Question 1

Consider an analog signal, $x_a(t) = \sin(480\pi t) + 3\sin(720\pi t)$.

(a) Determine the minimum sampling frequency that can be used to sample and then perfectly reconstruct $x_a(t)$.

$\sin(480\pi t)$ $\quad$ $\sin(720\pi t)$

$\downarrow$ $\quad$ $\downarrow$

240 cycles/time $\quad$ 360 cycles/time

$\downarrow$ $\quad$ $\downarrow$

480 samples/time $\quad$ $\boxed{\text{720 samples/time}}$

This one is bigger

(b) What is the maximum analog frequency that can be reconstructed from an analog signal sampled at $F_s = 600$ Hz?

$F_s = 600\text{ Hz} \longrightarrow$ can reconstruct $\frac{1}{2}F_s = 300$ cycles/time

(c) Suppose that $x_a(t)$ is sampled at $F_s = 600$ Hz. What is the sampled signal $x(n)$? Determine whether aliasing occurs in $x(n)$; make sure to justify your answer.

$$X(n) = X_a\left(t = \frac{n}{F_s}\right)$$

$$= \sin\left(480\pi \frac{n}{600}\right) + 3\sin\left(720\pi \frac{n}{600}\right)$$

$$= \sin(0.8\pi n) + 3\sin(1.2\pi n)$$

aliasing $\rightarrow$ $$= \sin(0.8\pi n) - 3\sin(0.8\pi n)$$

$$= -2\sin(0.8\pi n)$$

(d) If $x(n)$ is passed through an ideal D/A converter designed for $F_s = 600$ Hz, then what is the reconstructed signal in continuous time?

$\sin(480\pi t)$ wouldn't be aliased $\rightarrow$ $\sin(480\pi t)$

But $X(n) = -2\sin(0.8\pi n)$, which is $\sin(480\pi t)$ sampled, then multiplied by -2.

Therefore, the reconstructed signal is $-2\sin(480\pi t)$.

Question 2

Consider a discrete signal $x(n)$ defined as follows,

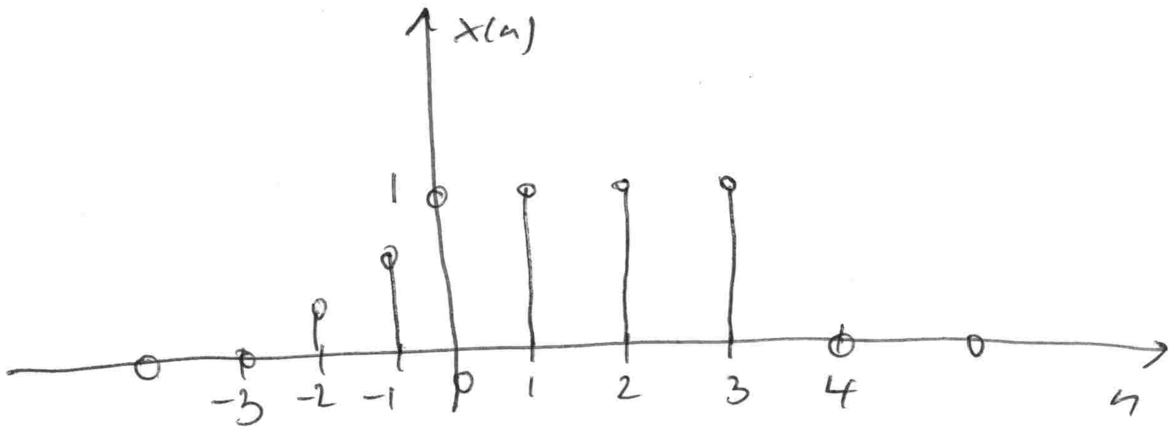
$$x(n) = \begin{cases} 1 + \frac{n}{3}, & -3 \le n \le -1 \\ 1, & 0 \le n \le 3 \\ 0, & else \end{cases} .$$

(a) Determine indices n where $x(n)$ is non-zero, and compute its value for those indices.

n	-3	-2	-1	0	1	2	3	4
$x(n)$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	1	1	1	0

