Signals and Systems

Tags	NSTU Year 2 Term 2
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Resources:

- https://www.youtube.com/playlist?list=PLVoS_Csfrfm6ZspJ43Sc0JpwqwP77y6Af (Bangla)
- https://www.youtube.com/watch?v=x5qRAihZRks&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=2 (Hindi)
- https://www.youtube.com/playlist?list=PLXOYj6DUOGrrAlYxrAu5U2tteJTrSe5Gt (Problem Solving)
- Euler Equation :

  https://www.physicsforums.com/threads/solve-e-frac-i-pi-4.336356/

  https://proofwiki.org/wiki/Euler's_Formula/Examples/e^i_pi_by_4

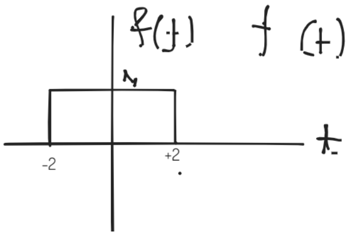
Skipped:

- https://www.youtube.com/watch?v=Rew03iHJGhk&list=PL9RcWoqXmzaIG-RWneeqDJ-FCt66S15pl&index=9&pp=iAQB (Complex Exponentiation)

Signals

Physical quantity that contains information. Signals are expressed mathematically as function of independent variable, which is usually time.

Example:

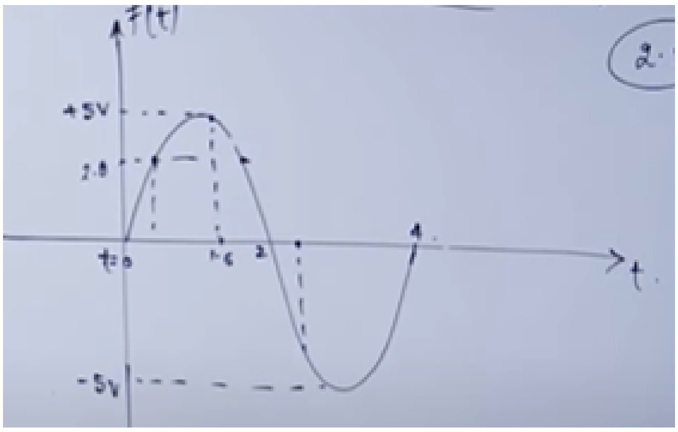


$$f(t) = \begin{cases} 1, & -2 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Basic Types of Signals:

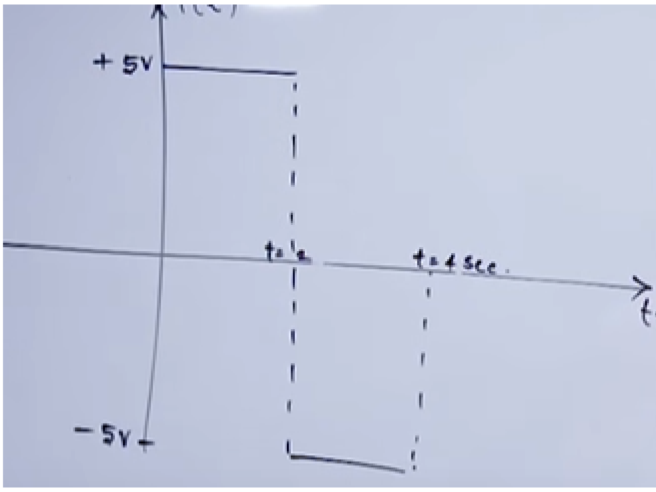
<u>Based on Continuous and Discrete:</u>

- **Continuous in Time Signal & Continuous in Value Signal**
  - A continuous-time signal has values for all points in time in some interval.
  - A continuous-value signal is all possible value within an interval will be available in a signal.



Has values for **0≤t≤4** and all values are available in the signal with **-5≤v≤+5**

- **Continuous in time but discrete in value signal**
  - A continuous-time signal has values for all points in time in some interval.
  - All values within a range is not available in the signal.

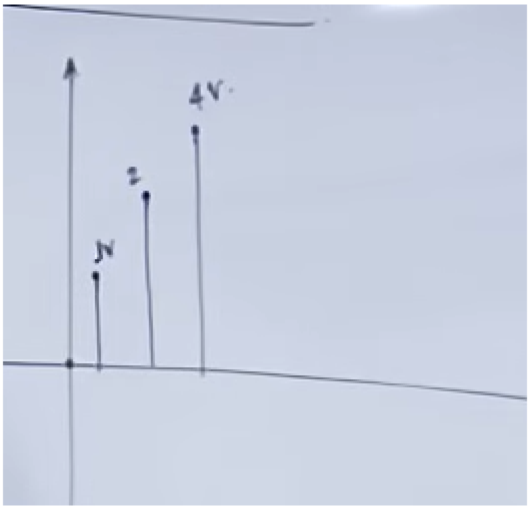


Have value for time within 0≤t≤4 but all values within -5≤v≤5 is not available

- **Continuous in value but discrete in time signal**
    - Haven’t values for all points in time within an interval
    - All values within an range are available in a signal

All values are available, but some points in time haven’t values

- **Discrete in time and discrete in value signal**

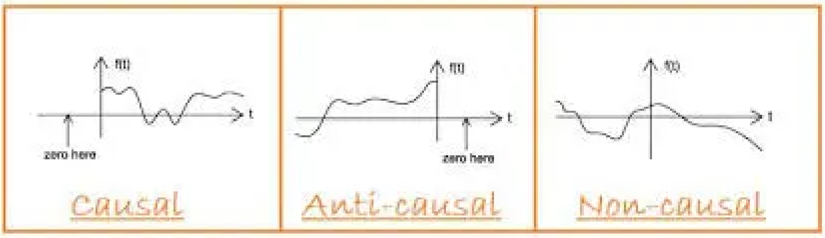


Analog Signal: Continuity in any of the domain (time or value)

Digital Signal: Discrete in both time and value

<u>**Based on Causal, Anti-Causal, Non-Causal**</u>

- **Causal Signals:**
  - 0 for all negative value/time
  - $x(t) = \begin{cases} x(t) > 0 & t \geq 0 \\ 0 & t < 0 \end{cases}$
- **Non-Causal Signals**
  - A signal that have positive amplitude for both positive and negative instance of time
- **Anti-Causal Signal**
  - 0 for all positive value/time
  - $x(t) = \begin{cases} x(t) > 0 & t \leq 0 \\ 0 & t > 0 \end{cases}$



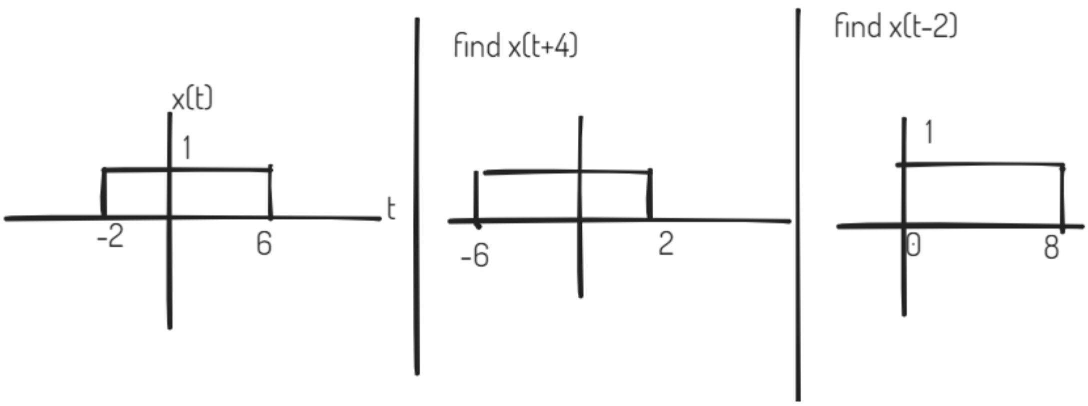
Causal, Anti-Causal, Non-Causal

Operations on Signals

- **Time Shifting Operation**

  $f(t)$ is be given. $f(t \pm t_0) = ?$

  - t_0 is a constant
  - $+$ → **advance :** Shift the signal towards left by t_0
  - $-$ → **delay :** Shift the signal towards right by t_0
  - Amplitude doesn't change for shifting

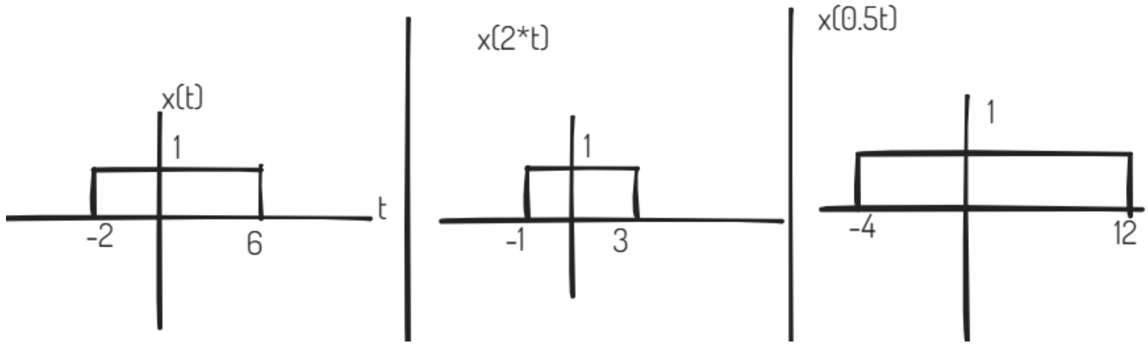


- **Time Scaling Operation**

  $x(t)$ is given. $f(\alpha t) = ?$

  - α : scaling factor
  - $\alpha > 1$ → **Signal Compression (Increasing Speed) :** Divide the existing limit by α

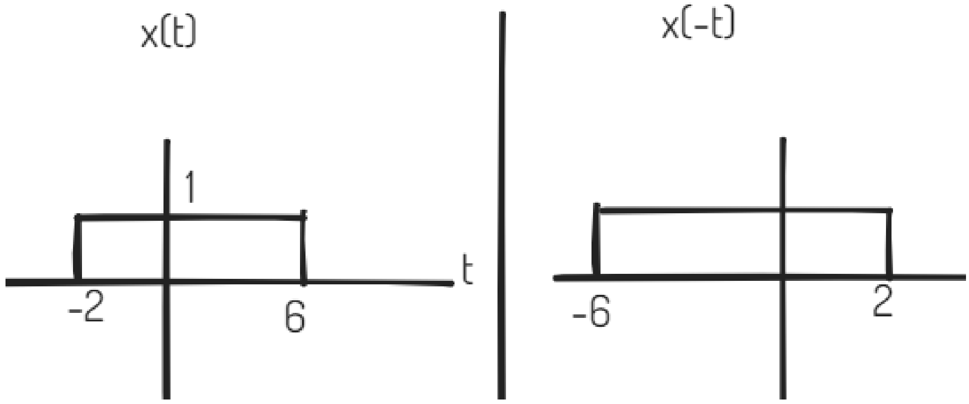
- $\alpha < 1$ → **Signal Expansion (Decreasing Speed) :** Divide the existing limit by
- Amplitude doesn’t change for this operation



