UNIT - 7: FIR Filter Design

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FIR Filter Design:[1, 2, 3, 4]

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- All the slides are prepared based on the reference material
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- This material is prepared based on Digital Signal Processing for ECE/TCE course as per Visvesvaraya Technological University (VTU) syllabus (Karnataka State, India).

Unit 7: FIR Filter Design:

PART-B-Unit 7: FIR Filter Design:

- Introduction to FIR Filters
- Design of FIR Filters using
  1. Rectangular window
  2. Hamming window
  3. Hanning window
  4. Bartlet window
  5. Kaiser window
- Design of FIR Filter using frequency sampling technique.

Advantages of the FIR digital filter

- Relatively easy to design and computationally more efficient.
- FIR filters are implemented in hardware or software.
- The phase response is linear. Linear phase property implies that the phase is a linear function of the frequency. FIR filter output is delayed by the same amount of time for all frequencies, thereby eliminating the phase distortion (Group delay).
- FIR filters are always stable i.e. for a finite input, the output is always finite.
- In linear phase, for the filter of length N the number of operations are of the order of N/2.

Disadvantages of the FIR digital filter (compared to IIR filters)

- They require more memory and/or calculation to achieve a given filter response characteristic. Also, certain responses are not practical to implement with FIR filters.
- For a desired frequency response, with tight constraints on the passband, transition band and the stopband, a FIR filter may have large number of coefficients, thereby have more arithmetic operations and hardware components.

An LTI system is causal iff

- Input/output relationship: $y[n]$ depends only on current and past input signal values.
- Impulse response: $h[n] = 0$ for $n < 0$
- System function: number of finite zeros $\leq$ number of finite poles.

- An ideal lowpass filter is given by

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

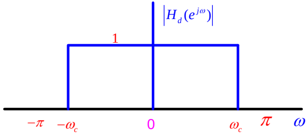


Figure 1: Ideal low pass filter

- The impulse response is given by

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H(\omega)e^{j\omega n}d\omega = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} & n \neq 0 \end{cases}$$

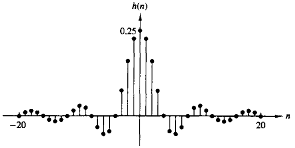


Figure 2: Unit sample response

Paley-Wiener Theorem:
If $h(n)$ has finite energy and $h(n) = 0$ for $n < 0$ then

$$\int_{-\pi}^{\pi} |\ln|H(\omega)||d\omega < \infty$$

- $H(\omega)$ can be zero at some frequencies. but it cannot be zero over any finite of frequencies, since the integral then becomes infinite.

$H(\omega)$ cannot be exactly zero over any band of frequencies. (Except in the trivial case where h[n] = 0.) Furthermore, $|H(\omega)|$ cannot be flat (constant) over any finite band.

Magnitude Characteristic of FIR filter

The magnitude response can be expressed as

$$Magnitude = \begin{cases} 1-\delta_1 \leq |H(\omega)|1+\delta_1 & for\ 0 \leq \omega \leq \omega_p \\ 0 \leq |H(\omega)\delta_2 & for\ \omega_s \leq \omega \leq \pi \end{cases}$$

Approximate formula for order N is

$$N = \frac{-10log_{10}(\delta_1\delta_2)-15}{14\Delta f}$$

where $\Delta f = \frac{\omega_s-\omega_p}{2\pi} = f_s - f_p$
Approximate formula for order N is

$$N = k\frac{2\pi}{\omega_s-\omega_p}$$

The width of the main lobe is
$N = k\frac{2\pi}{M}$

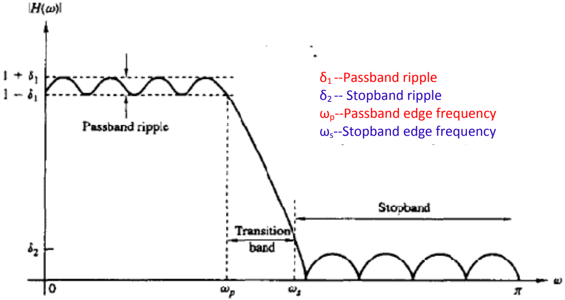


Figure 3: Magnitude Specification of FIR

- Ideal filters are noncausal, hence physically unrealizable for real time signal processing applications.
- Causality implies that the frequency response characteristic $H(\omega)$ of the filter cannot be zero, except at finite set of points in the frequency range. And also $H(\omega)$ cannot have an infinitely sharp cutoff from passband to stopband, that is $H(\omega)$ cannot drop from unity to zero abruptly.
- It is not necessary to insist that the magnitude be constant in the entire passband of the filter. A small amount of ripple in the passband is usually tolerable.
- The filter response may not be zero in the stopband, it may have small nonzero value or ripple.
- The transition of the frequency response from passband to stopband defines transition band.
- The passband is usually called bandwidth of the filter.
- The width of transition band is $\omega_s - \omega_p$ where ω_p defines passband edge frequency and ω_s defines stopband edge frequency.
- The magnitude of passband ripple is varies between the limits $1 \pm \delta_1$ where δ_1 is the ripple in the passband
- The ripple in the stopand of the filter is denoted as δ_2

FIR Filter Design

- An FIR system does not have feedback. Hence $y(n-k)$ term is absent in the system. FIR output is expressed as

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

- If there are M coefficients then

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

- The coefficients are related to unit sample response as

$$h(n) = \begin{cases} b_n & for \quad 0 \leq n \leq M-1 \\ 0 & otherwise \end{cases}$$

- Expanding the summation

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots b_{(M-1)} x(n-M+1)$$

- Since $h(n) = b_n$ then y(n) is

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

Symmetric and Antisymmetric FIR Filters Linear Phase FIR structure

- Linear phase is a property of a filter, where the phase response of the filter is a linear function of frequency. The result is that all frequency components of the input signal are shifted in time (usually delayed) by the same constant amount, which is referred to as the phase delay. And consequently, there is no phase distortion due to the time delay of frequencies relative to one another.
- Linear-phase filters have a symmetric impulse response.
- The FIR filter has linear phase if its unit sample response satisfies the following condition:

$$h(n) = h(M-1-n) \qquad n = 0, 1, 2, \ldots, N-1$$

- The Z transform of the unit sample response is given as

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

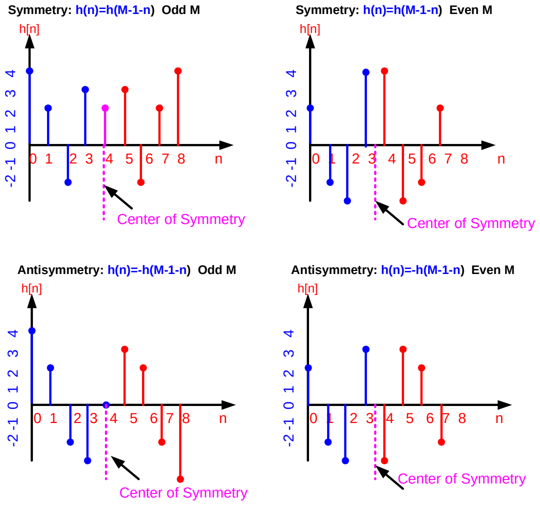
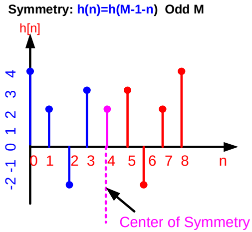


Figure 4: Symmetric and antisymmetric responses

- The unit sample response of FIR filter is symmetric if $h(n) = h(M-1-n)$
- The unit sample response of FIR filter is antisymmetric if $h(n) = -h(M-1-n)$

Frequency response of Linear Phase FIR Filter: Symmetric with M=odd

$$H(z)=\sum_{n=0}^{M-1} h(n)z^{-n}$$



Symmetric impulse response with M=odd Then
$h(n)=h(M-1-n)$ and $(z=e^{j\omega})$

$$H(z)=h\left(\frac{M-1}{2}\right)z^{\left(\frac{M-1}{2}\right)}+\sum_{n=0}^{(M-3)/2} h(n)\left[z^{-n}+z^{-(M-1-n)}\right]$$

$$H(e^{j\omega})=h\left(\frac{M-1}{2}\right)e^{-j\omega\left(\frac{M-1}{2}\right)}+\sum_{n=0}^{\frac{M-3}{2}} h(n)\left[e^{-j\omega n}+e^{-j\omega(M-1-n)}\right]$$

$$\begin{aligned} e^{-j\omega n} &= e^{-j\omega n}e^{j\omega(\frac{M-1}{2})}e^{-j\omega(\frac{M-1}{2})}=e^{j\omega(\frac{M-1}{2}-n)}.e^{-j\omega(\frac{M-1}{2})} \\ e^{-j\omega(M-1-n)} &= e^{-j\omega(M-1)}e^{j\omega n}=e^{-j\omega(\frac{M-1}{2})}.e^{-j\omega(\frac{M-1}{2})}e^{j\omega n}=e^{-j\omega(\frac{M-1}{2})}.e^{-j\omega(\frac{M-1}{2}-n)} \end{aligned}$$

$$\begin{aligned} e^{-j\omega n}+e^{-j\omega(M-1-n)} &= e^{-j\omega\left(\frac{M-1}{2}\right)}\left[e^{j\omega\left(\frac{M-1}{2}-n\right)}+e^{-j\omega\left(\frac{M-1}{2}-n\right)}\right] \\ &= e^{-j\omega\left(\frac{M-1}{2}\right)}2cos\omega\left(\frac{M-1}{2}-n\right) \end{aligned}$$

$$\begin{aligned}
H(e^{j\omega}) &= h(\frac{M-1}{2})e^{-j\omega(\frac{M-1}{2})} + \sum_{n=0}^{\frac{M-3}{2}} h(n)[e^{-j\omega n} + e^{-j\omega(M-1-n)}] \\
&= h\left(\frac{M-1}{2}\right)e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega\left(\frac{M-1}{2}\right)} 2cos\omega\left(\frac{M-1}{2} - n\right) \\
&= e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\ \omega\left(\frac{M-1}{2} - n\right)\right]
\end{aligned}$$

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)}$$

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\ \omega\left(\frac{M-1}{2} - n\right)$$

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & for\ |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & for\ |H(\omega)| < 0 \end{cases}$$

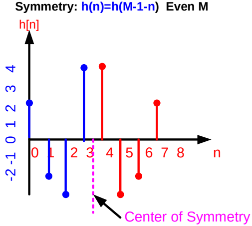
Frequency response of Linear Phase FIR Filter: Symmetric with M=Even

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\ \omega\left(\frac{M-1}{2}-n\right)\right]$$

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)}$$

$$|H(\omega)| = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\ \omega\left(\frac{M-1}{2}-n\right)$$

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & for\ |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & for\ |H(\omega)| < 0 \end{cases}$$



Design of linear-phase FIR filters using windows

Design steps for Linear Phase FIR Filter (Fourier Series method)

1. Based on the desired frequency response specification $H_d(e^{j\omega})$ determine the corresponding unit sample response $h_d(n)$.

$$H_d(e^{j\omega}) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n}$$

2. Obtain the impulse response $h_d(n)$ for the desired frequency response $H_d(\omega)$ by evaluating the inverse Fourier transform.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

3. In general the sample response $h_d(n)$ is infinite in duration and must be truncated at some point to get an FIR filter of length M. Truncation is achieved by multiplying $h_d(n)$ by window function.

$$h(n) = h_d(n)w(n)$$

   where $w(n)$ is window function

4. Obtain the $H(z)$ for $h(n)$ by taking z transform
5. Obtain the magnitude response $|H(e^{j\omega})|$ and phase response $\theta(\omega)|$

- Low-pass filter is used to eliminate high-frequency fluctuations (eg. noise filtering, demodulation, etc.)
- High-pass filter is used to follow small-amplitude high-frequency perturbations in presence of much larger slowly-varying component (e.g. recording the electrocardiogram in the presence of a strong breathing signal)
- Band-pass is used to select a required modulated carrier signal (e.g. radio)
- Band-stop filter is used to eliminate single-frequency (e.g. mains) interference (also known as notch filtering)

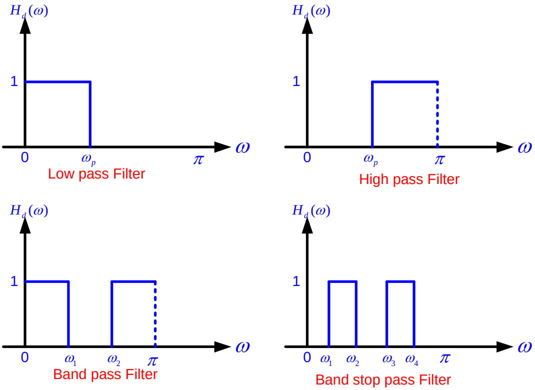


Figure 5: Frequency response characteristic of different types of filters

Different Types of Windows

- Rectangular:
- Hanning
- Hamming:
- Blackman:
- Bartlett (Triangular) Window
- Kaiser window

