Signals and Systems

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Tags	NSTU Year 2 Term 2

Resources:

- https://www.youtube.com/watch?v=x5qRAihZRks&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=2

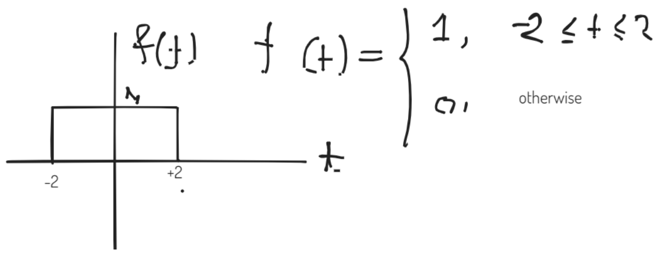
Skipped:

- https://www.youtube.com/watch?v=Rew03iHJGhk&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=9&pp=iAQB (Complex Exponentiation)

Signals

Physical quantity that contains information. Systems are expressed mathematically as function of independent variable, which is usually time.

Example:

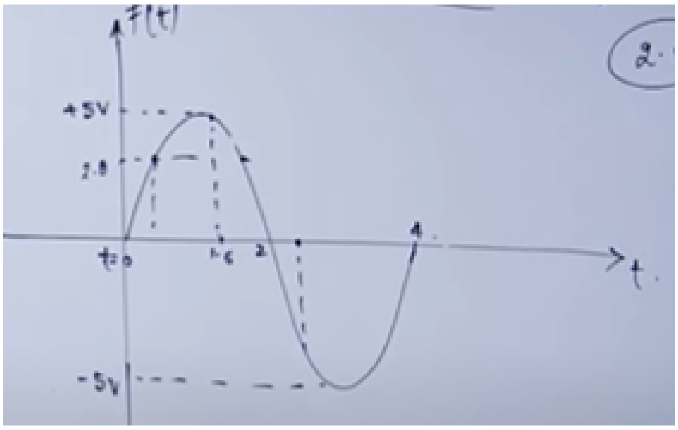


Basic Types of Signals:

Based on Continuous and Discrete:

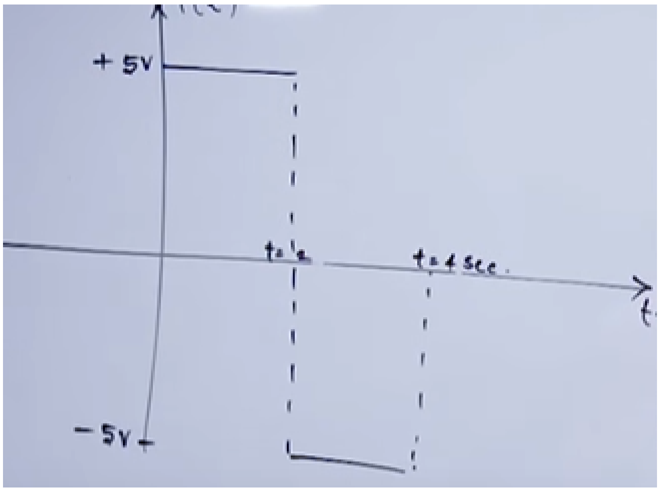
- **Continuous in Time Signal & Continuous in Value Signal**
  - A continuous-time signal has values for all points in time in some interval.

- A continuous-value signal is all possible value within an interval will be available in a signal.



Has values for **0≤t≤4** and all values are available in the signal with **-5≤v≤+5**

- **Continuous in time but discrete in value signal**
  - A continuous-time signal has values for all points in time in some interval.
  - All values within a range is not available in the signal.

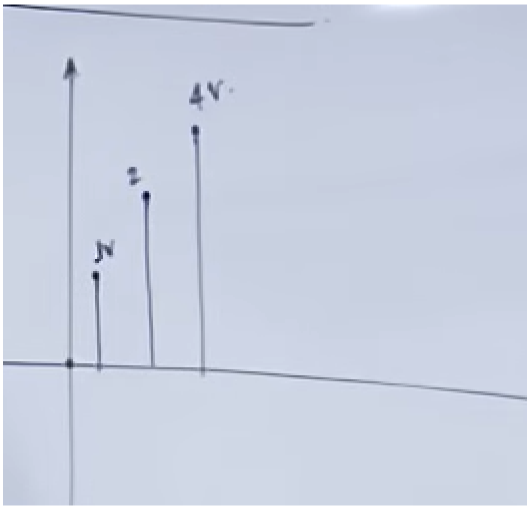


Have value for time within 0≤t≤4 but all values within -5≤v≤5 is not available

- **Continuous in value but discrete in time signal**
  - Haven’t values for all points in time within an interval
  - All values within an range are available in a signal

All values are available, but some points in time haven't values

- **Discrete in time and discrete in value signal**



Analog Signal: Continuity in any of the domain (time or value)

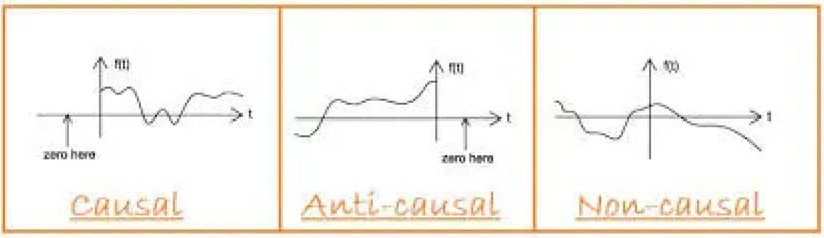
Digital Signal: Discrete in both time and value

<u>**Based on Causal, Anti-Causal, Non-Causal**</u>

- **Causal Signals:**
    - 0 for all negative value/time

- $x(t) = \begin{cases} x(t) > 0 & t \geq 0 \\ 0 & t < 0 \end{cases}$

- **Non-Causal Signals**
  - A signal that have positive amplitude for both positive and negative instance of time
- **Anti-Causal Signal**
  - 0 for all positive value/time
  - $x(t) = \begin{cases} x(t) > 0 & t \leq 0 \\ 0 & t > 0 \end{cases}$

