DSP: Z-Transform and Basic of Filters

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Tag	Year 3 Term 1

Z Transform

- Discrete time LTI system z transform is used.
- Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z.
- Biliteral/Two sided
  $X(z) = Z(x(n)) = \sum_{n=-\infty}^{n=\infty} x(n)Z^{-n}$
- Uniliteral/One sided
  $X(z) = Z(x(n)) = \sum_{n=0}^{n=\infty} x(n)Z^{-n}$
- if $x[n] = 0$ for $n < 0$, then Biliteral = Uniliteral

Geometric Series Sum formula

- Infinite
  - $\sum_{-\infty}^{\infty} c^n = \frac{1}{1-c}$

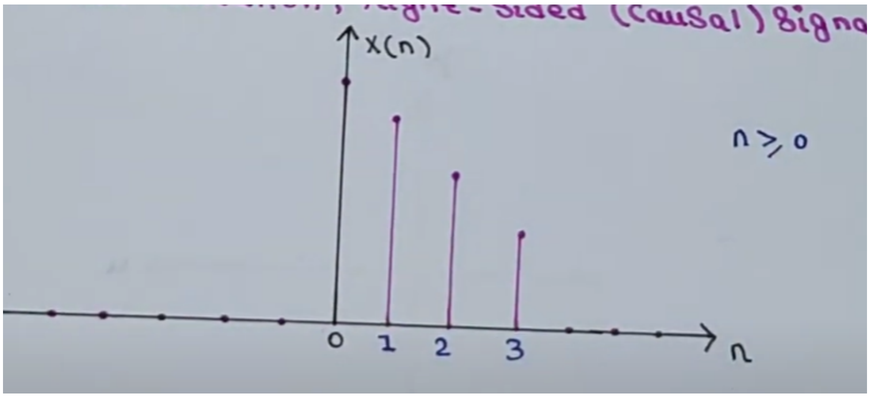
  $0 < |c| < 1$
- Finite
  - $c \neq 1$
    $\sum_0^{N-1} c^n = \frac{c^N-1}{c-1}, \sum_0^N c^n = \frac{c^{N+1}-1}{c-1}$
  - $c = 1$
    $\sum_0^{N-1} c^n = N, \sum_0^N c^n = N+1$

  (c is a complex constant)

ROC

- The set of Z for which $X(z)$ is converges (gives finite value)/the set of points in Z-plane for which $X(z)$ is converges is called *Region Of Converges.*
- If there's no value of z where $X(z)$ is converges, then sequence of $x(n)$ is said to be having no Z transform.

- **ROC for finite duration signal, right-sided (causal) signal**



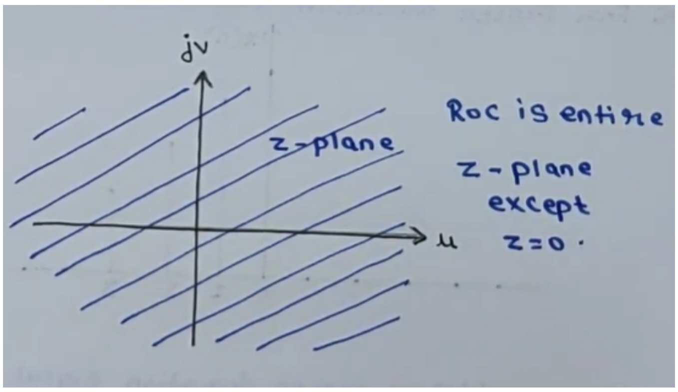
N samples, $0 \le n \le N-1$

$x(n) = \{x(0), x(1), ...x(n)\}$

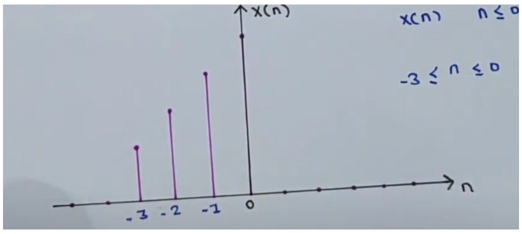
$X(z) = \sum_{n=0}^{n=\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + ...x(n-1)z^{-(n-1)}$

if $z = 0$, $X(z)$ is infinite.

$X(z)$ exists for all values of z, except $z = 0$



- **ROC for finite duration, left sided (anti causal)**

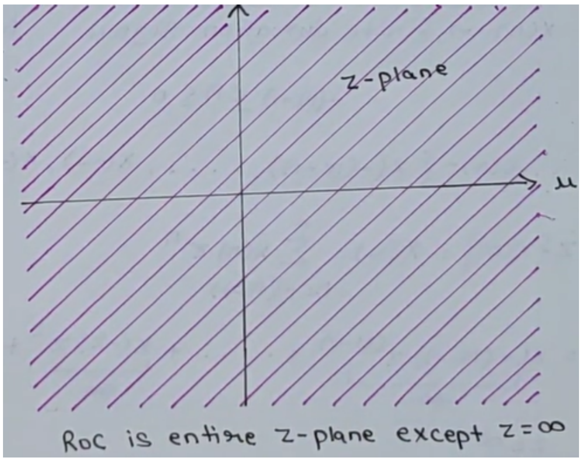


Defined : $-(N-1) \leq n \leq 0$

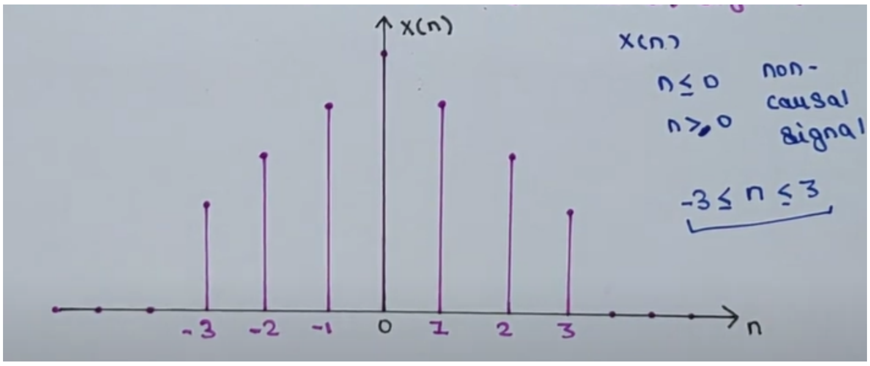
$x(n) = \{x(-(n-1)...x(0)\}$

$X(z) = \sum_{n=-(N-1)}^{0} x(n)z^{-n} = x(-(N-1))z^{N-1} + ... + x(0)$

if $z = \infty$, $X(z)$ is infinite. $X(z)$ exists for all values of z, except $z = \infty$.