- **Time Reversal or folding Operation**

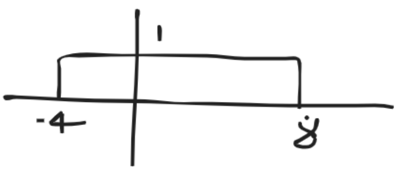
  $x(t)$ is given. $x(-t) = ?$

  - The sign of the limit will be changed



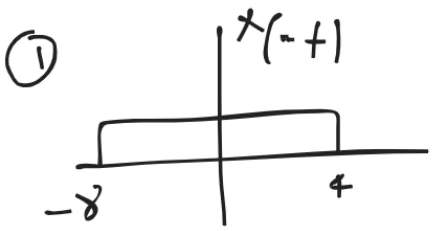
Example on Operation:

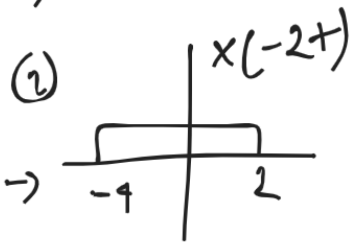
* $x(t)$

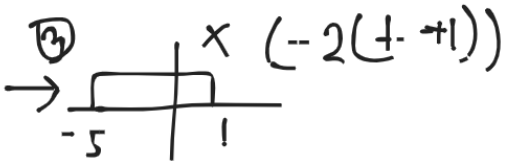


$x(-2t-2) = ?$

$\Rightarrow x(-2(t+1))$ - 1. the coefficient of t must be one



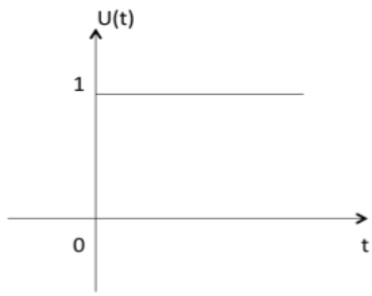




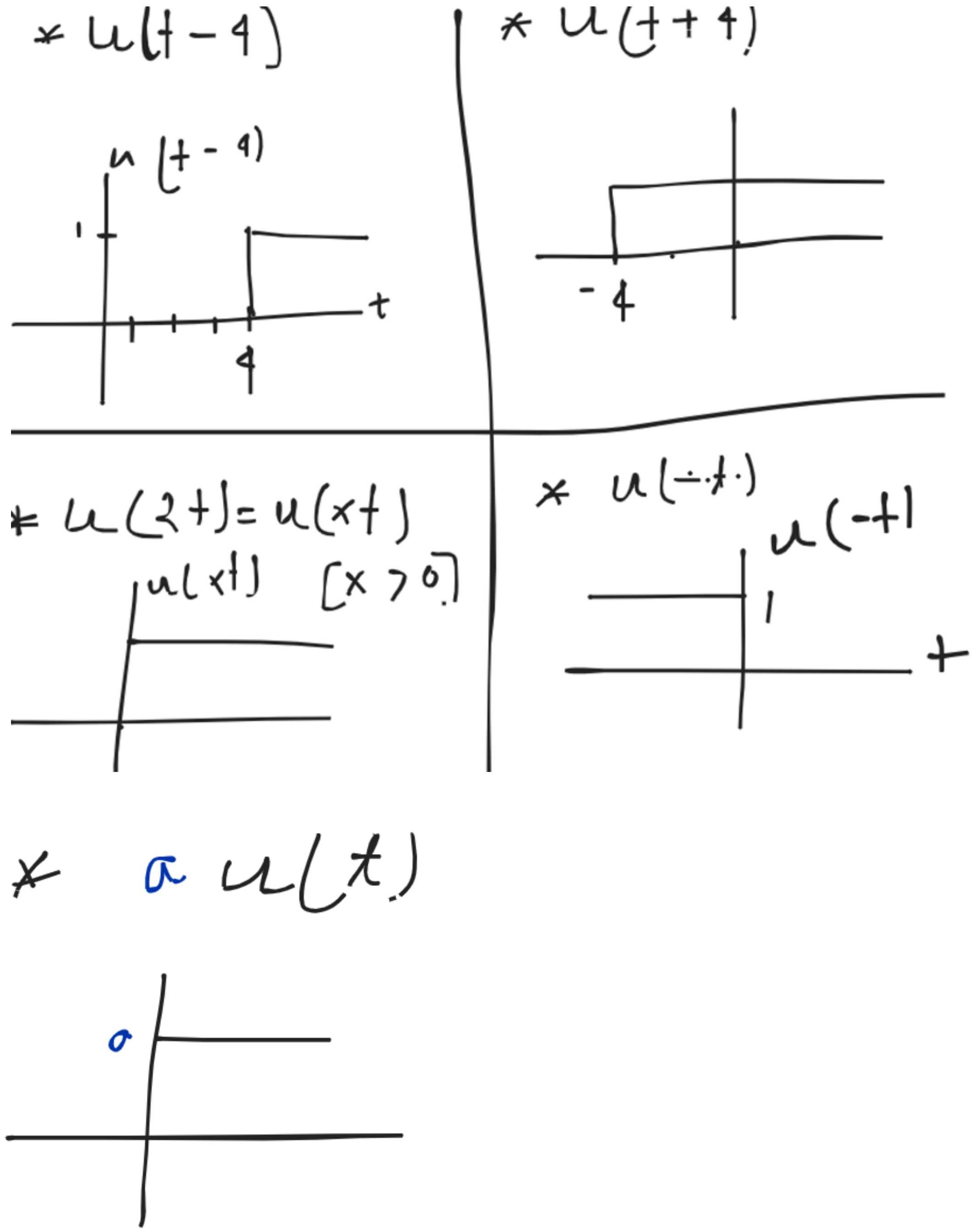
Elementary Signals

- **Unit Step Signal**
  - Also known as **Heaviside Step Function**
  - Unit step function is denoted by $u(t)$

    $$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



  - Amplitude = coefficient of $u(t)$
  - Non-Causal Signal$*$Unit Step Function $=$ Causal Signal
  - **Operations on Unit Step Signal:**

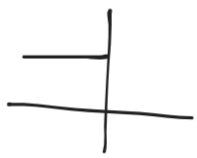


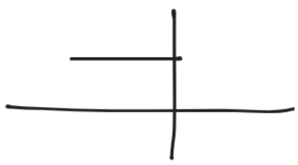
- **Example:**

* $u(-2t + 4)$

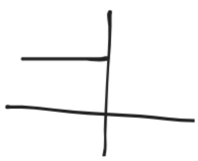
$\Rightarrow u(-2(t-2))$

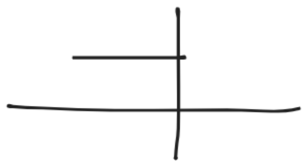
① $u(-t)$ ② $u(-2t)$



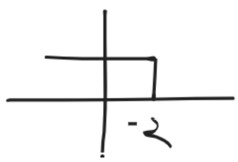


① $u(-t)$ ② $u(-2t)$





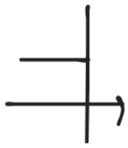
③ $u(-2(t-2))$

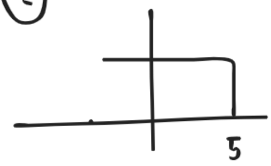


$$\cancel{x}\, u(-t+5)$$

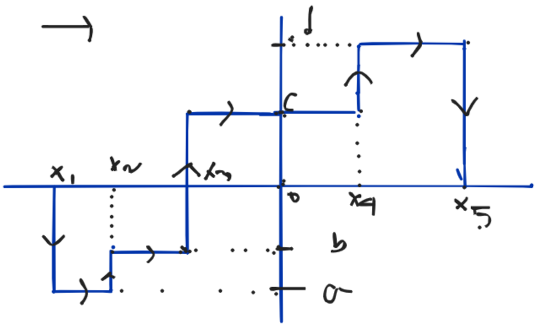
$$\Rightarrow u(-(t-5))$$

① $u(-t)$ ②



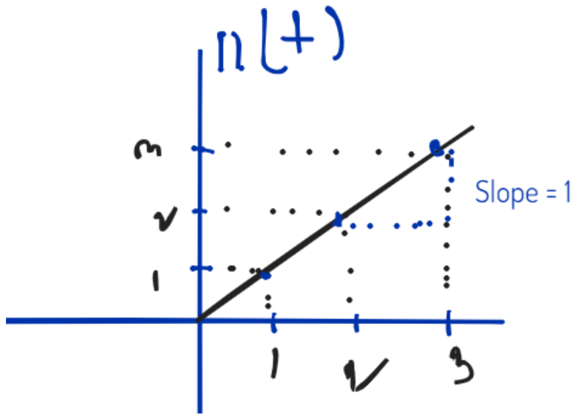


- **Expressing by unit step function**

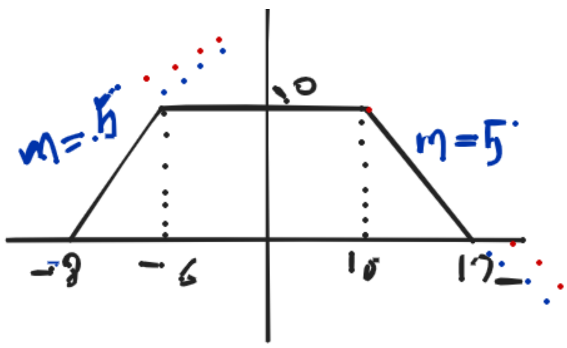


$$x(t) = (-a-0)\, u(t+x_1) + [-b-(-a)]\, u(t+x_2) + [c-(-b)]\, u(t+x_3) + [d-c]\, u(t-x_4) + (0-d)\, u(t-x_5)$$

- **Unit Ramp Signal**
  - $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$
  - Slope = Coefficient of $r(t)$



- $r(t) = t \cdot u(t)$
- $\frac{d}{dt} r(t) = u(t)$
- $\frac{d}{dt}[A \cdot r(t)] = A \cdot u(t)$
- **Expressing into Ramp Signal**

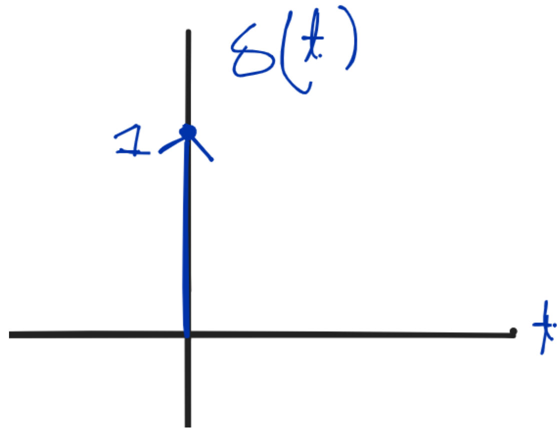


$$= +5r(t+8) - 5r(t+6) - 5r(t-10) + 5r(t-12)$$

- **Impulse Function**
  - An ideal impulse signal is a signal that is zero everywhere but at the origin (t = 0), it is infinitely high. Although, the area of the impulse is

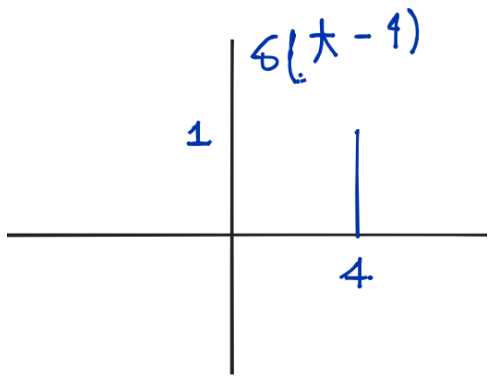
finite.

