Properties of regular languages - The pumping lemma

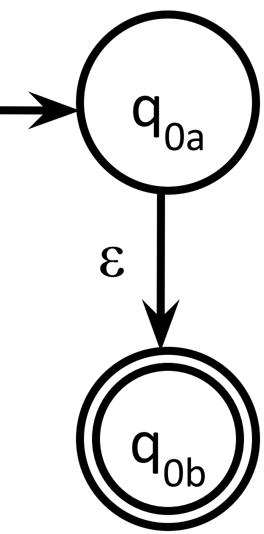
CHAPTER-4 (PART 1)

Regular Languages

- Regular languages are the languages which are accepted by a Finite Automaton.
- Not all languages are regular

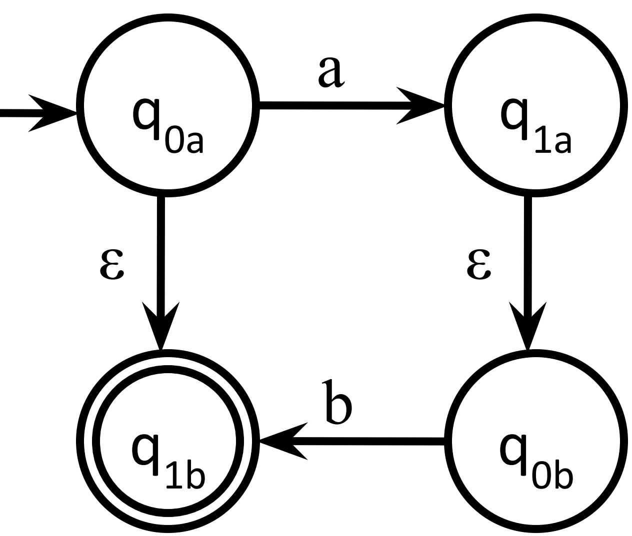
Non-Regular Languages

- $L_0 = \{a^k b^k : k \leq 0\} = \{\varepsilon\}$ is a regular language



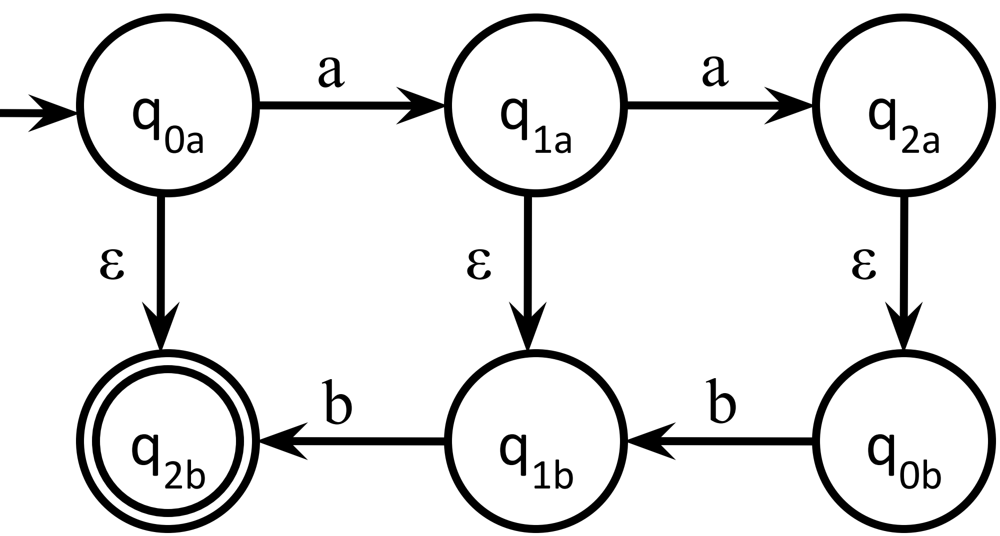
Non-Regular Languages

- $L_1 = \{a^k b^k : k \leq 1\} = \{\varepsilon, ab\}$ is regular



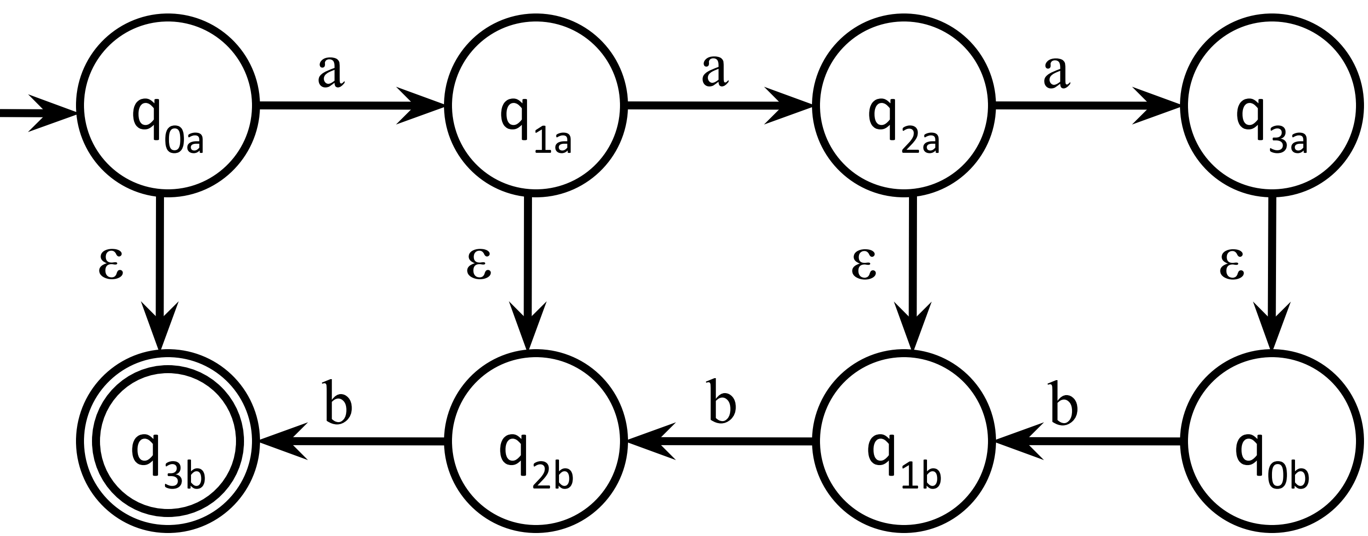
Non-Regular Languages

- $L_2 = \{a^k b^k : k \leq 2\} = \{\varepsilon, ab, aabb\}$ is regular



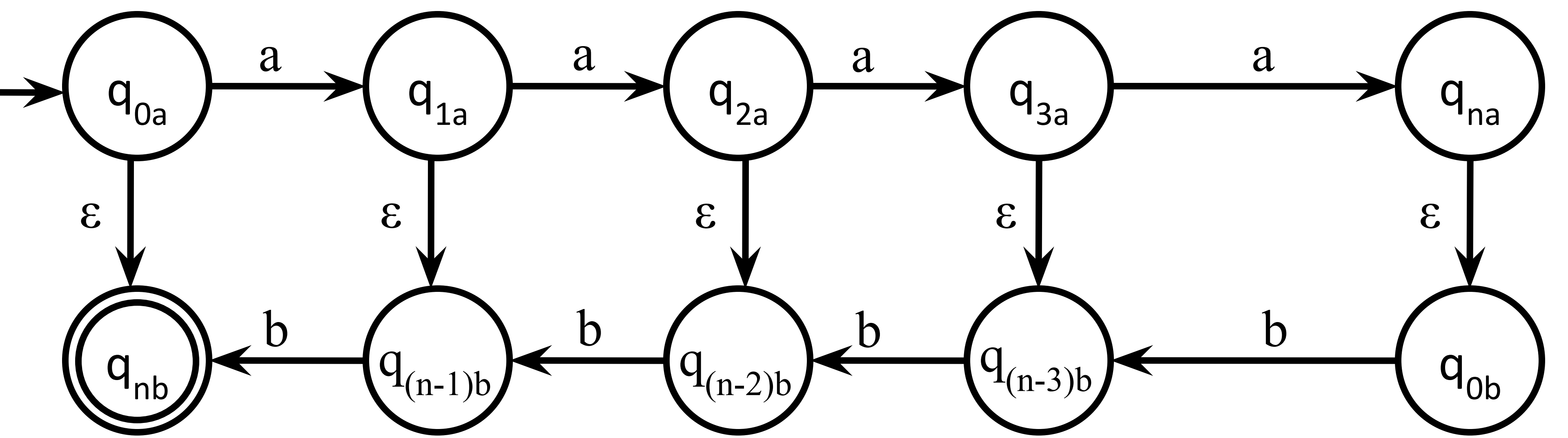
Non-Regular Languages

- $L_3 = \{a^k b^k : k \leq 3\}$ is regular



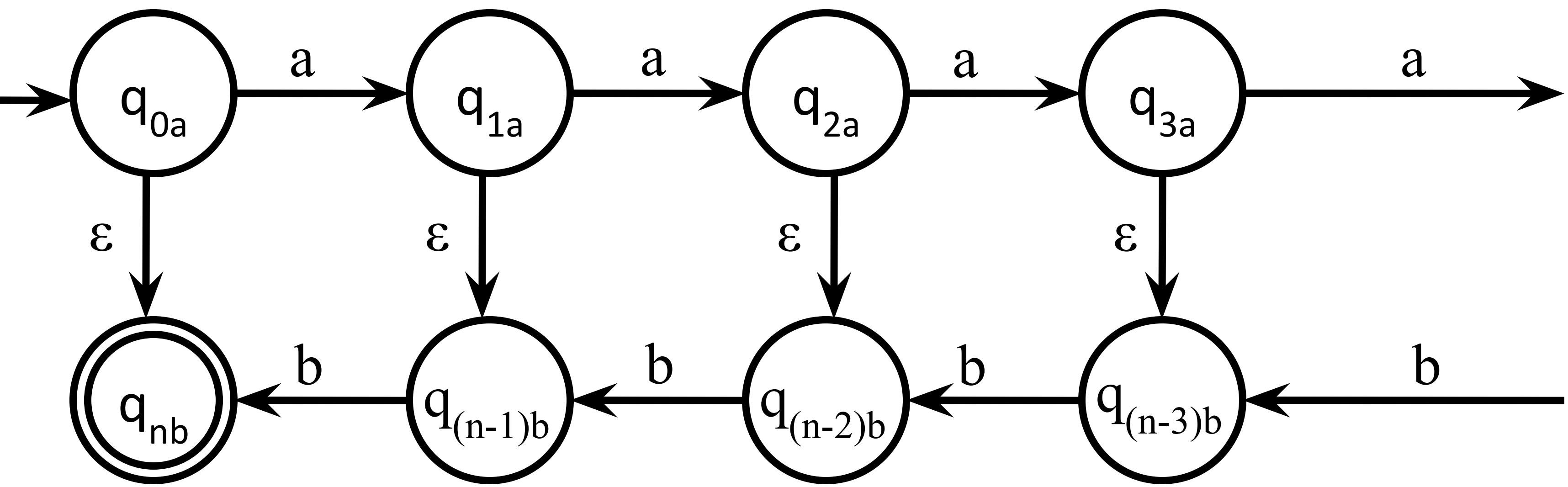