- **ROC for finite duration, two sided (non causal signal)**

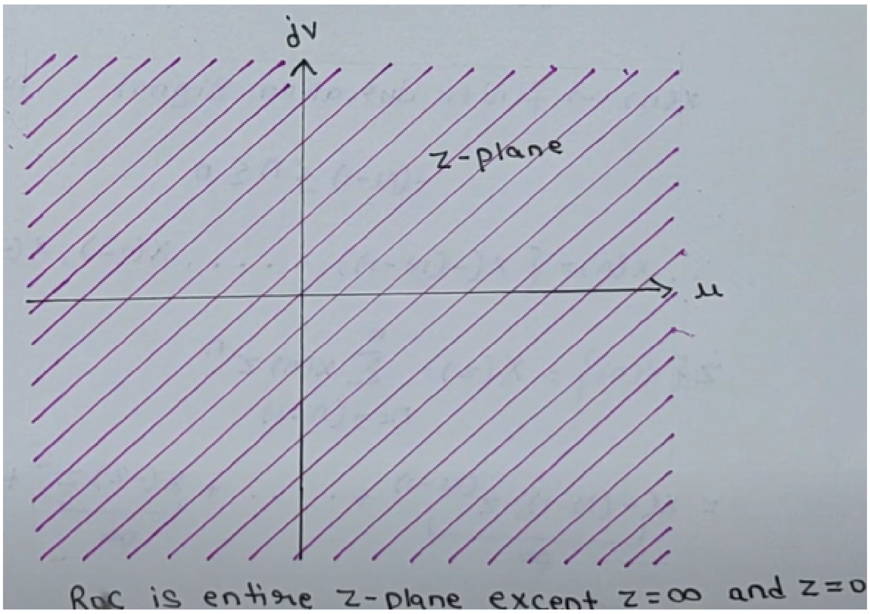


$-m \leq n \leq +m$, $m = \frac{N-1}{2}$, N samples

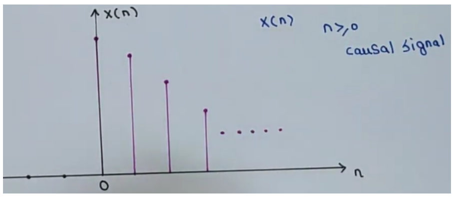
$x(n) = \{x(-m), ..x(0)...x(m)\}$

$X(z) = \sum_{-m}^{+m} x(n)z^{-n} = x(-m)z^{m} + ...x(0)... + x(m)z^{-m}$

$z = 0, \infty$, there are no finite value. ROC = entire $Z-plane$ except $Z = 0, \infty$



- **ROC for infinite duration, right-sided (causal) signal**



Let, $x(n) = r^n; n > 0$

$X(z) = \sum_{n=0}^{\infty} (rz^{-1})^n$

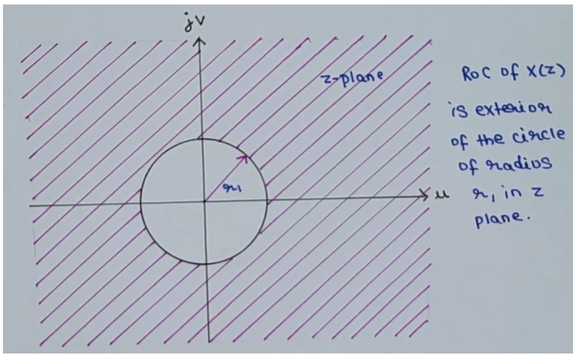
if $0 < |rz^{-1}| < 1,$

$X(z) = \frac{1}{1-rz^{-1}}$

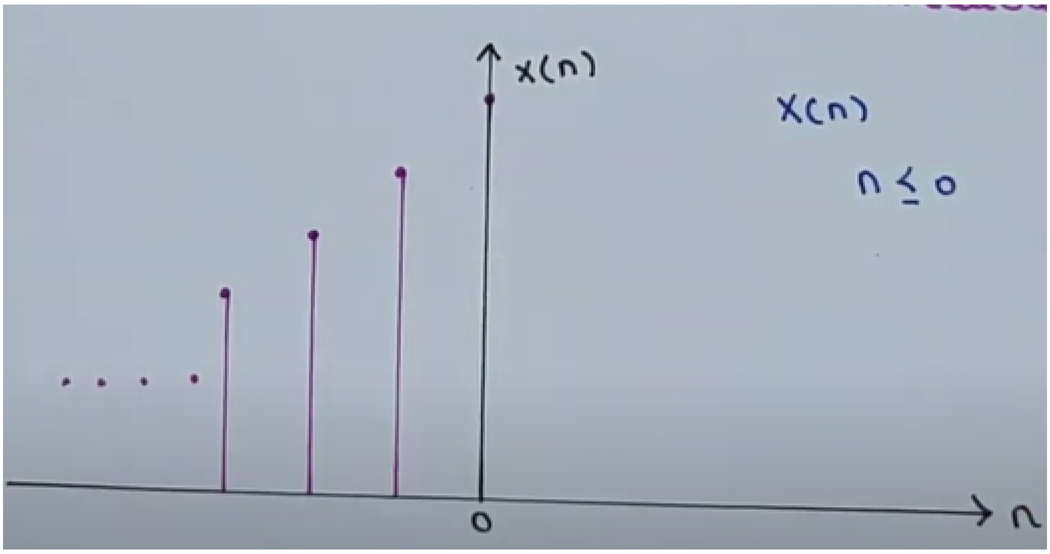
The condition to be satisfied for the convergence of $X(z)$

$0 < |rz^{-1}| < 1$

$|rz^{-1}| < 1 \Rightarrow |z| > |r|$



- **ROC for infinite duration, left-sided(anti causal) signal**



$X(z) = \sum_{n=-\infty}^{0} (r_2 z^{-1})^n = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$

