

Theory of Computation

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NB: There are many mistakes. **Read at your own risk.**

What is TOC?

- A branch of theoretical CS
- Whether and how efficiently a problem can be solved on computational model, using an algorithm

TOC has major 3 branches.

- Automata theory: deals with various definition and properties of mathematical model
- Computability theory: what can and cannot be computed
- Computational Complexity theory: it groups computable problem based on hardness

Model of Computation: mathematical abstraction of Computer

Definition: describe a object and notations.

Theorem: mathematical statement basis on previously established statement.

Proof: convincing logical argument that statement is true.

Lemma: A minor result (theorem, it's the lemma) to prove another theorem.

Corollaries: A result in which the proof relies heavily.

THEOREM

If m and n are any two whole numbers and

- $a = m^2 - n^2$
- $b = 2mn$
- $c = m^2 + n^2$

then $a^2 + b^2 = c^2$

Proof:

$$\begin{aligned}a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\&= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\&= m^4 + 2m^2n^2 + n^4 \\&= (m^2 + n^2)^2 \\&= c^2\end{aligned}$$

Deductive Proof: If H (hypothesis), then C (conclusion)

- Sequence of statement
- hypothesis to conclusion

Contradiction Proof:

- Assume the theorem is false, this assumption leads to false
Example: $\sqrt{2}$ is irrational.

Induction proof:

- All elements of infinite set have a specific property
- 2 Steps
 - Base case

- Induction Step (Assume $S(k)$, then $S(k+1)$)
- Example: $\sum n = n(n+1)/2$

Contraposition Proof:

- $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$
- Assume $\neg Q$ is true, prove $\neg P$ is true
- Example : If n is even, n^2 is even

Theorem: If n^2 is even, then n is even

Proof by Contrapositive:

If n is not even, n^2 is not even

Assume n is odd

$$\exists k_1 \in \mathbb{Z}, n = 2k_1 + 1$$

$$\text{So } n^2 = (2k_1 + 1)^2 =$$

$$4k_1^2 + 4k_1 + 1 =$$

$$2(2k_1^2 + 2k_1) + 1$$

$$\text{Let } k_2 = 2k_1^2 + 2k_1.$$

$$\text{So } n^2 = 2k_2 + 1$$

Thus n^2 is odd.

Counter Example proof:

- Show an example to disprove the claim

Automata theory

- Study of abstract machines and computational problems that can be solved by these machines

Automata:

- abstract machine/computing device
- mathematical model of a system that involves with input, output, state, transition etc.

Consists of

- States: Circle, description of the status of the system
- Transition: Arrow, input, one state to another

Basic Definition:

1. Symbols

- a. Symbols are indivisible objects or entity that cannot be defined.

2. Alphabets

- a. Finite set of symbols
- b. Σ

3. String

- a. Finite sequence of symbols
- b. Denoted by w, z, y, z

4. Empty String

- a. Denoted by ϵ
- b. The length of the empty string is 0

5. Length of a string

- a. Denoted by $|W|$

6. Power of Alphabets

a. Set of string length k , Σ^k

i. $\Sigma^0 = \{\epsilon\}$

b. Set of all string including empty, $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \dots \cup \Sigma^n$

c. Set of all String w/o empty string, $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \dots \cup \Sigma^n$

7. Concatenation of a string

a. $x = 01$, $y = 10$ concatenation of x and y , $xy = 0110$, $yx = 1001$

8. Language

a. A language over an alphabet is a set of strings over that alphabet.

b. Set of all Σ^*

c. Empty language \emptyset

Membership Problem:

- Given a word and a Language, we want to check word belongs to the language or not, this is called membership problem.

3 requirements of automata:

- Taking input
- Producing output
- May have Temporary storage
- Control unit: can change state according to transition function

Finite automata

- An abstract computing device
- No temporary storage
- Used to recognize pattern
- Accept or reject input depending on pattern

2 types

- DFA (Deterministic Finite Automata)
- NFA (Non-Deterministic Finite Automata)

DFA	NFA
one state transition in DFA	May have more than one
Cannot ϵ	Can use ϵ
Understand as one machine	Multiple machine
have max. one possible next state for one input	May have multiple next possible states for one input
Difficult to construct	Easier
Time less	Executing time more
All DFA = NFA	All NFA \neq DFA
$\delta: Q \times \Sigma \rightarrow Q$	$\delta: Q \times \Sigma \rightarrow 2^Q$

State:

- Description of the Status of system waiting to execute a transition
- Denoted by Circle/Vertex

Transition:

- set of actions to execute when a condition is fulfilled or an event received.
- Denoted by Arrow/Edge

DFA

Deterministic finite automata (or DFA) are finite state machines that accept or reject strings of characters by parsing them through a sequence that is **uniquely determined by each string**.

