Solution of non-linear equation

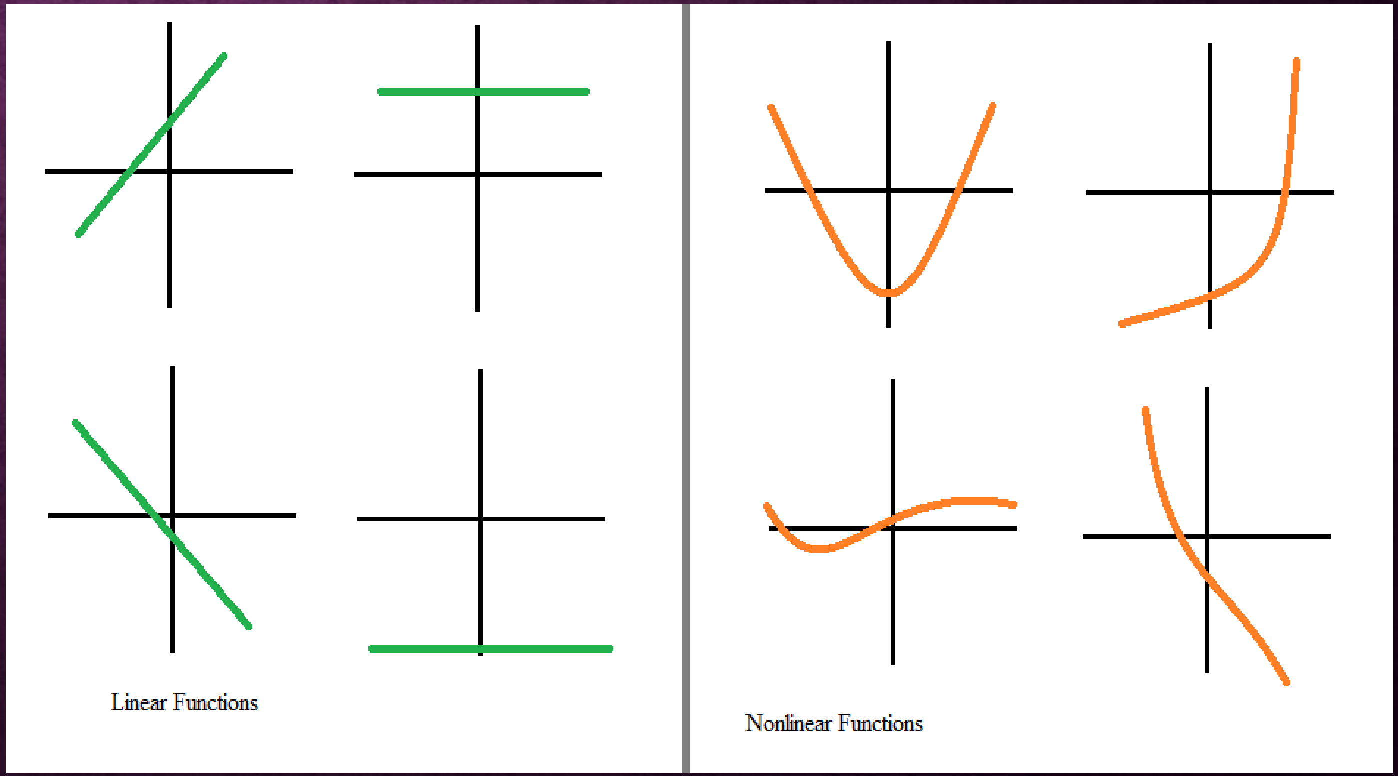
Solution of non-linear equation

Any non-linear equation can be formed like the following

$f(x)=0$ [Ex. $y=f(x)=2x+5=0$]

To solve the equation means to determine the value of x when $f(x)=0$

There are some linear and non-linear equation Graph is given to understand the graphical view of the equations.