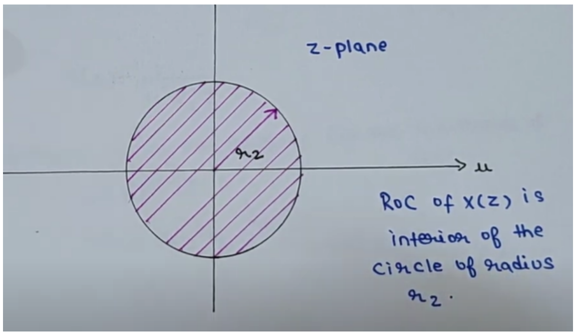
if, $0 < |r_2^{-1} z| < 1$

$X(z) = \frac{1}{1-r_2^{-1} z}$

Condition to be satisfied for the convergence,

$|r_2^{-1} z| < 1$

⇒ $|z| < |r_2|$



- **ROC for infinite duration, two-sided (non causal) signal**

Let, $x(n) = r_1^n u(n) + r_2^n y(-n), -\infty \leq n \leq +\infty$

$X(z) = \sum_{n=-\infty}^{\infty} [r_1^n u(n) + r_2^n y(-n)] z^{-n}$

$= \sum_{n=-\infty}^{0} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^n$

$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$

$= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1 z^{-1})^n$

If $0 < |r_2^{-1}z| < 1$ and $0 < |r_1z^{-1}| < 1$ then

$X(z) = \frac{1}{1 - r_2^{-1}z} + \frac{1}{1 - r_1z^{-1}}$

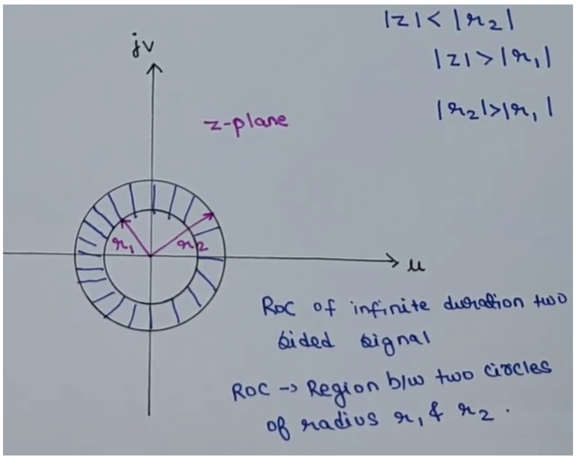
Converges if,

$|r_1z^{-1}| < 1$

⇒ $|r_1| < |z|$

$|r_2^{-1}z| < 1$

⇒ $|z| < |r_2|$



Properties of Z Transform

- **Linearity Property :** The linearity property of Z-transform states that the Z-transform of a weighted sum of two discrete time signals is equal to the weighted sum of individual Z-transform of that system.

$a_1x_1(n) + a_2x_2(n) \leftrightarrow a_1X(z) + a_2X(z)$

- **Shifting Property :** shifting of m-units obtained by multiplying z^m

$x(n) \leftrightarrow X(z), -\infty < n < +\infty$

$x(n-m) \leftrightarrow X(z)z^{-m}$

$x(n+m) \leftrightarrow X(z)z^{m}$

- **Shifting Property of One-sided**

$x(n) \leftrightarrow X(z), 0 < n < +\infty$

$x(n-m) \leftrightarrow z^{-m}X(z) + \sum_{i=1}^{m} x(-i)z^{-(m-i)}$

$x(n+m) \leftrightarrow z^{m}X(z) - \sum_{i=1}^{m} x(i)z^{(m-i)}$

- **Multiplication property**

$x(n) \leftrightarrow X(z)$, then

$nx(n) \leftrightarrow -z\frac{d}{dx}X(z)$

- **Multiplication by an exponential sequence property**

$x(n) \leftrightarrow X(z)$, then

$a^n x(n) \leftrightarrow X(\frac{z}{a})$

- **Time Reversal Property**

$x(n) \leftrightarrow X(z)$, then

$x(-n) \leftrightarrow X(z^{-1})$

- **Conjugation Property**

$x(n) \leftrightarrow X(z)$, then

$x^*(n) \leftrightarrow X^*(z^*)$

- **Convolution Property**

$x_1(n) \leftrightarrow X_1(z), x_2(n) \leftrightarrow X_2(z)$

$x_1(n) * x_2(n) \leftrightarrow X_1(z)X_2(z)$

$[\text{General convulotion} : x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)]$

- **Correlation property**

$x(n) \leftrightarrow X(z), y(n) \leftrightarrow Y(z)$ then

Z transform of $r_{xy}(m) \leftrightarrow X(z)Y(z^{-1})$

$[r_{xy} = \sum_{m=-\infty}^{\infty} x(n)y(n-m)]$

Finding Z transform

- $x(n) = u(n)$

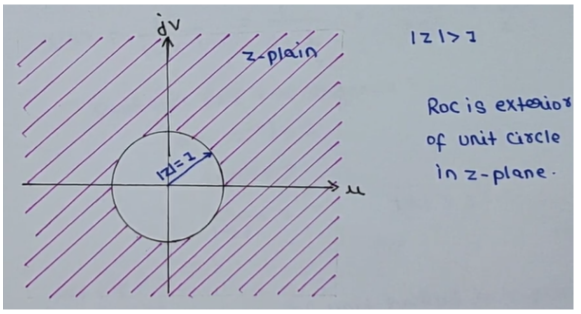
$$x(n) = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

$X(z) = \sum_{n=0}^{\infty} \mu(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$

$0 < |z^{-1}| < 1,$

$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

Condition for converges, $|z^{-1}| < 1$ ⇒ $|z| > 1,$



- $x(n) = (0.3)^n u(n)$

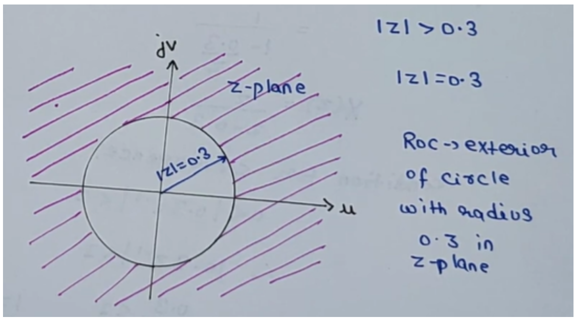
$$x(n) = \begin{cases} 0.3^n; n \geq 0 \\ 0; n < 0 \end{cases}$$

$X(z) = \sum_{n=0}^{\infty} 0.3^n z^{-n}$
$= \sum_{n=0}^{\infty} (0.3z^{-1})^n$

if $0 < |0.3z^{-1}| < 1,$

$X(z) = \frac{1}{1-(0.3z^{-1})} = \frac{z}{z-0.3}$

Condition for converges. $|0.3z^{-1}| < 1$ ⇒ $|z| > 0.3$



- $x(n) = (0.8)^n u(-n-1)$

$$u(n) = \begin{cases} 1; -n-1 \geq 0 \\ 0; n \leq 0 \end{cases}$$

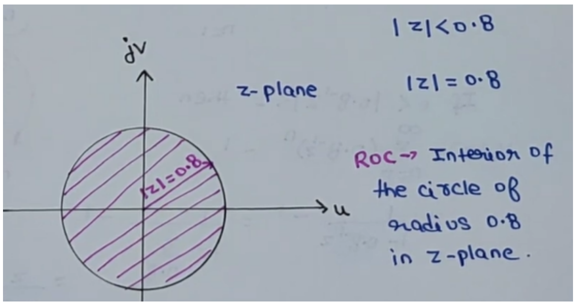
$$x(n) = \begin{cases} 0.8^n; n \leq -1 \\ 0; n \leq 0 \end{cases}$$

$X(z) = \sum_{n=-\infty}^{-1} 0.8^n z^{-n} = \sum_{n=1}^{\infty} 0.8^{-n} z^n = \sum_{n=1}^{\infty} (0.8^{-1} z)^n$