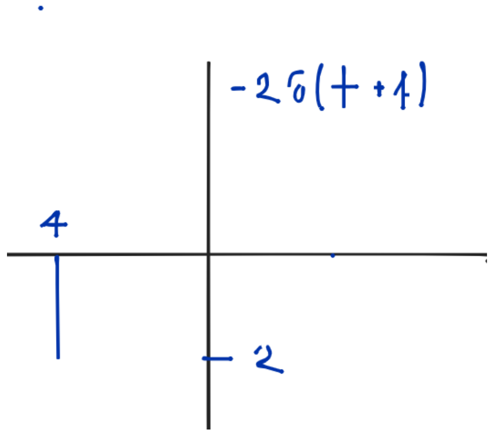
- $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$
- $A \cdot \delta(t)$, **Here A is the area of this impulse function.**



- $\int_{-\infty}^{\infty} \delta(t) = 1$ or $\int_{0^-}^{0^+} \delta(t) = 1$
- $\int_a^b \delta(t - t_0)$, it exists $if\ a \leq t_0 \leq b$, otherwise 0
- $\delta(t) = \frac{d}{dt} u(t)$

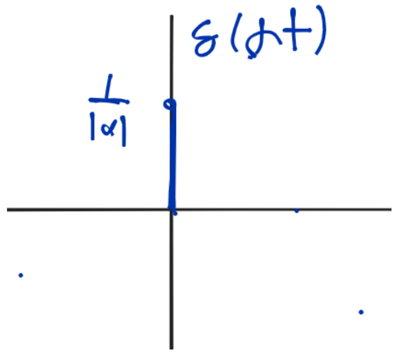
- **Operations:**





- **Properties:**
  - **Time Scaling Property:**

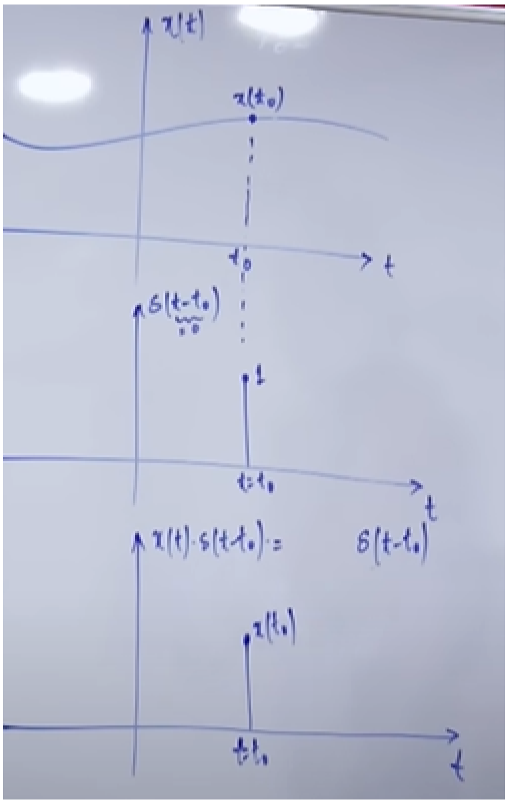
    $\delta(\alpha t)$ = $\frac{1}{|\alpha|}\delta(t)$



  - **Product Property**

    $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$

    $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$



- **Shifting Property**

  $(i) \int_{-\infty}^{\infty} x(t) \cdot \delta(t)$

  $= \int_{-\infty}^{\infty} x(0) \cdot \delta(t)$

  $= x(0) \int_{-\infty}^{\infty} \cdot \delta(t)$

  $= x(0)$

  $(ii) \int_{-\infty}^{\infty} x(t) \cdot \delta(t - t_0)$

  $= \int_{-\infty}^{\infty} x(t_0) \cdot \delta(t)$

  $= x(t_0) \int_{-\infty}^{\infty} \cdot \delta(t)$

  $= x(t_0)$

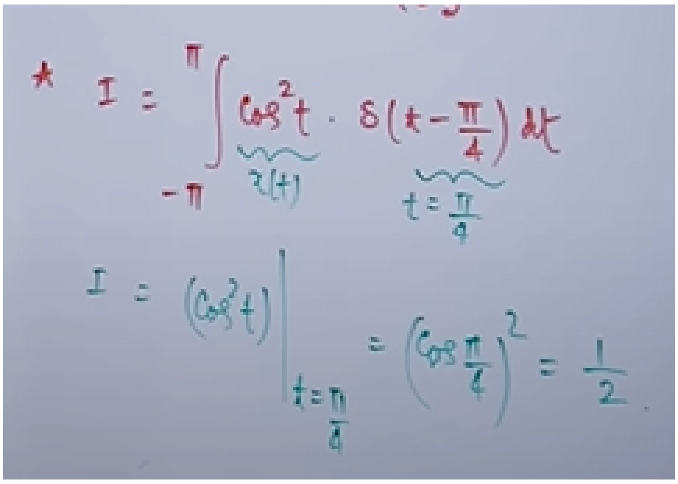
$$I = \int_{-6}^{8} (t^2 + 4)\, \delta(t-3)\, dt$$

$$x(t) = t^2 + 4.$$

$$x(3) = 9 + 4 = 13$$

$$I = \int_{-6}^{8} x(t_3)\, \delta(t-3)\, dt$$

$$= x(t_3) \times 1 = 13$$



$$I = \int_{-6}^{5} \overbrace{(t-2)}^{x(t)} \; \delta(2t-4)$$

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$$

$$\delta\{2(t-2)\} = \frac{1}{2}\,\delta(t-2)$$

$$I = \frac{1}{2}\int_{-6}^{5} x(t)\;\delta(t-2)$$

$$= \frac{1}{2}\cdot x(2) = 0$$

- **Extension Properties**

  $\int_a^b x(t)\cdot\delta(g(t))dt$

  $\delta(g(t)) = \frac{\delta(t-t_0)}{|g'(t_0)|}[if\ g\ has\ a\ real\ root\ at\ t=t_0]$

  If g has more than one real root at $t_0, t_1, ..., t_i$

  $s(g(t)) = \sum_i \frac{\delta(t-t_i)}{|g'(t_i)|}$

  Example:

  $I = \int_{-10}^{10}(t^2+10)\cdot\delta(t^2-16)dt$

  $t^2-16, t = +4, -4$

  $I = \int_{-10}^{10}(t^2+10)\cdot\frac{\delta(t-4)}{|2\cdot(-4)|}dt + \int_{-10}^{10}(t^2+10)\cdot\frac{\delta(t+4)}{|2\cdot(4)|}dt$

  $= \frac{13}{4}$

- **Exponential Function**

  $x(t) = Ae^{(\alpha t)}$

$A =$ Amplitude at $t = 0$

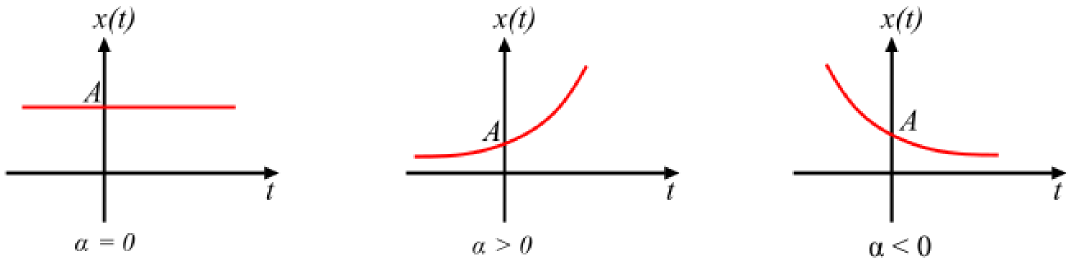


Figure-1

Signal Classification on Even and Odd

- **Even/Symmetric Signal**

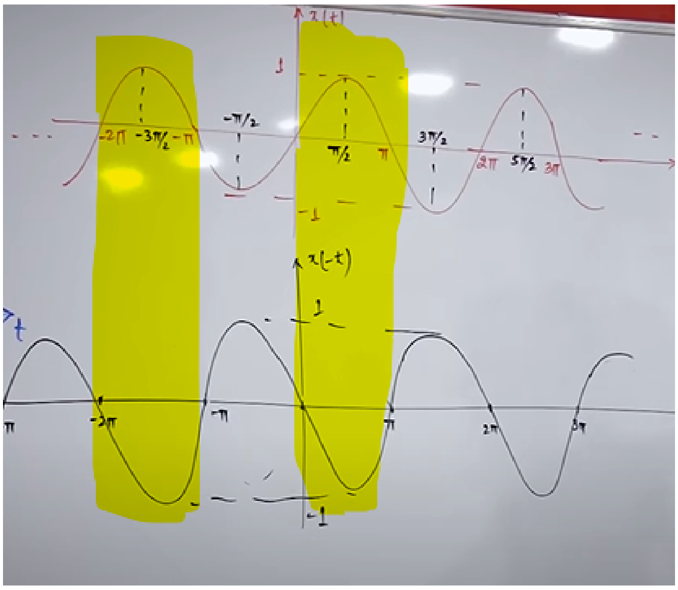
  $x(-t) = x(t)$

  Example : $x(t) = \begin{cases} 2 & -a \le t \le a \\ 0 & otherwise \end{cases}, x(t) = t^2, ..., t^{2n}$

- **Odd/Anti-symmetric Signal**

$x(-t) = -x(t)$

Example: $x(t) = t^3, t^5, .., t^{2n+1}$



Check a function even or odd:

$x(t)$ is given. Find $x_1(t) = x(-t) = ...$

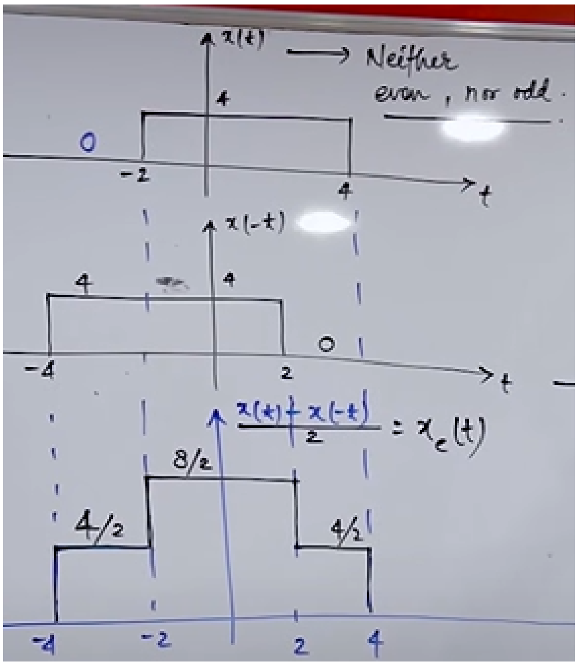
if $x(1) == x_1(1)$, it is even, otherwise odd.

