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Graduation qualification thesis

Algorithmic Aspects of Alexandrov’s Uniqueness Theorem

Bachelor’s program
„Mathematics“
Specialization and code: 01.03.01 „Mathematics“
Cipher of the EP: CB.5000.2015

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Saint Petersburg
2019

Introduction

One of the directions in computational geometry is designing algorithms that provide constructive proofs of geometric theorems that are well-known but have only non-constructive proofs. This thesis deals with such situation, in particular it concerns a constructive proof of Alexandrov Theorem.

This theorem implies that if the boundary of a convex polyhedron P is cut into a finite number of flat polygons $T_1 \dots T_n$ (it is not necessary that these polygons are faces of P) and it is known how exactly the boundaries of the polygons were glued to each other initially, then there is a single convex polyhedron that can be obtained by gluing $T_1 \dots T_n$ with respect to the given rules—and it is P itself.

To state this theorem formally we need the following

Definition 1 ([1])**.** A *net* is a set of polygons equipped with a number of rules describing the way edges of these polygons must be glued to each other.

Theorem 1 (Alexandrov, 1950, [1])**.** *If a net is homeomorphic to a sphere and the sum of angles at each of its vertices is at most* 360° *then there is a single convex polyhedron P that can be glued from this net.*

However, the algorithmic question of *constructing* the polyhedron P, its 1-dimensional skeleton and coordinates of the vertices, remains open for around 60 years.

Problem 1. Given a net, calculate the coordinates of the vertices of the convex polyhedron P that corresponds to it and list the edges of P and the vertices they connect.

It is known that Problem 1 can be reduced to a system of partial differential equations [3, 9]. However, this method does not produce the exact answer. Moreover, there is no known algorithm for it that works faster than pseudopolynomial in n, where n is the number of vertices.

If the graph structure of P is known, Problem 1 can be reduced to solving a system of equalities and inequations of degrees 2 and 3. The number of variables in the system is $3n$, where n is number of vertices, and the number of equalities and inequations is $O(|V| + |E|)$. The construction of this system is shown in Demaine, O'Rourke [6]: the equations have form

$$|uv| = d$$

for every pair u, v of vertices of the same face and d known from the net that P corresponds to, and the inequalities have form

$$\det \begin{pmatrix} \overrightarrow{a-u} & \overrightarrow{v-u} & \overrightarrow{b-u} \end{pmatrix} \geq 0$$

for every edge uv and vertices a, b opposite to it (they state that each edge is in the „convex position"). It is also known that sometimes there is no closed-form expression for the coordinates of vertices of polyhedron in terms of edge lengths and coordinates of vertices of $T_1 \dots T_n$ [7].

To get closer to a solution of Problem 1 one can solve some of its restricted cases. One way is to consider a single polygon of a particular type, e.g. the Latin cross [5] or a regular k-gon [6]. Another way is to consider only *edge-to-edge* gluings, where an edge of a polygon T_i needs to be glued to an entire other edge of another polygon T_j (possibly $j = i$) [6, 11].

Recently a third way of restricting the setting was proposed [10] that on one hand is a combination of these two, and on another hand is their generalization from one single polygon to many. We wish to glue together several copies of the same regular polygon edge-to-edge. For the case of regular k-gons with $k > 6$, the only two possibilities are: two k-gons glued into a doubly covered k-gon, or one k-gon folded in half (if k is even) [10]. The first interesting case when $k = 6$ was also studied in [10]—one of the reasons that did not allow the authors to complete the characterization is that they did not have a rigorous way of justifying that a given gluing corresponds to a given graph structure.

0.1 Edge-to-edge gluings when $m = 5$: the list of all possible shapes

Edge-to-edge gluings of a number of regular pentagons are thoroughly studied in Arseneva et al. [2]. It was proved that a convex polyhedron can be made of at most 12 congruent pentagons. All possible ways to glue edges of pentagons together were considered (their number is exponential on the number of polygons)—after it one obtained the full list of the nets consisting of regular pentagons that satisfy the conditions of Theorem 1.

The algorithm described in Bobenko, Izmestiev [3] was implemented in the MSc thesis of Sechelmann [15], the resulting program allows for calculating the approximate coordinates of vertices of polyhedra that are glued from a number of polygons. Approximate coordinates of the polyhedra glued from regular pentagons were obtained with the use of this program that could visualize these polyhedra and provide a better understanding of their structure.

The aim of this work is to establish the true graph structure of these polyhedra. To achieve it we did the following:

1) Estimated the difference between the location of edges and vertices of *approximate* polyhedra obtained by the algorithm and the true ones arising from Theorem 1.

2) Developed automatic procedure to check whether two vertices of a polyhedron are connected by an edge or not.

This procedure is not sufficient to achieve the goal: when there are non-triangular faces in the polyhedron, the procedure can not *prove* the absence of an edge between two vertices. Thus in this work for some polyhedra we geometrically prove that they have quadrilateral faces that appear to be either trapezoids or parallelograms.

We also show that geomertical methods can be applied to prove the *existence* of edges of polyhedra—as an example we consider the polyhedron glued from six regular pentagons.

0.2 Edge-to-edge gluings when $m = 3, 4, 6$: isomorphism of gluings.

The cases of triangles, quadrilaterals and hexagons are different from the case of pentagons in that the internal angles of these polygons are multiple of 360°. It means that flat vertices (the sum of angles at which is 360°) can be obtained when gluing them, and thus there are infinitely many ways to glue a number of such polygons into a convex polyhedron. Thus it is not possible to consider *all* the ways to do so.

Definition 2. Let us call two polyhedra *combinatorially different* if they have different graph structures—i. e. different 1-dimensional skeleta.

Gluings of regular hexagons were studied in [10]. The authors have identified the 15 possible graph structures of combinatorially different polyhedra that can be glued, 5 of which are in fact doubly-covered 2D polygons. They characterized the latter ones fully, and gave examples for 5 non-flat shapes, including all simplicial ones. It is open whether the remaining 5 shapes can be realized in $\mathbb{R}^3$.

In order to classify the polyhedra obtained by gluing several polygons let us introduce the notion of isomorphic nets.

Definition 3. Two nets $(T_1 \dots T_n, \sim_1)$, $(T'_1 \dots T'_n, \sim_2)$ satisfying the conditions of Theorem 1 are called *isomorphic* if convex polyhedra corresponding to them are homothetic.

The aim of this work is to propose a method to check whether two nets are isomorphic if given are graph structures of the polyhedra corresponding to them—for example, as a result provided by the algorithm [3].

For a net it is easy to find out which of its vertices are going to be the vertices of the glued polyhedron—those are the ones sum of angles at which is strictly less than 360°. It can be done in time linear on the number of vertices.

Let two nets have the same number of vertices (if not, they can not be isomorphic). In this case the isomorphism between nets can be established by

1) Calculating the length of edges of the graph structures,

2) Making them „normal"—for example, making the sum of edge lengths the same for both nets,

3) Checking the isomorphism of planar weighted graphs obtained that can be done in polynomial time (shown in Hopcroft, Wong [8]).

To conclude, the main part of checking the isomorphism of two nets is calculating the lengths of the shortest paths between the vertices of the nets.

0.3 Shortest paths on the boundaries of polyhedra

The problem of finding shortest paths on the boundaries of the polyhedra is widely studied. Similar to the shortest path is the notion of the *geodesic path*.

Definition 4 ([12])**.** A geodesic path is a path which is locally optimal, i. e. can not be shortened by a slight perturbation.

There are two lemmas of great importance when searching for geodesic and shortest paths that are proved in Mitchell, Mount, Papadimitriou [12].

Lemma 2 ([12], 3.2)**.** *Let π be a geodesic shortest path on the surface of a polyhedron P. The intersection between π and a face f of P is a (possibly empty) line segment.*

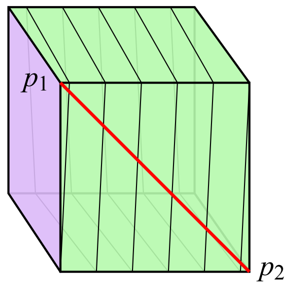

Figure 1: Shortest path $\pi = p_1p_2$ crosses parallelogramm T (colored green) seven times.

Lemma 3 ([12], 3.3)**.** *If π is a geodesic path which crosses an edge sequence $e_1 \dots e_m$, then the planar unfolding of π along $e_1 \dots e_m$ is a straight line segment.*

There are three well-known algorithms for computing the shortest paths, see the table below:

№	The algorithm	Time complexity	Space complexity	Time to resonstruct the shortest path to any point
1)	«Continuous Dijkstra» [12]	$O(n^2 \log n)$	$O(n^2)$	$O(\log n)$
2)	Chen—Han Algorithm [4]	$O(n^2)$	$O(n)$	$O(\log n)$
3)	Optimal-time algorithm [14]	$O(n \log n)$	$O(n \log n)$	$O(\log n)$

One of the goals of this work is to develop an algorithm that finds the shortest paths between points on an arbitrary net satisfying the conditions of the Alexandrov's Theorem. Algorithms 1–3 solve this problem with an additional restriction that $T_1 \dots T_n$ are faces of the polyhedron P.