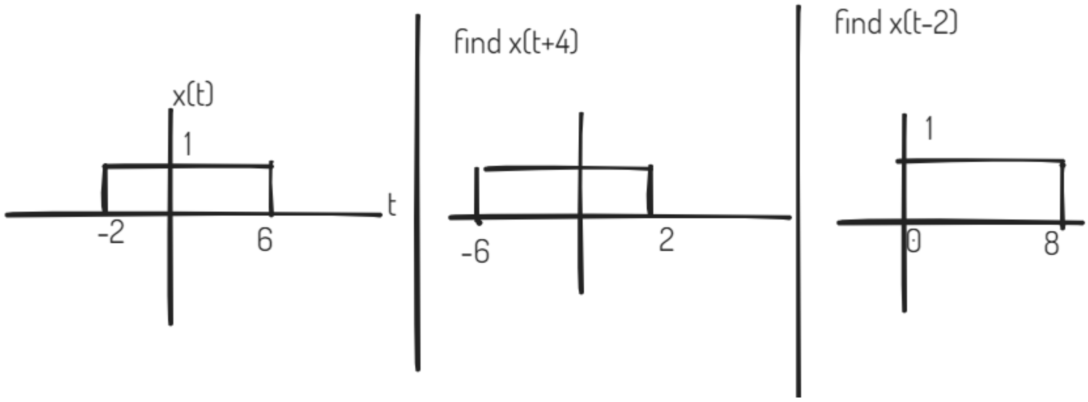


Causal, Anti-Causal, Non-Causal

Operations on Signals

- **Time Shifting Operation**

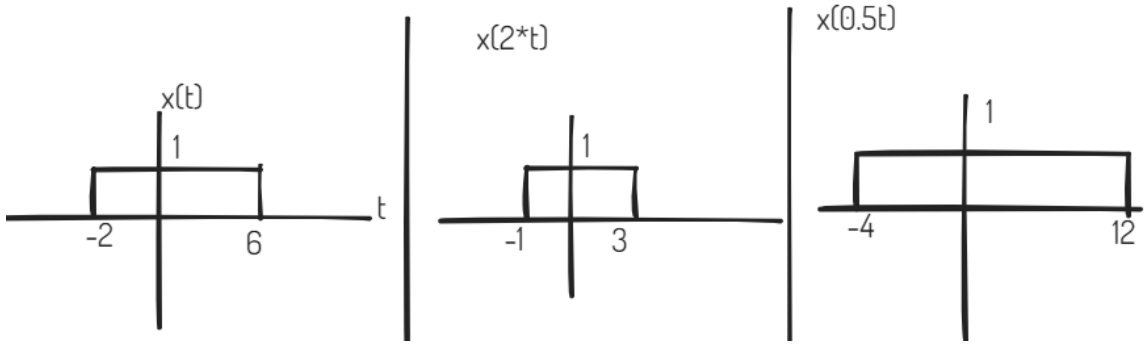
  $f(t)$ is be given. $f(t \pm t_0) = ?$
  - t_0 is a constant
  - $+ \rightarrow$ **advance :** Shift the signal towards left by t_0
  - $- \rightarrow$ **delay :** Shift the signal towards right by t_0
  - Amplitude doesn't change for shifting



- **Time Scaling Operation**

  $x(t)$ is given. $f(\alpha t) = ?$

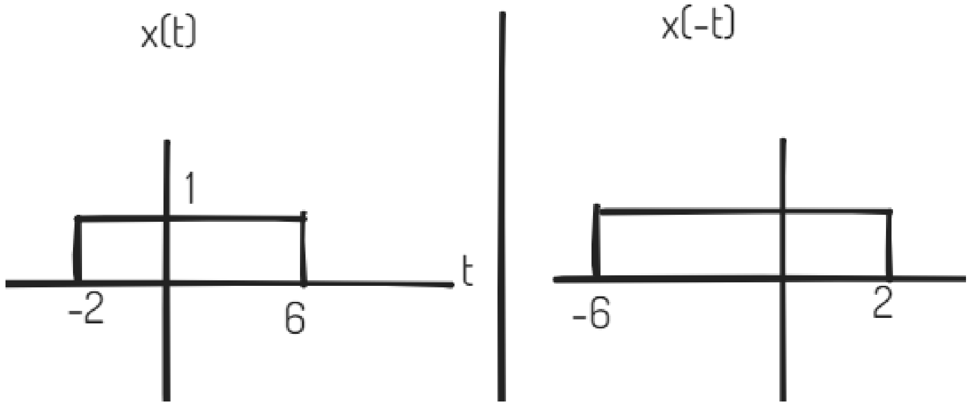
  - α : scaling factor
  - $\alpha > 1$ → **Signal Compression (Increasing Speed) :** Divide the existing limit by α
  - $\alpha < 1$ → **Signal Expansion (Decreasing Speed) :** Divide the existing limit by
  - Amplitude doesn't change for this operation



