Noakhali Science & Technology University

Assignment Title: Matrices, Vector Analysis and Co-ordinate Geometry

Course Code: MATH 2105

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Date: 04/07/2023

Green's Theorem :

Statement : It's a fundamental result in vectors calculus that retakes a line integral around a simple closed curve in a plane to a double integral of the region bounded by the curve.

Proof : Let, 'c' be positively oriented piecewise smooth, simple closed curve plane, and let 'D' be the region bounded by 'c'. If $P(x,y)$ & $Q(x,y)$ have continous first-order partial derivatives on an open region containing 'D'. then the line integral of the vector field.

$F = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$ around c is equal to the double integral of the curl of F over D.

$$\oint F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot dA$$

Where dr is a vector tangent to c with magnitude equal to differential arc length & $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are the partial derivatives of Q and P with respect to x & y respectively.

Example :

Suppose we have a vector field $F(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$ & we want to calculate the line integral of F around the curve c, which is the circle centered at the origin with a radius of 2. We need to find the curl of F & the area bounded by the curve.

The curl of F is given by,

$$\nabla \times F = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \quad \text{where } P(x,y) = -y$$
$$Q(x,y) - x$$

The partial derivatives, we get,

$$\frac{\partial P}{\partial y} = -1 \quad \& \quad \frac{\partial Q}{\partial x} = 1$$

The curl of F is $(1 - (-1)) = 2$

The area bounded by the curve,

$$\pi r^2 = 4\pi$$

Applying Green's theorem,

The curl of $F = \oint_c F \cdot dr = \iint_D (\nabla \times F) \cdot dA$

where D is the region bounded by c.

Since, the curl of F is constant and equal to 2. The double integral simplifies to,

$$\iint_D 2\,dA = 2 \iint_D dA = 2$$

Area of $D = 2 \cdot 4\pi = 8\pi$

The line integral of F around the circle c is 8π.

Stokes Theorem :-

Statement: Stoke's theorem states that the circulation of a vector field around a closed curve is equal to the flux of the curl of the vector field through the surface bounded by that curve.

Proof: Let, S be an oriented, piecewise smooth & bounded surface in a ~~there~~ three dimensional space. & Let C be its positively oriented boundary curve. If F is a vector field with continous first order partial derivatives on an open region containg S, then the circulation of F around C is equal to the flux of the curl of F through S.

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot ds$$

Where. $\nabla \times F$ denotes the curl of F, dr is a vector tangent to C with magnitude equal to the differential arc length, & with magnitude equal to the differential area.

Example:

consider a vector field $F(x,y,z) = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix}$ & let S be the surface of the hemisphere defined by $x^2+y^2+z^2 = 4$ with $z=0$ we want to calculate flux of $\nabla \times F$ through S.

1. $\nabla \times F = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{bmatrix}$

we find. $\nabla \times F = \begin{bmatrix} 2z - 0 \\ 2z - 0 \\ 2y - 2x \end{bmatrix} = \begin{bmatrix} 2z \\ 2z \\ 2y - 2x \end{bmatrix}$

$$S = \iint_S (\nabla \times F) \cdot ds = \iint_S \begin{bmatrix} 2z \\ 2z \\ 2y-2x \end{bmatrix} \cdot n \cdot ds$$

where n is the unit of outward normal vector to the surface S and ds the differential area element n, so,

The normal vector n is simply $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The flux integral simplifies to,

$$\iint_S (\nabla \times F) \cdot ds = \iint_S \begin{bmatrix} 2z \\ 2z \\ 2y-2x \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot ds$$

$$= \iint_S 2z \cdot ds$$

To evaluate this integral we need to parameterise the surface e, we can use spherical coordinates, $x = 2\sin\theta \cdot \cos\phi$ and $z = 2\cos\theta$ with θ ranging 0 to $\pi/2$ and ϕ ranging 0 to 2π.

Then,

$$\iint_S 2z \cdot ds = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\theta \cdot 2\sin\theta \cdot 2\,d\theta \cdot d\phi$$

$$= 8\int_0^{2\pi}\int_0^{\pi/2} (\cos\theta \cdot \sin\theta)\,d\theta \cdot d\phi$$

$$= 8\int_0^{2\pi}\left[-\frac{1}{2}\cos^2\theta\right]_0^{\pi/2} d\phi$$

$$= 8\int_0^{2\pi}\left(-\frac{1}{2}\right) d\phi$$

$$= -8\pi$$

Hence, the hemisphere surface S is -8π.

Gauss Divergence Theorem :

Statement: The divergence theorem states that the flux of a vector field through a closed surface is equal to the net amount of the vector field's source or sink within the volume enclosed by the surface.

Proof:

Let V be a bounded region in three dimensional space with a closed surface s that encloses V and let F be a vector field with continous first order partial derivatives defined on an open region enclosing vector. Then the flux of F through S is equal to the triple integral of the divergence of F over V.

$$\iint_S F \cdot ds = \iiint_V \nabla \cdot E \cdot dV.$$

Example: Use Divergence theorem to evaluate $\int_s \vec{A} \cdot \vec{ds}$ where $A = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$.
Where s is the surface of the sphere, $x^2+y^2+z^2 = a^2$

Here,

$$\iint_s \vec{A} \cdot \vec{ds} = \iiint \text{div} A \cdot dv$$

$$= \iiint \text{div } A \cdot dv$$

$$= \iiint \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3 i + y^3 j + z^3 k)\, dv$$

$$= \iiint (3x^2 + 3y^2 + 3z^2)\, dv$$

$$= 3\iiint (x^2+y^2+z^2)\, dv$$

On putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ we get,

$$3\iiint r^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = 3 \times 8 \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= 24\,(\phi)_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\,\left[\frac{r^5}{5}\right]_0^a$$

$$= \frac{12\pi a^5}{5}.$$ (Ans).