Solution of Non-linear Equation

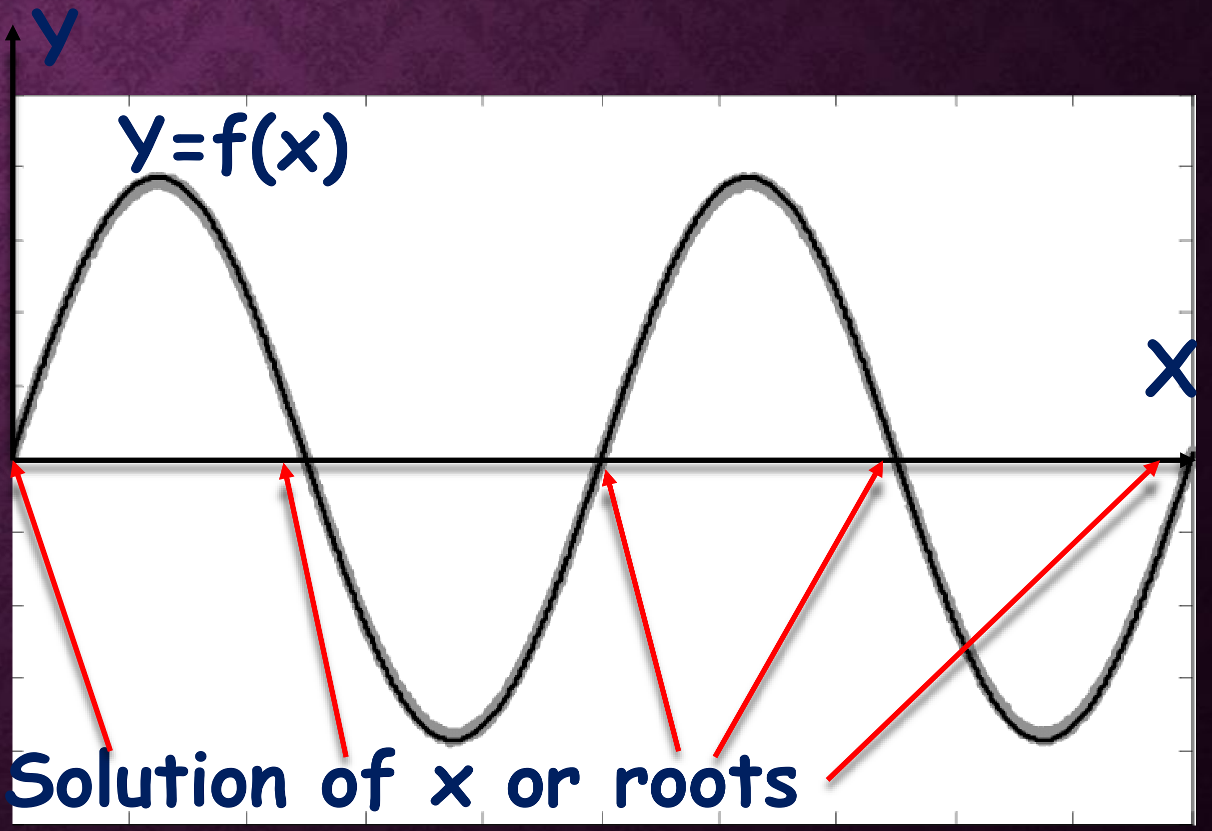
Graphical interpretation

The meaning of solving f(x)=0 is that determine the value of x where y=f(x) curve cuts the x axis

Ex. f(x)= sin(x)

Sol. X=0,180,360…….