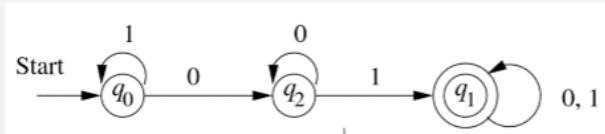
- A formalism for defining languages, consisting of:
 1. A finite set of *states* (Q , typically).
 2. A finite set of *input symbols* (Σ , typically).
 3. A *transition function* (δ , typically).
 4. A *start state* (q_0 , one of the states in Q , typically).
 1. Only one start state
 5. A set of *final states* ($F \subseteq Q$, typically).
 - “Final” and “accepting” are synonyms.
 - May have multiple final states
- So, A DFA is a *five-tuple* notation:

$$A = (Q, \Sigma, \delta, q_0, F)$$

where **A** is the name of the DFA.

Lecture 3:

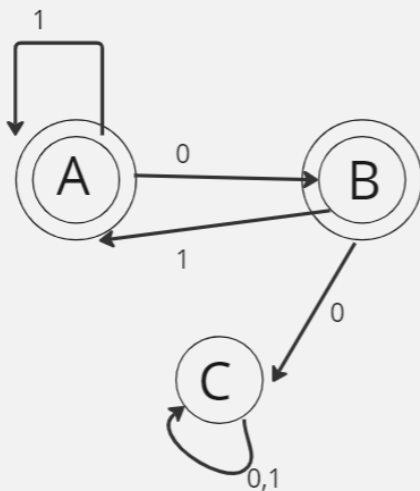
1. Show a transition table for the following diagram/automata:



	0	1
→ q0	q2	q1
q2	q2	q1
*q1	q1	q1

2. Draw a transition diagram from the following table:

	0	1
→ *B	C	A
C	C	C



Extended Transition Functions

- Denoted by $\hat{\delta}$
- Takes state q and string w (where Transition function usually takes only alphabet)
- Induction steps

A recursive algorithm is used to reach the final state, which is as follows:

1. Base condition:

$$\hat{\delta}(q, \epsilon) \rightarrow q$$

2. Recursion rule:

$$\hat{\delta}(q, xa) \rightarrow \delta(\hat{\delta}(q, x), a)$$

Here, $x \in \Sigma^*$ and $a \in \Sigma$. Also, x is a string of characters belonging to the set of the input symbols and a is a single character.

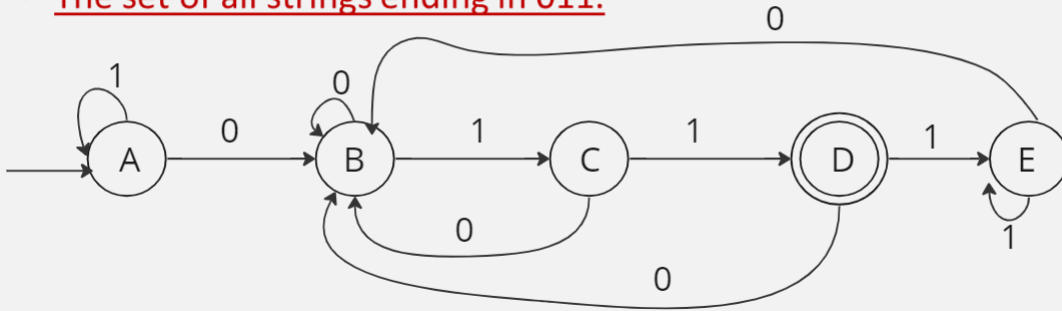
Handwritten derivation of the extended transition function for the string 110101 starting from state q_0 :

$$\begin{aligned} & \hat{\delta}(q_0, 110101) \\ &= \delta(\hat{\delta}(q_0, 11010), 1) \\ &= \delta(\delta(\hat{\delta}(q_0, 1101), 0), 1) \\ &= \delta(\delta(\delta(\hat{\delta}(q_0, 110), 1), 0), 1) \\ &= \delta(\delta(\delta(\delta(\hat{\delta}(q_0, 11), 0), 1), 0), 1) \\ &= \delta(\delta(\delta(\delta(\delta(\hat{\delta}(q_0, 1), 1), 0), 1), 0), 1) \\ &= \delta(\delta(\delta(\delta(\delta(\delta(q_1, 1), 0), 1), 0), 1) \\ &= \delta(\delta(\delta(\delta(q_2, 1), 0), 1) \\ &= \delta(\delta(q_3, 0), 1) \\ &= \delta(q_1, 1) \\ &= q_0 \end{aligned}$$