- **Time Reversal or folding Operation**

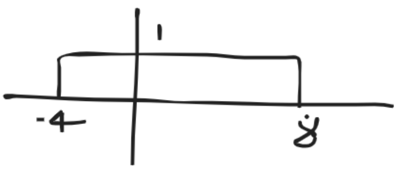
  $x(t)$ is given. $x(-t) = ?$

  - The sign of the limit will be changed



<u>Example on Operation:</u>

$* \; x(t)$

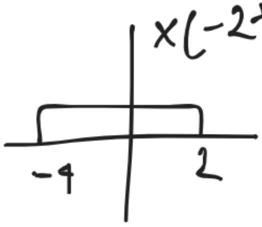


$x(-2t-2) = \,?$

$\Rightarrow x(-2(t+1))$ - 1. the coefficient of t must be one

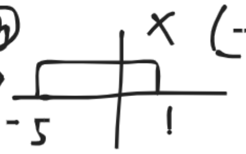
① $x(-t)$

$\rightarrow$ ② $x(-2t)$



③ $x(-2(t+1))$

$\rightarrow$

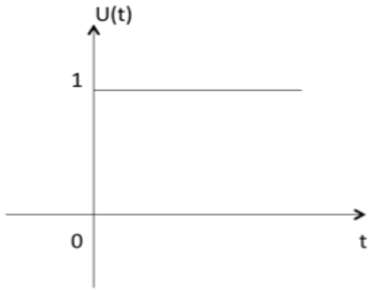


Elementary Signals

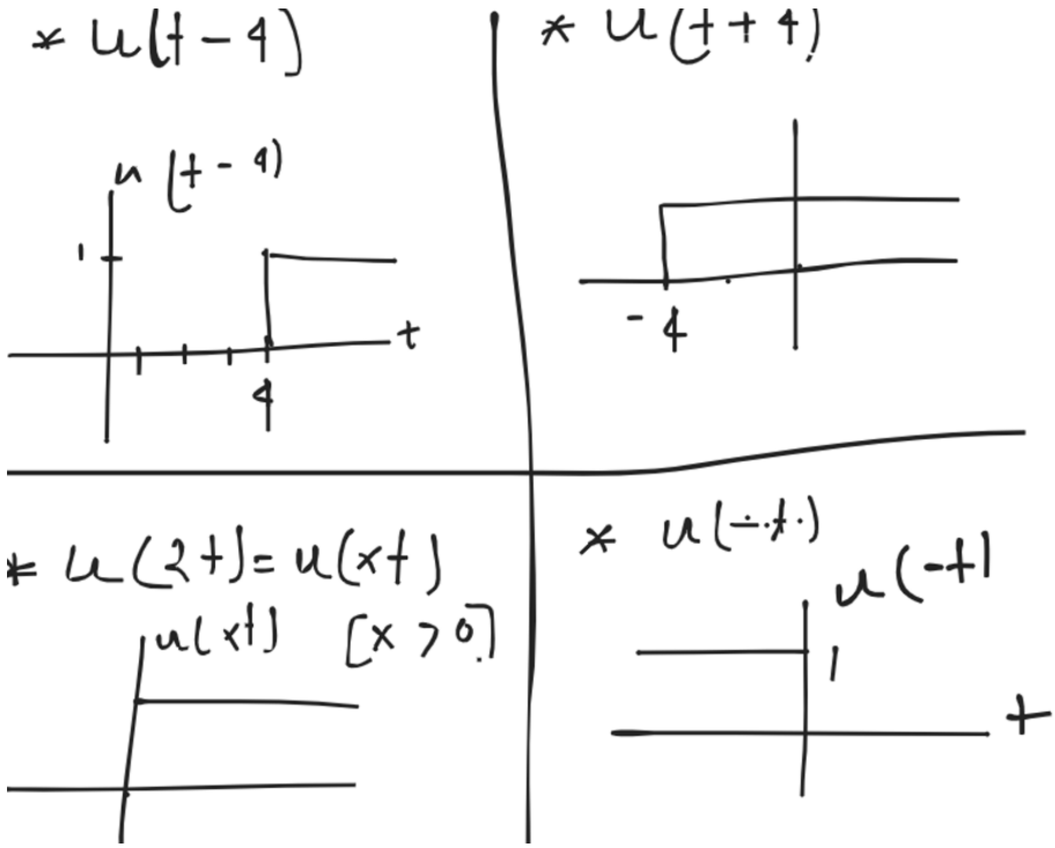
- **Unit Step Signal**
  - Also known as **Heaviside Step Function**

- Unit step function is denoted by $u(t)$

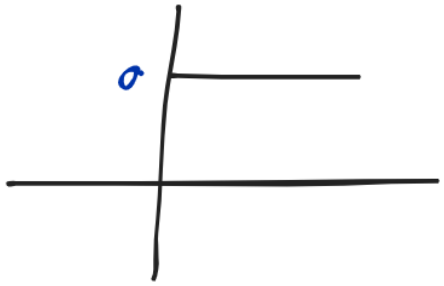
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- Amplitude = coefficient of $u(t)$
- Non-Causal Signal$*$Unit Step Function $=$ Causal Signal
- **<u>Operations on Unit Step Signal:</u>**



