d\vec{y}}{dt} = M\vec{y}(t) \\ \vec{y}(0) = \vec{y}_0 \end{cases}$$

note: $\boxed{e^{ai} = \cos(a) + i\sin(a)}$

2) Method 1: Eigenvalues and Eigenvectors

when $M \in \mathbb{C}^{n \times n}$ is diagonalizable, it has n lin-ind eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ corresponding to n eigenvalues $\lambda_1, \dots, \lambda_n$ such that $M\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1, \dots, n$

then $M = V\Lambda V^{-1}$ where

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$\vec{y}_i(t) = e^{\lambda t}\vec{v}_i$ for $i = 1, \dots, n$ are n lin-ind solutions of $\frac{d\vec{y}(t)}{dt} = M\vec{y}$

ex: solve the following ODE

$$\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = x - 2y \end{cases} \quad \text{with } x(0) = 8,\ y(0) = 4$$

rewrite in matrix form

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

find the eigenvalues

$$\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 1(-3) = \lambda^2 - 4 + 3 = \lambda^2 - 1 \Rightarrow \lambda = -1, 1$$

$\lambda = -1$

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = y \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$$

$$x = 3y \Rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$\Rightarrow c_1 + 3c_2 = 8,\ c_1 + c_2 = 4$

$\Rightarrow c_1 = 2,\ c_2 = 2$

$$\vec{y}(t) = e^{-t} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{t} \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

ex: solve the following ODE

$$\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = x + y \end{cases}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 \Rightarrow \lambda = 1 \pm i$$

$\lambda = 1 + i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = 1 - i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \Rightarrow \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{y}(t) &= c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= c_1 e^{t} e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{t} e^{-it} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= c_1 e^{t} \begin{bmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -i\cos(-t) + \sin(-t) \\ \cos(-t) + i\sin(-t) \end{bmatrix} \\ &= c_1 e^{t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_1 e^{t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} i + c_2 e^{t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} - c_2 e^{t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \\ &= (c_1 + c_2) e^{t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + (c_1 - c_2) i e^{t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

$\Rightarrow$ the general real solution is

$$\vec{y}(t) = \tilde{c}_1 e^{t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + \tilde{c}_2 e^{t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \quad \text{where } \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$$

2) Method 2: Exponential Method

when $\frac{d\vec{y}}{dt} = M\vec{y}(t)$ then $\vec{y}(t) = e^{Mt}\vec{y}(0)$ is the solution

where $e^{Mt} = Je^{Dt}J^{-1}$ if $M = JDJ^{-1}$ is diagonalizable

proof: $e^{x} = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$e^{Mt} = \sum_{n=1}^{\infty} \frac{t^n M^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n (JDJ^{-1})^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n JD^n J^{-1}}{n!} = J\left(\sum_{n=1}^{\infty} \frac{t^n D^n}{n!}\right)J^{-1} = Je^{Dt}J^{-1}$$

Asymptotic Properties of ODE solutions

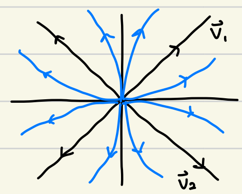
let $\frac{dx}{dt} = ax + by$; $\frac{dy}{dt} = cx + dy$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(M - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = ad - a\lambda - d\lambda + \lambda^2 - bc = \lambda^2 - (a+d)\lambda + ad - bc$$

set $p = a+d$, $q = ad - bc$ $\Rightarrow \Delta = p^2 - 4q$

if $\Delta > 0$, the eigenvalues are real (node or saddle points)

p>0, q>0

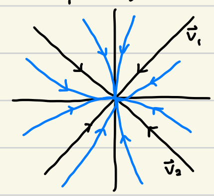


a) $p>0$, $q>0$: two positive real roots

$\Rightarrow \lim_{t\to+\infty} e^{\lambda_1 t} = +\infty$ and $\lim_{t\to+\infty} e^{\lambda_2 t} = +\infty$ b/c $\lambda_1, \lambda_2 > 0$

$\Rightarrow$ solution is unstable

p<0, q>0

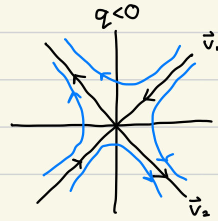


b) $p<0$, $q>0$: two negative real roots

$\Rightarrow \lim_{t\to+\infty} e^{\lambda_1 t} = 0$ and $\lim_{t\to+\infty} e^{\lambda_2 t} = 0$ b/c $\lambda_1, \lambda_2 < 0$