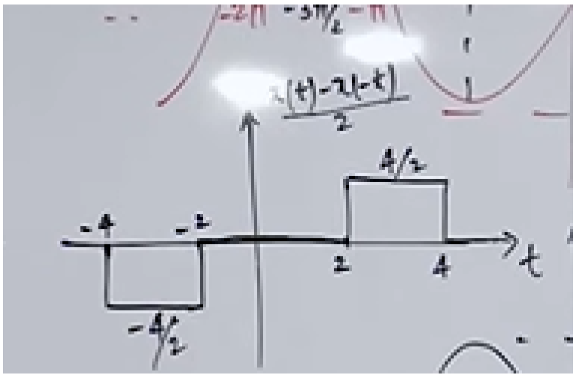
Converting to Odd or Even Signal

To convert any arbitrary signal which is neither even nor odd into equivalent even or odd parts:

$x_e(t) = \frac{x(t)+x(-t)}{2}$

$x_o(t) = \frac{x(t)-x(-t)}{2}$





- **$x(t)$ is complex**

  $x(t) = a + ib$

  - **Even**

    $x(t) = \overline{x(-t)}$

  - **Odd**

    $x(t) = -\overline{x(-t)}$

  Conjugate symmetric Part of $x(t)$

  $x_e(t) = \frac{x(t)+\overline{x(-t)}}{2}$

  Conjugate anti-symmetric part of $x(t)$

$x_o(t) = \frac{x(t) - \overline{x(-t)}}{2}$

- **Some Operations**

  $E = Even, O = Odd$

  $O \pm O = O$

  $E \pm E = E$

  $O \pm E = Neither\ Odd\ nor\ Even$

  $E \times E = E$

  $E \times O = O$

  $O \times O = E$

  $E/E = E$

  $O/O = E$

  $E/O = O$

  $O/E = O$

  $\int O = E$

  $\int E = O$

  $\frac{d}{dt}E = O$

  $\frac{d}{dt}O = E$

Periodic and Aperiodic Signal

- **Periodic Signal**
  - A signal is said to periodic, if it satisfies following two properties

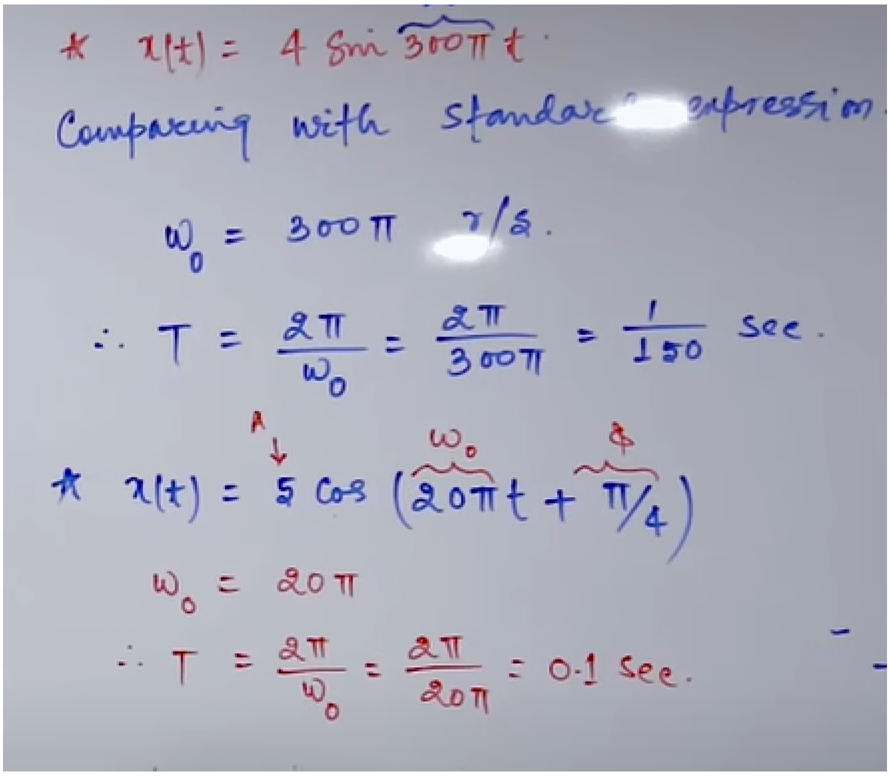
        - It must be exist for $-\infty \leq t \leq \infty$
        - It must repeat itself after some constant amount of time T, which is called **Fundamental Time Period**
            - $T = \frac{2\pi}{w_0}$
            - $w_0 =$ Fundamental frequency $rad\ s^{-1} = 2\pi f$
            - $T = \frac{1}{f}$
            - **Frequency must be real number. If frequency is not real number, it's not periodic.**
    - **Types**
        - **Sinusoidal Signals**
            - **Representation**

              $x(t) = Asin(w_0 t + \theta)$

              $A =$ Amplitude

              $w_0 t =$ Phase Angle

              $\theta =$Phase shift (+ → Advance, - delay)
            - $\frac{d}{dt}(Phase) = \frac{d}{dt}(w_0 t) = w_0 = freequency$
            - Shifting effect doesn't effect on periodicity, T
            - $x(t \pm kT) = x(t)$
            - Comparing with $x(t) = Asin(n\pi t + \theta)$**, if t is not square root of t, then the signal must be periodic.**



- **Combination of periodic signals**

  $x(t) = Asin(w_0t) + Bcos(w_1t) + Csin(w_2t) + ...$

  - $x(t)$ **will be periodic if ratio of individual time period is a rational number**
    - **Rational Number**
      - Can be expressed by $\frac{p}{q}$. p, q are co-prime.
      - The value of $\frac{p}{q}$ should terminating or repeating decimal.

        Example : 3.3333.., 2.5, 5.20202020..

    Example: $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_1}{T_3} = Rational\ Number$

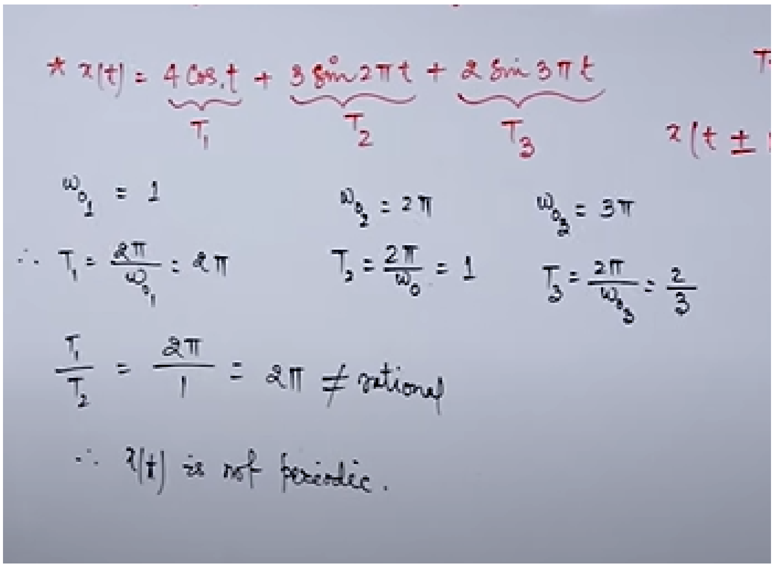
  - **Time Period of Resultant Signal**

    $T = LCM(T_1, T_2, ...)$

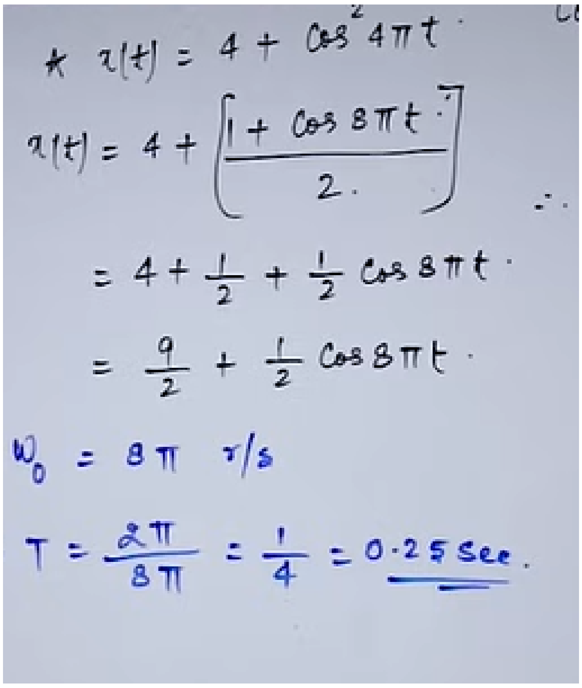
  - **Frequency of Resultant Signal**

$$w_0' = GCD(w_0, w_1, ...)$$

- If all $w_0, w_1, ...$ have π, then it is periodic. Or if all $w_0, w_1, ...$ haven't π, it is periodic too. If some of them have π and rest of them not, then it isn't periodic.



- **DC/Constant Signal :** independent from time. It doesn't effect on frequency, time. So, it is not countable.



4 is a constant Signal

Energy and Power Signals

- **Energy Signals**

  Energy of $x(t)$ is given as

  $E_x = \lim_{t \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$
  $\text{[No need to take lim if the signal is aperiodic]}$

  if $0 < E_x < \infty$ (Finite), then $x(t)$ is said to be $Energy\ Signal.$

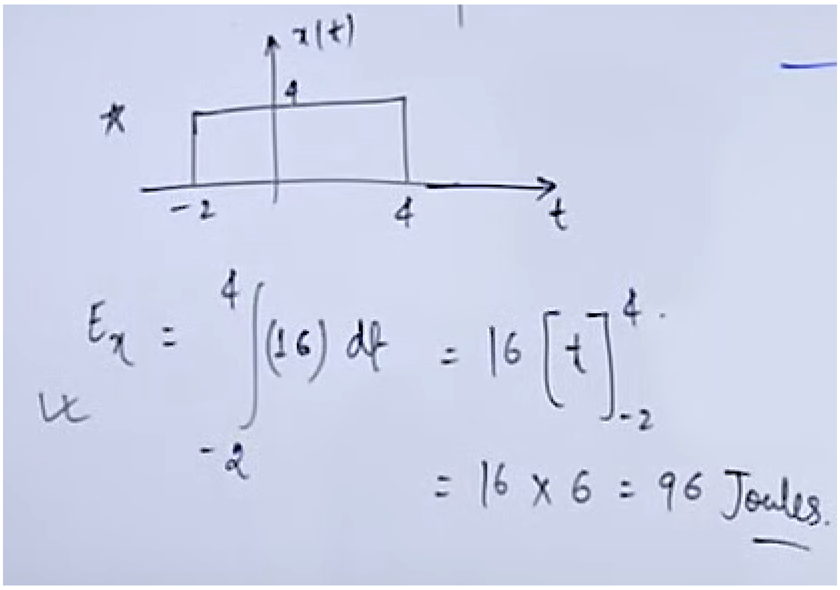
  Example:

  $(i)\ x(t) = e^{-4t} \cdot u(t)$

  $E_x = \int_{-\infty}^{\infty} [e^{-4t} \cdot u(t)]^2 \, dt$

  $= \int_0^{\infty} e^{-8t} dt \ \ \text{[u(t) exists only 0 to } \infty]$

  $= \frac{1}{8}$



  - **When a signal $x(t)$ will be energy signal**
    - If $x(t)$ is existing for infinite direction and decreasing in value.