* $a\,u(t)$

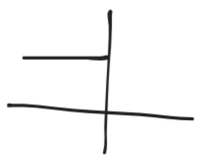


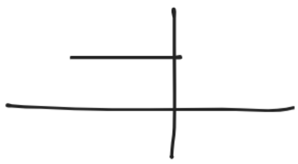
- **Example:**

* $u(-2t+4)$

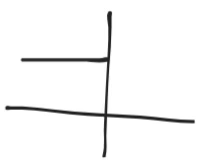
$\Rightarrow u(-2(t-2))$

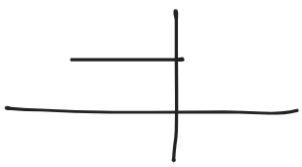
① $u(-t)$ ② $u(-2t)$



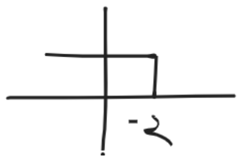


① $u(-t)$ ② $u(-2t)$





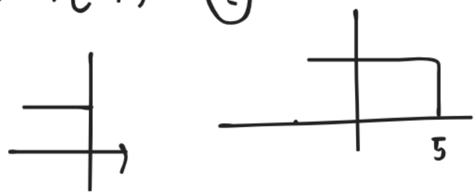
③ $u(-2(t-2))$



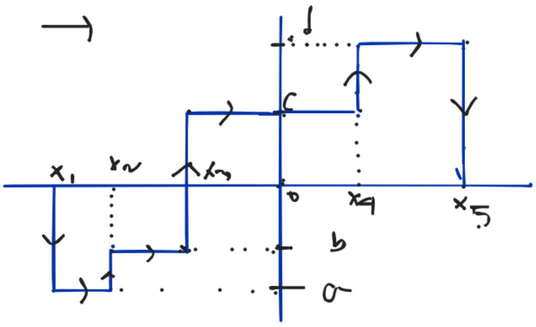
$$x\,u(-t+5)$$

$$\Rightarrow u(-(t-5))$$

① $u(-t)$ ②

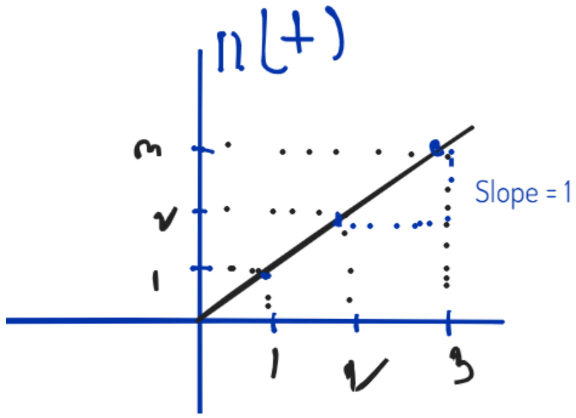


- **Expressing by unit step function**

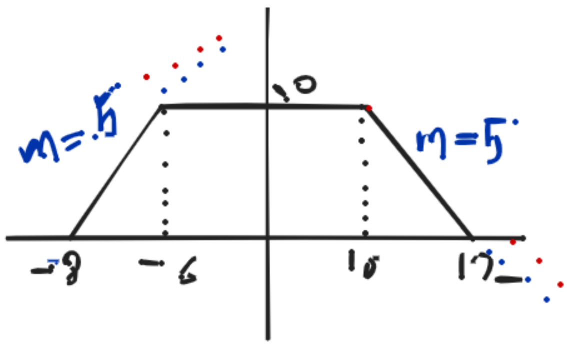


$$x(t) = (-a-0)\,u(t+x_1) + [-b-(-a)]\,u(t+x_2)$$
$$+ [c-(-b)]\,u(t+x_3) + [d-c]\,u(t-x_4)$$
$$+ (0-d)\,u(t-x_5)$$

- **Unit Ramp Signal**
  - $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$
  - Slope = Coefficient of $r(t)$



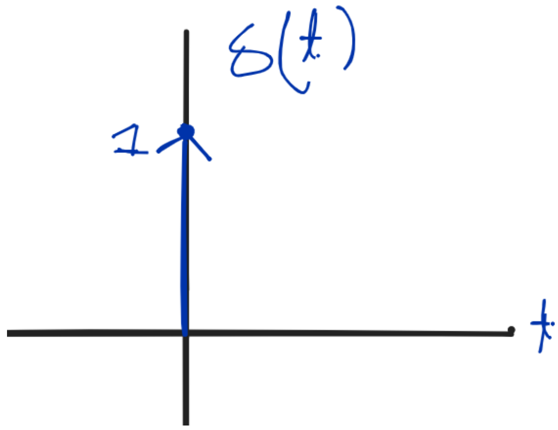
- $r(t) = t \cdot u(t)$
- $\frac{d}{dt}r(t) = u(t)$
- $\frac{d}{dt}[A \cdot r(t)] = A \cdot u(t)$
- **Expressing into Ramp Signal**



$$= +5r(t+8) - 5r(t+6) - 5r(t-10) + 5r(t-12)$$

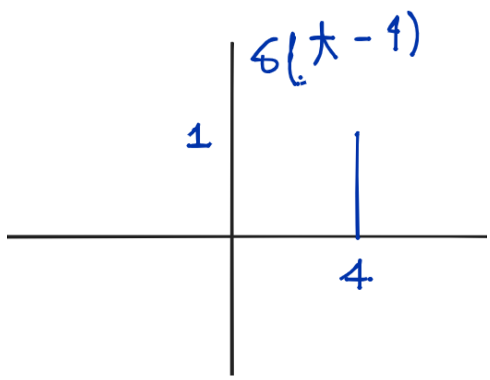
- **Impulse Function**
  - An ideal impulse signal is a signal that is zero everywhere but at the origin (t = 0), it is infinitely high. Although, the area of the impulse is finite.

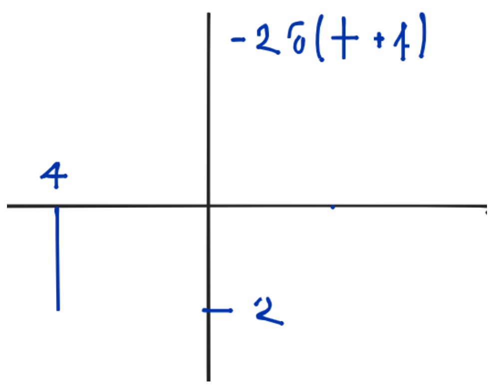
- $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$
- $A \cdot \delta(t)$, **Here A is the area of this impulse function.**



- $\int_{-\infty}^{\infty} \delta(t) = 1$ or $\int_{0^-}^{0^+} \delta(t) = 1$
- $\int_a^b \delta(t - t_0)$, it exists $if\ a \leq t_0 \leq b$, otherwise 0
- $\delta(t) = \frac{d}{dt} u(t)$

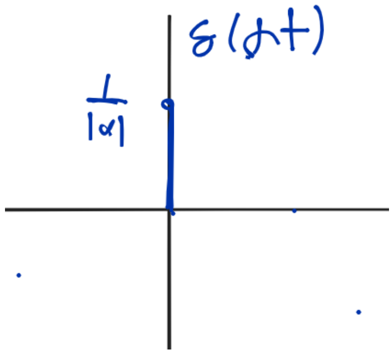
- **Operations:**





- **Properties:**
  - **Time Scaling Property:**

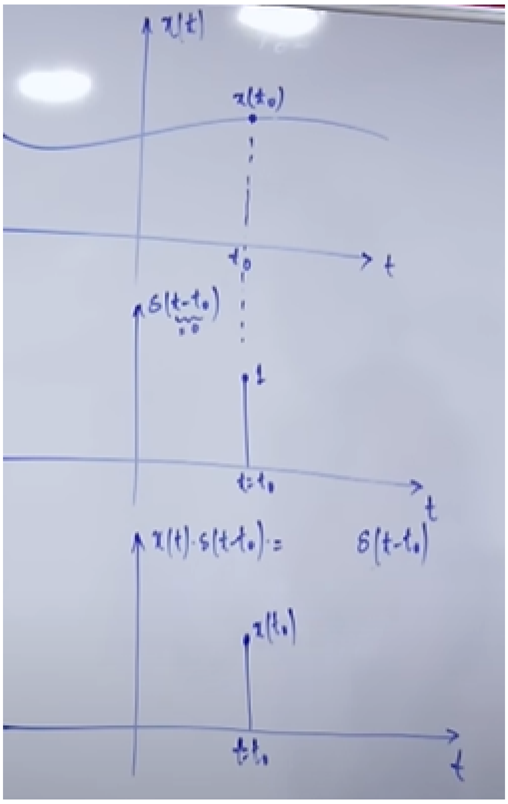
    $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$



