Homework 4

David Abeiku Saah

MATH212: Linear Algebra

Dr Ayawoa Dagbovie

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Problem 1

Given matrix:

$$A = \begin{bmatrix} 12 & 10 & -6 & 8 & 4 & -14 \\ -7 & -6 & 4 & -5 & -7 & 9 \\ 9 & 9 & -9 & 9 & 9 & -18 \\ -4 & -3 & -1 & 0 & -8 & 1 \\ 8 & 7 & -5 & 6 & 1 & -11 \end{bmatrix}$$

The reduced row echelon form of A using software (python) is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $1, 2, 3,$ and 5. Hence, let matrix $\mathrm{B} = \begin{bmatrix} 12 & 10 & -6 & 4 \\ -7 & -6 & 4 & -7 \\ 9 & 9 & -9 & 9 \\ -4 & -3 & -1 & -8 \\ 8 & 7 & -5 & 1 \end{bmatrix}$

Solving $B\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} 12 & 10 & -6 & 4 \\ -7 & -6 & 4 & -7 \\ 9 & 9 & -9 & 9 \\ -4 & -3 & -1 & -8 \\ 8 & 7 & -5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The row echelon form of B using software (python) is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0.$

Problem 2

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 9 \\ 15 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$$

a.

For v_3 to be in Span$\{v_1, v_2\}$, then v_3 must be a linear combination of v_1 and v_2. This is the same as checking for consistency of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ where $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \\ -5 & 15 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$.

$$\text{Augmented matrix: } \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -5 \\ -5 & 15 & h \end{bmatrix}$$

Row reducing:

$$R_3 \to R_3 + 5R_1 \text{ and } R_2 \to R_2 + 3R_1$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+10 \end{bmatrix}$$

$\therefore$ From the echelon form above, the matrix equation, $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent ($0 \neq 1$ in R_2) no matter the value of h. Hence, v_3 is not in Span$\{v_1, v_2\}$.

b.

To check for the value(s) of h that makes the set $\{v_1, v_2, v_3\}$ linearly dependent, we need to check for the value(s) of h that makes the matrix equation, $A\boldsymbol{x} = \boldsymbol{0}$ have a non-trivial solution.

Where $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -5 \\ -5 & 15 & h \end{bmatrix}$.

Augmented matrix:

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -3 & 9 & -5 & 0 \\ -5 & 15 & h & 0 \end{bmatrix}$$

Taking the echelon form of A from part (a) above, we have:

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

For the system above to be consistent, $h+10=0 \implies h=-10$. Hence, the set $\{v_1, v_2, v_3\}$ is linearly dependent if and only if $h=-10$. The existence of a free variable, x_2 implies the existence of non-trivial solutions.

Problem 3

a.

$$T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$$
$$A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}$$

b.

$$T(\boldsymbol{x}) = (0, -1, -4)$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$$

Augumenting the matrix above and row reducing:

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1 \text{ and } R_2 \to R_2 + \frac{3}{2}R_1$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & -2 & -4 \end{bmatrix}$$

$$R_2 \to -2R_2 \text{ and } R_1 \to \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2 \text{ and } R_1 \to R_1 + \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

From the reduced echelon form above, $x_1 = 1$ and $x_2 = 2$.

$$\therefore \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c.

Row reduced echelon form of the standard matrix from part b:

$$\begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There is a pivot position in every column. Hence, the columns of the standard matrix of T are linearly independent. Therefore, T is one-to-one.

There is not a pivot position in every row. Hence, the rows of the standard matrix of T are linearly dependent. Therefore, T is not onto.