Homework 5

David Abeiku Saah

MATH212: Linear Algebra

Dr Ayawoa Dagbovie

October 27, 2023

Problem 1

Column space of A is the set of pivot columns in A.

$$ColA = \left\{ \begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix} \right\}$$

Finding the null space of A. Solving $[\boldsymbol{A}\ \boldsymbol{0}]$:

$$\begin{bmatrix} 1 & 2 & -4 & 3 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to \frac{-1}{5} R_3$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to -3R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to 4R_2 + R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Equations: } \begin{cases} x_1 + 2x_4 - 5x_5 = 0 \\ x_2 - 2x_5 = 0 \\ x_3 = 0 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_4 + 5x_5 \\ 2x_5 \\ 0 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore NulA = \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Problem 2

Let

$$\boldsymbol{A} = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

$$\det \boldsymbol{A} = \sum_{j=1}^{n} (-1)^{2+j} a_{2j} \boldsymbol{A}_{2j} \text{ where } n = \text{ number of columns in } \boldsymbol{A}$$

$$= -0 + 0 - 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} + 0 - 0$$

$$\det \boldsymbol{A} = \sum_{i=1}^{n} (-1)^{i+2} a_{i2} \boldsymbol{A}_{i2} \text{ where } n = \text{ number of columns in submatrix}$$

$$= -2\left(-0 + 3\begin{vmatrix}4 & 3 & -5\\ 5 & 2 & -3\\ 0 & -1 & 2\end{vmatrix} - 0 + 0\right)$$

$$\det \boldsymbol{A} = \sum_{j=1}^{n}(-1)^{3+j}a_{3j}\boldsymbol{A}_{3j} \text{ where } n = \text{ number of columns in submatrix}$$

$$= -6\left(0 - 1\begin{vmatrix}4 & -5\\ 5 & -3\end{vmatrix} + 2\begin{vmatrix}4 & 3\\ 5 & 2\end{vmatrix}\right)$$

Determinant of a 2 x 2 matrix, $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$

$$\det \boldsymbol{A} = -6(0 - 1(-12 - (-25)) + 2(8 - 15))$$

$$= -6(0 - 1(13) + 2(-7))$$

$$= -6(-13 - 14)$$

$$= -6(-27)$$

$$= 162$$

$\therefore$ The determinant of $\boldsymbol{A}$ is 162.

Problem 3

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 4 x 4 matrices such that:

$$\det \boldsymbol{A} = -3, \det \boldsymbol{B} = -1$$

a)

$$\det(\boldsymbol{AB}) = (\det \boldsymbol{A})(\det \boldsymbol{B})$$

$$\therefore \det(\boldsymbol{AB}) = (-3)(-1) = 3$$

The determinant of a product of matrices is equal to the product of the determinants of the matrices.

b)

$$\det 2\boldsymbol{A} = (2^n)(\det \boldsymbol{A}), \text{ where } n = \text{ number of rows in } \boldsymbol{A}$$

$$\therefore \det 2\boldsymbol{A} = (2^4)(-3) = -48$$

$2\boldsymbol{A}$ is equivalent to multiplying all the rows of $\boldsymbol{A}$ by 2. One of the properties of determinants is that if one row is multiplied by a scalar, k, then the determinant of the new matrix is $k \cdot \det \boldsymbol{A}$. For multiplying all the rows, the determinant is $k^n \cdot \det \boldsymbol{A}$, where n in the number of rows.

c)

$$\det(\boldsymbol{A}^T\boldsymbol{B}\boldsymbol{A}) = (\det \boldsymbol{A})(\det \boldsymbol{B})(\det \boldsymbol{A})$$

$$\therefore \det(\boldsymbol{A}^T\boldsymbol{B}\boldsymbol{A}) = (-3)(-1)(-3) = -9$$

The determinant of a product of matrices is equal to the product of the determinants of the matrices. Also, the determinant of a transpose of a matrix, say $\boldsymbol{A}^T$ is equal to the determinant of the original matrix, $\boldsymbol{A}$.

d)

$$\det(\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}) = \left(\frac{1}{\det \boldsymbol{A}}\right)(\det \boldsymbol{A})(\det \boldsymbol{B})$$

$$\therefore \det(\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}) = \left(\frac{1}{-3}\right)(-3)(-1) = -1$$

The determinant of a product of matrices is equal to the product of the determinants of the matrices. Also, the determinant of the inverse of a matrix, say $\boldsymbol{A}^{-1}$ is equal to the reciprocal of the determinant of the original matrix, $\boldsymbol{A}$.