MATH-2105: Matrices, Vector Analysis and Coordinate Geometry

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Physical quantities can be divided into two main groups, scalar quantities and vector quantities:

Scalar Quantities

A Physical Quantity which has magnitude only is called as a Scalar. Example: Time, Temperature, Mass, Density, Volume, Energy, Distance, Speed, Specific Heat etc. are examples of scalars. That is, the measurement of years, months, weeks, days, hours, minutes, seconds, and even milliseconds, A temperature of 15°C, A mass of 0.2 kg, etc.

Vector Quantities

A Physical Quantity which has both magnitude and direction is called as Vector Examples: velocity, displacement, acceleration, force, Weight, Momentum, Magnetic Field Intensity etc.

Some Examples:

01. A speed of 10 km/h is a scalar quantity, but a velocity of 10 km/h due north is a vector quantity.

02. A temperature of 1000 c is a scalar quantity.

03. The weight of a 7 kg mass is a vector quantity. $[w = mg]$

Magnitude and Sign Convention of Vectors

Two dimensional vector is

$\overrightarrow{OA}$ or $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

Length or magnitude is $|\overrightarrow{OA}|$ or $|\vec{A}| = \sqrt{{A_x}^2 + {A_y}^2}$

Three dimensional vector is

$\overrightarrow{OA}$ or $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$

Length or magnitude is $|\overrightarrow{OA}|$ or $|\vec{A}| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$

Example

Determine the vector having initial point $P(1, 2, 3)$ and terminal point $Q(5, 3, -1)$ and find its magnitude.

Solution

$$\overrightarrow{PQ} = (5-1)\hat{\imath} + (3-2)\hat{\jmath} + (-1-1)\hat{k} = 4\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{21}$$

Physical Significance of The scalar or dot product: Work done

We know

Work = Force applied × displacement happened

$$W = \vec{F}.\vec{S}$$

Example

A block of mass m moves from point A to B along a smooth plane surface under the action of force as shown in the figure. Find the work done.

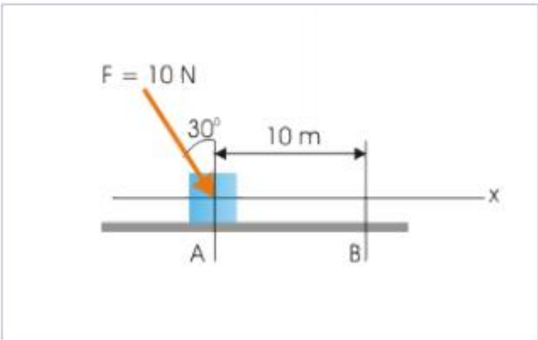


Figure 31

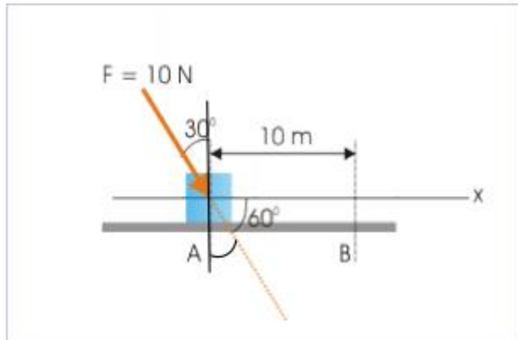
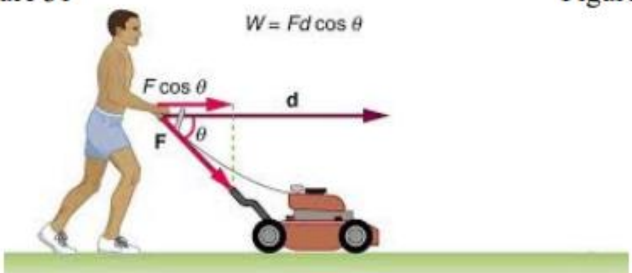


Figure 32



Solution

Workdone $W = \vec{F}.\vec{S} = |\vec{F}||\vec{S}|\cos\theta = 10.10.\cos 60° = 10.10.\frac{1}{2} = 50$ joule

Dot Product of Two Vectors

If $\vec{A}$ and $\vec{B}$ are two vectors, then the dot product is defined by

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then $\vec{A}.\vec{B}$ can be found by

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Note:

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ and $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{\imath} = \hat{\imath}.\hat{k} = \hat{k}.\hat{\imath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\jmath} = 0$