Rectangular window

- This is the simplest window function but provides the worst performance from the viewpoint of stopband attenuation.
- The width of main lobe is $4\pi/N$

$$\omega_R(n) = \begin{cases} 1 & \textit{for } n = 0, 1, M-1 \\ 0 & \textit{otherwise} \end{cases}$$

- Magnitude response of rectangular window is

$$|W_R(\omega)| = \frac{|\sin(\frac{\omega M}{2})|}{|\sin(\frac{\omega}{2})|}$$

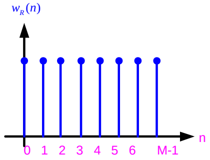


Figure 6: Rectangular window

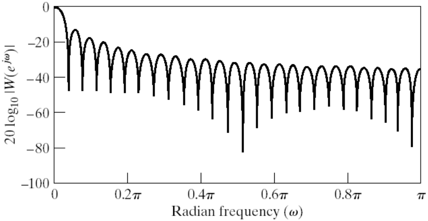


Figure 7: Rectangular window

Bartlett (Triangular) Window

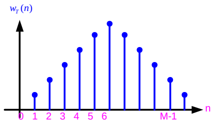


Figure 8: Bartlett window

- Bartlett Window is also Triangular window.
- The width of main lobe is $8\pi/M$

$$\omega_T(n) = 1 - \frac{2|n - \frac{M-1}{2}|}{M-1}$$

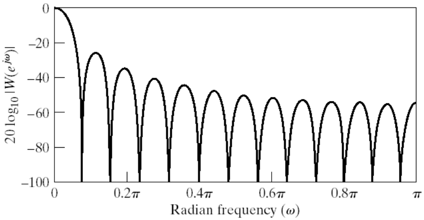


Figure 9: Bartlett window

Hanning window

- This is a raised cosine window function given by:

$$w(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M-1}\right)\right]$$

$$W(\omega) \approx 0.5W_R(\omega) + 0.25\left[W_R(\omega - \frac{2\pi}{M}) + W_R(\omega + \frac{2\pi}{M})\right]$$

The width of main lobe is: $\frac{8\pi}{M}$

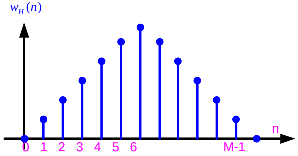


Figure 10: Hanning window

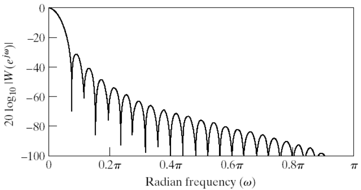


Figure 11: Hanning window

Hamming window

This is a modified version of the raised cosine window

$$w(n) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)\right]$$

$$W(\omega) \approx 0.54 W_R(\omega) + 0.23\left[W_R(\omega - \frac{2\pi}{M}) + W_R(\omega + \frac{2\pi}{M})\right]$$

The width of main lobe is: $\frac{8\pi}{M}$

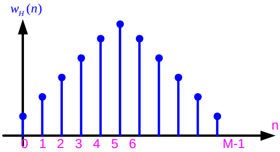


Figure 12: Hamming window

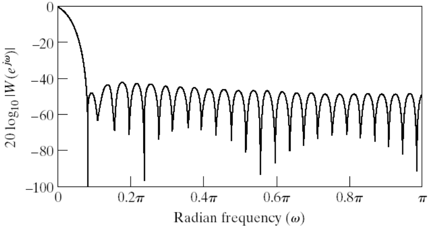


Figure 13: Hamming window

Blackman window

This is a 2^{nd}-order raised cosine window.

$$w(n) = \left[0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right)\right]$$

$$W(\omega) \approx 0.42W_R(\omega) + 0.25\left[W_R(\omega - \tfrac{2\pi}{M}) + W_R(\omega + \tfrac{2\pi}{M})\right]$$
$$+0.04\left[W_R(\omega - \tfrac{4\pi}{M}) + W_R(\omega + \tfrac{4\pi}{M})\right]$$

The width of main lobe is: $\frac{12\pi}{M}$

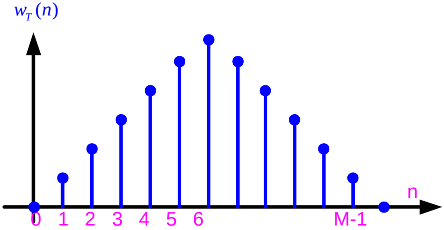


Figure 14: Blackman window

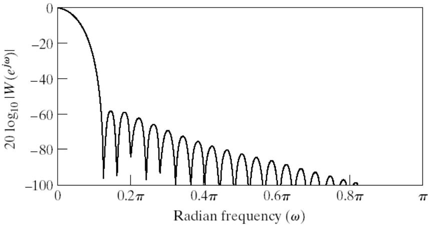


Figure 15: Blackman window

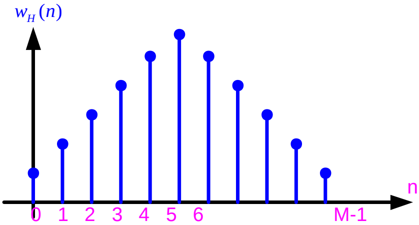
Kaiser window

This is one of the most useful and optimum windows.

$$w(n) = \frac{I_0\left(\beta\sqrt{1-\left(1-\frac{2n}{M-1}\right)^2}\right)}{I_0(\beta)}$$

Where $I_0(X)$ is the modified zero-order Bessel function, and is a parameter that can be chosen to yield various transition widths and stop band attenuation. This window can provide different transition widths for the same N.

$$\beta = 0 \rightarrow \text{rectangularwindow}$$
$$\beta = 5.44 \rightarrow \text{Hammingwindow}$$
$$\beta = 8.5 \rightarrow \text{Blackmanwindow}$$



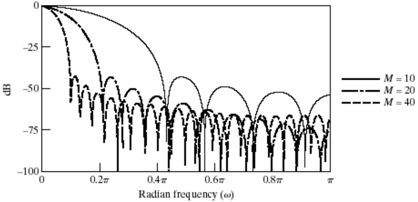


Table 1: Window and its functions

Window name	Window Function
Rectangular	$\omega_R(n) = \begin{cases} 1 & for\ 0 \leq n \leq M-1 \\ 0 & otherwise \end{cases}$
Triangular (Bartlet)	$\omega_T(n) = 1 - \frac{2\lvert n - \frac{M-1}{2} \rvert}{M-1}$
Hamming	$w(n) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right]$
Hanning	$w(n) = \left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right]$
Blackman	$w(n) = \left[0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)\right]$

Table 2: Summary of window function characteristics

Window name	Transition width of main lobe	Min. stopband attenuation	Peak value of side lobe
Rectangular	$\frac{4\pi}{M+1}$	-21 dB	-21 dB
Hanning	$\frac{8\pi}{M}$	-44 dB	-31 dB
Hamming	$\frac{8\pi}{M}$	-53 dB	-41 dB
Bartlett	$\frac{8\pi}{M}$	-25 dB	-25 dB
Blackman	$\frac{12\pi}{M}$	-74 dB	-57 dB

Gibbs Phenomenon

- The magnitude of the frequency response $H(\omega)$ is as shown in Figure. Large oscillations or ripples occur near the band edge of the filter. The oscillations increase in frequency as M increases, but they do not dimmish in amplitude.
- These large oscillations are due to the result of large sidelobes existing in the frequency characteristic $W(\omega)$ of the rectangular window.
- The truncation of the Fourier series is known to introduce ripples in the frequency response characteristic $H(\omega)$ due to the nonuniform convergence of the Fourier series at a discontinuity.
- The oscillatory behavior near the band edge of the filter is called the **Gibbs Phenomenon**.
- To alleviate the presence of large oscillations in both the passband and the stopband window function is used that contains a taper and decays toward zero gradually .

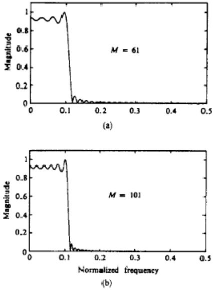


Figure 18: LPF designed with rectangular window M=61 and 101

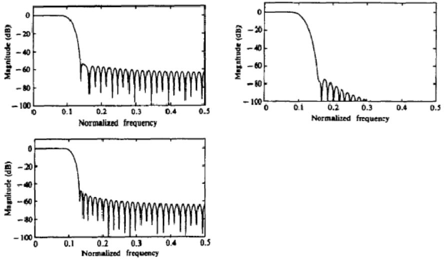


Figure 19: LPF designed with Hamming, Hanning and Blackman window M=61

Design a LPF using rectangular window for the desired frequency response of a low pass filter given by $\omega_c = \frac{\pi}{2}$ rad/sec, and take M=11. Find the values of $h(n)$. Also plot the magnitude response.

Solution:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\omega_c \le \omega \le \omega_c \\ 0 & -\pi \le \omega \le -\omega_c \\ 0 & \omega_c \le \omega \le \pi \end{cases}$$

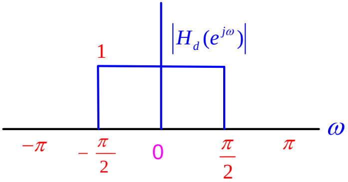


Figure 20: Frequency response of LPF

$$\tau = \frac{M-1}{2} = 5$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\omega_c \le \omega \le \omega_c \\ 0 & Otherwise \end{cases}$$

By taking inverse Fourier transform

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}d\omega \end{aligned}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi(n-\tau)}\left[e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}\right] \\ &= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right] \end{aligned}$$

$$h_d(n) = \frac{sin\left[\omega_c(n-\tau)\right]}{\pi(n-\tau)}$$

for $n \neq 5$ and $\omega_c = \frac{\pi}{2}$, $\tau = \frac{M-1}{2} = 5$

$$h_d(n) = \frac{sin\left[\omega_c(n-5)\right]}{\pi(n-5)} = \frac{sin\left[\frac{\pi(n-5)}{2}\right]}{\pi(n-5)}$$

for n=5 $h_d(n) = \frac{0}{0}$. Using L Hospital's Rule

$$\lim_{\theta \to 0} \frac{\sin B\theta}{\theta} = B$$

$$\lim_{n \to 5} \frac{\sin \frac{\pi}{2}(n-5)}{\pi(n-5)} = \frac{\pi/2}{\pi} = 0.5$$

where $\pi = 3.1416$

$$h_d(0) = \frac{sin\frac{\pi(0-5)}{2}}{\pi(0-5)} = 0.0637$$

$$h_d(1) = \frac{sin\frac{\pi(1-5)}{2}}{\pi(1-5)} = 0$$

$$h_d(2) = \frac{sin\frac{\pi(2-5)}{2}}{\pi(2-5)} = -0.106$$

$$h_d(3) = \frac{sin\frac{\pi(3-5)}{2}}{\pi(3-5)} = 0$$

$$h_d(4) = \frac{sin\frac{\pi(4-5)}{2}}{\pi(4-5)} = .318$$

$$h_d(5) = \frac{sin\frac{\pi(5-5)}{2}}{\pi(5-5)} = .5$$

$$h_d(6) = \frac{sin\frac{\pi(6-5)}{2}}{\pi(6-5)} = .318$$

$$h_d(7) = \frac{sin\frac{\pi(7-5)}{2}}{\pi(7-5)} = 0.0$$

$$h_d(8) = \frac{sin\frac{\pi(8-5)}{2}}{\pi(8-5)} = -.106$$

$$h_d(9) = \frac{sin\frac{\pi(9-5)}{2}}{\pi(9-5)} = 0$$

$$h_d(10) = \frac{sin\frac{\pi(10-5)}{2}}{\pi(10-5)} = .063$$

The given window is rectangular window

$$\omega(n) = \begin{cases} 1 & for\ 0 \leq n \leq 10 \\ 0 & Otherwise \end{cases}$$

This is rectangular window of length M=11. $h(n) = h_d(n)\omega(n) = h_d(n)\ for\ 0 \leq n \leq 10$

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n} = \sum_{n=0}^{10} h(n)z^{-n}$$

