Homework 2

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MATH212: Linear Algebra

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Problem 1

$$\boldsymbol{a_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{a_2} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \quad \boldsymbol{a_3} = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

For $\mathbf{b}$ to be a linear combination of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, and $\boldsymbol{a_3}$, the augmented matrix, $\begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{b} \end{bmatrix}$ must be consistent.

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

$$R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$$

The matrix in echelon form above is consistent because all columns except the rightmost one (the last column) are pivot columns.

$$\therefore \boldsymbol{b} \text{ is a linear combination of } \boldsymbol{a_1}, \boldsymbol{a_2}, \text{ and } \boldsymbol{a_3}.$$

Problem 2

$$\boldsymbol{v_1} = \begin{bmatrix} 50 \\ 56 \\ 34 \end{bmatrix}, \quad \boldsymbol{v_2} = \begin{bmatrix} 35 \\ 14 \\ 123 \end{bmatrix}$$

(a)

$3\boldsymbol{v_2}$ represents the output of Farm #2 after operating for 3 months.

(b)

$$\text{Let } \boldsymbol{b} = \text{target output} = \begin{bmatrix} 830 \\ 728 \\ 1358 \end{bmatrix}$$

Vector equation whose solution gives the number of months each farm should operate in order to produce the target output:

$$x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} = \boldsymbol{b}$$

$$\implies x_1 \begin{bmatrix} 50 \\ 56 \\ 34 \end{bmatrix} + x_2 \begin{bmatrix} 35 \\ 14 \\ 123 \end{bmatrix} = \begin{bmatrix} 830 \\ 728 \\ 1358 \end{bmatrix}$$

Problem 3

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Augmented matrix for $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & 2b_1 + b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_1$$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & 2b_1 + b_2 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{7}{2}R_2$$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & 2b_1 + b_2 \\ 0 & 0 & 0 & b_3 - 4b_1 + \frac{7}{2}(2b_1 + b_2) \end{bmatrix}$$

Simplifying:

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & 2b_1 + b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{7}{2}b_2 + b_3 \end{bmatrix}$$

(a)

The third entry in column 4 equals $3b_1 + \frac{7}{2}b_2 + b_3$. The equation, $A\boldsymbol{x} = \boldsymbol{b}$ is not consistent for every $\boldsymbol{b}$ because some choices of $\boldsymbol{b}$ can make $3b_1 + \frac{7}{2}b_2 + b_3$ nonzero. For example:

If $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $3b_1 + \frac{7}{2}b_2 + b_3 = 3(1) + \frac{7}{2}(1) + 1 = \frac{15}{2} \neq 0$.

$\therefore$ The equation, $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution for all possible $\boldsymbol{b}$.

(b)

For the equation, $A\boldsymbol{x} = \boldsymbol{b}$ to have a solution, all the entries of $\boldsymbol{b}$ must satisfy the equation:

$$3b_1 + \frac{7}{2}b_2 + b_3 = 0$$

$\therefore$ The set of all possible $\boldsymbol{b}$ is the set of all vectors in $\mathbb{R}^3$ that satisfy the equation:

$$3b_1 + \frac{7}{2}b_2 + b_3 = 0$$