$\Rightarrow$ solution is stable

q<0

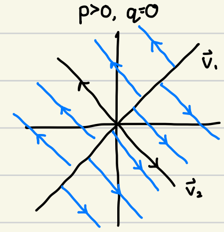


c) $q<0$: one positive and one negative root

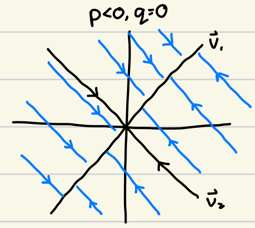
$\Rightarrow \lim_{t\to+\infty} e^{\lambda_1 t} = 0$ b/c $\lambda_1 < 0$; $\lim_{t\to+\infty} e^{\lambda_2 t} = +\infty$ b/c $\lambda_2 > 0$

$\Rightarrow$ solution is unstable

p>0, q=0



p<0, q=0



d) $p>0$, $q=0$: one positive and one zero root

$\Rightarrow \lim_{t\to+\infty} e^{\lambda_1 t} = 1$ b/c $\lambda_1 = 0$; $\lim_{t\to+\infty} e^{\lambda_2 t} = +\infty$ b/c $\lambda_2 > 0$

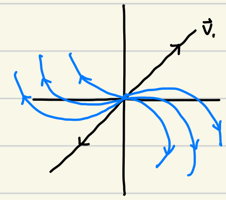
$\Rightarrow$ solution is unstable

e) $p<0$, $q=0$: one negative and one zero root

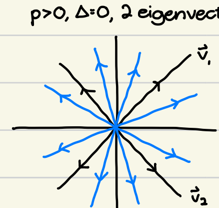
$\Rightarrow \lim_{t\to+\infty} e^{\lambda_1 t} = 1$ b/c $\lambda_1 = 0$; $\lim_{t\to+\infty} e^{\lambda_2 t} = 0$ b/c $\lambda_2 < 0$

$\Rightarrow$ solution is stable

p>0, Δ=0, 1 eigenvector



p>0, Δ=0, 2 eigenvectors

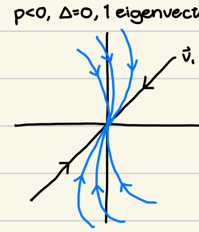


if $\Delta = 0$, there are repeated real eigenvalues (improper node)

f) $p>0$, not diagonalizable: one positive root w/ one eigenvector

$\Rightarrow$ solution is unstable

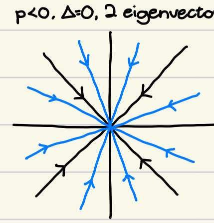
p<0, Δ=0, 1 eigenvector



g) $p>0$, diagonalizable: one positive root w/ two eigenvectors

$\Rightarrow$ solution is unstable

p<0, Δ=0, 2 eigenvectors



h) $p<0$, not diagonalizable: one negative root w/ one eigenvector

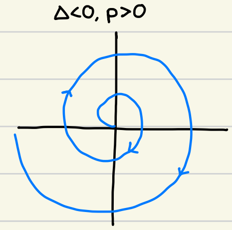
$\Rightarrow$ solution is stable

i) $p<0$, diagonalizable: one negative root w/ two eigenvectors

$\Rightarrow$ solution is stable

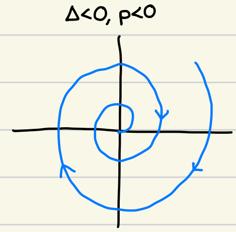
?) $p=0$: both roots are 0, solution is unstable

Δ<0, p>0



if $\Delta < 0$, the eigenvalues are complex (spiral); $\lambda = \mu \pm vi$

Δ<0, p<0

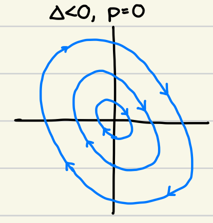


j) $p>0$: real component $\mu > 0$

$\Rightarrow \lim_{t\to+\infty} e^{\mu t}(c_1 \sin(vt)\vec{v}_1 + c_2 \cos(vt)\vec{v}_2) = +\infty$

$\Rightarrow$ solution is unstable

Δ<0, p=0



k) $p<0$: real component $\mu < 0$

$\Rightarrow \lim_{t\to+\infty} e^{\mu t}(c_1 \sin(vt)\vec{v}_1 + c_2 \cos(vt)\vec{v}_2) = 0$

