

# DSP: Z-Transform and Basic of Filters

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## Z Transform

- Discrete time LTI system z transform is used.
- Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z.
- Biliteral/Two sided

$$X(z) = Z(x(n)) = \sum_{n=-\infty}^{n=\infty} x(n)Z^{-n}$$

- Unilateral/One sided

$$X(z) = Z(x(n)) = \sum_{n=0}^{n=\infty} x(n)Z^{-n}$$

- if  $x[n] = 0$  for  $n < 0$ , then Biliteral = Unilateral

## Geometric Series Sum formula

- Infinite

- $\sum_{-\infty}^{\infty} c^n = \frac{1}{1-c}$

$$0 < |c| < 1$$

- Finite

- $c \neq 1$

$$\sum_0^{N-1} c^n = \frac{c^N - 1}{c - 1}, \sum_0^N c^n = \frac{c^{N+1} - 1}{c - 1}$$

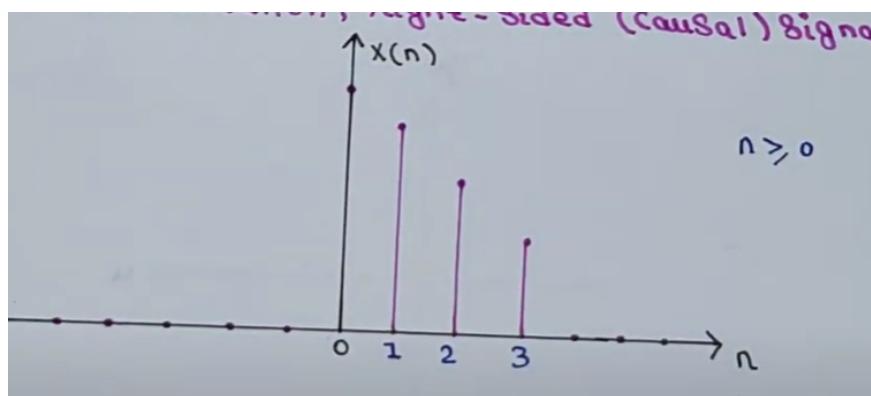
- $c = 1$

$$\sum_0^{N-1} c^n = N, \sum_0^N c^n = N + 1$$

( $c$  is a complex constant)

## ROC

- The set of  $Z$  for which  $X(z)$  is converges (gives finite value)/the set of points in  $Z$ -plane for which  $X(z)$  is converges is called *Region Of Converges*.
- If there's no value of  $z$  where  $X(z)$  is converges, then sequence of  $x(n)$  is said to be having no  $Z$  transform.
- ROC for finite duration signal, right-sided (causal) signal**



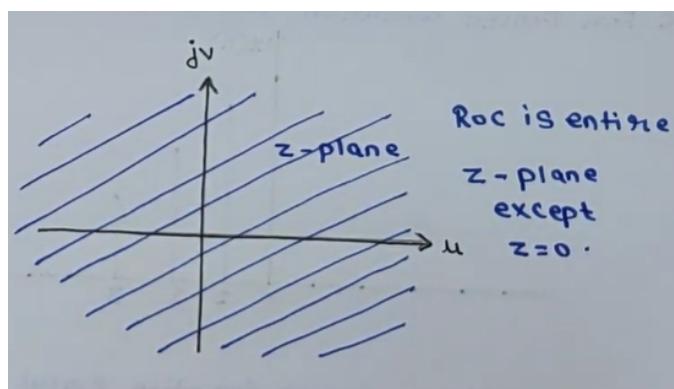
$N$  samples,  $0 \leq n \leq N - 1$

$$x(n) = \{x(0), x(1), \dots, x(N-1)\}$$

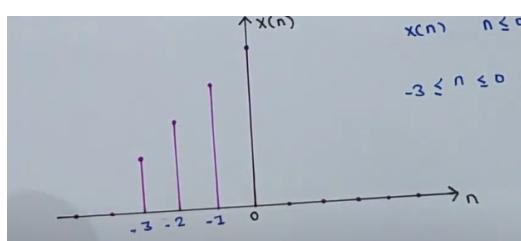
$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n} = x(0) + x(1)z^{-1} + \dots + x(N-1)z^{-(N-1)}$$

if  $z = 0$ ,  $X(z)$  is infinite.

$X(z)$  exists for all values of  $z$ , except  $z = 0$



- ROC for finite duration, left sided (anti causal)**

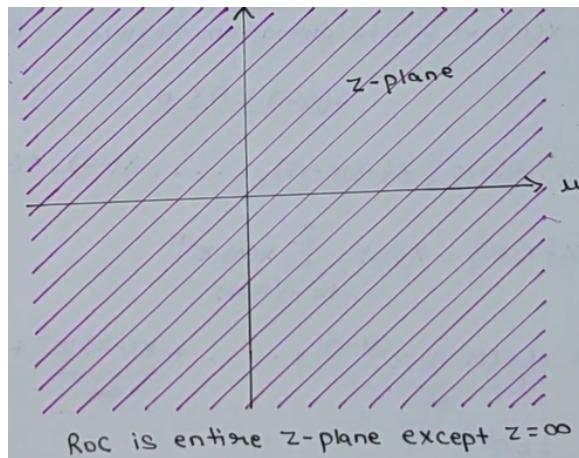


Defined :  $-(N-1) \leq n \leq 0$

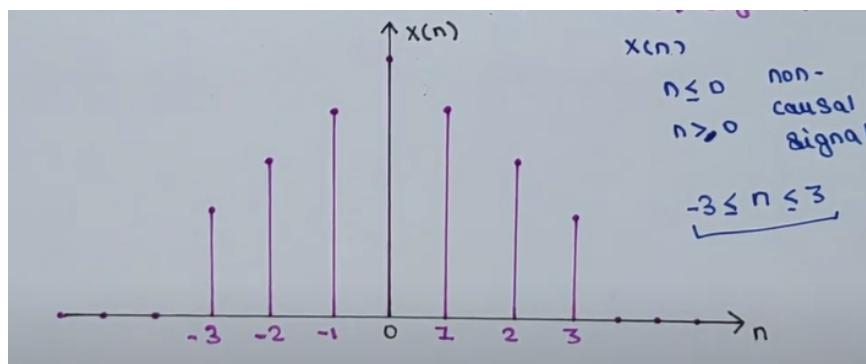
$$x(n) = \{x(-(n-1), \dots, x(0)\}$$

$$X(z) = \sum_{n=-(N-1)}^0 x(n)z^{-n} = x(-(N-1))z^{N-1} + \dots + x(0)$$

if  $z = \infty$ ,  $X(z)$  is infinite.  $X(z)$  exists for all values of  $z$ , except  $z = \infty$ .