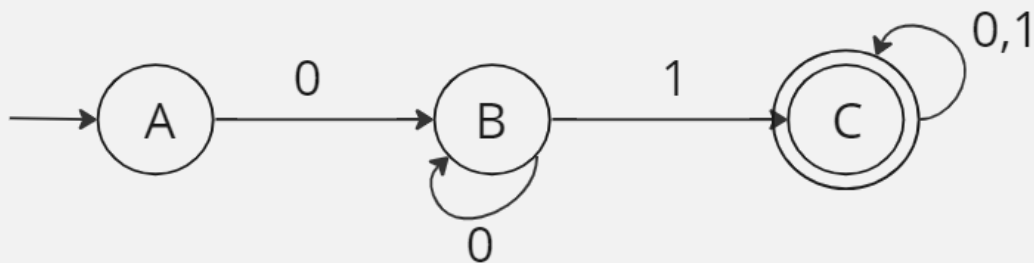
From : Pollock bhai

Lecture 4:

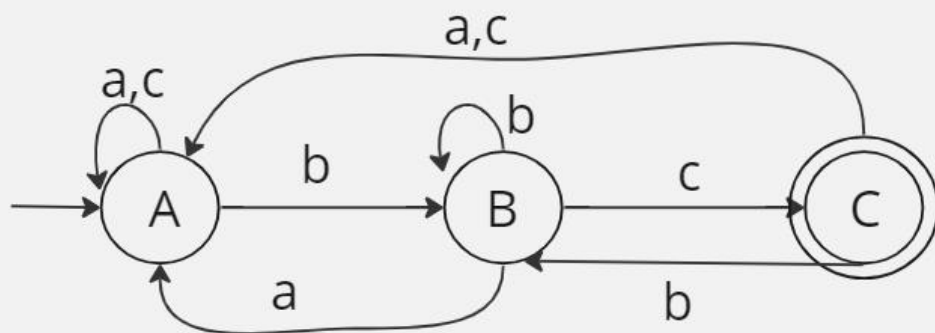
- The set of all strings ending in 011.



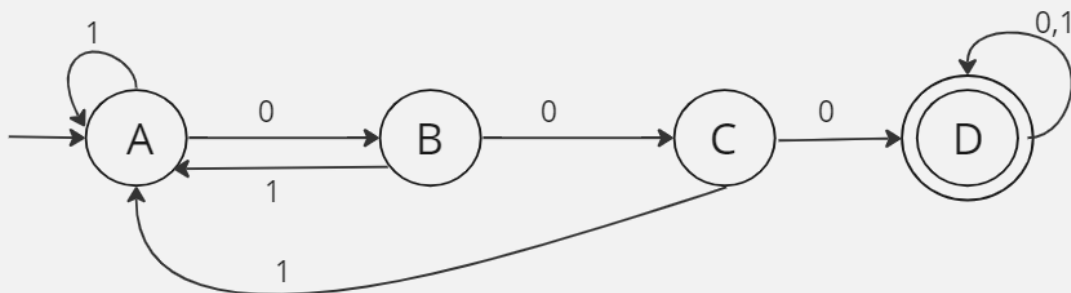
- The set of all string with 01 as a substring.



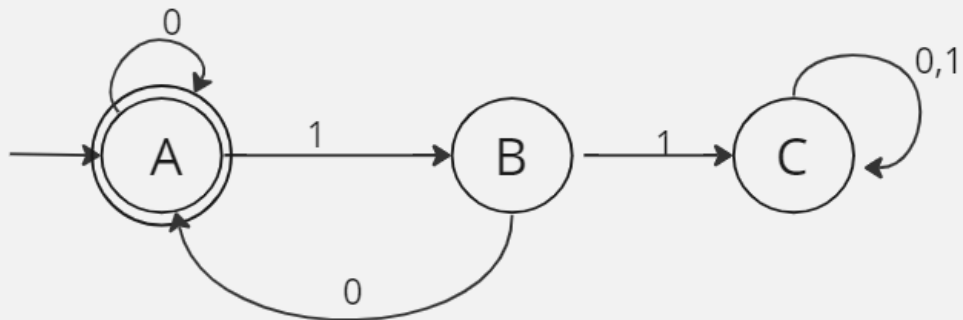
- Draw a DFA for the following language over $\Sigma = \{a,b,c\}$
 - The set of all strings ending in bc.