If $0 < |0.8^{-1} z| < 1$, then

$X(z) = \frac{1}{1-0.8^{-1}z} = \frac{0.8}{0.8-z}$

Condition for converges, $|0.8^{-1} z| < 1$ ⇒ $|z| < 0.8$



Poles and Zeroes of Rational Function of Z

$X(z) = \frac{N(z)}{D(z)}$, if $X(z)$ expressed as a ratio of two polynomials $z \text{ or } z^{-1}$, then $X(z)$ is called rational function of z.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+....+b_mz^{-m}}{a_0+a_1z^{-1}+....+a_nz^{-n}} = \frac{b_0}{a_0}\frac{1+\frac{b_1}{b_0}z^{-1}+...+\frac{b_m}{b_0}z^{-m}}{1+\frac{a_1}{a_0}z^{-1}+...+\frac{a_m}{a_0}z^{-n}} \;.... (i)$$

$$= G\frac{z^{-m}(z^m+\frac{b_1}{b_0}z^{m-1}+...+\frac{b_m}{b_0})}{z^{-n}(z^n+\frac{a_1}{a_0}z^{n-1}+...+\frac{a_n}{a_0})}, [\text{Scaling Factor, G} = \frac{b_0}{a_0}]$$

$$= G\frac{(z-z_1)(z-z_2)...(z-z_n)}{(z-p_1)(z-z_2)...(z-p_n)} \; [n=m]$$

$z_1, z_2, ..., z_n$: roots of numerator polynomial, zeros of $X(z)$, marked by ○

$p_1, p_2, .., p_n$: roots of denominator polynomial, poles of $X(z)$, marked by X

- **Inverse Z Transform :**
  - $Z^{-1}(x(z)) = x(n) = \frac{1}{2\pi\hat{j}} \oint_c X(z) z^{n-1} dz$

Questions:

1.	a)	Write the advantages of Z-transform system over DTFT system. Show the transfer function in Z-transform with equations. Distinguish between zeros and poles in Z-plane.	8
	b)	If $(Z) = \frac{Z^2+Z+1}{Z^2+0.5Z+0.75}$, Find out: a). its transfer function representation, b). its difference equation representation.	4
	c)	If X(z) = $\frac{2+Z+3Z^2}{5+9Z+0.5Z^3}$, find the values of input and output coefficients.	2

(a)

Advantages of Z-transform over DTFT system:

- $\delta(n), u(n)$ can't be analyzed by DTFT but Z.
- The transient response of a system **due to initial condition** or due to changing inputs cannot be computed by using DTFT but Z.
- The Z-transform might **exist anywhere in the Z-plane**; the DTFT can only exist on the **unit circle**.
- Z transform allows for the discrete time system **analysis in the frequency domain**. This facilitates the analysis of system characteristics such as **stability, causality, linearity and time and frequency** response.
- Z transform allows the representations of signals and systems in the **frequency domain**.
- It becomes possible to analyze frequency content of signals **determine spectral properties, apply filtering and modulation.**
- Useful for **determining the stability** of DT system.
- It **has convolution property** that simplifies the analysis of linear time invariant system.
- **Simplifies mathematical** operation.

The Transfer Function

The transfer function $H(z)$ of a DT linear system is defined as the ration of the Z-transform of the output signal $Y(z)$ to the Z-transform of the input signal $X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{n=-\infty}^{+-\infty} b_n z^{-n}}{\sum_{m=-\infty}^{+-\infty} a_m z^{-m}}$

$= \frac{b_0+b_1 z^{-1}+...+b_m z^{-m}}{a_0+a_1 z^{-1}+...+a_n z^{-n}}$

Where, $b_0, b_1, ...$ are input coefficient and $a_0, a_1 ...$ are output.

if $a_o = 1$, then this called the standard form of transfer function.

If we can convert, $H(z) = G\frac{(z-z_1)(z-z_2)...(z-z_n)}{(z-p_1)(z-p_2)...(z-p_n)}$

Then, $z_1, z_2 ...$ are called zeros and $p_1, p_2 ....$ are called poles.

Distinguish between Zeros and Poles

Poles	Zeros
A pole of the transfer function H(z) is value of z that makes the denominator zero.	A zero of the transfer function H(z) is a value of z that makes the numerator zero.
Mathematical : H(z) = N(z)/D(z) If D(z) = 0, at z=zp, then H(z) → infinity as z→ zp	Mathematical : H(z) = N(z)/D(z) If N(z) = 0, at z=zs, then H(z) → 0 as z→ zs
Influence system stability and response	Influence frequency response
Poles outside the unit circle indicate instability	Zeros do not affect stability directly
Represented by "X" in the Z-plane	Represented by "O" in the Z-plane

(b) *Transfer function representation :* $X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1 z^{-1}+....+b_m z^{-m}}{1+a_1 z^{-1}+....+a_n z^{-n}} = \frac{b_0+b_1 z^{-1}+....+b_m z^{-m}}{a_0+a_1 z^{-1}+....+a_n z^{-n}}$ *[*$a_0 = 1$ *means normalized]* $.z = e^{j\omega}$

$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}+0.75z^{-2}}$ [Dividing by z^2]

$H(e^{j\omega}) = \frac{e^{2j\omega}+e^{2\omega}+1}{e^{2j\omega}+0.5e^{j\omega}+0.75}$ [Replace $z = e^{j\omega}$ on given $X(z)$ of the question]

Difference equation representation : Transfer function → Inverse Z-transform → $y(n)$

$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}+0.75z^{-2}}$, [Transfer function, $H(z) = \frac{Y(z)}{X(z)} = ..$]

$X(z) + z^{-1}X(z) + z^{-2}X(z) = Y(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z)$ *[cross product]*

⇒ $x[n] + x[n-1] + x[n-2] = y[n] + 0.5y[n-1] + 0.75y[n-2]$ *[Inverse Z-function]*

⇒ $y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]]$

(c)

Idea : $X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+....+b_mz^{-m}}{a_0+a_1z^{-1}+....+a_nz^{-n}}$, *make given* $X(z)$ *this format and compare with this (* $a_0 = 1$ *) where* $b_0, ..., b_n$ *= input and* $a_0 + ...a_n =$ *output coefficient.*

$X(z) = \frac{2+z+3z^2}{0.5z^3(10z^{-3}+18z^{-2}+1)} = \frac{4z^{-3}+2z^{-2}+6z^{-1}}{10z^{-3}+18z^{-2}+1}$ [Normalized]

Input coefficient : 4, 2, 6

Output Coefficient : 1, 0, 18, 10

Show the transfer function in Z-transform with equations. Compare zeros and poles in Z-plane.

If $H(Z) = \frac{Z+1}{Z^2 - 0.9Z + 0.81}$, Find out: a). its transfer function representation, b). its difference equation representation.

If $X(z) = \frac{2+z^2+3z^4}{3+4z+z^2}$, find the values of input and output coefficients.

(a) Check the answer above.