- **ROC for finite duration, two sided (non causal signal)**

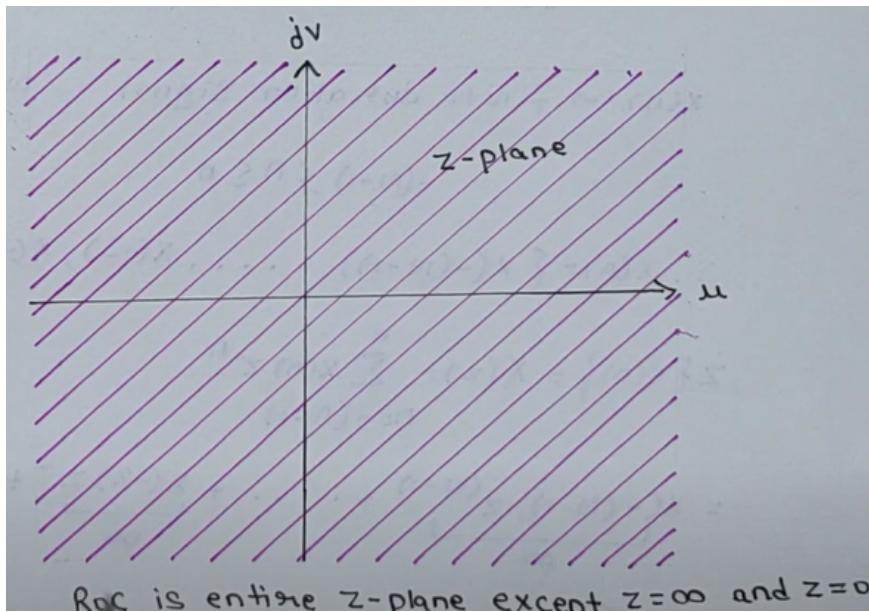


$$-m \leq n \leq +m, m = \frac{N-1}{2}, N \text{ samples}$$

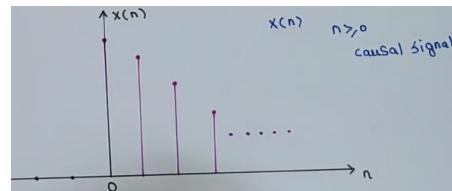
$$x(n) = \{x(-m), \dots, x(0), \dots, x(m)\}$$

$$X(z) = \sum_{-m}^{+m} x(n)z^{-n} = x(-m)z^m + \dots + x(0) + \dots + x(m)z^{-m}$$

$z = 0, \infty$ , there are no finite value. ROC = entire  $Z - plane$  except  $Z = 0, \infty$



- ROC for infinite duration, right-sided (causal) signal



$$\text{Let, } x(n) = r^n; n > 0$$

$$X(z) = \sum_{n=0}^{\infty} (rz^{-1})^n$$

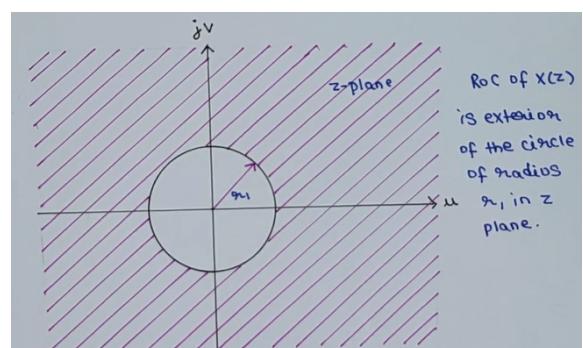
if  $0 < |rz^{-1}| < 1$ ,

$$X(z) = \frac{1}{1-rz^{-1}}$$

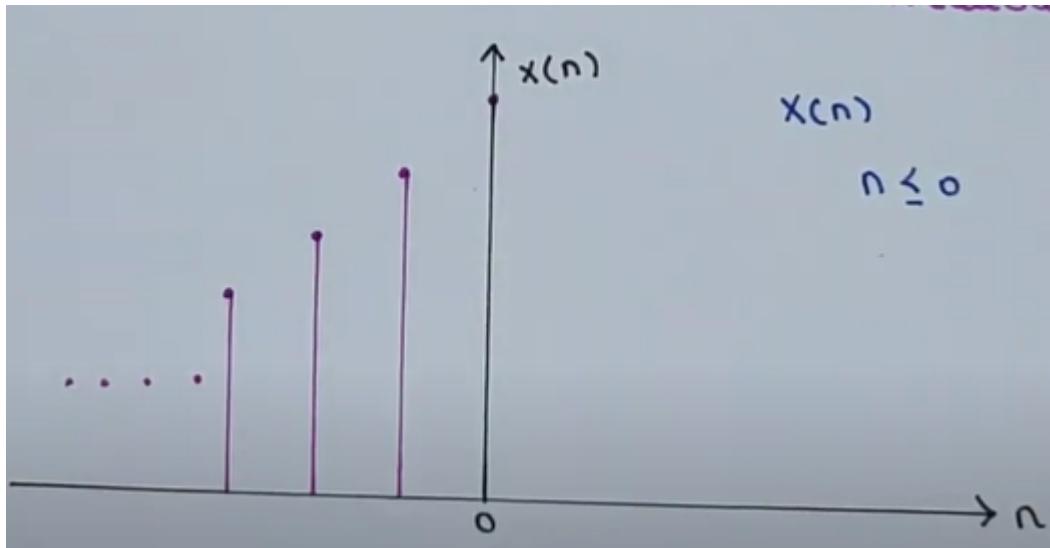
The condition to be satisfied for the convergence of  $X(z)$

$$0 < |rz^{-1}| < 1$$

$$|rz^{-1}| < 1 \Rightarrow |z| > |r|$$



- ROC for infinite duration, left-sided(anti causal) signal



$$X(z) = \sum_{n=-\infty}^0 (r_2 z^{-1})^n = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$$

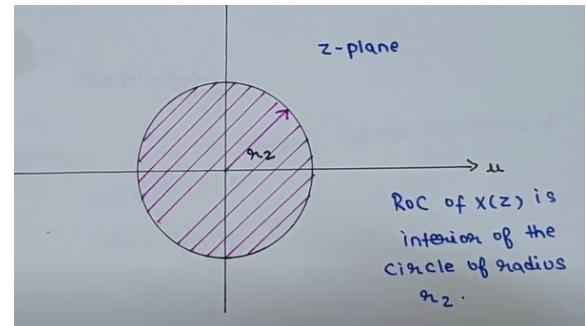
if,  $0 < |r_2^{-1} z| < 1$

$$X(z) = \frac{1}{1 - r_2^{-1} z}$$

Condition to be satisfied for the convergence,

$$|r_2^{-1} z| < 1$$

$$\Rightarrow |z| < |r_2|$$



- ROC for infinite duration, two-sided (non causal) signal

$$\text{Let, } x(n) = r_1^n u(n) + r_2^n y(-n), -\infty \leq n \leq +\infty$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} [r_1^n u(n) + r_2^n y(-n)] z^{-n} \\ &= \sum_{n=-\infty}^0 r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^n \\ &= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n} \\ &= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1 z^{-1})^n \end{aligned}$$