Non-Regular Languages

- $\forall$ $n \geq 0$, $L_n = \{a^k b^k : k \leq n\}$ is regular



Non-Regular Languages

- However $L = \{a^n b^n : n \geq 0\} = \cup_{n \geq 0} L_n$ doesn't seem to be a regular language at all!
- We need an infinite number of states to build this automaton!



Non-Regular Languages

- However $L = \{a^n b^n : n \geq 0\} = \bigcup_{n \geq 0} L_n$ doesn't seem to be a regular language at all!
- We need an infinite number of states to build this automaton!
- (Observe that you cannot use the fact that regular languages are closed under union because we have an infinite union)

Is this a proof?

NO! In fact consider:

L’ = {s : s contains equal number of a and b}

L’’ = {s : s contains equal number of ab and ba}

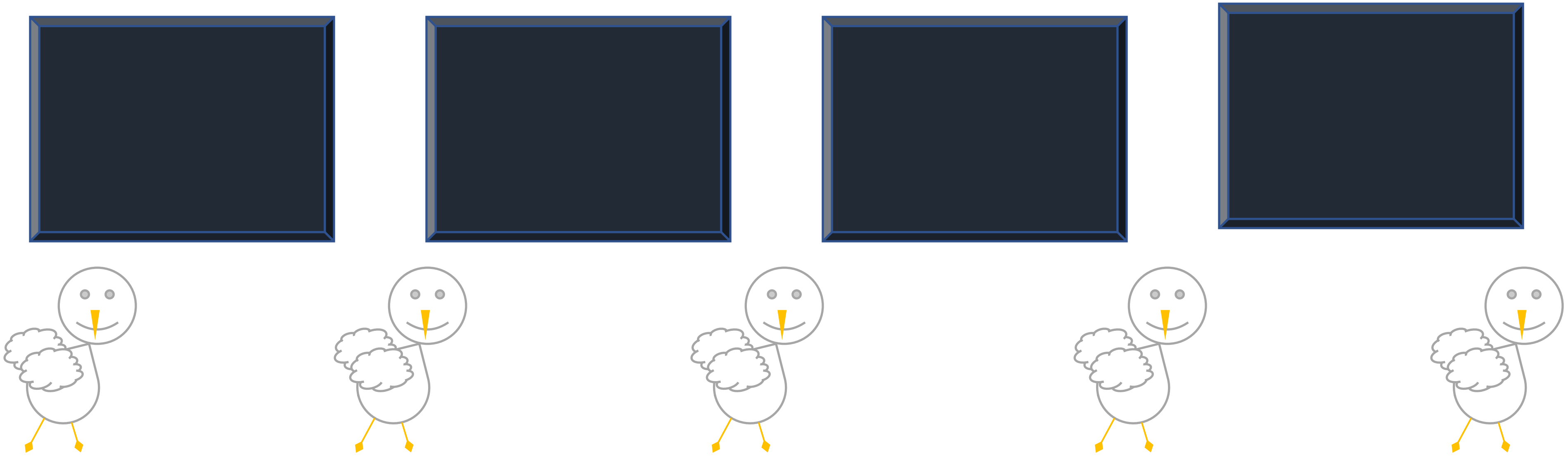
L’ is indeed not regular but L’’ is regular!