A system can have single or multiple roots or solution. Numerically one solution can be obtain at a time and it depends on the initial guesses and the method applied



Solution on non-linear equation

Numerical Method of Solution of non-linear equation

1. **Close or Bracketing Method**
    1. Bisection Method
    2. False Position Method

2. **Open Method**
    1. Fixed Point Iteration Method
    2. Newtown Raphson Method
    3. Secant Method
    4. Modefied Secant Method

Close or Bracketing Method

Feature of Close or Bracketing Method

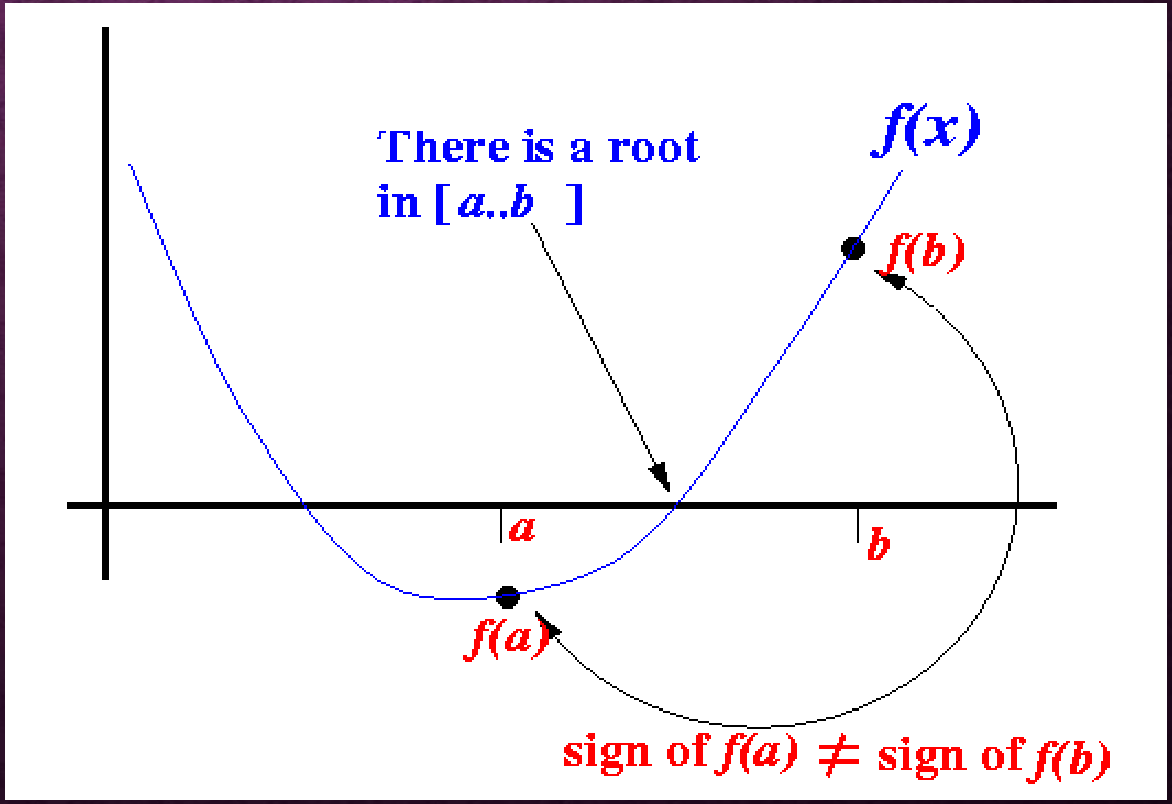
1. Requires two initial guesses
2. These guesses must bracket the root i.c. for these two value of X the function value must have the opposite sign.

If two guesses are Xup and Xlow so the conditions are

f(Xup) = +
f(Xlow) = -

OR

f(Xup) = -
f(Xlow) = +



Feature of Close or Bracketing Method

3. Always convergent, as steps increases error will always decreases
4. Relatively slow and complicated
5. Root determining equation $X_r = (X_{up}+X_{low})/2$