The impulse response is symmetric with M=odd=11

$$\begin{aligned} H(z) &= h\left(\frac{M-1}{2}\right)z^{\frac{M-3}{2}} + \sum_{n=0}^{\frac{M-3}{2}} h(n)[z^{-n} + z^{(M-1-n)}] \\ &= h(5)z^{-5} + h(0)[z^{-0} + z^{-10}] + h(1)[z^{-1} + z^{-9}] + h(2)[z^{-2} + z^{-8}] + \\ &= +h(3)[z^{-3} + z^{-7}] + h(4)[z^{-4} + z^{-6}] \end{aligned}$$

$$\begin{aligned} |H(e^{j\omega})| &= h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\ \omega\left(\frac{M-1}{2} - n\right) \\ &= h(5) + 2\sum_{n=0}^{4} h(n)cos\ \omega(5-n) \\ &= h(5) + 2h(0)cos\ 5\omega + 2h(1)cos\ 4\omega + 2h(2)cos\ 3\omega + 2h(3)cos\ 2\omega + 2h(4)cos\ \omega \\ &= 0.5 + 0.127cos\ 5\omega - 0.212cos\ 3\omega + 0.636cos\ \omega \end{aligned}$$

$$|H(e^{j\omega})| = 0.5 + 0.127cos\ 5\omega - 0.212cos\ 3\omega + 0.636cos\ \omega$$

$$|H(e^{j\omega})|_{dB} = 20log|H(e^{j\omega})|$$

ω	$\|H(e^{j\omega})\|$	$\|H(e^{j\omega})\|_{dB}$
0	1.0151	-0.44
0.1π	0.9808	-0.17
0.2π	0.9535	-0.41
0.3π	1.0758	0.63
0.4π	0.9952	-0.04
0.5π	0.5	-6.02
0.6π	0.0048	-46.37
0.7π	0.0758	-22.41
0.8π	0.0467	-26.65
0.9π	0.0192	-34.35
1.0π	0.0512	-25.74

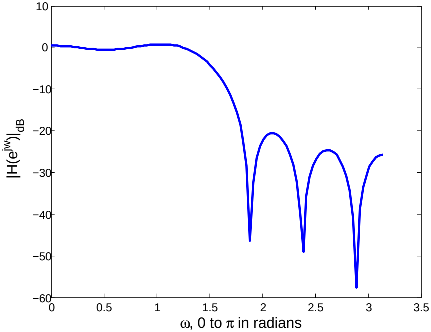


Figure 21: Frequency response of LPF

The desired frequency response of low pass filter is given by

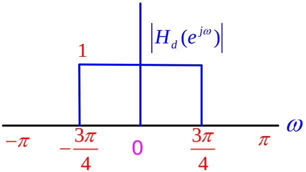
$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & -\frac{3\pi}{4} \leq \omega \leq \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

Determine the frequency response of the FIR if Hamming window is used with N=7 June-2015, Dec-2014, June-2012

Solution:

$$\tau = \frac{M-1}{2} = 3$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right] \end{aligned}$$



$$h_d(n) = \frac{sin\omega_c(n-\tau)}{\pi(n-\tau)}$$

$n \neq 3$ $\omega_c = \frac{3\pi}{4}$ $\tau = \frac{M-1}{2} = 3$

$$h_d(n) = \frac{sin\left[\frac{3\pi(n-3)}{4}\right]}{\pi(n-3)}$$

for n=3 $h_d(n) = \frac{0}{0}$. Using L Hospital's Rule

$$\lim_{n\to 3} \frac{\sin\left[\frac{3\pi}{4}(n-3)\right]}{\pi(n-3)} = \frac{3\pi/4}{\pi} = 0.75$$

$$h_d(0) = \frac{sin\left(\frac{3\pi(0-3)}{4}\right)}{\pi(0-3)} = 0.075$$

$$h_d(1) = \frac{sin\left(\frac{3\pi(1-3)}{4}\right)}{\pi(1-3)} = -0.159$$

$$h_d(2) = \frac{sin\left(\frac{3\pi(2-3)}{4}\right)}{\pi(2-3)} = 0.225$$

$$h_d(3) = \frac{sin\left(\frac{3\pi(3-3)}{4}\right)}{\pi(3-3)} = 0.75$$

$$h_d(4) = \frac{sin\left(\frac{3\pi(4-3)}{4}\right)}{\pi(4-3)} = 0.225$$

$$h_d(5) = \frac{sin\left(\frac{3\pi(5-3)}{4}\right)}{\pi(5-3)} = -0.159$$

$$h_d(6) = \frac{sin\left(\frac{3\pi(6-3)}{4}\right)}{\pi(6-3)} = 0.075$$

The given window is Hamming window

$$
\begin{aligned}
w(n) &= 0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right) \\
\omega(0) &= 0.54 - 0.46cos\left(\frac{0}{6}\right) = 0.08 \\
\omega(1) &= 0.54 - 0.46cos\left(\frac{2\pi}{6}\right) = .31 \\
\omega(2) &= 0.54 - 0.46cos\left(\frac{4\pi}{6}\right) = .77 \\
\omega(3) &= 0.54 - 0.46cos\left(\frac{6\pi}{6}\right) = 1 \\
\omega(4) &= 0.54 - 0.46cos\left(\frac{8\pi}{6}\right) = 0.77 \\
\omega(5) &= 0.54 - 0.46cos\left(\frac{10\pi}{6}\right) = .31 \\
\omega(6) &= 0.54 - 0.46cos\left(\frac{12\pi}{6}\right) = .08
\end{aligned}
$$

To calculate the value of h(n)

$$h(n) = h_d(n)w(n)$$

$$
\begin{aligned}
h(0) &= h_d(0)w(0) = 0.075 \times 0.08 = 0.006 \\
h(1) &= h_d(1)w(1) = -0.159 \times 0.31 = -0.049 \\
h(2) &= h_d(2)w(2) = 0.225 \times 0.77 = 0.173 \\
h(3) &= h_d(3)w(3) = 0.750 \times 1 = 0.750 \\
h(4) &= h_d(4)w(4) = 0.225 \times 0.77 = 0.173 \\
h(5) &= h_d(5)w(5) = -0.159 \times 0.31 = -0.049 \\
h(6) &= h_d(6)w(6) = 0.075 \times 0.08 = 0.006
\end{aligned}
$$

The frequency response is symmetric with M=odd=7

$$
\begin{aligned}
|H(e^{j\omega})| &= h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\ \omega\left(\frac{M-1}{2} - n\right) \\
&= h(3) + 2\sum_{n=0}^{2} h(n)cos\ \omega(3-n) = h(3) + 2h(0)cos\ 3\omega + 2h(1)cos\ 2\omega + 2h(2)cos\ \omega \\
&= 0.75 + 0.012cos\ 3\omega - 0.098cos\ 2\omega + 0.346cos\ \omega
\end{aligned}
$$

$$|H(e^{j\omega})|_{dB} = 20log|H(e^{j\omega})|$$

ω	$\lvert H(e^{j\omega})\rvert$	$\lvert H(e^{j\omega})\rvert_{dB}$
0	1.0100	0.0864
0.1π	1.0068	0.0592
0.2π	0.9959	-0.0354
0.3π	1.9722	-0.2445
0.4π	0.9265	-0.6631
0.5π	0.8480	-1.4321
0.6π	0.7321	-2.7089
0.7π	0.5883	-4.6077
0.8π	0.4435	-7.0620
0.9π	0.3346	-9.5095
1.0π	0.2940	-10.6331

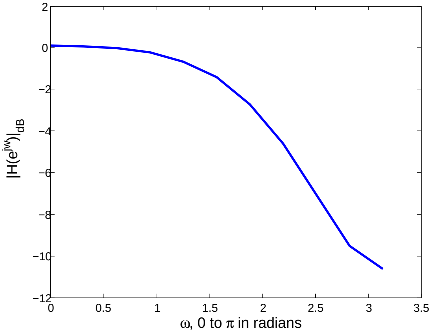


Figure 22: Frequency response of LPF

```
Matlab code
clc; clear all; close all;
M= input('enter the value of M:');
omega= input('enter the value of omega:');
tau=(M-1)/2 ;
for n=0:M-1;
   % c(n+1)=.5-.5*cos((2*pi*n)/(M-1));
    c(n+1)=.54-.46*cos((2*pi*n)/(M-1));
    if n==tau
         h(n+1)=omega/pi;
     else
    h(n+1)=sin(omega*(n-tau))/(pi*(n-tau));
    end
end
 h
 c
for n=1:M
   y=h(n)*c(n)'
end
```

```
clc; clear all; close all;
range=0;
%M= input('enter the value of M:');
for omega=0:.1*pi:pi
range=range+1;
H_omega=abs(0.75+.012*cos(3*omega)-.098*cos(2*omega)+.346*cos(omega));
%H_omega=abs(0.25+.45*cos(omega)+.318*cos(2*omega));
H_indB(range)=20*log10(H_omega)
end
 omega=0:.1*pi:pi;
 i=1:range;
 plot(omega, H_indB(i),'linewidth',2 )
 xlabel('\omega, 0 to \pi in radians','fontsize',13)
 ylabel(' |H(e^{jw})|_{dB}','fontsize',13)
```

Determine the filter coefficients $h_d(n)$ for the desired frequency response of a low pass filter given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-2j\omega} & \text{for } -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \text{for } \frac{\pi}{4} \le |\omega| \le -\pi \end{cases}$$

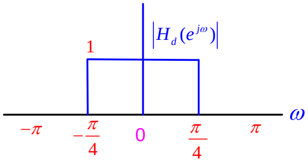
If we define the new filter coefficients by $h_d(n) = h_d(n)\omega(n)$ where

$$\omega(n) = \begin{cases} 1 & \text{for } 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine $h(n)$ and also the frequency response $H(e^{j\omega})$ July-2013, July-2011

Solution:

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j2\omega}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{-\pi/4}^{\pi/4} \end{aligned}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2j\pi(n-2)}\left[e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}\right] \\ &= \frac{1}{\pi(n-2)}\left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}}{2j}\right] \end{aligned}$$

$n \neq 2$

$$h_d(n) = \frac{sin\frac{\pi(n-2)}{4}}{\pi(n-2)}$$

for n=2 $h_d(n) = \frac{0}{0}$. Using L Hospital's Rule

$$\lim_{n \to 2} \frac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)} = \frac{\pi/4}{\pi} = 0.25$$

The given window function is

$$\omega(n) = \begin{cases} 1 & for\ 0 \leq n \leq 4 \\ 0 & Otherwise \end{cases}$$

This is rectangular window of length M=5.
In this case $h(n) = h_d(n)$ for $0 \leq n \leq 4$

n	$h_d(n)$	n	$h_d(n)$
0	0.159091	3	0.224989
1	0.224989	4	0.159091
2	0.25		

The frequency response is symmetric with M=odd=5

$$|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\ \omega\left(\frac{M-1}{2} - n\right)$$

$$= h(2) + 2\sum_{n=0}^{1} h\left(2-n\right) cos\ \omega n = h(2) + 2h(0)cos\ 2\omega + 2h(1)cos\ \omega$$

$$= 0.25 + 0.318cos\ 2\omega + 0.45cos\ \omega$$

$$|H(e^{j\omega})|_{dB} = 20log|H(e^{j\omega})|$$

ω	$\|H(e^{j\omega})\|$	$\|H(e^{j\omega})\|_{dB}$
0	1.0180	0.1550
0.1π	0.9352	-0.5815
0.2π	0.7123	-2.9464
0.3π	0.4162	-7.6132
0.4π	0.1318	-17.6023
0.5π	0.0680	-23.3498
0.6π	0.1463	-16.6936
0.7π	0.1128	-18.9561
0.8π	0.0158	-36.0322
0.9π	0.0793	-22.0154
1.0π	0.1180	-18.5624

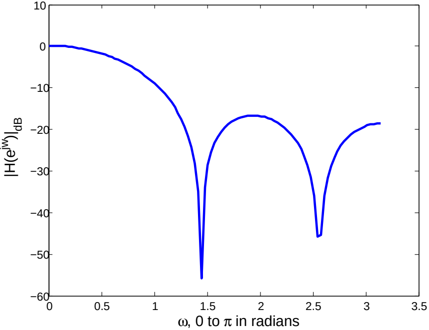


Figure 23: Frequency response of LPF

Design the symmetric FIR lowpass filter whose desired frequency response is given as

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ |\omega| \leq \omega_c \\ 0 & Otherwise \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Use rectangular window.

