Course code: EEE-453
Course title: Numerical Method
Lecture on
Curve Fitting Techniques

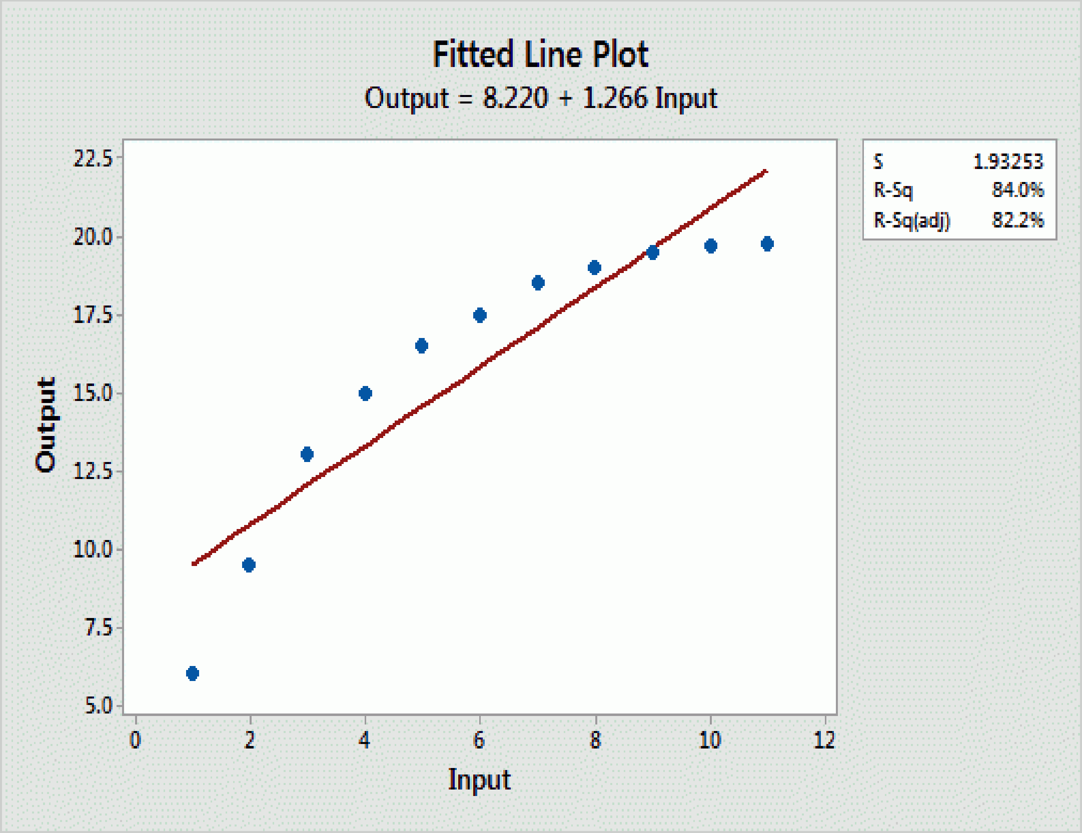
Curve Fitting Technique

Curve Fitting Technique

- ✓ Curve fitting technique is a process to determine a mathematical model with some data. Generally data are obtained from experimentally.

Necessity of Curve Fitting Technique

- ✓ It is a mathematical model for a system whose data are obtained.
- ✓ It gives or shows the pattern or trend of data.
- ✓ From it those value can be estimated for data were not taken.



Curve Fitting Technique

Curve Fitting Technique Types

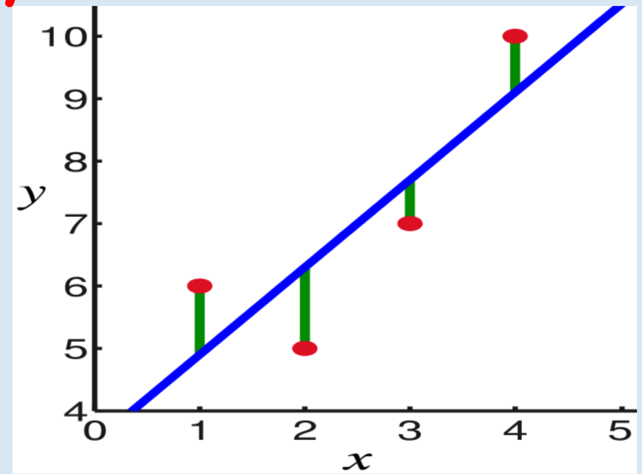
1. Least square regression
   i. Linear model
   ii. Quadrature model