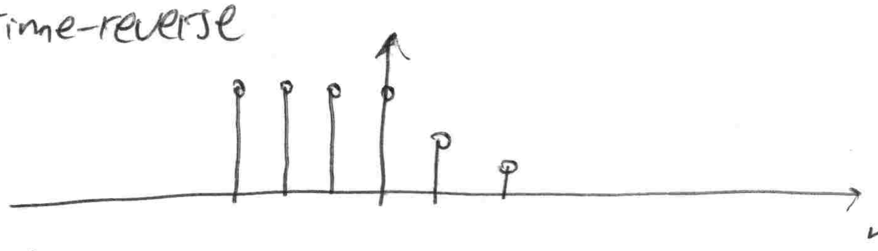
range of non-zeros

(b) Sketch the signal $x(n)$.

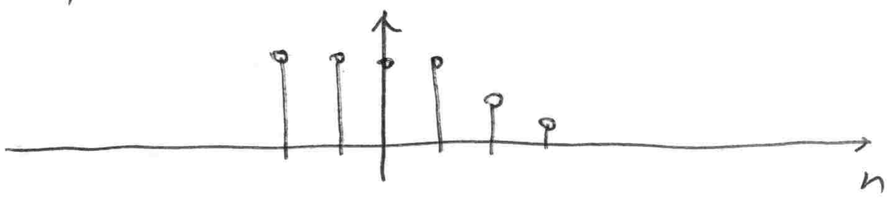


(c) Sketch the signals obtained if we (*i*) first time-reverse $x(n)$, and then delay the resulting signal by one sample; and (*ii*) first delay $x(n)$ by one sample, and then time-reverse the resulting signal.

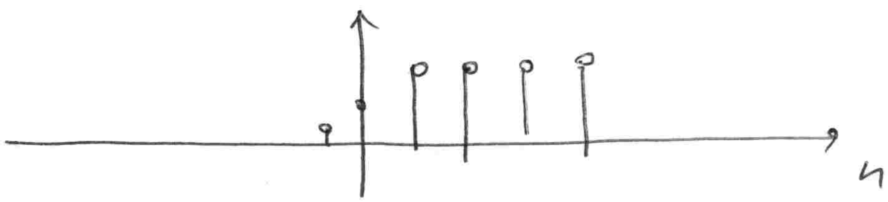
(i) Time-reverse



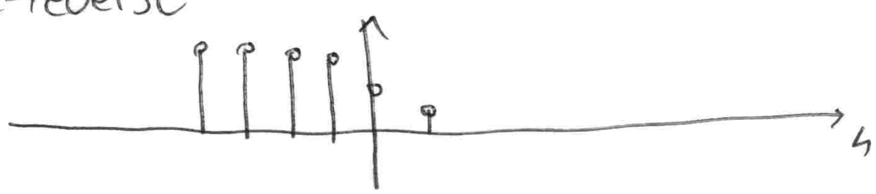
Delay



(ii) Delay



Time-reverse



Note The answers in parts (i) and (ii) are different.

Question 3

Compute the Fourier transform $X(\omega)$ for the following signal, $x(n) = (\frac{8}{5})^n u(-n)$.

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{8}{5}\right)^n e^{-j\omega n}$$

$l = -n$ $\quad = \sum_{l=0}^{\infty} \left(\frac{5}{8}\right)^l e^{+j\omega l}$

$\left(\frac{8}{5}\right)^n = \left(\frac{5}{8}\right)^l$

$e^{-j\omega n} = e^{+j\omega l}$

$$= \sum_{l=0}^{\infty} \left(\frac{5}{8} e^{j\omega}\right)^l$$

$$= \frac{1}{1 - \frac{5}{8} e^{j\omega}}$$

Question 4

Consider the following system, $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$.