$\Rightarrow$ solution is stable

l) $p=0$. real component $\mu = 0$

$\Rightarrow$ cyclic function $c_1 \sin(vt)\vec{v}_1 + c_2 \cos(vt)\vec{v}_2$

VI Predator Prey Model

1) factors: fish population F

shark population S

note: $aF-bF^2$ is logistic growth

assumptions: unlimited fish food $(b=0)$

no migration

relations: $\left.\begin{array}{l}\frac{dF}{dt}=aF-bF^2-cFS \\ \frac{dS}{dt}=-kS+\lambda FS\end{array}\right.$; $a,b,c,k,\lambda \geq 0$ $\Big\}$ Lotka-Volterra Model

2) solve

$$\left.\begin{array}{l}\frac{dF}{dt}=(a-cS)F=0 \\ \frac{dS}{dt}=(-k+\lambda F)S=0\end{array}\right] \Rightarrow \begin{array}{l}(F,S)=(0,0) \\ (F,S)=(\frac{k}{\lambda},\frac{a}{c})\end{array}$$

$$\begin{bmatrix}\frac{dF_1}{dt} \\ \frac{dS_1}{dt}\end{bmatrix} = M\cdot\begin{bmatrix}F_1 \\ S_1\end{bmatrix}$$

3) evaluate the stability

let $g(F,S)=aF-cFS$

$f(F,S)=-kS+\lambda FS$

use Taylor expansion around equilibrium points (F_e,S_e)

ie. use $F_1(t)$ and $S_1(t)$ to approximate $F(t)$ and $S(t)$ at (F_e,S_e)

let $F(t)=F_e+\varepsilon F_1(t)$

$S(t)=S_e+\varepsilon S_1(t)$

then $\frac{dF}{dt}=g(F_e+\varepsilon F_1(t), S_e+\varepsilon S_1(t))$

$$=\underbrace{\cancel{g(F_e,S_e)}}_{\approx 0}+\frac{\partial}{\partial F}(g(F_e,S_e)\cdot\varepsilon F_1(t))+\frac{\partial}{\partial S}(g(F_e+S_e)\cdot\varepsilon S_1(t))+\underbrace{\cancel{O(\varepsilon^2)}}_{\approx 0}$$

$\frac{dS}{dt}=f(F_e+\varepsilon F_1(t), S_e+\varepsilon S_1(t))$

$$=\underbrace{\cancel{f(F_e+S_e)}}_{\approx 0}+\frac{\partial}{\partial F}(f(F_e,S_e)\cdot\varepsilon F_1(t))+\frac{\partial}{\partial S}(g(F_e+S_e)\cdot\varepsilon S_1(t))+\underbrace{\cancel{O(\varepsilon^2)}}_{\approx 0}$$

and $\frac{dF}{dt}=\frac{d}{dt}(F_e+\varepsilon F_1(t))=\varepsilon\frac{dF_1}{dt}$

$\frac{dS}{dt}=\frac{d}{dt}(S_e+\varepsilon S_1(t))=\varepsilon\frac{dS_1}{dt}$

so $\cancel{\varepsilon}\frac{dF_1}{dt}=\cancel{\varepsilon}\frac{\partial}{\partial F}(g(F_e,S_e))F_1+\cancel{\varepsilon}\cdot\frac{\partial}{\partial S}(g(F_e,S_e))S_1 \Rightarrow \frac{\partial F_1}{\partial t}=\left(\frac{\partial g}{\partial F}\Big|_{(F_e,S_e)}\right)F_1+\left(\frac{\partial g}{\partial S}\Big|_{(F_e,S_e)}\right)S_1$

$\cancel{\varepsilon}\frac{dS_1}{dt}=\cancel{\varepsilon}\frac{\partial}{\partial F}(f(F_e,S_e))F_1+\cancel{\varepsilon}\cdot\frac{\partial}{\partial S}(f(F_e,S_e))S_1 \Rightarrow \frac{\partial S_1}{\partial t}=\left(\frac{\partial f}{\partial F}\Big|_{(F_e,S_e)}\right)F_1+\left(\frac{\partial f}{\partial S}\Big|_{(F_e,S_e)}\right)S_1$

Method 1: find the eigenvalues using F_1 and S_1

for $(0,0)$

$$M=\begin{bmatrix}\frac{\partial g}{\partial F}\Big|_{(0,0)} & \frac{\partial g}{\partial S}\Big|_{(0,0)} \\ \frac{\partial f}{\partial F}\Big|_{(0,0)} & \frac{\partial f}{\partial S}\Big|_{(0,0)}\end{bmatrix}=\begin{bmatrix}(a-cS)\big|_{(0,0)} & (-cF)\big|_{(0,0)} \\ (\lambda S)\big|_{(0,0)} & (-k+\lambda F)\big|_{(0,0)}\end{bmatrix}=\begin{bmatrix}a & 0 \\ 0 & -k\end{bmatrix}$$

