University of Asia Pacific

Department of EEE

Course Code: EEE 453

Course Title: Numerical Method

Lecture On

Solution of System of Linear equation

Solution of System of Linear equation

Solution of System of Linear equation

A linear system containing n-unknown variables and m-equation will

Look like as follow

$$a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\ldots\ldots+a_{2n}x_n=b_2$$
$$.$$
$$.$$
$$a_{m1}x_1+a_{m2}x_2+\ldots\ldots+a_{mn}x_n=b_m$$

If $m=n \longrightarrow$ unique solution is possible

$m>n \longrightarrow$ unique solution is possible

$m<n \longrightarrow$ no unique solution is possible

Solution of System of Linear equation

A System can be considered as three matrices

$$A=\begin{bmatrix} a11 & a12 & \dots\dots..a1n \\ a21 & a22 & \dots\dots..a2n \\ & . & \\ & . & \\ & . & \\ am1 & am2 & \dots\dots\dots amn \end{bmatrix} = \text{co-efficient matrix}$$

$$X=[x1 \quad x2 \quad \dots\dots\dots xn] = \text{unknown variable matrix}$$

$$B=\begin{bmatrix} b1 \\ b2 \\ . \\ bm \end{bmatrix} = \text{constant matrix}$$

Solution of System of Linear equation

So AX=B

G=$[A \quad B]$= Argument matrix

So

$$G=\begin{bmatrix} a11 & a12 & \ldots\ldots..a1n & b1 \\ a21 & a22 & \ldots\ldots..a2n & b2 \\ & & . & \\ & & . & \\ & & . & \\ am1 & am2 & \ldots\ldots\ldots amn & bm \end{bmatrix}$$

Solution of System of Linear equation

Solution method

1. Matrix Method
    1. Naïve Gauss Elimination Method
    2. Gauss Elimination Method with partial pivoting
    3. Gauss Jordan Method

2. Iterative Method
    1. Gauss Seidel Method
    2. Jacobi Method

1. Naïve Gauss Elimination Method

Feature of Naïve Gauss Elimination Method

It works in two steps

1. Elimination Step

   Here the argument matrix is converted to an upper triangular matrix

2. Back substitution Step

   Solution is done from the last equation and then backward Calculation is continued to find the others results.

1. Naïve Gauss Elimination Method

Problem 01

Solve the following linear equation using Naïve Gauss Elimination Method

$3I_2-4I_1+6I_3+7I_4=10$

$I_1+I_2+9I_4-7=0$

$2I_3+6I_2+I_1=2.5$

$I_2+6I_3+7I_4-2=0$

1. Naïve Gauss Elimination Method

Solution procedure

1. Prepare the equation (Maintain the serial of element)

$3I_2-4I_1+6I_3+7I_4=10$
$I_1+I_2+9I_4-7=0$
$2I_3+6I_2+I_1=2.5$
$I_2+6I_3+7I_4-2=0$

$-4I_1+3I_2+6I_3+7I_4=10$
$I_1+I_2+9I_4=7$
$I_1+6I_2+2I_3=2.5$
$I_2+6I_3+7I_4=2$

1. Naïve Gauss Elimination Method

2. Write in matrix form

$$A=\begin{bmatrix} -4 & 3 & 6 & 7 \\ 1 & 1 & 0 & 9 \\ 1 & 6 & 2 & 0 \\ 0 & 1 & 6 & 7 \end{bmatrix}$$

$$X=[I_1 \quad I_2 \quad I_3 \quad I_4]$$

$$B=\begin{bmatrix} 10 \\ 7 \\ 2.5 \\ 2 \end{bmatrix}$$

$$G=[A \quad B]$$

So the argument matrix

$$G=\begin{bmatrix} -4 & 3 & 6 & 7 & 10 \\ 1 & 1 & 0 & 9 & 7 \\ 1 & 6 & 2 & 0 & 2.5 \\ 0 & 1 & 6 & 7 & 2 \end{bmatrix}$$

Pivoting row=first row

Pivoting element =A11=-4

1.Naïve Gauss Elimination Method

3.Elimination process

By doing elimination process G matrix is converted to an upper

Triangular matrix means, G= $\begin{bmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \end{bmatrix}$

Elimination:1st time

Keep the 1st row as usual and do necessary change for making a21=0

G= $\begin{bmatrix} -4 & 3 & 6 & 7 & 10 \\ 0 & 1.75 & 1.5 & 10.75 & 9.5 \\ 0 & 6.75 & 3.5 & 1.75 & 5 \\ 0 & 1 & 6 & 7 & 2 \end{bmatrix}$

R2-R1*a21/a11=1-(-4)*1/(-4)=0

R3-R1*a31/a11

R4-R1*a41/a11

In calculator type A-B*(1/(-4)) then press solve and each time change value of A and B to calculate each element.