If  $0 < |r_2^{-1}z| < 1$  and  $0 < |r_1z^{-1}| < 1$  then

$$X(z) = \frac{1}{1-r_2^{-1}z} + \frac{1}{1-r_1z^{-1}}$$

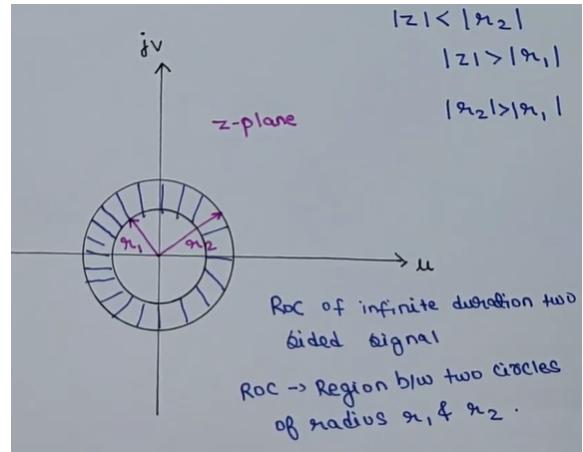
Converges if,

$$|r_1z^{-1}| < 1$$

$$\Rightarrow |r_1| < |z|$$

$$|r_2^{-1}z| < 1$$

$$\Rightarrow |z| < |r_2|$$



## Properties of Z Transform

- Linearity Property :** The linearity property of Z-transform states that the Z-transform of a weighted sum of two discrete time signals is equal to the weighted sum of individual Z-transform of that system.

$$a_1x_1(n) + a_2x_2(n) \leftrightarrow a_1X(z) + a_2X(z)$$

- Shifting Property :** shifting of  $m$ -units obtained by multiplying  $z^m$

$$x(n) \leftrightarrow X(z), -\infty < n < +\infty$$

$$x(n-m) \leftrightarrow X(z)z^{-m}$$

$$x(n+m) \leftrightarrow X(z)z^m$$

- Shifting Property of One-sided**

$$x(n) \leftrightarrow X(z), 0 < n < +\infty$$

$$x(n-m) \leftrightarrow z^{-m}X(z) + \sum_{i=1}^m x(-i)z^{-(m-i)}$$

$$x(n+m) \leftrightarrow z^mX(z) - \sum_{i=1}^m x(i)z^{(m-i)}$$

- **Multiplication property**

$x(n) \leftrightarrow X(z)$ , then

$$nx(n) \leftrightarrow -z \frac{d}{dx} X(z)$$

- **Multiplication by an exponential sequence property**

$x(n) \leftrightarrow X(z)$ , then

$$a^n x(n) \leftrightarrow X\left(\frac{z}{a}\right)$$

- **Time Reversal Property**

$x(n) \leftrightarrow X(z)$ , then

$$x(-n) \leftrightarrow X(z^{-1})$$

- **Conjugation Property**

$x(n) \leftrightarrow X(z)$ , then

$$x^*(n) \leftrightarrow X^*(z^*)$$

- **Convolution Property**

$x_1(n) \leftrightarrow X_1(z), x_2(n) \leftrightarrow X_2(z)$

$$x_1(n) * x_2(n) \leftrightarrow X_1(z)X_2(z)$$

[General convolution :  $x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)$ ]

- **Correlation property**

$x(n) \leftrightarrow X(z), y(n) \leftrightarrow Y(z)$  then

Z transform of  $r_{xy}(m) \leftrightarrow X(z)Y(z^{-1})$

$$[r_{xy} = \sum_{m=-\infty}^{\infty} x(n)y(n-m)]$$

## Finding Z transform

- $x(n) = u(n)$

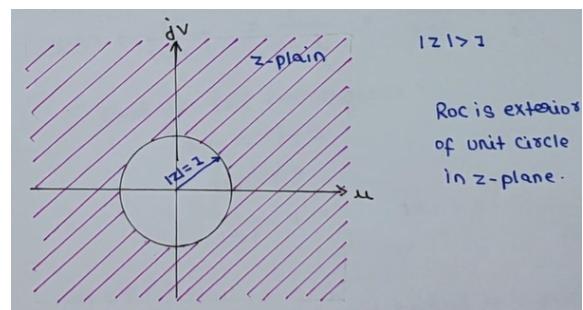
$$x(n) = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} \mu(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$0 < |z^{-1}| < 1,$$

$$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Condition for converges,  $|z^{-1}| < 1$   
 $\Rightarrow |z| > 1$ ,



- $x(n) = (0.3)^n u(n)$

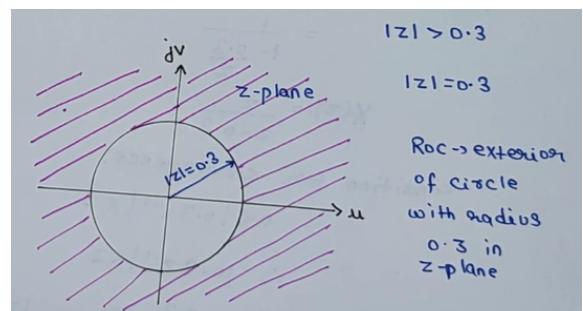
$$x(n) = \begin{cases} 0.3^n; n \geq 0 \\ 0; n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} 0.3^n z^{-n} = \sum_{n=0}^{\infty} (0.3z^{-1})^n$$

$$\text{if } 0 < |0.3z^{-1}| < 1,$$

$$X(z) = \frac{1}{1-(0.3z^{-1})} = \frac{z}{z-0.3}$$

Condition for converges.  $|0.3z^{-1}| < 1 \Rightarrow |z| > 0.3$



- $x(n) = (0.8)^n u(-n-1)$

$$u(n) = \begin{cases} 1; & -n - 1 \geq 0 \\ 0; & n \leq 0 \end{cases}$$

$$x(n) = \begin{cases} 0.8^n; & n \leq -1 \\ 0; & n \leq 0 \end{cases}$$

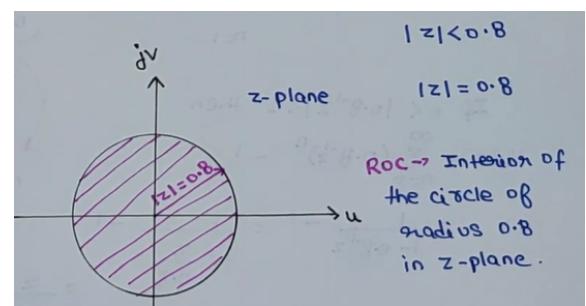
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{-1} 0.8^n z^{-n} = \\ &\sum_{n=1}^{\infty} 0.8^{-n} z^n = \\ &\sum_{n=0}^{\infty} (0.8^{-1}z)^n - 1 \end{aligned}$$