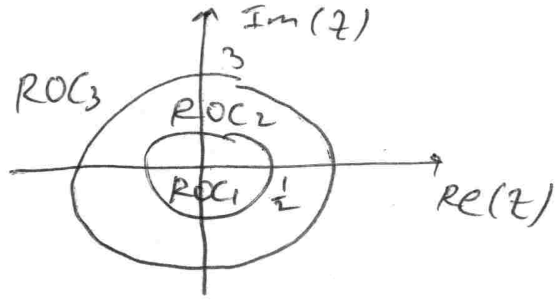
(a) Describe all possible regions of convergence (ROC's).

The possible borders are $|z| = \frac{1}{2}$, $|z| = 3$

$ROC_1 = \{ |z| < \frac{1}{2} \}$

$ROC_2 = \{ \frac{1}{2} < |z| < 3 \}$

$ROC_3 = \{ |z| > 3 \}$



(b) What is the ROC in order for H be stable? Justify your answer.

Stable requires the unit circle to lie within the ROC. Only ROC_2 satisfies this.

$$ROC_H = \{ \tfrac{1}{2} < |z| < 3 \}$$

(c) Compute the impulse response $h(n)$ that corresponds to the stable system H in part (b).

Due to the ROC, $\frac{1}{1-\frac{1}{2}z^{-1}}$ is causal, and $\frac{2}{1-3z^{-1}}$ is anti-causal. Taking the inverse z transform yields

$$h(n) = \left(\tfrac{1}{2}\right)^n u(n) - 2 \cdot 3^n u(-n-1)$$

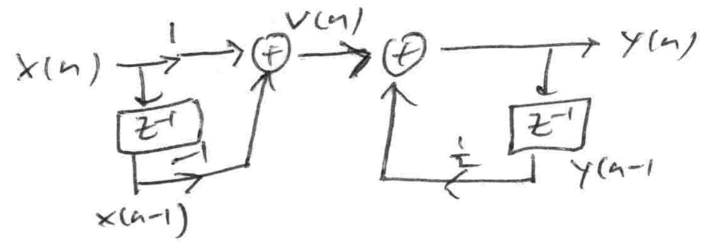
Question 5

Consider the following difference equation,

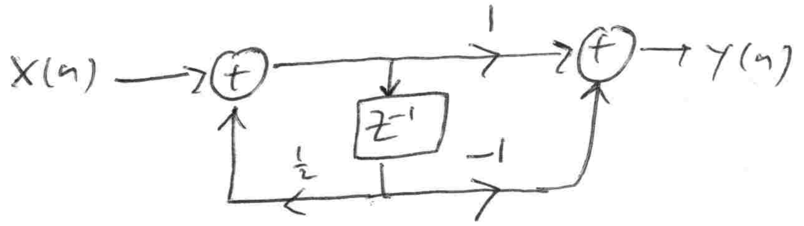
$$y(n) = \frac{1}{2}y(n-1) + x(n) - x(n-1).$$

(a) Sketch an implementation of the system using direct form II.

Form I:



Form II:



(b) Compute the transfer function $H(z)$ corresponding to this difference equation.

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)\left[1 - z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

(c) Consider an input $x(n) = u(n)$ with initial condition $y(-1) = -2$. Please compute $x(n)$ for $n \in \{0, 1, 2\}$, and show that the output is always zero. Please explain carefully why the output is zero in light of $H(z)$?
(Hint: it might help you to think of $x(n)$ as an exponential signal of the form $\alpha^n u(n)$, where $\alpha = 1$.)

$\underline{n=0}$ $\quad y(0) = \frac{1}{2}y(-1) + x(0) - x(-1)$

$= \frac{1}{2}(-2) + 1 - 0 = 0$

$\underline{n=1}$ $\quad y(1) = \frac{1}{2}y(0) + x(1) - x(0)$

$= \frac{1}{2} \cdot 0 + 1 - 1 = 0$

$\underline{n=2}$ $\quad y(2) = \frac{1}{2}y(1) + x(2) - x(1)$

$= \frac{1}{2} \cdot 0 + 1 - 1 = 0$

In $H(z)$ we have a zero at $\alpha = 1$.
The input has the form $x(n) = \alpha^n u(n)$,
and this exponent is attenuated by zero.

Note also that we chose the initial conditions such that they would not impact the output.