MATH-2105: Matrices, Vector Analysis and Coordinate Geometry

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Lecture Sheet 2

Determinant of a matrix

A determinant is a function of a square matrix that reduces it to a single number. For example

The determinant of the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 4 \times 2 - 1 \times 1 = 7$$

The determinant of the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is given by

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 1 \times (-1) - 2 \times (-3) + 3 \times (-2)$$

$$= -1 + 6 - 6$$

$$= -1$$

Singular matrix

A matrix having determinant zero is said to be singular matrix, i.e. $|A| = 0$. For example

The matrix $A = \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$ is singular as

$$|A| = \begin{vmatrix} -4 & -6 \\ 2 & 3 \end{vmatrix} = (-4) \times 3 - 2 \times (-6) = 0$$

The matrix $B = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ singular as

$$|B| = \begin{vmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{vmatrix} = 3 \times \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 8 \times \begin{vmatrix} -4 & 1 \\ -4 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} -4 & 1 \\ -4 & 1 \end{vmatrix} = 0$$

- The inverse of a singular matrix does not exist.
- To have an inverse, a matrix must be non-singular.
- Singular matrix is also known as non-invertible matrix.

Invertible or non-singular matrix

A matrix having non-zero determinant is said to be non-singular or invertible matrix, *i.e.* $|A| = 0$. For example

The matrix $A = \begin{bmatrix} 2 & -6 \\ 2 & 3 \end{bmatrix}$ is singular as

$$|A| = \begin{vmatrix} 2 & -6 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times (-6) = 18 \neq 0$$

The matrix $B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 4 & 6 \end{bmatrix}$ singular as

$$|B| = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 4 & 6 \end{vmatrix} = 2 \times \begin{vmatrix} 4 & 0 \\ 4 & 6 \end{vmatrix} - 0 \times \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} + 0 \times \begin{vmatrix} 3 & 4 \\ 5 & 4 \end{vmatrix} = 48 - 0 + 0 = 48 \neq 0$$

- There exist the inverse of a non-singular matrix.
- To have an inverse, a matrix must be non-singular.
- Non-ingular matrix is also known as invertible matrix.

Example

If $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 0, & when\ i < j \\ i + j, & when\ i = j \\ 2i - j, & when\ i > j \end{cases}$

Construct a 3×3 matrix and identify the type of the matrix A. Also check whether it is singular or not.

Solution

A 3×3 matrix is $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Using the conditions $a_{ij} = \begin{cases} 0, & when\ i < j \\ i+j, & when\ i = j \\ 2i-j, & when\ i > j \end{cases}$, we get

$a_{11} = i + j = 1 + 1 = 2$, $a_{12} = 0, a_{13} = 0$

$a_{21} = 2i - j = 2 \times 2 - 1 = 3, a_{22} = i + j = 2 + 2 = 4, a_{23} = 0$

$a_{31} = 2i - j = 2 \times 3 - 1 = 5,\ a_{32} = 2i - j = 2 \times 3 - 2 = 4, a_{33} = i + j = 3 + 3 = 6$

$$A = [a_{ij}] = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 4 & 6 \end{bmatrix}$$

This matrix is a square matrix and a lower triangular matrix.

The determinant of A is

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 4 & 6 \end{vmatrix} = 2 \times \begin{vmatrix} 4 & 0 \\ 4 & 6 \end{vmatrix} - 0 \times \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} + 0 \times \begin{vmatrix} 3 & 4 \\ 5 & 4 \end{vmatrix} = 48 - 0 + 0 = 48 \neq 0$$

So the matrix A is not singular.

Example

If $A = [a_{ij}]$, where $a_{ij} = \begin{cases} 0, & when\ i \neq j \\ C, & when\ i = j \end{cases}$

Construct a 3×3 order matrix and identify the type of matrix. Also test the matrix A is orthogonal or not, where C is the sum of the 1st digit and the last digit of your ID.

Solution

Solve yourself.

Example

Construct a 4×4 matrix A having:

$$A = (a_{ij}) = \begin{cases} 2i - j, & when\ i < j \\ 0, & when\ i = j \\ 3i + j, & when\ i > j \end{cases}$$

Solution

Solve yourself.