If  $0 < |0.8^{-1}z| < 1$ , then

$$X(z) = \frac{1}{1-0.8^{-1}z} - 1 = \frac{z}{0.8-z}$$

Condition for converges,  $|0.8^{-1}z| <$

$$1 \Rightarrow |z| < 0.8$$



### Poles and Zeroes of Rational Function of Z

$X(z) = \frac{N(z)}{D(z)}$ , if  $X(z)$  expressed as a ratio of two polynomials  $z$  or  $z^{-1}$ , then  $X(z)$  is called rational function of  $z$ .

$$\begin{aligned} X(z) &= \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}} = \frac{b_0}{a_0} \frac{1 + \frac{b_1}{b_0} z^{-1} + \dots + \frac{b_m}{b_0} z^{-m}}{1 + \frac{a_1}{a_0} z^{-1} + \dots + \frac{a_n}{a_0} z^{-n}} \dots (i) \\ &= G \frac{z^{-m} (z^m + \frac{b_1}{b_0} z^{m-1} + \dots + \frac{b_m}{b_0})}{z^{-n} (z^n + \frac{a_1}{a_0} z^{n-1} + \dots + \frac{a_n}{a_0})}, [\text{Scaling Factor, } G = \frac{b_0}{a_0}] \\ &= G \frac{(z-z_1)(z-z_2)\dots(z-z_n)}{(z-p_1)(z-p_2)\dots(z-p_n)} [n = m] \end{aligned}$$

$z_1, z_2, \dots, z_n$  : roots of numerator polynomial, zeros of  $X(z)$ , marked by  $\circ$

$p_1, p_2, \dots, p_n$  : roots of denominator polynomial, poles of  $X(z)$ , marked by  $\times$

- **Inverse Z Transform :**

- $Z^{-1}(x(z)) = x(n) = \frac{1}{2\pi j} \oint_c X(z)z^{n-1}dz$

### Questions:

1 .	a)	Write the advantages of Z-transform system over DTFT system. Show the transfer function in Z-transform with equations. Distinguish between zeros and poles in Z-plane.	8
	b)	If $(Z) = \frac{z^2+z+1}{z^2+0.5z+0.75}$ , Find out: a). its transfer function representation, b). its difference equation representation.	4
	c)	If $X(z) = \frac{2+z+3z^2}{5+9z+0.5z^3}$ , find the values of input and output coefficients.	2

(a)

#### **Advantages of Z-transform over DTFT system:**

- $\delta(n), u(n)$  can't be analyzed by DTFT but Z.
- The transient response of a system **due to initial condition** or due to changing inputs cannot be computed by using DTFT but Z.
- The Z-transform might **exist anywhere in the Z-plane**; the DTFT can only exist on the **unit circle**.
- Z transform allows for the discrete time system **analysis in the frequency domain**. This facilitates the analysis of system characteristics such as **stability, causality, linearity and time and frequency** response.
- Z transform allows the representations of signals and systems in the **frequency domain**.
- It becomes possible to analyze frequency content of signals **determine spectral properties, apply filtering and modulation**.
- Useful for **determining the stability** of DT system.
- It **has convolution property** that simplifies the analysis of linear time invariant system.
- **Simplifies mathematical** operation.

### **The Transfer Function**

The transfer function  $H(z)$  of a DT linear system is defined as the ratio of the Z-transform of the output signal  $Y(z)$  to the Z-transform of the input signal  $X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{n=-\infty}^{+\infty} b_n z^{-n}}{\sum_{m=-\infty}^{+\infty} a_m z^{-m}}$$

$$= \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}}$$

Where,  $b_0, b_1, \dots$  are input coefficient and  $a_0, a_1 \dots$  are output.

if  $a_o = 1$ , then this called the standard form of transfer function.

$$\text{If we can convert, } H(z) = G \frac{(z-z_1)(z-z_2)\dots(z-z_n)}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

Then,  $z_1, z_2 \dots$  are called zeros and  $p_1, p_2 \dots$  are called poles.

### **Distinguish between Zeros and Poles**

Poles	Zeros
A pole of the transfer function $H(z)$ is value of $z$ that makes the denominator zero.	A zero of the transfer function $H(z)$ is a value of $z$ that makes the numerator zero.
Mathematical : $H(z) = N(z)/D(z)$ If $D(z) = 0$ , at $z=z_p$ , then $H(z) \rightarrow \infty$ as $z \rightarrow z_p$	Mathematical : $H(z) = N(z)/D(z)$ If $N(z) = 0$ , at $z=z_s$ , then $H(z) \rightarrow 0$ as $z \rightarrow z_s$
Influence system stability and response	Influence frequency response
Poles outside the unit circle indicate instability	Zeros do not affect stability directly
Represented by "X" in the Z-plane	Represented by "O" in the Z-plane

**(b) Transfer function representation :**  $X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}}$  [ $a_0 = 1$  means normalized] .  $z = e^{j\omega}$

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{1 + 0.5z^{-1} + 0.75z^{-2}} \quad [\text{Dividing by } z^2]$$

$$H(e^{j\omega}) = \frac{e^{2j\omega} + e^{2\omega} + 1}{e^{2j\omega} + 0.5e^{j\omega} + 0.75} \quad [\text{Replace } z = e^{j\omega} \text{ on given } X(z) \text{ of the question}]$$

*Difference equation representation : Transfer function  $\rightarrow$  Inverse Z-transform  $\rightarrow y(n)$*

$$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}+0.75z^{-2}}, [\text{Transfer function}, H(z) = \frac{Y(z)}{X(z)} = ..]$$

$$X(z) + z^{-1}X(z) + z^{-2}X(z) = Y(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z) [\text{cross product}]$$

$$\Rightarrow x[n] + x[n-1] + x[n-2] = y[n] + 0.5y[n-1] + 0.75y[n-2] [\text{Inverse Z-function}]$$

$$\Rightarrow y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]]$$

**(c)**

Idea :  $X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1z^{-1} + \dots + b_mz^{-m}}{a_0 + a_1z^{-1} + \dots + a_nz^{-n}}$ , make given  $X(z)$  this format and compare with this ( $a_0 = 1$ ) where  $b_n = \text{input}$  and  $a_0 + \dots + a_n = \text{output coefficient}$ .

$$X(z) = \frac{2+z+3z^2}{0.5z^3(10z^{-3}+18z^{-2}+1)} = \frac{4z^{-3}+2z^{-2}+6z^{-1}}{10z^{-3}+18z^{-2}+1} [\text{Normalized}]$$

Input coefficient : 4, 2, 6

Output Coefficient : 1, 0, 18, 10

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~~1~~ Show the transfer function in Z-transform with equations. Compare zeros and poles in Z-plane.

~~2~~ If  $H(z) = \frac{Z+1}{Z^2 - 0.9Z + 0.81}$ , Find out: a). its transfer function representation, b). its difference equation representation.

~~3~~ If  $X(z) = \frac{2+z^2+3z^4}{3+4z+z^2}$ , find the values of input and output coefficients.

