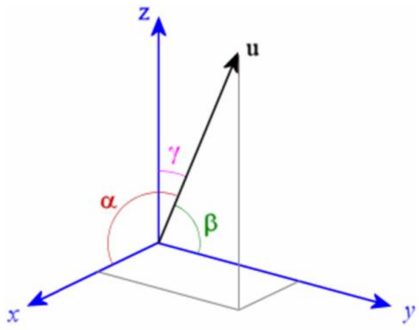
1. Explain direction cosines of a line. If the angle between two straight lines is $\theta$ and their direction cosines are $l_1, m_1, n_1$ and $l_2, m_2, n_2$ then show that

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

Hence develop this relation for $\sin\theta$.

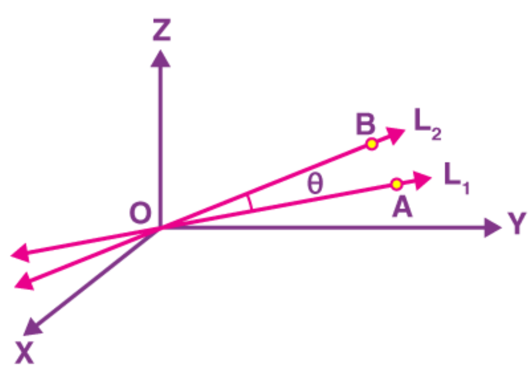
**Direction cosines of a line**

If a given line makes angles $\alpha, \beta, \gamma$ with the positive direction of $x, y$ and $z$ axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the line and are generally denoted by $l, m, n$ respectively. The angles $\alpha, \beta, \gamma$ are called the direction angle of the line.




**Angle between two straight lines**

Let OA and OB be lines through the origin parallel to the given lines so that the cosines of the angles which OA and OB make with the axes are $l_1, m_1, n_1$ and $l_2, m_2, n_2$ respectively and the angle between the lines is the angle between OA and OB. Let the angle be $\theta$.




The projection of the line OB joining O(0, 0, 0) and B$(x_2, y_2, z_2)$ on the line OA whose direction cosines are $l_1, m_1, n_1$ is

$$s = (x_2 - 0)l_1 + (y_2 - 0)m_1 + (z_2 - 0)n_1$$
$$= l_1 x_2 + m_1 y_2 + n_1 z_2$$

But the projection of OB on OA is $\text{OB}\cos\theta$

$\therefore \text{OB}\cos\theta = l_1 x_2 + m_1 y_2 + n_1 z_2$

But $x_2 = l_2\text{OB}$, $y_2 = m_2\text{OB}$, $z_2 = n_2\text{OB}$, then

$\text{OB}\cos\theta = l_1 l_2\text{OB} + m_1 m_2\text{OB} + n_1 n_2\text{OB}$

$$\therefore \quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$\sin\theta = \sqrt{1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2}$$

2. Explain shortest distance. Find the equation of the line of shortest distance and evaluate the length of the shortest distance between the lines
   $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-5}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

   **Shortest Distance**

   When two lines do not intersect and are parallel as well, that is, they do not lie in the same plane, then these lines are said to be non-intersecting lines. The straight line which is perpendicular to each of these non-intersecting lines is called the line of shortest distance and the length of this line intercepted between the given lines is called the shortest distance of those lines.

   **Length of the shortest distance**

   Let $l, m, n$ be the direction cosines of the line of shortest distance. As it is perpendicular to both the lines given

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-5}{4}$$
$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

We get $2l + 3m + 4n = 0$ and $3l + 4m + 5n = 0$

Solving simultaneously we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$$

Giving $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = \frac{1}{\sqrt{6}}$, thus $l = -\frac{1}{\sqrt{6}}$, $m = \frac{2}{\sqrt{6}}$ and $n = -\frac{1}{\sqrt{6}}$

The magnitude of the shortest distance is the projection of the line joining $(1, 2, 5)$ and $(2, 4, 5)$

$\therefore$ Shortest Distance $= (2-1)\left(-\frac{1}{\sqrt{6}}\right) + (4-2)\left(\frac{2}{\sqrt{6}}\right) + (5-5)\left(-\frac{1}{\sqrt{6}}\right) = \frac{3}{\sqrt{6}}$

Now, the equation of the plane containing the first of the two given lines and the line of shortest distance is

$$\begin{vmatrix} x-1 & y-2 & z-5 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0$$
$$11x + 2y - 7z + 13 = 0$$

Also the equation of the plane containing the second line and the shortest distance is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix} = 0$$
$$7x + y - 5z + 7 = 0$$

Therefore, the equation of the line of shortest distance is

$$11x + 2y - 7z + 13 = 0 = 7x + y - 5z + 7$$

3. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are coplanar. Find their intersection point and the equation of the plane in which they lie.

The condition for the lines
$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ to be coplanar is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Now

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

So the given lines are coplanar.
The equation of the plane in which they lie is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

$$21x - 19y + 22z + 125 = 0$$