For an arbitrary net of a polyhedron and a shortest path between two points on it Lemma 3 obviously still holds. However, Lemma 2 does not: the cube glued from two squares and a parallelogramm (see Figure 1) is a counterexample. Shortest path connecting points p_1 and p_2 crosses the parallelogramm seven times. One can obtain an unbounded number of intersections between the shortest path and one of the polygons.

However, previously it was assumed that the Chen—Han algorithm that is designed for nets consisting of faces of a polyhedron can also be applied to arbitrary nets without any change [6, 13]. Still, if the number of intersections between the shortest path and a polygon from the set $T_1 \dots T_n$ is not bounded, one can not find out when to stop running the algorithm to be sure that after we stop itwe obtain correct answer.

In this work we prove for some types of polygons that the number of intersections between them and an optimal path on the boundary of P is bounded from above. In particular, our results imply that the modified Chen—Han algorithm runs in still the same $O(n^2)$ time on arbitrary nets consisting of regular triangles, hexagons, or squares.

0.4 The Chen—Han Algorithm

When developing an algorithm that takes the boundary of a polyhedron as an input it is usually thought that this polyhedron lies in the most general class possible:

Definition 5. *Simplicial polyhedron* in $\mathbb{R}^3$ is a polyhedron all whose faces are triangles.

Chen—Han algorithm [4] is a decently effective algorithm for computing the shortest paths from a given vertex S to all other vertices of a given simplicial polyhedron. Moreover, in this work it is Chen—Han algorithm that will be modified to be applied for arbitrary nets. This is why in this section we explain how this algorithm works in detail.

This algorithm constructs the *sequence tree*: each node of it, apart from the root, is a triple $n' = (e, I, Proj_e^I)$ where

1) e is an edge of the polyhedron,
2) I is the location of the vertex S (in the corresponding flat system of coordinates) after all the edges in the ancestors of n' are unfolded,
3) $Proj_e^I$ is a segment on the edge e consisting of endpoins of all the segments starting from I and intersecting all the edges in the ancestors of n'.

The number of levels in the sequence tree is at most the number of faces of the polyhedron due to Lemma 2. On the i-th step, $1 \le i \le n$, for each node $(e_k, I_k, Proj_{e_k}^{I_k})$ on the i-th level, the algorithm does the following:

1) „Unfolds" the edge e_k of the polyhedron, i. e. makes two faces incident to e_k and all the faces unfolded before complanar.
2) For each of two edges e_1', e_2' of the „new" face calculates $Proj_{e_j'}^{I_k}$ in accordance with Lemma 3. This set consists of the endpoints of all the segments starting at S and intersecting $Proj_{e_k}^{I_k}$. If this set is non-empty, the algorithm inserts node
$$\left(e_j', I_k, Proj_{e_j'}^{I_k}\right) \tag{1}$$
into the tree as a child of $(e_k, I_k, Proj_{e_k}^{I_k})$.

However, the tree as described can have exponentially many nodes. This can make the time complexity of the algorithm exponential. This is why in reality Chen—Han algorithm does not insert into the tree *all* the nodes described in (2). To bound the number of nodes in the sequence tree, Chen and Han proved the following

Lemma 4 ([4], 1)**.** *Given a face ABC of the polyhedron P and two nodes n_1, n_2 of the sequence tree on the same edge BC. If each of this nodes has a non-empty child on both of edges CA, AB, then at most one of the nodes n_1, n_2 can have two children which can be used to define shortest sequences.*

Lemma 4 bounds from above by $O(n)$ the number of nodes on any level of the sequence tree (including its leaves). This is because any edge of the polyhedron can have at most one node on it with two children that can define shortest paths. This allows the algorithm to run in $O(n^2)$ time.

This is how Chen—Han algorithm looks in its final form:

Algorithm 1 Chen—Han

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root $:= S$ ▷ The root of tree is the given vertex
for all edges e opposite to S **do**
    Insert (e, S, e) into the tree as a root's child;
end for
for $i := 1$ to n **do** ▷ n is the number of faces
    for all the nodes $n' = (e, I, Proj_e^I)$ on the i-th level **do**
        Unfold the edge e;
        ▷ Let us denote by ABC the face behind e so that $e = BC$
        Calculate $Proj_{AB}^I$, $Proj_{CA}^I$ in accordance with Lemma 3;
        if both *Proj*-s are non-empty and thus both contain A **then**
            if $|IA| < |I'A|$, where I' is stored in the node n'' that occupied A previously **then**
                Delete one of the children of n'' in accordance with Lemma 4;
                Delete subtree rooted at this child;
                Insert $(AB, I, Proj_{AB}^I)$, $(CA, I, Proj_{CA}^I)$ into the tree as children of n';
                Now node n' *occupies* A
            else
                $e' :=$ the only edge of face ABC that in accordance with Lemma 4
                  can define shortest paths;
                Insert $(e', I, Proj_{e'}^I)$ into the tree as the sole child of n'
            end if
        else
            $e' :=$ Either AB or CA depending on which *Proj* is non-empty;
            Insert $(e', I, Proj_{e'}^I)$ into the tree as the sole child of n'
        end if
    end for
end for

0.5 Main results

In the section 1 we consider geodesic shortest paths on boundaries of polyhedra and intersections between them and polygons drawn on the boundary. We prove that

1) The intersection between a shortest path and a square drawn on the polyhedron consists of at most 7 segments (Theorem 9).

2) The intersection between a shortest path and a regular hexagon consists of at most 6 segments (Theorem 13).

3) The intersection between a shortest path and a regular r-gon, $r \geq 6$, consists of at most $r + 3$ segments (Theorem 12).

4) The intersection between a shortest path and a regular triangle, when the gluing is edge-to-edge, consists of at most 5 segments (Theorem 14).

These results imply that Algorithm 1 can be applied to arbitrary nets consisting of regular triangles, hexagons, or squares that are glued edge-to-edge. The depth of the sequence tree can be bounded above by $5n$, $6n$, $7n$ correspondingly. The time complexity of the algorithm remains $O(n^2)$.

In Sections 2–3 we solve the problem of establishing the graph structure of convex polyhedra that are glued rom regular pentagons edge-to-edge. The following results are presented in Section 2:

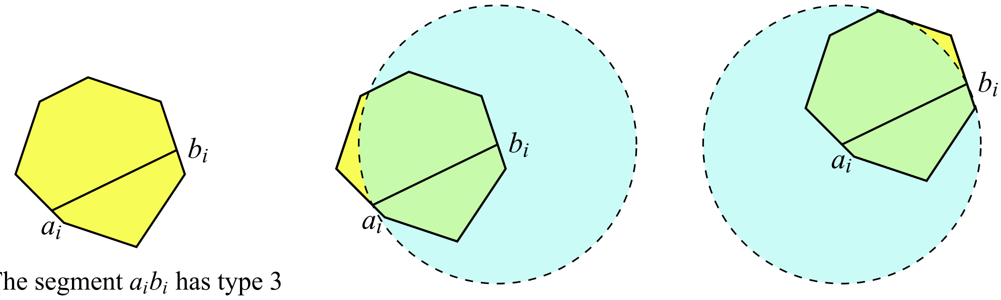
1) The discrepancy in vertex coordinates between the polyhedron arising from the Alexandrov's Theorem and a given approximate polyhedron is estimated (Theorem 16).
2) The sufficient condition for the existence of the edge connecting two vertices of the polyhedron arising from Alexandrov Theorem is found (Theorem 19).
3) The program is developed that checks this condition for the approximate polyhedra we obtained (Section 2.3).

In Section 3 we apply geometric methods for establishing the graph structures. We prove that

1) Two of the polyhedra glued from regular pentagons have quadrilateral faces (Theorems 20–21).
2) On the other side, the existence of one of the edges in the polyhedron glued from 6 regular pentagons was ensured geometrically (Theorem 22).

One of the outcomes of this work is the full list of the polyhedra that are obtained by edge-to-edge gluings of regular pentagons (Section 3.4).

Contents



(a) The segment a_ib_i has type 3 which is the maximum possible type of a segment inside a heptagon.

(b) No points of $a_1b_1 \dots a_{i-1}b_{i-1}$ can lie in this disk.

(c) No points of $a_{i+1}b_{i+1} \dots a_mb_m$ can lie in this disk.

Figure 2: Type of a segment and Lemma 6.

1 Shortest paths on various tiles

Let P be a convex polyhedron, T be a convex polygon drawn on the surface of P, and p_1, p_2 be two points on the surface of P. Also let π be a geodesic shortest path connecting them.