  - **Product Property**

    $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$

    $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$



- **Shifting Property**

$(i) \int_{-\infty}^{\infty} x(t) \cdot \delta(t)$

$= \int_{-\infty}^{\infty} x(0) \cdot \delta(t)$

$= x(0) \int_{-\infty}^{\infty} \cdot \delta(t)$

$= x(0)$

$(ii) \int_{-\infty}^{\infty} x(t) \cdot \delta(t - t_0)$

$= \int_{-\infty}^{\infty} x(t_0) \cdot \delta(t)$

$= x(t_0) \int_{-\infty}^{\infty} \cdot \delta(t)$

$= x(t_0)$

$$I = \int_{-6}^{8} (t^2 + 4)\, \delta(t-3)\, dt$$

$$x(t) = t^2 + 4.$$

$$x(3) = 9 + 4 = 13$$

$$I = \int_{-6}^{8} x(t_3)\, \delta(t-3)\, dt$$

$$= x(t_3) \times 1 = 13$$

$$\star \quad I = \int_{-\pi}^{\pi} \underbrace{\cos^2 t}_{x(t)} \cdot \underbrace{\delta\left(t - \frac{\pi}{4}\right)}_{t = \frac{\pi}{4}} dt$$

$$I = \left(\cos^2 t\right)\Big|_{t=\frac{\pi}{4}} = \left(\cos\frac{\pi}{4}\right)^2 = \frac{1}{2}.$$

$$I = \int_{-6}^{5} \overset{x(t)}{(t-2)} \; \delta(2t-4)$$

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$$

$$\delta\{2(t-2)\} = \frac{1}{2}\delta(t-2)$$

$$I = \frac{1}{2}\int_{-6}^{5} x(t)\,\delta(t-2)$$

$$= \frac{1}{2}\cdot x(2) = 0$$

- **Extension Properties**

  $\int_a^b x(t) \cdot \delta(g(t)) dt$

  $\delta(g(t)) = \frac{\delta(t-t_0)}{|g'(t_0)|}[if\ g\ has\ a\ real\ root\ at\ t = t_0]$

  If g has more than one real root at $t_0, t_1, ..., t_i$

  $s(g(t)) = \sum_i \frac{\delta(t-t_i)}{g'(t_i)}$

  Example:

  $I = \int_{-10}^{10} (t^2 + 10) \cdot \delta(t^2 - 16) dt$

  $t^2 - 16, t = +4, -4$

  $I = \int_{-10}^{10} (t^2 + 10) \cdot \frac{\delta(t-4)}{|2\cdot(-4)|} dt + \int_{-10}^{10} (t^2 + 10) \cdot \frac{\delta(t+4)}{|2\cdot(4)|} dt$

  $= \frac{13}{4}$

- **Exponential Function**

  $x(t) = Ae^{(\alpha t)}$

$A =$ Amplitude at $t = 0$

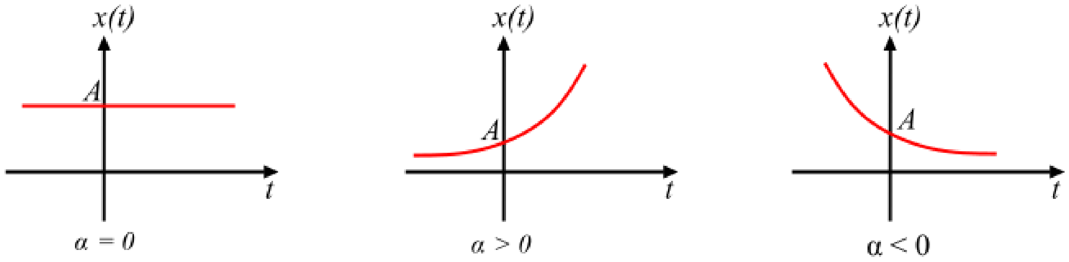


Figure-1

Signal Classification on Even and Odd

- **Even/Symmetric Signal**

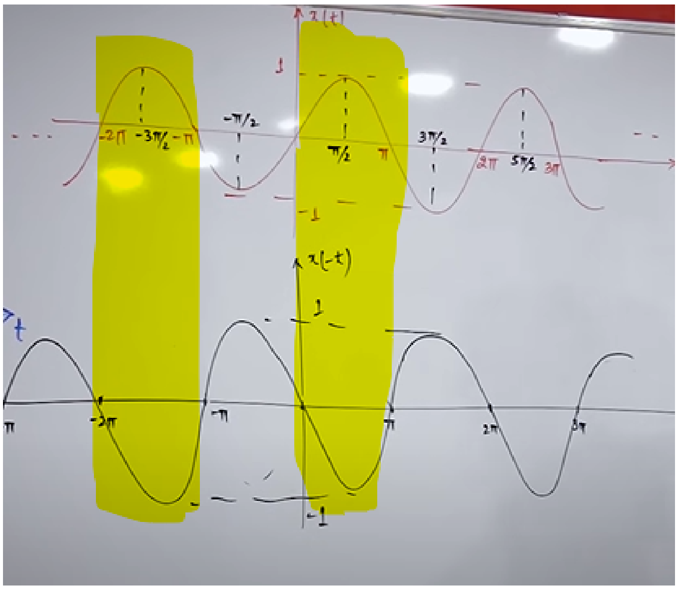
  $x(-t) = x(t)$

  <u>**Example :**</u> $x(t) = \begin{cases} 2 & -a \leq t \leq a \\ 0 & otherwise \end{cases}$, $x(t) = t^2, ..., t^{2n}$