Solution:

- Desired frequency response $H_d(\omega)$
- Length of the filter M=7
- Cut-off frequency $\omega_c = 1$ radians/sample.
- Unit sample response is defined as

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

- Given $H_d(\omega)$ is

$$H_d(\omega) = \begin{cases} e^{-\omega\tau} & for \ -1 \leq \omega \leq 1 \\ 0 & Otherwise \end{cases}$$

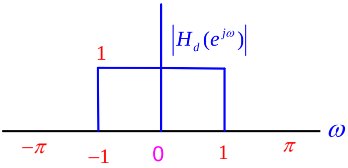


Figure 24: Frequency response of LPF

$$
\begin{aligned}
h_d(n) &= \frac{1}{2\pi}\int_{-1}^{1} e^{-\omega\tau} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-\tau)} d\omega \\
&= \frac{1}{2\pi}\left[\frac{e^{\omega(n-\tau)}}{j(n-\tau)}\right]_{-1}^{1} \\
&= \frac{sin(n-\tau)}{\pi(n-\tau)} \quad for\ n \neq \tau
\end{aligned}
$$

for $n = \tau$ $h_d(n) = \frac{0}{0}$. Using L Hospital's Rule

$$\lim_{n\to\tau} \frac{sin(n-\tau)}{\pi(n-\tau)} = \frac{1}{\pi}$$

Thus $h_d(n)$ is

$$h_d(n) = \begin{cases} \frac{sin(n-\tau)}{\pi(n-\tau)} & for\ n \neq \tau \\ \frac{1}{\pi} & for\ n = \tau \end{cases}$$

Determine the value of τ

$$\tau = \frac{M-1}{2} = 3$$

$$h_d(n) = \begin{cases} \frac{sin(n-3)}{\pi(n-3)} & for\ n \neq \tau \\ \frac{1}{\pi} & for\ n = \tau \end{cases}$$

This is rectangular window of length M=7. In this case $h(n) = h_d(n).w(n) = h_d(n)$

n	$h(n)$	n	$h(n)$
0	0.015	4	0.2678
1	0.1447	5	0.14472
2	0.2678	6	0.15
3	0.3183		

This is the unit sample response of required FIR filter. The filter is symmetric and satisfies $h(n) = h(M-1-n)$

Design the FIR filter using Hanning window

Solution:For M=7

$$\omega(n) = 0.5(1 - cos\frac{2\pi n}{M-1})$$
$$\omega(n) = 0.5(1 - cos\frac{2\pi n}{6})$$

$$\omega(0) = 0.0$$
$$\omega(1) = 0.5(1 - cos\frac{2\pi}{6}) = .25$$
$$\omega(2) = 0.5(1 - cos\frac{4\pi}{6}) = .75$$
$$\omega(3) = 0.5(1 - cos\frac{6\pi}{6}) = 1$$
$$\omega(4) = 0.5(1 - cos\frac{8\pi}{6}) = .75$$
$$\omega(5) = 0.5(1 - cos\frac{10\pi}{6}) = .25$$
$$\omega(6) = 0.5(1 - cos\frac{12\pi}{6}) = 0$$

To calculate the value of h(n)

$$h(n) = h_d(n)w(n)$$

$$h(0) = h_d(0)w(0) = 0.01497 \times 0 = 0$$
$$h(1) = h_d(1)w(1) = 0.014472 \times 0.25 = 0.03618$$
$$h(2) = h_d(2)w(2) = 0.26785 \times 0.75 = 0.20089$$
$$h(3) = h_d(3)w(3) = 0.31831 \times 1 = 0.31831$$
$$h(4) = h_d(4)w(4) = 0.26785 \times 0.75 = 0.20089$$
$$h(5) = h_d(5)w(5) = 0.14472 \times 0.25 = 0.03618$$
$$h(6) = h_d(6)w(6) = 0.014497 \times 0.0 = 0$$

Design a lowpass digital filter to be used in an A/D Hz D/A structure that will have a -3 dB cut-off at 30 π rad/sec and an attenuation of 50 dB at 45 π rad/sec. The filter is required to have linear phase and the system will use sampling rate of 100 samples/second.

Solution:
3 dB cut-off at 30 π rad/sec
$\omega_c = 30\pi rad/sec$
Sampling frequency $F_{SF} = 100$ Hz
Stopband attenuation of 50 dB at 45 π rad/sec
A_s =50 dB for $\omega_s = 45\pi rad/sec$
$\omega = \frac{\Omega}{F_{sf}}$

$$\omega_1 = \frac{\Omega_1}{F_{sf}} = \frac{30\pi}{100} = 0.3\pi \ rad/sample$$
$$\omega_2 = \frac{\Omega_2}{F_{sf}} = \frac{45\pi}{100} = 0.45\pi \ rad/sample$$

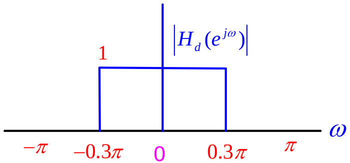


Figure 25: Frequency response of LPF

3 dB attenuation at $\omega_1 = 0.3\pi \ rad/sample$
50 dB attenuation at $\omega_2 = 0.45\pi \ rad/sample$

Type of window is

The stopband attenuation of 50 dB is provided by the Hamming window which of -53 dB. Hence Hamming window is selected for the given specifications.

To determine the order of the filter

The width of the main lobe in Hamming window is $\frac{8\pi}{M}$

$$k\frac{2\pi}{M} = \frac{8\pi}{M}$$

$$M = \frac{8\pi}{\omega_2 - \omega_1}$$

The order of the filter M is:

$$M = \frac{8\pi}{0.45\pi - 0.3\pi} = 53.33$$

Assume linear phase FIR filter of odd length Hence select next odd integer length of 55.

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ -\omega_c \leq \omega \leq \omega_c \\ 0 & Otherwise \end{cases}$$

$\tau = (M-1)/2 = 55 - 1/2 = 27$
$\omega_c = 0.3\pi$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-0.3\pi}^{0.3\pi} e^{j\omega(n-27)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(n-27)}}{j(n-27)}\right]_{-0.3\pi}^{0.3\pi} \\ &= \frac{sin[\omega_c(n-27)]}{\pi(n-27)} \quad for \ n \neq 27 \end{aligned}$$

when $n = 27$

$$h_d(n) = \frac{\omega_c}{\pi} = 0.3$$

The selected window is Hamming M=27

$$w(n) = 0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$$
$$= 0.54 - 0.46cos\left(\frac{\pi n}{18}\right)$$

The value of $h(n)$

$$h(n) = h_d(n)w(n)$$

for M= 27

$$h(n) = \frac{sin[0.3\pi(n-27)]}{\pi(n-27)}\left[0.54 - 0.46cos\left(\frac{\pi n}{18}\right)\right]$$

for $n \neq 27$

$$h(n) = 0.3\left[0.54 - 0.46cos\left(\frac{\pi n}{18}\right)\right]$$

n	$h(n)$	n	$h(n)$
0	0.0	28	0.2567
1	0.0	29	0.1495
2	-0.0012	30	0.0319
3	0.0	31	-0.0445
4	0.0	32	-0.0588
5	0.0021	33	- 0.0278
6	0.0023	34	0.012
7	0.0	35	0.0308
8	-0.0036	36	0.0220
9	-0.0052	37	-0.0
10	-0.0021	38	-0.0157
11	0.0048	39	-0.0156
12	0.0098	40	-0.0043
13	0.0069	41	0.0069
14	-0.0043	42	0.0098
15	-0.0156	43	0.0048
16	-0.0157	44	-0.0021
17	0.0	45	-0.0052
18	0.0220	46	-0.0036
19	-0.0308	47	0.0
20	-0.0120	48	0.0023
21	-0.0278	49	0.0021
22	-0.0588	50	0.0
23	-0.0445	51	0.0
24	0.0319	52	-0.0012
25	0.1495	53	0.0
26	0.2567	54	0.0
27	0.3		

An analog signal contains frequencies upto 10 KHz. The signal is sampled at 50 KHz. Design an FIR filter having linear phase characteristic and transition band of 5 KHz. The filter should provide minimum 50 dB attenuation at the end of transition band.

Solution:
3 dB cut-off at 30 π rad/sec
$\Omega_p = 2\pi \times 10 \times 10^3 rad/sec$
$\Omega_s = 2\pi \times (10+5) \times 10^3 rad/sec$
Sampling frequency $F_{SF} = 100$ Hz
Stopband attenuation of 50 dB at 45 π rad/sec
A_s =50 dB for $\omega_s = 45\pi rad/sec$
$\omega = \frac{\Omega}{F_{sf}}$

$$\omega_p = \frac{\Omega_p}{F_{sf}} = \frac{2\pi \times 10 \times 10^3}{50 \times 10^3} = 0.4\pi$$
$$\omega_s = \frac{\Omega_s}{F_{sf}} = \frac{2\pi \times (10+5) \times 10^3}{50 \times 10^3} = 0.6\pi$$

$\omega_p = 0.4\pi \; rad/sample$
$\omega_s = 0.6\pi \; rad/sample$

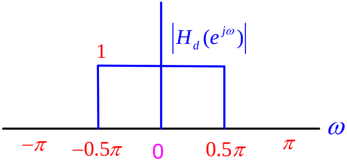


Figure 26: Frequency response of LPF

Type of window is

The stopband attenuation of 50 dB is provided by the Hamming window which of -53 dB. Hence Hamming window is selected for the given specifications.

To determine the order of the filter

The width of the main lobe in Hamming window is $\frac{8\pi}{M}$

$$k\frac{2\pi}{M} = \frac{8\pi}{M}$$

$$M \geq \frac{8\pi}{\omega_s - \omega_p}$$

The order of the filter M is:

$$M \geq \frac{8\pi}{0.6\pi - 0.4\pi} \geq 40$$

Assume linear phase FIR filter of odd length Hence select next odd integer length of 41.

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ -\omega_c \leq \omega \leq \omega_c \\ 0 & Otherwise \end{cases}$$

$\tau = (M-1)/2 = 41 - 1/2 = 20$
$\omega_c = \omega_p + \frac{\Delta\omega}{2} = 0.4\pi + \frac{0.2\pi}{2}$
$\omega_c = 0.5\pi$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-0.5\pi}^{0.5\pi} e^{j\omega(n-20)} d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(n-27)}}{j(n-20)}\right]_{-0.5\pi}^{0.5\pi} \\ &= \frac{sin[\omega_c(n-20)]}{\pi(n-20)} \quad for \ \ n \neq 20 \end{aligned}$$

when $n = 20$

$$h_d(n) = \frac{\omega_c}{\pi} = \frac{0.5\pi}{\pi} = 0.5$$

The selected window is Hamming M=41

$$w(n) = 0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$$
$$= 0.54 - 0.46cos\left(\frac{2\pi n}{40}\right)$$

The value of $h(n)$

$$h(n) = h_d(n)w(n)$$

for M= 41 $n \neq 20$

$$h(n) = \frac{sin[0.5\pi(n-20)]}{\pi(n-20)}\left[0.54 - 0.46cos\left(\frac{2\pi n}{20}\right)\right]$$

for $n = 20$

$$h(n) = 0.5\left[0.54 - 0.46cos\left(\frac{2\pi n}{20}\right)\right]$$

n	$h(n)$	n	$h(n)$
0	0.0	21	0.3148
1	-0.00146	22	0.0
2	0.0	23	-0.1
3	-0.00247	24	0.0
4	0	25	0.055
5	-0.00451	26	0.0
6	0.0	27	-0.0337
7	0.0079	28	0.0
8	0.0	29	0.0213
9	-0.0136	30	0.0
10	0.0	31	-0.0136
11	0.002135	32	0.0
12	0.0	33	0.0079
13	-0.03375	34	0
14	0.0	35	-0.0045
15	0.05504	36	0.0
16	0.0	37	0.0024
17	-0.1006	38	0.0
18	0.0	39	-0.0014
19	0.3148	40	0.0
20	0.5		

Design an FIR filter (lowpass) using rectangular window with passband gain of 0 dB, cutoff frequency of 200 Hz, sampling frequency of 1 kHz. Assume the length of the impulse response as 7.