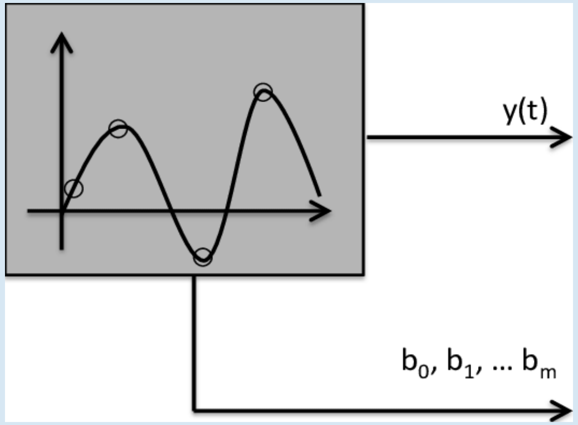
2. Interpolation
   i. Polynomial Interpolation
      a. Newton's Interpolating Polynomial
      b. LaGrange's Interpolating Polynomial
   ii. Spline Interpolation

Curve Fitting Technique

Difference between Curve Fitting Technique Types

Least square regression	Interpolation
1.Data is fitted to a pre-defined curve in such way that error will be minimum	1.Single or multiple curve are obtained using all data point
2.The curve may or may not pass through all data point	2.The curve must pass all the point so error is zero
3.It is applicable when data have less accuracy	3. It is applicable when data have high accuracy and reliability





Curve Fitting Technique

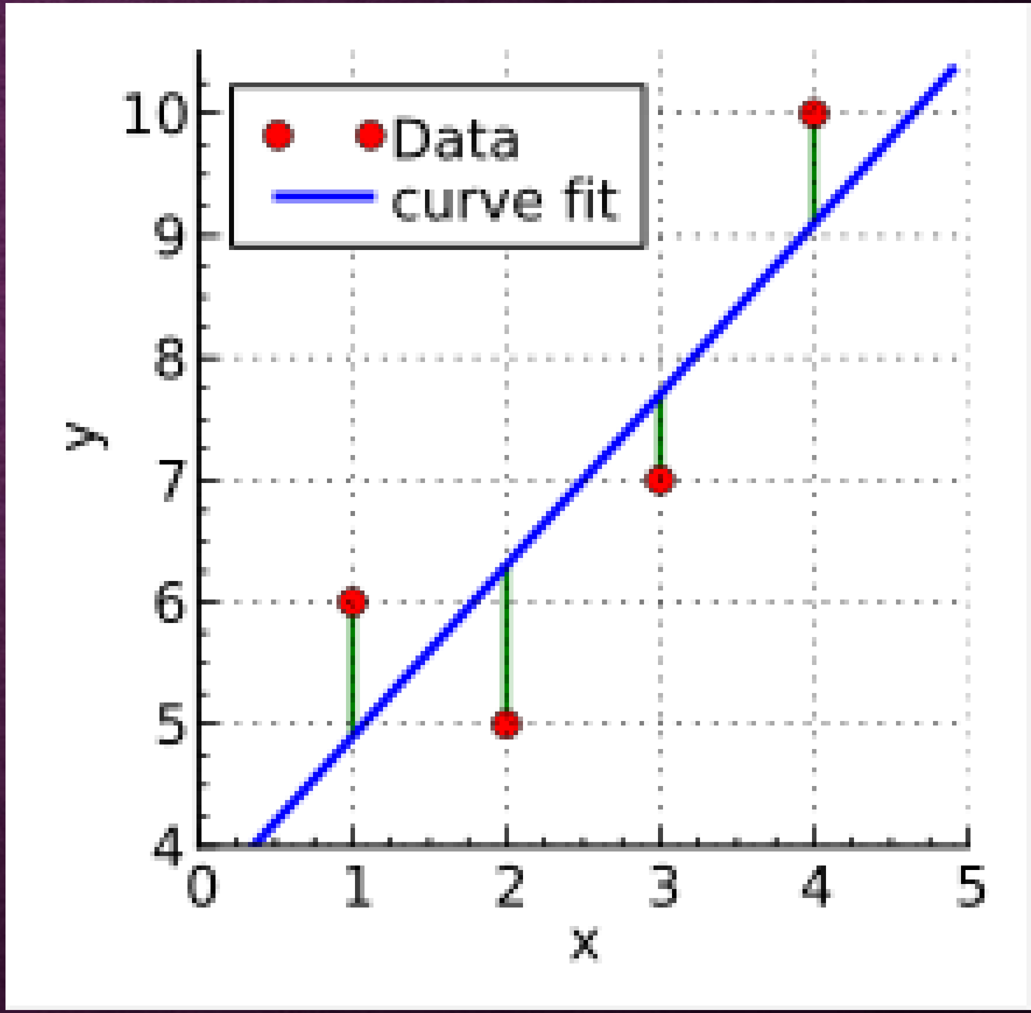
Least Square Regression

In this process data is fitted to a pre-defined curve in such way that the sum of square of individual error has to be minimum

$$Sr = \sum_{i=1}^{n} \{e_i^2\}$$

$$Sr = e_1^2 + e_2^2 + e_3^2 + e_4^2$$

e_i^2 = fitted data value- actual data value



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Principle of Least square regression

- ✓ Linear model

$$a_0+a_1x=y$$

$$na_0+(\Sigma x)a_1 =\Sigma y$$

$$(\Sigma x)a_0+(\Sigma x^2)a_1=\Sigma xy$$

n=no. of data point

x=independent variable

y=dependent variable

- ✓ Quadratic Model

$$a_0+a_1x+a_2x^2=y$$

$$na_0+(\Sigma x)a_1 ++(\Sigma x^2)a_2 = \Sigma y$$

$$(\Sigma x)a_0+(\Sigma x^2)a_1 +(\Sigma x^3)a_2 = \Sigma xy$$

$$(\Sigma x^2)a_0+(\Sigma x^3)a_1 +(\Sigma x^4)a_2 = \Sigma x^2y$$

Curve Fitting Technique: Least square regression

Problem 01

The capacitance of a varactor diode used in VCO(voltage control oscillator) varies with its input voltage. A sample varactor diode is tested for various voltage V in volts and the following capacitance C(μF)is founded.

V(volt)=x	0	1	3	6	8	12
C(μF)=y	2	2.5	2.75	5	9.5	20

a. Using principle of Least square regression fit the above data in a linear model like $C=C_0+C_1V$ to determine C_0 and C_1.
b. Fit it to a model like $C=C_0+C_1V+C_2V^2$ to determine C_0,C_1 and C_2.

Curve Fitting Technique: Least square regression

Solution

a. Here

n=data point=6

$C_1 = a_1$ and $C_0 = a_0$

Linear model

$$a_0 + a_1 x = y$$

$$n a_0 + (\Sigma x) a_1 = \Sigma y$$

$$(\Sigma x) a_0 + (\Sigma x^2) a_1 = \Sigma xy$$

To fit data in linear model we need

Σx, Σy, Σx^2 and Σxy

Curve Fitting Technique: Least square regression

x	y	x^2	xy
0	2	0	0
1	2.5	1	2.5
3	2.75	9	8.25
6	5	36	30
8	9.5	64	76
12	20	144	240
$\Sigma x=30$	$\Sigma y=41.75$	$\Sigma x^2 = 254$	$\Sigma xy=356.75$

So, $6a_0+30a_1 = 41.75$

$30a_0+254a_1=356.75$

$C_0 = a_0 = -0.1571$

$C_1 = a_1 = 1.4231$

Curve Fitting Technique: Least square regression

b. Here

n=data point=6

$C_2=a_2$, $C_1=a_1$ and $C_0= a_0$

Quadratic Model

$$a_0+a_1x+a_2x^2=y$$

$$na_0+(\Sigma x)a_1 ++(\Sigma x^2)a_2 = \Sigma y$$

$$(\Sigma x)a_0+(\Sigma x^2)a_1 +(\Sigma x^3)a_2 = \Sigma xy$$

$$(\Sigma x^2)a_0+(\Sigma x^3)a_1 +(\Sigma x^4)a_2 = \Sigma x^2y$$

To fit data in linear model we need

Σx, Σx^2, Σx^3, Σx^4, Σy, Σxy and Σx^2y

Curve Fitting Technique: Least square regression

x	y	x^2	x^3	x^4	xy	x^2y
0	2	0	0	0	0	
1	2.5	1	1	1	2.5	
3	2.75	9	9	27	8.25	
6	5	36			30	
8	9.5	64			76	
12	20	144			240	
$\Sigma x=30$	$\Sigma y=41.75$	$\Sigma x^2 = 254$	$\Sigma x^3=$	$\Sigma x^4=$	$\Sigma xy=356.75$	$\Sigma x^2y=$

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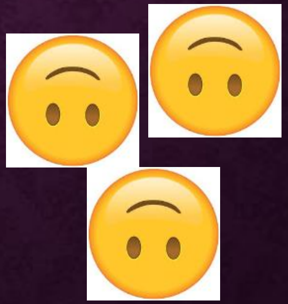
So,

$$6a_0+30a_1 ++254a_2 = 41.75$$
$$30a_0+254a_1 +(\Sigma x^3)a_2 = 356.75$$
$$254a_0+(\Sigma x^3)a_1 +(\Sigma x^4)a_2 = \Sigma x^2y$$

$C0= a_{0=}$ = ……………………

$C1=a_{1=}$ ………………… and

$C2=a_{2=}$ ……………………...



Curve Fitting Technique: Least square regression

Linearization of non-linear relationship

Here an exponential relation is linearized

$y=\alpha e^{\beta x}$

$\ln(y)=\ln(\alpha e^{\beta x})$

$\ln(y)=\ln(\alpha)+\ln(e^{\beta x})$

$\ln(y)=\ln(\alpha)+ \beta x$

$\boxed{y}$ $\quad a_0 \quad a_1 \boxed{x}$

So

$a_0 = \ln(\alpha)$

$\alpha = e^{a_0}$ and $a_1 = \beta$

Linear model

$a_0+a_1x=y$

$na_0+(\Sigma x)a_1 = \Sigma y$

$(\Sigma x)a_0+(\Sigma x^2)a_1=\Sigma xy$

Curve Fitting Technique: Least square regression

Linearization of non-linear relationship

Non-linear relation	Linearization	Determination of coefficient
1. Exponential $y=\alpha e^{\beta x}$	$\ln(y)=\ln(\alpha)+\beta x$ $Y = a_0+a_1X$	$\alpha = e^{a_0}$ and $\beta=a_1$
2. Power relation $y=\alpha x^{\beta}$	$\ln(y)=\ln(\alpha)+\beta\ln(x)$ $Y = a_0+a_1X$	$\alpha = e^{a_0}$ and $\beta=a_1$
3. Rational $y=\frac{alpha}{\beta+x}$	$\frac{1}{y}=\frac{\beta}{alpha}+\frac{x}{alpha}$ $Y = a_0+a_1X$	$alpha=\frac{1}{a_1}$ and $\beta=a_0 alpha$

Curve Fitting Technique: Least square regression

Problem 02

The capacitance of a varactor diode used in VCO(voltage control oscillator) varies with its input voltage. A sample varactor diode is tested for various voltage V in volts and the following capacitance C(μF)is found.

V(volt)=x	0	1	3	6	8	12
C(μF)=y	2	2.5	2.75	5	9.5	20

a. Using principle of Least square regression fit the above data in a exponential relationship like $C=C_0e^{C_1V}$ to determine C_0 and C_1.

Curve Fitting Technique: Least square regression

Solution

a. Here

n=data point=6

$C_1 = a_1$ and $C_0 = a_0 = \ln(a)$ or $a = e^{a_0}$

$$C = C_0 e^{C_1 V}$$
$$y = a e^{\beta x}$$
$$\ln(y) = \ln(a e^{\beta x})$$
$$\ln(y) = \ln(a) + \ln(e^{\beta x})$$
$$\ln(y) = \ln(a) + \beta x$$
$$Y \quad a_0 \quad + \quad a_1 X$$

To fit data in linear model we need

ΣX, ΣY, ΣX^2 and ΣXY

Linear model

$$a_0 + a_1 x = y$$
$$n a_0 + (\Sigma x) a_1 = \Sigma y$$
$$(\Sigma x) a_0 + (\Sigma x^2) a_1 = \Sigma xy$$

Curve Fitting Technique: Least square regression

$X=x$	y	$Y=\ln(y)$	X^2	xY
0	2	0.6931	0	0
1	2.5	0.9163	1	0.9163
3	2.75	1.0116	9	3.0348
6	5	1.6094	36	9.6564
8	9.5	2.2513	64	18.0104
12	20	2.9957	144	35.9484
$\Sigma X=30$	$\Sigma y=41.75$	$\Sigma Y=9.4779$	$\Sigma X^2 = 254$	$\Sigma xY=67.5663$

Curve Fitting Technique: Least square regression

$$na_0+(\Sigma x)a_1 = \Sigma y$$
$$(\Sigma x)a_0+(\Sigma x^2)a_1=\Sigma xy$$

[Here $\boxed{Y}$=ln(y)]

So,

$$6a_0+30a_1 = 9.4774$$
$$30a_0+254a_1=67.5663$$

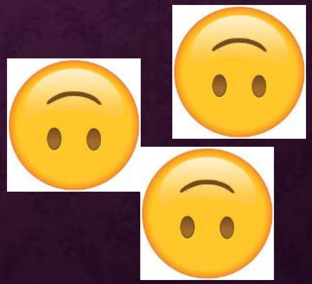
$a_0 = 0.6094$

$a_1 = 0.1940$

So, $\beta = a_1 = 0.1940$

$\alpha = e^{a_0} = 1.8393$

$C = C_0 e^{C_1 V} = 1.8393e^{0.1940V}$



Assignment

Curve Fitting Technique: Interpolation

Interpolation

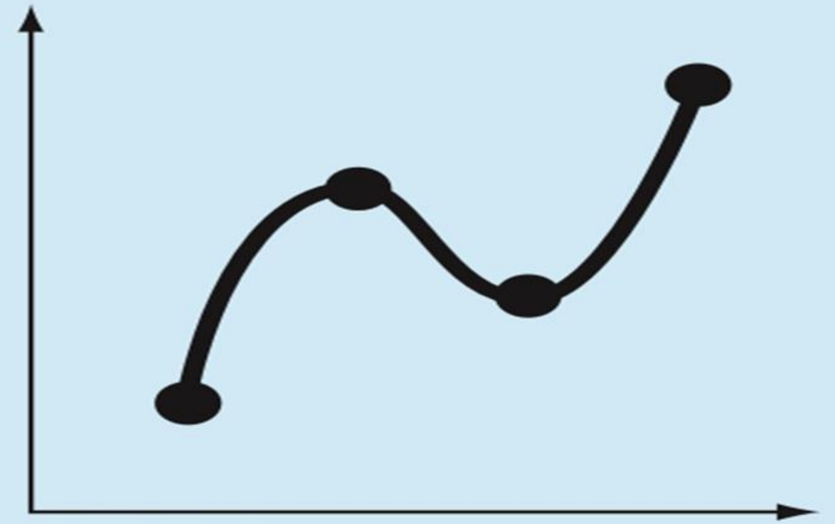
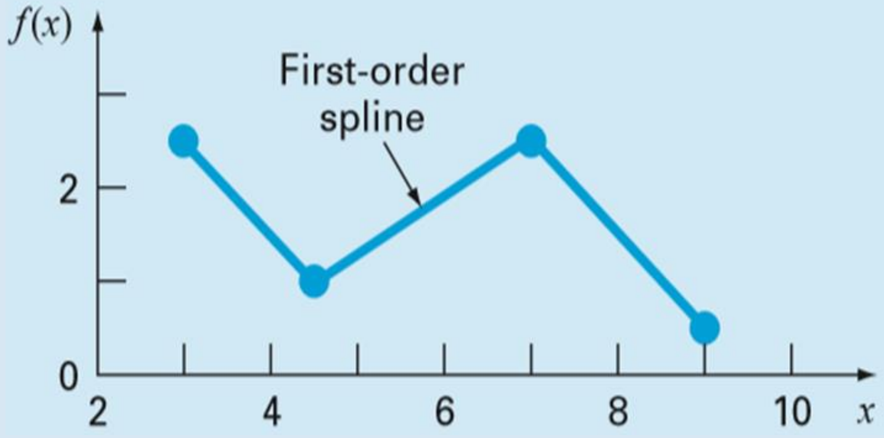
- ✓ In interpolation a single or multiple curves are fitted such that those pass through all data points.
- ✓ Interpolation is applied when data have high accuracy.

Interpolation Types

1. Polynomial Interpolation
    1. Newton's Polynomial Interpolation
    2. Lagrange's Polynomial Interpolation
2. Spline Interpolation

Curve Fitting Technique: Interpolation

Difference between Interpolation Technique Types

Polynomial Interpolation	Spline Interpolation
1.A single curve is fitted to passing through all data points	1.A separate curve is fitted with every two consecutive data points
2.The order of the fitted function is decided by the no. of data point. A nth order polynomial is formed with n+1 data points	2.The order of the curve is pre-decided means the curve may be linear, quadrature, cubic etc.Ex- linear spline, cubic spline etc.
	

Curve Fitting Technique: Polynomial Interpolation

Polynomial Interpolation

N+1 data points will form a polynomial of nth order. So the fitted data will have the function of

$$y=f(x)= a_0+a_1X +a_2X^2 +a_3X^3 \ldots\ldots\ldots\ldots +a_{n-1}X^{n-1}$$

To fit the data a_0 ,a_1 ,a_2 ,a_3 …… $a_{n-1,}$ a_n has to be determined

Two forms of Polynomial Interpolation

1. Newton's Polynomial Interpolation
2. Lagrange's Polynomial Interpolation

Curve Fitting Technique: Polynomial Interpolation

Newton's Polynomial Interpolation

For n+1 data points,

$\{(x_{1,}y_1), (x_{2,}y_2), (x_{3,}y_3)\ldots\ldots\ldots (x_{n+1,}y_{n+1})\}$

The polynomial has the form of

$Y=f(x)= b_0 + b_1(x- x_1) + b_2(x- x_1)(x- x_2) + b_3(x- x_1)(x- x_2)(x-x_3) + b_4(x- x_1)(x- x_2)(x- x_3)(x- x_4)\ldots\ldots\ldots+ b_n(x- x_1)(x- x_2)\ldots..(x- x_n)$

b_0, b_1, b_2, b_3, b_4………… b_n are the coefficient of the function those are determine by "Divided Difference Method"

Newton's Polynomial Interpolation

Problem 03

Suppose the demand of electricity in a small town is surveyed from 2000-2015 in regular interval and the following data is obtained.

Year	2000	2003	2007	2009	2013	2015	2020
Time(t)	0	3	7	9	13	15	20
Electricity Demand(MW)	10	15	18	22	27	35.5	?

Use Newton's Polynomial Interpolation to estimate the demand in Electricity in 1999(t=-1) and 2020(t=20) in that town.

Newton's Polynomial Interpolation

Solution Here

Time(t=x)	0	3	7	9	13	15	20
y	10	15	18	22	27	35.5	?

$Y=f(x)=b_0+b_1(x-0)+b_2(x-0)(x-3)+b_3\,(x-0)(x-3)(x-7)+b_4\,(x-0)(x-3)(x-7)(x-9)+b_5(x-0)(x-3)(x-7)(x-9)(x-13)$

X= -1(for 1999) and 20(for 2020)

Now coefficient $b_0, b_1, \ldots\ldots b_5$ should be calculated in Divided difference method.

