

Signals and Systems

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Resources:

- <https://www.youtube.com/watch?v=x5qRAihZRks&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=2> (Hindi)
- <https://www.youtube.com/playlist?list=PLXOYj6DUOGrrAIYxrAu5U2tteJTrSe5Gt> (Problem Solving)
- Euler Equation :

<https://www.physicsforums.com/threads/solve-e-frac-i-pi-4.336356/>

https://proofwiki.org/wiki/Euler's_Formula/Examples/e^i_pi_by_4

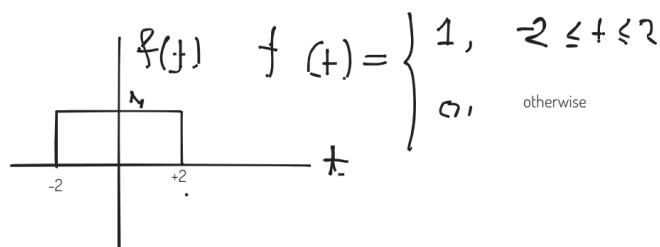
Skipped:

- <https://www.youtube.com/watch?v=Rew03iHJGhk&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=9&pp=iAQB> (Complex Exponentiation)

Signals

Physical quantity that contains information. Signals are expressed mathematically as function of independent variable, which is usually time.

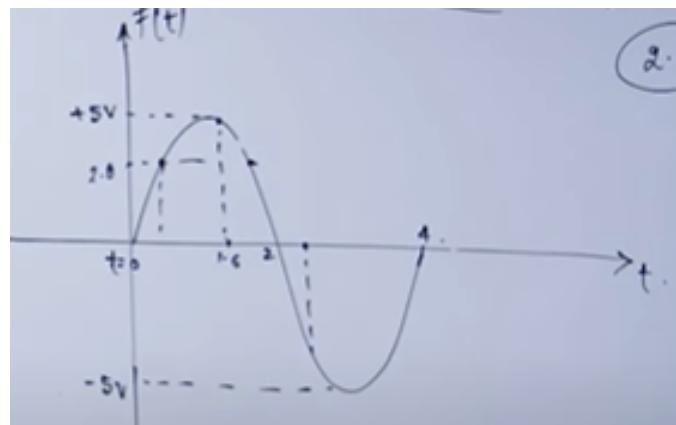
Example:



Basic Types of Signals:

Based on Continuous and Discrete:

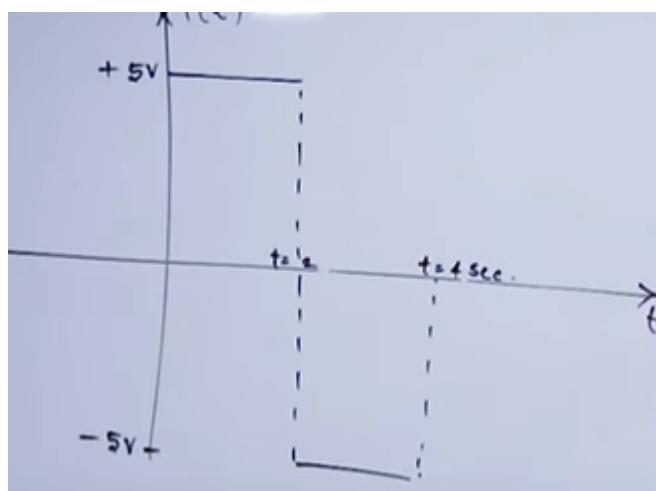
- **Continuous in Time Signal & Continuous in Value Signal**
 - A continuous-time signal has values for all points in time in some interval.
 - A continuous-value signal is all possible value within an interval will be available in a signal.



Has values for $0 \leq t \leq 4$ and all values are available in the signal with $-5 \leq v \leq +5$

- **Continuous in time but discrete in value signal**

- A continuous-time signal has values for all points in time in some interval.
- All values within a range is not available in the signal.



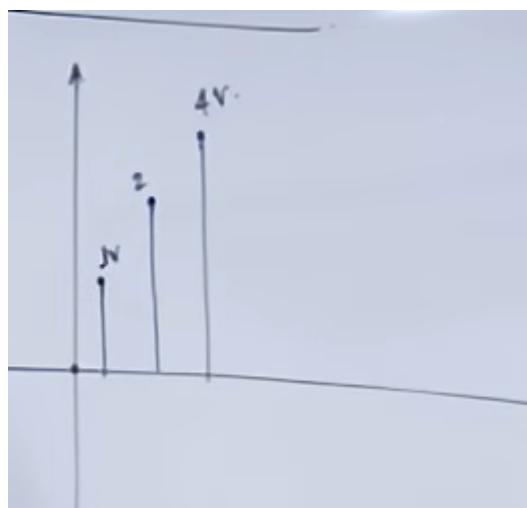
Have value for time within $0 \leq t \leq 4$ but all values within $-5 \leq v \leq 5$ is not available

- **Continuous in value but discrete in time signal**
 - Haven't values for all points in time within an interval
 - All values within an range are available in a signal



All values are available, but some points in time haven't values

- **Discrete in time and discrete in value signal**



Analog Signal: Continuity in any of the domain (time or value)

Digital Signal: Discrete in both time and value

Based on Causal, Anti-Causal, Non-Causal

- **Causal Signals:**

- 0 for all negative value/time

- $$x(t) = \begin{cases} x(t) > 0 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

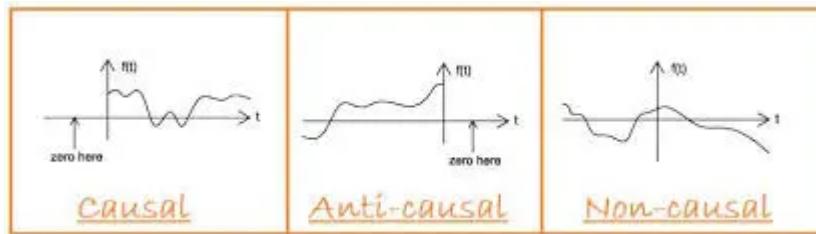
- **Non-Causal Signals**

- A signal that have positive amplitude for both positive and negative instance of time

- **Anti-Causal Signal**

- 0 for all positive value/time

- $$x(t) = \begin{cases} x(t) > 0 & t \leq 0 \\ 0 & t > 0 \end{cases}$$



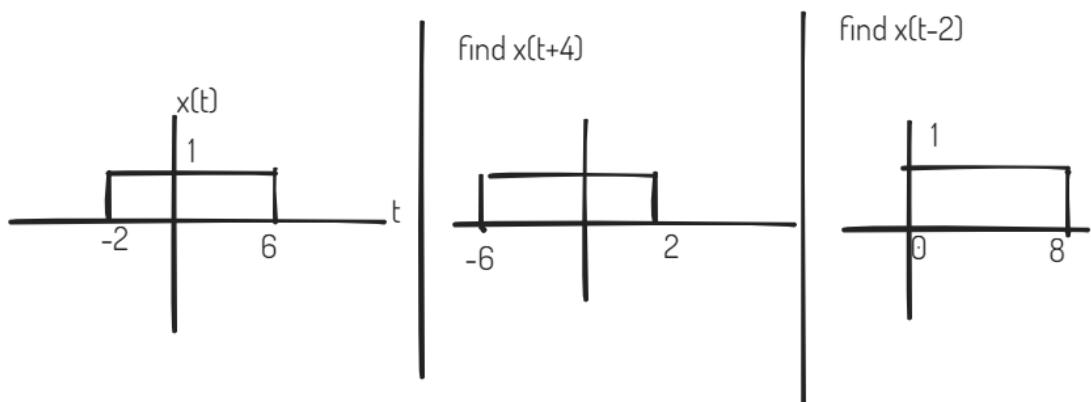
Causal, Anti-Causal, Non-Causal

Operations on Signals

- **Time Shifting Operation**

$f(t)$ is be given. $f(t \pm t_0) = ?$

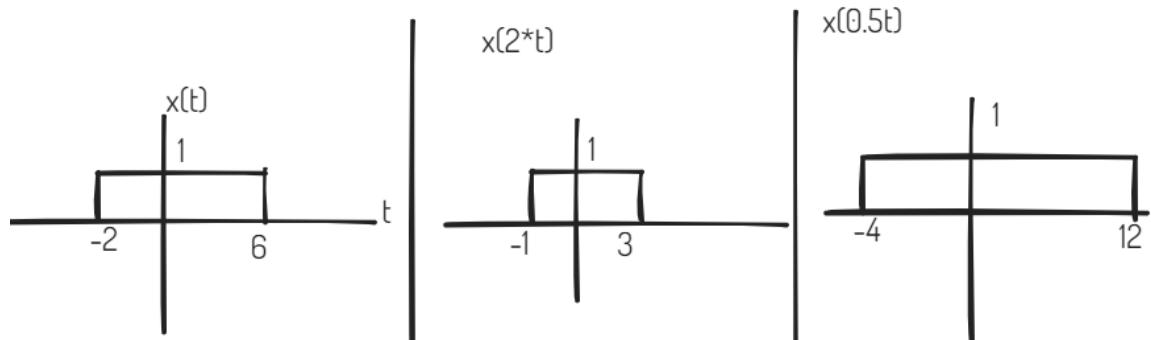
- t_0 is a constant
- $+ \rightarrow \text{advance}$: Shift the signal towards left by t_0
- $- \rightarrow \text{delay}$: Shift the signal towards right by t_0
- Amplitude doesn't change for shifting