1. Naïve Gauss Elimination Method

Elimination: 2nd time

Keep the 1st row, 2nd row and 1st column as usual and do necessary change for making a32=0

$$G=\begin{bmatrix} -4 & 3 & 6 & 7 & 10 \\ 0 & 1.75 & 1.5 & 10.75 & 9.5 \\ 0 & 0 & -2.286 & -39.714 & -31.643 \\ 0 & 0 & 5.143 & 0.857 & -3.429 \end{bmatrix}\begin{array}{l} \\ \\ \text{R3-R2*a32/a22} \\ \text{R4-R2*a42/a22} \end{array}$$

Elimination: 3rd time

$$G=\begin{bmatrix} -4 & 3 & 6 & 7 & 10 \\ 0 & 1.75 & 1.5 & 10.75 & 9.5 \\ 0 & 0 & -2.286 & -39.714 & -31.643 \\ 0 & 0 & 0 & -88.491 & -74.619 \end{bmatrix}\begin{array}{l} \\ \\ \\ \text{R4-R3*a43/a33} \end{array}$$

1. Naïve Gauss Elimination Method

4. Back calculation from matrix

$-4I_1+3I_2+6I_3+7I_4=10$

$1.75I_2+1.5I_3+10.75I_4=9.5$

$-2.286I_3-39.714I_4=31.643$

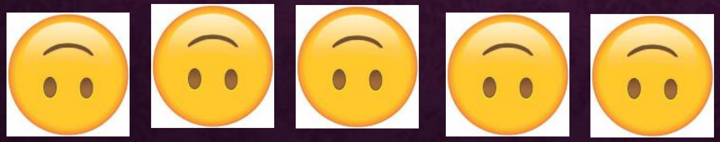
$-88.491I_4=-74.619$

So $I_4= -74.619/-88.491 =$…………..

$-2.286I_3-39.714I_4=31.643$……………………. $I_3=$…………………..

$1.75I_2+1.5I_3+10.75I_4=9.5$……………………….. $I_2=$……………..

$-4I_1+3I_2+6I_3+7I_4=10$…………………………………. $I_1=$……………………



1. Naïve Gauss Elimination Method

Problems of Naïve Gauss Elimination Method

- ✓ Dividing by zero problem
  Such problem occurs when at any step the pivoting element becomes zero (aii=0) for i-th element step.
  In such cases the system gives no result due to having divide by zero.

Solution

- ✓ Partial pivoting before every elimination

Assignment of Naïve Gauss Elimination Method

2.Gauss Elimination Method with partial pivoting

Problem 02

Solve the following linear equation using Gauss Elimination Method with partial pivoting

0.3b+5c+30=0

-7a+4b-9c=-2.5

6a+20b-3c+7.5=0

Solution procedure

1.Prepare the equation

0.3b+5c=-30

-7a+4b-9c=-2.5

6a+20b-3c=-7.5

2.Gauss Elimination Method with partial pivoting

2.Write in matrix form

A= $\begin{bmatrix} 0 & 0.3 & 5 \\ -7 & 4 & -9 \\ 6 & 20 & -3 \end{bmatrix}$

X= $[a \quad b \quad c]$

B= $\begin{bmatrix} -30 \\ -2.5 \\ -7.5 \end{bmatrix}$

G= $[A \quad B]$

So the argument matrix

G= $\begin{bmatrix} 0 & 0.3 & 5 & -30 \\ -7 & 4 & -9 & -2.5 \\ 6 & 20 & -3 & -7.5 \end{bmatrix}$

Pivoting row=first row
Pivoting element =A11=0
Dividing by zero problem so need partial pivoting

