MATH-2105: Matrices, Vector Analysis & Coordinate Geometry

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Lecture Sheet 1

Matrix

Matrix is an arrangement of numbers, objects or any other data. A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$.

For example

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is an $m \times n$ matrix.

A particular entry is referenced by its row index (labeled i) and its column index (labeled j).

A matrix is basically a collection of vectors. A vector can be expressed as a column matrix:

$$\vec{V} = (V_x, V_y, V_z) \text{ or } \vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Any matrix contained with three vectors $\vec{U} = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}$, $\vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$ and $\vec{W} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}$ can be expressed as

$$A = [\vec{U} \quad \vec{V} \quad \vec{W}] = \begin{bmatrix} U_x & V_x & W_x \\ U_y & V_y & W_y \\ U_z & V_z & W_z \end{bmatrix}$$

Square Matrix

Matrix having same number of rows and columns is called a square matrix.

For example, a n$\times$ n square matrix is as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$1 \times 1 \quad [2], \qquad 2 \times 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad 3 \times 3 \quad \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 3 \end{bmatrix}$

Rectangular Matrix

Matrix having different number of rows and columns is called a square matrix.

For example, a m $\times$ n square matrix is as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$1 \times 2 \quad \begin{bmatrix} 2 & 3 \end{bmatrix}, \qquad 2 \times 3 \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}, \qquad 3 \times 2 \quad \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}$

Row Matrix

Matrix having only one row is called a row matrix.

For example, a 1$\times$ n row matrix is as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

Column Matrix

Matrix having only one column is called a column matrix.

For example, a $n \times 1$ column matrix is as

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

Null or Zero Matrix

Matrix having all elements zero is called a Null matrix or a zero matrix.

For example, a 2×3 zero matrix is as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Unit or Identity Matrix

A square matrix having all diagonal elements unity and the rests zero is called a unit matrix or an identity matrix (I).

For example, a 1×1 , 2×2 and 3×3 identity matrix is as

$$I_1 = [1] \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal Matrix

A square matrix having all diagonal (principle diagonal) elements non-zero and the rests zero is called a Diagonal matrix.

For example, a 3×3 diagonal matrix is as

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Scalar Matrix

A square matrix having all diagonal elements constant and the rests zero is called a Scalar matrix.

For example, a 3×3 scalar matrix is as

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5I$$

Tri-Diagonal Matrix

A square matrix having all elements of the three major diagonals non-zero and the rests zero is called a Tri-Diagonal matrix.

For example, a 3×3 Tri-Diagonal matrix is as

$$\begin{bmatrix} 5 & 2 & 0 \\ -9 & -1 & 1 \\ 0 & 7 & 8 \end{bmatrix}$$

Example of a 4×4 Tri-Diagonal matrix is as

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{bmatrix}$$

Triangular Matrix

A square matrix having all the elements below the major diagonal zero and the rests non-zero is called upper triangular matrix. If all the elements above the major diagonal zero and the rests non-zero is called lower triangular matrix.

For example, a 3×3 trangular matrix is as

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} \qquad \begin{bmatrix} 5 & 0 & 0 \\ -9 & -1 & 0 \\ 1 & 7 & 8 \end{bmatrix}$$

Upper triangular Lower triangular

Equality of Matrices

Two matrices of the same orders are said to be mutually equal if all of their corresponding elements are equal. For example, if

$$\begin{bmatrix} 5 & a & 0 \\ 0 & -1 & 0 \\ 0 & a+2b & 2c^2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 1 & 8 \end{bmatrix}$$

then obviously $a = 0,\ b = \frac{1}{2},\ c = \pm 2$ and $\alpha = -1$

Transpose of a matrix

The transpose of a matrix is created by converting its rows into columns; that is, row 1 becomes column 1, row 2 becomes column 2 etc. The transpose of matrix $A = [a_{ij}]$ is $A^T = [a_{ji}]$. For example,

If $A = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$

Symmetric and Skew-symmetric Matrix

If for a square matrix A, satisfy $A^T = A$ *i.e.* $\text{Transpose}(A) = A$

Then A is said to be symmetric.