- **Time Scaling Operation**

$x(t)$ is given. $f(\alpha t) = ?$

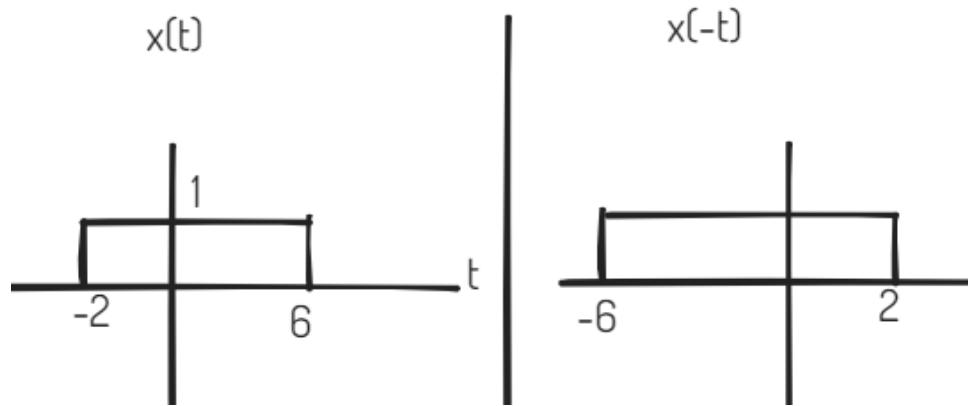
- α : scaling factor
- $\alpha > 1 \rightarrow \text{Signal Compression (Increasing Speed)}$: Divide the existing limit by α
- $\alpha < 1 \rightarrow \text{Signal Expansion (Decreasing Speed)}$: Divide the existing limit by
- Amplitude doesn't change for this operation



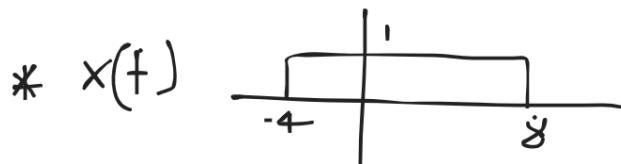
- **Time Reversal or folding Operation**

$x(t)$ is given. $x(-t) = ?$

- The sign of the limit will be changed

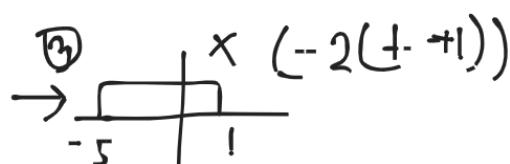
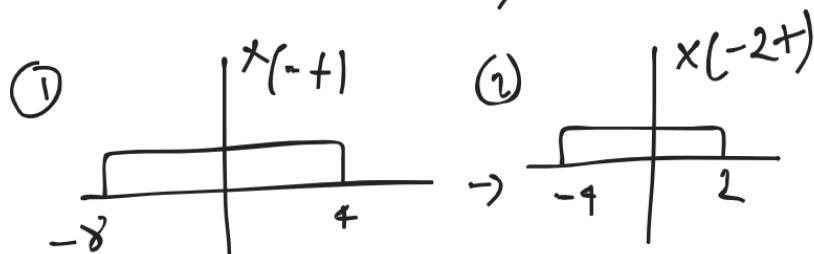


Example on Operation:



$$x(-2 + -2) = ?$$

$$\Rightarrow x(-2(++1)) \quad \text{-1. the coefficient of } t \text{ must be one}$$

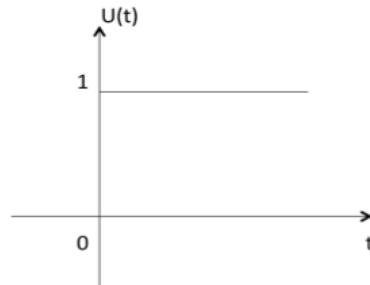


Elementary Signals

- Unit Step Signal
 - Also known as **Heaviside Step Function**

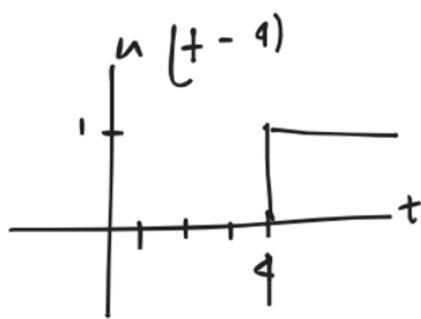
- Unit step function is denoted by $u(t)$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

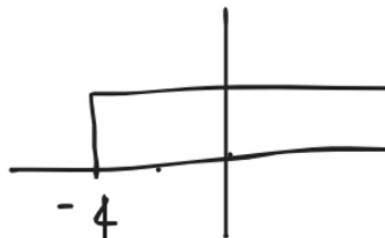


- Amplitude = coefficient of $u(t)$
- Non-Causal Signal * Unit Step Function = Causal Signal
- Operations on Unit Step Signal:**

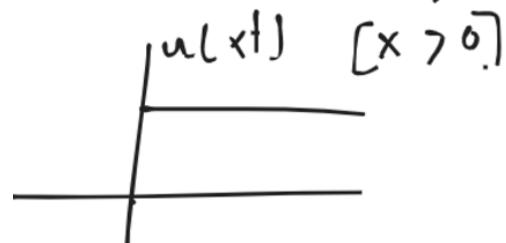
* $u(t - 4)$



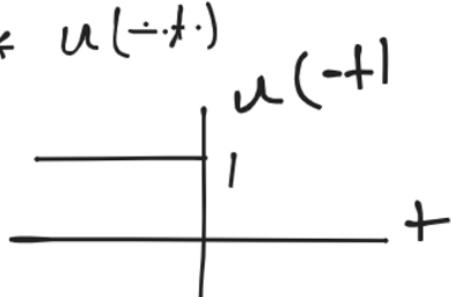
* $u(t + 4)$



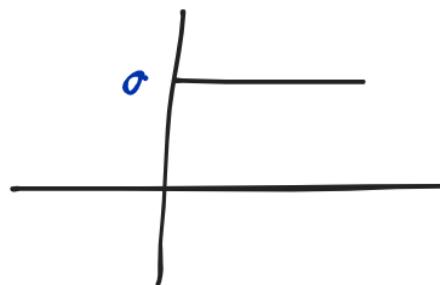
* $u(2t) = u(xt)$



* $u(-t)$ $u(-t)$



$$* \alpha u(t)$$

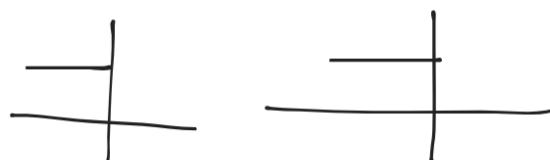


◦ Example:

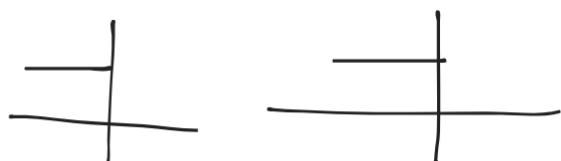
$$* u(-2t+4)$$

$$\Rightarrow u(-2(+ - 2))$$

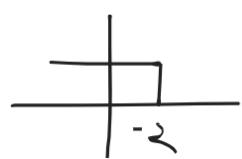
$$\textcircled{1} u(-+) \quad \textcircled{2} u(-2+)$$



$$\textcircled{1} u(-+) \quad \textcircled{2} u(-2+)$$



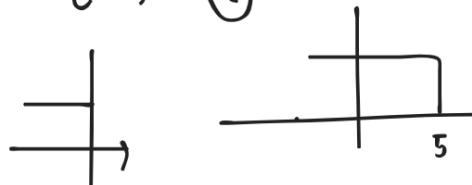
$$\textcircled{3} u(-2(+ - 2))$$



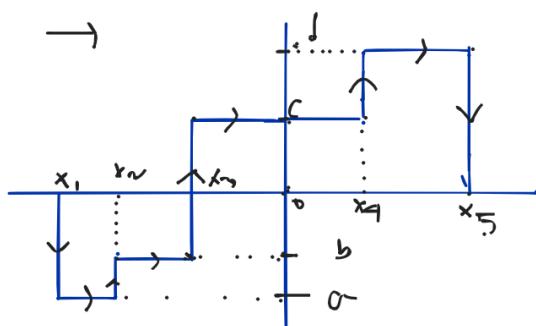
$$\cancel{u(-+ + t_5)}$$

$$\Rightarrow u(-(+-t_5))$$

① $u(-+)$ ②



- Expressing by unit step function

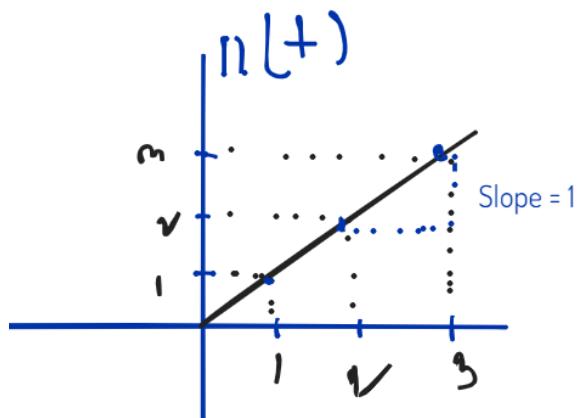


$$\begin{aligned}
 y(t) = & [(-b - 0)] u(t + x_1) + [-b - (-a)] u(t + x_2) \\
 & + [c - (-b)] u(t + x_3) + [d - c] u(t - x_4) \\
 & + (e - d) u(t - x_5)
 \end{aligned}$$