      $\lim_{t \to \infty} f(t) = 0$
    - If $x(t)$ exists for finite direction and value of $x(t)$ is finite finite at all points, $x(t)$ is energy signal.

- **Power Signals**

$Power\ Signal$ of $x(t)$ is given as

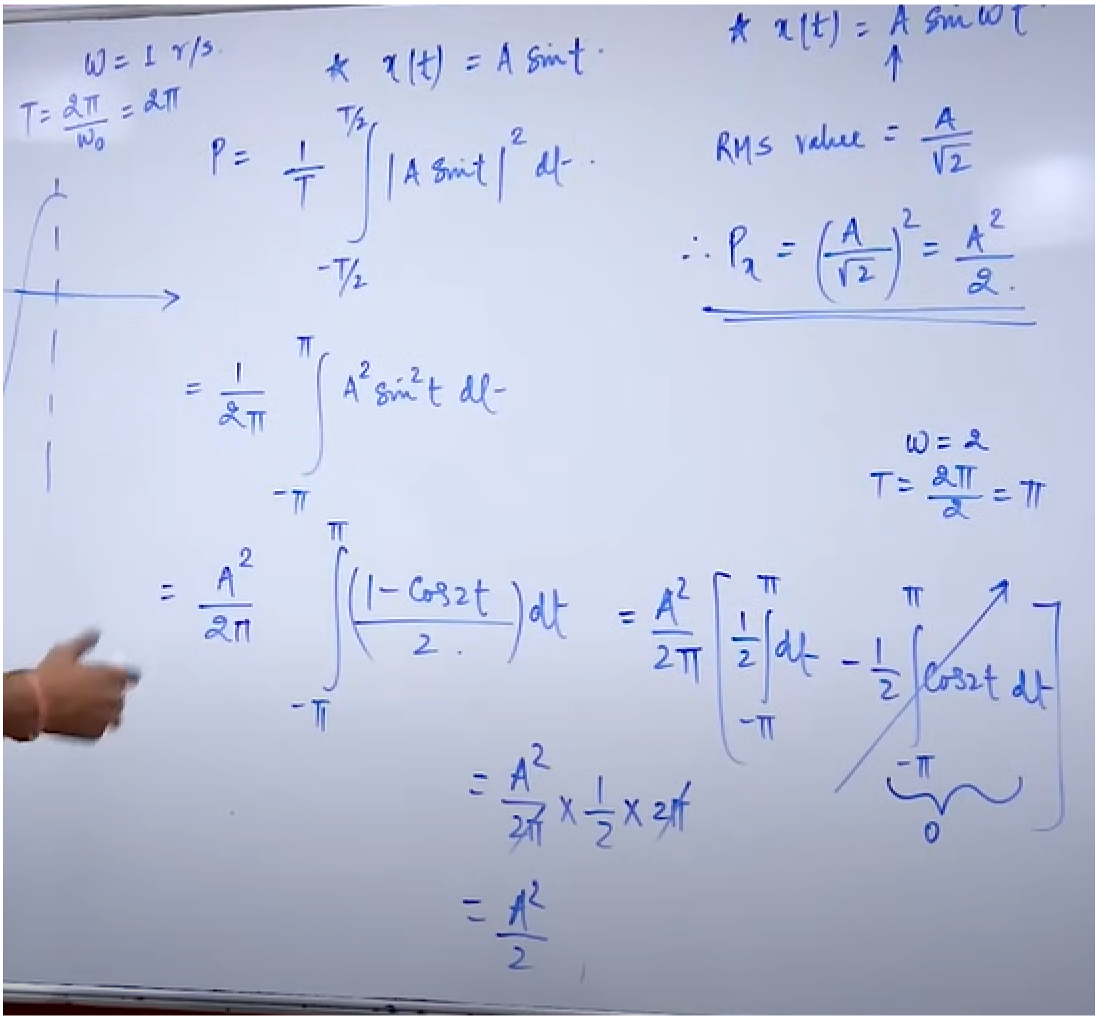
$P_x = \lim\limits_{T \to \infty} \frac{1}{T} \cdot E$

$= \lim\limits_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$

$= \lim\limits_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$
[No need to take lim if the signal is aperiodic]

If $0 < P_x < \infty$ (Finite), then $x(t)$ is said to be Power Signal

- $\text{Power} = RMS^2$
- $x(t) = Asinwt,\ RMS = \frac{A}{\sqrt{2}}, P = RMS^2 = \frac{A^2}{2}$



- **When a signal will be Power Signal**

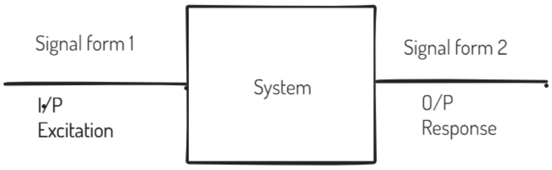
        - All periodic signal are power signal but converse is not true
        - If $x(t)$ is not a periodic signal and follows the conditions
            - $\lim_{t \to \infty} f(t) \neq 0$
            - $\lim_{t \to \infty} f(t) \neq \infty$
- A signal can't be Energy and Power Signals together.
    - If E_x is finite, then P_x is Zero . Vice-Versa.
- **Operations**
    - Time shifting has no effect on power and energy of signal.
      $Power\ x(t) = Power\ x(t - \frac{T}{2})$
      $Energy\ x(t) = Energy\ x(t-4)$
    - Time Scaling doesn't effect on Time Periodic but in Time period, Energy.
        - $\text{For } x(t) T, E_x$

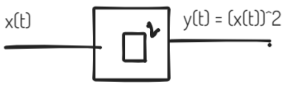
          $x'(t) = x(\alpha t), T' = \frac{T}{\alpha}, E'_x = \frac{E_x}{\alpha}$
        - Power remains same.
    -

Systems

System is a interconnection of different physical components which is used to convert one form of signal to others.

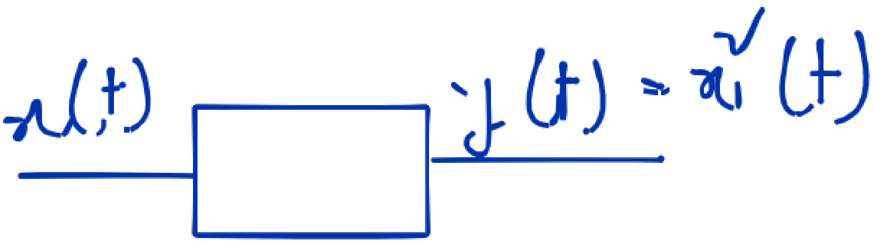
Example:



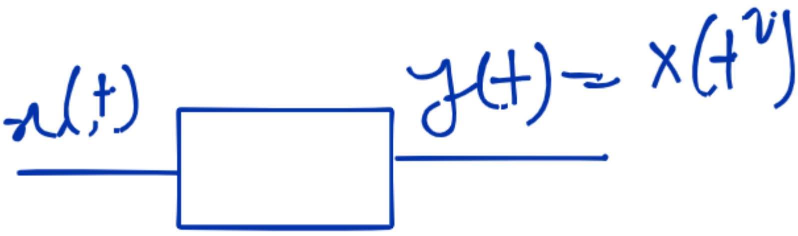


Properties of Systems

- Static and Dynamic System
    - Static
        - Memoryless
        - if present o/p depends on only present i/p



    - Dynamic
        - With Memory
        - if present o/p depends on past or future i/p



$y(2) = x(2^2) = x(4)$, $t = 2$'s output depends of $t = 4$. That means present o/p depends on future i/p;

- Causal, Noncausal, Anticausal
  - Causal
    - Present o/p depends on
      - present i/p or
      - Present + Past i/p
    - Static system are always causal
    - $h(t) = 0 \qquad t < 0$
      $h[n] = 0 \qquad t < 0$
  - Noncausal
    - Present o/p depends on
      - present + future or
      - present + past + future or
      - past + future
    - $h(t) \neq 0, \quad t < 0$
      $h[n] \neq 0, \quad t < 0$

    $$y[n] = x[n^2]$$
    $$y[0] = x[0]$$
    $$y[1] = x[1] \text{ depends on present}$$
    $$y[2] = x[4] \text{ depends on future}$$
  - Anticausal
    - Present o/p depends on
      - only future i/p

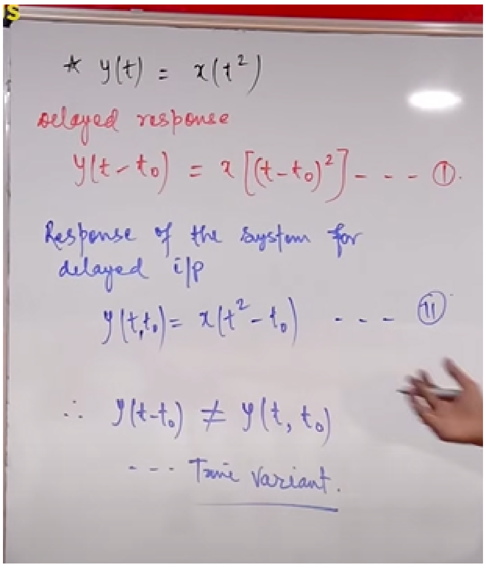
      $$y(t) = x(t^2 + 1)$$
      $$y(0) = x(1)$$
      $$y(-1) = x(0)$$

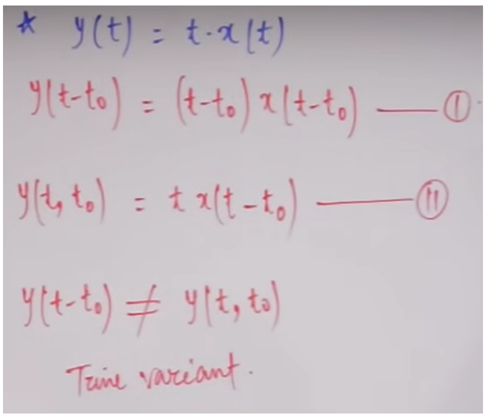
      depends on only future
    - $h(t) = 0 \quad t > 0$

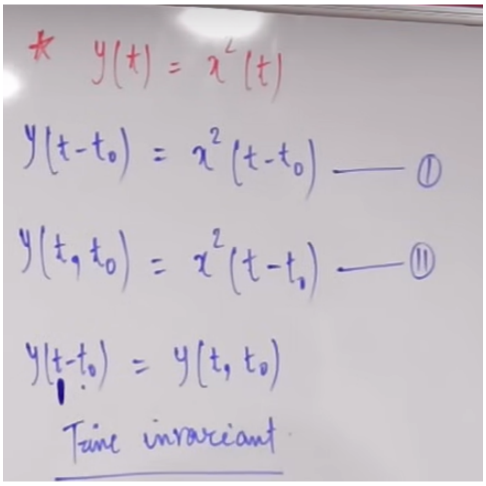
$$h[n] = 0 \quad t > 0$$

- Time variant and Invariant System
    - Time Invariant
        - If time shift in i/p results identical time shift in o/p without changing the nature of the output.
    - To check
        - Find $y(t)$ delay, $y(t - t_0)$, replace t with $t - t_0$
        - Find $x(t - t_0) = y(t, t_0)$, response of the system for delayed i/p

          $y(t, t_0) = \text{write the y(t) just add/minus } t_0$
        - if $y(t - t_0) = y(t, t_0)$, it is time variant.