<u>WE NEED A MATHEMATICAL PROOF!!!</u>

A proof that there is no FA that accepts L or L’.

Pigeonhole Principle

- If we have n holes and m pigeons (m>n) then there is a hole with at least two pigeons.



Pigeonhole Principle

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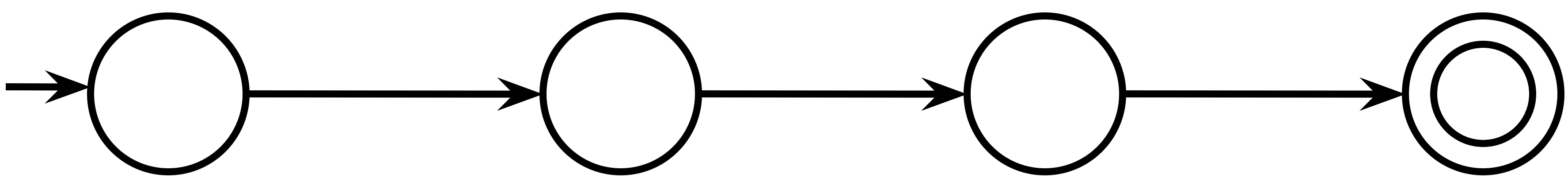
PP and Finite Automata

- If an automaton with n states accepts a string with length m (m≥n) then for every accepting path there should be at least one repeating state.



$$s = aaba$$

PP and Finite Automata

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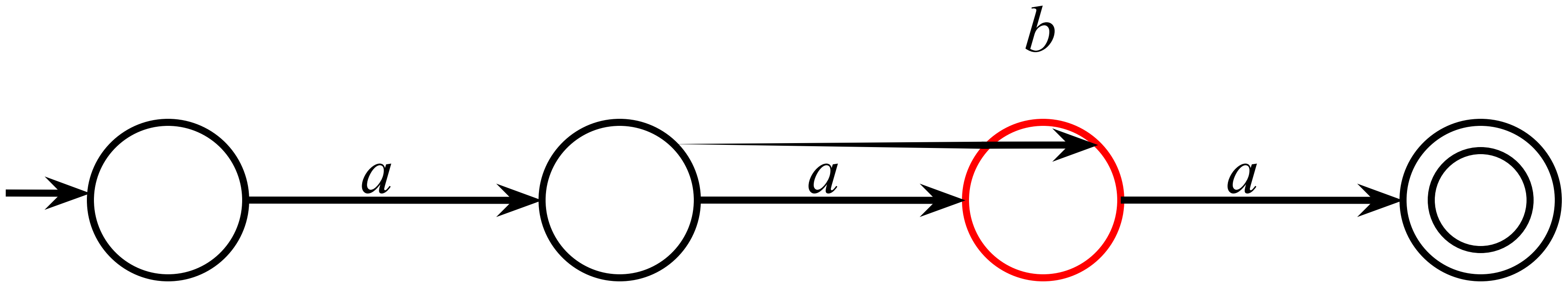