$\Rightarrow$ eigenvalue $a>0 \Rightarrow$ unstable

for $(\frac{k}{\lambda},\frac{a}{c})$

$$M=\begin{bmatrix}(a-cS)\big|_{(\frac{k}{\lambda},\frac{a}{c})} & (-cF)\big|_{(\frac{k}{\lambda},\frac{a}{c})} \\ (\lambda S)\big|_{(\frac{k}{\lambda},\frac{a}{c})} & (-k+\lambda F)\big|_{(\frac{k}{\lambda},\frac{a}{c})}\end{bmatrix}=\begin{bmatrix}0 & -\frac{ck}{\lambda} \\ \frac{\lambda a}{c} & 0\end{bmatrix}$$

$\det(M-tI)=t^2+ak=0 \Rightarrow t=\pm\sqrt{ak}\,i$ are complex eigenvalues

$\Rightarrow (\frac{k}{\lambda},\frac{a}{c})$ is algebraically unstable

This method doesn't give you much information about the original model

Method 2: study $\frac{dF}{dS}$ near $(0,0)$

note: $F(t) = F_0 + \varepsilon F_1(t) \Rightarrow F_1(t) = \frac{F(t)}{\varepsilon}$

$S(t) = S_0 + \varepsilon S_1(t) \Rightarrow S_1(t) = \frac{S(t)}{\varepsilon}$

$$\frac{dF}{dS} = \frac{dF/dt}{dS/dt} = \frac{dF_1/dt}{dS_1/dt} = \frac{aF_1}{-kS_1} = \frac{aF}{-kS}$$

$\Rightarrow \frac{dF}{F} = -\frac{a}{k}\frac{dS}{S}$

$\Rightarrow \ln F = -\frac{a}{k}\ln S + c$

$\Rightarrow F = \tilde{c} \cdot S^{-a/k}$

$\Rightarrow \lim_{S\to+\infty} F = 0, \ \lim_{S\to 0} F = +\infty$

Method 3: solve for F and S near $(\frac{k}{\lambda}, \frac{a}{c})$ since they're periodic

since $\frac{dF_1}{dt}$ is a function of S_1

and $\frac{dS_1}{dt}$ is a function of F_1

$\frac{dF}{dt} = \varepsilon \frac{dF_1}{dt} = -\varepsilon \frac{ck}{\lambda} S_1$ and $\frac{dS}{dt} = \varepsilon \frac{dS_1}{dt} = \varepsilon \frac{\lambda a}{c} F_1$

note: $\frac{d^2F_1}{dt^2} = \frac{d}{dt}\left(-\frac{ck}{\lambda}S_1\right) = -\frac{ck}{\lambda}\left(\frac{dS_1}{dt}\right) = -\frac{ck}{\lambda}\cdot\frac{a\lambda}{c}F_1 = -akF_1$

$\Rightarrow \frac{d^2F_1}{dt^2} + akF_1 = 0 \Rightarrow t^2 = \pm\sqrt{ak}\,i$ are the eigenvalues

$\Rightarrow F_1 = c_1\cos(\sqrt{ak}\,t) + c_2\sin(\sqrt{ak}\,t)$

similarly, $S_1 = c_3\cos(\sqrt{ak}\,t) + c_4\sin(\sqrt{ak}\,t)$

solve for $c_1, \dots, c_4$ using the initial conditions

$\Rightarrow$ F and S are periodic near $(\frac{k}{\lambda}, \frac{a}{c})$

Method 4: use substitution to study the relation between F and S

$\frac{dF}{dt} = aF - cFS = aF(1 - \frac{c}{a}S)$

$\frac{dS}{dt} = -kS + \lambda FS = kS(-1 + \frac{\lambda}{k}F)$

let $u = \frac{c}{a}S, \ v = \frac{\lambda}{k}F$

$\frac{du}{dt} = \frac{c}{a}\frac{dS}{dt} = \frac{c}{a}kS(-1 + \frac{\lambda}{k}F) = ku(-1+v)$

$\frac{dv}{dt} = \frac{\lambda}{k}\frac{dF}{dt} = \frac{\lambda}{k}aF(1 - \frac{c}{a}S) = av(1-u)$