Difference between Determinant and Matrices

Determinant	Matrix
Determinant of A is denoted by $\|A\|$	Matrix can be denoted by [ ], ( ) or ‖ ‖
Number of rows and columns are always equal. Example $\begin{vmatrix} -4 & -6 \\ 2 & 3 \end{vmatrix}$	Number of rows and columns are not always equal. Example $\begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}$
It has a definite value although it is an arrangement of numbers	It has no definite value, it is merely an arrangement of numbers
If two rows (or columns) are identical, a determinant vanishes	Identical rows (or columns) may occur in a matrix
Rows and columns can be interchanged	Rows and columns cannot be interchanged
If a determinant is multiplied by a number m, every element of any one row (or any one column) is multiplied by m. Example 2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix}$	If a matrix is multiplied by m, every element of the entire matrix is multiplied by m Example 2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$
A determinant can be resolved into two determinants by using only one row (or column)	A matrix cannot be resolved into two matrices using only one row (or column)
The sign of a determinant changes if two rows (or columns) are interchanged	Two rows (or columns) cannot be interchanged at all
The product of two determinants does not change the order	The product of two matrices may change the order

Example

Show that $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$ are inverses of each other.

Solution

Given matrices are $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

Now the product of the matrices A and B is

$$AB = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-11) + 0(-4) + 2.6 & 1.2 + 0.0 + 2(-1) & 1.2 + 0.1 + 2(-1) \\ 2(-11) - 1(-4) + 3.6 & 2.2 - 1.0 + 3(-1) & 2.2 - 1.1 + 3(-1) \\ 4(-11) + 1(-4) + 8.6 & 4.2 + 1.0 + 8(-1) & 4.2 + 1.1 + 8(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Since $AB = I$, the matrices A and B are the inverses of each other.

Algebraic operations of Matrices

1. Addition ($A + B$)
2. Subtraction ($A - B$)
3. Scalar Multiplication ($2A$, $5A$ or $\frac{5}{7}A$)
4. Multiplication (AB)
5. Combination ($2A - 3B + 2AB$ & so on)
6. Inversion (A^{-1})

Example

Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$, find $A + B$, $A - B$ and $2A - 3B$.

Solution

$$A + B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 11 & 4 & -14 \end{bmatrix}$$

$$2A - 3B = 2\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix} - 3\begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 6 \\ 8 & 10 & -12 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 6 \\ -21 & 3 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -4 & 0 \\ 29 & 7 & -36 \end{bmatrix}$$

Example

Find x, y, z, t using the concept of equality of matrices, where

$$3\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2t \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+t & 3 \end{bmatrix}$$

Solution

We write each side as a single matrix

$$\begin{bmatrix} 3x & 3y \\ 3z & 3t \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ z+t-1 & 2t+3 \end{bmatrix}$$

Equating the corresponding terms, we get

$3x = x + 4$ giving $x = 2$

$3y = x + y + 6$ giving $y = 4$

$3t = 2t + 3$ giving $t = 3$

$3z = z + t - 1$ giving $z = 1$

So, the solution is $(x, y, z, t) = (2, 4, 1, 3)$.

Example

Find AB, where $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{bmatrix}$.

Solution

Given matrices are $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 + 3.1 - 1.4 & 2(-1) + 3.3 - 1.1 & 2.0 + 3(-5) - 1(-2) & 2.6 + 3.1 - 1.2 \\ 4.2 - 2.1 + 5.4 & 4(-1) - 2.3 + 5.1 & 4.0 - 2(-5) + 5(-2) & 4.6 - 2.1 + 5.2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -13 & 13 \\ 26 & -5 & 0 & 32 \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$, then find AB. Is BA exist? Create an argument.

Solution

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 3 & 2 \\ 13 & 12 \end{bmatrix}$$

BA doesn't exist.

Argument: As the number of the columns in the matrix B (2) and the number of the rows in the matrix A (3) are not the same, so BA isn't possible to calculate.

Example

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB and show that $AB \neq BA$.

Assignment

Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, find

(i) $5A + 3B$

(ii) $2A - B$

(iii) AB

(iv) A^2

(v) B^3

(vi) $A^2 + 2AB + B^2$

(vii) $(A + B)^2$

(viii) the matrix C, such that $A + 2C = B$

Solution

Part (viii)

Given that $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

Now, $A + 2C = B$ implies $2C = B - A$ implies $C = \frac{1}{2}(B - A)$

$$\therefore \quad C = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 5/2 \\ -1/2 & 1 & 3/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Example

Solve the following equations for A and B

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

$$2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Solution

Given matrix equations are

$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ ---- (i)

$2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ ---- (ii)

(i)−2×(ii) $\rightarrow$

$$-5B = \begin{bmatrix} 3-8 & -3-2 & 0-10 \\ 3+2 & 3-8 & 2+8 \end{bmatrix} = \begin{bmatrix} -5 & -5 & -10 \\ 5 & -5 & 10 \end{bmatrix}$$

$$\therefore \quad B = -\frac{1}{5}\begin{bmatrix} -5 & -5 & -10 \\ 5 & -5 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Again, 2×(i)+(ii) $\rightarrow$

$$5A = \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$

$$\therefore \quad A = \frac{1}{5}\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$