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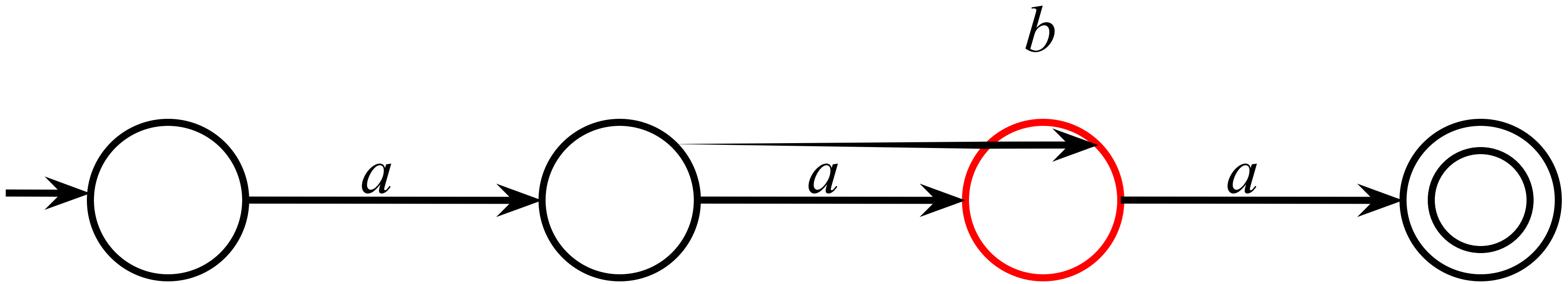
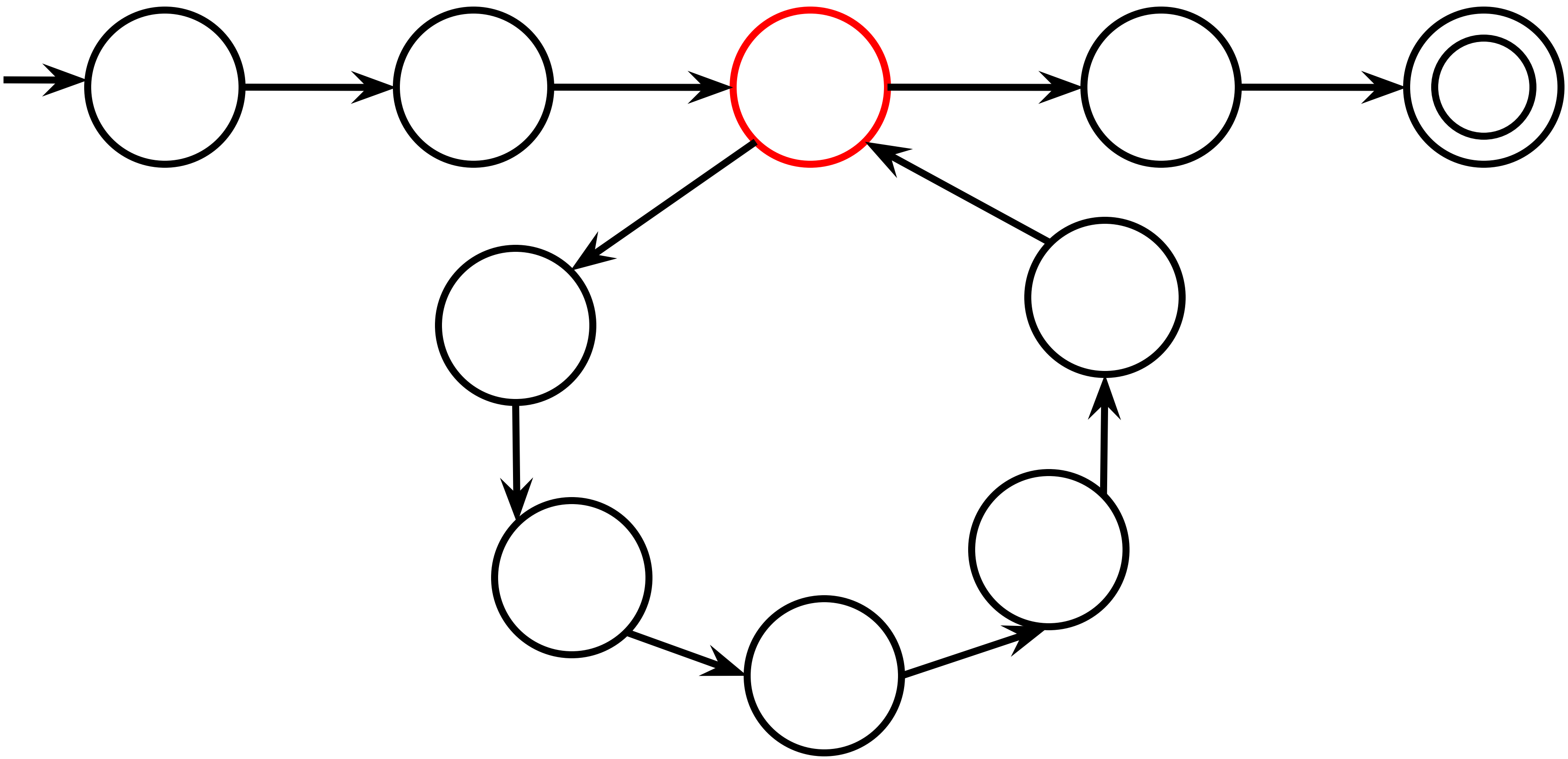
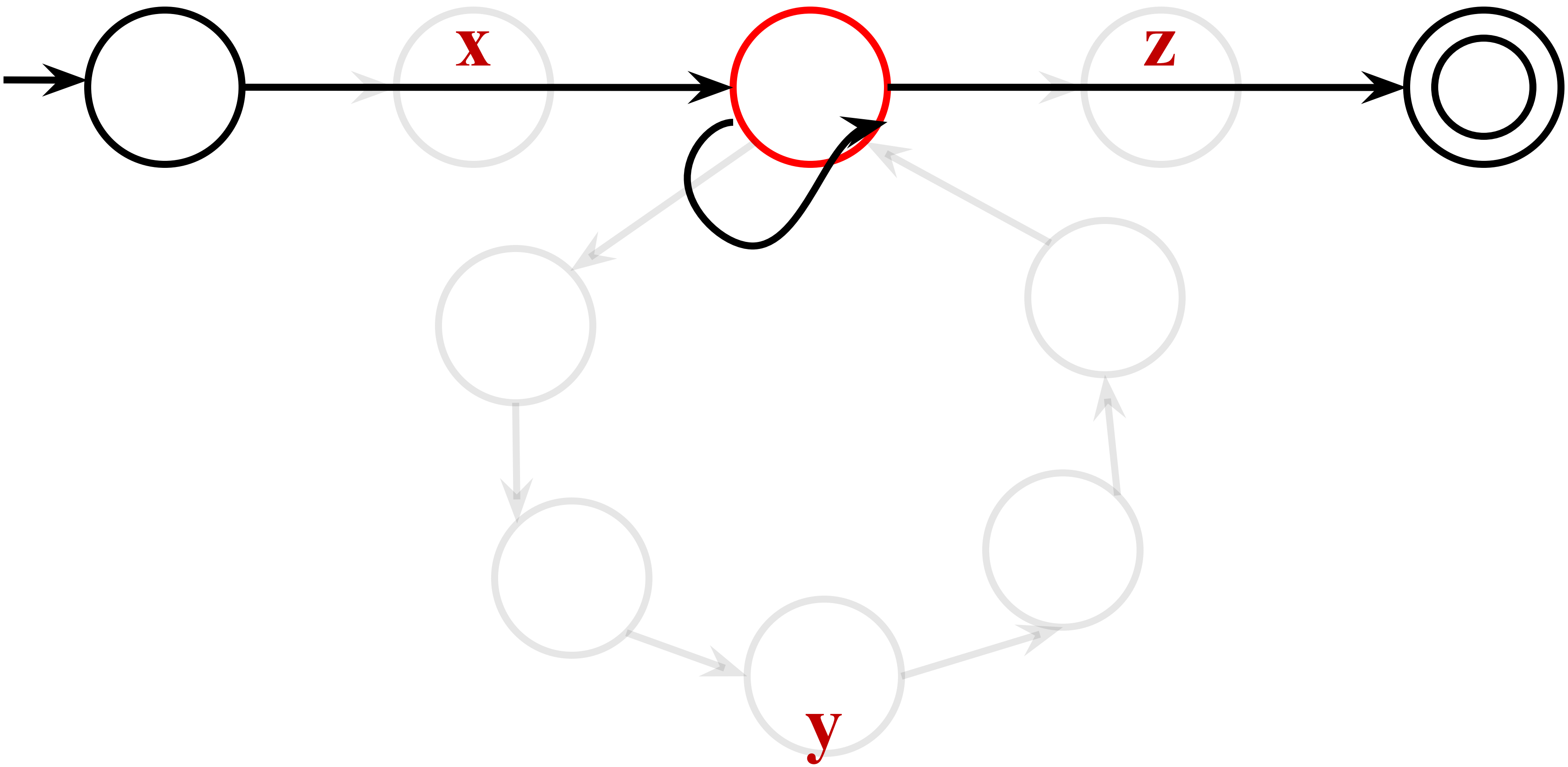