(b) *Transfer function representation:*

$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$

$H(z) = \frac{e^{j\omega+1}}{e^{2j\omega}+0.9e^{j\omega}+0/81}$

Difference equation representation

$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} = \frac{Y(z)}{X(z)}$

⇒ $z^{-2}X(z) + z^{-1}X(z) = Y(z) + 0.9z^{-1}Y(z) + 0.82z^{-2}Y(z)$

⇒ $x[n-2] + x[n-1] = y[n] + 0.9y[n-1] + 0.81y[n-2]$

(c) $X(z) = \frac{3+z^{-2}+2z^{-4}}{z^{-2}+4z^{-3}+3z^{-4}}$

Input Coefficients: 3, 0, 1, 0,2

Output Coefficients: 0, 0, 1, 4, 3

- **Given difference equation, find transfer function representation.**

$y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]]$

⇒ $Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z)$

⇒ $\frac{Y(z)}{X(z)} = ...$

⇒ $H(z) = \frac{Y(z)}{X(z)} = ...$

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2. a) What is meant by region of convergence? Write the properties of ROC. 3

b) Determine the Z-transform and ROC of the signal $x(n) = \left(-\frac{1}{2}\right)^n u[-n-1]$ where $n \leq -1$. 4

c) Shortly describe unite circle in z-domain. A causal LTI system has impulse response h[n] for which the z-transform is $H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$, find the region of convergence of $H(z)$. 1+2

(a) The set of Z for which $X(z)$ is converges (gives finite value)/the set of points in Z-plane for which $X(z)$ is converges is called *Region Of Converges.*

Properties:

Properties of ROC :

(1) For a finite duration right-sided signal (causal signal), the ROC will be entire z-plane except $z=0$

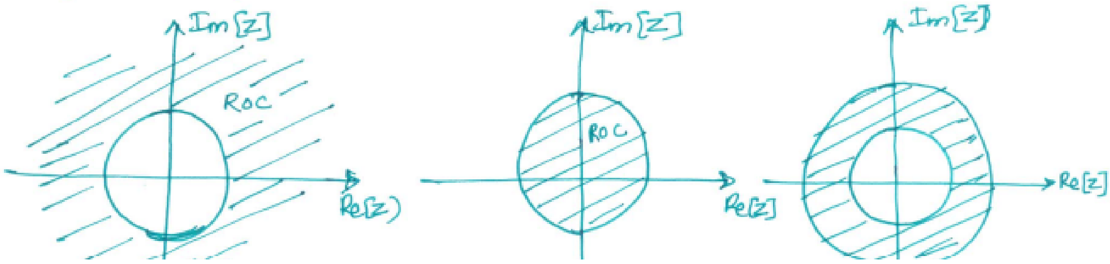
(2) For a finite duration left-sided signal (anticausal signal), the ROC will be entire z-plane except $z=\infty$

(3) For a finite duration two sided signal, the ROC will be the entire z-plane except $z=0$ and $z=\infty$

(4) If $x[n]$ is right sided and of infinite duration (causal signal), then ROC is outside a circle whose radius is equal to the largest pole magnitude (Fig. 1)

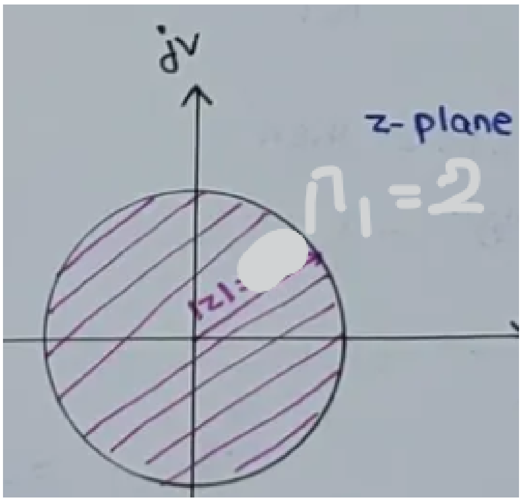
(5) If $x[n]$ is left sided and of infinite duration (anticausal signal), then ROC is inside a circle whose radius is equal to the smallest pole magnitude (Fig. 2)

(6) If $x[n]$ is two sided and of infinite duration, then ROC will be a ring in the z-plane bounded by smallest & largest pole magnitudes (Fig 3)



Source

(b) $x(n) = \begin{cases} (-\frac{1}{2}); n \leq 1 \\ 0; n \geq 0 \end{cases}$



$X(z) = \sum_{n=-\infty}^{1} (-0.5)^n z^{-n} =$
$\sum_{n=-\infty}^{-1} (-0.5z^{-1})^n =$
$\sum_{n=1}^{\infty} (-0.5^{-1}z)^n$

If $0 < |-0.5^{-1}z| < 1$, then

$X(z) = \frac{1}{1-(-0.5^{-1}z)} = \frac{0.5}{0.5-z}$

Condition for converges,

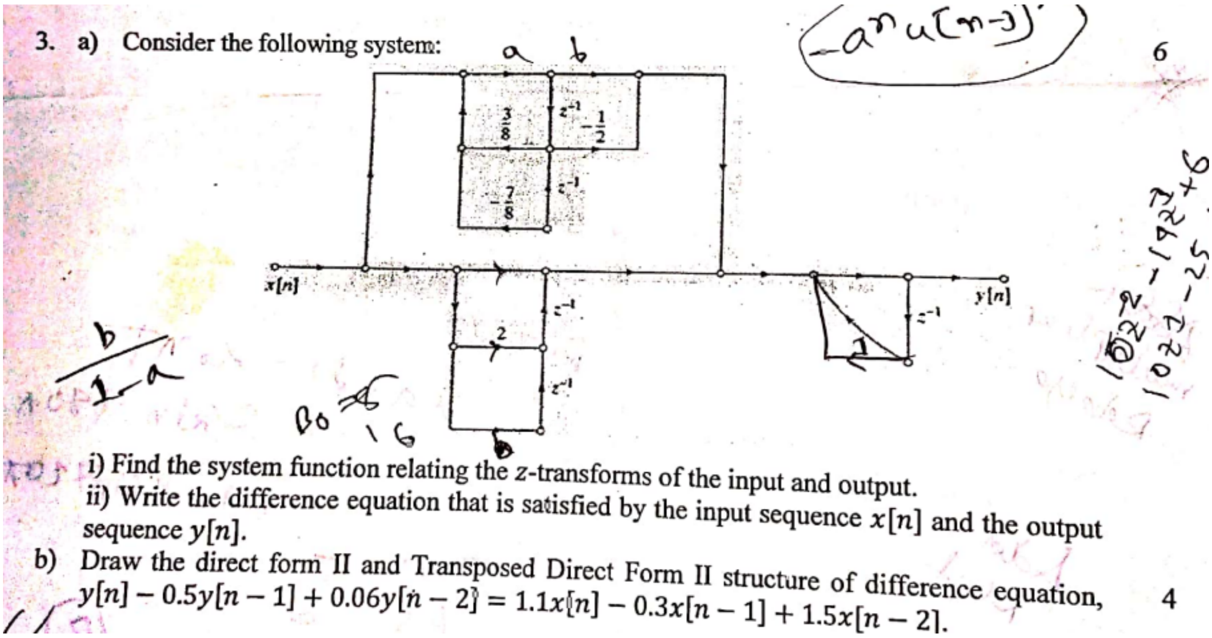
$|-0.5^{-1}z| < 1 \Rightarrow |z| < 2$

a) Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response is $H(z) = \frac{3-7z^{-1}+5z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}}$

If $x[n]$ is the input to the system is a unit step sequence. Find the output $y[n]$ by computing the inverse z-transform of $Y(z)$

b) Define Cauchy residue theorem. Evaluate the inverse z-transform of $X(z) = \frac{1}{1-bz^{-1}}$ where $|z| > |b|$, using the complex inversion integral.