Divided difference method$=\Delta Y/\Delta x=(Y2-Y1)/(x2-x1)$

Newton's Polynomial Interpolation

x	Y(bo)	b1	b2	b3	b4	b5
0	10	$\frac{15-10}{3-0}=1.6667$	$\frac{0.75-1.6667}{7-0}=-0.0310$	$\frac{0.2083-0.1309}{9-0}$	$\frac{-0.0333-0.0377}{13-0}$	$\frac{0.00928+0.00546}{15-0}$
3	15	$\frac{18-15}{7-3}=0.75$	$\frac{2-0.75}{9-3}$			
7	18	2	$\frac{1.25-2}{13-7}$			
9	22	1.25	$\frac{4.25-1.25}{15-9}$			
13	27	4.25				
15	35.5					

Newton's Polynomial Interpolation

Now

$Y=f(x)=b_0+b_1(x-0)+b_2(x-0)(x-3)+b_3\ (x-0)(x-3)(x-7)+b_4\ (x-0)(x-3)(x-7)(x-9)+b_5(x-0)(x-3)(x-7)(x-9)(x-13)$

For 1999 $X=-1$ and $b0=10$, $b1=1.6667$, $b2=-0.1310$, $b3=0.0377$, $b4=-0.0055$, $b5=0.0010$

So $Y=b_0+b_1(x-0)+b_2(x-0)(x-3)+b_3\ (x-0)(x-3)(x-7)+b_4\ (x-0)(x-3)(x-7)(x-9)+b_5(x-0)(x-3)(x-7)(x-9)(x-13)$

$Y =$ 

Similarly for 2020 $x=20$

$Y=$ 

Curve Fitting Technique: Polynomial Interpolation

Lagrange's Polynomial Interpolation

- ✓ Another form of Newton's Polynomial Interpolation
- ✓ The polynomial has the form of

$$Y=f(x)=\sum_{i=1}^{n} Li\ f(Xi)$$

- ✓ n = Number of data points
- ✓ Li = $\prod_{i=1}^{n} \frac{x-xj}{xi-xj}$ and [j ≠ i]
- ✓ For $L1 = \frac{x-x1}{x1-x1} * \frac{x-x2}{x1-x2} * \frac{x-x3}{x1-x3} \cdots\cdots \frac{x-xn}{x1-xn}$

$$L3 = \frac{x-x1}{x3-x1} * \frac{x-x2}{x3-x2} * \frac{x-x3}{x3-x3} \cdots\cdots \frac{x-xn}{x3-xn}$$

Lagrange's Polynomial Interpolation

Problem 04

Suppose the demand of electricity in a small town is surveyed from 2000-2015 in regular interval and the following data is obtained.

Year	2000	2003	2007	2009	2013	2015	2020
Time(t)=x	0	3	7	9	13	15	20
Electricity Demand(MW)=y	10	15	18	22	27	35.5	?

Use Lagrange's Polynomial Interpolation to estimate the demand in Electricity in 1999(t=-1) and 2020(t=20) in that town.

Solution

Lagrange's Polynomial Interpolation

Time(t=x)	0	3	7	9	13	15	20
y	10	15	18	22	27	35.5	?

$$y=10\frac{(x-3)(x-7)(x-9)(x-13)(x-15)}{(0-3)(0-7)(0-9)(0-13)(0-15)}+15\frac{(x-0)(x-7)(x-9)(x-13)(x-15)}{(3-0)(3-7)(3-9)(3-13)(3-15)}+$$

$$18\frac{(x-0)(x-3)(x-9)(x-13)(x-15)}{(7-3)(7-0)(7-9)(7-13)(7-15)}+22\frac{(x-0)(x-3)(x-7)(x-13)(x-15)}{(9-3)(9-7)(9-0)(9-13)(9-15)}+$$

$$27\frac{(x-0)(x-3)(x-7)(x-9)(x-15)}{(13-3)(13-7)(13-9)(13-0)(13-15)}+$$

$$35.5\frac{(x-0)(x-3)(x-7)(x-9)(x-13)}{(15-0)(15-7)(15-9)(15-13)(15-3)}\ldots\ldots\ldots\ldots(i)$$

Lagrange's Polynomial Interpolation

So,

For 1999 x=-1

From equation (i)

Y=………… 🙃 🙃

For 2020 x=20

Y=…………………….🙃🙃

Spline Interpolation

Spline Interpolation(Linear)

- ✓ Multiple curve can be obtain from neighboring two point
- ✓ For linear spline function can be look like