$\frac{du}{dv} = \frac{ku(-1+v)}{av(1-u)}$

$\frac{1-u}{u}du = \frac{k}{a}\frac{v-1}{v}dv$

$\ln(u) - u = \frac{k}{a}(v - \ln(v)) + c$

$ue^{-u} = \tilde{c}\,v^{-\frac{k}{a}}e^{\frac{k}{a}v}$

$\frac{c}{a}Se^{-\frac{c}{a}S} = \tilde{c}\left(\frac{\lambda}{k}\right)^{-\frac{k}{a}}F^{-\frac{k}{a}}e^{\frac{\lambda}{a}F}$

$F^{-\frac{k}{a}}e^{\frac{\lambda}{a}F} = \tilde{\tilde{c}}\,Se^{-\frac{c}{a}S}$

$F^{-k}e^{\lambda F} = \hat{c}\,S^{a}e^{-cS} = z$

study relationship of F and S to z

$\frac{dz}{dF} = -kF^{-k-1}e^{\lambda F} + F^{-k}(\lambda e^{\lambda F})$

$= F^{-k}e^{\lambda F}\left(\frac{k}{F} + \lambda\right) = 0$

$\Rightarrow F = \frac{k}{\lambda}$

$\Rightarrow$ when $F > \frac{k}{\lambda}$, $\frac{dz}{dF} > 0$

when $F < \frac{k}{\lambda}$, $\frac{dz}{dF} < 0$

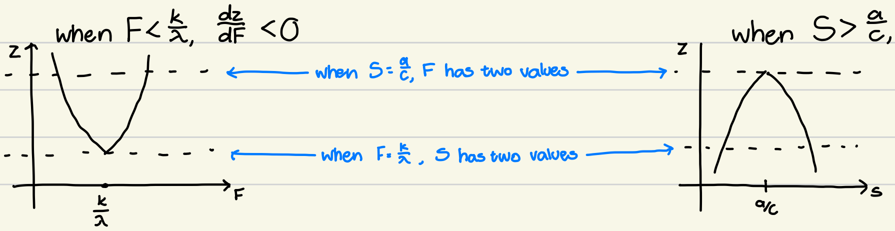
$\frac{dz}{dS} = (-c)\hat{c}e^{-cS}S^{a} + \hat{c}e^{-cS}\cdot aS^{a-1}$

$= \hat{c}e^{-cS}S^{a}(-c + \frac{a}{S}) = 0$

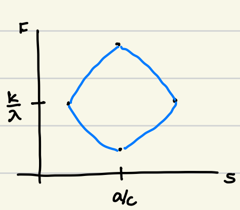
$\Rightarrow S = \frac{a}{c}$

$\Rightarrow$ when $S < \frac{a}{c}$, $\frac{dz}{dS} < 0$

when $S > \frac{a}{c}$, $\frac{dz}{dS} > 0$



when z is the same, you can find the relationship b/w F and S



VII Cooperation Model

1) factors: total population N
percentage of cooperators x
percentage of cheaters y
number of random interactions n
benefit b
cost c
proportional constant k
death rate d

assumptions: cooperators work together
cheaters do nothing

if cooperator + cooperator: costs $\frac{c}{2}$ each
benefits b each

cooperator + cheater: costs c for cooperator, costs 0 for cheater
benefits b each

cheater + cheater: costs 0 each
benefits 0 each

payoff of an interaction = benefit − cost

total payoff = weighted payoffs of interactions

total payoff to cooperators = #cooperators $\cdot (b-\frac{c}{2})$ + #cheaters $\cdot (b-c)$

total payoff to cheaters = # cooperators $\cdot (b-0)$ + # cheaters $\cdot (0-0)$

birth rate $\propto$ total payoff $\cdot$ population

birth rate of cooperators = $k \cdot$ total payoff to cooperaters $\cdot$ # of cooperators

birth rate of cheaters = $k \cdot$ total payoff to cheater $\cdot$ # of cheater

payoff matrix:

$$\begin{bmatrix} \text{payoff to cooperators in cooperator + cooperator} & \text{payoff to cooperators in cooperator + cheater} \\ \text{payoff to cheater in cooperator + cheater} & \text{payoff to cheaters in cheaters + cheaters} \end{bmatrix}$$

relationships:

$\frac{d}{dt}(Nx) = k \cdot Nx \cdot (nx(b-\frac{c}{2}) + ny(b-c)) - Nx \cdot d$

$\frac{d}{dt}(Ny) = k \cdot Ny(nx(b) + ny(0)) - Ny \cdot d$

$x+y=1 \Rightarrow \frac{dx}{dt} + \frac{dy}{dt} = 0$

2) solve:

$\frac{dx}{dt} + \frac{dy}{dt} = 0$

$\Rightarrow x(k(nx(b-\frac{c}{2}) + ny(b-c)) - d) + y(k(nx(b-0) + ny(0)) - d) = 0$