7. a) Let a system $H(z) = \frac{1-0.5z^{-1}}{1-0.3z^{-1}}$. Describe stability and causality for both the system and inverse system using ROC. show both regions are overlapped. 6



3\. a) Consider the following system: 6

i) Find the system function relating the z-transforms of the input and output.

ii) Write the difference equation that is satisfied by the input sequence $x[n]$ and the output sequence $y[n]$.

b) Draw the direct form II and Transposed Direct Form II structure of difference equation, 4

$y[n] - 0.5y[n-1] + 0.06y[n-2] = 1.1x[n] - 0.3x[n-1] + 1.5x[n-2]$.

- **Differences between Z-transform and Laplace Transform**

Z Transform	Laplace Transform
Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z.	an integral transform that converts a function of a real variable (usually , in the time domain) to a function of a complex variable
Used to analyze discrete time signal	continuous time signal
Uses the complex variable Z	uses the complex variable s
The set of points in z-plane for which X(z) converges is called the ROC of X(z)	The set of points in s-plane for which X(s) converges is called the ROC of X(s)
Used for discrete time linear system such as digital filters and sampled data control system	Used for the analysis of CT system such as control system and differential equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$F(s) = \int_0^\infty f(t) e^{-st} t'$$

where $s = \sigma + i\omega$

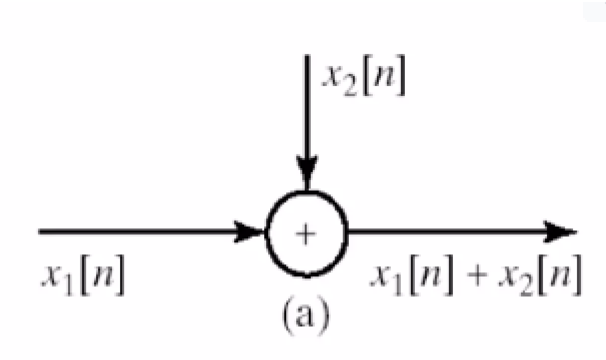
Digital Filters

A digital filter is a system that performs mathematical operations on a sampled discrete time signal to reduce or enhance certain aspects of that signal.

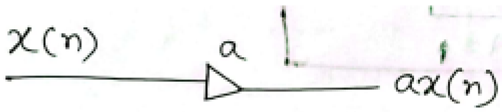
Types : FIR, IIR

Basic element to design Digital Filter

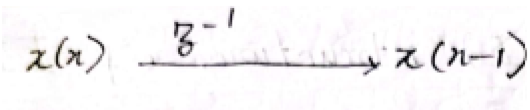
- Adder :



- Multiplexer



- Delay Element



IIR (Infinite Duration Impulse Response) : If the impulse response exists infinitely, it is an IIR Filter.

- **Frequency Response of IIR:**

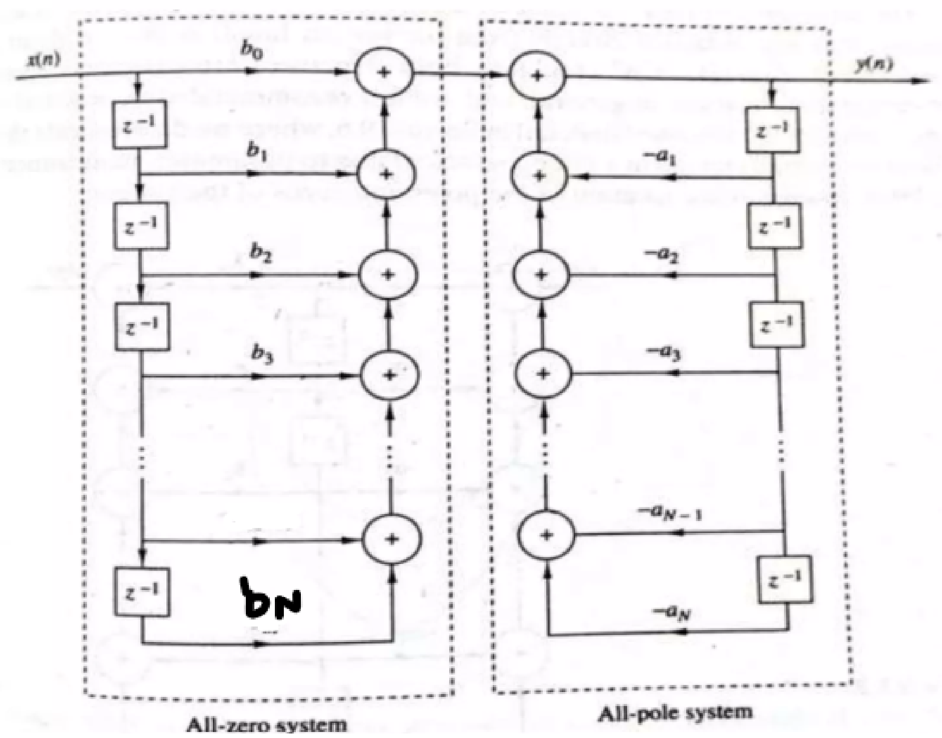
  $H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1+\sum_{n=1}^{N} a_n z^{-n}}, [a_0 = 1]$