Solution:
$F_c = 200$ Hz, $F_s = 1000$ Hz,
$f_c = \frac{F_c}{F_s} \frac{200}{1000} = 0.2 cycles/sample$
$\omega_c = 2\pi * f_c = 2\pi \times 0.2 = 0.4\pi rad$
M=7

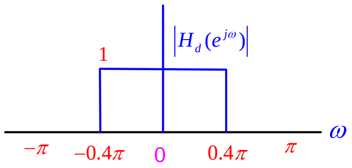


Figure 27: Frequency response of LPF

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ -\omega_c \leq \omega \leq \omega_c \\ 0 & Otherwise \end{cases}$$

$\tau = (M-1)/2 = 7 - 1/2 = 3$
$\omega_c = 0.4\pi$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-0.4\pi}^{0.4\pi} e^{j\omega(n-3)} d\omega \\ &= \frac{1}{2\pi} \left[\frac{e^{\omega(n-3)}}{j(n-3)} \right]_{-0.4\pi}^{0.4\pi} \end{aligned}$$

when $n \neq 3$

$$h_d(n) = \frac{sin[0.4\pi(n-3)]}{\pi(n-3)}$$

when $n = 3$

$$h_d(n) = \frac{0.4\pi}{\pi} = 0.4$$

Determine the value of $h(n)$

Since it is rectangular window $h(n) = w(n) = h_d(n) = h(n)$

For M=7

n	$h(n)$	n	$h(n)$
0	-0.062341	4	-0.062341
1	0.093511	5	0.093511
2	0.302609	6	0.302609
3	0.4		

Using rectangular window design a lowpass filter with passband gain of unity, cutoff frequency of 1000 Hz, sampling frequency of 5 kHz. The length of the impulse response should be 7.
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Solution:
$F_c = 1000$ Hz, $F_s = 5000$ Hz,
$f_c = \frac{F_c}{F_s} \frac{1000}{5000} = 0.2 cycles/sample$
$\omega_c = 2\pi f_c = 2 \times \pi \times 0.2 = 0.4\pi rad$
M=7
The filter specifications (ω_c and M=7) are similar to the previous example. Hence same filter coefficients are obtained.
h(0)=-0.062341, h(1)=0.093511, h(2)=0.302609
h(3)=0.4, h(4)=0.302609, h(5)=0.093511, h(6)=-0.062341

Design a normalized linear phase FIR low pass filter having phase delay of $\tau = 4$ and at least 40 dB attenuation in the stopband. Also obtain the magnitude/frequency response of the filter.

Solution: The linear phase FIR filter is normalized means its cut-off frequency is of $\omega_c = 1 rad/sample$
The length of the filter with given τ is related by

$$\tau = \frac{M-1}{2}$$

For $\tau = 4$ M=9
Desired unit sample response $h_d(n)$ is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-4)} d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(n-4)}}{j(n-4)}\right]_{-1}^{1} \end{aligned}$$

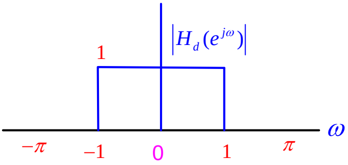


Figure 28: Frequency response of LPF

when $n \neq 4$

$$h_d(n) = \frac{sin[(n-4)]}{\pi(n-4)}$$

when $n = 4$

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} 1 d\omega = \frac{\omega}{\pi} = \frac{1}{\pi}$$

$$h(n) = h_d(n)w(n)$$

The stopband attenuation required for this filter is 40 dB. From the table the Hanning window satisfies this requirement. The Hanning window function given by:

$$w(n) = 0.5 - 0.5cos\left(\frac{2\pi n}{M-1}\right)$$

for $n \neq 4$ and $M = 9$

$$h(n) = \frac{sin(n-4)}{\pi(n-4)}\left[0.5 - 0.5cos\left(\frac{\pi n}{4}\right)\right]$$

for $n = 4$

$$h(n) = \frac{1}{\pi}\left[0.5 - 0.5cos\left(\frac{\pi 4}{4}\right)\right] = \frac{1}{\pi}$$

for $n = 0$ *to* 8

$$\begin{aligned}
h(0) &= \frac{sin(0-4)}{\pi(0-4)}\left[0.5 - 0.5cos\left(\frac{\pi 0}{4}\right)\right] = 0.0000 \\
h(1) &= \frac{sin(1-4)}{\pi(1-4)}\left[0.5 - 0.5cos\left(\frac{\pi 1}{4}\right)\right] = 0.0022 \\
h(2) &= \frac{sin(2-4)}{\pi(2-4)}\left[0.5 - 0.5cos\left(\frac{2\pi}{4}\right)\right] = 0.0724 \\
h(3) &= \frac{sin(3-4)}{\pi(3-4)}\left[0.5 - 0.5cos\left(\frac{3\pi}{4}\right)\right] = 0.2286 \\
h(4) &= \frac{sin(4-4)}{\pi(4-4)}\left[0.5 - 0.5cos\left(\frac{4\pi}{4}\right)\right] = 0.3183 \\
h(5) &= \frac{sin(5-4)}{\pi(5-4)}\left[0.5 - 0.5cos\left(\frac{5\pi}{4}\right)\right] = 0.2286 \\
h(6) &= \frac{sin(6-4)}{\pi(6-4)}\left[0.5 - 0.5cos\left(\frac{6\pi}{4}\right)\right] = 0.0724 \\
h(7) &= \frac{sin(7-4)}{\pi(7-4)}\left[0.5 - 0.5cos\left(\frac{7\pi}{4}\right)\right] = 0.0022 \\
h(8) &= \frac{sin(8-4)}{\pi(8-4)}\left[0.5 - 0.5cos\left(\frac{8\pi}{4}\right)\right] = 0.0000
\end{aligned}$$

The frequency response is symmetric with M=odd=9

$$
\begin{aligned}
|H(e^{j\omega})| &= h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h\left(\frac{M-1}{2} - n\right) cos\ \omega \\
&= h(4) + 2\sum_{n=0}^{3} h(n) cos\ \omega\,(4-n) \\
&= h(4) + 2h(0)cos\ 4\omega + 2h(1)cos\ 3\omega + 2h(2)cos\ 2\omega + 2h(3)cos\ \omega \\
&= 0.3183 + 0.044 cos\ 3\omega + 0.1448 cos\ 2\omega + 0.4572 cos\ \omega
\end{aligned}
$$

$$|H(e^{j\omega})|_{dB} = 20log|H(e^{j\omega})|$$

ω	$\|H(e^{j\omega})\|$	$\|H(e^{j\omega})\|_{dB}$
0	1.0180	0.1550
0.1π	0.9352	-0.5815
0.2π	0.7123	-2.9464
0.3π	0.4162	-7.6132
0.4π	0.1318	-17.6023
0.5π	0.0680	-23.3498
0.6π	0.1463	-16.6936
0.7π	0.1128	-18.9561
0.8π	0.0158	-36.0322
0.9π	0.0793	-22.0154
1.0π	0.1180	-18.5624

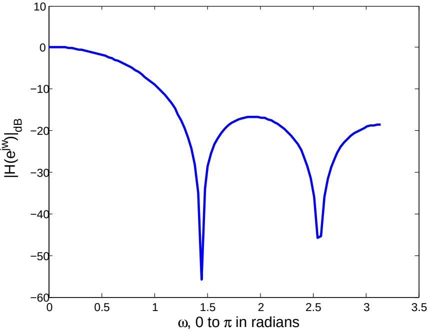


Figure 29: Frequency response of LPF

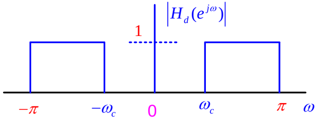
Design a HPF using Hamming window. Given that cutoff frequency the filter coefficients $h_d(n)$ for the desired frequency response of a low pass filter given by $\omega_c = 1\text{rad/sec}$, and take M=7. Also plot the magnitude response.

Solution:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\pi \le \omega \le -\omega_c \\ e^{-j\omega\tau} & \omega_c \le \omega \le -\pi \\ 0 & -\omega_c \le \omega \le \omega_c \end{cases}$$

$$\tau = \frac{M-1}{2} = 3$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\pi \le -\omega \le \omega_c \\ 0 & Otherwise \end{cases}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}[\int_{-\pi}^{-\omega_c} e^{j\omega(n-\tau)}d\omega + \int_{\omega_c}^{\pi} e^{j\omega(n-\tau)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{\omega_c}^{\pi} \\ &= \frac{1}{\pi(n-\tau)}\left[\frac{e^{-j\omega_c(n-\tau)} - e^{-j\pi(n-\tau)} + e^{j\pi(n-\tau)} - e^{j\omega_c(n-\tau)}}{2j}\right] \end{aligned}$$

$$h_d(n) = \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\pi(n-\tau)} - e^{-j\pi(n-\tau)} - \left[e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}\right]}{2j}\right]$$
$$= \frac{1}{\pi(n-\tau)}\left[sin\pi(n-\tau) - sin\omega_c(n-\tau\right]$$

$\tau = 3$ $\omega_c = 1$ $h_d(n) = \frac{1}{\pi(n-3)}\left[sin\pi(n-3) - sin(n-3)\right]$
when $n = \tau$ using L Hospital rule

$$h_d(n) = \frac{1}{\pi}\left[\frac{sin\pi(n-3)}{(n-3)} - \frac{sin\omega_c(n-3)}{(n-3)}\right] = \frac{1}{\pi}\left[\pi - \omega_c\right] = \frac{1}{\pi}\left[\pi - 1\right]$$

The given window function is Hamming window. In this case $h(n) = h_d(n)\omega(n))$ *for* $0 \leq n \leq 6$

$$w(n) = 0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$$

$$h(n) = \frac{1}{\pi(n-3)}\left[sin\pi(n-3) - sin(n-3)\right] \times \left[0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)\right]$$

n	$h(n)$	n	$h(n)$
0	-0.00119	4	-0.00119
1	-0.00448	5	-0.00448
2	-0.2062	6	-0.2062
3	0.6816		

The magnitude response of a symmetric FIR filter with M=odd is

$$|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{(M-3)}{2}} h(n)cos\omega\left(\frac{M-1}{2} - n\right)$$

For M=7

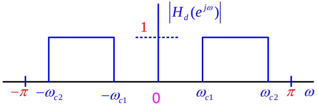
$$\begin{aligned}
|H(e^{j\omega})| &= h(3) + 2\sum_{n=0}^{2} h(n)cos\omega(3-n) \\
&= h(3) + 2h(0)cos3\omega + 2h(1)cos2\omega + 2h(2)cos\omega \\
&= 0.6816 - 0.000238cos3\omega - 0.0896cos2\omega - 0.4214cos\omega
\end{aligned}$$

Design the bandpass linear phase FIR filter having cut off frequencies of $\omega_{c_1} = 1$rad/sample and $\omega_{c_2} = 2$rad/sample. Obtain the unit sample response through following window.

$$\omega(n) = \begin{cases} 1 & \text{for } 0 \le n \le 6 \\ 0 & \text{Otherwise} \end{cases}$$

Solution:

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \omega_{c1} \le |\omega_c| \le \omega_{c_2} \\ 0 & \text{Otherwise} \end{cases}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{-j\omega\tau}e^{j\omega n}d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\tau}e^{j\omega n}d\omega\right] \\ &= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega(n-\tau)} + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega(n-\tau)}d\omega\right] \\ &= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega(n-\tau)}}{(n-\tau)}\right]_{-\omega_{c_2}}^{-\omega_{c_1}} + \left[\frac{e^{j\omega(n-\tau)}}{(n-\tau)}\right]_{\omega_{c_1}}^{\omega_{c_2}}\right] \\ &= \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)} \quad for\ n \neq \tau \end{aligned}$$

$$h_d(n) = \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)}$$
$$= \frac{\omega_{c_2} - \omega_{c_1}}{\pi} \quad for \;\; n = \tau$$

$$h_d(n) = \begin{cases} \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)} & for \;\; n \neq \tau \\ \frac{\omega_{c_2} - \omega_{c_1}}{\pi} & for \;\; n = \tau \end{cases}$$

The linear phase FIR filter is normalized means its cut-off frequency is of $\omega_c = 1 rad/sample$

The length of the filter with given τ is related by

$$\tau = \frac{M-1}{2} = \frac{7-1}{2} = 3$$

and $\omega_{c_2} = 2$ rad/sample $\omega_c = 1$rad/sample

$$h_d(n) = \begin{cases} \frac{sin2(n-3) - sin(n-3)}{\pi(n-\tau)} & for \;\; n \neq 3 \\ \frac{1}{\pi} \quad for \;\; n = 3 & for \end{cases}$$

The given window is rectangular hence

$$h(n) = h_d(n)w(n) = h_d(n)$$

n	$h(n)$	n	$h(n)$
0	-0.044	4	0.0215
1	-0.165	5	0.265
2	0.215	6	-0.044
3	0.3183		

For n=0,1,2..6 estimate the FIR filter coefficients h(n).
For M=7 The magnitude response of the FIR filter is given by

$$H(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) cos\ \omega\left(\frac{M-1}{2} - n\right)$$

$$H(\omega) = h(3) + 2\sum_{n=0}^{2} h(n) cos\ \omega\,(n-3)$$

Estimate the $H(\omega)$ by substituting the required values in the above equation.

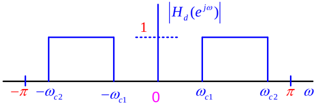
Design an ideal bandpass filter having frequency response

$$H_d e^{(j\omega)} \quad for \ \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4}$$

Use rectangular window with N=11 in your design

Solution:

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} \omega_{c1} \leq |\omega_c| \leq \omega_{c_2} \\ 0 \qquad Otherwise \end{cases}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{-j\omega\tau} e^{j\omega n} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\tau} e^{j\omega n} d\omega\right] \\ &= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega(n-\tau)} + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega(n-\tau)} d\omega\right] \\ &= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega(n-\tau)}}{(n-\tau)}\right]_{-\omega_{c_2}}^{-\omega_{c_1}} + \left[\frac{e^{j\omega(n-\tau)}}{(n-\tau)}\right]_{\omega_{c_1}}^{\omega_{c_2}}\right] \\ &= \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)} \quad for \ n \neq \tau \end{aligned}$$

$$h_d(n) = \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)}$$
$$= \frac{\omega_{c_2} - \omega_{c_1}}{\pi} \quad for \;\; n = \tau$$

$$h_d(n) = \begin{cases} \frac{sin\omega_{c_2}(n-\tau) - sin\omega_{c_1}(n-\tau)}{\pi(n-\tau)} & for \;\; n \neq \tau \\ \frac{\omega_{c_2} - \omega_{c_1}}{\pi} \qquad for \;\; n = \tau & for \end{cases}$$

The length of the filter with given τ is related by

$$\tau = \frac{M-1}{2} = \frac{11-1}{2} = 5$$

and $\omega_{c_2} = \frac{\pi}{4}$ rad/sample $\omega_c = \frac{3\pi}{4}$rad/sample

$$h_d(n) = \begin{cases} \frac{sin\left[\frac{3\pi(n-5)}{4}\right] - sin\left[\frac{\pi(n-5)}{4}\right]}{\pi(n-5)} & for \;\; n \neq 5 \\ \frac{\frac{3\pi}{4} - \frac{\pi}{4}}{\pi} \qquad for \;\; n = 5 & for \end{cases}$$

The given window is rectangular hence

$$h(n) = h_d(n)w(n) = h_d(n)$$

For n=0,1,2..10 estimate the FIR filter coefficients h(n).

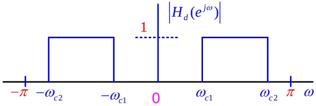
Design a BPF using Hanning window with M=7. Given that lower cutoff frequency $\omega_{c1} = 2$rad/sec and $\omega_{c2} = 3$rad/sec.

Solution:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & for \quad -\omega_{c2} \leq \omega \leq -\omega_{c1} \\ e^{-j\omega\tau} & for \ \omega_{c1} \leq \omega \leq \omega_{c2} \\ 0 & \quad for \quad -\omega_c - \omega_{c1}\omega \leq \omega_{c1} \end{cases}$$

$$\tau = \frac{M-1}{2} = 3$$

- The inverse transform of the $H_d(e^{\omega})$ is



$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}[\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\tau)}d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\tau)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{\omega_{c2}}^{\omega_{c1}} \\ &= \frac{1}{\pi(n-\tau)}[sin\omega_{c2}(n-\tau) - sin\omega_{c1}(n-\tau)] \end{aligned}$$

$\tau = \frac{M-1}{2} = \frac{7-1}{2} = 3$
$\omega_{c1} = 2$ rad/sec $\omega_{c2} = 3$ rad/sec
for $n \neq 3$

$$h_d(n) = \frac{1}{\pi(n-3)}\left[sin3(n-3) - sin2(n-3)\right]$$

for $n = \tau$

$$h_d(n) = \frac{1}{\pi}\left[\lim_{n\to\tau}\frac{sin\omega_{c2}(n-\tau)}{(n-\tau)} - \lim_{n\to\tau}\frac{sin\omega_{c1}(n-\tau)}{(n-\tau)}\right]$$

$$h_d(n) = \frac{1}{\pi}\left[\omega_{c2} - \omega_{c1}\right] = \frac{1}{\pi}$$

n	$h(n)$	n	$h(n)$
0	0	4	0
1	0.0189	5	0.0189
2	-0.01834	6	-0.01834
3	0.3183		

The given window function is Hanning window

$$\omega(n) = 0.5 - 0.5cos\frac{2\pi n}{M-1} \qquad 0 \leq n \leq M-1$$

This is rectangular window of length M=11. In this case $h(n) = h_d(n)\omega(n) = h_d(n)$
for $0 \leq n \leq 6$

$$h(n) = \left[\frac{sin3(n-\tau) - sin2(n-\tau)}{\pi(n-3)}\right]\left[0.5 - 0.5cos\frac{2\pi n}{M-1}\right]$$

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{M-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n} \\
&= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \\
&= 0 + 0.0189z^{-1} - 0.1843z^{-2} + 0.3183z^{-3} - 0.1834z^{-4} + 0.0189z^{-5} + 0
\end{aligned}
$$

The magnitude response of a symmetric FIR filter with M=odd is

$$
|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + \sum_{n=1}^{(M-1)/2} 2h\left(\frac{M-1}{2} - n\right) cos\omega n
$$

For M=7

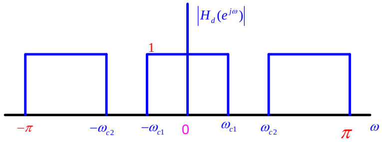
$$
|H(e^{j\omega})| = h\left(3\right) + \sum_{n=1}^{3} 2h\left(3 - n\right) cos\omega n
$$

$$
\begin{aligned}
|H(e^{j\omega})| &= h\left(3\right) + \sum_{n=1}^{3} 2h\left(3 - n\right) cos\omega n \\
&= h(3) + \sum_{n=1}^{5} 2h\left(5 - n\right) cos\omega n \\
&= h(3) + 2h(2)cos\omega + 2h(1)cos2\omega + 2h(0)cos3\omega \\
&= 0.3183 - 0.3668cos\omega + 0.0378cos2\omega
\end{aligned}
$$

Design a bandstop filter to reject the frequencies from 2 to 3 rad/sec using rectangular window with M=5. Find the frequency response.

Solution:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & for \ -\pi \leq \omega \leq -\omega_{c2} \\ e^{-j\omega\tau} & for \ -\omega_{c1} \leq \omega \leq \omega_{c1} \\ e^{-j\omega\tau} & for \ \omega_{c2} \leq \omega \leq \pi \\ 0 & for \ \omega_{c1} \leq |\omega| \leq \omega_{c2} \end{cases}$$



$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}[\int_{-\pi}^{-\omega_{c2}} e^{j\omega(n-\tau)}d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega(n-\tau)}d\omega + \int_{\omega_{c2}}^{\pi} e^{j\omega(n-\tau)}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{\omega_{c2}}^{\pi} \\ &= \frac{1}{\pi(n-\tau)}[sin\omega_{c1}(n-\tau) + sin\pi(n-\tau - sin\omega_{c2}(n-\tau)] \end{aligned}$$

- The inverse transform of the $H_d(e^\omega)$ is

$\tau = \frac{M-1}{2} = \frac{5-1}{2} = 2$ $\omega_{c1} = 2$ rad/sec $\omega_{c2} = 3$ rad/sec

$$h_d(n) = \frac{1}{\pi(n-2)}\left[sin2(n-2) + sin\pi(n-2) - sin3(n-2)\right] \qquad for\ \ n \neq 2$$

for $n = \tau$

$$h_d(n) = \frac{1}{\pi}\left[\lim_{n\to\tau}\frac{sin\omega_{c1}(n-\tau)}{(n-\tau)} + \lim_{n\to\tau}\frac{sin\pi(n-\tau)}{(n-\tau)} - \lim_{n\to\tau}\frac{sin\omega_{c2}(n-\tau)}{(n-\tau)}\right]$$

$$h_d(n) = \frac{1}{\pi}\left[\omega_{c1} + \pi - \omega_{c2}\right] = \frac{1}{\pi}\left[\pi - 1\right]$$

The given window function is Rectangular window

$$\omega(n) = 1 \qquad 0 \leq n \leq M-1$$

This is rectangular window of length M=5.
In this case $h(n) = h_d(n)\omega(n) = h_d(n)$ *for* $0 \leq n \leq 4$

$$h_d(n) = \frac{1}{\pi(n-2)}\left[sin2(n-2) + sin\pi(n-2) - sin3(n-2)\right] \qquad for\ \ n \neq 2$$

$$n=0 \quad h(0) = \left[\frac{sin2(0-2)+sin\pi(0-2)-sin3(0-2)}{\pi(0-2)}\right] = -0.0759$$

$$n=1 \quad h(1) = \left[\frac{sin2(1-2)+sin\pi(1-2)-sin3(1-2)}{\pi(1-2)}\right] = 0.2445$$

$$n=2 \quad h(2) = \frac{1}{\pi}[\pi-1] = 0.6817$$

$$n=3 \quad h(3) = \left[\frac{sin2(3-2)+sin\pi(3-2)-sin3(3-2)}{\pi(3-2)}\right] = 0.2445$$

$$n=4 \quad h(4) = \left[\frac{sin2(4-2)+sin\pi(4-2)-sin3(4-2)}{\pi(4-2)}\right] = -0.0759$$

$$\begin{aligned} H(z) &= \sum_{n=0}^{M-1} h(n)z^{-n} = \sum_{n=0}^{4} h(n)z^{-n} \\ &= h(0)+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+h(4)z^{-4} \\ &= -0.0759+0.2445z^{-1}+0.6817z^{-2}+0.2445z^{-3}-0.0759z^{-4} \end{aligned}$$

The magnitude response of a symmetric FIR filter with M=odd is

$$|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + \sum_{n=1}^{(M-1)/2} 2h\left(\frac{M-1}{2} - n\right) cos\omega n$$

For M=5

$$|H(e^{j\omega})| = h\left(2\right) + \sum_{n=1}^{2} 2h\left(2-n\right) cos\omega n$$

$$\begin{aligned}
|H(e^{j\omega})| &= h\left(3\right) + \sum_{n=1}^{3} 2h\left(3-n\right) cos\omega n \\
&= h(2) + 2h(1)cos\omega + 2h(0)cos2\omega \\
&= 0.6817 + 2(0.2445)cos\omega + 2h(-0.0759)cos2\omega \\
&= 0.6817 + 0.4890cos\omega - 0.1518cos2\omega
\end{aligned}$$

FIR Filter Design Using Kaiser Window

- The Kaiser window is parametric and its bandwidth as well as its sidelobe energy can be designed.
- Mainlobe bandwidth controls the transition characteristics and sidelobe energy affects the ripple characteristics.
- The Kaiser window function is given by

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{M-1}\right)^2}\right]}{I_0(\alpha)}$$

where M is the order of the filter, $I_0(x)$ is a zeroth Bessel function of the first kind

$$\begin{aligned} I_0(x) &= 1+\sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right] \\ &= 1+\frac{0.25x^2}{(1!)^2}+\frac{(0.25x^2)^2}{(2!)^2}+\frac{(0.25x^2)^3}{(3!)^2}+ \end{aligned}$$

$$\begin{aligned} \alpha &= 0 && \text{if A} < 21 \\ &= 0.5842(A-21)^{0.4}+0.07886(A-21) && \text{if } 21 \le \text{A} \le 50 \text{ dB} \\ &= 0.1102(A-8.7)) && \text{if A} > 50 \text{ dB} \end{aligned}$$

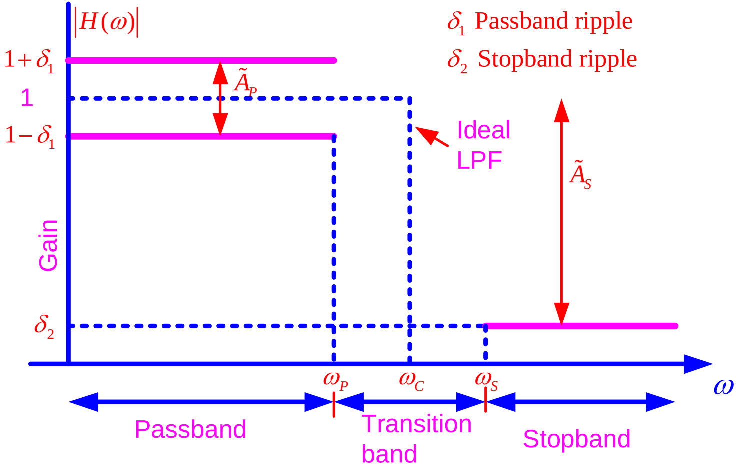


Figure 30: Frequency response of LPF

Kaiser Window Design Equations

1. Determine ideal frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 \;\; for \;\; |\omega| \leq \omega_c \\ 0 \;\; for \;\; \omega_c \leq |\omega| \leq \pi \end{cases}$$

   where $\omega_c = \frac{1}{2}(\omega_p + \omega_s)$

2. Chose δ such that the actual passband ripple, A_p is equal to or less than the specified passband ripple $\tilde{A}_p$, and the actual minimum stopband attenuation A is equal or greater than the specified minimum stop attenuation $\tilde{A}_s$

$$\delta = min(\delta_p, \delta_s)$$

   where $\delta_p = \frac{10^{0.05\tilde{A}_p}-1}{10^{0.05\tilde{A}_p}+1}$ and $\quad \delta_s = 10^{-0.05\tilde{A}_s}$

3. The actual stopband attenuation is

$$A = -20log_{10}\delta$$

4. The parameter α is

$$\alpha = \begin{cases} 0 & for \;\; A \leq 21 \\ 0.5842(A-21)^{0.4} + 0.07886(A_a - 21) & for \; 21 < A \leq 50 \\ 0.1102(A-8.7) & for \;\; A > 50 \end{cases}$$

5. The value of M is found by
$$M \geq \frac{A - 7.95}{14.36\Delta f}$$
where $\Delta f = \frac{\Delta\omega}{2\pi} = \frac{\omega_s - \omega_p}{2\pi}$ and $\Delta\omega$ is the width of transition band
6. Obtain impulse response by multiplying Kaiser window function
$$h(n) = h_d(n)w_k(n)$$
7. Obtain the causal finite impulse response
8. The system function is given by
$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

Design a lowpass filter with a cutoff frequencies from $w_c = \frac{\pi}{4}$ $\Delta\omega = 0.02\pi$ and a stopband ripple $\delta_s = 0.01$. Use Kaiser window

Solution:

$$H_d(e^{j\omega}) = \begin{cases} 1 \ for \ |\omega| \leq \omega_c \\ 0 \ for \ \omega_c \leq |\omega| \leq \pi \end{cases}$$

The inverse transform of the $H_d(e^{j\omega})$ is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi n}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right] \\ &= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] \\ &= \frac{sin\omega_c n}{\pi n} \end{aligned}$$

$$A = -20log\delta_s = -20log(0.01) = 40 \ dB$$

$$\begin{aligned} \alpha &= 0.5842(A-21)^{0.4} + 0.07886(A-21) \\ &= 0.5842(40-21)^{0.4} + 0.07886(40-21) \\ &= 3.4 \end{aligned}$$

$$\Delta f = \frac{0.02\pi}{2\pi} = 0.01$$

$$M \geq \frac{A - 7.95}{14.36\Delta f} \geq 223.189 \simeq 225$$

$$\tau = \frac{225-1}{2} = 112$$

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{M-1}\right)^2}\right]}{I_0(\alpha)} \qquad 0 \leq n \leq M-1$$

$$w_k(n) = \frac{I_0\left[3.4\sqrt{1-\left(\frac{2n}{224}\right)^2}\right]}{I_0(3.4)} \qquad 0 \leq n \leq M-1$$

$$h(n) = h_d \times w_k(n) = \frac{1}{\pi n}\left[sin\omega_c n\right] \times \frac{I_0\left[3.4\sqrt{1-\left(\frac{2n}{224}\right)^2}\right]}{I_0(3.4)}$$

where

$$\omega_c^{'} = \omega_c + \frac{\Delta\omega}{2} = 0.25\pi + \frac{0.02\pi}{2} = 0.26\pi$$

Find an expression for the impulse response $h(n)$ of a linear phase lowpass FIR filter using Kaiser window to satisfy the following magnitude response specifications for the equivalent analog filter.

- Stopband attenuation: 40 dB
- Passband ripple: 0.01 dB
- Transition width: 1000 π rad/sec
- Ideal cutoff frequency: 2400 π rad/sec
- Sampling frequency: 10 KHz

Solution:

$$A = -20 log \delta_s = 40 \;\; dB$$

$$log \delta_s = -2 \Rightarrow \delta_s = 0.01$$

$$20 log(1 + \delta_p) = 0.01$$

$$log(1 + \delta_p) = 0.0005$$

$$\delta_p = 0.00115$$

$$\delta min(\delta_p, \delta_s) = 0.00115$$

$$A = -20 log(0.00115) = 58.8 \;\; dB$$

$$\Delta\omega = \frac{\Delta\Omega}{f_s} = \frac{1000\pi}{10 \times 10^3} = 0.1\pi \;\; rad$$

$$\Delta f = \frac{0.1\pi}{2\pi} = 0.05$$

$$M \geq \frac{A - 7.95}{14.36 \Delta f} \geq \frac{58.8 - 7.95}{14.36 \times 0.05} = 70.82 \simeq 71$$

$$\tau = \frac{71 - 1}{2} = 35$$

The attenuation is 58.8 dB then the parameter α is

$$\begin{aligned} \alpha &= 0.1102(A-8.7) \\ &= 0.1102(58.8-8.7) \simeq 5.5 \end{aligned}$$

$$w_k(n) = \frac{I_0\left[5.5\sqrt{1-\left(\frac{2n}{70}\right)^2}\right]}{I_0(5.5)}$$

$$\begin{aligned} \omega_c &= \Omega_c \times T = 2400\pi \times \frac{1}{10\times 10^3} \\ &= 0.24\pi \end{aligned}$$

$$\begin{aligned} \omega'_c &= \omega_c + \frac{\Delta\omega}{2} \\ &= 0.24\pi + \frac{0.1\pi}{2} = 0.29\pi \end{aligned}$$

$$H_d(e^{j\omega}) = \begin{cases} 1 \ for \ |\omega| \le \omega_c \\ 0 \ for \ \omega_c \le |\omega| \le \pi \end{cases}$$

The inverse transform of the $H_d(e^{j\omega})$ is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi n}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right] \\ &= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] \\ &= \frac{sin\omega_c n}{\pi n} \end{aligned}$$

$$h(n) = h_d \times w_k(n) = \frac{1}{\pi(n-\tau)}\left[sin\omega_c(n-\tau)\right] \times \frac{I_0\left[5.48\sqrt{1-\left(\frac{2n}{224}\right)^2}\right]}{I_0(5.5)}$$

Find an expression for the impulse response $h(n)$ of a linear phase Design a lowpass FIR filter satisfying the following specifications using Kaiser window

- $\alpha_p \leq 0.1\ dB \quad \alpha_s \geq 44\ dB$
- $\omega_p = 20\ rad/sec \quad \omega_s = 30\ rad/sec \quad \omega_{sf} = 100\ rad/sec$

Solution:

$$H_d(e^{j\omega}) = \begin{cases} 1 \ \ for\ \ |\omega| \leq \omega_c \\ 0 \ \ for\ \ \omega_c \leq |\omega| \leq \pi \end{cases}$$

The inverse transform of the $H_d(e^{j\omega})$ is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi n}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right] \\ &= \frac{sin\omega_c n}{\pi n} \end{aligned}$$

$$\Delta\omega = \omega_s - \omega_p = 10 rad/sec$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s) = 25 rad/sec$$

$$\omega_c(in\ discrete\ and\ radian) = \frac{25}{100}(2\pi) = \frac{\pi}{2} rad$$

$$\delta_s = 10^{-0.05A_s} = 10^{-0.05\times 44} = 6.3096 \times 10^{-3}$$

$$\delta_p = \frac{10^{0.05A_p} - 1}{10^{0.05A_p} + 1} = \frac{10^{0.05\times 0.1} - 1}{10^{0.05\times 0.1} + 1} = 5.7563\times 10^{-3}$$

$$\delta = min(\delta_p, \delta_s) = 5.7563 \times 10^{-3}$$

$$A = -20log_{10}(\delta) = 44.797dB$$

$$\begin{aligned}\alpha &= 0.5842(A-21)^{0.4}+0.07886(A-21)\\ &= 0.5842(44.797-21)^{0.4}+0.07886(44.797-21)\\ &= 3.9524\end{aligned}$$

$$\Delta f = \frac{\Delta\omega}{\omega_{sf}} = \frac{10}{100} = 0.1$$

$$M \geq \frac{A-7.95}{14.36\Delta f} \geq \frac{44.797-7.95}{14.36\times 0.1} \geq 25.66 \simeq 27$$

$$\tau = \frac{27-1}{2} = 13$$

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{M-1}\right)^2}\right]}{I_0(\alpha)} \qquad 0 \leq n \leq M-1$$

$$w_k(n) = \frac{I_0\left[3.9524\sqrt{1-\left(\frac{2n}{27}\right)^2}\right]}{I_0(3.9524)} \qquad 0 \leq n \leq M-1$$

$$h(n) = h_d \times w_k(n) = \frac{1}{\pi n}\left[sin\omega_c n\right] \times \frac{I_0\left[3.9524\sqrt{1-\left(\frac{2n}{27}\right)^2}\right]}{I_0(3.9524)}$$

Design a high pass digital satisfying the following specifications using Kaiser window

- Passband cut-off frequency $f_p = 3200$Hz Stopband cut-off frequency $f_s = 1600$Hz
- Passband ripple $\alpha_p \leq 0.1$dB Stopband ripple $\alpha_s \geq 40$dB
- Sampling frequency $F = 10000$Hz

Solution:

$$H_d(e^{j\omega}) = \begin{cases} 0 \ \ for \ |\omega| \leq \omega_c \\ 1 \ \ for \ \omega_c \leq |\omega| \leq \pi \end{cases}$$

The inverse transform of the $H_d(e^{j\omega})$ is

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \int_{\omega_c}^{\pi} e^{j\omega n} d\omega\right] \\ &= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{-\omega_c} + \left(\frac{e^{j\omega n}}{jn}\right)_{\omega_c}^{\pi}\right] \\ &= \frac{1}{2j\pi n}\left[e^{-j\omega_c n} - e^{-j\pi n} + e^{j\pi n} - e^{j\omega_c n}\right] \\ &= \frac{sin\pi n - sin\omega_c n}{\pi n} \end{aligned}$$

$$\omega_p = 2\pi f_p = 6400\pi \ \ rad/sec$$

$$\omega_s = 2\pi f_s = 3200\pi \ \ rad/sec$$

$$\omega_{sf} = 2\pi F = 20000\pi \ \ rad/sec$$

$$\Delta\omega = \omega_p - \omega_s = 3200\pi \ \ rad/sec$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s) = 4800\pi \ \ rad/sec$$

$$\omega_c(discrete, radian) = \frac{4800}{20000}(2\pi) = 0.48\pi rad$$

$$\delta_s = 10^{-0.05A_s} = 10^{-0.05\times 40} = 0.01$$

$$\delta_p = \frac{10^{0.05A_p - 1}}{10^{0.05A_p + 1}} = \frac{10^{0.05\times 0.1 - 1}}{10^{0.05\times 0.1 + 1}} = 0.005756$$

$$\delta = min(\delta_p, \delta_s) = 5.756 \times 10^{-3}$$

$$A = -20log_{10}(\delta) = 44.797dB$$

$$\begin{aligned}\alpha &= 0.5842(A-21)^{0.4}+0.07886(A-21)\\ &= 0.5842(44.797-21)^{0.4}+0.07886(44.797-21)\\ &= 3.9524\end{aligned}$$

$$\Delta f=\frac{\Delta\omega}{\omega_{sf}}=\frac{3200\pi}{20000\pi}=0.16$$

$$M\geq\frac{A-7.95}{14.36\Delta f}\geq\frac{44.797-7.95}{14.36\times 0.16}\geq 16.03\simeq 17$$

$$\tau=\frac{17-1}{2}=8$$

$$w_k(n)=\frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{M-1}\right)^2}\right]}{I_0(\alpha)}\qquad 0\leq n\leq M-1$$

$$w_k(n)=\frac{I_0\left[3.9524\sqrt{1-\left(\frac{2n}{17}\right)^2}\right]}{I_0(3.9524)}\qquad 0\leq n\leq M-1$$

$$h(n)=h_d\times w_k(n)=\frac{sin\pi n-sin\omega_c n}{\pi n}\times\frac{I_0\left[3.9524\sqrt{1-\left(\frac{2n}{27}\right)^2}\right]}{I_0(3.9524)}$$

Design of FIR filter using Frequency Sampling

With necessary mathematical analysis explain the frequency sampling technique of FIR filter design

- In this method a set of M equally spaced samples in the interval $(0, 2\pi)$are taken in the desired frequency response $H_d(\omega)$.
- The continuous frequency ω is replaced by

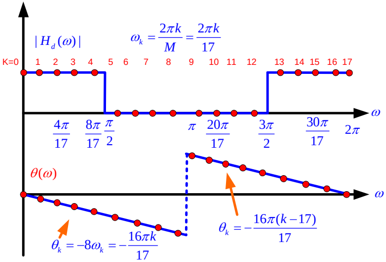
$$\omega = \omega_k = \frac{2\pi}{M}k \qquad k = 0, 1, \ldots M-1$$

- The discrete time Fourier transform (DTFT) is

$$\begin{aligned} H(k) &= H_d(\omega)|_{\omega=\omega_k} \\ &= H_d\left(\frac{2\pi}{M}k\right) \quad k = 0, 1, \ldots M-1 \end{aligned}$$