*Any string of the form aab*a should be accepted!*

PP and FA continued



PP and FA continued



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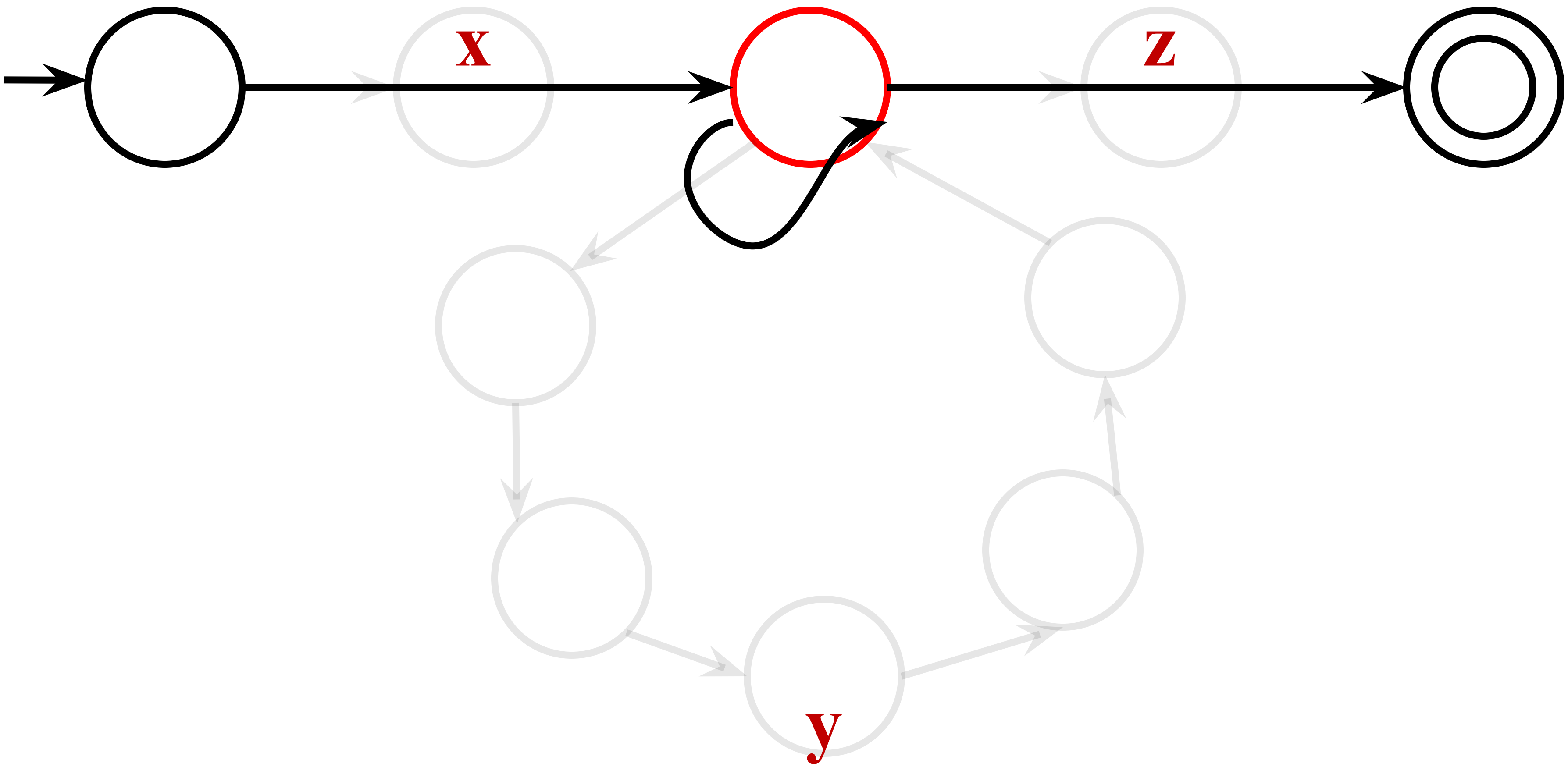
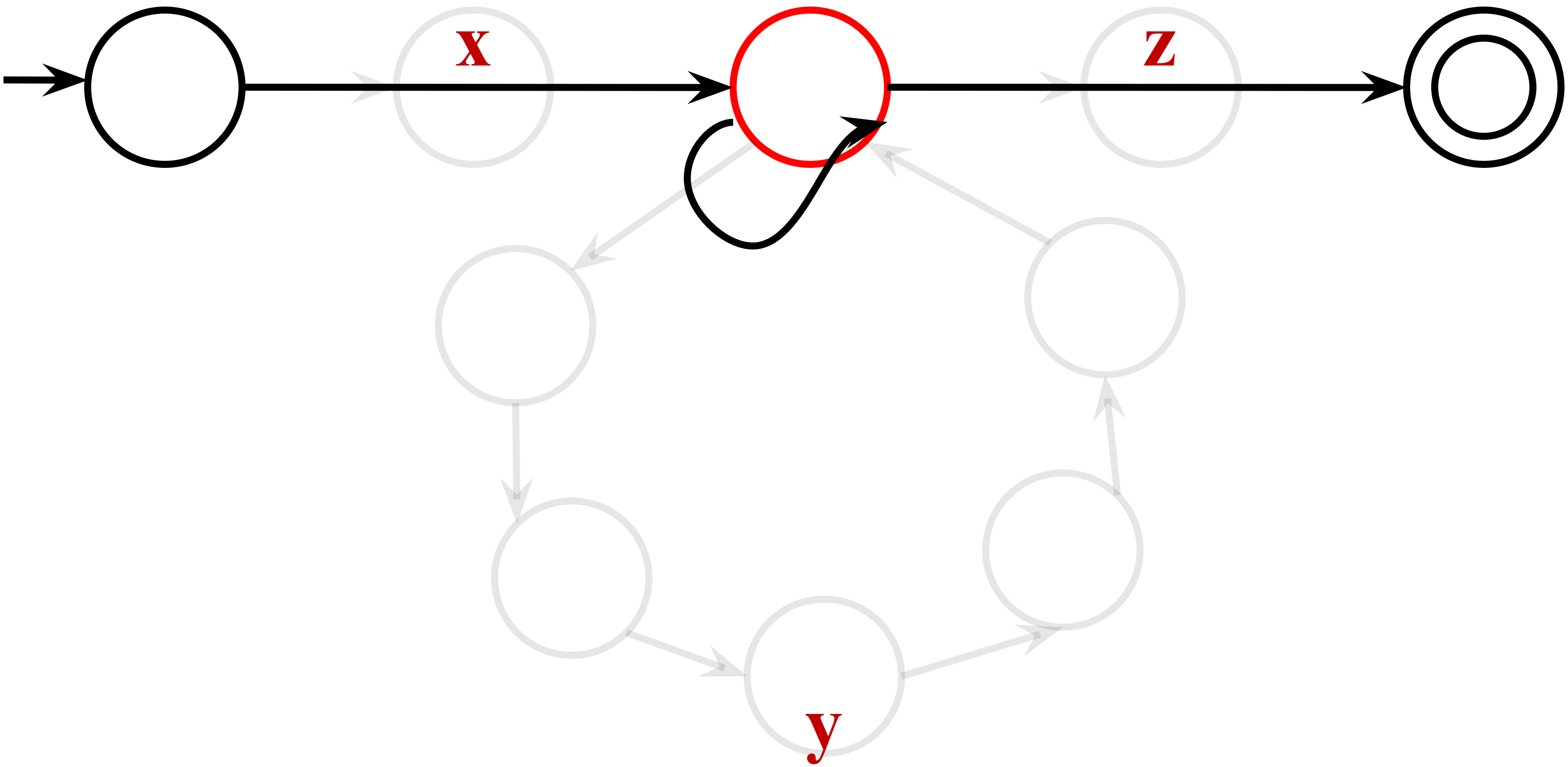


xyz is accepted

PP and FA continued



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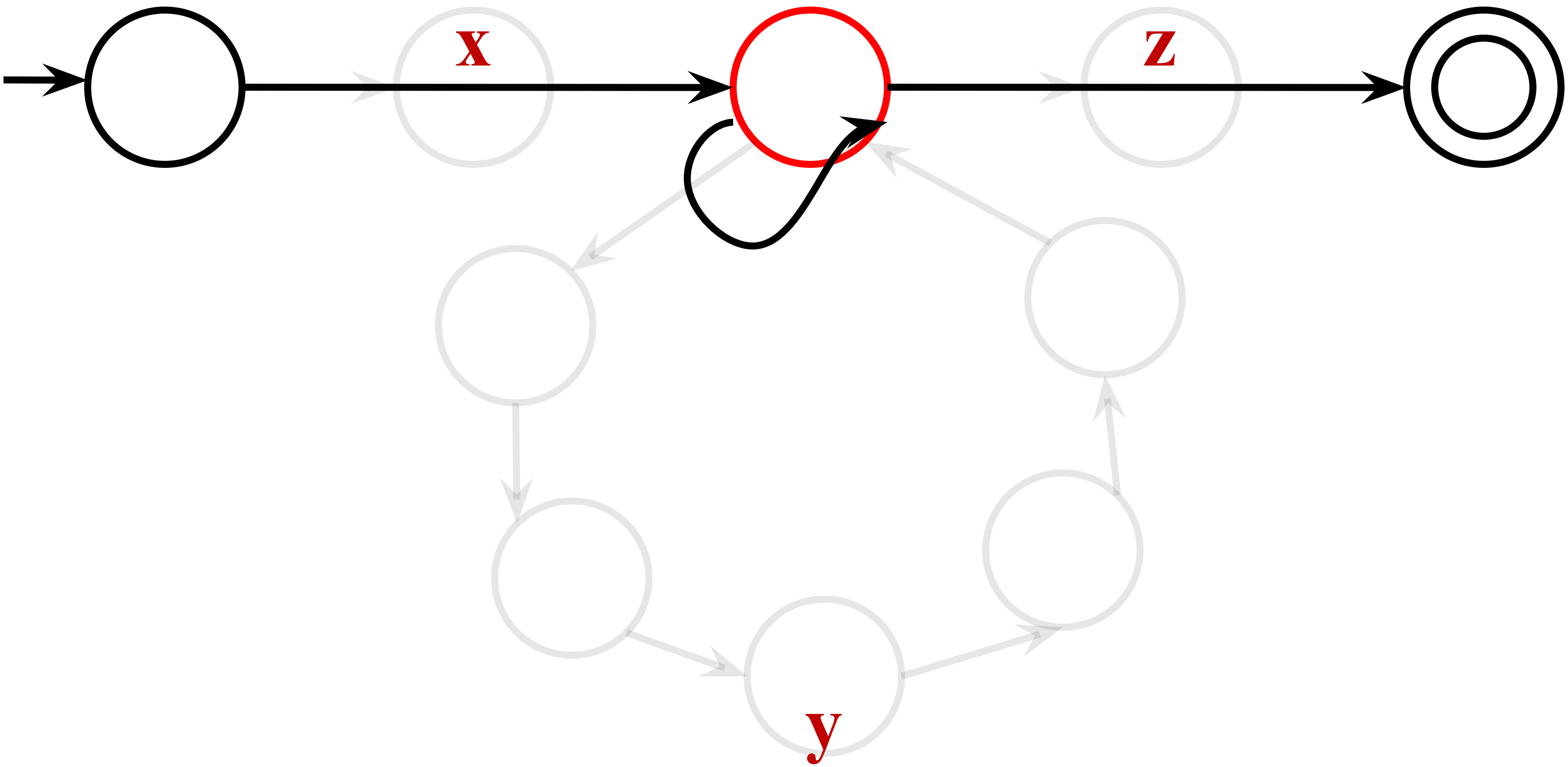
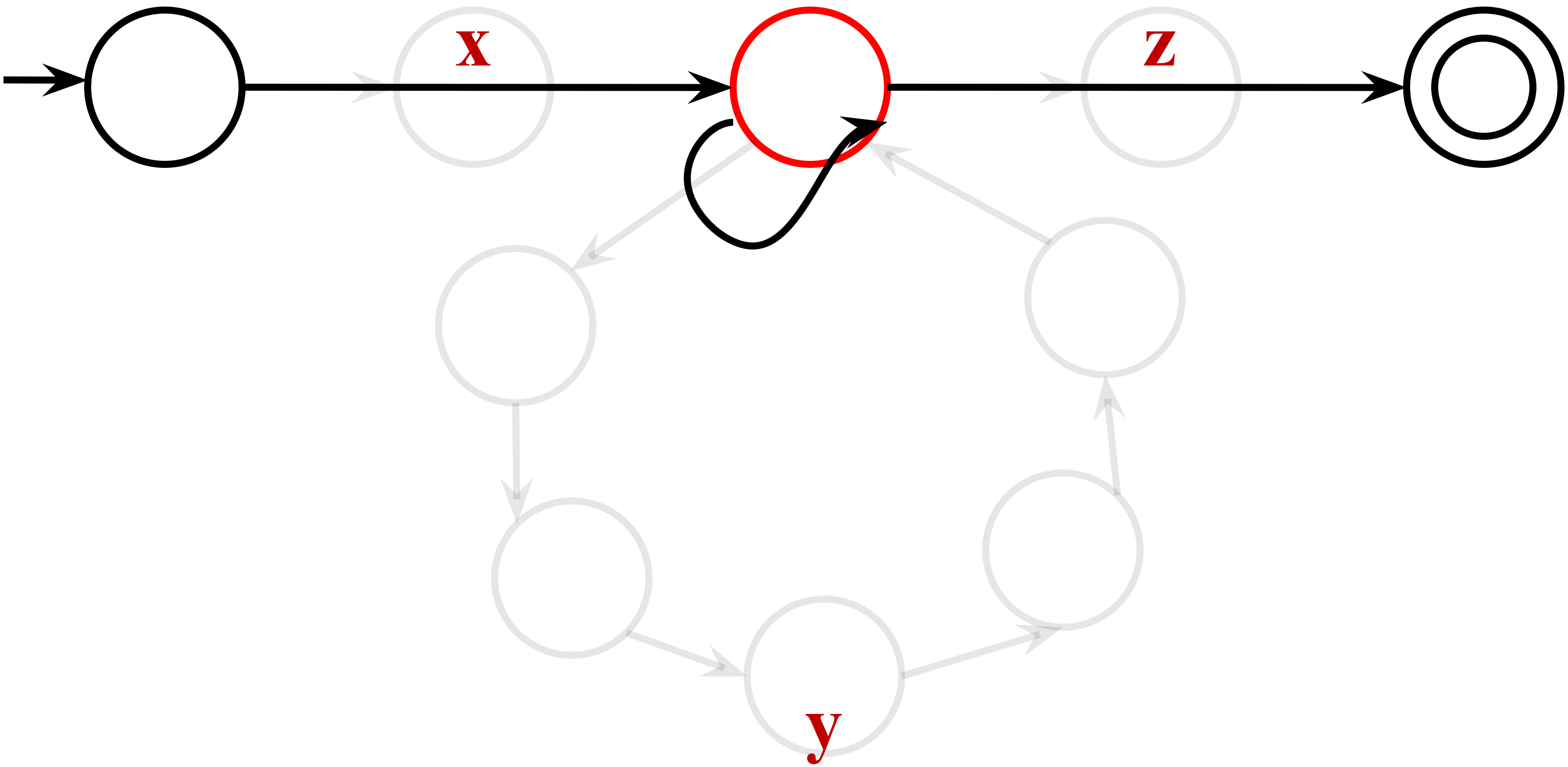


