Predicate Logic

Example:
All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

Note: We need logic laws that work for statements involving quantities like "some" and "all".

In English, the **predicate** is the part of the sentence that tells you something about the subject.

More on predicates

Example: Nate is a student at UT.
What is the subject? What is the predicate?

Example: We can form two different predicates.
Let P(x) be "x is a student at UT".
Let Q(x, y) be "x is a student at y".

Definition: A **predicate** is a property that a variable or a finite collection of variables can have. A predicate becomes a proposition when specific values are assigned to the variables. P$(x_1, x_2, ..., x_n)$ is called a predicate of n variables or n arguments.

Example: She lives in the city.
P(x,y): x lives in y.
P(Mary, Austin) is a proposition: Mary lives in Austin.

Example: Predicates are often used in if statements and loop conditions.
if($x > 100$)
then y:= $x * x$

predicate T(x): $x > 100$

Domains and Truth Sets

Definition: The **domain** or **universe** or **universe of discourse** for a predicate variable is the set of values that may be assigned to the variable.

Definition: If P(x) is a predicate and x has domain U, the **truth set** of P(x) is the set of all elements t of U such that P(t) is true, ie $\{t \in U | P(t) \text{ is true}\}$

Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
P(x): "x is even".
The truth set is: $\{2, 4, 6, 8, 10\}$

The Universal Quantifier: $\forall$

Turn predicates into propositions by assigning values to all variables:
Predicate P(x): "x is even"
Proposition P(6): "6 is even"

The other way to turn a predicate into a proposition: add a *quantifier* like "all" or "some" that indicates the number of values for which the predicate is true.

Definition: The symbol $\forall$ is called the **universal quantifier**. The **universal quantification** of P(x) is the statement "P(x) for all values x in the universe", which is written in logical notation as: $\forall x P(x)$ or sometimes $\forall x \in D, P(x)$.

Ways to read $\forall x P(x)$:
For every x, P(x)
For every x, P(x) is true
For all x, P(x)

More on the universal quantifier

Definition: A **counterexample** for $\forall x P(x)$ is any $t \in U$, where U is the universe, such that P(t) is false.

Some Examples

Example: P(x, y): x + y = 8
Assign x to be 1, and y to be 7. We get proposition P(1, 7) which is true.
Proposition P(2, 5) is false since $2 + 5 \neq 8$.

Example: $\forall x[x \geq 0]$
$U = \mathbb{N}$ (non-negative integers)
We could re-write this proposition as: $\forall x \in \mathbb{N}, x \geq 0$
Is the proposition true?
What if the universe is $\mathbb{R}$?

Example: $\forall x \forall y[x + y > x]$
Is this proposition true if:

1. If $U = \mathbb{N}$?
2. If $U = \mathbb{R}$?

Example: $\forall x \forall y[x > y]$
True if:
universe for x = the non-negative integers
universe for y = the non-positive integers

The Existential Quantifier: $\exists$

Definition: The symbol $\exists$ is call the **existential quantifier** and represents the phrase "there exists" or "for some". The **existential quantification** of P(x) is the statement "P(x) for some values x in the universe", or equivalently, "There exists a value for x such that P(x) is true", which is written $\exists x P(x)$.

Note: If P(x) is true for at least one element in the domain, then $\exists x P(x)$ is true. Otherwise it is false.

Note: Let P(x) be a predicate and $c \in U$ (U = domain).
The following implications are true:
$\forall x P(x) \rightarrow P(c)$
$P(c) \rightarrow \exists x P(x)$

Example: $\exists x$ [x is prime] where $U = \mathbb{Z}$
Is this proposition true or false?

Example: $\exists x[x^2 < 0]$ where $U = \mathbb{R}$
True or false?

Exercises: True or false? Prove your answer.

1. $\exists n[n^2 = n]$ where $U = \mathbb{Z}$.
2. $\exists n[n^2 = n]$ where $U = \{4, 5, 6, 7\}$.

Translating Quantified Statements

Translate the following into English.

1. $\forall x[x^2 \geq 0]$ where $U = \mathbb{R}$.
2. $\exists t[(t > 3) \wedge (t^3 > 27)]$ where $U = \mathbb{R}$.
3. $\forall x[(2|x) \vee (2 \not| x)]$ where $U = \mathbb{N}$

Translate the following into logic statements.

1. There is an integer whose square is twice itself.
2. No school buses are purple.
3. If a real number is even, then its square is even.

Note: Let $U = \{1, 2, 3\}$.
Proposition $\forall x P(x)$ is equivalent to $P(1) \wedge P(2) \wedge P(3)$.
Proposition $\exists x P(x)$ is equivalent to $P(1) \vee P(2) \vee P(3)$.