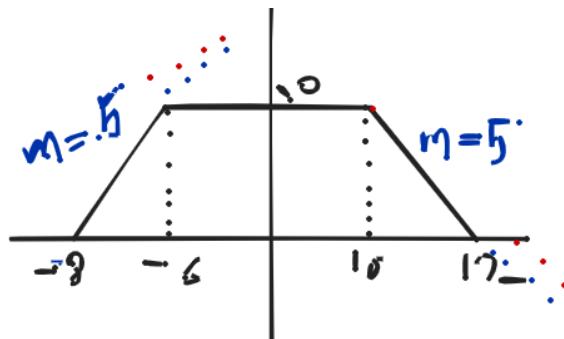
- Unit Ramp Signal

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Slope = Coefficient of $r(t)$



- $r(t) = t \cdot u(t)$
- $\frac{d}{dt}r(t) = u(t)$
- $\frac{d}{dt}[A \cdot r(t)] = A \cdot u(t)$
- **Expressing into Ramp Signal**



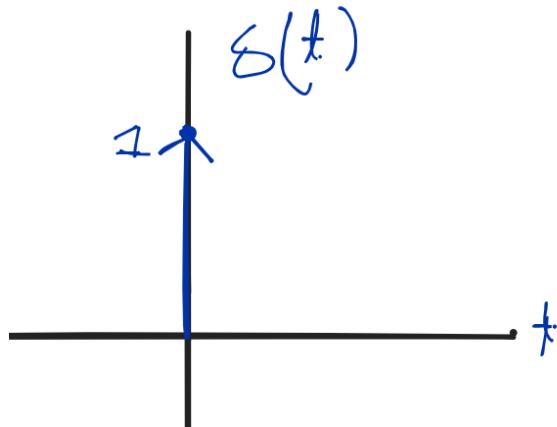
$$\begin{aligned}
 &= +5 \pi(t+8) - 5 \pi(t+6) \\
 &\quad - 5 \pi(t-10) + 5 \pi(t-12)
 \end{aligned}$$

- **Impulse Function**

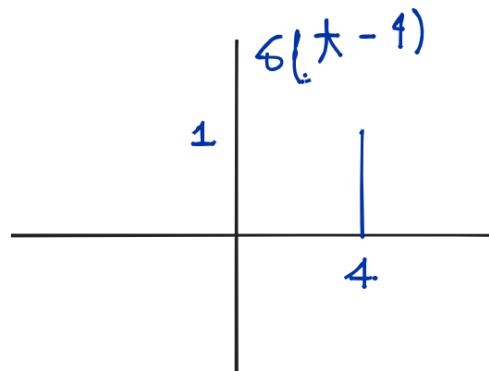
- An ideal impulse signal is a signal that is zero everywhere but at the origin ($t = 0$), it is infinitely high. Although, the area of the impulse is

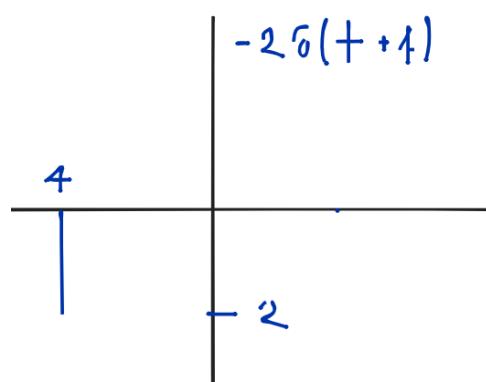
finite.

- $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$
- $A \cdot \delta(t)$, Here **A** is the area of this impulse function.



- $\int_{-\infty}^{\infty} \delta(t) dt = 1$ or $\int_{0^-}^{0^+} \delta(t) dt = 1$
- $\int_a^b \delta(t - t_0) dt$, it exists if $a \leq t_0 \leq b$, otherwise 0
- $\delta(t) = \frac{d}{dt} u(t)$
- **Operations:**

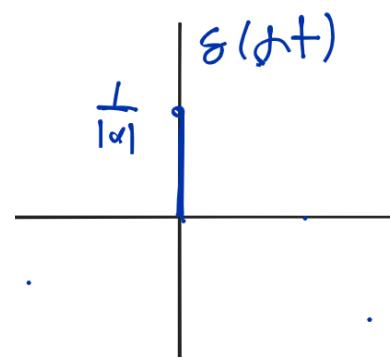




- **Properties:**

- **Time Scaling Property:**

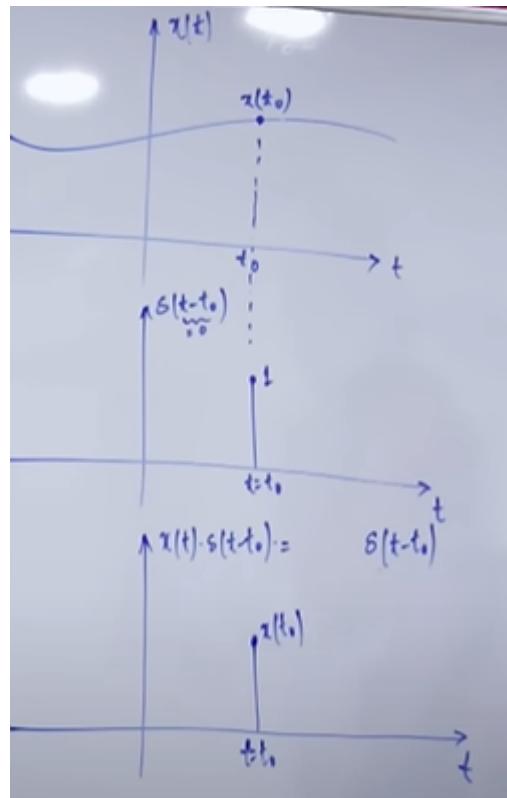
$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$



- **Product Property**

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$

$$x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$$



■ **Shifting Property**

$$(i) \int_{-\infty}^{\infty} x(t) \cdot \delta(t)$$

$$= \int_{-\infty}^{\infty} x(0) \cdot \delta(t)$$

$$= x(0) \int_{-\infty}^{\infty} \cdot \delta(t)$$

$$= x(0)$$

$$(ii) \int_{-\infty}^{\infty} x(t) \cdot \delta(t - t_0)$$

$$= \int_{-\infty}^{\infty} x(t_0) \cdot \delta(t)$$

$$= x(t_0) \int_{-\infty}^{\infty} \cdot \delta(t)$$

$$= x(t_0)$$

$$I = \int_{-6}^8 (t^3 + 4) \delta(t-3) dt$$

$$x(t) = t^3 + 4$$

$$x(3) = 9 + 4 = 13$$

$$\begin{aligned} I &= \int_{-6}^8 x(t) \delta(t-3) dt \\ &= x(3) \times 1 = 13 \end{aligned}$$

←

$$\star I = \int_{-\pi}^{\pi} \underbrace{\cos^2 t}_{x(t)} \cdot \underbrace{\delta(t - \frac{\pi}{4})}_{t = \frac{\pi}{4}} dt$$

$$I = (\cos^2 t) \Big|_{t=\frac{\pi}{4}} = \left(\cos \frac{\pi}{4}\right)^2 = \frac{1}{2}.$$

$$I = \int_{-6}^5 (t-1) \cdot \delta(2t-4)$$

$\xrightarrow{\quad}$

$$\delta(t-1) = \frac{1}{|2|} \delta(t)$$

$$\delta\{2(t-1)\} = \frac{1}{2} \delta(t-2)$$

$$I = \frac{1}{2} \int_{-6}^6 x(t) \cdot \delta(t-2)$$

$$= \frac{1}{2} \cdot x(2) = 0$$

■ Extension Properties

$$\int_a^b x(t) \cdot \delta(g(t)) dt$$

$$\delta(g(t)) = \frac{\delta(t-t_0)}{|g'(t_0)|} [if g has a real root at t = t_0]$$