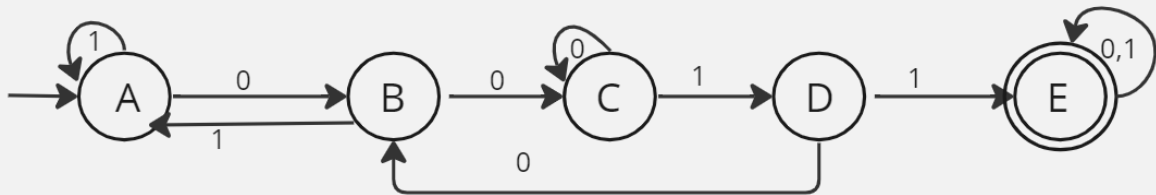
- The set of all strings with three consecutive 0's (not necessarily at the end).



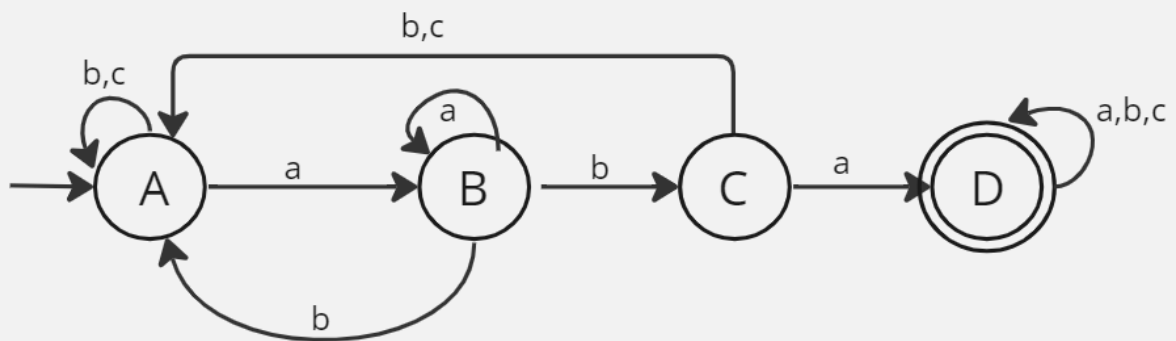
- The set of all strings with not having two consecutive 1's (not necessarily at end).



- The set of strings containing two consecutive zero's followed by two consecutive ones.



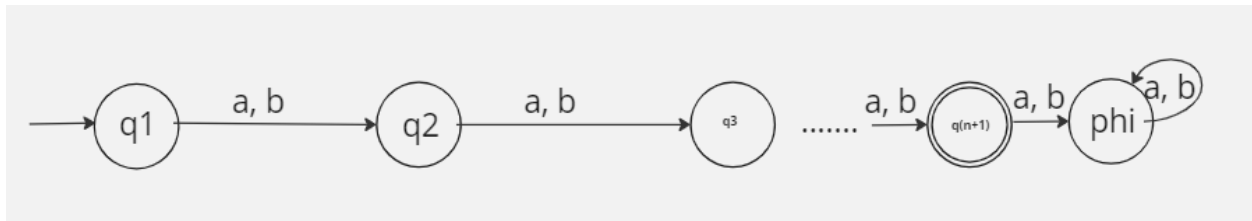
- The set of all strings with aba is a substring.