Lemma 5. *The intersection between π and T is a union of a finite number of segments $a_1b_1 \dots a_mb_m$; all the points a_i, b_i lie on the boundary of T, except possibly a_1 and b_m.*

Definition 6. Let a_ib_i be a segment of the intersection between π and T with $a_i, b_i \in \partial T$. The points a_i, b_i divide the boundary of T into two parts. Let there be n_1 and n_2 be the numbers of vertices of ∂T in the parts respectively. Then the segment a_ib_i has *type s* if

$$\min(n_1, n_2) = s.$$

If T is a convex r-gon then the possible types of a segment can range between 1 and $\lfloor r/2 \rfloor$ (see Figure 2a).

Lemma 6. *If a_ib_i is a segment of the intersection between π and T, then*

1) *no point of $a_1b_1, \dots, a_{i-1}b_{i-1}$ may lie in the disk centered at b_i with radius $|a_ib_i|$ (shown in Figure 2b),*

2) *no point of $a_{i+1}b_{i+1}, \dots, a_mb_m$ may lie in the disk centered at a_i with radius $|a_ib_i|$ (shown in Figure 2c),*

3) *no point of a_jb_j, $j \neq i$ may lie inside the disk having a_ib_i as its diameter.*

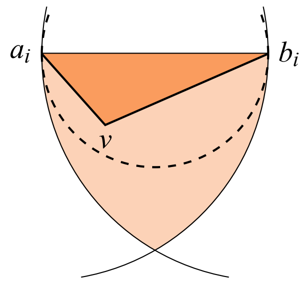


Figure 3: No points of path π lie inside the disk whose diameter is a_ib_i and thus, in turn, inside the triangle a_ivb_i.

Proof. Let us prove item (1). If there is a point p of a segment a_jb_j preceding a_ib_i inside that circle then (because T is convex) the segment pb_i lies entirely inside T and is shorter than a_ib_i. Then we can replace the path

$$\pi = \dots a_jpb_j \dots a_ib_i \dots$$

with

$$\dots a_jpb_i \dots,$$

which is shorter than π , thus π is not the shortest.

The proof of (2) is analogous. To prove (3) let us note that the intersection between the disks from (1) and (2) covers the smaller circle we are considering (see Figure 3)—thus no segments either from „the past“ or „the future“ of path π can lie inside it.

□

Lemma 7. *If the angle at a vertex v of polygon T is not less than* $90°$ *then there is at most one segment a_ib_i of the intersection $\pi \cap T$ of type 1 such that the vertex v lies between points a_i and b_i.*

Proof. Suppose there are multiple segments satisfying the statement of the lemma. Let us denote by a_ib_i the one whose end is the farthest from v. Due to Lemma 6 all the points of the segments $a_jb_j, j \neq i$ of the intersection $\pi \cap T$ must lie outside the disk D whose diameter is a_ib_i.

This disk covers all the points s such that $\measuredangle a_isb_i \geq 90°$, thus it covers the triangle a_ivb_i (see Figure 3) and consequently no other points of the path π can lie inside this triangle.

All the other segments satisfying the statement have at least some of their points lying inside a_ivb_i. Thus we arrive to a contradiction: π is not the shortest path.

□

1.1 Shortest paths crossing squares

Lemma 8. *Let $T = ABCD$ be a square, then there is at most one segment of type 2 in $\pi \cap T$.*

Proof. A segment of type 2 connects either AB with CD or BC with DA. Since π is a shortest path, segments that are parts of π can not intersect each other. Therefore, without loss of generality all the segments of type 2 connect AB with CD.

Among all these segments let us consider the segment a_ib_i of type 2 with the greatest possible i. Without loss of generality $a_i \in AB$, $b_i \in CD$. The disk centered at b_i with radius $|a_ib_i| \geq |CD|$

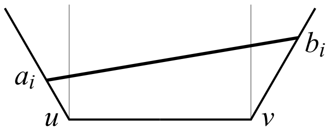


Figure 4: $a_i b_i$ is the only segment of type 2 corresponding to the vertices u, v.

covers CD—thus, by Lemma 6, there are no segments $a_j b_j, j < i$ with endpoints on CD. It means that $a_i b_i$ is the only segment of type 2, which completes the proof. □

Theorem 9. *If T is a square and π is a geodesic shortest path between two points p_1, p_2 then the intersection between π and T consists of at most 7 segments.*

Proof. A segment in the intersection can either have type 1, type 2 or contain p_1 or p_2.

1) By Lemma 7, there are at most 4 segments of type 1: at most one for each vertex;

2) By Lemma 8, there is at most one segment of type 2;

3) For each of the points p_1, p_2 at most one segment contains it.

This sums up to $4 + 1 + 2 = 7$ segments.

□

Remark. If p_1 and p_2 lie outside T or on its boundary, the number of segments in $\pi \cap T$ is at most 5.

1.2 Shortest paths crossing regular hexagons

Lemma 10. *If T is a regular hexagon and u, v is a pair of its vertices adjacent to each other, then there is at most one segment of type 2 with vertices u, v between its endpoints.*

Proof. Let us consider a segment $a_i b_i$ of type 2 with vertices u, v between its endpoints with largest possible i, i. e. the last segment traversing T. Clearly, $|a_i b_i| \geq |uv|$ as the part of $a_i b_i$ bounded by the perpendiculars raised from u and v (shown in Figure 4) is not shorter than uv.

Therefore the circle centered at b_i of radius $|a_i b_i|$ covers the side of T that contains b_i (as T is regular and uv is longer than any of its sides). Thus, by Lemma 6, no segments $a_j b_j$ with $j < i$ can intersect with that side, which finishes the proof.

□

Theorem 11. *If T is a regular hexagon and π is a geodesic shortest path between two points p_1, p_2 then the intersection between π and T consists of at most 12 segments.*

Proof. A segment in the intersection can either have type 1, type 2, type 3 or contain p_1 or p_2.

1) According to Lemma 7, there are at most 6 segments of type 1: at most one for each vertex;

2) According to Lemma 10, there are at most 3 segments of type 2: there can not be more than 3 pairs of adjacent vertices such that the corresponding segments of type 2 do not intersect;

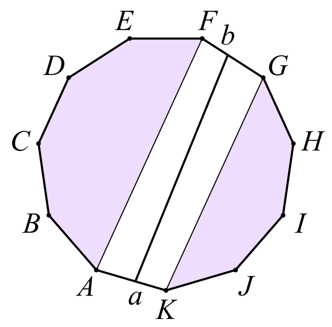


Figure 5: Shortest neighboring diagonal for segment ab is KG.

3) Finally, there is at most one segment of type 3: the proof for this is analogous to one of Lemma 8;

4) At most one segment can contain each of the points p_1, p_2.

This totals in $6+3+1+2=12$ segments.

$\square$

1.3 Shortest paths crossing regular n–gons

The results above can be generalized to polygons with more vertices:

Theorem 12. *If $T = v_1 v_2 \dots v_n$ is a regular n–gon with $n > 5$ and π is a geodesic shortest path between two points p_1, p_2 then the intersection between π and T consists of at most $n+3$ segments.*

Proof. Let us first establish that there is at most one segment of „maximum“ type—$\lfloor \frac{n}{2} \rfloor$. If n is even then there is only one pair of sides of T that can be connected by non-intersecting segments of maximum type. If n is odd, then there can be two such pairs of sides sharing a common edge, say—

$$V_n V_1, \quad V_{\frac{n+1}{2}-1} V_{\frac{n+1}{2}} \quad \text{and} \quad V_n V_1, \quad V_{\frac{n+1}{2}} V_{\frac{n+1}{2}+1}.$$

Let us consider the segment $a_i b_i$ of maximum type with the largest i. Its length is bounded below by the length of the shortest among two neighboring diagonals of T (see Figure 5), which is equal to

$$2 \cdot \sin\left(\frac{360^\circ}{n} \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \frac{1}{2}\right),$$

which is more than the radius needed to cover two contiguous sides of T, equal to

$$2 \cdot \sin\left(\frac{360^\circ}{n}\right).$$

Therefore the circle centered at b_i with radius $|a_i b_i|$ covers the sides of T adjacent to the one containing b_i (including that side too). Applying Lemma 6 we can finish the proof with $a_i b_i$ being the only segment of the maximum type.

Let us now bound the number of segments of lesser types. We will establish that if $a_i b_i$ is a segment of type strictly less than $\lfloor \frac{n}{2} \rfloor$, than for any vertex v between its endpoints

$$\measuredangle a_i v b_i \geq 90^\circ.$$

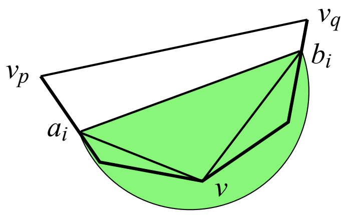


Figure 6: $\measuredangle a_i v b_i \leq 90°$. The circle where no other segments are allowed covers all the sides of T_k between a_i and b_i.