$\Rightarrow kx(nxb - nx\frac{c}{2} + nyb - nyc) - xd + ky(nxb) - yd = 0$

$\Rightarrow kx(nxb - \frac{1}{2}nxc + n(1-x)b - n(1-x)c) - xd + k(1-x)(nxb) - (1-x)d = 0$

$\Rightarrow knx(b-\frac{c}{2})(2-x) = d \geq 0$

$\Rightarrow b \geq \frac{c}{2} \geq 0$

substitute $d = knx(b-\frac{c}{2})(2-x)$ back into the original equation

substitute $y = 1-x$, $x = 1-y$ to get $\frac{dx}{dt} = g(x)$ and $\frac{dy}{dt} = f(y)$

3) solve $g(x) = 0$

plot $\frac{dx}{dt}$ by x to get stability

VIII Stochastic Growth Processes

1) factors: regular subintervals: Δt

total time interval: $[0, T]$

birth rate: b

size of population N

initial size of a population N_0

probability a population is size N at time t: $P_N(t)$

assumptions: probability of a birth per unit of time $= b \cdot \Delta t$

probability of k births from N individuals per unit time $= \underbrace{\overbrace{\frac{N!}{(N-k)!k!}}^{\binom{N}{k}}}_{\text{number of ways to choose k out of N individuals}} \underbrace{(b \cdot \Delta t)^k (1 - b \cdot \Delta t)^{N-k}}_{\text{probability of k births (and N-k not-births)}}$

relationships: $P_N(t + \Delta t) = (1 - b\Delta t)^N P_N(t) + \binom{N-1}{1}(1 - b\Delta t)^{N-2}(b\Delta t)P_{N-1}(t) + \binom{N-2}{2}(1 - b\Delta t)^{N-4}(b\Delta t)^2 P_{N-2}(t) + \ldots$

$\approx (1 - Nb\Delta t)P_N(t) + (N-1)(b\Delta t)P_{N-1}(t)$

2) solve:

use Taylor's expansion

$$P_N(t + \Delta t) = P_N(t) + \frac{dP_N}{dt}\Delta t + \frac{1}{2}\frac{d^2P_N(t)}{dt^2}\Delta t^2 + \ldots$$

$$\approx P_N(t) + \frac{dP_N}{dt}\Delta t$$

$$P_N(t) + \frac{dP_N}{dt}\Delta t = (1 - Nb\Delta t)P_N(t) + (N-1)(b\Delta t)P_{N-1}(t)$$

$$\frac{dP_N}{dt}\cancel{\Delta t} = -Nb\cancel{\Delta t}\, P_N(t) + (N-1)(b\cancel{\Delta t})P_{N-1}(t)$$

$$\Rightarrow \boxed{\frac{dP_N}{dt} = -Nb\, P_N(t) + (N-1)\, b\, P_{N-1}(t)}$$

initial condition: $P_{N_0}(0) = 1$, $P_{N \neq N_0}(0) = 0$

use induction to find the general solution

$$\boxed{P_N(t) = (1 - e^{-bt})^{N-1} e^{-bt}}$$

note: $P_N(t)$ is called the probability mass function

1) $\sum_{N=1}^{\infty} P_N(t) = 1$, i.e. a population must be some finite size N

2) $E(t) = \sum_{N=1}^{\infty} N \cdot P_N(t) = e^{bt}$, i.e. expected population (mean) at time t is e^{bt}

IX Flow: Random Walks

1) factors: time: $t_k = t_0 + k\Delta t$

length of a step: Δl

average position: $\bar{x}_k = \mathbb{E}(x_k)$

probability of moving right: p

directional distance traversed in $[t_k, t_{k+1})$: d_{k+1}

relationships: $d_{k+1} = \begin{cases} \Delta l & \text{with probability } p \\ -\Delta l & \text{with probability } (1-p) \end{cases}$

$$x_{k+1} = x_k + d_{k+1}$$

2) find the expectation:

$$\mathbb{E}(x_{k+1}) = \mathbb{E}(x_k + d_{k+1}) = \mathbb{E}(x_k) + \mathbb{E}(d_{k+1})$$
$$= \mathbb{E}(x_k) + p(\Delta l) - (1-p)(\Delta l)$$
$$= \mathbb{E}(x_k) + (2p-1)\Delta l$$
$$= k(2p-1)\Delta l$$

find the variance.

