MATH-2105: Matrices, Vector Analysis and Coordinate Geometry

Course Instructor:

Abdul Karim
Lecturer, Department of Applied Mathematics, NSTU.
Cell: 01737184545
E-mail: akarim.amth@nstu.edu.bd

Lecture Sheet 3

Row Echelon Form

A matrix is in row echelon form if

1. The leading term in any nonzero row is 1 and all terms below this leading 1 are zero.
2. The leading 1 in any non-zero row occurs to the right of the leading 1 in any previous row.
3. The non-zero rows appear before any zero rows.
4. All rows of zeros are at the bottom of the matrix

Example: $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rank of a Matrix

The number of non-zero rows of a matrix given in the echelon form is its rank. The number of linear independent rows of the matrix is called its rank.

In other words, the rank of a matrix is the maximum number of its linearly independent column vectors (or row vectors). From this definition it is obvious that the rank of a matrix cannot exceed the number of its rows (or columns).

In control theory, the rank of a matrix can be used to determine whether a linear system is controllable, or observable. In the field of communication complexity, the rank of the communication matrix of a function gives bounds on the amount of communication needed for two parties to compute the function.

Basically, when the rank of a matrix is n, then it implies there will be at least one set of consecutive n elements in a particular row or column of that matrix that cannot be converted to 0s (all 0s) by using any no. of valid elementary row or column transformation.

For example, if a 3×3 matrix has a rank 3 then it means that there will be at least one such row or column in that matrix which you cannot convert into all zeros by using any no. of elementary row or column transformations. At the most, you can convert it into a 3×3 identity matrix. In this case, no row or column will be converted into all 0s.You can try on your own!!

This explanation of rank comes from the properties of determinants.

Question

Find the rank of the matrix $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$

Solution

We perform some row operations on the given matrix:

$$A \sim \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ({r'}_2 = r_2 - 2r_1,\ {r'}_3 = r_3 - 3r_1)$$

Then the rank of matrix is one, as there is only one non-zero row.

Question

Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

We perform some row and column operations on the given matrix:

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 3 & 10 \end{bmatrix} ({r'}_2 = 4r_1 - r_2,\ {r'}_3 = 2r_1 - r_3)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & -4 \end{bmatrix} ({r''}_3 = {r'}_2 - {r'}_3)$$

$\text{Rank}(A) = 3$, as there are three non-zero rows.

System of Linear Equations

A system of linear equations is a list of linear equations with the same set of unknowns. The standard form of a system of m linear equations $L_1, L_2, \dots, L_m$ in n unknowns $x_1, x_2, \dots, x_n$ is as follows

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{gathered}$$

Matrix Representation of a System of Linear Equations

The system of linear equations mentioned earlier can be expressed in a matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

More precisely $AX = B$

Where $A =$ Coefficient Matrix (Stiffness)

$X =$ Set of variables (Unknowns)

$B =$ Constant Vectors (External Load / Force / Voltage)

For example, let us consider a SLE with four variables

$$\begin{gathered} x_1 + 2x_2 - 4x_3 + 7x_4 = 4 \\ 3x_1 - 5x_2 + 6x_3 - 8x_4 = 8 \\ 4x_1 - 3x_2 - 2x_3 + 6x_4 = 11 \end{gathered}$$

The equivalent matrix form of this system is

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 3 & -5 & 6 & -8 \\ 4 & -3 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix}$$

Let us consider another SLE with five variables

$$\begin{gathered} x_1 + 2x_2 = 40 \\ -x_1 - 2x_2 + 6x_3 = 11 \\ -5x_2 + 6x_3 - 8x_4 = 11 \\ -2x_3 + 6x_4 - x_5 = 11 \\ 2x_4 - 7x_5 = 200 \end{gathered}$$

The equivalent matrix form of this system is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ -1 & -2 & 6 & 0 & 0 \\ 0 & -5 & 6 & -8 & 0 \\ 0 & 0 & -2 & 6 & -1 \\ 0 & 0 & 0 & 2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 40 \\ 11 \\ 11 \\ 11 \\ 200 \end{bmatrix}$$

Vector Representation of a System of Linear Equations

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Homogeneous and non-homogeneous System of Linear Equations

The system of linear equations for which all the constant terms are zero i.e. if

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

the system is said to be homogeneous.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Or $AX = 0$

Otherwise, the system $AX = B$ is said to be non-homogeneous.