- The inverse M point DFT (IDFT) h(n) is

$$\begin{aligned} h(n) &= \frac{1}{M}\sum_{n=0}^{M-1} H(k)e^{j\omega n} \\ &= \frac{1}{M}\sum_{n=0}^{M-1} H(k)e^{j\frac{2\pi kn}{M}} \quad n = 0, 1, \ldots M-1 \end{aligned}$$



Magnitude frequency response is symmetric about π, while ideal phase response is antisymmetric about π

- For the FIR filter to be realizable the coefficients h(n) must be real. This is possible if all complex terms appear in complex conjugate pairs. Consider the term $H(M-k)e^{j2\pi n(M-k)/M}$

$$H(M-k)e^{j2\pi n(M-k)/M} = H(M-k)e^{j2\pi n}e^{-j2\pi kn/M}$$

$$H(M-k)e^{j2\pi n(M-k)/M} = H(M-k)e^{-j2\pi kn/M} \quad \because \quad e^{j2\pi n} = cos(2\pi n) + jsin(2\pi n) = 1$$

- substituting the $|H(M-k)| = |H(k)|$

$$H(M-k)e^{j2\pi n(M-k)/M} = H(k)e^{-j2\pi kn/M}$$

- The term $H(k)e^{-j2\pi kn/M}$ is complex conjugate of $H(k)e^{j2\pi kn/M}$.
- Hence $H(M-k)e^{j2\pi n(M-k)/M}$ is complex conjugate of $H(k)e^{-j2\pi kn/M}$

$$H(M-k) = H^{*}(k)$$

- If $H(M-k) = H*(k)$ then $h(n)$

$$h(n) = \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{P} Re\left[H(k)e^{j2\pi kn/M}\right]\right)$$

- where P is

$$P = \begin{cases} \frac{M-1}{2} & if \;\; M \;\; is \;\; odd \\ \frac{M}{2} - 1 & if \;\; M \;\; is \;\; even \end{cases}$$

- This equation is used to compute the coefficients of FIR filter.
- H(z) is

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

Design a lowpass FIR filter using frequency sampling technique having cut-off frequency of $\pi/2$ rad/sample. The filter should have linear phase and length of 17.

Solution:

- The Ideal LPF frequency response $H_d(\omega)$ for the linear phase is

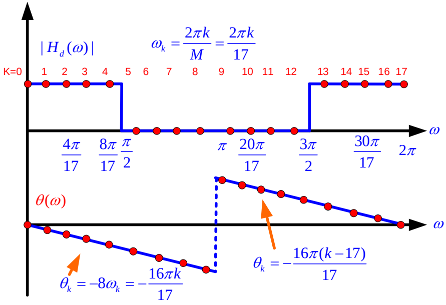
$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j8\omega} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

- To sample put $\omega = \frac{2\pi k}{M} = \frac{2\pi k}{17}$

$$H_d(\omega) = \begin{cases} e^{-j\frac{2\pi k}{17}8} & 0 \le \frac{2\pi k}{17} \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \frac{2\pi k}{17} \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j\frac{16\pi k}{17}} & 0 \le k \le \frac{17}{4} \\ 0 & \frac{17}{4} \le k \le \frac{17}{2} \end{cases}$$



The range of k is

$\frac{2\pi k}{17} = \frac{\pi}{2} \quad k = \frac{17}{4} \simeq 4$

$\frac{2\pi k}{17} = \pi \quad k = \frac{17}{2} \simeq 8$

The range of k is $0 \leq k \leq \frac{17}{4}$
k is an integer.
Hence the range is $0 \leq k \leq 4$
Similarly $\frac{17}{4} \leq k \leq \frac{17}{2} = 4.25 \leq k \leq 8.5$
The range $5 \leq k \leq 8$

$$|H(k)| = \begin{cases} 1 & 0 \leq k \leq 4 \\ 0 & 5 \leq k \leq 8 \\ 1 & 13 \leq k \leq 16 \end{cases}$$

The value of h(n)is given by

$$\begin{aligned} h(n) &= \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{8} Re\left[H(k)e^{j2\pi kn/17}\right]\right) \end{aligned}$$

$|H(k)| = 1 \quad 0 \leq k \leq 4$

$$\begin{aligned} h(n) &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} Re\left[e^{-j\frac{16\pi k}{17}} e^{j2\pi kn/17}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} Re\left[e^{j2\pi k(n-8)/17}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} cos\left[\frac{2\pi k(n-8)}{17}\right]\right) \end{aligned}$$

Determine the impulse response h(n) of a filter having desired frequency response

$$H_d(w) = \begin{cases} e^{-j\left(\frac{(M-1)\omega}{2}\right)} & 0 \le |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

M=7 use frequency sampling approach.

Solution:

- The Ideal LPF frequency response $H_d(\omega)$ is

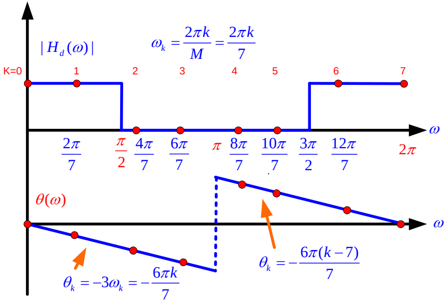
$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

- To sample put $\omega = \frac{2\pi k}{M} = \frac{2\pi k}{7}$

$$H_d(\omega) = \begin{cases} e^{-j\frac{2\pi k}{7}3} & 0 \le \frac{2\pi k}{7} \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \frac{2\pi k}{7} \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j\frac{6\pi k}{7}} & 0 \le k \le \frac{7}{4} \\ 0 & \frac{7}{4} \le k \le \frac{7}{2} \end{cases}$$



The range of k is

$\frac{2\pi k}{7} = \frac{\pi}{2} \quad k = \frac{7}{4} \simeq 1$

$\frac{2\pi k}{7} = \pi \quad k = \frac{7}{2} \simeq 3$

The range of k is $0 \leq k \leq \frac{7}{4}$
k is an integer.
Hence the range is $0 \leq k \leq 1$
Similarly $\frac{7}{4} \leq k \leq \frac{7}{2} = 1.75 \leq k \leq 3.5$
The range $2 \leq k \leq 3$

$$|H(k)| = \begin{cases} 1 & 0 \leq k \leq 1 \\ 0 & 2 \leq k \leq 3 \\ 1 & k = 6 \end{cases}$$

n	$h(n)$	n	$h(n)$
0	-0.1146	4	321
1	0.0793	5	0.0793
2	0.321	6	-0.1146
3	0.4283		

The value of h(n)is given by

$$h(n) = \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right)$$
$$= \frac{1}{7}\left(1 + 2\sum_{k=1}^{3} Re\left[H(k)e^{j2\pi kn/7}\right]\right)$$

$$|H(k)| = 1 \quad 0 \leq k \leq 1$$

$$h(n) = \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} Re\left[e^{-j\frac{6\pi k}{7}} e^{j2\pi kn/7}\right]\right)$$
$$= \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} Re\left[e^{j2\pi k(n-3)/7}\right]\right)$$
$$= \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} cos\left[\frac{2\pi k(n-3)}{7}\right]\right)$$

Determine the filter coefficients h(n) obtained by frequency sampling $H_d(w)$ given by

$$H_d(w) = \begin{cases} e^{-j3\omega} & 0 \leq |\omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq \omega \leq \pi \end{cases}$$

Also obtain the frequency response $H(w)$. Take N=7. DEC 2011

Proakis Exercise 8.6

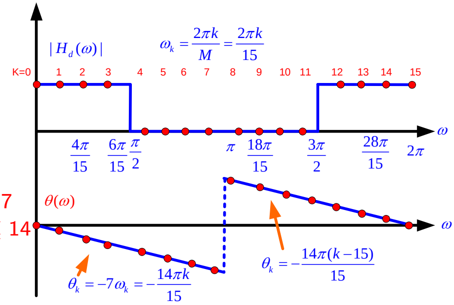
Determine the filter coefficients h(n) of a linear phase FIR of length M=15 which has symmetric unit impulse response and the frequency response that satisfies the coefficients.

$$H\left(\frac{2\pi}{15}k\right) = \begin{cases} 1 & k = 0,1,2,3 \\ 0 & k = 4,5,6,7 \end{cases}$$

Solution:

$$|H(k)| = \begin{cases} 1 & 0 \le k \le 3 \\ 0 & 4 \le k \le 11 \\ 1 & 12 \le k \le 14 \end{cases}$$

$$\theta(k) = \begin{cases} -\frac{14}{15}\pi k & 0 \le k \le 7 \\ 14\pi - \frac{14}{15}\pi k = -\frac{14}{15}\pi(k-15) & 8 \le k \le 14 \end{cases}$$



The value of h(n)is given by

$$h(n) = \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right)$$
$$= \frac{1}{15}\left(1 + 2\sum_{k=1}^{7} Re\left[H(k)e^{j2\pi kn/7}\right]\right)$$

$|H(k)| = 1 \quad 0 \leq k \leq 3$

$$h(n) = \frac{1}{715}\left(1 + 2\sum_{k=1}^{3} Re\left[e^{-j\frac{17\pi k}{15}} e^{j2\pi kn/15}\right]\right)$$
$$= \frac{1}{15}\left(1 + 2\sum_{k=1}^{3} Re\left[e^{j2\pi k(n-7)/15}\right]\right)$$
$$= \frac{1}{15}\left(1 + 2\sum_{k=1}^{3} cos\left[\frac{2\pi k(n-7)}{15}\right]\right)$$

n	$h(n)$	n	$h(n)$
0	-0.05	8	0.3188
1	0.041	9	0.034
2	0.066	10	-0.108
3	-0.036	11	-0.036
4	-0.108	12	0.066
5	0.034	13	0.041
6	0.3188	14	-0.05
7	0.466		

Proakis Exercise 8.7

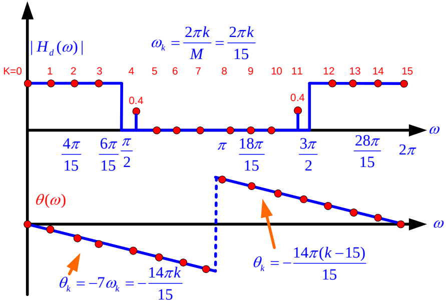
Determine the filter coefficients h(n) of a linear phase FIR of length M=15 which has symmetric unit impulse response and the frequency response that satisfies the coefficients.

$$H\left(\frac{2\pi}{15}k\right) = \begin{cases} 1 & k = 0,1,2,3 \\ 0.4 & k = 4 \\ 0 & k = 5,6,7 \end{cases}$$

Solution:

$$|H(k)| = \begin{cases} 1 & 0 \le k \le 3 \\ 0.4 & k = 4 \\ 0 & 5 \le k \le 10 \\ 0.4 & k = 11 \\ 1 & 12 \le k \le 14 \end{cases}$$

$$\theta(k) = \begin{cases} -\frac{14}{15}\pi k & 0 \le k \le 7 \\ -\frac{14}{15}\pi(k-15) & 8 \le k \le 14 \end{cases}$$



The value of h(n)is given by

$$
\begin{aligned}
h(n) &= \frac{1}{M}\left(H(0)+2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right)\\
&= \frac{1}{15}\left(1+2\sum_{k=1}^{7} Re\left[H(k)e^{j2\pi kn/7}\right]\right)
\end{aligned}
$$

$|H(k)| = 1 \quad 0 \leq k \leq 3$
$|H(k)| = 0.4 \quad k = 4\&11$

$$
\begin{aligned}
h(n) &= \frac{1}{715}\left(1+2\sum_{k=1}^{3} Re\left[e^{-j\frac{17\pi k}{15}}e^{j2\pi kn/15}\right]\right)\\
&= \frac{1}{15}\left(1+2\sum_{k=1}^{3} Re\left[e^{j2\pi k(n-7)/15}\right]+2Re\left[0.4e^{j2\pi 4(n-7)/15}\right]\right)\\
&= \frac{1}{15}\left(1+2\sum_{k=1}^{3} cos\left[\frac{2\pi k(n-7)}{15}\right]+0.8cos\left[\frac{8\pi(n-7)}{15}\right]\right)
\end{aligned}
$$

n	$h(n)$	n	$h(n)$
0	-0.0143	8	0.313
1	-0.002	9	-0.0181
2	0.04	10	-0.091
3	0.0122	11	0.0122
4	-0.091	12	0.04
5	-0.0181	13	-0.002
6	0.313	14	-0.0143
7	0.520		

References

- J. G. Proakis and D. G. Monalakis, *Digital signal processing Principles Algorithms & Applications*, 4th ed. Pearson education, 2007.
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