**(a)** Check the answer above.

**(b)** Transfer function representation:

$$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

$$H(z) = \frac{e^{j\omega+1}}{e^{2j\omega}+0.9e^{j\omega}+0/81}$$

Difference equation representation

$$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow z^{-2}X(z) + z^{-1}X(z) = Y(z) + 0.9z^{-1}Y(z) + 0.82z^{-2}Y(z)$$

$$\Rightarrow x[n-2] + x[n-1] = y[n] + 0.9y[n-1] + 0.81y[n-2]$$

$$(c) X(z) = \frac{3+z^{-2}+2z^{-4}}{z^{-2}+4z^{-3}+3z^{-4}}$$

**Input Coefficients: 3, 0, 1, 0, 2**

**Output Coefficients: 0, 0, 1, 4, 3**

- Given difference equation, find transfer function representation.

$$y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]$$

$$\Rightarrow Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \dots$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \dots$$

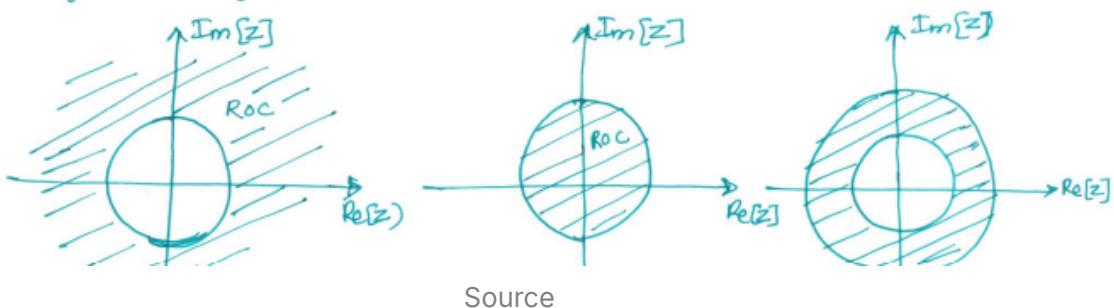
2. a) What is meant by region of convergence? Write the properties of ROC. 3
- b) Determine the Z-transform and ROC of the signal  $x(n) = \left(-\frac{1}{2}\right)^n u[-n-1]$  where  $n \leq -1$ . 4
- c) Shortly describe unit circle in z-domain. A causal LTI system has impulse response  $h[n] = 1 + 2$   
for which the z-transform is  $H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$ , find the region of convergence of  $H(z)$ .

(a) The set of  $Z$  for which  $X(z)$  is converges (gives finite value)/the set of points in  $Z$ -plane for which  $X(z)$  is converges is called *Region Of Converges*.

**Properties:**

### Properties of ROC :

- (1) For a finite duration right-sided signal (causal signal), the ROC will be entire  $z$ -plane except  $z=0$
- (2) For a finite duration left-sided signal (anticausal signal), the ROC will be entire  $z$ -plane except  $z=\infty$
- (3) For a finite duration two sided signal, the ROC will be the entire  $z$ -plane except  $z=0$  and  $z=\infty$
- (4) If  $x[n]$  is right sided and of infinite duration (causal signal), then ROC is outside a circle whose radius is equal to the largest pole magnitude (Fig.1)
- (5) If  $x[n]$  is left-sided and of infinite duration (anticausal signal), then ROC is inside a circle whose radius is equal to the smallest pole magnitude (Fig.2)
- (6) If  $x[n]$  is two sided and of infinite duration, then ROC will be a ring in the  $z$ -plane bounded by smallest & largest pole magnitudes (Fig.3)



$$(b) x(n) = \begin{cases} (-\frac{1}{2}); & n \leq 1 \\ 0; & n \geq 0 \end{cases}$$

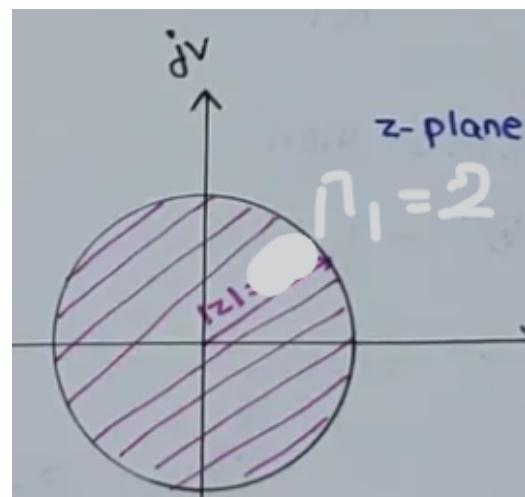
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^1 (-0.5)^n z^{-n} = \\ &\sum_{n=-\infty}^{-1} (-0.5z^{-1})^n = \\ &\sum_{n=1}^{\infty} (-0.5^{-1}z)^n = \\ &\sum_{n=0}^{\infty} (-0.5^{-1}z)^n - 1 \end{aligned}$$

If  $0 < |-0.5^{-1}z| < 1$ , then

$$X(z) = \frac{1}{1-(-0.5^{-1}z)} - 1 = \frac{z}{0.5-z}$$

Condition for converges,

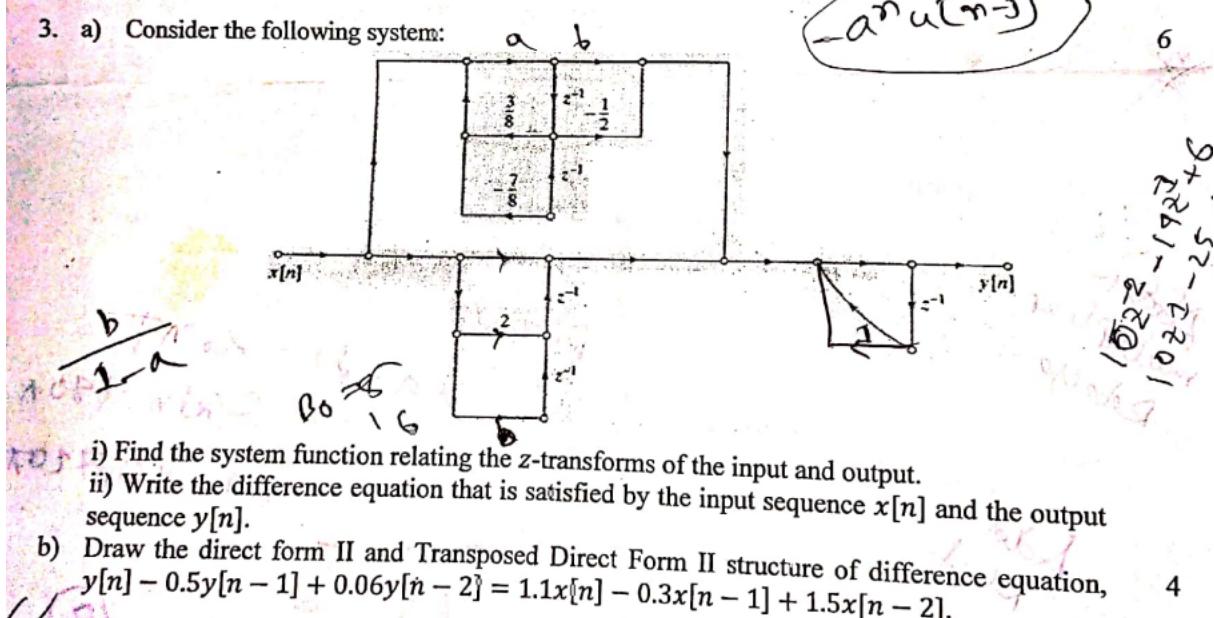
$$|-0.5^{-1}z| < 1 \Rightarrow |z| < 2$$