Example

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then show that

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Solution

Given vectors are $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$$\vec{A}.\vec{B} = A_xB_x(\hat{\imath}.\hat{\imath}) + A_xB_y(\hat{\imath}.\hat{\jmath}) + A_xB_z(\hat{\imath}.\hat{k}) + A_yB_x(\hat{\jmath}.i) + A_yB_y(\hat{\jmath}.\hat{\jmath}) + A_yB_z(\hat{\jmath}.k) + A_zB_x(\hat{k}.\hat{\imath}) + A_zB_y(\hat{k}.\hat{\jmath}) + A_zB_z(\hat{k}.\hat{k})$$

$$= A_xB_x + A_yB_y + A_zB_z$$

Orthogonal (Perpendicular) Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other, their dot product is always zero, that is

$$\vec{A}.\vec{B} = 0$$

Since

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta = |\vec{A}||\vec{B}|\cos 90° = 0$$

Example

Determine whether the vectors $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and $\vec{B} = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ are perpendicular.

Solution

Given that $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and $\vec{B} = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$

Now $\vec{A}.\vec{B} = (3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}).(2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}) = 3.2 + 5.(-2) + (-2).(-2) = 0$

Since $\vec{A}.\vec{B} = 0$, the vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Cross Product of Two Vectors

If $\vec{A}$ and $\vec{B}$ are two vectors, then the dot product is defined by

$$\vec{A} \times \vec{B} = \hat{n}|\vec{A}||\vec{B}|\sin\theta$$

where $\hat{n}$ is the unit vector which will indicate the direction of $\vec{A} \times \vec{B}$.

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then $\vec{A} \times \vec{B}$ can be found by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then show that

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Solution

Given vectors are $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$$\vec{A} \times \vec{B} = A_xB_x(\hat{\imath} \times \hat{\imath}) + A_xB_y(\hat{\imath} \times \hat{\jmath}) + A_xB_z(\hat{\imath} \times \hat{k}) + A_yB_x(\hat{\jmath} \times i) + A_yB_y(\hat{\jmath} \times \hat{\jmath})$$

$$+A_yB_z(\hat{\jmath} \times k) + A_zB_x(\hat{k} \times \hat{\imath}) + A_zB_y(\hat{k} \times \hat{\jmath}) + A_zB_z(\hat{k} \times \hat{k})$$

$$= A_xB_x.0 + A_xB_y\hat{k} + A_xB_z(-\hat{\jmath}) + A_yB_x(-\hat{k}) + A_yB_y.0 + A_yB_z\hat{\imath} + A_zB_x\hat{\jmath} +$$

$$A_zB_y(-\hat{\imath}) + A_zB_z.0$$

$$= \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Parallel Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are parallel to each other, their cross product is always zero, that is

$$\vec{A} \times \vec{B} = 0$$

Since

$$\vec{A} \times \vec{B} = \hat{n}|\vec{A}||\vec{B}| \sin\theta = \hat{n}|\vec{A}||\vec{B}| \sin 0° = 0$$

Example

Determine whether the vectors $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and $\vec{B} = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ are parallel.

Solution

Given that $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and $\vec{B} = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$

Now $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & -2 \\ 2 & -2 & -2 \end{vmatrix}$$

$$= (-10-4)\hat{\imath} + (-4+6)\hat{\jmath} + (-6-10)\hat{k}$$

$$= -14\hat{\imath} + 2\hat{\jmath} - 16\hat{k}$$

$$\neq 0$$

Hence the given vectors are not mutually parallel.

Example

Find the angle between the vectors $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{B} = 4\hat{\imath} + \hat{\jmath} - 3\hat{k}$.

Solution

Given that $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{B} = 4\hat{\imath} + \hat{\jmath} - 3\hat{k}$. Now

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

$$\cos\theta = \frac{2.4 + (-3).1 + 1.(-3)}{\sqrt{2^2 + (-3)^2 + 1^2}\sqrt{4^2 + 1^2 + (-3)^2}}$$

$$\cos\theta = \frac{2}{\sqrt{14}\sqrt{26}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}\sqrt{26}}\right)$$

Example

Determine the angles α, β, γ which the vector $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ makes with the positive directions of the coordinate axes and show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$.