Bound and Free Variables

Definition: All variables in a predicate must be **bound** to turn a predicate into a proposition. We **bind** a variable by assigning it a value or quantifying it. Variables which are not bound are **free**.

Note: If we bind one variable in a predicate $P(x, y, z)$ with 3 variables, say by setting $z = 4$, we get a predicate with 2 variables: $P(x, y, 4)$.

Example: Let $U = \mathbb{N}$.
$P(x, y, z) : x + y = z \leftarrow$ 3 free variables
Let $Q(y, z) = P(2, y, z) : 2 + y = z \leftarrow$ 2 free variables

Examples with Quantifiers

Example: $U = \mathbb{Z}$
N(x): x is a non-negative integer
E(x): x is even
O(x): x is odd
P(x): x is prime
Translate into logical notation.

1. There exists an even integer.
2. Every integer is even or odd.
3. All prime integers are non-negative.
4. The only even prime is 2.
5. Not all integers are odd.
6. Not all primes are odd.
7. If an integer is not odd, then it is even.

Examples with Nested Quantifiers

Note about nested quantifiers: For predicate $P(x, y)$:
$\forall x \forall y P(x, y)$ has the same meaning as $\forall y \forall x P(x, y)$.
$\exists x \exists y P(x, y)$ has the same meaning as $\exists y \exists x P(x, y)$.
We can **not** interchange the position of $\forall$ and $\exists$ like this!

Example: U = set of married people. True or false?

1. $\forall x \exists y$[x is married to y]
2. $\exists y \forall x$[x is married to y]

Example: $U = \mathbb{Z}$. True or false?

1. $\forall x \exists y[x + y = 0]$
2. $\exists y \forall x[x + y = 0]$

Exercise: $U = \mathbb{N}$.
$L(x, y) : x < y$
S(x, y, z): x + y = z
$P(x, y, z) : xy = z$
Rewrite the following in logic notation.

1. For every x and y, there is a z such that $x + y = z$.
2. No x is less than 0.
3. For all x, $x + 0 = x$.
4. There is some x such that $xy = y$ for all y.

Negating Quantified Statements

Precedence of logical operators

1. $\forall, \exists$
2. $\neg$
3. $\wedge, \vee$
4. $\rightarrow, \leftrightarrow$

Example: Statement: "All dogs bark."
Negation: "One or more dogs do not bark" or "some dogs do not bark".
NOT "No dogs bark".
If at least one dog does not bark, then the original statement is false.

One example of DeMorgan's laws for quantifiers:

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

Example: Some cats purr.
Negation: No cats purr.
I.e., if it is false that some cats purr, then no cat purrs.

DeMorgan's laws for quantifiers:

$\neg \forall x P(x) \equiv \exists x \neg P(x)$
$\neg \exists x P(x) \equiv \forall x \neg P(x)$

More Examples - Negating Statements with Quantifiers

Example: Write the statements in logical notation. Then negate the statements.

1. Some drivers do not obey the speed limit.
2. All dogs have fleas.

Example: Using DeMorgan's laws to push negation through multiple quantifiers:

$$\begin{aligned} \neg\exists x \forall y \forall z P(x, y, z) &\equiv \forall x \neg\forall y \forall z P(x, y, z) \\ &\equiv \forall x \exists y \neg\forall z P(x, y, z) \\ &\equiv \forall x \exists y \exists z \neg P(x, y, z). \end{aligned}$$

Example: Write the following statement in logical notation and then negate it.
For every integer x and every integer y, there exists an integer z such that $y - z = x$.
Logical notation:
Negation (apply DeMorgan's laws):
Let $U = \mathbb{N}$. Show the original statement is false by showing the negation is true.

Some Definitions

Definition: Let U be the universe of discourse and $P(x_1, ..., x_n)$ be a predicate. If $P(x_1, ..., x_n)$ is true for every choice of $x_1, ..., x_n \in U$, then we say P is **valid** in universe U. If $P(x_1, ..., x_n)$ is true for some (not necessarily all) choices of arguments from U, then we say that P is **satisfiable** in U. If P is not satisfiable in U, we say P is **unsatisfiable** in U.

Definition: The **scope** if a quantifier is the part of a statement in which variables are bound by the quantifier.

Example: $R \vee \exists (P(x) \vee Q(x))$
Scope of $\exists$: $P(x) \vee Q(x)$.

Note: We can use parentheses to change the scope, but otherwise the scope is the smallest expression possible.