- a) Consider an LTI system that is stable and for which  $H(z)$ , the z-transform of the impulse response is  $H(z) = \frac{6-7z^{-1}+5z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}}$ . If  $x[n]$  is the input to the system is a unit step sequence. Find the output  $y[n]$  by computing the inverse z-transform of  $Y(z)$
- b) Define Cauchy residue theorem. Evaluate the inverse z-transform of  $X(z) = \frac{1}{1-bz^{-1}}$  where  $|z| > |b|$ , using the complex inversion integral.

7. a) Let a system  $H(z) = \frac{1-0.5z^{-1}}{1-0.3z^{-1}}$ . Describe stability and causality for both the system and inverse system using ROC. show both regions are overlapped. 6

3. a) Consider the following system:



- Differences between Z-transform and Laplace Transform**

Z Transform	Laplace Transform
Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z.	an integral transform that converts a function of a real variable (usually, in the time domain) to a function of a complex variable
Used to analyze discrete time signal	continuous time signal
Uses the complex variable Z	uses the complex variable s
The set of points in z-plane for which $X(z)$ converges is called the ROC of $X(z)$	The set of points in s-plane for which $X(s)$ converges is called the ROC of $X(s)$
Used for discrete time linear system such as digital filters and sampled data control system	Used for the analysis of CT system such as control system and differential equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$F(s) = \int_0^{\infty} f(t)e^{-st}t' \quad \text{where } s = \sigma + i\omega$$

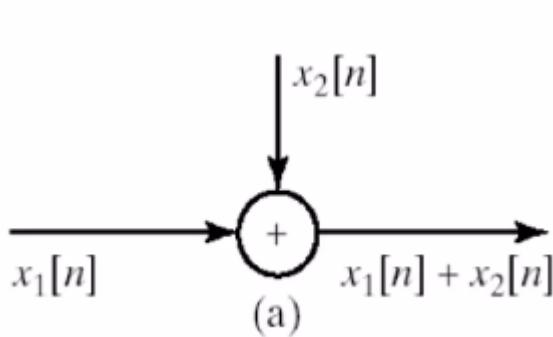
## Digital Filters

A digital filter is a system that performs mathematical operations on a sampled discrete time signal to reduce or enhance certain aspects of that signal.

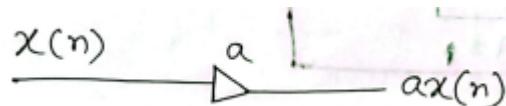
**Types : FIR, IIR**

### Basic element to design Digital Filter

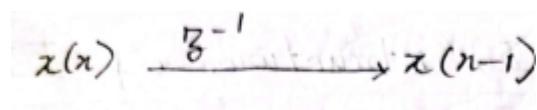
- Adder :



- Multiplexer



- Delay Element



**IIR (Infinite Duration Impulse Response) :** If the impulse response exists infinitely, it is an IIR Filter.

- **Frequency Response of IIR:**

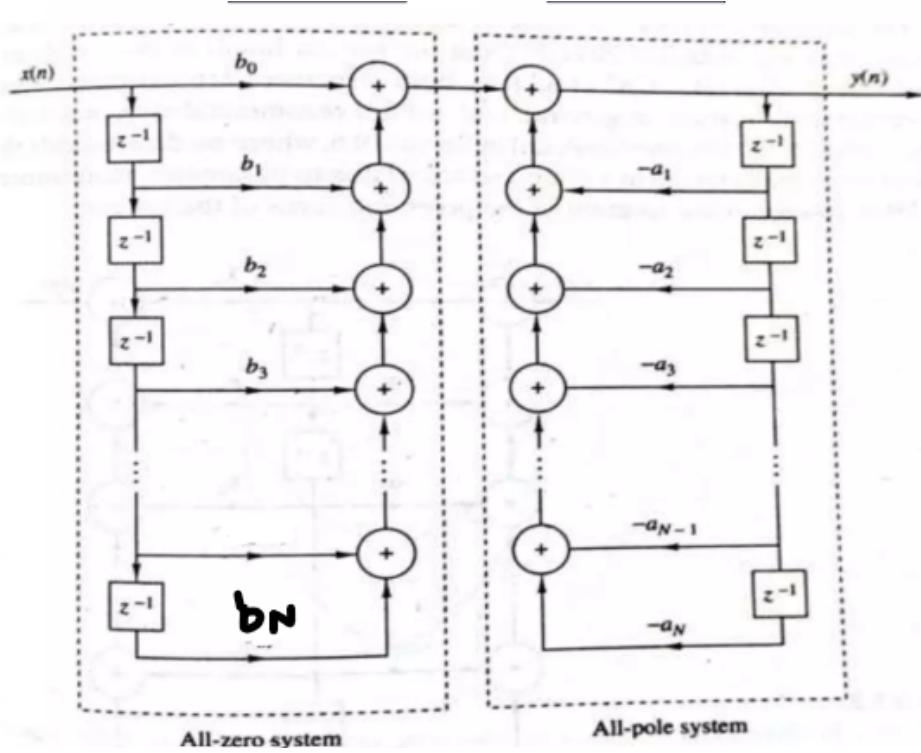
$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^M b_m z^{-m}}{1 + \sum_{n=1}^N a_n z^{-n}}, [a_0 = 1]$$