$y=a_0+a_1x \quad ; \; X1 \leq X \leq X2$

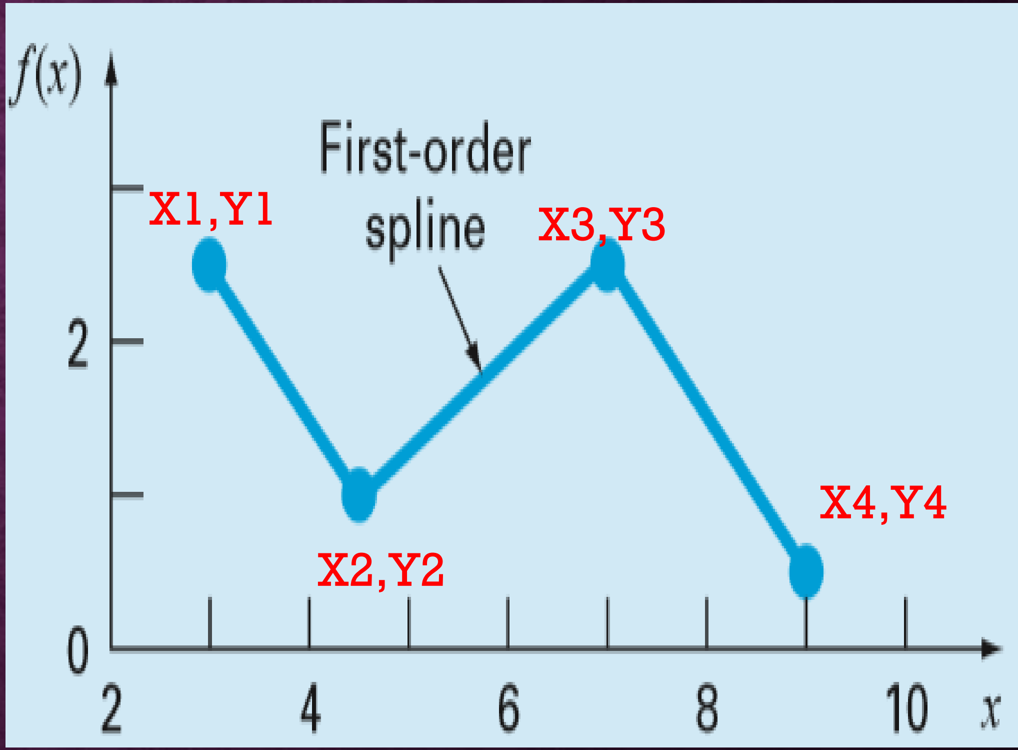
$y=a_2+a_3x \quad ; \; X2 \leq X \leq X3$

$y=a_4+a_5x \quad ; \; X3 \leq X \leq X4$

- ✓ Interpolation-- x in between data points
- ✓ Extrapolation--x beyond data points like

$y= \quad X4 \leq X$

$y= \quad X \leq X1$



Spline Interpolation

Problem 05

Suppose the demand of electricity in a small town is surveyed from 2000-2015 in regular interval and the following data is obtained.

Year	2000	2003	2007	2009	2013	2015	2020
Time(t)=x	0	3	7	9	13	15	20
Electricity Demand(MW)=y	10	15	18	22	27	35.5	?

Use Spline Interpolation to estimate the demand in Electricity in 1999(t=-1) and 2020(t=20) in that town.

Spline Interpolation

Solution
Here

Time(t=x)	0	3	7	9	13	15	20
y	10	15	18	22	27	35.5	?
	(0,10)	(3,15)	(7,18)	(9,22)	(13,27)	(15,35.5)	

Interpolation

When x is within limit like x=10

$9 \leq x \leq 13 \quad \frac{y-27}{22-27} = \frac{x-13}{9-13}$ $\quad [\frac{y-y1}{y2-y1} = \frac{x-x1}{x2-x1}]$

y=(...)x+(...)

So x=10 Y=(...)10+(...)

Spline Interpolation

Extrapolation

When x is not within the limit

like x=20

$$9 \leq x \leq 13 \qquad \frac{y-22}{22-27} = \frac{x-13}{13-9}$$

y=(…)x+(…)

So x=20

Y=(…)20+(…)

So for x=-1

Y=(…)(-1)+(…)

Assignment