Example: $\forall x P(x) \wedge Q(x)$
Scope of $\forall$: $P(x)$.
Note that this is a predicate, not a proposition, since the variable in $Q(x)$ is not bound. It is confusing to have 2 variables which are both denoted x. Rewrite as: $\forall x P(x) \wedge Q(z)$.

Quantifiers plus $\wedge$ and $\vee$

Example: Show $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$. (That is, show that no matter what the domain is, these 2 propositions always have the same truth value).

Proof: First assume that $\forall x(P(x) \wedge Q(x))$ is true. So for all x, $P(x)$ is true and $Q(x)$ is true. Therefore $\forall x P(x)$ is true, and $\forall x Q(x)$ is true. Therefore $\forall x P(x) \wedge \forall x Q(x)$ is true.

Now assume $\forall x P(x) \wedge \forall x Q(x)$ is true. So $\forall x P(x)$ is true and $\forall x Q(x)$ is true. So for all x, $P(x)$ is true and for all x, $Q(x)$ is true. Therefore, for all x, $P(x) \wedge Q(x)$ is true. So $\forall x(P(x) \wedge Q(x))$ is true.

Therefore $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$. $\square$

Terminology: We say that $\forall$ distributes over $\wedge$.

Distributing $\exists$ over $\wedge$

Note: The existential quantifier $\exists$ does **not** distribute over $\wedge$. That is, $\exists x(P(x) \wedge Q(x)) \not\equiv \exists x P(x) \wedge \exists x Q(x)$.

Proof: We must find a counterexample - a universe and predicates P and Q such that one of the propositions is true and the other is false.

Let $U = \mathbb{N}$. Set P(x): "x is prime" and Q(x): "x is composite" (ie not prime). Then $\exists x(P(x) \wedge Q(x))$ is false, but $\exists x P(x) \wedge \exists x Q(x)$ is true. $\square$

Note: The following is true though:
$\exists x(P(x) \wedge Q(x)) \rightarrow \exists x P(x) \wedge \exists x Q(x)$.

Proof: exercise

Note: With $\vee$, the situation is reversed. $\exists$ distributes over $\vee$, but $\forall$ does not.

Distributing the Existential Quantifier

Recall: $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

This rule holds for arbitrary P and Q. Replace P by $\neg S$ and Q by $\neg R$ and negate both sides to see that:

$\exists x(S(x) \vee R(x)) \equiv \exists x S(x) \vee \exists x R(x)$.

Exercise: Show that
1. $(\forall x P(x) \vee \forall x Q(x)) \rightarrow \forall x(P(x) \vee Q(x))$ is true.
2. $\forall x P(x) \vee \forall x Q(x) \not\equiv \forall x(P(x) \vee Q(x))$

$\exists$ does not distribute over $\rightarrow$

Note: $\exists$ does not distribute over $\rightarrow$. I.e.,
$\exists x(P(x) \rightarrow Q(x)) \not\equiv \exists x P(x) \rightarrow \exists x Q(x)$.

Proof:
$\exists x(P(x) \rightarrow Q(x)) \equiv \exists x(Q(x) \vee \neg P(x))$ by implication
$\equiv \exists x Q(x) \vee \exists x \neg P(x)$ by distributivity of $\exists$ over $\vee$
$\equiv \exists x Q(x) \vee \neg \forall x P(x)$ by DeMorgan's law
$\equiv \forall x P(x) \rightarrow \exists x Q(x)$ by implication law

So we need to show that $\forall x P(x) \rightarrow \exists x Q(x)$ is not logically equivalent to $\exists x P(x) \rightarrow \exists x Q(x)$. Note that if $\exists x Q(x)$ is false, $\forall x P(x)$ is false, and $\exists x P(x)$ is true, then we would have a counterexample, since one of the implications is true and the other is false. So let $U = \mathbb{N}$, and set P(x) to be "x is even" and Q(x) to be "x is negative". In this case $\forall x P(x) \rightarrow \exists x Q(x)$ and $\exists x P(x) \rightarrow \exists x Q(x)$ have different truth values.

Logical Relationships with Quantifiers

Law	Name
$\neg \forall x P(x) \equiv \exists x \neg P(x)$ $\neg \exists x P(x) \equiv \forall x \neg P(x)$	DeMorgan's laws for quantifiers
$\forall x P(x) \wedge \forall x Q(x) \equiv \forall x (P(x) \wedge Q(x))$ $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$	distributivity of $\forall$ over $\wedge$ distributivity of $\exists$ over $\vee$

Compact Notation

Example: For every $x > 0$, P(x) is true.
Current notation: $\forall x[(x > 0) \rightarrow P(x)]$.
More compact notation: $\forall x_{x>0} P(x)$ (or $\forall x > 0, P(x)$).

