Homework 3

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MATH212: Linear Algebra

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Problem 1

$$x_1 - 2x_2 + 3x_3 = 0 \tag{1}$$

Solving for x_1 in terms of x_2 and x_3:

$$x_1 = 2x_2 - 3x_3$$

x_1 and x_2 are free. Representing the solution set in vector form:

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{Let } \boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$x = x_2\boldsymbol{u} + x_3\boldsymbol{v}$$

$$x_1 - 2x_2 + 3x_3 = 4 \tag{2}$$

Solving for x_1 in terms of x_2 and x_3:

$$x_1 = 2x_2 - 3x_3 + 4$$

x_1 and x_2 are free. Representing the solution set in vector form:

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_2 - 3x_3 + 4 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Substituting $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$x = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2\boldsymbol{u} + x_3\boldsymbol{v}$$

$\therefore$ The solution of $eqn(1)$ and $eqn(2)$ are in $\mathbb{R}^3$. The line $x_1 - 2x_2 + 3x_3 = 0$ passes through the origin and spans $\boldsymbol{u}$ and $\boldsymbol{v}$. The line $x_1 - 2x_2 + 3x_3 = 4$ is parallel to the line $x_1 - 2x_2 + 3x_3 = 0$ and passes through the point $(4, 0, 0)$.

Problem 2

1. Let A be Agriculture.
2. Let E be Energy.
3. Let M be Manufacturing.
4. Let T be Transportation.

(a)

A	**E**	**M**	**T**	**Purchased By**
0.65	0.3	0.3	0.2	**A**
0.1	0.1	0.15	0.1	**E**
0.25	0.35	0.15	0.3	**M**
0	0.25	0.4	0.4	**T**

(b)

Let p_a, p_e, p_m, p_t be the amount of money spent on Agriculture, Energy, Manufacturing and Transportation respectively.

$$\text{Systems of equations: } \begin{cases} -0.35p_a + 0.3p_e + 0.3p_m + 0.2p_t = 0 \\ 0.1p_a - 0.9p_e + 0.15p_m + 0.1p_t = 0 \\ 0.25p_a + 0.35p_e - 0.85p_m + 0.3p_t = 0 \\ 0p_a + 0.25p_e + 0.4p_m - 0.6p_t = 0 \end{cases}$$

After computing the augmented matrix using software, the reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & -2.0279 & 0 \\ 0 & 1 & 0 & -0.5311 & 0 \\ 0 & 0 & 1 & -1.1681 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equivalent system of equations:

$$
\begin{aligned}
p_a - 2.0279t &= 0 \\
p_e - 0.5311t &= 0 \\
p_m - 1.1681t &= 0 \\
p_t &= p_t
\end{aligned}
$$

The general solution for equilibrium prices in the economy: $\begin{cases} p_a = 2.0279t \\ p_e = 0.5311t \\ p_m = 1.1681t \\ p_t = p_t \end{cases}$

Vector form:

$$
p = \begin{bmatrix} p_a \\ p_e \\ p_m \\ p_t \end{bmatrix} = \begin{bmatrix} 2.0279p_t \\ 0.5311p_t \\ 1.1681p_t \\ p_t \end{bmatrix} = p_t \begin{bmatrix} 2.0279 \\ 0.5311 \\ 1.1681 \\ 1 \end{bmatrix}
$$

If $p_t = \$1,000,000$, then $p_a = \$2,027,900$, $p_e = \$531,100$, and $p_m = \$1,168,100$.

$\therefore$ The incomes and expenditures of each sector will be equal if the output of Agriculture is priced at \$2,027,900, Energy is \$531,100, Manufacturing is priced at \$1,168,100 and that of Transportation is \$1,000,000 (using the example above).

Problem 3

Intersection	Flow in	Flow out
A	x_1	$x_2 + 100$
B	$x_2 + 50$	x_3
C	x_3	$x_4 + 120$
D	$x_4 + 150$	x_5
E	x_5	$x_6 + 80$
F	$x_6 + 100$	x_1

System of linear equations:

$$\begin{aligned} x_1 - x_2 &= 100 \\ x_2 - x_3 &= -50 \\ x_3 - x_4 &= 120 \\ x_4 - x_5 &= -150 \\ x_5 - x_6 &= 80 \\ x_6 - x_1 &= -100 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

Reduced row echelon form (using software):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{The general solution of the network is: } \begin{cases} x_1 = 100 + x_6 \\ x_2 = x_6 \\ x_3 = 50 + x_6 \\ x_4 = -70 + x_6 \\ x_5 = 80 + x_6 \\ x_6 = x_6 \end{cases}$$

Distance cannot be negative so the smallest possible value for x_6 is 0.