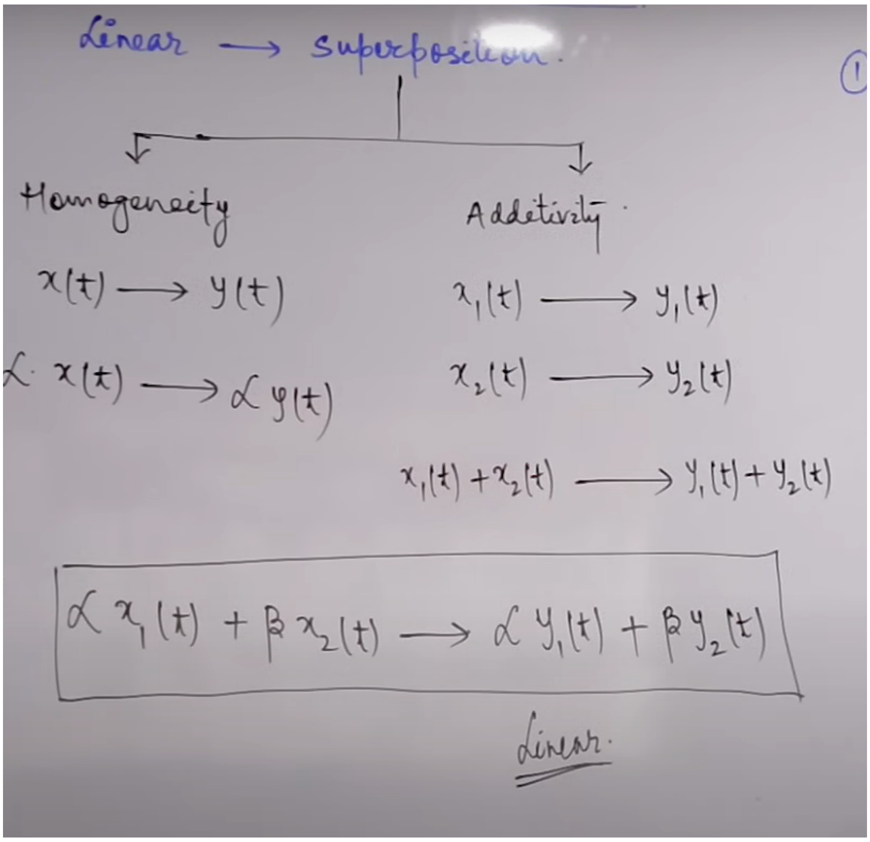






- For a system to be time invariant
  - There must not be any scaling in $x(t) \text{ or } y(t)$
  - Coefficient must not function of $time$
  - Any extra term except $x(t) \text{ or } y(t)$ must be zero or constant

- **Linear and Nonlinear System**
  - Linear
    - If the system follows principle of superposition



- **To check linearity**
  - Graph between output and input must be throughout a **straight** line **passing through origin** without having **saturation or dead zone**
  - If system is represented by a linear differential equation, then the equation must be linear.
  - System must be follow zero input zero output criteria.
    - $x(t) \to y(t)$
      $\alpha x(t) \to \alpha y(t)$
      $0 = 0, \alpha = 0$
- **Mathematical way to prove a system is linear or not**

  $\text{Given, } x(t) \to y(t) = tx(t)$

  $x_1(t) \to y_1(t) = tx_1(t)$

  $x_2(t) \to y_2(t) = tx_2(t)$

  $y_3(t) = y_1(t) + y_2(t) = t(x_1(t) + x_2(t))...(1)$

  $x_3'(t) = x_1(t) + x_2(t)$

  $y_3'(t) = tx_3(t) = t(x_1(t) + x_2(t))...(2)$

$(1) = (2),$ it follows the additivity.

$\alpha x(t) \rightarrow \alpha y(t) = \alpha t x(t)$

$y'(t) = t[\alpha x(t)] = \alpha t x(t) ...(3)$

$\alpha y(t) = \alpha t x(t) ...(4)$

$(3) = (4),$ it follows the homogeneity rule

- **Checking homogeneity and additivity at once**

  $x(t) \rightarrow y(t)$

  $\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t),$ the system will be lineary

  Given, $y(t) = x[sint]$

  $y_1(t) = x_1[sint], y_2[t] = x_2[sint]$

  $y_3(t) = \alpha y_1(t) + \beta y_2(t) = \alpha x_1(sint) + \beta x_2(sint)$

  $x_3'(t) = \alpha x_1(t) + \beta x_2(t)$

  $y_3'(t) = x_3'(sint) = \alpha x_1(sint) + \beta x_2(sint)$

- **Invertible and Non-invertible**
  - Invertible
    - If distinct input produces distinct outputs
    - if input can be determined by observing output
    - a inverse system can be design, overall gain = 1

- **Stable and Unstable System**
  - **Stable**
    - Follows bounded input bounded output(BIBO)
    - The output of the system must be bounded for bounded input

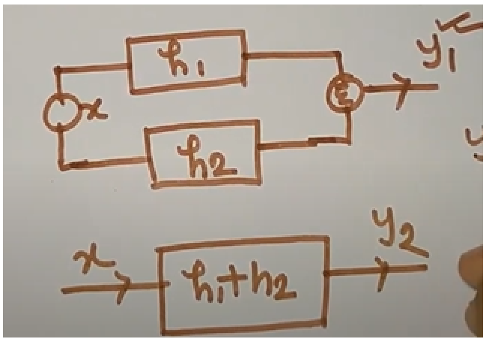
- $0 \leq |x(t)| < \infty$, then $0 \leq |y(t)| < \infty$
    - BIBO implies that impulse response must tend to zero , as time tends to infinity.

- **LTI System**
  - **Properties**
    - The commutative property of LTI System
      - **Discrete Time Signal**

        $x[n] * h[n] = h[n] * x[n]$
        or, $\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
      - **Continuous Time Signal**

        $x(t) * h(t) = x(t) * h(t)$
        or, $\int_{-\infty}^{\infty} x(\tau)h(t-\tau) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)$
    - **The distributive Property of LTI System**

      Summing of the outputs of of two systems is equivalent to a system with an impulse response equal to the sum of the impulse response of the two individual system.

      $y_1 = y_2$

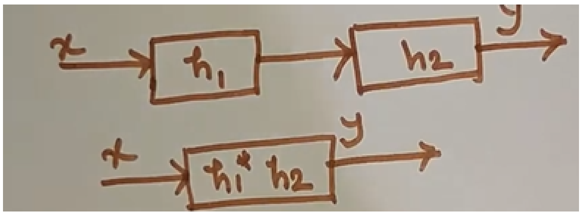
      $y_1 = x * h_1 + x * h_2$

      $y_2 = x * (h1 + h2)$

      Hence, $x * h_1 + x * h_2 = x * (h_1 + h_2)$

    - **Associative Property**

The change of order of the cascaded system will not affect the response.



$$y = x * (h1 * h2) = (x * h_1) * h_2$$

- **LTI System with and without memory**

  Memoryless → Output(t) → input(t), Current output depends on only current input

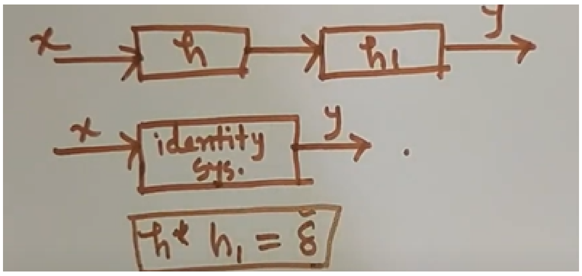
  DTS: $h[n] = 0 \text{ for } n \neq 0$

  CTS: $h[t] = 0 \text{ for } t \neq 0$

  Otherwise, with memory.

- **Invertibility of LTI System**

  A system S is invertible if and only if there exists an inverse system S^{-1} such that S^{-1} is an identity system.



$$h * h_1 = \delta$$

- **Causality Property**

  DTS: $h(n) = 0, \ n < 0$

  CTS: $h(t) = 0, \ t < 0$

- **Stability Property**

$\sum_{k=\infty}^{\infty}|h(k)| < \infty$, for DTS
$\int_{-\infty}^{\infty}|h(t)| < \infty$, for CTS

- **Check Causality and Stability for $h(n) = \frac{1}{2}^n u(n)$ (DTS : Discrete Time Signal)**

  $u(n) = 0,\ n < 0$

  $h(n) = \frac{1}{2}^n \times 0;\ n < 0$
  $= 0$, Causal;

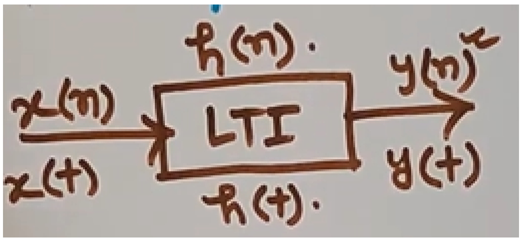
  $\sum_{k=-\infty}^{\infty} h(k) = \sum_{k=-\infty}^{0} \frac{1}{2}^n u(k) + \sum_{k=0}^{\infty} \frac{1}{2}^n u(k) = 0 + \frac{1}{1-\frac{1}{2}} = 2 < \infty$, Stable

- **Convolution**

  Convolution is a mathematical way of **"combining two signals to form a third signal"**

  $*$ is the convolution sign.