- **Difference equation of IIR Filter**

  $y(n) = \sum_{m=0}^{M} b_m x(n-m) - \sum_{m=1}^{N} a_m y(n-m)$

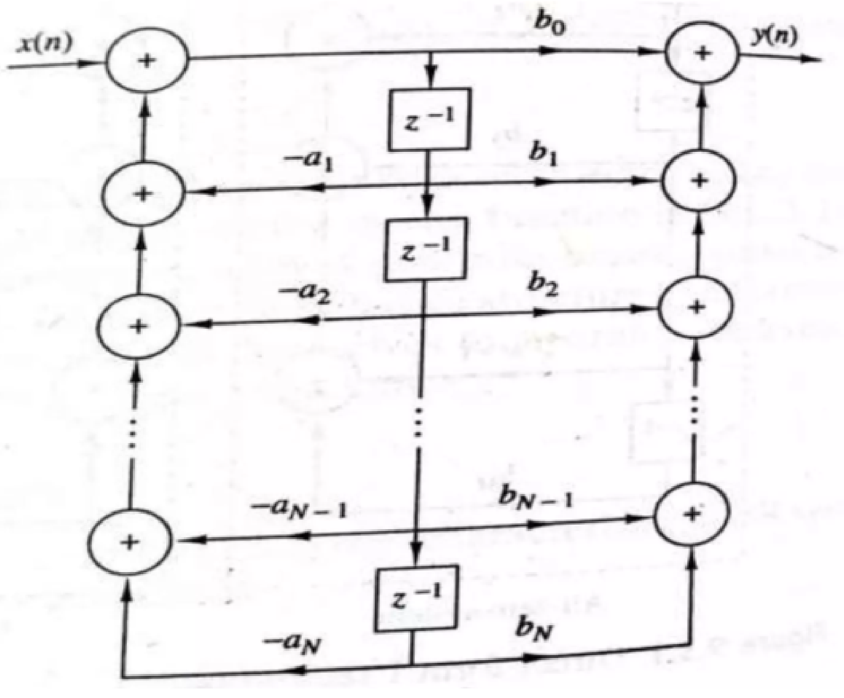
- $N-th$ **order Direct form (I)**

$$y(n) = b_0x(n) + b_1x(n-1)... + b_N(n-N) - (a_1x(n-1) + ... + a_Nx(n-N))$$



- $N-th$ **order direct form (II)**

  $$y(n) = b_0x(n) + b_1x(n-1)... + b_N(n-N) - (a_1x(n-1) + ... + a_Nx(n-N))$$



- **Cascade form :**

  In this form the frequency response, $H(z)$ is factored into smaller second section, called bi-quads. The frequency response is then represented as a product of these bi-quads section.

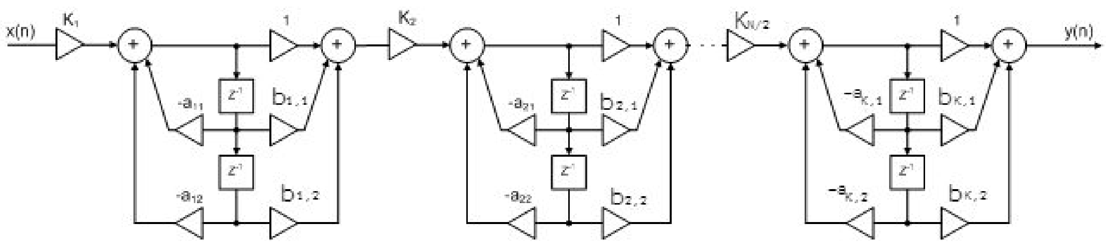
  $H(z) = \frac{B(z)}{A(z)} = b_0 \frac{1+\frac{b_1}{b_0}z^{-1}+....+\frac{b_N}{b_0}z^{-N}}{1+a_1z^{-1}+....+a_Nz^{-N}}$

  $= b_0 \prod_{k=1}^{k} \frac{1+B_{k,1}z^{-1}+B_{k,2}z^{-2}}{1+A_{k,1}z^{-1}+A_{k,2}z^{-2}}$ [for second-order]

  $k = \frac{N}{2}$ [Always]

Cascade form for $N-th$ order

$K = \frac{N}{2}$



FIR (Finite Duration Impulse Response): The filters have a finite impulse response function which has finite length of time.

- **Transfer Function**

  M : length of filter

  N : Order of filter $= M-1$

  $H(z) = \sum_{n=0}^{M-1} b_n Z^{-n}$

  $h(n) = \begin{cases} b_n, 0 \le n \le M-1 \\ 0, \text{otherwise} \end{cases}$

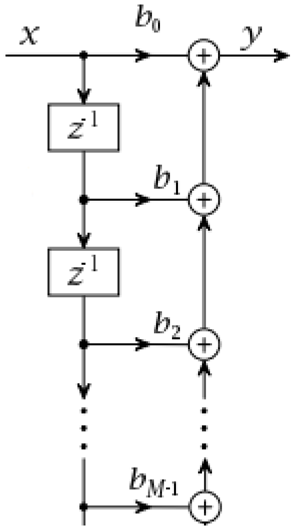
- **Difference equation**

  $y[n] = \sum_{m=0}^{M-1} b_m x(n-m)$

- **Direct form I if $M = M$**

  Order of filter, $N = M-1$

$N-th$ order, $y(n)=b_0x(n)+b_1x(n-1)+...+b_Nx(n-N)$



- **Cascade FIR**

  For second order section,

  $H(z)=H(z)=\sum_{n=0}^{M-1}b_nZ^{-n}=b_0+b_1z^{-1}+....+b_{M-1}z^{-M+1}$

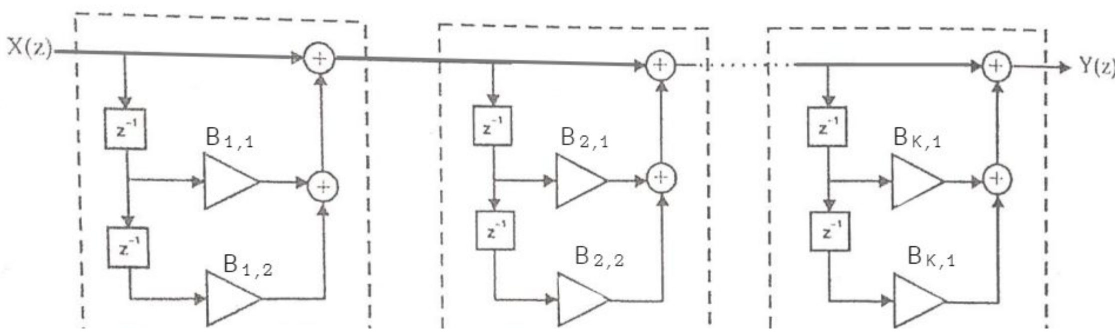
  $=b_0(1+\frac{b_1}{b_0}z^{-1}+....+\frac{b_{M-1}}{b_0}z^{-M+1})$

  $=b_0\prod_{k=1}^{k}(1+B_{k,1}z^{-1}+B_{k,2}z^{-2})$

  $K=\lfloor\frac{M}{2}\rfloor$

  Design cascade form of FIR where filter length is M

  $K=\lfloor\frac{M}{2}\rfloor$



- **Properties of Linear-phase FIR filter**

Let, $h(n), 0<n<M-1$, be the impulse response of length M. The frequency response,