Example: $\begin{bmatrix} 1 & 4 \\ 4 & -2 \end{bmatrix}$ or $\begin{bmatrix} 1 & -i & 2i \\ -i & -3 & 7 \\ 2i & 7 & 5 \end{bmatrix}$

If for a square matrix A, satisfy $A^T = -A$ *i.e.* $\text{Transpose}(A) = -A$

Then A is said to be skew-symmetric.

Example: $\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -5 & 2i \\ 5 & 0 & -i \\ -2i & i & 0 \end{bmatrix}$

Example

If the matrix $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, find the value of x and hence find the matrix A.

Solution

Given matrix is $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$

Transpose of the matrix A is

$$A^T = \begin{bmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{bmatrix}$$

Since the matrix A is symmetric, so $A^T = A$

$$\begin{bmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{bmatrix} = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$$

Equating the corresponding term from both side, we get

$2x - 3 = x + 2$ giving $x = 5$

Hence, the matrix A is $A = \begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$.

Orthogonal Matrix

If for a square matrix A, satisfy $A.A^T = A^T.A = I$

Then A is said to be orthogonal. For example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -5 & 2i \\ 5 & 0 & -i \\ -2i & i & 0 \end{bmatrix}$$

For an orthogonal matrix A, $A^{-1} = A^T$

Idempotent and Nilpotent Matrix

If for a square matrix A, satisfy $A^2 = A$

Then A is said to be idempotent.

If $A^k = A$, then it is said to be periodic with period k

Example: $\begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$ or $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

If for a square matrix A, satisfy $A^k = 0$

Then A is said to be nilpotent.

Example: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ or $\begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$

Example

Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent.

Solution

$$A^2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} . \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 2 + (-2) \times (-1) + (-4) \times 1 & 2 \times (-2) + (-2) \times 3 + (-4) \times (-3) & 2 \times (-4) + (-2) \times 4 + (-4 \\ (-1) \times 2 + 3 \times (-1) + 4 \times 1 & (-1) \times (-2) + 3 \times 3 + 4 \times (-2) & (-1) \times (-4) + 3 \times 4 + 4 \\ 1 \times 2 + (-2) \times (-1) + (-3) \times 1 & 1 \times (-2) + (-2) \times 3 + (-3) \times (-2) & 1 \times (-4) + (-2) \times 4 + (-3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= A$$

As the given matrix A, satisfy $A^2 = A$, it is idempotent.

Involutory Matrix

If for a square matrix A that satisfy $A^2 = I$ is said to be involutory.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -5 & 2i \\ 5 & 0 & -i \\ -2i & i & 0 \end{bmatrix}$$

- For an involutory matrix A, $A^{-1} = A$, i.e. the matrix is its own inverse.
- All identity matrices are involutory.

Example

Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an involutory matrix.

Solution

$$A^2 = A.A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 0 \times 0 & 1 \times 0 + 0 \times (-1) \\ 0 \times 1 + (-1) \times 0 & 0 \times 0 + (-1) \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

As the given matrix A, satisfy $A^2 = I$, it is involutory.

Complex Matrix and Complex Conjugate

If the elements of any matrix A are complex numbers, A is said to be complex. Matrix containing the complex conjugate of all the elements of A is called the complex conjugate matrix, usually denoted by $\bar{A}$.

$$\begin{bmatrix} 0 & -5+i & 2i \\ 5 & 0 & 2-i \\ -3i & 1+i & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -5-i & -2i \\ 5 & 0 & 2+i \\ 3i & 1-i & 0 \end{bmatrix}$$

Matrix A Conjugate of A

Hermitian Matrix

$$\bar{A}^T = A$$

Skew-hermitian Matrix

$$\bar{A}^T = -A$$

Example

Prove that the matrix $A = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$ is a Hermitian matrix.

$$\bar{A} = \begin{bmatrix} 2 & 2+3i & 3-5i \\ 2-3i & 3 & -i \\ 3+5i & i & 5 \end{bmatrix}$$

Now, $\bar{A}^T = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$

It is clear that $\bar{A}^T = A$.

Therefore, the given matrix is Hermitian.