  - **Characterizing a LTI System**

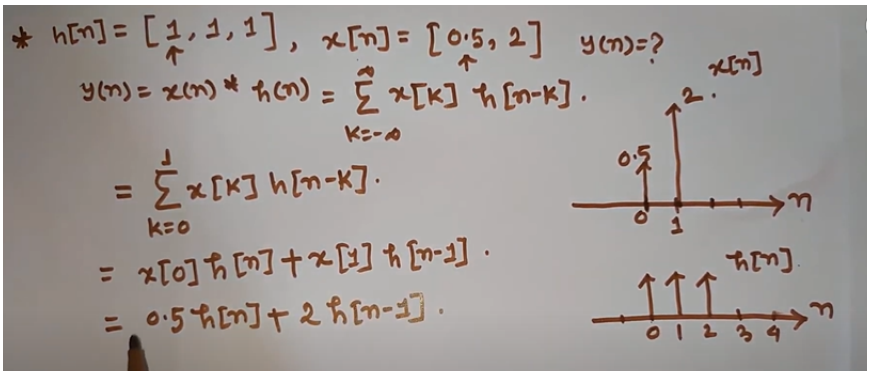


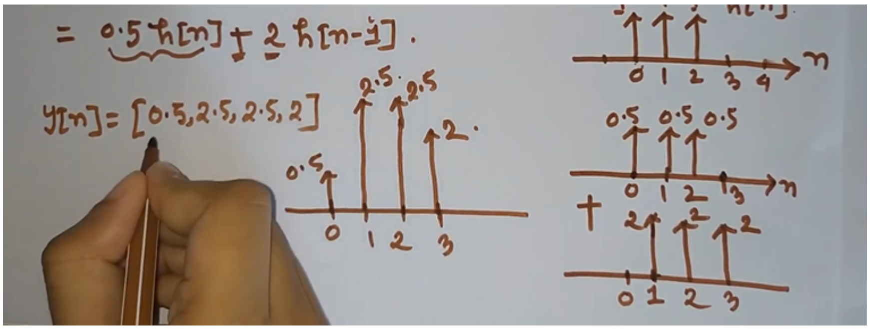
$y(n) = x(n) * h(n) = h(n) * x(n)$

$y(t) = x(t) * h(t) = h(t) * x(t)$

DTS(Discrete Time Signal):

$y(n) = x(n) * h(n)$
$= h(n) * x(n)$
$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$
$= \sum_{k=-\infty}^{\infty} x(n-k)h(n)$



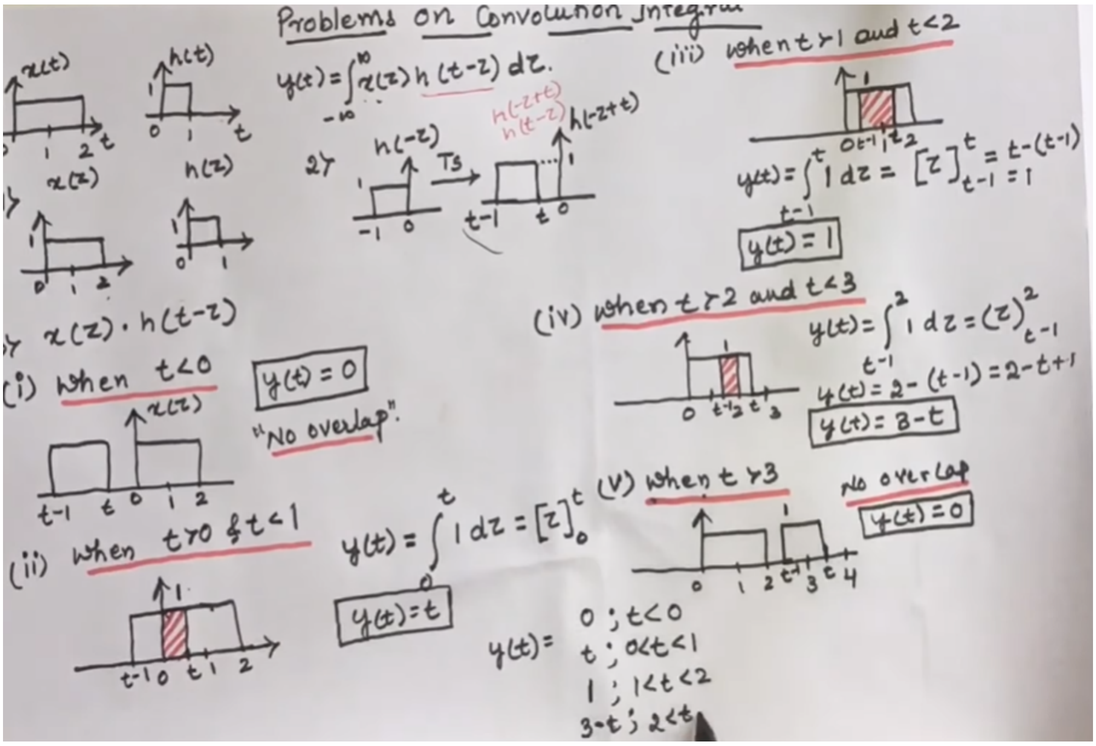


CTS (Continuous Time Signal):

$y(t) = h(t) * x(t) = x(t) * h(t)$
$= \int_0^t x(\lambda)h(t-\lambda)d\lambda = \int_0^t h(\lambda)x(t-\lambda)d\lambda$

Steps:

- Folding: $h(\lambda) = h(-\lambda)$
- Shifting: $h(-\lambda + t) \rightarrow h(t - \lambda)$
- Multiplication: $x(\lambda), h(t-\lambda)$
- Integrating: $\int_{t_1}^{t_2} x(\lambda) * h(t-\lambda)$



Laplace Transform

Laplace transform is an integral transformation of a function $f(t)$ from time domain into the frequency domain.

$f(t) \rightarrow F(s)$ $\quad\quad\quad\quad$ $s = a + jw$, **real and imaginary part**

Laplace of f(t), $\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$

$y(t) = x(t) * h(t)$ convulotion in time domain
$Y(s) = X(s) \times H(s)$
simple multiplication frequency domain

$\mathscr{L}[u(t)] = \int_0^\infty u(t)e^{-st}dt = \frac{1}{s}$

$\mathscr{L}[coswt] = \int_0^\infty \frac{e^{jwt}+e^{-jwt}}{2}e^{-st}dt = \frac{s}{s^2+w^2}$

$\mathscr{L}[sinwt] = \int_0^\infty \frac{1}{2j}(e^{jwt} - e^{-jwt})e^{-st}dt = \frac{w}{s^2+w^2}$

$\mathscr{L}[tu(t)] = \int_0^\infty tu(t)e^{-st} = t\int_0^\infty e^{-st} - \int_0^\infty \frac{dt}{dt}\int_0^\infty e^{-st} = \frac{1}{s^2}$

Laplace transform condition/Dirichlet's conditions

- The function must be integrable in the given interval of time
  $\int_{-\infty}^{\infty} f(t)dt < \infty$
- The function should have finite number of maxima and minima.

  Example: Noise signal doesn't follow this condition
- There must be finite number of discontinuities in the signal in the given interval of time.

Properties

Linearity Property

If $x_1(t), x_2(t)$ are two signal, then the will follow the superposition theorem.

$a_1x_1(t) + a_2x_2(t) \rightarrow a_1X_1(s) + a_2X_2(s)$

$Z_1(t) = a_1x_1(t) + a_2x_2(t)$

$X_1(s) = \mathscr{L}[x_1(t)] = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt$
$X_2(s) = \mathscr{L}[x_2(t)] = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt$

$Z_1(s) = \mathscr{L}[Z_1(t)] = a_1X_1(s) + a_2X_2(s)$

Time Shifting Property

$x(t) \leftrightarrow X(s)$

$x(t - t_0) \leftrightarrow e^{-st_0}X(s)$

Time Scaling Property

$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{s}{a})$

Freequency Property

$X(s - s_0) \rightarrow e^{s_0t}x(t)$

Time Reversal

$x(t) \leftrightarrow X(s)$

$x(-t) \leftrightarrow -X(-s)$

Differentiation & Integration Property

$\frac{dx(t)}{dt} \leftrightarrow SX(S)$

$\int x(t)dt \leftrightarrow \frac{X(s)}{s}$

Multiplication and Convolution Property

$x_1(t), x_2(t)$

$x_1(t) * x_2(t) \leftrightarrow X_1(s) \times X_2(s)$

$x_1(t) \times x_2(t) \leftrightarrow X_1(s) * X_2(s)$

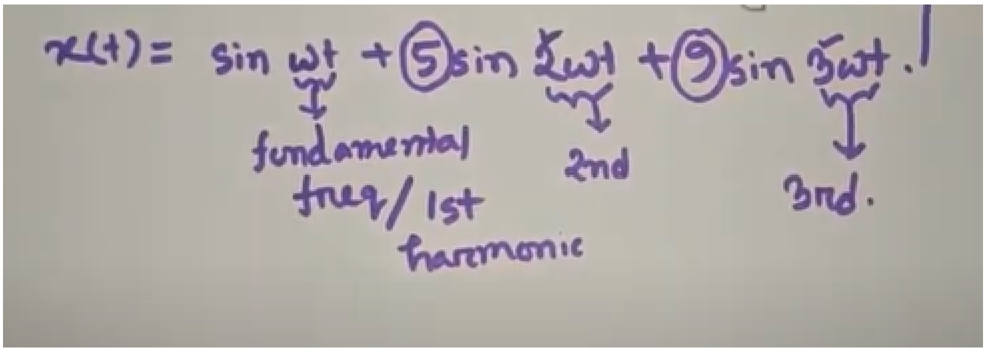
Fourier Series

Fourier Series is a sum that represents a "periodic function" as a sum of $sine$ and $cosine$ waves in terms of their harmonics.

$x(t) = x(x \pm T)$, T : Time period, $x(t)$ is periodic.

Frequency, $\texttt{f} = \texttt{no of cycle/sec}$ / rate of change

Harmonics



3rd Harmonics will dominant 2nd one because it's magnitude is greater than the 2nd one. (9 > 5)

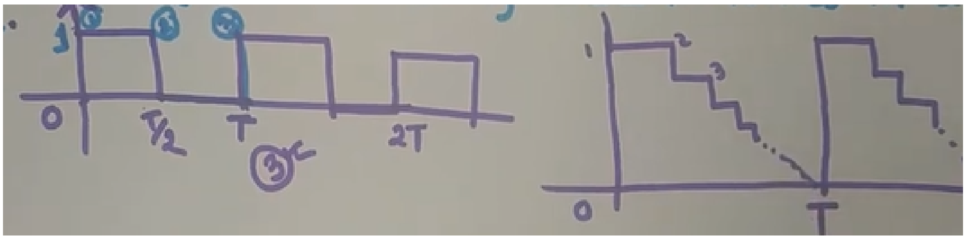
Dirichlet conditions of Fourier Series

- Function $f(t)$ is single valued everywhere.

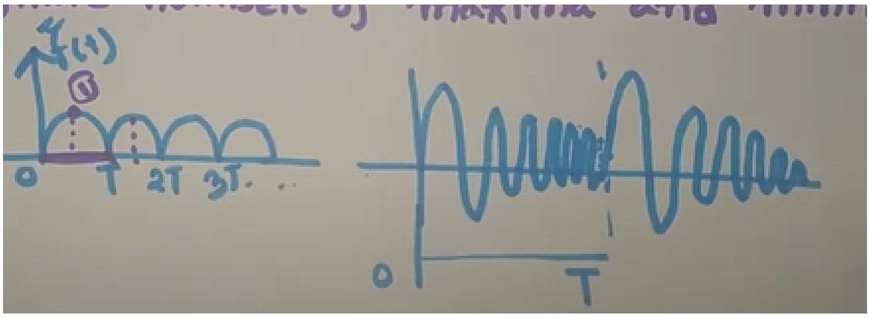
  $f(x) = x^2;$

  For every x there is only one $f(x)\ or\ y$

- Function has a finite number of discontinuities in any time period.