$$var(x_{k+1}) = \mathbb{E}((x_{k+1} - \mathbb{E}(x_{k+1}))^2) = \mathbb{E}(x_{k+1}^2) - 2\mathbb{E}(x_{k+1})\mathbb{E}(x_{k+1}) + \mathbb{E}(x_{k+1})^2 = \mathbb{E}(x_{k+1}^2) - \mathbb{E}(x_{k+1})^2$$
$$= \mathbb{E}(x_k^2 + 2d_{k+1} + d_{k+1}^2) - \mathbb{E}(x_k + d_{k+1})^2$$
$$= \mathbb{E}(x_k^2) + 2\mathbb{E}(d_{k+1}) + \mathbb{E}(d_{k+1}^2) - \mathbb{E}(x_k)^2 - \cancel{2\mathbb{E}(x_k)\mathbb{E}(d_{k+1})} - \mathbb{E}(d_{k+1})^2$$

$\to 0$ b/c independent

$$= \mathbb{E}(x_k^2) - \mathbb{E}(x_k)^2 + \mathbb{E}(d_{k+1}^2) - \mathbb{E}(d_{k+1})^2 + 2\mathbb{E}(d_{k+1})$$
$$= var(x_k) + var(d_{k+1}) + (2p-1)\Delta l \qquad = var(x_k) + p(\Delta l)^2 + (1-p)(-\Delta l)^2 - ((2p-1)\Delta l)^2 + (2p-1)\Delta l$$
$$= k \cdot var(d_{k+1}) + k(2p-1)\Delta l \qquad = k\Delta l^2 - k((2p-1)\Delta l)^2 + k(2p-1)\Delta l$$

find the probability of being at location d at time t_N: $P_N(d)$

let d_1 = # of steps taken to the right

d_2 = # of steps taken to the left

then $d_1 + d_2 = N$

$$d_1 \cdot \Delta l - d_2 \cdot \Delta l = d$$
$$\Rightarrow d_1 = \frac{N + d/\Delta l}{2}, \quad d_2 = \frac{N - d/\Delta l}{2}$$
$$\Rightarrow d_1 = \frac{N}{2} + \frac{d}{2\Delta l}, \quad d_2 = \frac{N}{2} - \frac{d}{2\Delta l}$$

therefore, $P_N(d) = \binom{N}{\frac{N}{2} + \frac{d}{2\Delta l}} (p)^{\frac{N}{2} + \frac{d}{2\Delta l}} (1-p)^{\frac{N}{2} - \frac{d}{2\Delta l}}$

X Flow: Diffusion

1) factors: time: t density: $\rho(x,t)$

distance from the origin: x flux: $q(x,t)$

relationships: for some interval with length Δx around point x, i.e. $\left[x-\frac{\Delta x}{2}, \hat{x}+\frac{\Delta x}{2}\right]$

$\Delta x \cdot \rho(x,t)$ = the total number of individuals in $\left[x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right]$ at time t

$\rho(x,t)$ = density = the average number of individuals over $\left[x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right]$ at time t

$\Delta t \cdot q(x,t)$ = the total number of individuals that visit x over time $[t, t+\Delta t]$

$q(x,t)$ = flux = the average number of individuals at x over time $[t, t+\Delta t]$

2) solve

$(\rho(x,t+\Delta t)-\rho(x,t))\Delta x$ = the change in the number of individuals in $\left[x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right]$ over time $[t, t+\Delta t]$

$(q(x-\frac{\Delta x}{2},t)-q(x+\frac{\Delta x}{2},t))\Delta t$ = the change in the number of individuals in $\left[x-\frac{\Delta x}{2}, x+\frac{\Delta x}{2}\right]$ over time $[t, t+\Delta t]$

$$\Rightarrow \frac{\rho(x,t+\Delta t)-\rho(x,t)}{\Delta t} = -\frac{q(x+\frac{\Delta x}{2},t)-q(x-\frac{\Delta x}{2},t)}{\Delta x}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} \Rightarrow \boxed{\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=0}$$ continuity equation

let $p(x,t)$ be the probability that an individual is at position $x=n\cdot\Delta x$ at time $t=m\cdot\Delta t$

for some constants n, m

then $p(x,t+\Delta t)=\frac{1}{2}p(x-\Delta x,t)+\frac{1}{2}p(x+\Delta x,t)$, assuming there's an equal probability of moving left or right

use Taylor's expansion

1) $p(x,t+\Delta t)=p(x,t)+\frac{\partial p}{\partial t}\cdot\Delta t+\frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2+\dots$

$\approx p(x,t)+\frac{\partial p}{\partial t}\cdot\Delta t$

2) $p(x-\Delta x,t)=p(x,t)-\frac{\partial p}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(-\Delta x)^2+\dots$