Example

Show the following system of linear equations as its equivalent matrix form and as linear combination of vectors:

$$x_1 + 2x_2 - 4x_3 + 7x_4 = 4$$

$$3x_1 + x_2 + 6x_3 - 8x_4 = 12$$

$$4x_1 - 5x_2 - 3x_3 + 7x_4 = 8$$

Solution

Matrix Representation:

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 3 & 1 & 6 & -8 \\ 4 & -5 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix}$$

Vector Representation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 6 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -8 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$, find a non-zero column vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $Au = 3u$. Describe all such vectors.

Solution

First we set up the matrix equation

$$Au = 3u$$

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x + 3y \\ 4x - 3y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$$\begin{bmatrix} x + 3y - 3x \\ 4x - 3y - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x + 3y \\ 4x - 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We set the corresponding elements to each other to obtain the system of equations

$$-2x + 3y = 0$$

$$4x - 6y = 0$$

which simplifies to $2x - 3y = 0$

The system reduces to a system with one equation and two unknowns. So it has infinite number of solutions. To obtain a non-zero solution, let say $y = 2$; then $x = 3$.

Thus, the non-zero column vector $u = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

To find the general solution, we set $y = a$, where a is a parameter. Substitute $y = a$ into $2x - 3y = 0$ to obtain $x = \frac{3}{2}a$

Thus, $u = \begin{bmatrix} \frac{3}{2}a \\ a \end{bmatrix}$ represents all such solutions.

Matrix method to solve system of linear equations

Any system of linear equations given in the form
$AX = B$, where A is the coefficient matrix and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the unknown vector, can be solved

$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

Question

Using matrix method, solve the following system of equations:

$$2x - y = 4$$

$$3x + 2y = 13$$

Solution

Matrix form of the given system of equations is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

We say $AX = B$, where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

Now $A^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

$$\therefore \quad X = A^{-1}B$$

$$X = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 13 \end{bmatrix} = \begin{bmatrix} -31 \\ 30 \end{bmatrix}$$

Therefore, the solution is $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -31 \\ 30 \end{bmatrix}$, i.e. $x = -31$, $y = 30$.

Question

Using matrix method, solve the following system of equations:

$$x + y + z = 6, \;\; x - y + z = 2, \;\;\; 2x + y - z = 1$$

Solution

Matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

We say $AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

Now $A^{-1} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}$

$$\therefore \;\; X = A^{-1}B$$

$$X = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -6 \end{bmatrix}$$

Therefore, the solution is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -6 \end{bmatrix}$, i.e. $x = 6$, $y = 12$ and $z = -6$.

Augmented matrix

If $A = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the augmented matrix of A and B

$$[A|B] = \begin{bmatrix} 1 & -2 & \vdots & 2 \\ 0 & 4 & \vdots & 3 \end{bmatrix}$$

is created by joining the columns of two matrices.

One of the most common uses of augmented matrix is to use them to solve the linear system of equations.

$$\begin{aligned} x + y - 2z &= 1 \\ x + 2y + z &= 12 \\ 5x + y + 3z &= 3 \end{aligned} \sim \left[\begin{array}{ccc:c} 1 & 1 & -2 & 1 \\ 1 & 2 & 1 & 12 \\ 5 & 1 & 3 & 3 \end{array}\right]$$

Another important use of augmented matrix is to define inverse matrix:

$$[A|I] \sim [I|A^{-1}]$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Example

Use Augmented matrix to solve the system of linear equations

$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

Solution

Solve yourself.