Solution

Given vector is $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$

Unit vector along the x-axis is $\hat{\imath}$

Thus, the angle α making by the vector $\vec{A}$ with the x-axis is given by

$$\alpha = \cos^{-1}\left(\frac{\vec{A}.\hat{\imath}}{|\vec{A}||\hat{\imath}|}\right)$$

$$= \cos^{-1}\left(\frac{2.1+(-3).0+1.0}{\sqrt{2^2+(-3)^2+1^2}.\sqrt{1^2+0^2+0^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$$

Unit vector along the y-axis is $\hat{\jmath}$

Thus, the angle β making by the vector $\vec{A}$ with the x-axis is given by

$$\beta = \cos^{-1}\left(\frac{\vec{A}.\hat{\jmath}}{|\vec{A}||\hat{\jmath}|}\right)$$

$$= \cos^{-1}\left(\frac{2.0+(-3).1+1.0}{\sqrt{2^2+(-3)^2+1^2}.\sqrt{0^2+1^2+0^2}}\right)$$

$$= \cos^{-1}\left(-\frac{3}{\sqrt{14}}\right)$$

Unit vector along the z-axis is $\hat{k}$

Thus, the angle γ making by the vector $\vec{A}$ with the x-axis is given by

$$\gamma = \cos^{-1}\left(\frac{\vec{A}.\hat{k}}{|\vec{A}||\hat{k}|}\right)$$

$$= \cos^{-1}\left(\frac{2.0+(-3).0+1.1}{\sqrt{2^2+(-3)^2+1^2}.\sqrt{1^2+0^2+0^2}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$$

Example

A particle acted on by constant forces $\overrightarrow{F_1} = 4\hat{\imath} + \hat{\jmath} - 3\hat{k}$ and $\overrightarrow{F_2} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$ (both measured in Newton), is displaced from the point $(1, 2,3)$ to the point $(5,4,1)$ (measured in meters). Find the total work done by the forces.

Solution

fsajh

Scalar triple product

If $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 0, & when\ i < j \\ i + j, & when\ i = j \\ 2i - j, & when\ i > j \end{cases}$

Example

Find a unit vector in the direction to the vector $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.

Solution

Given vector is $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

The magnitude of the vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{2^2 + 4^2 + (-5)^2} = \sqrt{45}$$

Thus, the unit vector parallel to the vector $\vec{a}$ is

$$\hat{e} = \frac{2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}}{\sqrt{45}} = \frac{2}{\sqrt{45}}\hat{\imath} + \frac{4}{\sqrt{45}}\hat{\jmath} - \frac{5}{\sqrt{45}}\hat{k}$$

Example

Find a unit vector parallel to the resultant of vectors $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Solution

the resultant of vectors $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ is given by

$$\vec{R} = \vec{a} + \vec{b}$$

$$= (2 + 1)\hat{\imath} + (4 + 2)\hat{\jmath} + (-5 + 3)\hat{k}$$

$$= 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$$

Magnitude of the resultant, $|\vec{R}| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{49} = 7$

Thus, the unit vector parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$$\hat{e} = \frac{\vec{R}}{|\vec{R}|} = \frac{3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}}{7} = \frac{3}{7}\hat{\imath} + \frac{6}{7}\hat{\jmath} - \frac{2}{7}\hat{k}$$

Example

Find a unit vector perpendicular to the vectors $\vec{a} = 3\hat{\imath} + \hat{\jmath}$ and $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

Solution

A vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 0 \\ -1 & 2 & 2 \end{vmatrix} = \hat{\imath}(2-0) - \hat{\jmath}(6-0) + \hat{k}(6+1) = 2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}$$

Now, a unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is obtained by dividing the vector $\vec{a} \times \vec{b}$ by its magnitude.

$$\therefore \quad \hat{e} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}}{\sqrt{2^2 + (-6)^2 + 7^2}} = \frac{2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}}{\sqrt{89}} = \frac{2}{\sqrt{89}}\hat{\imath} - \frac{6}{\sqrt{89}}\hat{\jmath} + \frac{7}{\sqrt{89}}\hat{k}$$

Example

Show that $\vec{A} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$, $\vec{B} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{C} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$ are coplanar.

Solution

The necessary and sufficient condition for three vectors $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$ and $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath} + C_z\hat{k}$ to be coplanar is

$$[\vec{A}\ \vec{B}\ \vec{C}] = 0 \quad \text{or} \quad \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0$$

Now $[\vec{A}\ \vec{B}\ \vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= 0$$

Therefore, the vectors $\vec{A}, \vec{B}, \vec{C}$ are coplanar.

Example

If $\vec{A}.\vec{B} = \sqrt{3}$ and $\vec{A} \times \vec{B} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$, find the angle between $\vec{A}$ and $\vec{B}$.

Solution

Given

Example

Find all vectors $\vec{V}$ such that $\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) \times \vec{V} = 3\hat{\imath} + \hat{\jmath} - 5\hat{k}$.

Solution

Given

Differentiation of Vectors: velocity and acceleration

Example

A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time.

(a) Determine its velocity and acceleration at any time

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

Solution

We perform some