$H(z)=\sum_{n=0}^{M-1}h(n)z^{-n}$

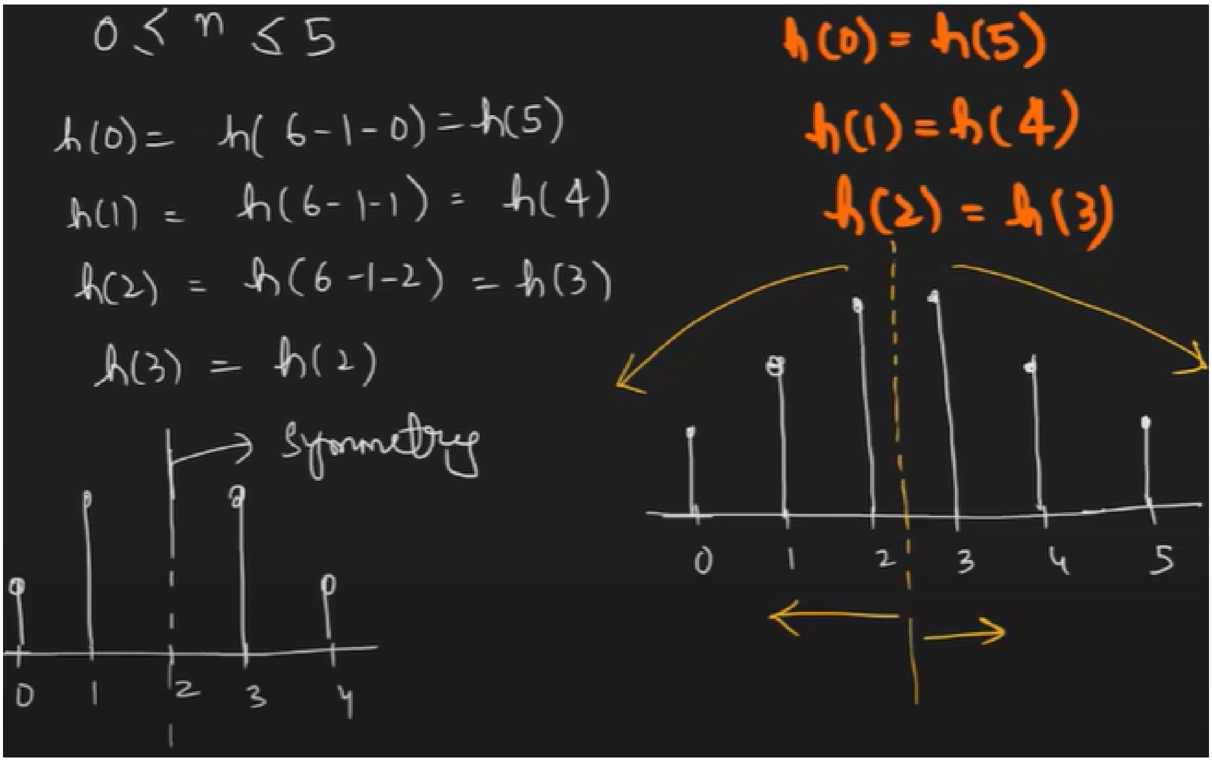
We know, $H(e^{jw}) = \sum_{n=0}^{M-1} h(n)e^{-jwn}, -\pi \leq w \leq \pi$

Where, $\angle H(e^{jw}) = -\alpha w$ (A system has linear phase if its phase response θ(ω) = ∠H(e^jω) = −cω for all ω and any constant c.)

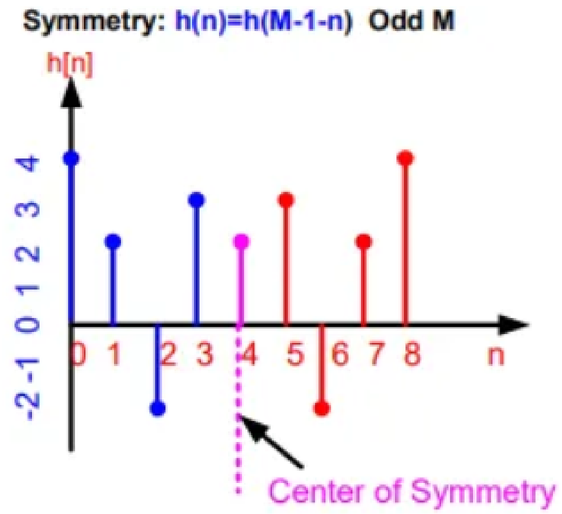
where, α is a constant phase delay, $h(n)$ must be symmetric, if $h(n) = h(M-1-n)$

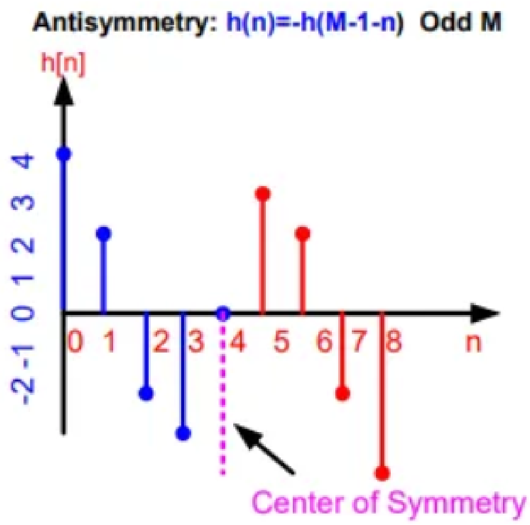
$0 \leq n \leq M-1$ with $\alpha = \frac{M-1}{2}$ if M is odd.

Anti symmetry : $h(n) = -h(M-1-n)$



b)	Draw the figures of symmetric and anti-symmetric impulse response h(n) of length M. where M is 9.	2





Linear phase system fall into one of 4 categories:

- **M odd, h[n] is symmetric (Type I)**

  $H(e^{jw}) = \sum_{n=0}^{M} h[n]e^{-jwn}$

  $= h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + ... + h[\frac{M}{2}]e^{-jw\frac{M}{2}}$

  $= e^{jwM/2}(h[0](e^{jwM/2} + e^{-jwM/2}) + ... + h[M/2])$

  $= e^{-jwM/2}(h[0].2cos(wM/w) + h[1].2cos(w(M/2-1) + ... + h[M/2])$

  $= e^{-jwM/2}\sum_{k=0}^{M/2} a_1[k]cos(wk)$

  As class note,

  $M = Odd, \alpha = \frac{M-1}{2}$

  $H(w) = \sum_{n=0}^{\frac{M-1}{2}} a(n)cos(wn)$

  ⇒ $H(e^{jw}) = e^{-jw\frac{M-1}{2}}\sum_{n=0}^{\frac{M-1}{2}} a(n)cos(wn)$

- **M even, h[n] is symmetric (Type II)**

  $H(e^{jw}) = h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + ... + h[\frac{M-1}{2}](e^{-jw\frac{M-1}{2}} + e^{-jw\frac{M+1}{2}})$

$$= e^{-jw\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} b(n)cos(w(k+\frac{1}{2}))$$

As class note,

$$H(e^{jw}) = e^{-jw\frac{M-1}{2}} \sum_{n=1}^{\frac{M}{2}} b(n)con\{w(n-\frac{1}{2})\}$$

- M even, h[n] is antisymmetric (Type III),
- M odd, h[n] is antisymmetric (type IV)