Characteristic Matrix

If a A be a square matrix, then the matrix $A - \lambda I$ is the characteristic matrix of A, where I is identity matrix and λ is a parameter.

For example, if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then

Characteristic polynomial

The polynomial $|A - \lambda I|$ is the characteristic polynomial of A, where I is identity matrix

Characteristic Equation

The equation $|A - \lambda I| = 0$ is characteristic equation of A, where I is identity matrix

Characteristic Root / Eigen Value

The roots of the equation $|A - \lambda I| = 0$ are characteristic roots or Eigen value of A, where I is identity matrix.

Question

Find the characteristic equation and all the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution

Firstly, the characteristic matrix

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Secondly, the characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)^2 = 0$$

This is the required characteristic equation.

The characteristic roots (Eigen values) are $\lambda = 1, 2, 2$.

Question

Find the Eigenvalues of the matrix $A = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans. $\lambda = 0,\ 3,\ 15$

Cayley-Hamilton theorem

Every square matrix satisfies its own characteristic equation.

Example

State the Cayley Hamilton theorem. Verify the theorem for the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and hence find A^{-1}.

Solution

Solve yourself.

Question

Define with example: inner product of two vectors, orthogonal vectors, orthonormal vectors and orthonormal vectors. Also, mention properties of inner product in $R_{n,1}$ and exemplify inner product space.

Inner Product of Vectors

The inner product of two vectors (also called the dot product or scalar product) defines multiplication of vectors. The inner product is only defined for vectors of the same dimension. The inner product of two vectors is denoted by $\langle \vec{U}, \vec{V} \rangle$ or $\vec{U}.\vec{V}$ and defined by

$\langle \vec{U}, \vec{V} \rangle = \vec{U}.\vec{V} = \sum_{i=1}^{n} U_i V_i$

For example, if $\vec{U} = (1, 6, 7, 4)$ or $\begin{bmatrix} 1 \\ 6 \\ 7 \\ 4 \end{bmatrix}$ and $\vec{V} = (3, 2, 8, 3)$ or $\begin{bmatrix} 3 \\ 2 \\ 8 \\ 3 \end{bmatrix}$, then

$\langle \vec{U}, \vec{V} \rangle = \vec{U}.\vec{V} = 1.3 + 6.2 + 7.8 + 4.3 = 83$

Orthogonality of vectors

Two vectors are orthogonal to each other if their inner product equals zero. In two-dimensional space this is equivalent to saying that the vectors are perpendicular, or that the only angle between them is $90°$ angle. For example, the vectors $\vec{U} = [2, 1, -2, 4]$ and $\vec{V} = [3, -6, 4, 2]$ are orthogonal because

$$\vec{U}.\vec{V} = [2, 1, -2, 4].[3, -6, 4, 2] = 2(3) + 1(-6) - 2(4) + 4(2) = 0$$

Normal Vector

A normal vector (or unit vector) is a vector of length 1. Any vector with an initial length > 0, can be normalized by dividing each component in it by the vectors length. For example, if $\vec{V} = (2, 4, 1, 2)$ then a normal vector parallel to $\vec{V}$ is

$$\vec{U} = \frac{\vec{V}}{|\vec{V}|} = \frac{(2, 4, 1, 2)}{\sqrt{2^2 + 4^2 + 1^2 + 2^2}} = \left(\frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5}\right)$$

Clearly $|\vec{U}| = \sqrt{2/5^2 + 4/5^2 + 1/5^2 + 2/5^2} = 1$, so $\vec{U}$ is a normal vector.