If g has more than one real root at t_0, t_1, \dots, t_i

$$s(g(t)) = \sum_i \frac{\delta(t-t_i)}{|g'(t_i)|}$$

Example:

$$I = \int_{-10}^{10} (t^2 + 10) \cdot \delta(t^2 - 16) dt$$

$$t^2 - 16, t = +4, -4$$

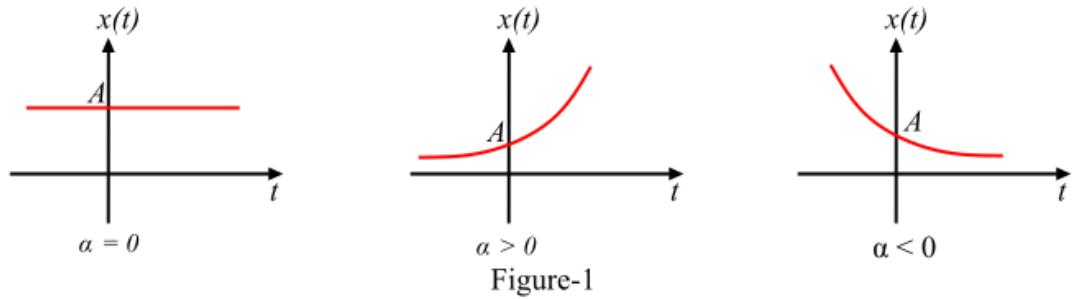
$$I = \int_{-10}^{10} (t^2 + 10) \cdot \frac{\delta(t-4)}{|2 \cdot (-4)|} dt + \int_{-10}^{10} (t^2 + 10) \cdot \frac{\delta(t+4)}{|2 \cdot (4)|} dt$$

$$= \frac{13}{4}$$

• Exponential Function

$$x(t) = A e^{(\alpha t)}$$

$A = \text{Amplitude at } t = 0$



Signal Classification on Even and Odd

- Even/Symmetric Signal

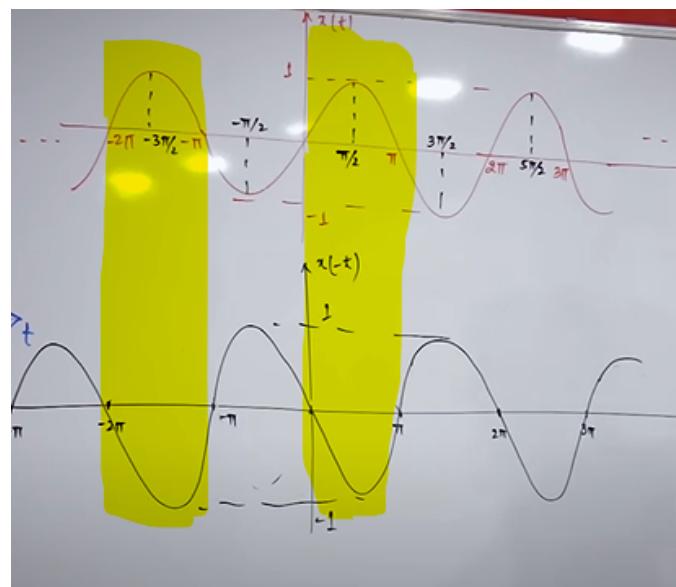
$$x(-t) = x(t)$$

Example : $x(t) = \begin{cases} 2 & -a \leq t \leq a \\ 0 & \text{otherwise} \end{cases}, x(t) = t^2, \dots, t^{2n}$

- Odd/Anti-symmetric Signal

$$x(-t) = -x(t)$$

Example: $x(t) = t^3, t^5, \dots, t^{2n+1}$



Check a function even or odd:

$x(t)$ is given. Find $x_1(t) = x(-t) = \dots$

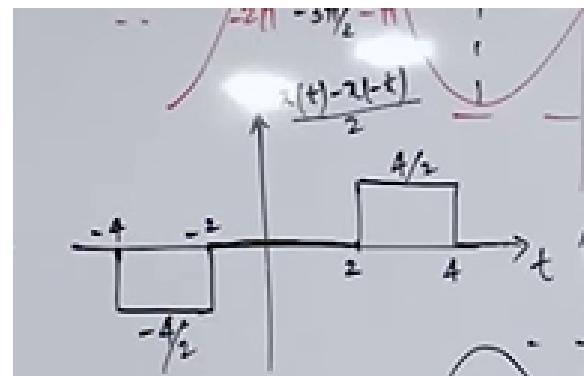
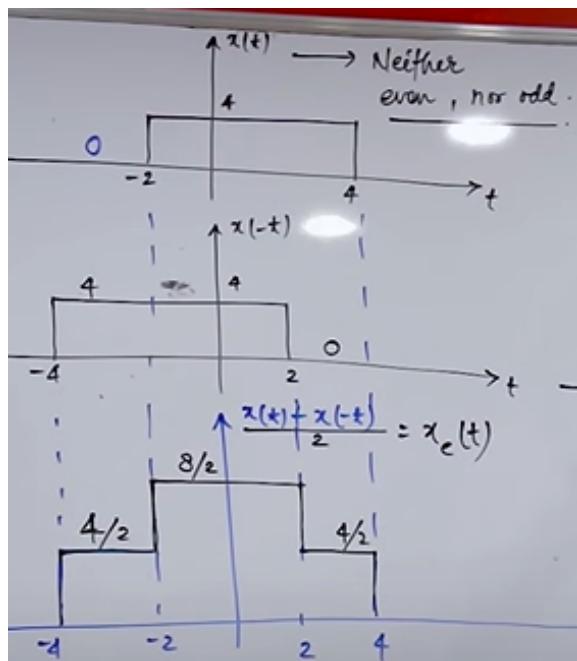
if $x(1) == x_1(1)$, it is even, otherwise odd.

Converting to Odd or Even Signal

To convert any arbitrary signal which is neither even nor odd into equivalent even or odd parts:

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$



- $x(t)$ is complex

$$x(t) = a + ib$$

- Even

$$x(t) = \overline{x(-t)}$$

- Odd

$$x(t) = -\overline{x(-t)}$$

Conjugate symmetric Part of $x(t)$

$$x_e(t) = \frac{x(t) + \overline{x(-t)}}{2}$$

Conjugate anti-symmetric part of $x(t)$

$$x_o(t) = \frac{x(t) - \overline{x(-t)}}{2}$$

- **Some Operations**

$$E = Even, O = Odd$$

$$O \pm O = O$$

$$E \pm E = E$$

$$O \pm E = Neither\ Odd\ nor\ Even$$

$$E \times E = E$$

$$E \times O = O$$

$$O \times O = E$$

$$E/E = E$$

$$O/O = E$$

$$E/O = O$$

$$O/E = O$$

$$\int O = E$$

$$\int E = O$$

$$\frac{d}{dt} E = O$$

$$\frac{d}{dt} O = E$$

Periodic and Aperiodic Signal

- **Periodic Signal**

- A signal is said to periodic, if it satisfies following two properties

- It must be exist for $-\infty \leq t \leq \infty$
- It must repeat itself after some constant amount of time T , which is called **Fundamental Time Period**
 - $T = \frac{2\pi}{w_0}$
 - w_0 = Fundamental frequency $\text{rad s}^{-1} = 2\pi f$
 - $T = \frac{1}{f}$
 - **Frequency must be real number. If frequency is not real number, it's not periodic.**
- Types
 - **Sinusoidal Signals**
 - **Representation**
 - $x(t) = A \sin(w_0 t + \theta)$
 - A = Amplitude
 - $w_0 t$ = Phase Angle
 - θ = Phase shift (+ → Advance, - delay)
 - $\frac{d}{dt}(\text{Phase}) = \frac{d}{dt}(w_0 t) = w_0 = \text{frequency}$
 - Shifting effect doesn't effect on periodicity, T
 - $x(t \pm kT) = x(t)$
 - Comparing with $x(t) = A \sin(n\pi t + \theta)$, if t is not square root of t , then the signal must be periodic.

$$\star x(t) = 4 \sin \overbrace{300\pi t}^{\omega_0}$$

Comparing with standard expression.

$$\omega_0 = 300\pi \text{ rad/s.}$$

$$\therefore T = \frac{2\pi}{\omega_0} = \frac{2\pi}{300\pi} = \frac{1}{150} \text{ sec.}$$

$$\star x(t) = \underbrace{5}_{\omega_0} \cos \left(\underbrace{20\pi t}_{\text{in rad}} + \underbrace{\pi/4}_{\text{in rad}} \right)$$

$$\omega_0 = 20\pi$$

$$\therefore T = \frac{2\pi}{\omega_0} = \frac{2\pi}{20\pi} = 0.1 \text{ sec.}$$

■ Combination of periodic signals

$$x(t) = A \sin(\omega_0 t) + B \cos(\omega_1 t) + C \sin(\omega_2 t) + \dots$$

- $x(t)$ will be periodic if ratio of individual time period is a rational number

- Rational Number

- Can be expressed by $\frac{p}{q}$. p, q are co-prime.
- The value of $\frac{p}{q}$ should be terminating or repeating decimal.

Example : 3.3333.., 2.5, 5.20202020..

Example: $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_1}{T_3} = \text{Rational Number}$

- Time Period of Resultant Signal

$$T = LCM(T_1, T_2, \dots)$$

- Frequency of Resultant Signal

$$w'_0 = \text{GCD}(w_0, w_1, \dots)$$

- If all w_0, w_1, \dots have π , then it is periodic. Or if all w_0, w_1, \dots haven't π , it is periodic too. If some of them have π and rest of them not, then it isn't periodic.

* $x(t) = \underbrace{4 \cos t}_T + \underbrace{3 \sin 2\pi t}_T + \underbrace{2 \sin 3\pi t}_T$

$$\omega_1 = 1 \quad \omega_2 = 2\pi \quad \omega_3 = 3\pi$$

$$\therefore T_1 = \frac{2\pi}{\omega_1} = 2\pi \quad T_2 = \frac{2\pi}{\omega_2} = 1 \quad T_3 = \frac{2\pi}{\omega_3} = \frac{2}{3}$$

$$\frac{T_1}{T_2} = \frac{2\pi}{1} = 2\pi \neq \text{rational}$$

$$\therefore x(t) \text{ is not periodic.}$$

- DC/Constant Signal**: independent from time. It doesn't effect on frequency, time. So, it is not countable.