xyyyz is accepted

PP and FA continued



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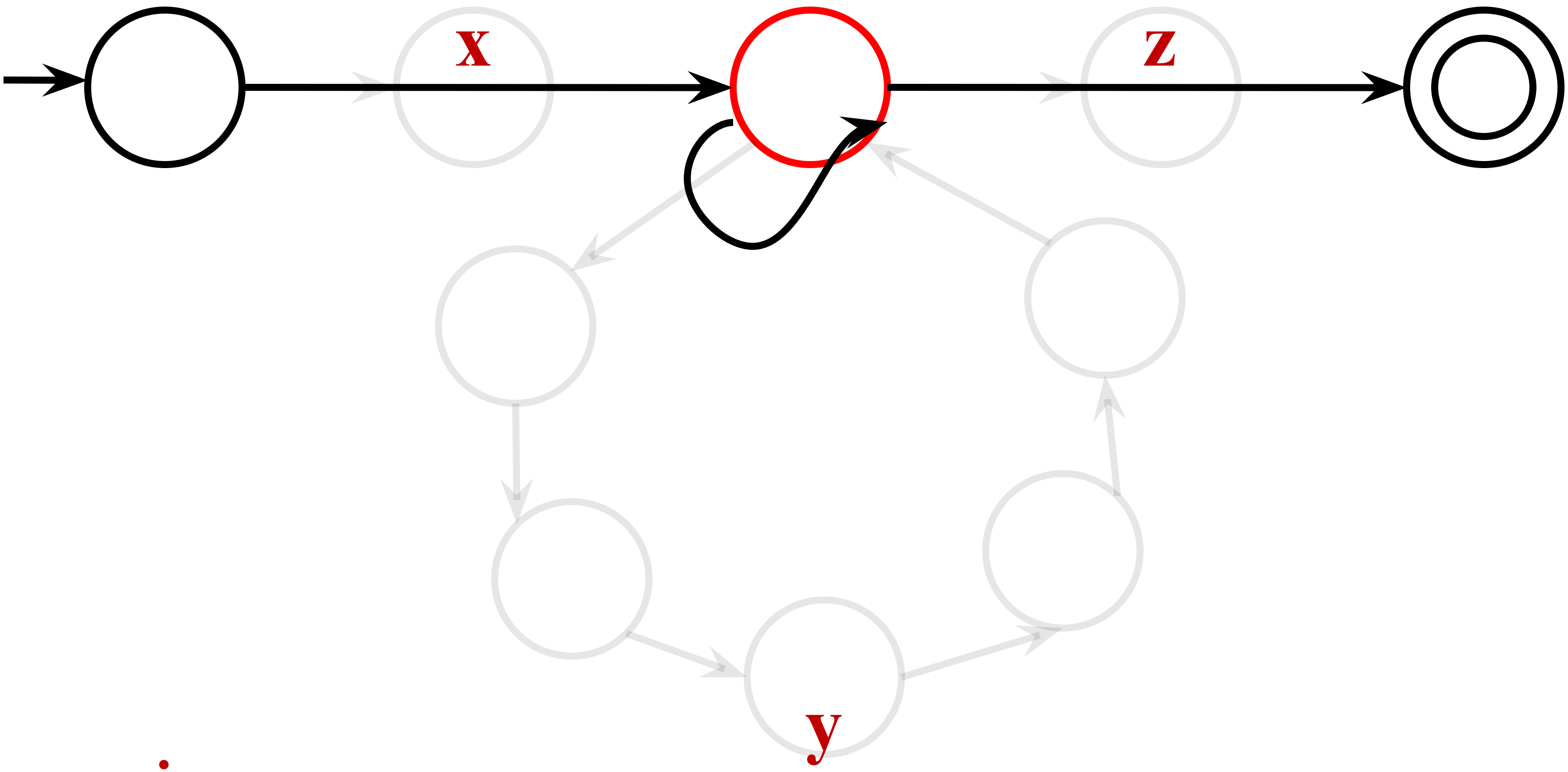
PP and FA continued



xy^iz**, for** $i \geq 0$ is accepted

The pumping lemma

For every <u>infinite</u> *regular* language L

there exists a pumping length $n > 0$ such that

for any string s in L with length $|s| \geq n$

we can write $s = xyz$

with $|xy| \leq n$ and $|y| \geq 1$

such that xy^iz *in L* for every $i \geq 0$.

Proof

- If L is regular then there exists a DFA M which accepts L. Set n to be M's number of states.
- L is infinite, there exists a string s with length greater than n.
- The number of states is n, the accepting path for s is of length at most n.
- The string is of length at least n, there is a part in the path that is repeated.

Proof

- Split s into 3 parts x, y, z with y being the first repeated part.
- Since we have n states the first repetition should take place ($|y| \geq 1$) in at most n transitions ($|xy| \leq n$).
- Since the path under y is a loop we can follow it as many times as we want (maybe none).
- Thus xy^iz for any $i \geq 0$ should lead us to the same accepting state as xyz.

Prove that L is not regular

- Given is an infinite language L.

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  (*What if L is finite?*)

Prove that L is not regular

- Given is an infinite language L
- If L is regular

Prove that L is not regular

- Given is an infinite language L
- If L is regular
- Pumping lemma holds:
  - *There exists* a pumping length n such that
  - *for all* <u>proper</u> strings s in L
  - *there is* a splitting of s in x,y,z (with the <u>desired properties</u>) such that
  - *for all* i xy^iz is in L.

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- If L is regular
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$$\boxed{|s| \geq n}$$

$$\boxed{|y| \geq 1 \text{ and } |xy| \leq n}$$

Prove that L is not regular

- Given is an infinite language L
- If L is regular the pumping lemma holds.
- The negation of the pumping lemma:
  - *For all* pumping lengths n
  - *there exists* a proper string s in L such that
  - *for every possible* splitting of s in x,y,z (with the desired properties)
  - *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
  - *Fix* an *arbitrary* pumping length n.
  - *Specify* a proper string s in L.
  - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
  - *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

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- Contradiction!!!

Prove that L is not regular

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- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
  - *Fix* an *arbitrary* pumping length n.
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  - *there is* an i for which xy^iz is *not* in L.
- Contradiction!!! **L is not regular**

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Assume that L is regular. The pumping lemma holds!

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Fix an arbitrary pumping length *k* for L.

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

The string $s = a^k b^k$ should be in the language.

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

s is proper: $|s| = 2k$ is greater than k

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties:

- $|xy| \leq k$ and
- $|y| \geq 1$

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$$\overbrace{a\ a\ a\ a\ a\ a\ a\ a\ a}^{k}\ \overbrace{b\ b\ b\ b\ b\ b\ b\ b\ b}^{k}$$

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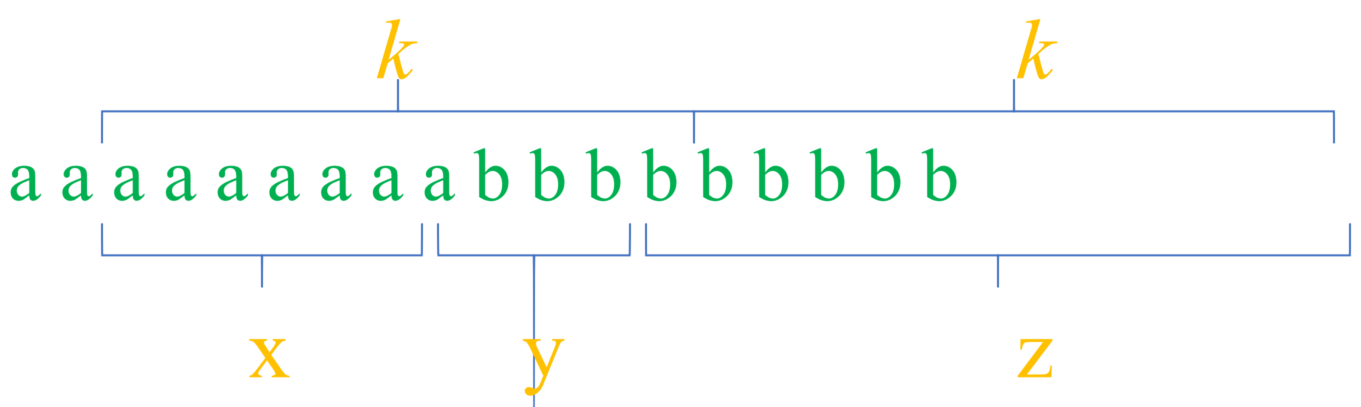