Orthonormal Vectors

Vectors of unit length that are orthogonal to each other are said to be orthonormal. For example, the vectors

$\vec{U} = \left(\frac{2}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{4}{5}\right)$ and $\vec{V} = \left(\frac{3}{\sqrt{65}}, -\frac{6}{\sqrt{65}}, \frac{4}{\sqrt{65}}, \frac{2}{\sqrt{65}}\right)$ are orthonormal because

$$|\vec{U}| = \sqrt{\frac{2}{5}^2 + \frac{1}{5}^2 + \left(-\frac{1}{5}\right)^2 + \frac{4}{5}^2} = 1$$

$$|\vec{V}| = \sqrt{\frac{3}{\sqrt{65}}^2 + \frac{4}{5}^2 + \left(-\frac{6}{\sqrt{65}}\right)^2 + \frac{2}{\sqrt{65}}^2} = 1$$

$$\begin{aligned}\vec{U}.\vec{V} &= \left(\frac{2}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{4}{5}\right).\left(\frac{3}{\sqrt{65}}, -\frac{6}{\sqrt{65}}, \frac{4}{\sqrt{65}}, \frac{2}{\sqrt{65}}\right) \\ &= \frac{2}{5}.\frac{3}{\sqrt{65}} + \frac{1}{5}.\left(-\frac{6}{\sqrt{65}}\right) + \left(-\frac{2}{5}\right).\frac{4}{\sqrt{65}} + \frac{4}{5}.\frac{2}{\sqrt{65}} = 0\end{aligned}$$

Gram-Schmidt Orthonormalization Process

The Gram-Schmidt orthonormalization process is a method for converting a set of vectors into a set of orthonormal vectors. It basically begins by normalizing the first vector under consideration and iteratively rewriting the remaining vectors in terms of themselves minus a multiplication of the already normalized vectors.

Question

Consider the following set of vectors in R2 (with the conventional inner product)

$$S = \left\{V_1 = \begin{bmatrix}3\\1\end{bmatrix}, V_2 = \begin{bmatrix}2\\2\end{bmatrix}\right\}$$

Now, perform Gram-Schmidt, to obtain an orthonormal set of vectors.

Solution

Orthogonalization:

$U_1 = V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$U_2 = V_2 -$ projection of V_2

$= V_2 - (V_2 . U_1)U_1$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - (2.3 + 2.1) \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 8 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 24 \\ 8 \end{bmatrix}$

$= \begin{bmatrix} -22 \\ -6 \end{bmatrix}$

Orthonormalization:

$$U_1 = \frac{U_1}{|U_1|} = \frac{1}{\sqrt{3^2 + 1^2}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$U_2 = \frac{U_2}{|U_2|} = \frac{1}{\sqrt{(-22)^2 + (-6)^2}} \begin{bmatrix} -22 \\ -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

Question

Apply Gram-Schmidt orthogonalization process to the vectors $V_1 = [1, 0, 1]$, $V_2 = [1, 0, -1]$ and $V_3 = [0, 3, 4]$ to obtain an orthogonal basis.

Solution

First normalize $V_1 = [1, 0, 1]$

$$U_1 = \frac{V_1}{||V_1||} = \frac{[1, 0, 1]}{\sqrt{1^2 + 0^2 + 1^2}} = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

Then normalize $V_2 = [1, 0, -1]$

$$U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|}$$

$$V_2 - (V_2.U_1)U_1 = [1, 0, -1] - \left([1, 0, -1].\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1] - \left(1.\frac{1}{\sqrt{2}} + 0.0 + (-1).\frac{1}{\sqrt{2}}\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1] - \left(1.\frac{1}{\sqrt{2}} + 0.0 + (-1).\frac{1}{\sqrt{2}}\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1]$$

$$\therefore\ U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|} = \frac{[1, 0, -1]}{\sqrt{1^2 + 0^2 + (-1)^2}} = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]$$

Now normalize $V_3 = [0, 3, 4]$

$$U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|}$$

$$\begin{aligned} V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2 \\ &= [0, 3, 4] - \left([0, 3, 4].\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right] \\ &- \left([0, 3, 4].\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right] \end{aligned}$$

$$= [0, 3, 4] - [2, 0, 2] - [2, 0, -2]$$

$$= [0, 3, 0]$$

$$\therefore\ \ U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|} = \frac{[0, 3, 0]}{\sqrt{0^2 + 3^2 + 0^2}} = [0, 1, 0]$$

Therefore, the orthonormal basis is $\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$, $\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]$, $[0, 1, 0]$

Question

Use Gram-Schmidt orthogonalization process to transform the following basis into an orthonormal basis: $V_1 = [1, 1, 1]$, $V_2 = [0, -1, 1]$ and $V_3 = [0, 1, 1]$. Hence verify your result whether they are orthonormal.