$p(x+\Delta x,t)=p(x,t)+\frac{\partial p}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2+\dots$

$\Rightarrow \frac{1}{2}p(x-\Delta x,t)+\frac{1}{2}p(x+\Delta x,t)\approx p(x,t)+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2$

$$\Rightarrow \cancel{p(x,t)}+\frac{\partial p}{\partial t}\cdot\Delta t=\cancel{p(x,t)}+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2$$

$$\Rightarrow \frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2 p}{\partial x^2}\underbrace{\lim_{\Delta x,\Delta t\to 0}\frac{(\Delta x)^2}{\Delta t}}_{=D}$$

note: $\rho(x,t)=\frac{N\cdot p(x,t)}{\Delta x} \Rightarrow p(x,t)=\frac{\Delta x}{N}\rho(x,t)$

$$\Rightarrow \frac{\partial}{\partial t}\left(\cancel{\frac{\Delta x}{N}}\rho(x,t)\right)=\frac{D}{2}\frac{\partial^2}{\partial x^2}\left(\cancel{\frac{\Delta x}{N}}\rho(x,t)\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t}=\tilde{D}\frac{\partial^2 \rho}{\partial x^2}$$

$$\Rightarrow \tilde{D}\frac{\partial^2 \rho}{\partial x^2}=-\frac{\partial q}{\partial x}$$ substitute the continuity equation

$$\Rightarrow \frac{\partial}{\partial x}\left(\tilde{D}\frac{\partial \rho}{\partial x}-q\right)=0$$

$$\Rightarrow \tilde{D}\frac{\partial \rho}{\partial x}-q=C(t)$$

if $C(t)=0$, then $\boxed{q=-\tilde{D}\frac{\partial \rho}{\partial x}}$ this is called Fick's law

use Fourier's Transformation on ρ with respect to x to solve $\boxed{\frac{\partial \rho}{\partial t}=D\frac{\partial^2 \rho}{\partial x^2}}$

$$\rho(x,t)=\frac{C}{2\sqrt{\pi D t}}e^{-\frac{x^2}{4tD}}$$

3) analyze

$\lim_{t\to 0^+}\rho(x,t)=\lim_{t\to 0^-}\rho(x,t)=+\infty$ all bacteria are at $x=0$ when $t=0$

$\lim_{t\to +\infty}\rho(x,t)=\lim_{t\to 0^-}\rho(x,t)=0$ all bacteria will leave $x=0$

$\rho(0,0)=\frac{C}{2\sqrt{\pi D t}}$ = decay of density at origin over time

XI Flow: Diffusion on a Bounded Domain

1) assumptions: $x \in [0, L]$

c = specific heat
ρ = material density
constant

T = temperature
$c\rho T$ = heat energy
q = flux

relationships: continuity equation: $\frac{\partial}{\partial t}(c\rho T) = -\frac{\partial q}{\partial x}$

Fick's law: $q(x,t) = -k\frac{\partial T}{\partial x}$

2) solve

$$\frac{\partial}{\partial t}(c\rho T) = k\frac{\partial^2 T}{\partial x^2}$$

$\boxed{\frac{\partial T}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 T}{\partial x^2}}$ where $0<x<L$, $t>0$, and $T(x,0) = T_0(x)$ initial condition is satisfied

ex: Dirichlet / fixed boundary condition: temperature on boundaries is constant

i.e. $T(0,t) = \alpha_0$, $T(L,t) = \alpha_L$

ex: Neumann / fixed-flux boundary condition: flux on boundaries is constant

i.e. $q(0,t) = \beta_0$ $\quad q(L,t) = \beta_L$

$\Rightarrow \left.\frac{\partial T}{\partial x}\right|_{(0,t)} = \gamma_0 \quad \left.\frac{\partial T}{\partial x}\right|_{(L,t)} = \gamma_L$ and $\beta_0\gamma_0 < 0$, $\beta_L\gamma_L < 0$

ex: perfect insulation: flux is 0, heat cannot enter or leave

i.e. $q(0,t) = q(L,t) = 0$

let $T_\infty(x) = \lim_{t\to+\infty} T(x,t)$ be the steady state at x (i.e. it is independent of t)

$\Rightarrow \frac{\partial}{\partial t}(T_\infty(x)) = 0$

$\Rightarrow D\cdot\frac{\partial^2}{\partial x^2}(T_\infty(x)) = 0$

$\Rightarrow \frac{\partial}{\partial x}T_\infty(x) = c_1$

$\Rightarrow T_\infty(x) = c_1x + c_2$

then substitute in the boundary conditions to find c_1 and c_2