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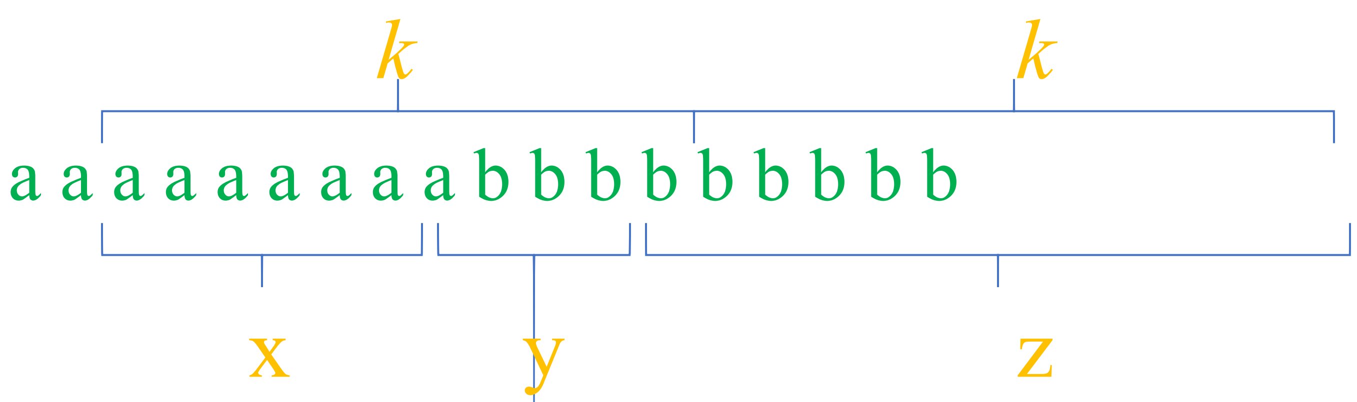
Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$



Example

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Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$
- for i =2, $xy^2z = a^{k+m}b^k$ is not in L!

a a a a a a a a a a a a b b b b b b b b b

x y y z

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- xy^2z is not in L!

CONTRADICTION!

a a a a a a a a a a a a b b b b b b b b b

x y y z

Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^jc^j \mid j \geq 0\}$$

- Members: a, abc, abbcc, abbbccc, …

Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^jc^j \mid j \geq 0\}$$

- Negation of the pumping lemma (reminder):
    - *Fix* an *arbitrary* pumping length n.
    - *Specify* a proper string s in L.
    - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
    - *there is* an i for which xy^iz is *not* in L.

How to use the pumping lemma

- The pumping lemma mentions that if L is a regular language then it can be pumped.
- The contrapositive is true: If L cannot be pumped then it shouldn't be regular!

Show that L cannot be pumped to prove that L is not regular.

How not to use the pumping lemma

- The pumping lemma mentions that if L is a regular language then it can be pumped.
- The converse is not true: If a language can be pumped this doesn't mean that it is regular!

Do not try to show that L can be pumped in order to prove that L is regular.

Pumping Lemma - Converse

- For example consider the language

$$L_p = \{a^i b^j c^k : i,j,k \geq 0, \text{ if } i=1 \text{ then } j=k\}.$$

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Proof:

- $L_p' = \{ab^j c^j : j \geq 0\}$ is not regular.

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Proof:

- $L_p' = \{ab^j c^j : j \geq 0\}$ is not regular.
- $L_p' = L_p \cap L(ab^*c^*)$.

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Proof:

- $L_p' = \{ab^j c^j : j \geq 0\}$ is not regular.
- $L_p' = L_p \cap L(ab^*c^*)$.
- If L_p was regular then L_p' would be regular too.

Pumping Lemma - Converse

- For example consider the language

$$L_p = \{a^i b^j c^k : i,j,k \geq 0, \text{ if } i=1 \text{ then } j=k\}.$$

2. L_p can be pumped

Pumping Lemma - Converse

- For example consider the language

$$L_p = \{a^i b^j c^k : i,j,k \geq 0, \text{ if } i=1 \text{ then } j=k\}.$$

2. L_p can be pumped

Proof:

Prove that: *There is* a pumping length n such that *for any* <u>proper</u> s in L_p *there exists* a splitting xyz of s with the <u>desired properties</u> such that *for all* i xy^iz is in L_p.

Pumping Lemma - Converse

- For example consider the language

$$L_p = \{a^i b^j c^k : i,j,k \geq 0, \text{ if } i=1 \text{ then } j=k\}.$$

2. L_p can be pumped

Proof:

Choose pumping length n = 2

Pumping Lemma - Converse

- For example consider the language

$$L_p = \{a^i b^j c^k : i,j,k \geq 0, \text{ if } i=1 \text{ then } j=k\}.$$

2. L_p can be pumped

Proof:

Choose pumping length n = 2

- For i=0, j=0, k ≥2 : $s = c^k$

  Set $x=\varepsilon$, $y = c$, $z = c^{k-1}$

  For every i≥0, xy^iz is in L_p

Pumping Lemma - Converse

- For example consider the language

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2. L_p can be pumped

Proof:

Choose pumping length $n = 2$

- For $i=0$, $j\geq 1$, $k \geq 0$: $s = b^j c^k$

    Set $x=\varepsilon$, $y = b$, $z = b^{j-1}c^k$

        For every $i\geq 0$, xy^iz is in L_p

Pumping Lemma - Converse

- For example consider the language

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2. L_p can be pumped

Proof:

Choose pumping length n = 2

- For i=1 and any j ≥ 1: $s = ab^jc^j$

Set $x=\varepsilon$, $y = a$, $z = b^jc^j$

For every $i\geq 0$, xy^iz is in L_p

Pumping Lemma - Converse

- For example consider the language

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2. L_p can be pumped

Proof:

Choose pumping length n = 2

- For i=2 and any j, k ≥ 0: $s = aab^jc^k$

Set $x=\varepsilon$, $y = aa$, $z = b^jc^k$

For every $i \geq 0$, xy^iz is in L_p

Pumping Lemma - Converse

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2. L_p can be pumped

Proof:

Choose pumping length n = 2

- For $i \geq 3$ and any $j, k \geq 0$: $s = aaa^{i-2}b^jc^k$

Set $x=\varepsilon$, $y = a$, $z\ aa^{i-2}b^jc^k$

For every $i \geq 0$, xy^iz is in L_p

The Universe