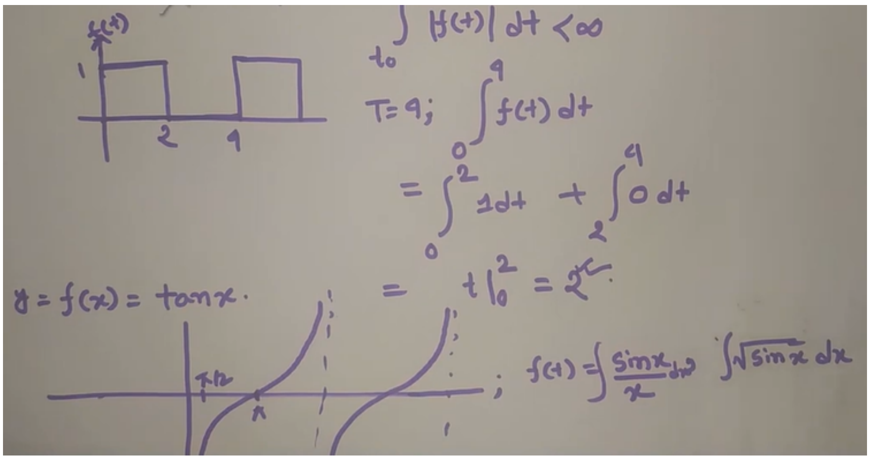


- Function has a finite number of maximum and minimum in any time period.



- Signal should be absolutely integrable in any time period.

  $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$



Why `sine and cosine` are special to Fourier Series ?

- The Fourier series uses cosine and sine functions **because they form a complete set of orthogonal functions, meaning they are independent and can be used to represent any periodic function**

Trigonometric Fourier Series

$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cosnw_0t + b_n sinnw_0t ...(i)$

$a_0, a_n, b_n = \text{fourier coneffecient}$

$a_0 = \frac{1}{T}\int_0^T f(t)dt = \text{average of function}$

Multiplying $cosmw_0t$ and integrating in (i),

$\int_0^T f(t)cosmw_0tdt = \int_0^T [a_0 + \sum_{n=1}^{\infty} a_n cosnw_0t + b_n sinnw_0t]cosmw_0tdt$

$= \int_0^t a_0 cosmw_0tdt + \sum_{n=1}^{\infty}(\int_0^t a_n cosmw_0t \cdot cosnw_0tdt + \int_0^t a_n cosmwt \cdot sinnw_0t)dt$

$= 0 + \sum_{n=1}^{\infty}\int_0^t a_n cosmw_0t \cdot cosnw_0tdt + 0$

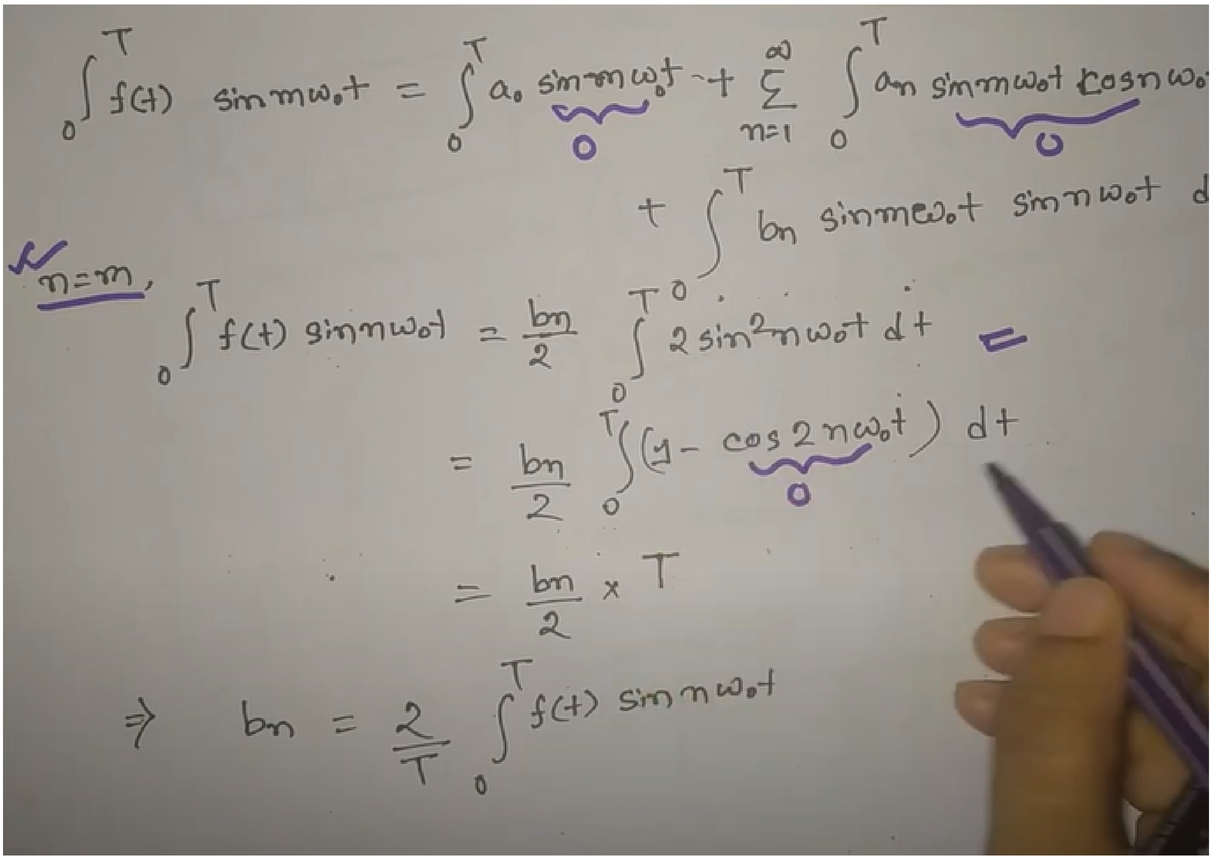
$\mathtt{Now, n = m},$

$\int_0^T f(t)cosnw_0tdt = \int_0^t a_n cosn^2w_0t = \int_0^T (1 + cos2nw_0t)dt = \frac{T}{2}a_n$

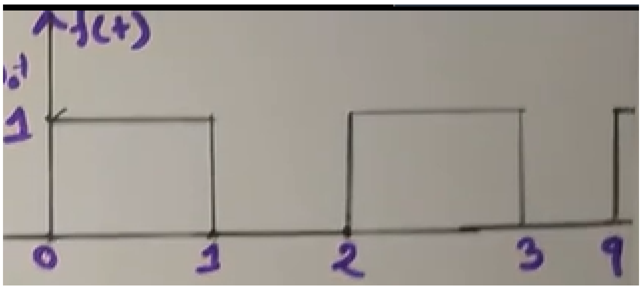
$a_n = \frac{2}{T}\int_0^T f(t)cosnw_0tdt$

Multiplying $sinmw_0t$ and integrating in (i),

$b_n = \frac{2}{T}\int_0^T f(t)sinnw_0tdt$



Fourier Series Expansion



$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

$T = 2$

$w_0 = \frac{2\pi}{T} = \pi$

$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}(\int_0^1 1dt + \int_1^2 0dt) = \frac{1}{2}$

$a_n = \frac{2}{T}\int_0^T f(t)cosnw_0tdt = \int_0^1 1 \cdot cosn\pi dt + \int_1^2 0 \cdot cosn\pi tdt$
$= \frac{sinn\pi t}{n\pi}|_0^1 = \frac{1}{n\pi}[sinn\pi - sin0] = 0$

Because, $sin\pi = sin2\pi ... = sinn\pi = 0$

$b_n = \frac{2}{T}\int_0^T f(t) \cdot sinnw_0tdt = \int_0^1 1 \cdot sinn\pi tdt + \int_1^2 0 \cdot sinn\pi tdt$
$= \frac{-cosn\pi t}{n\pi}|_0^1 = -\frac{1}{n\pi}[cosn\pi - cos0]$

$cosn\pi = (-1)^n$

$b_n = \frac{1}{n\pi}[1-(-1)^n] = \frac{2}{n\pi}\ [\mathbf{n} = \mathbf{odd}]$

$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} sin\pi t\ [\mathbf{n} = \mathbf{odd}]$
$= \frac{1}{2} + \frac{2}{\pi} sin\pi t + \frac{2}{3\pi} sin3\pi t + ...$

$f(t) = \begin{cases} 15, & 0 < t < 1 \\ -15, & 1 < t < 2 \end{cases}$

$T = 2$

$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$
$= \frac{1}{2}\left[\int_0^1 15 \cdot dt - \int_1^2 15\,dt\right]$
$= \frac{1}{2}\left[15 \times 1 - 15 \times 1\right]$
$= 0$

$w_0 = \frac{2\pi}{T} = \pi$

$a_n = \frac{2}{T}\int_0^2 f(t) \cos n\pi t \cdot dt$
$= 15 \cdot \frac{\sin n\pi t}{n\pi}\Big|_0^1 - 15 \cdot \frac{\sin n\pi t}{n\pi}\Big|_1^2$

$= 0$

$$b_n = \frac{1}{2}\int f(t).\sin n\pi t\, dt$$

$$= \frac{-15}{n\pi}\left[\cos n\pi - \cos 0\right] + \frac{15}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$$

$$= \frac{15}{n\pi}\left[\cos 2n\pi - 2\cos n\pi + 1\right]$$

$$= \frac{15}{n\pi}\left[2 - 2(-1)^n\right]$$

$$= \frac{30}{n\pi}\left[1 - (-1)^n\right]$$

$$= \frac{60}{n\pi}\left[n \text{ odd}\right]$$

Even Symmetry

If $f(t) = f(-t)$

$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt$
$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)cosnw_0tdt$
$b_n = 0$

Odd Symmetry

If $f(t) = -f(t)$

$a_0 = a_n = 0$

$f(t) = \sum_{n=1}^{\infty} b_n sinnw_0tdt$

Complex Exponential Fourier Series

$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0t}$

$\mathtt{c_n = complex\ exponential\ fourier\ coeffiecient}$

$c_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jnw_0t}dt$

$c_n = |c_n|e^{j\angle n}$

$$C_n^* = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt \quad \text{--- (i)} \qquad \boxed{a_n, b_n} \leftrightarrow C_n$$

1) Reversal: $(n = -n)$; $C_{-n} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jn\omega_0 t}\, dt$ --- (ii)

2) Conjugate: $C_{-n}^* = \frac{1}{T_0}\int_{T_0} x^*(t)\, e^{-jn\omega_0 t}$ ---- (iii)

If C_n is conjugate symmetry, $C_n = C_{-n}^*$ $\qquad \boxed{x(t) = x^*(t)} \rightarrow$ real

$$C_n = \underbrace{|C_n|}_{\text{real}}\ \underbrace{e^{j\angle C_n}}_{\text{img.}} \quad \text{--- (iv)}$$

1) $n \to -n$; $C_{-n} = |C_{-n}|\, e^{j\angle C_{-n}}$ --- (v)

$C_{-n}^* = |C_{-n}|\, e^{-j\angle C_{-n}}$ --- (vi)

If, $C_n = C_{-n}^* \Rightarrow |C_n|\, e^{j\angle C_n} = |C_{-n}|\, e^{-j\angle C_{-n}}$

$\Rightarrow$ * $|C_n| = |C_n| \rightarrow$ Even

* $\angle C_n = -\angle C_{-n} \rightarrow$ Od