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Enumerating All Convex Polyhedra Glued from Squares in Polynomial Time

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Abstract

We present a polynomial-time algorithm that allows to enumerate and classify all edge-to-edge gluings of squares that correspond to convex polyhedra. We show that the number of such gluings of n squares is polynomial in n. The methods we use to achieve this can be applied in several similar settings, including gluings of regular hexagons and triangles.

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1 Introduction

Given a collection of 2D polygons, a *gluing* or a *net* describes a closed surface by specifying how to glue each edge of these polygons onto another edge. Alexandrov's uniqueness theorem [1] states that any valid gluing that is homeomorphic to a sphere and that does not yield a total facial angle greater than 2π at any point, corresponds to the surface of a unique convex 3D polyhedron (doubly covered convex polygons are also polyhedra).

There is no known exact algorithm for reconstructing the 3D polyhedron [8, 9]. Enumerating all possible valid gluings is also not an easy task, as the number of gluings can be exponential even for a single polygon [5]. Complete enumerations of gluings and the resulting polyhedra are only known for very specific cases such as the Latin cross [6] and a single regular convex polygon [7].

The special case when the polygons to be glued together are all identical regular k-gons, and the gluing is *edge-to-edge* was studied recently for $k \geq 6$ [3] and $k = 5$ [2]. The aim of this paper is to study the case of $k = 4$: namely, to *enumerate* all valid gluings of squares and *classify* them up to isomorphism.

2 Chen—Han algorithm for gluings of squares

In [7] it is shown that polyhedra are isomorphic if the lengths of shortest paths between their vertices of nonzero curvature coincide. Thus, the problem of finding out if two gluings are isomorphic can be reduced to finding out the distances between vertices of a gluing.

Algorithm we are using for this is Chen—Han algorithm presented in [4]. The idea of the algorithm is to project a cone of all possible paths from the source onto the surface of the gluing. This algorithm runs in $O(n^2)$ time. It can be used when the polyhedron is cut into arbitrary flat parts. However, additional observations have to be made to show that the running time is preserved. In this section we make these for edge-to-edge gluings of squares that need not to be faces of the resulting polyhedron.

▶ **Theorem 1.** *If T is a square of the gluing and π is a geodesic shortest path between two vertices p_1, p_2 of the gluing then the intersection between π and T consists of at most 5 segments.*

Proof. To prove this theorem, we show that there is at most one segment for each pair of adjacent sides of T with endpoints on these sides, and at most one segment whose endpoints lie on opposite sides of T. ◀

▶ **Corollary 2.** *For an arbitrary gluing Γ of n squares and a vertex v of this gluing, Chen—Han algorithm for Γ with source v runs in $O(n^2)$ time.*

3 Bounds on the number of egde-to-edge gluings of squares

In this section, we prove that the number of edge-to-edge gluings of n squares is polynomial in n. This result allows to develop a polynomial algorithm to list all the valid gluings.

▶ **Theorem 3.** *There are $O\left(n^{36}\right)$ edge-to-edge gluings of at most n squares that satisfy Alexandrov's conditions.*

Proof. Triangulate the polyhedron corresponding to the net and draw its faces on the square grid. Note that an edge shared by two faces must have the same lengths of x- and y-projections on the drawings of these faces, see Figure 1. Then count the number of sets of triangles satisfying this restriction and taking up at most n squares.

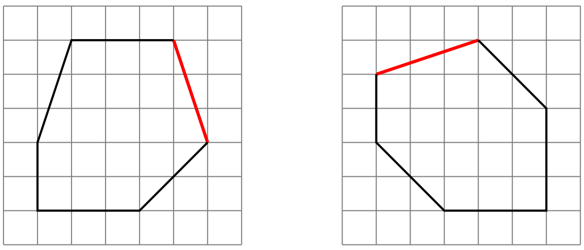


Figure 1 Highlighted edge has the same lengths of projections on the drawings of two faces

To do so, choose x- and y-projections (those do not exceed n in length) for each of at most 18 edges and note that there is at most two ways to place each edge such that the convexity of the face is preserved, those differ by $\frac{\pi}{2}$, see Figure 2. ◀