Bisection Method Procedure

1. Take Xup and Xlow
2. Determine $X_r = (X_{up}+X_{low})/2$
3. Determine $f(X_r)$, $f(X_{up})$ and $f(X_{low})$
4. Check----if $f(X_r)$ and $f(X_{up})$ sign same then $X_{up} = X_r$

   if $f(X_r)$ and $f(X_{low})$ sign same then $X_{low} = X_r$
5. Determine $E_r = \{(X_{r,new} - X_{r,old})/X_{r,new}\} * 100$
6. Check

   if $E_r < E_s$ then stop iteration and X_r = result

   if $E_r > E_s$ then next iteration where Xup and Xlow follow procedure 4

Bisection Method Procedure

Problem 01

In an environmental survey the water of a lake is observed and found that for a certain Pollution process the growth of bacteria C (in 1/cm3) is decayed with time t (in hours) like following equation $C = 75e^{-1.5t} + 20e^{-0.075t}$. Now determine the time bacteria growth decay to its safe limit C=15 (1/cm3) .Use Bisection method to solve the equation where initial guesses are 1 & 20 up to relative error (Er) 5%. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$C = 75\ e^{-1.5t} + 20\ e^{-0.075t}$

Assume t=x (for easy solution)

$C = 75\ e^{-1.5x} + 20\ e^{-0.075x}$

$15 = 75\ e^{-1.5x} + 20\ e^{-0.075x}$

$75\ e^{-1.5x} + 20\ e^{-0.075x} - 15 = 0$

So $f(X) = 75\ e^{-1.5x} + 20\ e^{-0.075x} - 15$

Here

Xup = 1

Xlow = 20

Xr = (Xup+Xlow)/2

= 10.5

Er = {(Xr,new –Xr,old)/Xr,new} *100

Bisection Method Procedure

2. Make a Table like following

No of iteration	Xup	Xlow	f(Xup)	f(Xlow)	Xr	f(Xr)	Er(%)
1.	1	20	20.2896	-10.5374 (rule 4)	10.5	-5.9004 (rule 4)	--------
2.	1	10.5	20.2896	-5.9004	5.75	-1.9926	82.61
3.	1	5.75	20.2896	-1.9926	3.375	1.0021	70.37
4.	3.375	5.75	1.0021	-1.9926	4.5625	-0.7157	26.62
5.	3.375	4.5625	1.0021	-0.7157	3.9688	0.0459	14.96
6.	3.9688	4.5625	0.0459	-0.7151	4.2657	-0.3512	6.7
7.	3.9688	4.2657	0.0459	-0.3512	4.1173	-0.1574	3.6

Bisection Method Procedure

3. As Er < Es so result=t=x=Xr=4.1171hrs (Approx)

Notes

In table Xr = (Xup+Xlow)/2 for each step

For 1st step there is no Xr,old so no errer(%) calculation but in 2nd iteration Xr,new=5.75 and Xr,old= 10.5

Assignment

False Position Method Procedure

False Position Method Procedure

1. Take Xup and Xlow
2. Determine Xr $= Xup - \frac{\text{Xup}-\text{Xlow}}{\text{f(Xup)}-\text{f(Xlow)}}$ *f(Xup)
3. Determine f(Xr), f(Xup) and f(Xlow)
4. Check

   if f(Xr) and f(Xup) sign same then Xup = Xr

   if f(Xr) and f(Xlow) sign same then Xlow = Xr
5. Determine Er = {(Xr,new –Xr,old)/Xr,new} *100
6. Check

   if Er < Es then stop iteration and Xr = result

   if Er > Es then next iteration where Xup and Xlow follow procedure 4

False Position Method Procedure

Problem 02

In an environmental survey the water of a lake is observed and found that for a certain Pollution process the growth of bacteria C (in 1/cm3) is decayed with time t (in hours) like following equation $C = 75\,e^{-1.5t} + 20\,e^{-0.075t}$. Now determine the time bacteria growth decay to its safe limit $C=15$ (1/cm3) .Use **False Position method** to solve the equation where initial guesses are 1 & 20 up to relative error (Er) 5%. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$C = 75\,e^{-1.5t} + 20\,e^{-0.075t}$

Assume t=x (for easy solution)