If n is even then there are diagonals of T which are diameters of the circumcircle of T. For each such diagonal $v_p v_q$ the angle $v_p v v_q$ is equal to $90°$. Let us find such v_p, v_q that a_i and b_i lie on the boundary of the half of T defined by the vertices $v_p \dots v \dots v_q$.

Since the half of a regular polygon is convex,

$$\measuredangle a_i v b_i \;\geq\; \measuredangle v_p v v_q \;=\; 90° \quad \text{(see Figure 6).}$$

The proof for odd n's is similar.

Since for any vertex v between a_i and b_i the angle $a_i v b_i \leq 90°$ the circle having $a_i b_i$ as its diameters covers all such vertices. Since both the regular n–gon and the circle are convex, all the edges between a_i and b_i are also covered.

According to Lemma 6, for every edge of T at most two segments of $\pi \cap T$ and of type less than $\lfloor \frac{n}{2} \rfloor$ can have their endpoints on this edge: the circle of the first one covering the left side of the edge, the circle of the second one coverting the right side of the edge. Thus there are at most $2n$ *endpoints* of the segments of lesser types in $\pi \cap T$, which results in at most n segments.

In total there are at most $n+1+2$ segments: 1 of the maximum type, n of the lesser types and 2 possibly containing p_1 and p_2, which finishes the proof.

□

Corollary 13. *If $T = ABCDEF$ is a regular hexagon, the points p_1, p_2 lie outside its interior, and π is the shortest path between p_1 and p_2, then the intersection $\pi \cap T$ consists of at most 6 segments.*

Proof. If there are no segments of type 3, then by Theorem 12 there are at most 6 segments. If there is a segment of type 3 (there is at most one due to Theorem 11), then we have to bound the number of segments of the lesser types.

Let the only segment of type 3 connect the sides BC and EF. First let us note that no segments of type 2 are possible in this case: if the vertices between the endpoints of a segment of type 2 are B, C or E, F then this segment intersects with $a_i b_i$.

If not, then this segment must intersect DA. Note that

$$|a_i A|, |b_i A|, |a_i D|, |b_i D| \leq \ell \leq |a_i b_i|,$$

where ℓ is the length of a side of the equilateral triangle ACE (see Figure 7). It means that the points A, D belong to both circles from Lemma 6, and according to this lemma no segments of π

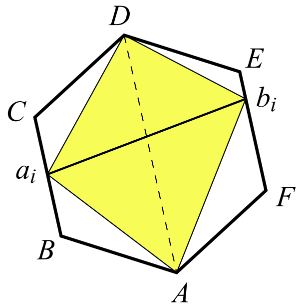


Figure 7: The segment a_ib_i is of type 3; no other points of the path π can belong to the tetragon Aa_iDb_i.

can intersect the convex hull Aa_iDb_i (since these four points belong to both circles and the circles are convex).

Thus no segments of type 2 are possible. Also not possible are the segments of type 1 with the vertices A or D between their endpoints. Finally we have the upper bound of

$$1\,(\text{of type 3}) \;+\; 4\,(\text{of type 1 corresponding to } B, C, E, F) \;=\; 5\,\text{segments},$$

which finishes the proof.

□

1.4 Shortest paths crossing regular triangles

Theorem 14. *Let P be a convex polyhedron glued edge-to-edge from a number of equal regular triangles. Let p_1 and p_2 be two points on the surface of P, and π be the geodesic shortest path from p_1 to p_2. Then for every tile T_k the intersection $\pi \cap T_k$ consists of not more than 5 segments.*

Proof. Without loss of generality, the side of T_k and all the other tiles is equal to 1, thus the diameter of T_k is equal to 1. Let q_1 be the first point of π that belongs to T_k, and q_2 be the last point of π that belongs to T_k. The distance between q_1 and q_2 as points of T_k does not exceed 1, this means that the distance between them on the path π also does not exceed 1.

Let us consider the part of the path π between q_1 and q_2 and unfold on the plane all the tiles it passes through. Since P is glued edge-to-edge and due to the classical theorem stating that the unfolded path is a part of straight line, we obtain a segment of length $l \leq 1$ lying inside a subset of the triangular grid.

This subset can be looked at as a sequence, that is every triangle except the last has exactly one triangle following it, and every triangle except the first has exactly one triangle preceding it. Some triangles in the sequence represent T_k, one triangle for every segment of the intersection $\pi \cap T_k$. Let us also note that five triangles in this sequence can not have a common vertex (see Figure 8), since any segment connecting the first triangle with the last triangle does not intersect the three triangles in between, as the convex hull of the two triangles does not intersect them.

We will now prove that a segment of length $l \leq 1$ can not intersect more than 5 trangles of the sequence. If there are at most five triangles between q_1 and q_2 then of course there are at most five segments in the intersection $\pi \cap T_k$.

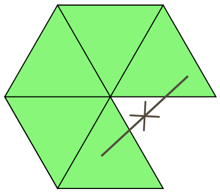


Figure 8: Five consequent triangles with a common vertex can not be present in the unfolding.

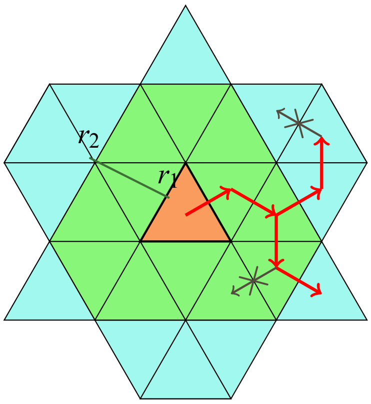


Figure 9: The triangles close to the initial one and the possible unfoldings along the path.

Let us consider the *initial* triangle that q_1 belongs to (depicted orange in Figure 9) and the triangular grid itself. There are 12 triangles such that the distance from them to the initial triangle is equal to 0 (depicted green) and 9 triangles such that the distance from them to the initial triangle is equal to $\sqrt{3}/2$ (depicted blue). Therefore the point q_2 can not lie outside these triangles, because the distance from the initial triangle to the outside exceeds 1.

Without loss of generality, the next triangle after initial in the sequence is the one to the right (otherwise we can rotate the whole picture). Only one option for the third triangle in the sequence can also be considered w. l. o. g. The red arrows in Figure 9 show possible unfoldings, and the brown arrows show sequences that can not be unfoldings since five triangles with common vertices emerge.

Thus all the unfoldings along the segment q_1q_2, $|q_1q_2| \leq 1$ consist of not more than 5 triangles, and the theorem is proved (the special case when q_1q_2 coincides with edges of the triangles is obvious). It remains to say that five triangles can be touched by a segment of length $l \leq 1$: see the segment r_1r_2 in Figure 9.

$\square$

2 Algorithmic methods to determine graph structure

2.1 Determining the shape out of the gluing

Consider a polyhedral metric M that satisfies the Alexandrov's conditions and thus corresponds to unique polyhedron $\mathcal{P}$. Suppose we have a simplicial polyhedron P whose edge lengths and facial angles are close to those of $\mathcal{P}$. In this section we show how to check whether the graph structures of P and $\mathcal{P}$ coincide.

We will be using the following notation: $v_1, v_2, v_3, \ldots$ for the vertices of $\mathcal{P}$; $u_1, u_2, u_3, \ldots$ for the vertices of P; V, E, F for the number of vertices, edges and faces of $\mathcal{P}$ respectively; $\mathcal{D}$ for the maximum degree of a vertex of P; L for the length of the largest edge of P; $B_r(u)$ for the ball in $\mathbb{R}^3$ of radius r centered at the point u.

Vertices of P correspond to cone points of metric M, and the edges are shortest paths between the corresponding vertices, thus for every edge u_iu_j of P we can calculate the discrepancy between its length and the corresponding length in M (which we call the *intended length* of that edge). Let μ be the maximum discrepancy for all edges of P.

Similarly, for every pair of adjacent edges u_iu_j and u_iu_k of P one can find the discrepancy between the facial angle $u_ju_iu_k$ and the corresponding angle $v_jv_iv_k$ in M. Let γ denote the maximum discrepancy between values of angles.

We base our check on an observation that each vertex v_i of $\mathcal{P}$ lies within r–ball centered at the corresponding vertex u_i of P, where

$$r = E^2 \cdot L \cdot 2\sin(\mathcal{D}\gamma/2) + E\mu. \tag{2}$$

We defer its proof to the Section 2.2, see Theorem 16.

Let u_iu_j be an edge of P and let u_a, u_b be the two vertices of P opposite to the edge u_iu_j. We want to check that there does not exist a plane intersecting all four r–balls centered at u_i, u_j, u_a, u_b respectively.

A plane containing a face of the convex polyhedron P divides $\mathbb{R}^3$ into two half-spaces, one of which contains the polyhedron. Assume without loss of generality that the plane passing through u_a, u_i, u_j is not vertical and that P lies below that plane (otherwise apply a rigid transformation to P so that it becomes true).

Consider three planes Π_1, Π_2, Π_3 tangent to $B_r(u_i)$, $B_r(u_j)$, $B_r(u_a)$ such that:

- Π_1 is below $B_r(u_i)$, $B_r(u_j)$ and above $B_r(u_a)$,
- Π_2 is below $B_r(u_i)$ and above $B_r(u_j)$, $B_r(u_a)$,
- Π_3 is below $B_r(u_j)$ and above $B_r(u_i)$, $B_r(u_a)$.

If u_b lies below Π_1, Π_2 and Π_3 and the distance from u_b to each of the planes Π_1, Π_2 and Π_3 is greater than r, then there must be the edge v_iv_j in $\mathcal{P}$.

This procedure has to be repeated for every edge u_iu_j of P. This implies the following

Theorem 15. *Given a metric M and a simplicial polyhedron P we can check in time $\mathcal{O}(E)$ that the convex polyhedron corresponding to metric M has the same graph structure as P without false-positive errors.*

False negative errors can occur if the precision is not sufficient, and a plane exists that intersects all four r–balls centered at the vertices of P even though there is an edge connecting two of the vertices. In such a case precision has to be increased by replacing P with a polyhedron that has smaller discrepancy in edge lengths and values of angles and repeating the procedure.

To obtain polyhedron P in practice one can use the algorithm developed by Kane et al. [9] or by Bobenko, Izmestiev [3]. It outputs a polyhedron P which is an approximation of $\mathcal{P}$. To measure the quality of an approximation the authors introduced the value called *residual curvature*. When running the algorithm, one can set the residual curvature not to exceed a given value $\varkappa$, the lower $\varkappa$ the better approximation will be obtained.

2.2 Precision of vertex location based on the approximation

In this section our aim is to find a small real number r such that each vertex of $\mathcal{P}$ lies within an r–ball centered at the corresponding vertex of P.

Without loss of generality, one of the vertices of both P and $\mathcal{P}$ (let them be u_1 and v_1 respectively) is located at $(0,0,0)$, one of the edges incident to this vertex (in $\mathcal{P}$ and in P, respectively) is aligned with x axis, and one of the faces incident to that vertex and that edge—both in $\mathcal{P}$ and P—lies on the horizontal plane $z = 0$.

We derive the following theorem, which is a key statement to prove the correctness of our main result.

Theorem 16. *Suppose* μ *is the maximum edge discrepancy between* P *and* $\mathcal{P}$*,* γ *is the maximum angle discrepancy between* P *and* $\mathcal{P}$*,* $\mathcal{D}$ *is the maximum degree of a vertex of* P*. If* $\mathcal{D}\gamma < \pi/2$*, then each vertex of* $\mathcal{P}$ *lies within an* r*–ball centered at the corresponding vertex of* P*, where*

$$r = E^2 \cdot L \cdot 2\sin(\mathcal{D}\gamma/2) + E\mu. \tag{3}$$

To prove this theorem we need two lemmas:

Lemma 17. *Let* pq*,* pq' *be line segments in* $\mathbb{R}^3$*. If there are two real numbers* ϵ*,* θ *with* $\epsilon > 0$ *and* $0 < \theta < \frac{\pi}{2}$ *such that*

(a) $|pq| - \epsilon \le |pq'| \le |pq| + \epsilon$,

(b) $\measuredangle qpq' \le \theta$,

then $\quad |qq'| \le 2|pq|\sin\frac{\theta}{2} + \epsilon. \quad (4)$

Proof. pq' can be obtained from pq, as shown in Figure 10, by a composition $\rho \circ \tau$ of

(1) rotation ρ around p by an angle less than θ,

(2) homothety τ with center p and ratio λ such that

$$\frac{l-\epsilon}{l} \le \lambda \le \frac{l+\epsilon}{l}.$$

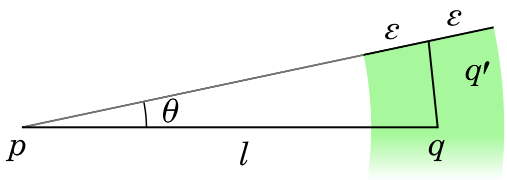


Figure 10: After a segment is rotated by at most θ and stretched by at most ϵ, its endpoint q moves by at most $l \cdot 2\sin\frac{\theta}{2} + \epsilon$.

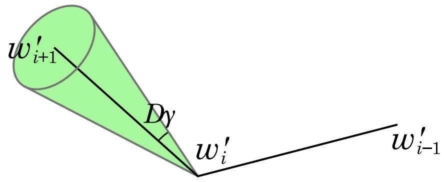


Figure 11: The angle between the edge of $\mathcal{P}$ and the edge of P is (locally) less than $\mathcal{D}\eta$.

In the first place, it is clear that

$$\operatorname{dist}(\rho(q),\ \tau(\rho(q))) \ \leq\ \epsilon.$$

Now we have to estimate $\operatorname{dist}(q, \rho(q))$. It is not more than $l \cdot 2\sin(\theta/2)$, which is the length of base of an isosceles triangle with sides equal to l and angle at the apex θ.

Combining estimations above and the triangle inequality concludes the proof.

□

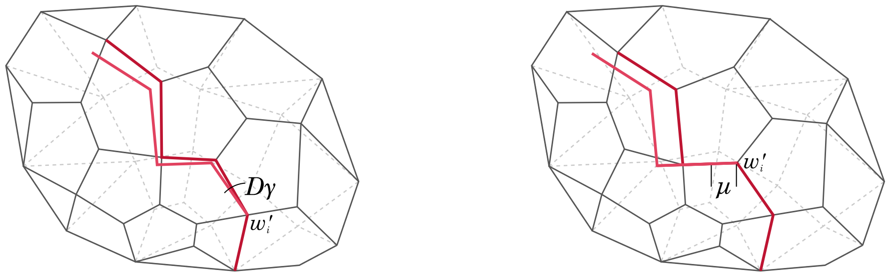
To find r we carry out the following procedure:

- Consider a vertex v of $\mathcal{P}$ and the shortest path $v_1 w_1 w_2 \ldots w_k v$ from v_1 to v in the graph structure of $\mathcal{P}$—it is comprised of edges of $\mathcal{P}$ and is not the geodesic shortest path from v_1 to v;
- Using Lemma 17, calculate the distance v moves when small changes are applied to edge lengths of this path and angles between its edges. This distance does not exceed the sum of the offsets generated by changes of a single edge or a single angle (see Figures 12a, 12b).

Lemma 18. *Let $p_0 \ldots p_m$ be a sequence of m edges sharing an endpoint such that* $\max |p_{i-1}p_i| \leq L$ *for some real L. Then, for any j from 0 to $m-1$, if the portion $p_j \ldots p_m$ of the path is rotated around p_j by an angle θ (such a rotation Φ is applied to it that $\measuredangle p_{j+1}p_j\Phi(p_{j+1}) = \theta$), then* $|p_m\Phi(p_m)| \leq 2mL\sin(\theta/2)$. *It can be said that p_m moves by at most this value.*

Proof. Apply Lemma 17 with $p = p_j$, $q = p_m$, q' being the image of p_m after rotation, $\epsilon = 0$. □

Proof. *(of Theorem 16)* Let us consider the path $u_1 w'_1 w'_2 \ldots w'_k u$ in the polyhedron P whose vertices correspond to the vertices of $v_1 w_1 w_2 \ldots w_k v$ in $\mathcal{P}$. It contains at most E edges and therefore its total length is at most EL.



(a) (b)

Figure 12: Illustration for the proof of Theorem 16: *(a)* Rotation by the angle less than $\mathcal{D}\gamma$ is applied to the residual path $w_i' \dots w_k'$. *(b)* The edge $w_i'w_{i+1}'$ is being lengthened or shortened by not more than μ.

Consider the vertex w_i' of the path. After we apply a rigid transformation to P and $\mathcal{P}$ so that w_i' coincides with w_i and edges $w_{i-1}'w_i'$, $w_{i-1}w_i$ lie on the same ray, then, by the triangle inequality, the angle between $w_i'w_{i+1}'$ and w_iw_{i+1} will not exceed $\mathcal{D}\gamma$.

Thus the region the edge w_iw_{i+1} of the polyhedron $\mathcal{P}$ can lie inside is a cone: the value of the angle between w_iw_{i+1} and $w_i'w_{i+1}'$ is at most $\mathcal{D}\gamma$ (see Figure 11).

Let us now make all the edges of the path in P parallel to the corresponding edges of the path in $\mathcal{P}$. We can do that by taking residual paths starting at w_1', then at w_2', ... and applying rotations to them, so that the entire path stays connected and its origin is intact (see Figure 12a). We can also preserve graph structure during the transformation.

By Lemma 18, every time we apply such rotation, the endpoint u of the path moves by at most

$$EL \cdot 2\sin(\mathcal{D}\gamma/2).$$

Since there are at most E vertices in the path and E rotations are applied, thus the endpoint u moves by at most

$$E^2 \cdot L \cdot 2\sin\left(\frac{\mathcal{D}\gamma}{2}\right). \tag{5}$$

Now that the directions of all the edges in the path are adjusted, we can make their lengths match the lengths of the corresponding edges in $\mathcal{P}$.

If the length of a single edge of a path in P is changed by at most μ, and other edges are not changed (as shown in Figure 12b), then the end of the path also moves by not more than μ. If we stretch any edge by at most μ, the total offset generated by that will be at most $E \cdot \mu$. (6)

Combining (5) and (6) implies that the total offset does not exceed

$$E^2 \cdot L \cdot 2\sin(\mathcal{D}\gamma/2) + E\mu. \tag{7}$$

This completes the proof.

□

2.3 Realisation on *Haskell*

Consider two adjacent faces pqr, sqr of the polyhedron P. Vertices of $\mathcal{P}$ corresponding to p, q, r, s can not lie on a common plane, if there does not exist a plane that intersects all four r–balls centered at p, q, r, s. If this is the case we can find out which of two edges—corresponding to qr or to ps—must be in the polyhedron $\mathcal{P}$. This can be done by finding out to which side the point s lies from the plane pqr.

Assume that the plane pqr is not vertical and that P lies below that plane (otherwise apply a rigid transformation to P so that it becomes true). Consider three planes Π_1, Π_2, Π_3 tangent to $B_r(p)$, $B_r(q)$, $B_r(r)$ such that:

- Π_1 is below $B_r(q)$, $B_r(r)$ and above $B_r(p)$,
- Π_2 is below $B_r(q)$ and above $B_r(r)$, $B_r(p)$,
- Π_3 is below $B_r(r)$ and above $B_r(q)$, $B_r(p)$.

Theorem 19. *If s lies below Π_1, Π_2 and Π_3 and the distance from s to each of the planes Π_1, Π_2 and Π_3 is greater than r, then there must be the edge qr in $\mathcal{P}$.*

Thus it is enough to implement the procedure that checks the condition of the theorem 19. In the first place we need to make sure that the plane pqr is not vertical. The most convenient way to do so is to make this plane horizontal by applying such an isometry of the space $\mathbb{R}^3$ to p, q, r, s that

1) p, q, r lie in the horizontal plane $z = 0$,

2) q is the origin,

3) $x_p < 0$ и $y_r > 0$.

It is easy to put p, q, r on the horizontal plane.

$$q' = (0, 0, 0) \tag{8}$$
$$r' = (0, |qr|, 0) \tag{9}$$
$$p' = (-|qp| \cdot \sin(\measuredangle pqr),\ |qp| \cdot \cos(\measuredangle pqr),\ 0) \tag{10}$$

Let us find the position of point s now. In order to do this we need to express the height of an arbitrary triangle pqr as a vector in $\mathbb{R}^3$ with endpoint p.We also need the expression for the projection of qp onto qr:

```
vectorProj p q r = (\a (x,y,z) -> (a*x , a*y , a*z))
  ((dotProd qr qp) / (d*d))
  qr
    where qp = triminus p q
          qr = triminus r q
          d = distance q r

vectorHeight p q r = triminus qp (vectorProj p q r)
  where qp = triminus p q
        qr = triminus r q
```

The angle between planes pqr and sqr is the angle between the heights of the corresponding triangles. Knowing that, we can express the coordinates of s':

```
fourthPoint p q r s = (-1 * l * c , pr , sig * l * sqrt(1-c*c))
  where l = fourthPointLeg p q r s
        c = planeAngle p q r s
        pr = fourthPointProjection p q r s
        sig = (-1) * signum (detOrder p q r s)
```

Here `detOrder p q r s` is the determinant

$$\det \begin{pmatrix} p-q & r-q & s-q \end{pmatrix}. \tag{11}$$

Now that three points lie in the horizontal plane, it is easy to construct three planes from Theorem 19 and check its statement.

```
firstPlane alpha p q r s = wellBelow alpha xs zs (xp + alpha) (zp + alpha)
  (-1 * alpha) (-1 * alpha)
    where xs = fsth $ fourthPoint p q r s
          zs = trdh $ fourthPoint p q r s
          xp = fsth $ firstPoint p q r s
          zp = trdh $ firstPoint p q r s

theresAnEdge alpha p q r s = (firstPlane alpha p q r s)
  && (secndPlane alpha p q r s)
  && (thirdPlane alpha p q r s)
```

3 Geometric methods to determine graph structure

The main goal of this work is to establish the graph structure of polyhedra that can be obtained as edge-to-edge gluings of regular pentagons. The procedure of Section 2 applied to a good enough approximation of the polyhedron helps us to confirm that a certain edge is present in it. Still, if such edge is not present, the procedure will not *find* it for any approximation, but we need to *prove* that the edge is not there.

In this section we:

- Prove that there are quadrilateral faces in polyhedrons 4-2, 4-3;
- Give an alternative geometric proof for the existence of an edge in polyhedron 6. This proof is specific for shape 6, so our automatic procedure remains to be the best way to establish that there is an edge between two vertices of a glued polyhedron.

3.1 Quadrilateral faces of Shape 4-2

Shape 4-2 (see Figures 13–14) is obtained by gluing four regular pentagons edge-to-edge. In this section we prove the following

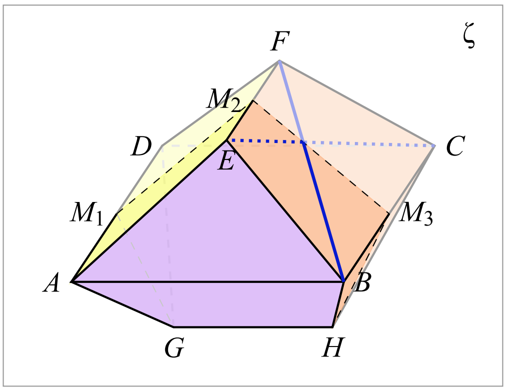


Figure 13: Shape 4-2 is symmetric with respect to 2 vertical planes, which yields four quadrilateral faces.

Theorem 20. *The polyhedron 4-3 has four faces each of which is a trapezoid:*

$$GHCD,\ ABHG,\ EFCB,\ ADFE.$$

Other four faces are isosceles triangles:

$$FDC,\ EBA,\ HBC,\ GDA.$$

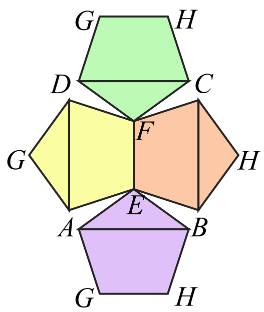


Figure 14: The net of Shape 4-2.

Proof. There are two vertical planes such that Shape 4-2 is symmetric with respect to both of them.

One of these planes, ζ, is shown in Figure 13: it passes through GH, a common side of two pentagons, and M_1, M_2, M_3, the midpoints of edges AD, EF, BC respectively. The reason the shape is symmetrical is that the segment HM_2 cuts the orange pentagon in half, and so the segment GM_2 does with the yellow pentagon.

The other plane passes through E, F, and midpoints of AB, GH and DC.

So, if, for example, BF is an edge of the polyhedron, then the segment EC must also be an edge, since these segments are symmetric with respect to ζ. It implies there must be a vertex of the polyhedron in the interior of a regular pentagon where EC and BF intersect, which can not occur. That means we found one quadrilateral face.

Three other quadrilateral faces can be found the same way, which completes the proof.

□

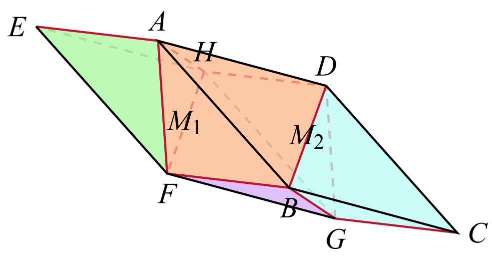


Figure 15: Shape 4-3 is symmetric with respect to the plane $EACG$. There are six faces which are all parallelograms.

3.2 Quadrilateral faces of Shape 4-3

Shape 4-3 (see Figures 15–16) is obtained by gluing four regular pentagons edge-to-edge. Each of vertices C and E is surrounded by a single pentagon folded along two of its diagonals. Our aim in this section is to prove the following.

Theorem 21. *The polyhedron 4-3 has six faces each of which is a parallelogram:*

$$EABF,\ EADH,\ CGFB,\ CGHD,\ ABCD,\ EFGH.$$

Thus it is a parallelepiped.

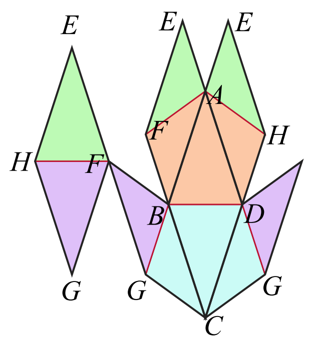


Figure 16: The net of Shape 4-3.

Proof. The pentagon $EAFHA$ is folded along its diagonals EF and EH and glued along its edge EA. The shape glued this way is symmetric with respect to the plane EAM_1, where M_1 is the midpoint of HF.

Let us now take another pentagon $AFBDH$ and glue one of its vertices to A. Place this pentagon in a way that the plane ADB is parallel to the plane EHF. To prove that it is possible to glue the two pentagons along the edges AF and AH without changing the position of the triangle ADB it suffices to prove that

$$\measuredangle FEA + \measuredangle EAF + \measuredangle FAB = 180^\circ.$$

Indeed, $\measuredangle FAB = \measuredangle AFE$ is the angle between the diagonal of a pentagon and the side of it, therefore the sum above is equal to the sum of angles of the triangle EAF, which is clearly 180°.

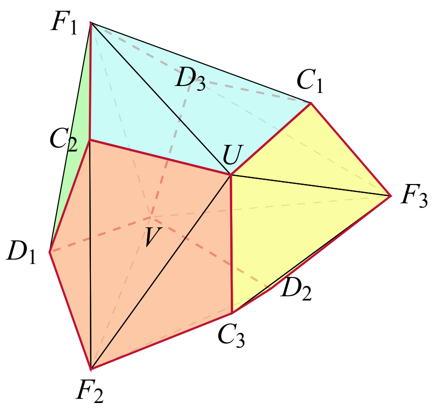


Figure 17: Shape 6.

So, $|AB| = |EF|$ and $AB \parallel EF$, which implies that $EABF$ is a parallelogram, and so are $EADH$, $CGFB$, and $CGHD$. Also, the shape obtained by gluing the pentagons $EAFHA$ and $AHDBF$ is still symmetric with respect to the plane EAM_1, and the planes EHF and ADB are parallel.

Now let us observe that $HDBF$ is a square. Indeed, all of its sides are equal as sides of a regular pentagon, and it has an axis of symmetry passing through the midpoints M_1 and M_2 of its opposite sides. Now if we glue the two halves of the shape along this common square, the triangles CDB and ADB will be coplanar, since

$$\measuredangle CM_2M_1 = \measuredangle EM_1M_2 \quad \text{and} \quad \measuredangle EM_1M_2 + \measuredangle M_1M_2A = 180^\circ.$$

Since $AD = DC = CB = BA$ as diagonals of a regular pentagon, $ADCB$ is a rhombus, and so is $EHGF$. Now the proof is complete.

□

3.3 There is an edge: Shape 6

Let us consider *Shape 6* (shown in Figure 17): it consists of 6 regular pentagons glued edge-to-edge. In the first place it is obvious that this polyhedron is preserved by a 120° rotation around vertical axis connecting vertices U and V.

To construct this polyhedron we have to at first divide six pentagons we have into two sets of three pentagons each, as shown on Figure 18. Three pentagons in each set are then glued in the only edge-to-edge way around vertices U and V. After this it only remains to glue the boundaries of two "caps" we just obtained along the broken line $F_1D_3C_1F_3D_2C_3F_2D_1C_2$.

Theorem 22. *$UF_1 \ldots UF_3$, $VF_1 \ldots VF_3$ are edges of Shape 6.*

Proof. Suppose that the length of edges of the pentagons is equal to 1. The quadrilateral $UC_2F_1C_1$ can be either

1) divided by an edge C_1C_2 connecting two closest vertices,

2) divided by an edge UF_1,

3) preserved flat without any edges.

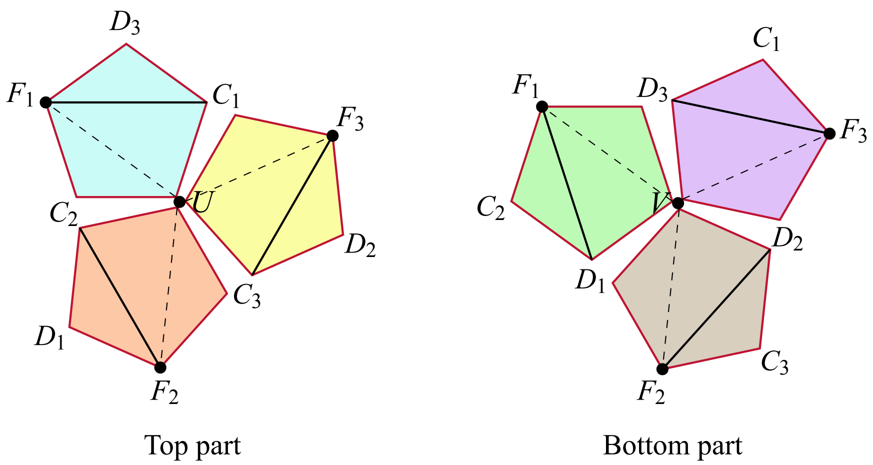


Figure 18: U is the *uppermost* vertex, $C_1 \dots C_3$ are *closest* vertices, $F_1 \dots F_3$ are *outermost* vertices. We are now aiming at finding out if the dashed lines are edges of the polyhedron.

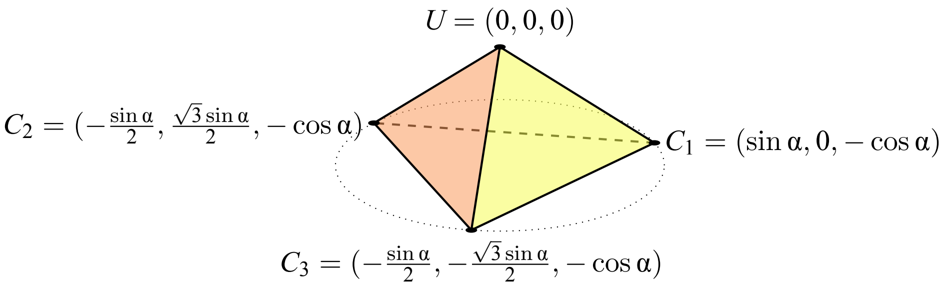


Figure 19: Coordinates of vertices U, $C_1 \dots C_3$. The lengths of C_1C_2, C_2C_3, C_3C_1 must be equal to $\left(1 + \sqrt{5}\right) / 2$.

We will prove that neither case 1 nor case 3 can hold. In these cases the triangle C_1UC_2 is flat, thus $\measuredangle C_1UC_2 = 108°$ (the angle of a regular pentagon) and $|C_1C_2| = (1 + \sqrt{5})/2$, same for triangles C_2UC_3 and C_3UC_1.

We can then find exact positions of C_1, C_2, C_3. We can place U at the origin $(0, 0, 0)$ and C_1 on the plane $y = 0$ in a way that

1) the axis UV is vertical,

2) the equilateral triangle $C_1C_2C_3$ is horisontal (since the vertices C_2, C_3 can be obtained from C_1 by rotation around UV).

Since $|UC_1| = 1$ and $y_{C_1} = 0$ we can find such an angle $0° \leq \alpha < 90°$ that $C_1 = (\sin\alpha, 0, -\cos\alpha)$ (see Figure 19). Applying to C_1 consequent $120°$ rotations around z axis we can get the coordinates of C_2 and C_3. Knowing that $\measuredangle C_1UC_2 = 108°$, we can find α.