Solution

First normalize $V_1 = [1, 1, 1]$

$$U_1 = \frac{V_1}{\|V_1\|} = \frac{[1,1,1]}{\sqrt{1^2+1^2+1^2}} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

Then normalize $V_2 = [0, -1, 1]$

$$U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|}$$

$$V_2 - (V_2.U_1)U_1 = [0,-1,1] - \left([0,-1,1].\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]\right)\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

$$= [0,-1,1] - \left(0.\frac{1}{\sqrt{3}} + (-1).\frac{1}{\sqrt{3}} + 1.\frac{1}{\sqrt{3}}\right)\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

$$= [0,-1,1]$$

$$\therefore\ U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|} = \frac{[0,-1,1]}{\sqrt{0^2+(-1)^2+1^2}} = \left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

Now normalize $V_3 = [0, 1, 1]$

$$U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|}$$

$$\begin{aligned} V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2 \\ &= [0,1,1] - \left([0,1,1].\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]\right)\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] \\ &- \left([0,1,1].\left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\right)\left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \end{aligned}$$

$$= [0,1,1] - \frac{2}{\sqrt{3}}.\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] - 0.\left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$= [0,1,1] - \left[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right] - [0,0,0]$$

$$= [-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}]$$

$$\therefore\ U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|} = \frac{[-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}]}{\sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}} = \left[-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

Therefore, the orthonormal basis is $\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$, $[-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}]$, $\left[-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$

Question

Use Gram-Schmidt orthogonalization process to convert the vectors $V_1 = [1, 0, 2, 1]$, $V_2 = [2, 2, 3, 1]$ and $V_3 = [1, 0, 1, 0]$ into an orthonormal vectors.

Solution

First normalize $V_1 = [1, 0, 2, 1]$

$$U_1 = \frac{V_1}{\|V_1\|} = \frac{[1, 0, 2, 1]}{\sqrt{1^2 + 0^2 + 2^2 + 1^2}} = \left[\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

Then normalize $V_2 = [2, 2, 3, 1]$

$$U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|}$$

$$V_2 - (V_2.U_1)U_1 = [2, 2, 3, 1] - \left([2, 2, 3, 1].\left[\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]\right)\left[\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

$$= [2, 2, 3, 1] - \frac{9}{\sqrt{6}}\left[\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

$$= \left[\frac{1}{2}, 2, 0, -\frac{1}{2}\right]$$

$$\therefore\ U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|} = \frac{\left[\frac{1}{2}, 2, 0, -\frac{1}{2}\right]}{\sqrt{\left(\frac{1}{2}\right)^2 + 2^2 + 0^2 + \left(-\frac{1}{2}\right)^2}} = \left[\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right]$$

Now normalize $V_3 = [1, 0, 1, 0]$

$$U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|}$$

$$V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2$$

$$= [1,0,1,0] - \left([1,0,1,0].\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]\right)\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$$

$$- \left([1,0,1,0].\left[0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]\right)\left[0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$$

$$= [1,0,1,0] - \frac{2}{\sqrt{3}}.\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right] - 0.\left[0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$$

$$= [1,0,1,0] - \left[\frac{2}{3},\frac{2}{3},\frac{2}{3}\right] - [0,0,0]$$

$$= [-\frac{2}{3},\frac{1}{3},\frac{1}{3}]$$

$$\therefore \quad U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|} = \frac{[-\frac{2}{3},\frac{1}{3},\frac{1}{3}]}{\sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}} = \left[-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right]$$

Therefore, the orthonormal basis is $\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, $[-\frac{2}{3},\frac{1}{3},\frac{1}{3}]$, $\left[-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right]$