$C = 75\,e^{-105x} + 20\,e^{-0.075x}$

$15 = 75\,e^{-1.5x} + 20\,e^{-0.075x}$

$75\,e^{-1.5x} + 20\,e^{-0.075x} - 15 = 0$

So $f(X) = 75\,e^{-1.5x} + 20\,e^{-0.075x} - 15$

Here

Xup = 1

Xlow = 20

$$Xr = Xup - \frac{Xup - Xlow}{f(Xup) - f(Xlow)} * f(Xup)$$

$= 13.5053$

Er = {(Xr,new –Xr,old)/Xr,new} *100

False Position Method Procedure

2. Make table like following

Iteration no	Xup	Xlow	f(Xup)	f(Xlow)	Xr	f(Xr)	Er(%)
1.	1	20	20.2896	- 10.5374	13.5053	-7.7367	------------
2.	1	13.5053	20.2896	-7.7367	10.0532	-5.5903	34.34
3.	1	10.0532	20.2896	-5.5903	8.0976	-4.1034	24.15
4.	1	8.0976	20.2896	-4.1034	6.9036	-3.0807	17.29
5.	1	6.9036	20.2896	-3.0807	6.1254	-2.3591	12.70
6.	1	6.1254	20.2896	-2.3591	5.5915	-1.8336	9.55
7.	1	5.5915	20.2896	-1.8336	5.2110	-1.4398	7.36
8.	1	5.2110	20.2896	-1.4398	4.9320	-1.1380	5.65
9.	1	4.9320	20.2896	-1.1380	4.7232	-0.9030	4.42

False Position Method Procedure

3. As Er < Es so result=t=x=Xr=4.7232hrs (Approx)

Notes

In table Xr = $Xup - \frac{Xup - Xlow}{f(Xup) - f(Xlow)}$ *f(Xup) for each step

For 1st step there is no Xr,old so no errer(%) calculation but in 2nd iteration Xr,new=10.0532 and Xr,old= 13.5053

Assignment

Comparison between Bisection & False Position Method

1. Both are close method
2. Root determining equation

Xr,bisection = (Xup+Xlow)/2

Xr,false position = $Xup - \frac{\text{Xup}-\text{Xlow}}{\text{f}(\text{Xup})-\text{f}(\text{Xlow})}$ *f(Xup)

3. False position method is more complex but faster than Bisection
4. False position method converges slowly so Bisection method is more efficient

Open Method

Feature of Open Method

1. Requires one initial guess or multiple unconditional initial guesses
2. Faster than close method
3. System may be convergent or divergent or even fluctuation depending upon
   - initial guesses
   - the process of handling the equation

Open Methods

1. Fixed Point Iteration Method
2. Newtown Raphson Method
3. Secant Method
4. Modefied Secant Method

Fixed point iteration method

Feature of Fixed Point Iteration Method

1. Require single initial guess
2. Simple and faster
3. Result is very much sensitive to initial guess and the process of handling equation
4. Root determining equation $Xr = f(Xi)+Xi$

Fixed Point Iteration Method Procedure

1. Take one initial guess
2. Determine $Xr = f(Xi)+Xi$
3. Determine $Er = \{(Xr,new - Xr,old)/Xr,new\} * 100$
4. Check

   if $Er<Es$ then Xr=result

   if $Er>Es$ then $Xi=Xr$ and repeat the process

Fixed point iteration method

Problem 03

In an environmental survey the water of a lake is observed and found that for a certain Pollution process the growth of bacteria C (in 1/cm3) is decayed with time t (in hours) like following equation $C = 75\, e^{-1.5t} + 20\, e^{-0.075t}$. Now determine the time bacteria growth decay to its safe limit $C=15$ (1/cm3) .Use Fixed point iteration method to solve the equation where initial guess is 7 up to relative error (Er) 5%. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$C = 75\, e^{-1.5t} + 20\, e^{-0.075t}$

Assume t=x (for easy solution)