2.Gauss Elimination Method with partial pivoting

3.Partial pivoting:1st time

Pivoting row is replaced by other row that has larger magnitude (sign is not considered) than that of pivoting element. So 1st row is replaced by 2nd row as 7 is greater than 6

$$G=\begin{bmatrix} -7 & 4 & -9 & -2.5 \\ 0 & 0.3 & 5 & -30 \\ 6 & 20 & -3 & -7.5 \end{bmatrix}$$

4.Elimination process:1st time

$$G=\begin{bmatrix} -7 & 4 & -9 & -2.5 \\ 0 & 0.3 & 5 & -30 \\ 0 & 23.428 & -10.714 & -9.643 \end{bmatrix}$$

R2-R1*a21/a11

R3-R1*a31/a11=6-(-7)*6/(-7)

=0

2.Gauss Elimination Method with partial pivoting

3.Partial pivoting:2nd time

Now Pivoting row is 2nd row and replaced by other row that has larger magnitude(sign is not considered) than that of pivoting element. So 2nd row is replaced by 3rd row as 20 is greater than 0.3

$$G=\begin{bmatrix} -7 & 4 & -9 & -2.5 \\ 0 & 23.428 & -10.714 & -9.642 \\ 0 & 0.3 & 5 & -30 \end{bmatrix}$$

4.Elimination process:2nd time

$$G=\begin{bmatrix} -7 & 4 & -9 & -2.5 \\ 0 & 23.428 & -10.714 & -9.642 \\ 0 & 0 & 5.137 & -29.876 \end{bmatrix}$$ R3-R2*a32/a22

2.Gauss Elimination Method with partial pivoting

4.Back calculation from matrix

-7a+4b-9c=-2.5
23.428b-10.713c=-9.642
5.137c=-29.876

So c=-29.876/5.137= -5.815

23.428b-10.713c=-9.642……………….b=-3.071

-7a+4b-9c=-2.5…………………………………..a=6.079

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Assignment of Gauss Elimination Method with partial pivoting

3.Gauss Jordan Method

Feature of Gauss Jordan Method

It works in three steps

1.Partial pivoting

Pivoting row is replaced by other row that has larger magnitude (sign is not considered) than that of pivoting element to avoid the divide by zero problem

2. Scaling

To make the pivoting element 1

3. Elimination Step

Here the argument matrix is converted to a unit matrix

It does not require back calculation

3.Gauss Jordan Method

Problem 03

Solve the following linear equation using Gauss Jordan Method

$0.3b+5c+30=0$

$-7a+4b-9c=-2.5$

$6a+20b-3c+7.5=0$

Solution procedure

1.Prepare the equation

$0.3b+5c=-30$

$-7a+4b-9c=-2.5$

$6a+20b-3c=-7.5$

3.Gauss Jordan Method

2.Write in matrix form

$$A=\begin{bmatrix} 0 & 0.3 & 5 \\ -7 & 4 & -9 \\ 6 & 20 & -3 \end{bmatrix}$$

$$X=[a \quad b \quad c]$$

$$B=\begin{bmatrix} -30 \\ -2.5 \\ -7.5 \end{bmatrix}$$

$$G=[A \quad B]$$

So the argument matrix

$$G=\begin{bmatrix} 0 & 0.3 & 5 & -30 \\ -7 & 4 & -9 & -2.5 \\ 6 & 20 & -3 & -7.5 \end{bmatrix}$$

Pivoting row=first row
Pivoting element =A11=0
Dividing by zero problem so need partial pivoting

3.Gauss Jordan Method

3.Partial pivoting:1st time

Pivoting row is replaced by other row that has larger magnitude (sign is not considered) than that of pivoting element. So 1st row is replaced by 2nd row as 7 is greater than 6

$$G=\begin{bmatrix} -7 & 4 & -9 & -2.5 \\ 0 & 0.3 & 5 & -30 \\ 6 & 20 & -3 & -7.5 \end{bmatrix}$$

4.Scaling:1st time

To make pivoting element 1.As pivoting element=-7 so row1 is divided by a11

$$G=\begin{bmatrix} 1 & -0.571 & 1.285 & 0.357 \\ 0 & 0.3 & 5 & -30 \\ 6 & 20 & -3 & -7.5 \end{bmatrix} R1/a11$$

3.Gauss Jordan Method

5.Elimination process:1st time

$$G=\begin{bmatrix} 1 & -0.571 & 1.285 & 0.357 \\ 0 & 0.3 & 5 & -30 \\ 0 & 23.428 & -10.714 & -9.642 \end{bmatrix}$$