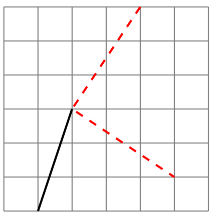
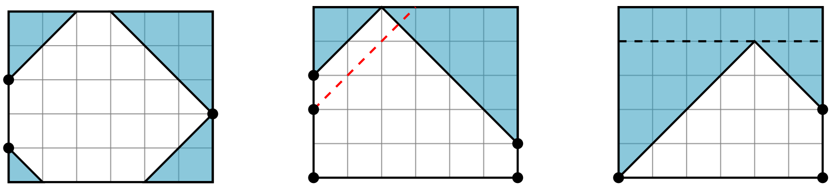


Figure 2 There are two ways to place each edge preserving convexity of the face

▶ **Theorem 4.** *There are* $\Omega\left(n^{\frac{5}{2}}\right)$ *edge-to-edge gluings of at most* n *squares that satisfy Alexandrov's conditions.*

Proof. To prove the theorem, we construct a series of such gluings. These gluings correspond to doubly-covered polygons, the polygons being obtained by cutting edges of a rectangle with sides no longer than $\frac{\sqrt{n}}{2}$, see Figure 3a.



(a) **(b)** **(c)**

Figure 3 (a) An example of a polygon produced by cutting angles of a rectangle. (b) Some pairs of points on short sides do not produce valid polygons. (c) A polygon can be obtained by cutting angles of several different rectangles.

If the length of the shorter side of the rectangle is equal to b, then there is $(b(b+1)/2)^2$ ways to choose how much of angles is cut. However, to count only valid gluings, we have to count in that in some cases the sides of angles we cut do not meet at a node of the grid, see Figure 3b, and that one polygon can be obtained from several rectangles, see Figure 3c. This yields the final formula. ◀

4 Enumerating the gluings

We implemented an algorithm that enumerates all the gluings of at most n squares for a given graph structure of a convex polyhedron. Note that one gluing can admit several ways to cut itself into flat polygons, see Figure 4. Thus it can appear in the list several times.

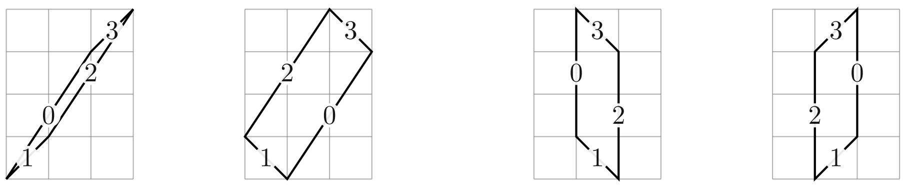


Figure 4 Doubly covered parallelogram can be cut into two flat quadrilaterals in two ways, the latter consisting of its faces

5 Algorithm to classify edge-to-edge gluings of squares

The algorithm is presented in Figure 5. Due to Theorem 3, generating $L(n)$ takes polynomial time, and $L(n)$ contains polynomial number of instances. Due to Theorem 1, it takes $O(n^2)$ time to compute $(M(n))_i$ and $O(n^3)$ time to compute $M(n)$. Since $M(n)$ contains matrices with at most 8 rows and 8 columns, it takes polynomial time to unicalize $M(n)$, which results in enumerating all non-isomorphic edge-to-edge gluings of at most n squares.

$L(n)$ = list of all edge-to-edge gluings of at most n squares
$M(n) = [\,]$ — list of matrices of pairwise distances between vertices of polyhedra
for all $\Gamma \in L(n)$ **do**
    for all v_i — vertices of Γ of nonzero curvature **do**
        $(M(n))_i = \text{Chen-Han}(\Gamma, v_i)$
    end for
end for
Unicalize $M(n)$ up to homothety and permutation of rows and columns, leave only corresponding elements of $L(n)$

Figure 5 Enumerating non-isomorphic edge-to-edge gluings of squares

6 Discussion

The cornerstone of the technique we have been using is the possibility to draw a face of a polyhedron glued from squares on a planar grid. It allows us to estimate the number of valid gluings. The same technique can seemingly be applied for the cases of regular hexagons and triangles, since these polygons also form a planar grid.

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