$C = 75\, e^{-1.5x} + 20\, e^{-0.075x}$

$15 = 75\, e^{-1.5x} + 20\, e^{-0.075x}$

$75\, e^{-1.5x} + 20\, e^{-0.075x} - 15 = 0$

So $f(X) = 75\, e^{-1.5x} + 20\, e^{-0.075x} - 15$

Here

$X_i = 7$

$X_r = f(X_i) + X_i$

$= 3.8332$

$E_r = \{(X_{r,new} - X_{r,old})/X_{r,new}\} * 100$

Fixed point iteration method

2.Prepare the table

No of iteration	Xi	Xr=f(Xi)+Xi	Er(%)
1.	7	3.8332	---------
2.	3.8332	4.0748	5.93
3.	4.0748	3.9745	2.52

3. As Er < Es so result=t=x=Xr=3.9745hrs (Approx.)

Notes

In table Xr = f(Xi)+Xi for each step

For 1st step there is no Xr,old so no errer(%) calculation but in 2nd iteration Xr,new=4.0748 and Xr,old= 3.8332

Fixed point iteration method

Open Method System may be convergent or divergent or even fluctuation depending upon initial guesses & the process of handling the equation.

Here is an example problem

Problem 04

Solve problem 03 only difference is that the equation is

$$f(X) = 15 - 75\, e^{-1.5x} - 20\, e^{-0.075x}$$

Now prepare the table

No of iteration	Xi	Xr=f(Xi)+Xi	Er(%)
1.	7	10.1668	--------
2.	10.1668	15.8369	35.8
3.	15.8369	24.7389	35.9

So we can see that Er is increasing so the system is divergent and very sensitive to the process of handling equation

Newton Raphson Method

Feature of Newton Raphson Method

1.Require single initial guess

2.**Advantages**---Faster and convergent than all other method

3.**Disadvantages**

i. Sometimes it is difficult to evaluate $f'(X)$

ii. Result is very much sensitive to initial guess because if Xi is such chosen that **$f'(Xi) = 0$** then **divide by zero problem** occur and **Xr goes infinity** and system is **fluctuating**, So the process needed to be stopped

4. Root determining equation **$Xr = Xi - f(Xi)/f'(Xi)$**

Newton Raphson Method Procedure

1.Take one initial guess Xi

2.Determine **$Xr = Xi - f(Xi)/f'(Xi)$**

3.Determine **$Er = \{(Xr,new - Xr,old)/Xr,new\} * 100$**

4. Check

if $Er<Es$ then Xr=result

if $Er>Es$ then $Xi=Xr$ and repeat the process

Newton Raphson Method

Problem 03

Suppose a power supply when is switched on gives current to the external circuit according to the following equation $i(t) = 9\, e^{-t} \cos(2\pi t)$, here t is the time is seconds. Now the external circuit can tolerate current as high as $i=3A$.You are asked to determine the time(sec) when the current reaches this safe limit for designing a protective device accordingly. Use Newton Raphson method to solve the equation where initial guess is 1 up to relative error (Er) 3%. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$i(t) = 9\, e^{-t} \cos(2\pi t)$

Assume t=x (for easy solution)

$i(x) = 9\, e^{-x} \cos(2\pi x)$

$3 = 9\, e^{-x} \cos(2\pi x)$

$9\, e^{-x} \cos(2\pi x) - 3 = 0$

So $f(X) = 9\, e^{-x} \cos(2\pi x) - 3$

Here

$X_i = 1$

$X_r = X_i - f(X_i)/f'(X_i)$

$= 1.0939$

$E_r = \{(X_{r,new} - X_{r,old})/X_{r,new}\} * 100$

Newton Raphson Method

So first derivative of f(x) is

$f'(X) = -9\, e^{-x} \cos(2\pi x) - 9\, e^{-x} \sin(2\pi x).2\pi$ [product rule of differentiation]