- **Odd/Anti-symmetric Signal**

$x(-t) = -x(t)$

<u>**Example:**</u> $x(t) = t^3, t^5, .., t^{2n+1}$



<u>**Check a function even or odd:**</u>

$x(t)$ is given. Find $x_1(t) = x(-t) = ...$

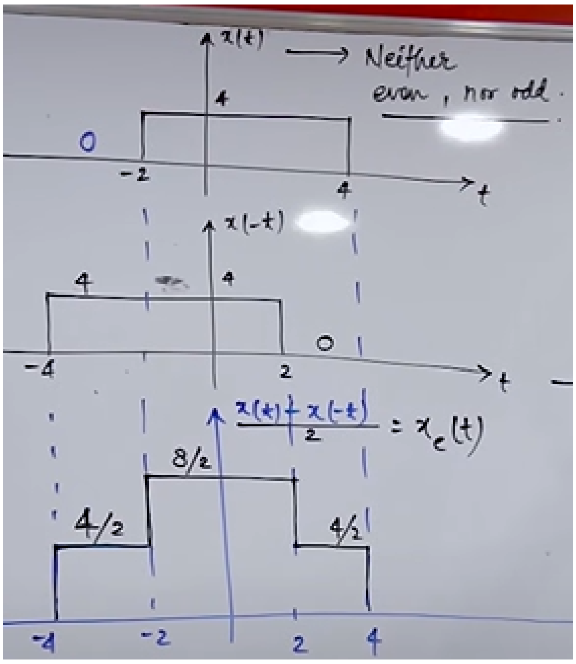
if $x(1) == x_1(1)$, it is even, otherwise odd.

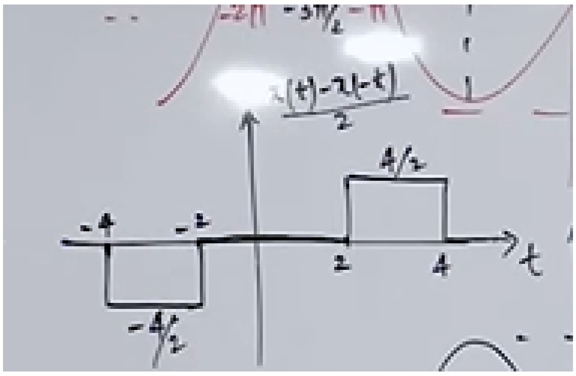
Converting to Odd or Even Signal

To convert any arbitrary signal which is neither even nor odd into equivalent even or odd parts:

$x_e(t) = \frac{x(t)+x(-t)}{2}$

$x_o(t) = \frac{x(t)-x(-t)}{2}$





- $x(t)$ **is complex**

  $x(t) = a + ib$

  - **Even**

    $x(t) = \overline{x(-t)}$

  - **Odd**

    $x(t) = -\overline{x(-t)}$

  Conjugate symmetric Part of $x(t)$

  $x_e(t) = \frac{x(t)+\overline{x(-t)}}{2}$

  Conjugate anti-symmetric part of $x(t)$

  $x_o(t) = \frac{x(t)-\overline{x(-t)}}{2}$

- **Some Operations**

  $E = Even, O = Odd$

  $O \pm O = O$

  $E \pm E = E$

  $O \pm E = Neither\ Odd\ nor\ Even$

  $E \times E = E$

  $E \times O = O$

  $O \times O = E$

  $E/E = E$

  $O/O = E$

  $E/O = O$

  $O/E = O$

  $\int O = E$

  $\int E = O$

  $\frac{d}{dt}E = O$

  $\frac{d}{dt}O = E$

Periodic and Aperiodic Signal

- **Periodic Signal**
  - A signal is said to periodic, if it satisfies following two properties
    - It must be exist for $-\infty \leq t \leq \infty$

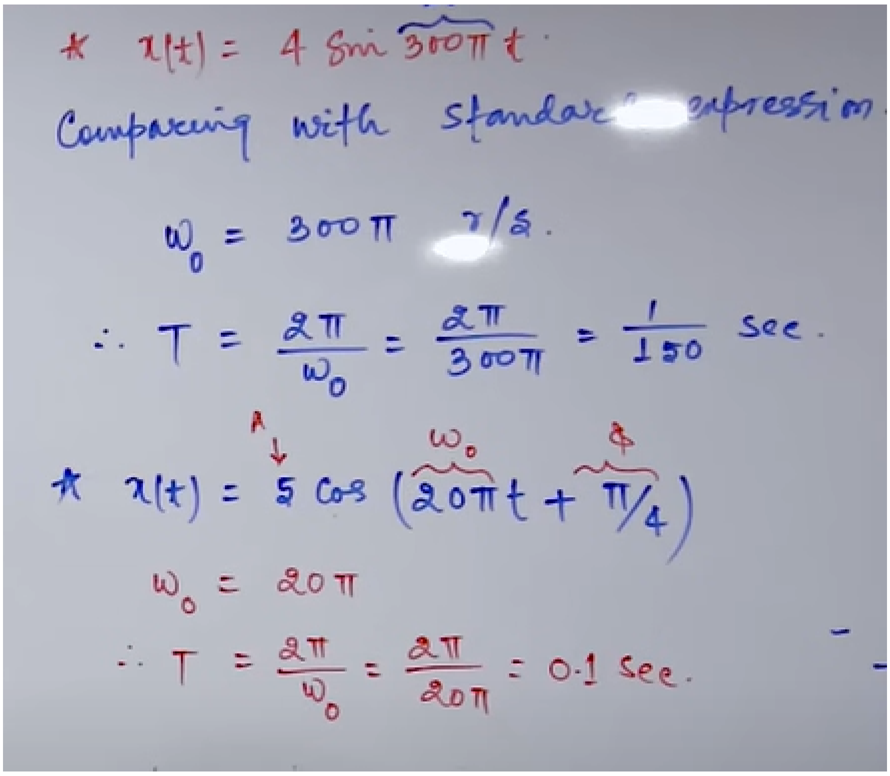
        - It must repeat itself after some constant amount of time T, which is called **Fundamental Time Period**
            - $T = \frac{2\pi}{w_0}$
            - $w_0 =$ Fundamental frequency $rad\ s^{-1} = 2\pi f$
            - $T = \frac{1}{f}$
            - Frequency must be real number. If frequency is not real number, it's not periodic.
    - **Types**
        - **Sinusoidal Signals**
            - **Representation**