We already know that

$$\frac{1 - \sqrt{5}}{4} = \cos\left(108°\right) = \langle UC_1, UC_2 \rangle . \tag{12}$$

The right-hand part of this equation can be expressed in terms of α:

$$\begin{aligned}\langle UC_1, UC_2\rangle = \sin\alpha \cdot \left(-\frac{\sin\alpha}{2}\right) + 0 \cdot \frac{\sqrt{3}\sin\alpha}{2} + \cos^2\alpha &= \\ = -\frac{\sin^2\alpha}{2} + \cos^2\alpha = \frac{3\cdot\cos^2\alpha - 1}{2} &= \\ = \frac{2 - 3\sin^2\alpha}{2}&.\end{aligned}$$

Substituting this into (12) we get:

$$\begin{aligned}\frac{2-3\sin^2\alpha}{2} &= \frac{1-\sqrt{5}}{4}\\ 6\cdot\sin^2\alpha &= 3+\sqrt{5}\\ \sin\alpha &= \sqrt{\frac{3+\sqrt{5}}{6}}.\end{aligned}$$

To check ourselves, let us calculate C_1C_2:

$$C_1C_2 = \sin(\alpha)\cdot 2\sin(120^\circ/2) = \sqrt{\frac{3+\sqrt{5}}{6}}\cdot 2\cdot\frac{\sqrt{3}}{2} = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

This is equal to $\frac{1+\sqrt{5}}{2}$:

$$\frac{1+\sqrt{5}}{2} = \sqrt{\frac{6+2\sqrt{5}}{4}} = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

Now that we know $\sin\alpha$, we can calculate $\cos\alpha$:

$$\cos\alpha = \sqrt{\frac{3-\sqrt{5}}{2}}.$$

So far we calculated the coordinates of C_1 and C_2 in both cases 1 and 3:

$$C_1 = \left(\sqrt{\frac{3+\sqrt{5}}{6}},\ 0,\ -\sqrt{\frac{3-\sqrt{5}}{6}}\right),$$

$$C_2 = \left(-\sqrt{\frac{3+\sqrt{5}}{24}},\ \sqrt{\frac{3\cdot\left(3+\sqrt{5}\right)}{24}},\ -\sqrt{\frac{3-\sqrt{5}}{6}}\right).$$

Now let us consider the pentagon $C_1UC_2F_1D_3$. The interection of diagonals of a regular pentagon divides them with the ratio of

$$\xi = \left(\frac{1}{2\cdot\cos(36^\circ)}\right)^2 = \frac{2}{3+\sqrt{5}}.$$

It means that (see Figure 20) in cases 1 and 3, when the triangle C_1UC_2 is flat,

$$UX = \xi\cdot UC_1 + (1-\xi)\cdot UC_2.$$

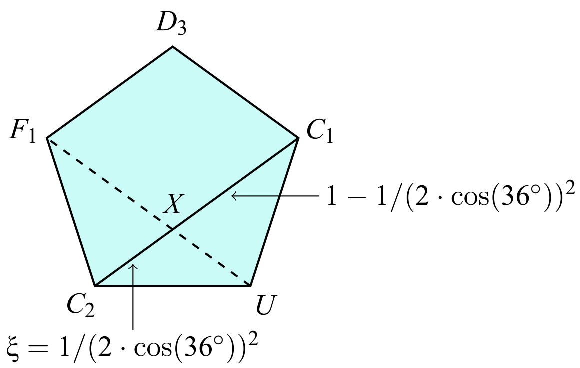


Figure 20: The ratio the diagonals are divided with.

Knowing that, we can find the coordinates of X by calculating this convex combination:

$$(1-\xi:) \qquad 1 - \frac{2}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{3+\sqrt{5}}.$$

$$(x \text{ coordinate:}) \qquad \frac{1}{\sqrt{3+\sqrt{5}}} \cdot \left(\frac{2}{\sqrt{6}} - \frac{1+\sqrt{5}}{\sqrt{24}} \right) =$$

$$= \frac{3-\sqrt{5}}{\sqrt{24}\sqrt{3+\sqrt{5}}} = \frac{1}{\sqrt{12}} \cdot \frac{3-\sqrt{5}}{1+\sqrt{5}}.$$

$$(y \text{ coordinate:}) \qquad \frac{1}{\sqrt{3+\sqrt{5}}} \cdot \sqrt{\frac{1}{8}} \cdot \left(1+\sqrt{5}\right) = \frac{1}{2}.$$

$$(z \text{ coordinate:}) \qquad \text{The same — } -\sqrt{\frac{3-\sqrt{5}}{6}}.$$

In case 3, when the quadrilateral $UC_2F_1C_1$ is flat,

$$UF_1 = \frac{1}{\xi} UX.$$

In case 1, when C_1C_2 is an edge, F_1 lies below the plane C_1UC_2 (since the polyhedron must be convex). Thus, in both cases

$$z_{F_1} \leq \frac{1}{\xi} \cdot z_X = -\frac{3+\sqrt{5}}{2} \cdot \sqrt{\frac{3-\sqrt{5}}{6}}.$$

The same for the remaining outermost vertices F_2 and F_3. Since the caps obtained from two sets of pentagons (see Figure 18) are congruent, we can estimate z_V:

$$z_V = 2 \cdot z_{F_1} \leq -\left(3+\sqrt{5}\right) \cdot \sqrt{\frac{3-\sqrt{5}}{6}}.$$

And z_{D_1}, which must be equal to $z_V - z_{C_1}$:

$$z_{D_1} \leq -\left(3+\sqrt{5}\right) \cdot \sqrt{\frac{3-\sqrt{5}}{6}} + \sqrt{\frac{3-\sqrt{5}}{6}} = -\left(2+\sqrt{5}\right) \cdot \sqrt{\frac{3-\sqrt{5}}{6}}.$$

Now note that (in cases 1 and 3:

$$|C_2D_1| \geq z_{C_2} - z_{D_1} \geq \left(1+\sqrt{5}\right) \cdot \sqrt{\frac{3-\sqrt{5}}{6}} =$$

$$= \frac{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)}{\sqrt{6}\sqrt{2}} > 1.$$

Which is a contradiction because $|C_2D_1| = 1$, as a side of a regular pentagon. Thus, only case 2 is possible: UF_1 is an edge. The same reasoning can be carried out for UF_2, UF_3, $VF_1 \ldots VF_3$. The theorem is proved.

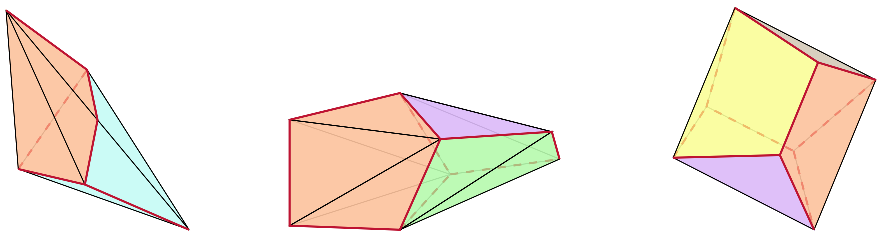
□

3.4 A complete list of all shapes obtained by gluing pentagons

Below is the list of all polyhedra that can be obtained by gluing regular pentagons. The list of all possible gluings is given in [2], thus we can run the program from [15] for each of them and obtain approximate polyhedra for the polyhedra we are seeking. Having these polyhedra, we can apply theorems we just proved to them.

For those polyhedra that are simplicial, their graph structure is confirmed by applyng method of Section 2, for the others the proof is geometric and is done in Section 3.

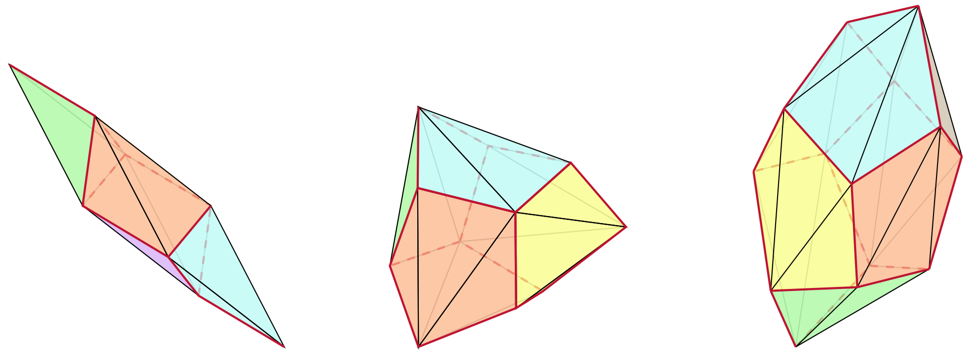
- 2 pentagons:
  - doubly-covered regular pentagon (we do not dedicate a figure to it, as it is widely known).
  - simplicial hexahedron with 5 vertices (3 vertices of degree 4, and 2 vertices of degree 3), see Figure 21a.



(a) (b) (c)

Figure 21: Simpler polyhedra

- 4 pentagons:
  - simplicial dodecahedron with 8 vertices (2 vertices of degree 5 and 6 vertices of degree 4), see Figure 21b.
  - octohedron with 8 vertices (4 vertices of degree 4 and 4 vertices of degree 3) and 4 quadrilateral and 4 triangular faces. It is a truncated biprism, see Figure 21c.



(a) (b) (c)

Figure 22: More complex polyhedra

  – hexahedron with 8 vertices each of degree 3 and 6 quadrilateral faces. This is a parallelepiped, see Figure 22a.

- 6 pentagons: simplicial decaoctohedron (18-hedron) with 11 vertices (5 vertices of degree 6, 6 vertices of degree 4), see Figure 22b.
- 8 pentagons: simplicial icositetrahedron (24-hedron) with 14 vertices (2 vertices of degree 6, 12 vertices of degree 5), see Figure 22c.
- 12 pentagons: regular dodecahedron with 20 vertices of degree 3 and 12 pentagonal faces (we do not dedicate a figure to it, as it is widely known).

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