$= -9\, e^{-x} [\cos(2\pi x) + 2\pi.\sin(2\pi x)]$

2. Now prepare the table

No. of iteration	Xi	f(Xi)	f'(Xi)	Xr	Er(%)
1.	1	0.3109	-3.3109	1.0939	----------
2.	1.0939	-0.4954	-13.0412	1.0559	3.6
3.	1.0559	-0.0602	-9.7080	1.0497	0.59

Newton Raphson Method

3. As Er < Es so result=t=x=Xr=1.0497sec (Approx.)

Notes

In table Xr = Xi – f(Xi)/f'(Xi) for each step

For 1st step there is no Xr,old so no error (%) calculation but in 2nd iteration Xr,new=1.0559 and Xr,old= 1.0939

Secant Method

Feature of Secant Method

1. Modified version of Newton Raphson Method
2. Take two initial guess Xi and Xi-1
3. No need of perform derivation
4. Slower than the Newton Raphson Method

Root determining equation of Secant Method

$$Xr = Xi - \frac{\{Xi - (Xi-1)\} * f(Xi)}{f(Xi) - f(Xi-1)}$$

Secant Method

Procedure of Secant Method

1. Take initial guesses Xi and Xi-1
2. Determine $Xr = Xi - \frac{\{Xi - (Xi-1)\} * f(Xi)}{f(Xi) - f(Xi-1)}$
3. Determine Er = {(Xr,new –Xr,old)/Xr,new} *100
4. Check

   if Er<Es then Xr=result

   if Er>Es then Xi=Xr and repeat the process

Secant Method

Problem 04

Suppose a power supply when is switched on gives current to the external circuit according to the following equation $i(t) = 9\, e^{-t} \cos(2\pi t)$, here t is the time is seconds. Now the external circuit can tolerate current as high as $i=3A$. You are asked to determine the time(sec) when the current reaches this safe limit for designing a protective device accordingly. Use Secant method to solve the equation where initial guesses are 1 and 0.5 up to relative error (Er) 3%. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$i(t) = 9\, e^{-t} \cos(2\pi t)$

Assume t=x (for easy solution)

$i(x) = 9\, e^{-x} \cos(2\pi x)$

$3 = 9\, e^{-x} \cos(2\pi x)$

$9\, e^{-x} \cos(2\pi x) - 3 = 0$

So $f(X) = 9\, e^{-x} \cos(2\pi x) - 3$

Here

$X_i = 1$ and $X_{i-1} = 0.5$

$$Xr = Xr = Xi - \frac{\{Xi - (Xi-1)\} * f(Xi)}{f(Xi) - f(Xi-1)}$$

$= 0.9823$

$Er = \{(Xr,new - Xr,old)/Xr,new\} * 100$

Secant Method

2.Prepare the table

No. of iteration	Xi	Xi-1	f(Xi)	f(Xi-1)	Xr	Er(%)
1.	1	0.5	0.3109	-8.4588	0.9823	----------
2.	0.9823	1	0.3492	0.3109	1.1437	14.1
3.	1.1437	0.9823	-1.2239	0.3402	1.0181	12.34
4.	1.0181	1.1437	0.2305	-1.2239	1.0380	1.9171

Secant Method

3. As Er < Es so result=t=x=Xr=1.0380sec (Approx.)

Notes

In table $Xr = Xi - \dfrac{\{Xi - (Xi-1)\} * f(Xi)}{f(Xi) - f(Xi-1)}$ for each step

For 1st step there is no Xr,old so no errer(%) calculation but in 2nd iteration Xr,new=1.1437 and Xr,old= 0.9823

Modified Secant Method

Feature of Modified Secant Method

1. Modified version of Secant Method
2. Take one initial guess Xi
3. No need of perform derivation
4. Faster than the Secant Method
5. One extra parameter is needed called interval(ΔX)