              $x(t) = Asin(w_0 t + \theta)$

              $A =$ Amplitude

              $w_0 t =$ Phase Angle

              $\theta =$Phase shift (+ → Advance, - delay)
            - $\frac{d}{dt}(Phase) = \frac{d}{dt}(w_0 t) = w_0 = freequency$
            - Shifting effect doesn't effect on periodicity, T
            - $x(t \pm kT) = x(t)$
            - Comparing with $x(t) = Asin(n\pi t + \theta)$**, if t is not square root of t, then the signal must be periodic.**



- **Combination of periodic signals**

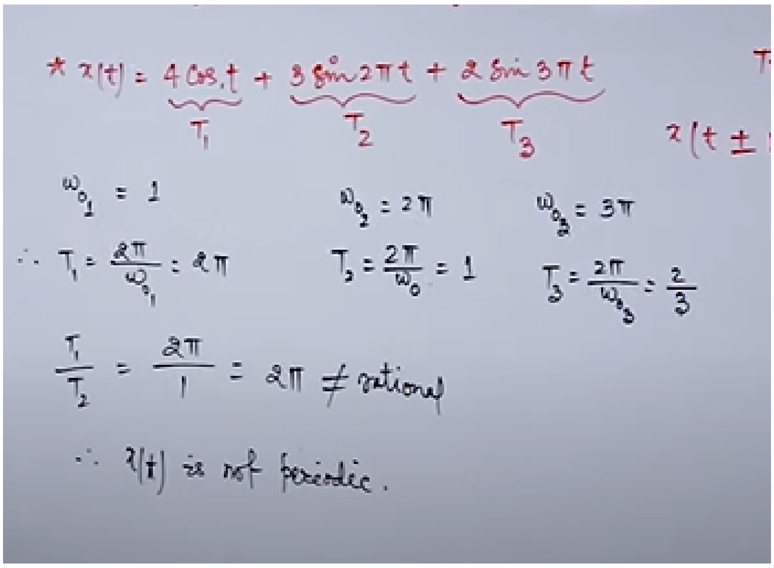
  $x(t) = Asin(w_0t) + Bcos(w_1t) + Csin(w_2t) + ...$

  - $x(t)$ **will be periodic if ratio of individual time period is a rational number**
    - **Rational Number**
      - Can be expressed by $\frac{p}{q}$. p, q are co-prime.
      - The value of $\frac{p}{q}$ should terminating or repeating decimal.
        Example : 3.3333.., 2.5, 5.20202020..

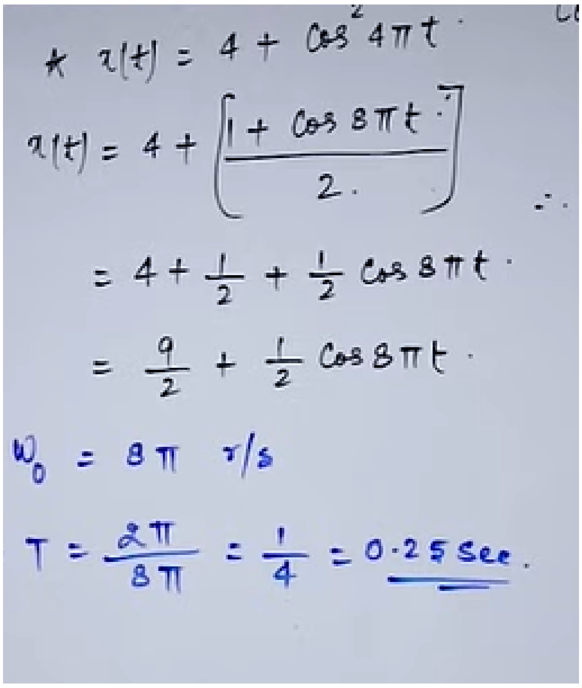
    Example: $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_1}{T_3} = Rational\ Number$

  - **Time Period of Resultant Signal**
    $T = LCM(T_1, T_2, ...)$
  - **Frequency of Resultant Signal**
    $w_0' = GCD(w_0, w_1, ...)$
  - If all $w_0, w_1, ...$ have π, then it is periodic. Or if all $w_0, w_1, ...$ haven't π, it is periodic too. If some of them have π and rest of them

not, then it isn't periodic.



- **DC/Constant Signal :** independent from time. It doesn't effect on frequency, time. So, it is not countable.



4 is a constant Signal

Energy and Power Signals

- **Energy Signals**

Energy of $x(t)$ is given as

$E_x = \lim\limits_{t \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$

$[\text{No need to take lim if the signal is aperiodic}]$

if $0 < E_x < \infty$ (Finite), then $x(t)$ is said to be $Energy \ Signal.$

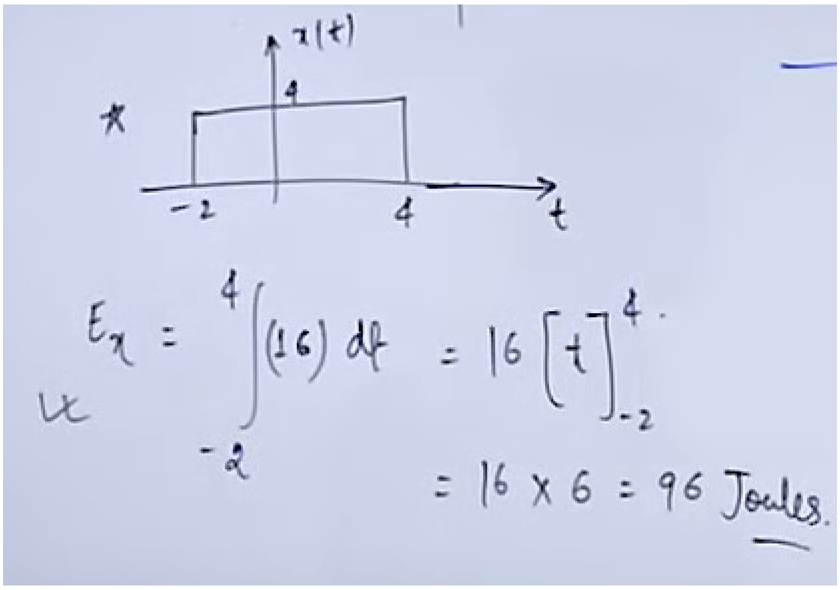
Example:

$(i) \ x(t) = e^{-4t} \cdot u(t)$

$E_x = \int_{-\infty}^{\infty} [e^{-4t} \cdot u(t)]^2 \, dt$

$= \int_0^{\infty} e^{-8t} dt \ \ [\text{u(t) exists only 0 to } \infty]$

$= \frac{1}{8}$



- When a signal $x(t)$ will be energy signal
  - If $x(t)$ is existing for infinite direction and decreasing in value.