* $x(t) = 4 + \cos^2 4\pi t$

$$x(t) = 4 + \left[1 + \cos 8\pi t \right]$$

$$= 4 + \frac{1}{2} + \frac{1}{2} \cos 8\pi t$$

$$= \frac{9}{2} + \frac{1}{2} \cos 8\pi t$$

$$\omega_0 = 8\pi \text{ r/s}$$

$$T = \frac{2\pi}{8\pi} = \frac{1}{4} = \underline{\underline{0.25 \text{ sec.}}}$$

4 is a constant Signal

Energy and Power Signals

- **Energy Signals**

Energy of $x(t)$ is given as

$$E_x = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

[No need to take lim if the signal is aperiodic]

if $0 < E_x < \infty$ (Finite), then $x(t)$ is said to be *Energy Signal*.

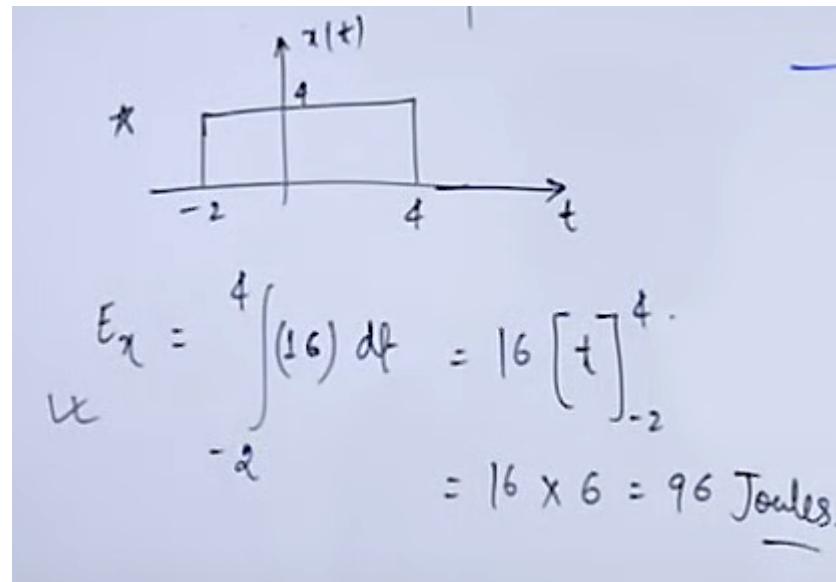
Example:

(i) $x(t) = e^{-4t} \cdot u(t)$

$$E_x = \int_{-\infty}^{\infty} [e^{-4t} \cdot u(t)]^2 dt$$

$$= \int_0^{\infty} e^{-8t} dt \quad [u(t) \text{ exists only } 0 \text{ to } \infty]$$

$$= \frac{1}{8}$$



- **When a signal $x(t)$ will be energy signal**

- If $x(t)$ is existing for infinite direction and decreasing in value.

$$\lim_{t \rightarrow \infty} f(t) = 0$$

- If $x(t)$ exists for finite direction and value of $x(t)$ is finite finite at all points, $x(t)$ is energy signal.

- **Power Signals**

Power Signal of $x(t)$ is given as

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot E$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

[No need to take lim if the signal is aperiodic]

If $0 < P_x < \infty$ (Finite), then $x(t)$ is said to be Power Signal

- Power = RMS^2
- $x(t) = A \sin \omega t$, $RMS = \frac{A}{\sqrt{2}}$, $P = RMS^2 = \frac{A^2}{2}$

Given $x(t) = A \sin \omega t$

$\omega = 1 \text{ rad/s}$

$T = \frac{2\pi}{\omega} = 2\pi$

$P = \frac{1}{T} \int_{-T/2}^{T/2} |A \sin \omega t|^2 dt$

$\therefore P_x = \left(\frac{A}{\sqrt{2}}\right)^2 = \frac{A^2}{2}$

$\therefore P_x = \frac{A^2}{2\pi} \int_{-\pi}^{\pi} A^2 \sin^2 t dt$

$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1 - \cos 2t}{2}\right) dt$

$= \frac{A^2}{2\pi} \left[\frac{1}{2} \int_{-\pi}^{\pi} dt - \frac{1}{2} \int_{-\pi}^{\pi} \cos 2t dt \right]$

$= \frac{A^2}{2\pi} \times \frac{1}{2} \times 2\pi$

$= \frac{A^2}{2}$

- When a signal will be Power Signal

- All periodic signal are power signal but converse is not true
- If $x(t)$ is not a periodic signal and follows the conditions
 - $\lim_{t \rightarrow \infty} f(t) \neq 0$
 - $\lim_{t \rightarrow \infty} f(t) \neq \infty$
- A signal can't be Energy and Power Signals together.
 - If E_x is finite, then P_x is Zero . Vice-Versa.
- **Operations**
 - Time shifting has no effect on power and energy of signal.
 $Power x(t) = Power x(t - \frac{T}{2})$
 $Energy x(t) = Energy x(t - 4)$
 - Time Scaling doesn't effect on Time Periodic but in Time period, Energy.
 - For $x(t)T, E_x$

$$x'(t) = x(\alpha t), T' = \frac{T}{\alpha}, E'_x = \frac{E_x}{\alpha}$$

- Power remains same.
 -

Systems

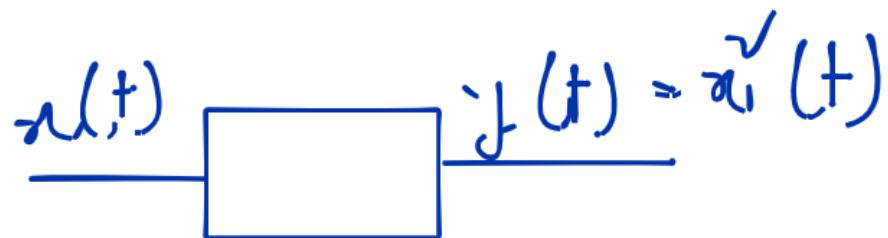
System is a interconnection of different physical components which is used to convert one form of signal to others.

Example:

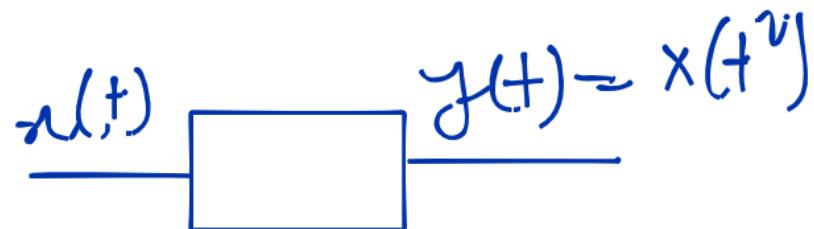


Properties of Systems

- Static and Dynamic System
 - Static
 - Memoryless
 - if present o/p depends on only present i/p



- Dynamic
 - With Memory
 - if present o/p depends on past or future i/p



$y(2) = x(2^2) = x(4)$, $t = 2$'s output depends of $t = 4$. That means present o/p depends on future i/p;

- Causal, Noncausal, Anticausal
 - Causal
 - Present o/p depends on
 - present i/p or
 - Present + Past i/p
 - Static system are always causal
 - $h(t) = 0 \quad t < 0$
 - $h[n] = 0 \quad t < 0$

- Noncausal
 - Present o/p depends on
 - present + future or
 - present + past + future or
 - past + future
 - $h(t) \neq 0, \quad t < 0$
 - $h[n] \neq 0, \quad t < 0$

$$y[n] = x[n^2]$$

$$y[0] = x[0]$$

$$y[1] = x[1] \text{ depends on present}$$

$$y[2] = x[4] \text{ depends on future}$$

- Anticausal
 - Present o/p depends on
 - only future i/p
 - $y(t) = x(t^2 + 1)$
 - $y(0) = x(1)$
 - $y(-1) = x(0)$
 - depends on only future
 - $h(t) = 0 \quad t > 0$

$$h[n] = 0 \quad t > 0$$

- Time variant and Invariant System
 - Time Invariant
 - If time shift in i/p results identical time shift in o/p without changing the nature of the output.
 - To check
 - Find $y(t)$ delay, $y(t - t_0)$, replace t with $t - t_0$
 - Find $x(t - t_0) = y(t, t_0)$, response of the system for delayed i/p
 $y(t, t_0) =$ write the $y(t)$ just add/minus t_0
 - if $y(t - t_0) = y(t, t_0)$, it is time invariant.

$\star y(t) = x(t^2)$
 delayed response
 $y(t-t_0) = x[(t-t_0)^2] \dots \text{①}$

Response of the system for
 delayed i/p
 $y(t, t_0) = x(t^2 - t_0) \dots \text{②}$

$\therefore y(t-t_0) \neq y(t, t_0)$
Time variant.

$\star y(t) = t \cdot x(t)$
 $y(t-t_0) = (t-t_0)x(t-t_0) \dots \text{①}$

$y(t, t_0) = t x(t-t_0) \dots \text{②}$

$y(t-t_0) \neq y(t, t_0)$
Time variant.

$$\star \quad y(t) = x^c(t)$$

$$y(t-t_0) = x^2(t-t_0) \quad \text{--- } \textcircled{I}$$

$$y(t, t_0) = x^2(t-t_0) \quad \text{--- } \textcircled{II}$$

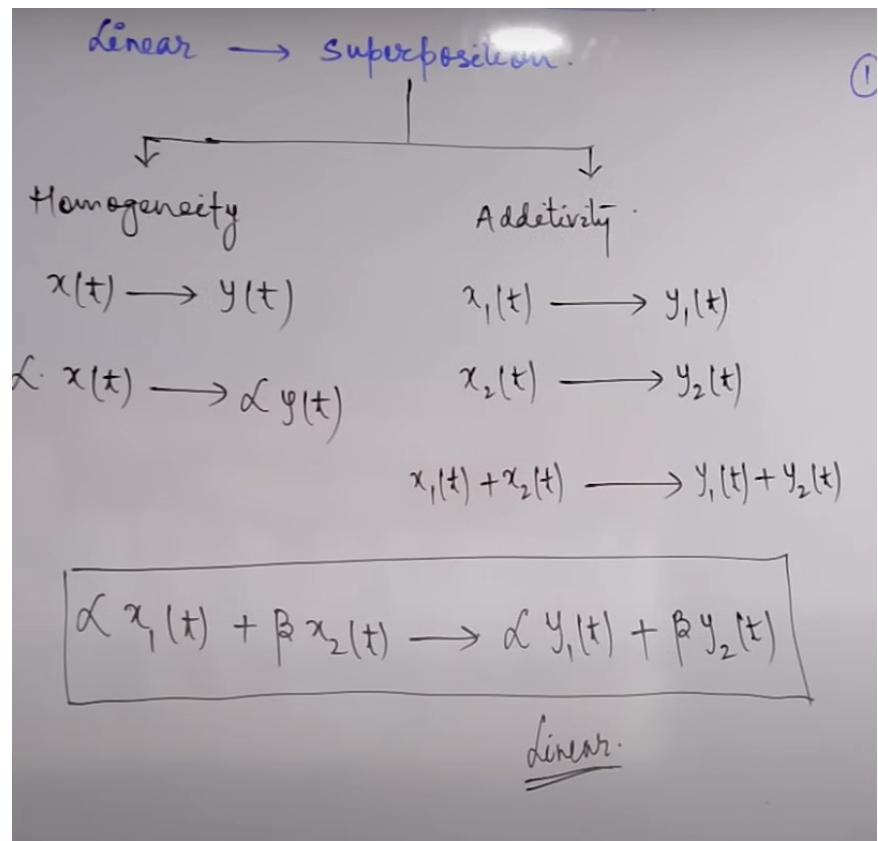
$$y(t-t_0) = y(t, t_0)$$

Time invariant

- For a system to be time invariant
 - There must not be any scaling in $x(t)$ or $y(t)$
 - Coefficient must not function of *time*
 - Any extra term except $x(t)$ or $y(t)$ must be zero or constant

- **Linear and Nonlinear System**

- Linear
 - If the system follows principle of superposition



- **To check linearity**

- Graph between output and input must be throughout a **straight line passing through origin** without having **saturation or dead zone**
- If system is represented by a linear differential equation, then the equation must be linear.
- System must be follow zero input zero output criteria.
 - $x(t) \rightarrow y(t)$
 - $\alpha x(t) \rightarrow \alpha y(t)$
 - $0 = 0, \alpha = 0$

- **Mathematical way to prove a system is linear or not**

$$\text{Given, } x(t) \rightarrow y(t) = tx(t)$$

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

$$y_3(t) = y_1(t) + y_2(t) = t(x_1(t) + x_2(t)) \dots (1)$$

$$x'_3(t) = x_1(t) + x_2(t)$$

$$y'_3(t) = tx_3(t) = t(x_1(t) + x_2(t)) \dots (2)$$

$(1) = (2)$, it follows the additivity.

$$\alpha x(t) \rightarrow \alpha y(t) = \alpha t x(t)$$

$$y'(t) = t[\alpha x(t)] = \alpha t x(t) \dots (3)$$

$$\alpha y(t) = \alpha t x(t) \dots (4)$$

$(3) = (4)$, it follows the homogeneity rule

- **Checking homogeneity and additivity at once**

$$x(t) \rightarrow y(t)$$

$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$, the system will be linear

Given, $y(t) = x[sint]$

$$y_1(t) = x_1[sint], y_2(t) = x_2[sint]$$

$$y_3(t) = \alpha y_1(t) + \beta y_2(t) = \alpha x_1(sint) + \beta x_2(sint)$$

$$x'_3(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y'_3(t) = x'_3(sint) = \alpha x_1(sint) + \beta x_2(sint)$$

- **Invertible and Non-invertible**

- Invertible

- If distinct input produces distinct outputs
- if input can be determined by observing output
- a inverse system can be design, overall gain = 1

- **Stable and Unstable System**

- **Stable**

- Follows bounded input bounded output(BIBO)
- The output of the system must be bounded for bounded input

- $0 \leq |x(t)| < \infty$, then $0 \leq |y(t)| < \infty$
- BIBO implies that impulse response must tend to zero , as time tends to infinity.

- **LTI System**

- **Properties**

- The commutative property of LTI System

- **Discrete Time Signal**

$$x[n] * h[n] = h[n] * x[n]$$

or, $\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

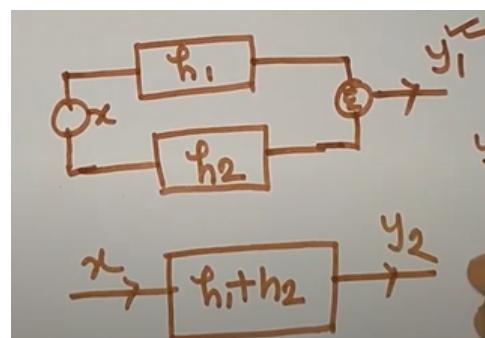
- **Continuous Time Signal**

$$x(t) * h(t) = x(t) * h(t)$$

or, $\int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau$

- **The distributive Property of LTI System**

Summing of the outputs of two systems is equivalent to a system with an impulse response equal to the sum of the impulse response of the two individual system.



$$y_1 = y_2$$

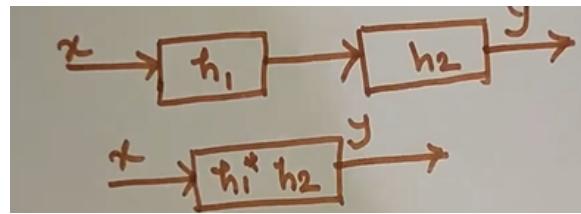
$$y_1 = x * h_1 + x * h_2$$

$$y_2 = x * (h_1 + h_2)$$

$$\text{Hence, } x * h_1 + x * h_2 = x * (h_1 + h_2)$$

- **Associative Property**

The change of order of the cascaded system will not affect the response.



$$y = x * (h_1 * h_2) = (x * h_1) * h_2$$

- **LTI System with and without memory**

Memoryless \rightarrow Output(t) \rightarrow input(t), Current output depends on only current input

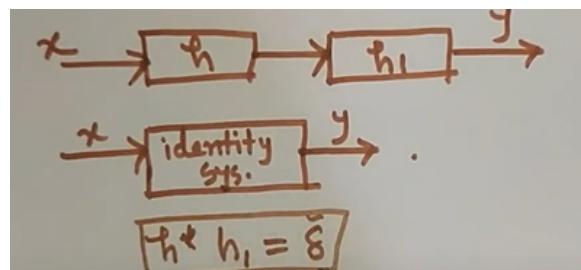
DTS: $h[n] = 0$ for $n \neq 0$

CTS: $h[t] = 0$ for $t \neq 0$

Otherwise, with memory.

- **Invertibility of LTI System**

A system S is invertible if and only if there exists an inverse system S^{-1} such that S^{-1} is an identity system.



$$h * h_1 = \delta$$

- **Causality Property**

DTS: $h(n) = 0, n < 0$

CTS: $h(t) = 0, t < 0$

- **Stability Property**

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty, \text{ for DTS}$$

$$\int_{-\infty}^{\infty} |h(t)| < \infty, \text{ for CTS}$$

- **Check Causality and Stability for $h(n) = \frac{1}{2}^n u(n)$ (DTS : Discrete Time Signal)**

$$u(n) = 0, n < 0$$

$$h(n) = \frac{1}{2}^n \times 0; n < 0$$

= 0, Causal;

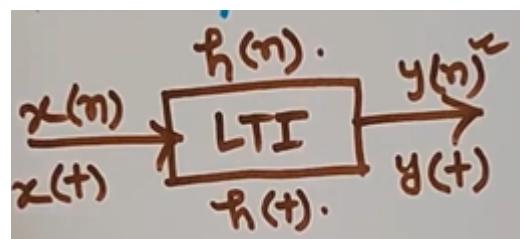
$$\sum_{k=-\infty}^{\infty} h(k) = \sum_{k=-\infty}^0 \frac{1}{2}^n u(k) + \sum_{k=0}^{\infty} \frac{1}{2}^n u(k) = 0 + \frac{1}{1-\frac{1}{2}} = 2 < \infty, \text{ Stable}$$

- **Convolution**

Convolution is a mathematical way of “**combining two signals to form a third signal**”

* is the convolution sign.

- **Characterizing a LTI System**



$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

DTS(Discrete Time Signal):

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= h(n) * x(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} x(n-k)h(n) \end{aligned}$$

$$\begin{aligned}
 * h[n] &= [1, 1, 1], \quad x[n] = [0.5, 2] \quad y[n] = ? \\
 y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]. \\
 &= \sum_{k=0}^1 x[k] h[n-k]. \\
 &= x[0] h[n] + x[1] h[n-1]. \\
 &= 0.5 h[n] + 2 h[n-1].
 \end{aligned}$$

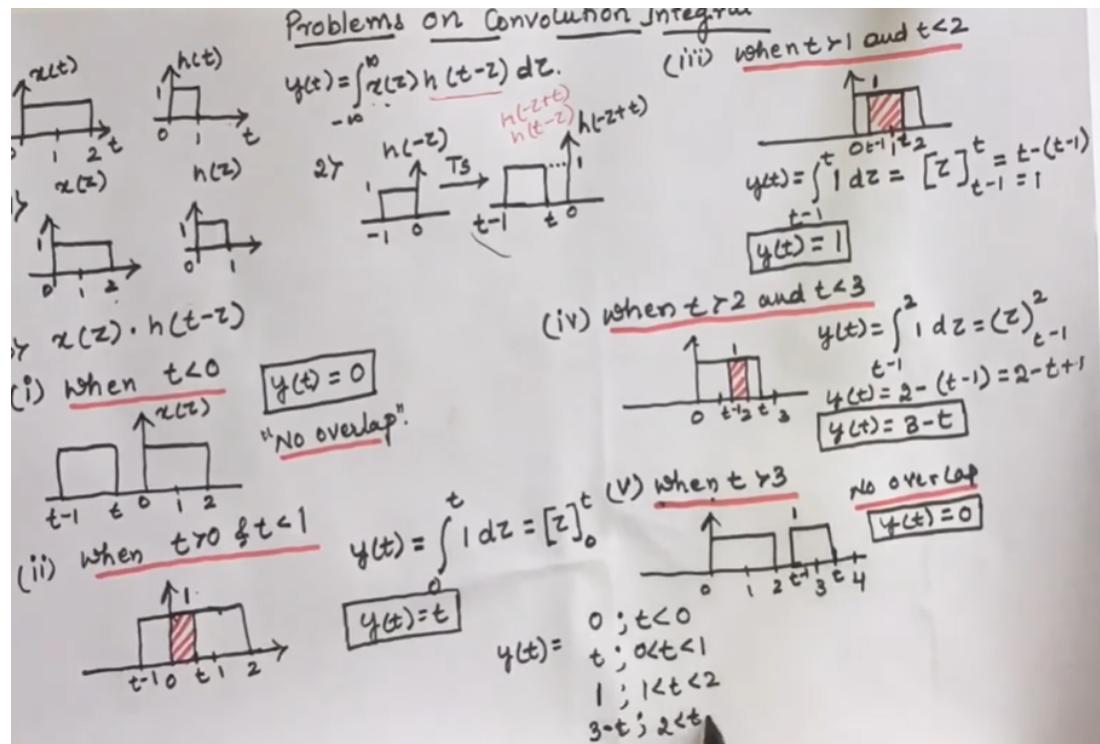
$$\begin{aligned}
 &= 0.5 h[n] + 2 h[n-1]. \\
 y[n] &= [0.5, 2.5, 2.5, 2]
 \end{aligned}$$

CTS (Continuous Time Signal):

$$\begin{aligned}
 y(t) &= h(t) * x(t) = x(t) * h(t) \\
 &= \int_0^t x(\lambda) h(t - \lambda) d\lambda = \int_0^t h(\lambda) x(t - \lambda) d\lambda
 \end{aligned}$$

Steps:

- Folding: $h(\lambda) = h(-\lambda)$
- Shifting: $h(-\lambda + t) \rightarrow h(t - \lambda)$
- Multiplication: $x(\lambda), h(t - \lambda)$
- Integrating: $\int_{t_1}^{t_2} x(\lambda) * h(t - \lambda)$



Laplace Transform

Laplace transform is an integral transformation of a function $f(t)$ from time domain into the frequency domain.

$$f(t) \rightarrow F(s) \quad s = a + jw, \text{ real and imaginary part}$$

$$\text{Laplace of } f(t), \mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st} dt$$

$y(t) = x(t) * h(t)$ convolution in time domain

$$Y(s) = X(s) \times H(s)$$

simple multiplication frequency domain

$$\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st} dt = \frac{1}{s}$$

$$\mathcal{L}[\cos wt] = \int_0^\infty \frac{e^{jwt} + e^{-jwt}}{2} e^{-st} dt = \frac{s}{s^2 + w^2}$$

$$\mathcal{L}[\sin wt] = \int_0^\infty \frac{1}{2j} (e^{jwt} - e^{-jwt}) e^{-st} dt = \frac{w}{s^2 + w^2}$$

$$\mathcal{L}[tu(t)] = \int_0^\infty tu(t)e^{-st} dt = t \int_0^\infty e^{-st} dt - \int_0^\infty \frac{dt}{dt} \int_0^\infty e^{-st} dt = \frac{1}{s^2}$$

Laplace transform condition/Dirichlet's conditions

- The function must be integrable in the given interval of time

$$\int_{-\infty}^{\infty} f(t)dt < \infty$$

- The function should have finite number of maxima and minima.

Example: Noise signal doesn't follow this condition

- There must be finite number of discontinuities in the signal in the given interval of time.

Properties

Linearity Property

If $x_1(t), x_2(t)$ are two signals, then they will follow the superposition theorem.

$$a_1x_1(t) + a_2x_2(t) \rightarrow a_1X_1(s) + a_2X_2(s)$$

$$Z_1(t) = a_1x_1(t) + a_2x_2(t)$$

$$X_1(s) = \mathcal{L}[x_1(t)] = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt$$

$$X_2(s) = \mathcal{L}[x_2(t)] = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$Z_1(s) = \mathcal{L}[Z_1(t)] = a_1X_1(s) + a_2X_2(s)$$

Time Shifting Property

$$x(t) \leftrightarrow X(s)$$

$$x(t - t_0) \leftrightarrow e^{-st_0}X(s)$$

Time Scaling Property

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

Frequency Property

$$X(s - s_0) \rightarrow e^{s_0 t}x(t)$$

Time Reversal

$$x(t) \leftrightarrow X(s)$$

$$x(-t) \leftrightarrow -X(-s)$$

Differentiation & Integration Property

$$\frac{dx(t)}{dt} \leftrightarrow SX(S)$$

$$\int x(t)dt \leftrightarrow \frac{X(s)}{s}$$

Multiplication and Convolution Property

$$x_1(t), x_2(t)$$

$$x_1(t) * x_2(t) \leftrightarrow X_1(s) \times X_2(s)$$

$$x_1(t) \times x_2(t) \leftrightarrow X_1(s) * X_2(s)$$

Fourier Series

Fourier Series is a sum that represents a "periodic function" as a sum of *sine* and *cosine* waves in terms of their harmonics.

$x(t) = x(x \pm T)$, T : Time period, $x(t)$ is periodic.

Frequency, $f = \text{no of cycle/sec} / \text{rate of change}$

Harmonics

$$x(t) = \sin \omega t + 5\sin 2\omega t + 9\sin 3\omega t.$$

fundamental freq/ 1st harmonic 2nd 3rd.

3rd Harmonics will dominate 2nd one because its magnitude is greater than the 2nd one. ($9 > 5$)

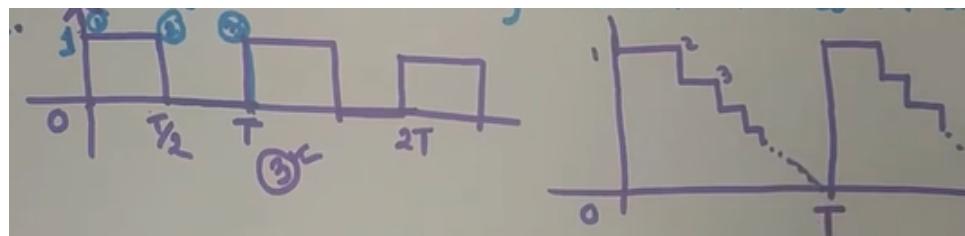
Dirichlet conditions of Fourier Series

- Function $f(t)$ is single valued everywhere.

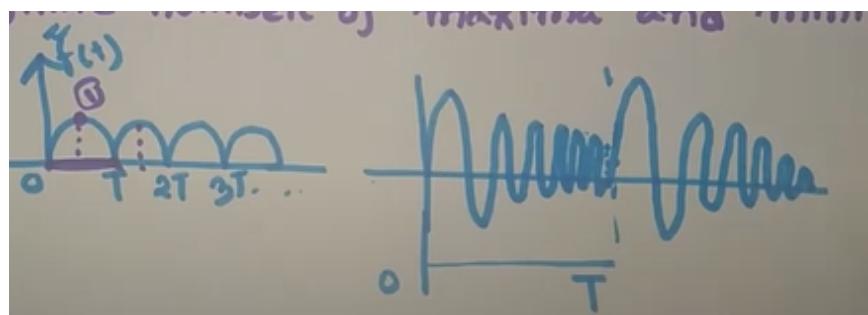
$$f(x) = x^2;$$

For every x there is only one $f(x)$ or y

- Function has a finite number of discontinuities in any time period.