Example: There exists an x such that $x \neq 0$ and P(x) is true.
Compact notation: $\exists x_{x \neq 0} P(x)$, instead of $\exists x[(x \neq 0) \wedge P(x)]$. The compact notation is more readable.

Example:
Definition: The **limit** of f(x) as x approaches c is k (denoted $lim_{x \rightarrow c} f(x) = k$) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $|x - c| < \delta$, then $|f(x) - k| < \varepsilon$.

Notation: $lim_{x \rightarrow c} f(x) = k$ if $\forall \varepsilon_{\varepsilon > 0} \exists \delta_{\delta > 0} \forall x[|x - c| < \delta \rightarrow |f(x) - k| < \varepsilon]$.

Arguments with Quantified Statements

Rules of Inference with Quantifiers

Rule of Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c)}$$ (where c is some element of P's domain)

Example: U = all men

All men are mortal.
Dijkstra is a man.
$\therefore$ Dijkstra is mortal.

P(x): x is mortal.
Argument:

$$\frac{\forall x P(x)}{\therefore \text{P(Dijkstra)}}$$

Universal Modus Ponens

$\forall x(P(x) \rightarrow Q(x))$
$\underline{P(c)}$
$\therefore$ Q(c)

Example:
All politicians are crooks.
Joe Lieberman is a politician.
$\therefore$ Joe Lieberman is a crook.

P(x): x is a politician, Q(x): x is a crook, U =all people.

Example: If x is an even number, then x^2 is an even number.
206 is an even number.
$\therefore 206^2$ is an even number.

Universal Modus Tollens

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \underline{\neg Q(c)} \\ \therefore \neg P(c) \end{array}$$

Example:

All dogs bark.
Otis does not bark.
$\therefore$ Otis is not a dog.

U = all living creatures, P(x): x is a dog, Q(x): x barks.

Universal Hypothetical Syllogism

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \underline{\forall x(Q(x) \to R(x))} \\ \therefore \forall x(P(x) \to R(x)) \end{array}$$

Example:

If integer x is even, then 2x is even.
If 2x is even, then $4x^2$ is even.
$\therefore$ If x is even, then $4x^2$ is even.

Universal Generalization

Universal Generalization:

$$\frac{P(c) \text{ for arbitrary c in } U}{\therefore \forall x P(x)}$$

Example: For arbitrary real number x, x^2 is non-negative. Therefore the square of any real number is non-negative.

Note: We use this rule to prove statements of the form $\forall x P(x)$. We assume that c is an arbitrary element of the domain, and prove P(c) is true. Then we conclude that $\forall x P(x)$.

Prove: The square of every even integer n is even.

Proof: Let c be an arbitrary even integer. Then c = 2k for some integer k, by the definition of even numbers. So $c^2 = (2k)^2 = 4k^2 = 2(2k^2)$, and $2k^2 \in \mathbb{Z}$. So c^2 is even. Therefore it follows that the square of every even integer is even. $\square$

Existential Instantiation and Existential Generalization

Existential Instantiation

$$\frac{\exists x P(x)}{\therefore \text{P(c) for some c}}$$

Existential Generalization

$$\frac{\text{P(c) for some element c}}{\therefore \exists x P(x)}$$

Arguments with Quantifiers

Def: An argument with quantifiers is **valid** if the conclusion is true whenever the premises are all true.

Example: A horse that is registered for today's race is not a throughbred. Every horse registered for today's race has won a race this year. Therefore a horse that has won a race this year is not a thoroughbred.

P(x): x is registered for today's race.
Q(x): x is a thoroughbred.
R(x): x has won a race this year.
U = all horses

$$\begin{array}{l} \exists x(P(x) \wedge \neg Q(x)) \\ \underline{\forall x(P(x) \to R(x))} \\ \therefore \exists x(R(x) \wedge \neg Q(x)) \end{array}$$

Proof:

Step	Reason
1. $\exists x(P(x) \wedge \neg Q(x))$	premise
2. $P(a) \wedge \neg Q(a)$ for some a	step 1, existential instantiation
3. $P(a)$	simplification, step 2
4. $\forall x(P(x) \to R(x))$	premise
5. $P(a) \to R(a)$	universal instantiation, step 4
6. R(a)	modus ponens, steps 3 and 5
7. $\neg Q(a)$	step 2, simplification
8. $R(a) \wedge \neg Q(a)$	conjunction, steps 6 and 7
9. $\exists x(R(x) \wedge \neg Q(x))$	existential generalization, step 8