- **Difference equation of IIR Filter**

$$y(n) = \sum_{m=0}^M b_m x(n-m) - \sum_{m=1}^N a_m y(n-m)$$

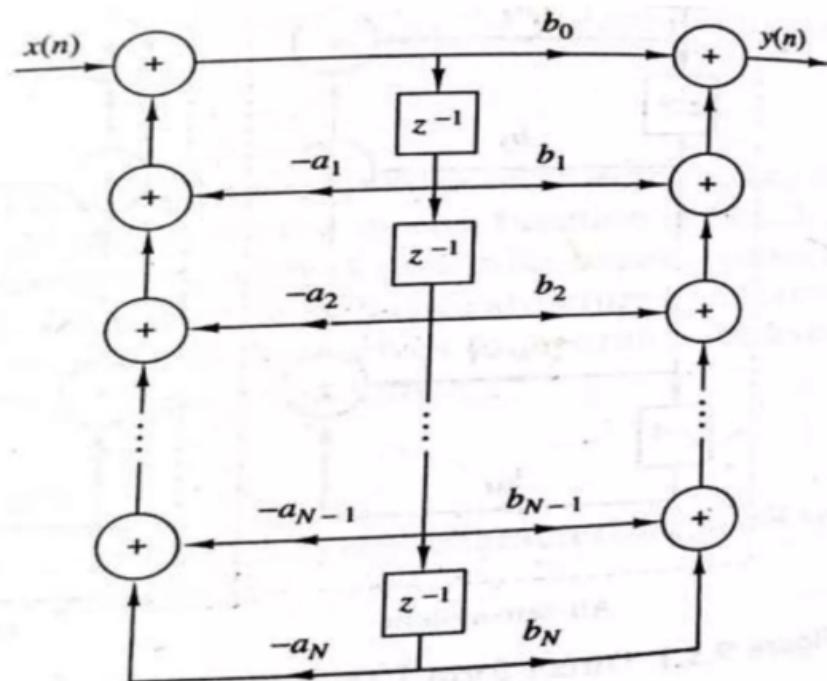
- **$N - th$  order Direct form (I)**

$$y(n) = b_0x(n) + b_1x(n-1) \dots + b_N(n-N) - (a_1x(n-1) + \dots + a_Nx(n-N))$$



- **$N - th$  order direct form (II)**

$$y(n) = b_0x(n) + b_1x(n-1) \dots + b_N(n-N) - (a_1x(n-1) + \dots + a_Nx(n-N))$$



- **Cascade form :**

In this form the frequency response,  $H(z)$  is factored into smaller second section, called bi-quads. The frequency response is then represented as a product of these bi-quads section.

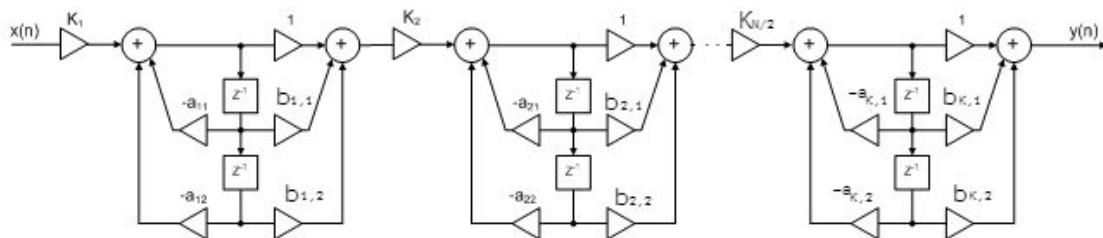
$$H(z) = \frac{B(z)}{A(z)} = b_0 \frac{1 + \frac{b_1}{b_0} z^{-1} + \dots + \frac{b_N}{b_0} z^{-N}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$= b_0 \prod_{k=1}^K \frac{1 + B_{k,1} z^{-1} + B_{k,2} z^{-2}}{1 + A_{k,1} z^{-1} + A_{k,2} z^{-2}} \text{ [for second-order]}$$

$$K = \frac{N}{2} \text{ [Always]}$$

*Cascade form for N – th order*

$$K = \frac{N}{2}$$



**FIR (Finite Duration Impulse Response):** The filters have a finite impulse response function which has finite length of time.

- **Transfer Function**

$M$  : length of filter

$N$  : Order of filter =  $M - 1$

$$H(z) = \sum_{n=0}^{M-1} b_n Z^{-n}$$

$$h(n) = \begin{cases} b_n, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

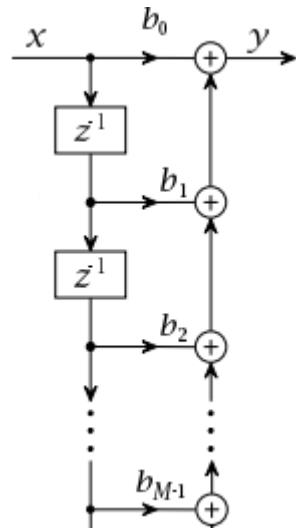
- **Difference equation**

$$y[n] = \sum_{m=0}^{M-1} b_m x(n-m)$$

- **Direct form I if  $M = M$**

Order of filter,  $N = M - 1$

$$N - \text{th order}, y(n) = b_0x(n) + b_1x(n-1) + \dots + b_Nx(n-N)$$



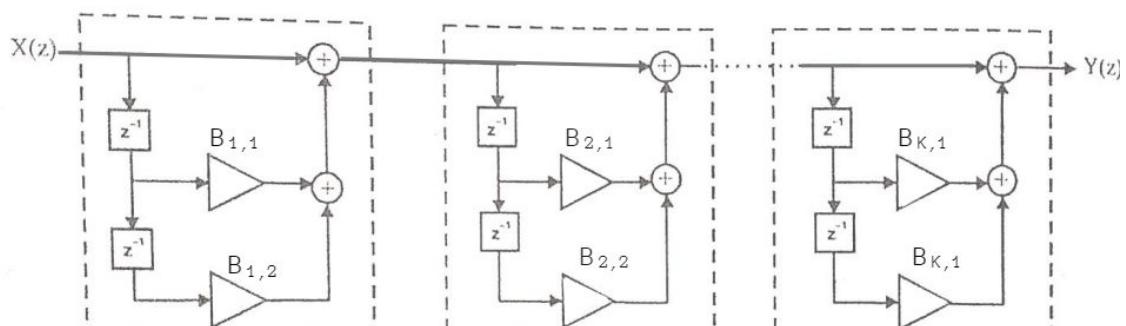
- **Cascade FIR**

For second order section,

$$\begin{aligned} H(z) &= H(z) = \sum_{n=0}^{M-1} b_n Z^{-n} = b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-M+1} \\ &= b_0 \left( 1 + \frac{b_1}{b_0} z^{-1} + \dots + \frac{b_{M-1}}{b_0} z^{-M+1} \right) \\ &= b_0 \prod_{k=1}^K (1 + B_{k,1} z^{-1} + B_{k,2} z^{-2}) \\ K &= \lfloor \frac{M}{2} \rfloor \end{aligned}$$

*Design cascade form of FIR where filter length is M*

$$K = \lfloor \frac{M}{2} \rfloor$$



- **Properties of Linear-phase FIR filter**

Let,  $h(n), 0 < n < M - 1$ , be the impulse response of length M. The frequency response,