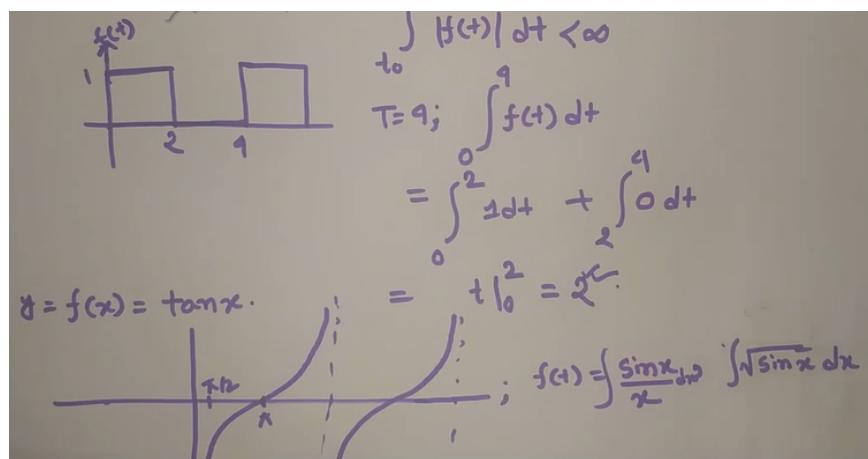


- Function has a finite number of maximum and minimum in any time period.



- Signal should be absolutely integrable in any time period.

$$\int_{t_0}^{t_0+T} |f(t)| dt < \infty$$



Why sine and cosine are special to Fourier Series ?

- The Fourier series uses cosine and sine functions **because they form a complete set of orthogonal functions, meaning they are independent and can be used to represent any periodic function**

Trigonometric Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \dots (i)$$

a_0, a_n, b_n = fourier coeefcient

$a_0 = \frac{1}{T} \int_0^T f(t) dt$ = average of function

Multiplying $\cos n\omega_0 t$ and integrating in (i),

$$\begin{aligned} \int_0^T f(t) \cos n\omega_0 t dt &= \int_0^T [a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \cos n\omega_0 t dt \\ &= \int_0^T a_0 \cos n\omega_0 t dt + \sum_{n=1}^{\infty} (\int_0^T a_n \cos n\omega_0 t \cdot \cos n\omega_0 t dt + \int_0^T a_n \cos n\omega_0 t \cdot \sin n\omega_0 t dt) \\ &= 0 + \sum_{n=1}^{\infty} \int_0^T a_n \cos n\omega_0 t \cdot \cos n\omega_0 t dt + 0 \end{aligned}$$

Now, $n = m$,

$$\begin{aligned} \int_0^T f(t) \cos n\omega_0 t dt &= \int_0^T a_n \cos n^2 \omega_0 t dt = \int_0^T (1 + \cos 2n\omega_0 t) dt = \frac{T}{2} a_n \\ a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt \end{aligned}$$

Multiplying $\sin n\omega_0 t$ and integrating in (i),

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t dt$$

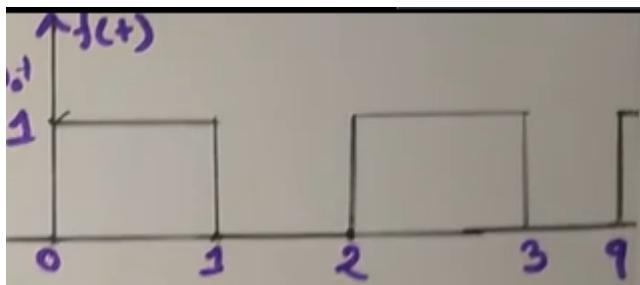
$$\int_0^T f(t) \sin n\omega_0 t = \int_0^T a_0 \sin n\omega_0 t + \sum_{n=1}^{\infty} \int_0^T a_n \sin n\omega_0 t \cos n\omega_0 t$$

\approx

$$\begin{aligned} \int_0^T f(t) \sin n\omega_0 t &= \frac{b_n}{2} \int_0^T 2 \sin^2 n\omega_0 t dt \\ &= \frac{b_n}{2} \int_0^T (1 - \cos 2n\omega_0 t) dt \\ &= \frac{b_n}{2} \times T \end{aligned}$$

$$\Rightarrow b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t$$

Fourier Series Expansion



$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 1 \leq t \leq 2 \end{cases}$$

$$T = 2$$

$$w_0 = \frac{2\pi}{T} = \pi$$

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} (\int_0^1 1 dt + \int_1^2 0 dt) = \frac{1}{2}$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t dt = \int_0^1 1 \cdot \cos n\pi t dt + \int_1^2 0 \cdot \cos n\pi t dt \\ &= \frac{\sin n\pi t}{n\pi} \Big|_0^1 = \frac{1}{n\pi} [\sin n\pi - \sin 0] = 0 \end{aligned}$$

Because, $\sin\pi = \sin 2\pi \dots = \sin n\pi = 0$

$$b_n = \frac{2}{T} \int_0^T f(t) \cdot \sin n\pi t dt = \int_0^1 1 \cdot \sin n\pi t dt + \int_1^2 0 \cdot \sin n\pi t dt \\ = \frac{-\cos n\pi t}{n\pi} \Big|_0^1 = -\frac{1}{n\pi} [\cos n\pi - \cos 0]$$

$$\cos n\pi = (-1)^n$$

$$b_n = \frac{1}{n\pi} [1 - (-1)^n] = \frac{2}{n\pi} \quad [\mathbf{n = odd}]$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t \quad [\mathbf{n = odd}] \\ = \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \dots$$

$$f(t) = \begin{cases} 15, & 0 < t < 1 \\ -15, & 1 < t < 2 \end{cases}$$

$$T=2$$

$$c_{L_n} = \frac{1}{T} \int_0^T f(t) dt \\ = \frac{1}{2} \left[\int_0^{15} dt - \int_{-15}^{15} dt \right] \\ = \frac{1}{2} [15 - (-15)] \\ = 0$$

$$\omega_n = \frac{2\pi}{T} = \pi$$

$$a_n = \frac{1}{T} \int_0^T f(t) \cos n\pi t dt \\ = 15 \cdot \frac{\sin n\pi t}{n\pi} \Big|_0^1 - 15 \cdot \frac{\sin n\pi t}{n\pi} \Big|_1^2 \\ = 0$$

$$\begin{aligned}
 b_n &= \frac{1}{2} \int_0^T f(t) \sin n\pi t dt \\
 &= \frac{15}{n\pi} [\cos n\pi - \cos 0] + \frac{15}{n\pi} \left[\frac{\cos 2n\pi}{2} - \frac{\cos 0}{2} \right] \\
 &= \frac{15}{n\pi} \left[\cos 2n\pi - 2 \cos n\pi + 1 \right] \\
 &= \frac{15}{n\pi} \left[2 - 2(-1)^n \right] \\
 &= \frac{30}{n\pi} \left[(-1)^n - 1 \right] \\
 &\therefore \frac{60}{n\pi} \left[n \text{ odd} \right]
 \end{aligned}$$

Even Symmetry

If $f(t) = f(-t)$

$$\begin{aligned}
 a_0 &= \frac{2}{T} \int_0^{\frac{T}{2}} f(t) dt \\
 a_n &= \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \cos nw_0 t dt \\
 b_n &= 0
 \end{aligned}$$

Odd Symmetry

If $f(t) = -f(-t)$

$$a_0 = a_n = 0$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nw_0 t dt$$

Complex Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

c_n = complex exponential fourier coefficient

$$c_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0 t} dt$$

$$c_n = |c_n| e^{j\angle n}$$

$$\underline{C}_n = \frac{1}{T_0} \int_{T_0}^{T_0 + \infty} x(t) e^{-jn\omega_0 t} dt \quad \text{--- (i) } [a_n, b_n] \xrightarrow{\text{jouw}} C_n$$

1) Reversal: ($n = -m$); $C_m = \frac{1}{T_0} \int_{T_0}^{T_0} x(t) e^{jn\omega_0 t} dt \dots \text{--- (ii)}$

2) Conjugate: $\underline{C}_m^* = \frac{1}{T_0} \int_{T_0}^{T_0} x^*(t) e^{-jn\omega_0 t} dt \dots \text{--- (iii)}$

If C_n is conjugate symmetric, $C_n = \underline{C}_{-n}$
 $x(t) = \underline{x}^*(t) \rightarrow \text{real}$

$$\underline{C}_n = |C_n| e^{j\angle C_n} \quad \text{--- (iv)}$$

1) $n \rightarrow -n$; $C_{-n} = |C_{-n}| e^{j\angle C_{-n}} \dots \text{--- (v)}$

$$\underline{C}_m^* = |C_{-n}| e^{-j\angle C_{-n}} \dots \text{--- (vi)}$$

∴, $C_n = C_{-n} \Rightarrow |C_n| e^{j\angle C_n} = |C_{-n}| e^{-j\angle C_n}$
 $\Rightarrow * |C_n| = |C_n| \rightarrow \text{Even}$
 $\Rightarrow \angle C_n = -\angle C_{-n} \rightarrow \text{odd}$