Root determining equation of Secant Method

$$Xr = Xi - \frac{\Delta X * f(Xi)}{f(Xi) - f(Xi - \Delta X)}$$

Modified Secant Method

Procedure of Modified Secant Method

1. Take initial guess Xi

2. Determine $Xr = Xi - \dfrac{\Delta X * f(Xi)}{f(Xi) - f(Xi - \Delta X)}$

3. Determine **Er = {(Xr,new –Xr,old)/Xr,new} *100**

4. Check

   if Er<Es then Xr=result

   if Er>Es then Xi=Xr and repeat the process

Modified Secant Method

Problem 05

Suppose a power supply when is switched on gives current to the external circuit according to the following equation $i(t) = 9\,e^{-t}\cos(2\pi t)$, here t is the time is seconds. Now the external circuit can tolerate current as high as i=3A.You are asked to determine the time(sec) when the current reaches this safe limit for designing a protective device accordingly. Use Modified Secant method to solve the equation where initial guess is 1 up to relative error (Er) 3%.Take interval of 0.1. Keep every calculation and result up to four significant decimal place.

Solution

1.Prepare the function, f(X)

$i(t) = 9\,e^{-t}\cos(2\pi t)$

Assume t=x (for easy solution)

$i(x) = 9\,e^{-x}\cos(2\pi x)$

$3 = 9\,e^{-x}\cos(2\pi x)$

$9\,e^{-x}\cos(2\pi x) - 3 = 0$

So $f(X) = 9\,e^{-x}\cos(2\pi x) - 3$

Here

$X_i = 1$ and $\Delta X = 0.1$

$$X_r = X_r = X_i - \frac{\Delta X * f(X_i)}{f(X_i) - f(X_i - \Delta X)}$$

$= 0.9113$

Er = {(Xr,new −Xr,old)/Xr,new} *100

Modified Secant Method

2.Prepare the table

No. of iteration	Xi	f(Xi)	f(Xi-ΔX)	Xr	Er(%)
1.	1	0.3109	--0.0397	0.9113	----------
2.	0.9113	0.0705	-1.4977	0.9068	0.49

3. As Er < Es so result=t=x=Xr=0.9068 sec (Approx.)

Modified Secant Method

Notes

In table $Xr = Xi - \frac{\Delta X * f(Xi)}{f(Xi) - f(Xi - \Delta X)}$ for each step

For 1st step there is no Xr,old so no errer(%) calculation but in 2nd iteration Xr,new=0.9068 and Xr,old= 0.9113

Solution of linear equation

Solution of linear equation

✓ Cramer's rule Procedure

1. Prepare the equation sequentially
2. Determine the determinants(D) of the matrix.
3. Determine Dx, Dy, Dz
4. Now x=Dx/D, y=Dy/D and Dz/D

Problem 01

Use Cramer's rule to find the solution of the following equation

$5x+6y+7z=0$

$2x+3y=0$

$3y+5z=0$

Solution of linear equation

Solution

$5x+6y+7z=5$

$2x+3y+0z=7$

$0x+3y+5z=9$

$$D=\begin{bmatrix}5 & 6 & 7\\ 2 & 3 & 0\\ 0 & 3 & 5\end{bmatrix}= 5*(15-0)-6*(10-0)+7*(6-0)=57$$

$$Dx=\begin{bmatrix}5 & 6 & 7\\ 7 & 3 & 0\\ 9 & 3 & 5\end{bmatrix}= 5*(15-0)-6*(35-0)+7*(21-27)=-177$$

Solution of linear equation

$$Dy=\begin{bmatrix}5 & 5 & 7\\ 2 & 7 & 0\\ 0 & 9 & 5\end{bmatrix} = 5*(35-0)-5*(10-0)+7*(18-0)= 251$$

$$Dz=\begin{bmatrix}5 & 6 & 5\\ 2 & 3 & 7\\ 0 & 3 & 9\end{bmatrix} = 5*(27-21)-6*(18-0)+5*(6-0)= -48$$

X=Dx/D =-177/57=……………. 

Y=Dy/D =………….. 

Z=Dz/D =………………