Lecture: 5

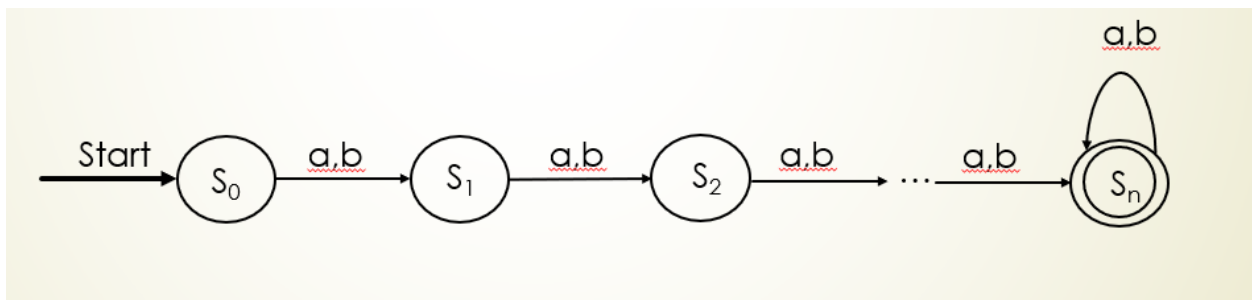
DFA with exactly n alphabets

- i. Number of states: $n+2$



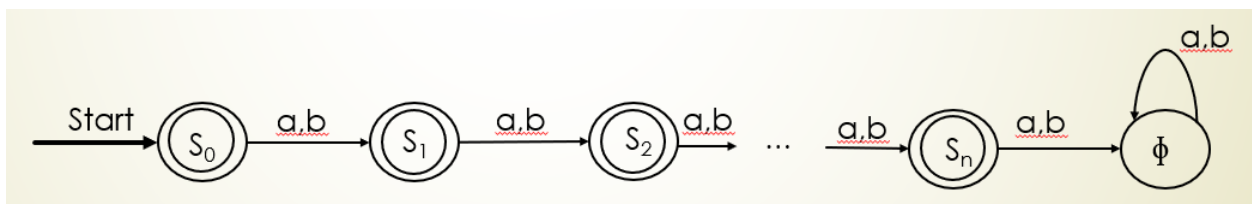
DFA with at least n alphabets

- i. Number of states: $n+1$

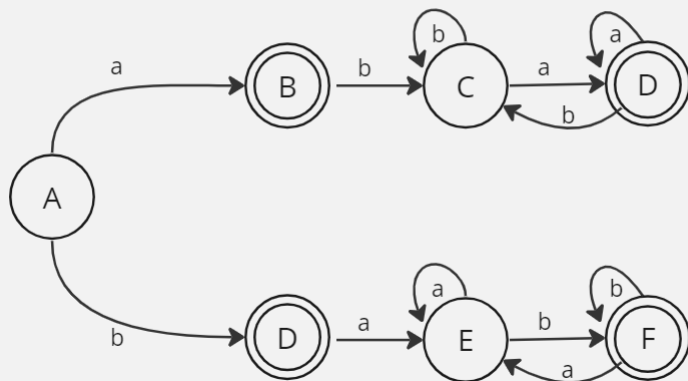


DFA with at most n alphabets

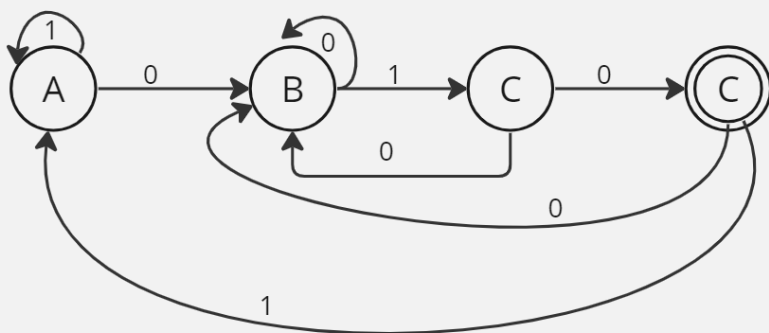
- i. Number of states: $n+2$



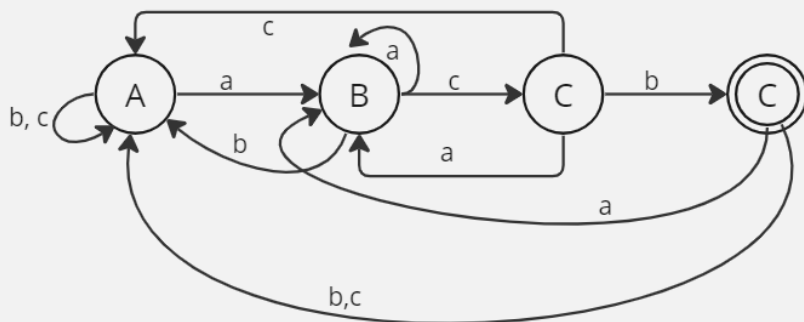
- Draw a DFA that accepts equal number of **ab** and **ba** over $\Sigma = \{a,b\}$.



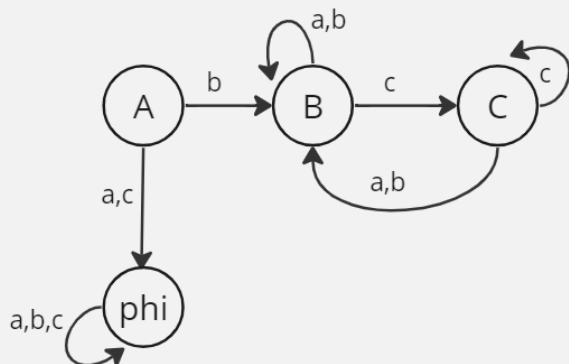
- Draw a DFA for the set of all strings ending in **010** over $\Sigma = \{0,1\}$.



- Draw a DFA for the set of all strings ending in **acb** over $\Sigma = \{a,b,c\}$.



- Draw a DFA for the set of all strings starting with **b** and ending with **c** over $\Sigma = \{a,b,c\}$.



Minimizing DFA

- Equivalence Theorem

Using equivalence theorem, we have

0 equivalence

$$P_0 = \{A, D, E\} \{B, C, G\}$$

1 equivalence

$$P_1 = \{A, D, E\} \{B, C\} \{G\}$$

2 equivalence

$$P_2 = \{A\} \{D, E\} \{B, C\} \{G\}$$

3 equivalence

$$P_3 = \{A\} \{D, E\} \{B, C\} \{G\}$$

Since $P_3 = P_2$, we stop.

	0	1
$\rightarrow A$	B	C
*B	D	E
*C	E	D
D	G	G
E	G	G
*G	G	G

	0	1
$\rightarrow \{A\}$	$\{B, C\}$	$\{B, C\}$
$\{D, E\}$	$\{G\}$	$\{G\}$
* $\{B, C\}$	$\{D, E\}$	$\{D, E\}$
* $\{G\}$	$\{G\}$	$\{G\}$

Table Filling Method / Myhill-Nerode Theorem

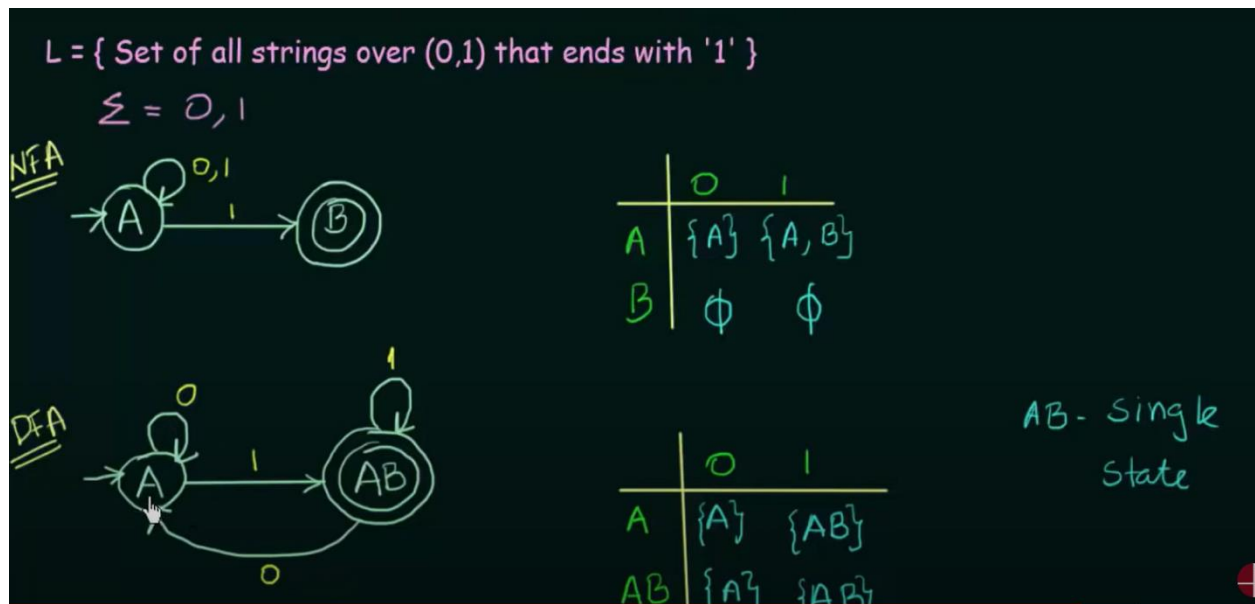
Steps:

- 1) Draw a table for all pairs of states (P,Q)
- 2) Mark all pairs where $P \in F$ and $Q \notin F$
- 3) If there are any Unmarked pairs (P,Q) such that $[\delta(P,x), \delta(Q,x)]$ is marked, then mark [P,Q] where 'x' is an input symbol
REPEAT THIS UNTIL NO MORE MARKINGS CAN BE MADE
- 4) Combine all the Unmarked Pairs and make them a single state in the minimized DFA

NFA

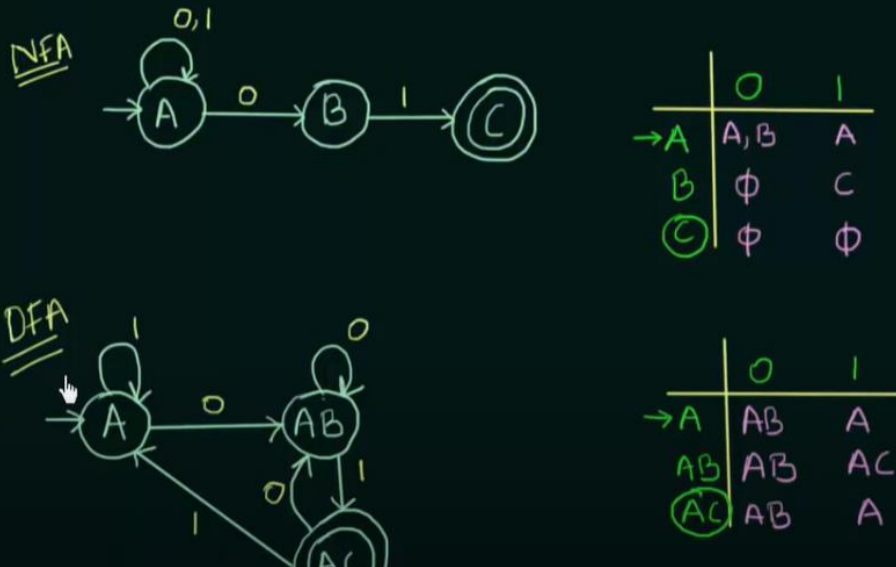
- Non-deterministic Finite Automata
- Transition from a state on an input symbol can be to any set of states
- DFA: $Q \times \Sigma \rightarrow Q$
- NFA: $Q \times \Sigma \rightarrow 2^Q$
- No need to add dead state/trap state
- Input can be empty

Conversion of NFA to DFA:



If there are any empty state/phi in NFA, a dead state should be added for that in NFA.

$L = \{ \text{Set of all strings over } (0,1) \text{ that ends with '01'} \}$. Construct its equivalent DFA



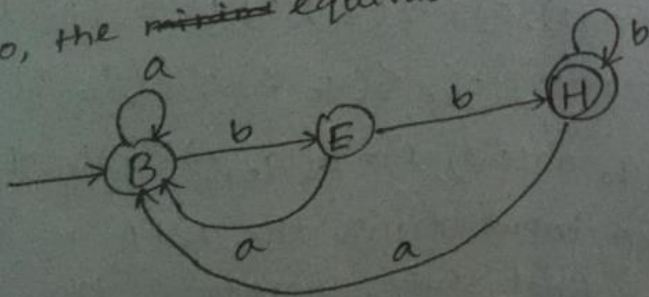
Set Construction Method:

State	a	b
\emptyset	\emptyset	\emptyset
→ $\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_1\}$	\emptyset	$\{q_2\}$
* $\{q_2\}$	\emptyset	\emptyset
$\{q_0, q_1\}$	$\{q_0\}$	$\{q_0, q_1, q_2\}$
* $\{q_0, q_2\}$	$\{q_0\}$	$\{q_0, q_1\}$
* $\{q_1, q_2\}$	\emptyset	$\{q_2\}$
* $\{q_0, q_1, q_2\}$	$\{q_0\}$	$\{q_0, q_1, q_2\}$

renaming the states →

State	a	b
A	A	A
→ B	B	E
C	A	D
* D	A	A
E	B	H
* F	B	E
* G	A	D
* H	B	H

So, the ~~min~~ equivalent DFA is as follows:



ϵ -transitions

- NFA allows go to next state w/o any input
- ϵ means empty

Formal Notation for an ϵ -NFA

□ **Definition:** an ϵ -NFA A is denoted by $A = (Q, \Sigma, \delta, q_0, F)$ where the transition function δ takes as arguments:

- a state in Q , and
- a member of $\Sigma \cup \{\epsilon\}$

Epsilon-Closure

- Denoted by ϵ^*
- All the states than be reached from particular state only by seeing the ϵ symbol

Epsilon-NFA to NFA:

- If a state (let x) goes to final state only by seeing ϵ in E-NFA, then the state (x) will be a final state in the NFA
- Using epsilon-closure

Epsilon-NFA to NFA

ϵ -NFA to NFA

ϵ -NFA

δ	a	b	c	ϵ
$\rightarrow q_0$	$\{q_0\}$	ϕ	ϕ	$\{q_1\}$
q_1	ϕ	$\{q_1\}$	ϕ	$\{q_2\}$
$* q_2$	ϕ	ϕ	$\{q_2\}$	ϕ

NFA

δ'	a	b	c
$\rightarrow q_0$	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
q_1	ϕ	$\{q_1, q_2\}$	$\{q_2\}$
q_2	ϕ	ϕ	$\{q_2\}$

$\epsilon\text{-closure}(q_0) = \{q_0, q_1, q_2\}$
 $\epsilon\text{-closure}(q_1) = \{q_1, q_2\}$
 $\epsilon\text{-closure}(q_2) = \{q_2\}$

$\delta'(q_0, a) = \epsilon\text{-closure}(\delta(\epsilon\text{-closure}(q_0), a))$
 $= \epsilon\text{-closure}(\delta(\{q_0, q_1, q_2\}, a))$
 $= \epsilon\text{-closure}(q_0)$

Epsilon NFA to DFA

ϵ -NFA

δ_ϵ	a	b	c	ϵ
$\rightarrow q_0$	$\{q_0\}$	ϕ	ϕ	$\{q_1\}$
q_1	ϕ	$\{q_1\}$	ϕ	$\{q_2\}$
$* q_2$	ϕ	ϕ	$\{q_2\}$	ϕ

$\epsilon\text{-closure}(q_0) = \{q_0, q_1, q_2\}$
 $\epsilon\text{-closure}(q_1) = \{q_1, q_2\}$
 $\epsilon\text{-closure}(q_2) = \{q_2\}$

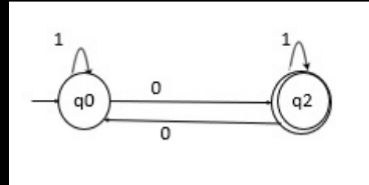
DFA

δ_D	a	b	c
$\rightarrow \{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
$\{q_1, q_2\}$	ϕ	$\{q_1, q_2\}$	$\{q_2\}$
$\{q_2\}$	ϕ	ϕ	$\{q_2\}$

$\delta_D(\{q_0, q_1, q_2\}, a) = \epsilon\text{-closure}(\delta_\epsilon(\{q_0, q_1, q_2\}, a))$
 $= \epsilon\text{-closure}(\delta_\epsilon(q_0, a) \cup \delta_\epsilon(q_1, a) \cup \delta_\epsilon(q_2, a))$
 $= \epsilon\text{-closure}(\{q_0\} \cup \phi \cup \phi)$
 $= \epsilon\text{-closure}(q_0)$
 $= \{q_0, q_1, q_2\}$

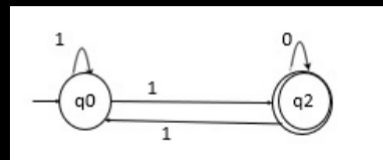
DFA accepting only odd number of 1's over an alphabet $\Sigma=\{0,1\}$

The language $L = \{1, 10, 1110, 1011, 10101, 100101, \dots\}$



DFA accepting only even number of 1's over an alphabet $\Sigma=\{0,1\}$

The language $L = \{11, 101, 11110, 10111, 101101, 1001011, \dots\}$



Now merge two transition diagrams to generate a machine accepting odd number of 0's or even number of 1's