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

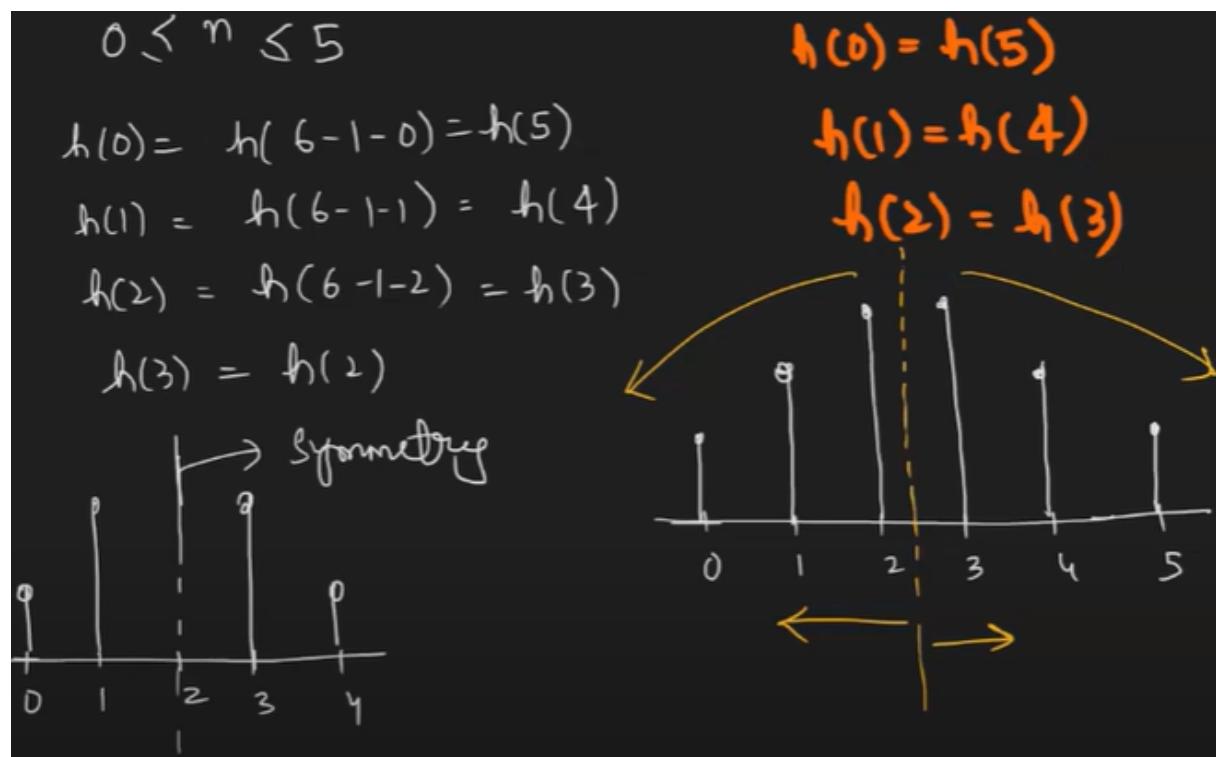
We know,  $H(e^{jw}) = \sum_{n=0}^{M-1} h(n)e^{-jwn}$ ,  $-\pi \leq w \leq \pi$

Where,  $\angle H(e^{jw}) = -\alpha w$  (A system has linear phase if its phase response  $\theta(\omega) = \angle H(e^{j\omega}) = -c\omega$  for all  $\omega$  and any constant  $c$ .)

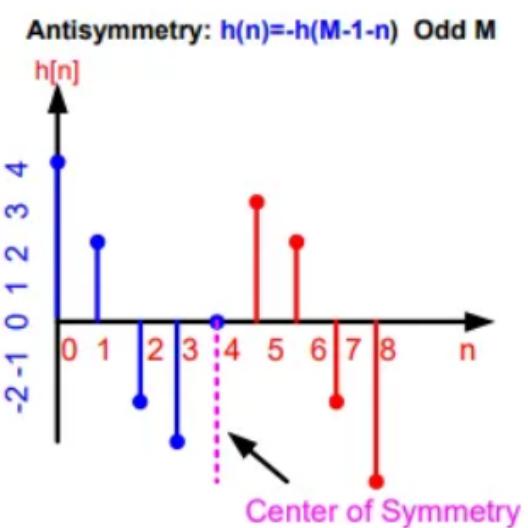
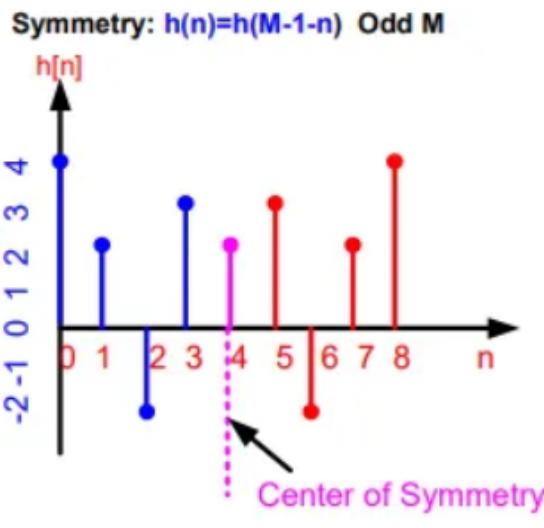
where,  $\alpha$  is a constant phase delay,  $h(n)$  must be symmetric, if  $h(n) = h(M-1-n)$

$0 \leq n \leq M-1$  with  $\alpha = \frac{M-1}{2}$  if  $M$  is odd.

Anti symmetry :  $h(n) = -h(M-1-n)$



- b) Draw the figures of symmetric and anti-symmetric impulse response  $h(n)$  of length  $M$  where  $M$  is 9. | 2



Linear phase system fall into one of 4 categories:

- **M odd,  $h[n]$  is symmetric (Type I)**

$$\begin{aligned}
 H(e^{jw}) &= \sum_{n=0}^M h[n]e^{-jwn} \\
 &= h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + \dots + \\
 &\quad h[\frac{M}{2}]e^{-jw\frac{M}{2}} \\
 &= e^{jwM/2}(h[0](e^{jwM/2} + e^{-jwM/2}) + \dots + h[M/2]) \\
 &= e^{-jwM/2}(h[0].2\cos(wM/w) + h[1].2\cos(w(M/2 - 1)) + \dots + \\
 &\quad h[M/2]) \\
 &= e^{-jwM/2} \sum_{k=0}^{M/2} a_1[k]\cos(wk)
 \end{aligned}$$

As class note,

$$M = \text{Odd}, \alpha = \frac{M-1}{2}$$

$$\begin{aligned}
 H(w) &= \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos(wn) \\
 \Rightarrow H(e^{jw}) &= e^{-jw\frac{M-1}{2}} \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos(wn)
 \end{aligned}$$

- **M even,  $h[n]$  is symmetric (Type II)**

$$\begin{aligned}
 H(e^{jw}) &= h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + \dots + \\
 &\quad h[\frac{M-1}{2}](e^{-jw\frac{M-1}{2}} + e^{-jw\frac{M+1}{2}})
 \end{aligned}$$

$$= e^{-jw\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} b(n) \cos(w(k + \frac{1}{2}))$$

As class note,

$$H(e^{jw}) = e^{-jw\frac{M-1}{2}} \sum_{n=1}^{\frac{M}{2}} b(n) \cos\{w(n - \frac{1}{2})\}$$

- M even,  $h[n]$  is antisymmetric (Type III),
- M odd,  $h[n]$  is antisymmetric (type IV)