R2-R1*a21/a11
R3-R1*a31/a11
=6-(1)*6/(1)
=0

6. Partial pivoting:2nd time

Now Pivoting row is 2nd row and replaced by other row that has larger magnitude(sign is not considered) than that of pivoting element. So 2nd row is replaced by 3rd row as 23.428 is greater than 0.3

$$G=\begin{bmatrix} 1 & -0.571 & 1.285 & 0.357 \\ 0 & 23.428 & -10.714 & -9.642 \\ 0 & 0.3 & 5 & -30 \end{bmatrix}$$

3.Gauss Jordan Method

4.Scaling:2nd time

To make pivoting element 1.As pivoting element=23.428 so row2 is divided by a22

$$G=\begin{bmatrix} 1 & -0.571 & 1.285 & 0.357 \\ 0 & 1 & -0.457 & -0.412 \\ 0 & 0.3 & 5 & -30 \end{bmatrix} \begin{matrix} \\ R2/a22 \\ \\ \end{matrix}$$

5. Elimination process:2nd time

$$G=\begin{bmatrix} 1 & 0 & 1.024 & 0.122 \\ 0 & 1 & -0.457 & -0.412 \\ 0 & 0 & 5.137 & -29.876 \end{bmatrix} \begin{matrix} R1\text{-}R2*a12/a22 \\ \\ R3\text{-}R2*a32/a22 \end{matrix}$$

3.Gauss Jordan Method

4.Scaling:3rd time

To make pivoting element 1.As pivoting element5.137 so row3 is divided by a33

$$G=\begin{bmatrix} 1 & 0 & 1.024 & 0.122 \\ 0 & 1 & -0.457 & -0.412 \\ 0 & 0 & 1 & -5.815 \end{bmatrix} R3/a33$$

5. Elimination process:3rd time

$$G=\begin{bmatrix} 1 & 0 & 0 & 6.079 \\ 0 & 1 & 0 & -3.071 \\ 0 & 0 & 1 & -5.815 \end{bmatrix} \begin{matrix} R1-R3*a13/a33 \\ R2-R3*a23/a33 \\ \end{matrix}$$

So a=6.079

b=-3.071

c=-5.815

Assignment of Gauss Jordan Method

Iteration Method

Feature of Iteration Method

1. Matrix is subjected to round off error. This can be minimized by Iteration method.
2. Iteration method is suitable for a system where no. of equation is very large
3. Totally depends on the initial guesses. Depending upon initial guesses sch method can be divergent, convergent or fluctuating.
4. Has two type
    1. Gauss Seidel Method
    2. Jacobi Method

Gauss Seidel Method

Procedure of gauss Seidel Method

1. Identify the initial guesses and percentage error
2. Find the equation of one root from one equation

   For example if you have an equation like

   2a+3b-4c=20 → then $a=\frac{20-3b+4c}{2}$

   5a-3b+2c=10 → $b=\frac{10-5a,new-2c}{-3}$

   a+2b-3c=2 → $c=\frac{2-a,new-2b,new}{-3}$

3. Using initial value(ai,bi,ci) and value calculated from the above equation ar, br and cr are calculated
4. Then for next iteration ai=ar, bi=br and ci=cr. And continue up to error limit.
5. Error, Er $=\frac{ar,new-ar,old}{ar,new}*100$

Gauss Seidel Method

Problem 04

Solve the following linear equation using Gauss Seidel Method with initial guesses {a, b, c}={0,0,0} and error {5%, 3%, 0.0050%}

$3a-0.1b-0.2c=7.85$

$0.1a+7b-0.3c=-19.3$

$0.3a-0.2b+10c=71.4$

Solution Procedure

1. Initial value a=0, b=0,c=0 and Er,a=5%, Er,b= 3% and Er,c=0.005%

2. Root equation

$3a-0.1b-0.2c=7.85$

$0.1a+7b-0.3c=-19.3$

$0.3a-0.2b+10c=71.4$

So $a=\dfrac{7.85+0.1b+0.2c}{3}$

$b=\dfrac{-19.3-0.1a_{,new}+0.3c}{7}$

$c=\dfrac{71.4-0.3a_{,new}+0.2b_{,new}}{10}$

Gauss Seidel Method

3. Calculate ar, br, cr

No. of iteration	ai	bi	ci	ar	br	cr	Er,a	Er,b	Er,c
1.	0	0	0	2.616	-2.794	7.005	-	-	-
2.	2.616	-2.794	7.005	2.990	-2.499	7.000	12.5	11.80	0.071
3.	2.990	-2.499	7.000	3.000	-2.500	7.000	0.333	0.04	0.000
4.									