    $\lim\limits_{t \to \infty} f(t) = 0$
  - If $x(t)$ exists for finite direction and value of $x(t)$ is finite finite at all points, $x(t)$ is energy signal.

- **Power Signals**

  $Power \ Signal$ of $x(t)$ is given as

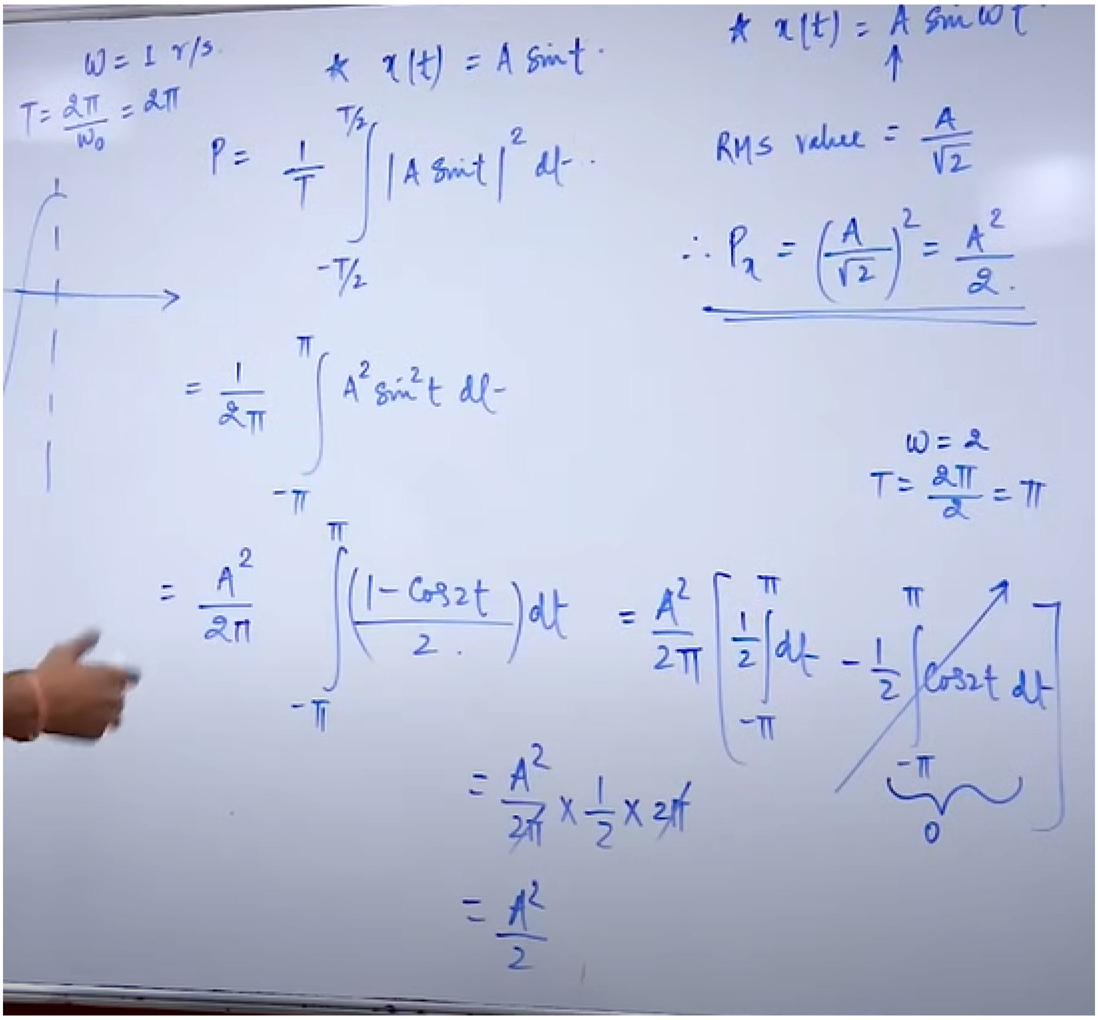
  $P_x = \lim\limits_{T \to \infty} \frac{1}{T} \cdot E$

$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$

$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$
$[\text{No need to take lim if the signal is aperiodic}]$

If $0 < P_x < \infty$ (Finite), then $x(t)$ is said to be Power Signal

- $\text{Power} = RMS^2$
- $x(t) = Asinwt,\ RMS = \frac{A}{\sqrt{2}}, P = RMS^2 = \frac{A^2}{2}$



- **When a signal will be Power Signal**
  - All periodic signal are power signal but converse is not true
  - If $x(t)$ is not a periodic signal and follows the conditions

- $\lim_{t \to \infty} f(t) \neq 0$
- $\lim_{t \to \infty} f(t) \neq \infty$

- A signal can't be Energy and Power Signals together.
  - If E_x is finite, then P_x is Zero . Vice-Versa.
- **Operations**
  - Time shifting has no effect on power and energy of signal.$Power\ x(t) = Power\ x(t - \frac{T}{2})$
    $Energy\ x(t) = Energy\ x(t-4)$
  - Time Scaling doesn't effect on Time Periodic but in Time period, Energy.
    - $\text{For } x(t) T, E_x$

      $x'(t) = x(\alpha t), T' = \frac{T}{\alpha}, E'_x = \frac{E_x}{\alpha}$
    - Power remains same.
  -

Systems

System is a interconnection of different physical components which is used to convert one form of signal to others.

Example: