MATH-2105: Matrices, Vector Analysis and Coordinate Geometry

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Gradient of a scalar field

Gradient measures the rate of change in a scalar field; the gradient of a scalar field is a vector field. The derivative/differentiation/rate of change of a scalar field result in a vector field called the gradient. It computes the gradient of a scalar function. That is, it finds the Gradient, the slope, how fast you change, in any given direction. A gradient is applied to a scalar quantity that is a function of a 3D vector field: position. The gradient measures the direction in which the scalar quantity changes the most, as well as the rate of change with respect to position. A common example of this is height as a function of latitude and longitude, often applied to mountain ranges. A measure of the slope, and direction of the slope, is often called the gradient.

The vector differential operator is $\vec{\nabla}= \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$.

If $\phi(x, y, z)$ is a scalar function, then the gradient of ϕ is defined as

$$grad(\phi) = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$$

Note that, while ϕ is a scalar function, $grad\phi$ is a vector function.

Example

If $\varphi(x, y, z) = 3x^2y - y^3z^2$, find the gradient of φ or $grad(\varphi)$ or $\vec{\nabla}\phi$ at the point $(1, -2, -1)$.

Solution

$$Grad(\phi) = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{\imath} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{\jmath} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= (6xy - 0)\hat{\imath} + (3x^2 - 3y^2z^2)\hat{\jmath} + (0 - 2y^3z)\hat{k}$$

$$= 6xy\hat{\imath} + (3x^2 - 3y^2z^2)\hat{\jmath} - 2y^3z\hat{k}$$

at the point $(1, -2, -1)$, the gradient is

$$Grad(\phi) = 6.1.(-2)\hat{\imath} + (3.1^2 - 3(-2)^2.(-1)^2)\hat{\jmath} - 2.(-2)^3.(-1)\hat{k}$$

$$= -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

Example

Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Solution

A normal to the surface $x^2y + 2xz = 4$ is

$$\vec{\nabla}\phi = \vec{\nabla}(x^2y + 2xz - 4)$$

$$= \frac{\partial}{\partial x}(x^2y + 2xz - 4)\hat{\imath} + \frac{\partial}{\partial y}(x^2y + 2xz - 4)\hat{\jmath} + \frac{\partial}{\partial z}(x^2y + 2xz - 4)\hat{k}$$

$$= (2xy + 2z)\hat{\imath} + x^2\hat{\jmath} + 2x\hat{k}$$

Thus, the normal at the point $(2, -2, 3)$ is

$$\vec{\nabla}\phi = -2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$$

Therefore, the unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$ is obtained by

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{|-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}|} = \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{\sqrt{(-2)^2 + 4^2 + 4^2}} = \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{6} = -\frac{1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$$

Example

Find the directional derivative of the function $\phi = x^2z + 2xy^2 + yz^2$ at the point $(1, 2, -1)$ in the direction of the vector $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$.

Solution

We start with finding the gradient

$$\vec{\nabla}\phi = \vec{\nabla}(x^2z + 2xy^2 + yz^2)$$

$$= \frac{\partial}{\partial x}(x^2z + 2xy^2 + yz^2)\hat{\imath} + \frac{\partial}{\partial y}(x^2z + 2xy^2 + yz^2)\hat{\jmath} + \frac{\partial}{\partial z}(x^2z + 2xy^2 + yz^2)\hat{k}$$

$$= (2xz + 2y^2)\hat{\imath} + (4xy + z^2)\hat{\jmath} + (x^2 + 2yz)\hat{k}$$

at the point $(1,2,-1)$

$$\vec{\nabla}\phi = 6\hat{\imath} + 9\hat{\jmath} - 3\hat{k}$$

Then, the directional derivative in the direction of the vector $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ will be

$$\hat{a}.\vec{\nabla}\phi = \frac{\vec{A}}{|\vec{A}|}.\vec{\nabla}\phi$$

$$= \left(\frac{2\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{2^2+3^2+1^2}}\right).\left(6\hat{\imath} + 9\hat{\jmath} - 3\hat{k}\right)$$

$$= \left(\frac{2}{\sqrt{14}}\hat{\imath} + \frac{3}{\sqrt{14}}\hat{\jmath} + \frac{1}{\sqrt{14}}\hat{k}\right).\left(6\hat{\imath} + 9\hat{\jmath} - 3\hat{k}\right)$$

$$= \frac{36}{\sqrt{14}}$$

Divergence of a vector field

Divergence measures a vector field's tendency to originate from or convergent upon a given point. It computes the divergence of a vector function. That is, it finds how much "stuff" is leaving a point in space. A divergence is applied to a vector as a function of position, yielding a scalar. The divergence actually measures how much the vector function is spreading out. If you are at a location from which the vector field tends to point away in all directions, you will definitely have a positive divergence. If the field points inward toward a point, the divergence in and near that point is negative. If just as much of the vector field points in as out, the divergence will be approximately zero.

If we form the scalar (dot) product of $\vec{\nabla}$ with a vector function

$$\vec{A}(x,y,z) = A_x(x,y,z)\hat{\imath} + A_y(x,y,z)\hat{\jmath} + A_z(x,y,z)\hat{k}$$

we get a scalar result called the divergence of $\vec{A}$.

$$div\vec{A} = \vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

If the divergence of a vector field function is zero i.e. $div\vec{A} = \vec{\nabla}.\vec{A} = 0$, the vector field function $\vec{A}$ is said to be solenoidal.

Example

If $\vec{V}(x,y,z) = xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}$, find the divergence of $\vec{V}$ or *div* $\vec{V}$ or $\vec{\nabla}.\vec{V}$.

Solution

Given vector field is $\vec{\mathrm{V}}(x,y,z) = xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}$

The divergence of $\vec{\mathrm{V}}$ is given by

$$\vec{\nabla}.\vec{\mathrm{V}} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

Curl of a vector field

In vector calculus, the curl (or rotor) is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point. The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation. If the vector field represents the flow velocity of a moving fluid, then the curl is the circulation density of the fluid. A vector field whose curl is zero is called irrotational. The curl is a form of differentiation for vector fields. Curl measures a vector field's tendency to rotate about a point; the curl of a vector field is another vector field. It computes the rotational aspects of a vector function, maybe people thought how vectors "curl" around a center point, like wind curling around a low pressure on a weather map. A curl measures just that, the curl of a vector field. Unlike the divergence, a curl yields a vector. A vector field that tends to point around an axis, such as vectors pointing tangential to a circle, will yield a non-zero curl with the axis around which the curl occurs as the direction. Another example is the velocity field of motion spiraling in or out, such as a whirlpool. Point your right-hand thumb along the direction of the curl. Curl your fingers around this axis. They will curl in the same direction as the vector field. I do not know the names of the texts, but I know there are books available with vector fields to illustrate both divergence and curl.

The curl of a vector field $\vec{A}(x,y,z) = A_x(x,y,z)\hat{\imath} + A_y(x,y,z)\hat{\jmath} + A_z(x,y,z)\hat{k}$ is defined by

$$curl\vec{A} = \vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

A vector field $\vec{A}$ for which the curl vanishes i.e. $curl\vec{A} = \vec{\nabla}\times\vec{A} = 0$ is said to be irrotational field or conservative vector field.

Example

If $\vec{V}(x,y,z) = xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}$, find the curl of $\vec{V}$ or *curl* $\vec{V}$ or $\vec{\nabla} \times \vec{V}$.

Solution

Given vector field is $\vec{V}(x,y,z) = xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}$

The curl of $\vec{V}$ is given by

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]$$

$$= -(2y + xy)\hat{\imath} + x\hat{\jmath} + yz\hat{k}$$

Example

If $\vec{A}(x,y,z) = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$, find $\vec{\nabla} \times \vec{A}$ (or curl A) at the point $(1,-1,1)$.

Solution

Given vector field is $\vec{A}(x,y,z) = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$

The curl of $\vec{A}$ is given by

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= (2z^4 + 2x^2y)\hat{\imath} - 3xz^2\hat{\jmath} - 4xyz\hat{k}$$

at the point $(1,-1,1)$

$$\vec{\nabla} \times \vec{A} = 3\hat{\jmath} + 4\hat{k}$$

Solenoidal and Irrotational vector field

If any vector field has the divergence zero, it is said to be solenoidal. That is, for a solenoidal vector field $\vec{\nabla}.\vec{A} = 0$.

If the curl of any vector field vanishes, it is said to be irrotational or conservative vector field. That is, for an irrotational or conservative vector field $\vec{\nabla} \times \vec{A} = 0$.

Example

Determine the force field $\vec{F}(x, y, z) = x^2y\hat{\imath} + xyz\hat{\jmath} - x^2y^2\hat{k}$ is a conservative vector field or not.

Solution

All that we need to do is compute the curl and see if we get the zero vector or not.

The given force field is

$$\vec{F}(x, y, z) = x^2y\hat{\imath} + xyz\hat{\jmath} - x^2y^2\hat{k}$$

Now, curl of $\vec{F}$ is $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$

$$= (-2x^2y - xy)\hat{\imath} + 2xy^2\hat{\jmath} + (yz - x^2)\hat{k}$$

$$\neq 0$$

The curl isn't the zero vectors and so this vector field is not conservative.

Example

If $\varphi(x, y, z)$ is any solution of the Laplace equation, then show that $\vec{\nabla}\varphi$ is a vector both solenoidal and irrotational.

Solution

For a solenoidal vector field $\vec{\nabla}.\vec{\nabla}\varphi = 0$.

For an irrotational or conservative vector field $\vec{\nabla} \times \vec{\nabla}\varphi = 0$.

Since $\varphi(x, y, z)$ is a solution of the Laplace equation, we have

$$\nabla^2\varphi = 0$$

which implies that

$$\vec{\nabla}.\vec{\nabla}\varphi = 0$$

Therefore, the vector $\vec{\nabla}\varphi$ is solenoidal.

Again $\vec{\nabla} \times \vec{\nabla}\varphi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$

$$= \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)\hat{\imath} + \left(\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial z\partial x}\right)\hat{\jmath} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right)\hat{k}$$

$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$= 0$$

Hence, the vector $\vec{\nabla}\varphi$ is irrotational.