So ar=………..br=………….cr=………………..

Assignment of Gauss Seidel Method

Jacobi Method

Procedure of Jacobi Method

1. Identify the initial guesses and percentage error
2. Find the equation of one root from one equation

   For example if you have an equation like

   2a+3b-4c=20 then $a=\frac{20-3b+4c}{2}$

   5a-3b+2c=10 $b=\frac{10-5a-2c}{-3}$

   a+2b-3c=2 $c=\frac{2-a-2b}{-3}$

3. Using only initial value(ai,bi,ci) to calculate ar, br and cr .
4. Then for next iteration ai=ar, bi=br and ci=cr. And continue up to error limit.
5. Error, Er $=\frac{ar,new-ar,old}{ar,new}$*100

Problem 05

Jacobi Method

Solve the following linear equation using Jacobi Method with initial guesses {a, b, c}={0,0,0} and error {5%, 3%, 0.050%}

$3a-0.1b-0.2c=7.85$

$0.1a+7b-0.3c=-19.3$

$0.3a-0.2b+10c=71.4$

Solution Procedure

1. Initial value a=0, b=0, c=0 and

Er,a=5%, Er,b= 3% and Er,c=0.005%

2. Root equation

$3a-0.1b-0.2c=7.85$

$0.1a+7b-0.3c=-19.3$

$0.3a-0.2b+10c=71.4$

So $a=\dfrac{7.85+0.1b+0.2c}{3}$

$b=\dfrac{-19.3-0.1a+0.3c}{7}$

$c=\dfrac{71.4-0.3a+0.2b}{10}$

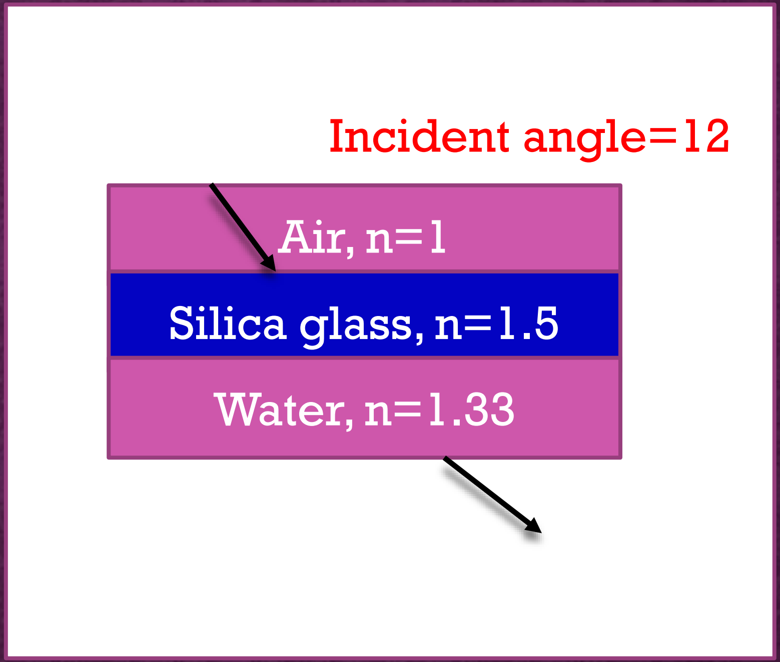
Jacobi Method

3. Calculate ar, br, cr

No. of iteration	ai	bi	ci	ar	br	cr	Er,a	Er,b	Er,c
1.	0	0	0	2.616	-2.757	7.14	-	-	-
2.	2.616	-2.757	7.14	3.000	-2.488	7.006	12.77	10.81	1.91
3.	3.000	-2.488	7.006	3.001	-2.4997	7.000	0.030	0.4	0.085
4.	3.001	-2.4997	7.000	3.000	-2.500	6.999	0.033	0.012	0.014

Jacobi Method

Assignment of Jacobi Method


Incident angle=12
Air, n=1
Silica glass, n=1.5
Water, n=1.33