Geodesics, Shortest Paths, and Non-Isomorphic Nets

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1 Introduction

Given a collection of 2D polygons, a *gluing* or a *net* describes a closed surface by specifying how to glue (a part of) each edge of these polygons onto (a part of) another edge. Alexandrov's uniqueness theorem [1] states that any valid gluing that is homeomorphic to a sphere and that does not yield a total facial angle greater than 2π at any point, corresponds to the surface of a unique convex 3D polyhedron (doubly covered convex polygons are also regarded as polyhedra). Note that the original polygonal pieces might need to be folded to obtain this 3D surface.

Unfortunately, the proof of Alexandrov's theorem is highly non-constructive. The only known approximation algorithm to find the vertices of this polyhedron [9] has a (pseudopolynomial) running time really large in n, where n is the total complexity of the gluing.

There is no known exact algorithm for reconstructing the 3D polyhedron, and in fact the coordinates of the vertices of the polyhedron might not even be expressible as a closed formula [8].

Enumerating all possible valid gluings is also not an easy task, as the number of gluings can be exponential even for a single polygon [5]. However one valid gluing can be found in polynomial time using dynamic programming [7, 10]. Complete enumerations of gluings and the resulting polyhedra are only known for very specific cases such as the Latin cross [6] and a single regular convex polygon [7].

The special case when the polygons to be glued together are all identical regular k-gons, and the gluing is *edge-to-edge* was studied recently for $k \geq 6$ [3] and $k = 5$ [2]. The aim of this project is to study the case of $k = 4$: namely, to *enumerate* all valid gluings of squares and *classify* them up to isomorphism.

2 Methods

2.1 Interpreting and listing valid gluings

Due to Alexandrov's theorem, for every gluing of squares satisfying Alexandrov's conditions there is a convex polyhedron corresponding to it. It is possible to draw each of its faces on square grid, the vertices of the face being also the vertices of the grid.

Due to Gauss—Bonnet formula, the number of vertices of the polyhedron is at most 8, which yields that the number of faces is at most 12. When there is a constant number of faces and edges, any kind of check can be carried for them in a short matter of time.

Square grid gives a convenient way to represent gluing as a set of faces draw on the grid, it is easy to check if Alexandrov's conditions are satisfied for a gluing if it is represented as a set of planar faces on the grid. This representation allows for estimating the number of valid gluings and for development of an algortihm with polynomial running time that lists all the gluings, which will be done later in this paper.

We have to note however, that a gluing can comply with different nets, and with one net in several different ways. See Figure 9 for an example: the net consisting of two parallelograms clearly corresponds to a doubly covered rectangle 1×4. This is still our a-priori knowlegde. It is not possible to determine which net is exactly the net of the polyhedron corresponding to the gluing, because of non-constructiveness of Alexandrov's theorem.

2.2 Chen—Han algorithm

Chen—Han algorithm is presented in [4]. Estimation for the running time is proved in [11].

3 Bounds on the number of egde-to-edge gluings of squares

In this section, we prove that the number of edge-to-edge gluings of n squares is polynomial in n. Note that the layout of these theorems allow to develop a polynomial algorithm to list the nets, one just needs to follow all the cases that we describe, and also estimate how much output this algorithm will produce.

Theorem 1. *There are $O\left(n^{36}\right)$ edge-to-edge gluings of at most n squares that satisfy Alexandrov's conditions.*

Proof. Since we're considering gluings that satisfy Alexandrov's conditions, there is a polyhedron corresponding to each of them. Consider this polyhedron and triangulate it. Since this polyhedron is glued from squares, it is possible to draw each of its faces on square grid, the vertices of the face being also the vertices of the grid.

What is more, an edge shared by two faces must look the same on the drawings of these faces; an example can be seen in Figure 1. That means, it must have the same length and the tilt angle that is the same or differing by $\frac{\pi}{2}$.

The graph of the polyhedron is planar and has at most 8 vertices. Due to Euler's formula, it can have at most 18 edges. For each edge, we can choose its projections on x and y axes once it is drawn on the grid. Since the edge is a part of a flat face, all the squares that intersect the edge are distinct. There is at most n of them, which yields that both projections are at most n, so there is at most n^2 ways to choose the edge.

Once the projections of the edges are known, let us draw the faces on the grid. At every vertex, there is at most two ways to place the next edge, such that the convexity of the face is preserved, those differ by $\frac{\pi}{2}$, see Figure 2 for an example.

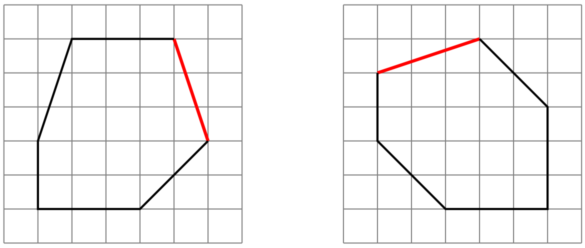


Figure 1: Highlighted edge looks the same on the drawings of two faces it belongs to

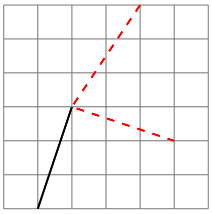


Figure 2: There are two ways to place each edge preserving convexity of the face

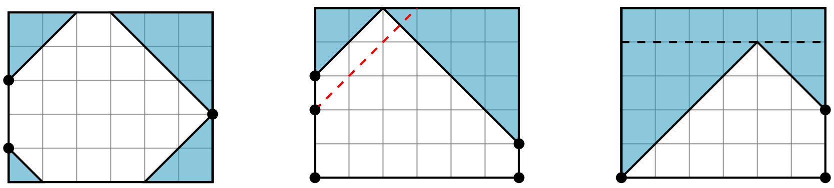
This adds at most $2^{2\cdot 18}$ ways to draw the net once the edges are known, which gives the total of at most $(2n)^{36}$ gluings. $\square$

Theorem 2. *There are* $\Omega\left(n^{\frac{5}{2}}\right)$ *edge-to-edge gluings of at most* n *squares that satisfy Alexandrov's conditions.*

Proof. We will construct a family of $\Omega\left(n^{\frac{5}{2}}\right)$ distinct polyhedra that can be glued from squares edge-to-edge. Those will be doubly-covered convex polygons whose edges are either parallel to the axes or inclined by $\frac{\pi}{4}$, and vertices are vertices of the grid.

Consider a square $\left\lfloor \frac{\sqrt{n}}{2} \right\rfloor \times \left\lfloor \frac{\sqrt{n}}{2} \right\rfloor$ drawn on the square grid. If a polygon is drawn inside this square, and it satisfies the conditions of the previous paragraph, then doubly covered version of this polygon can be glued from squares and consists of at most n squares.

The polygons will be constructed the following way: we will take a rectangle and cut it angles off. More formally, we will pick two points on either of short sides of the rectangle and shoot lines inclined by $\frac{\pi}{4}$ from them. The points can correspond, in this case there is one vertex of the polyhedron on the corresponding side of the rectangle. An example can be seen in Figure 3a: the points we picked are marked, and the areas that are cut off are highlighted.



(a) (b) (c)

Figure 3: (a) An example of a polygon produced by cutting angles of a rectangle. (b) Some pairs of points on short sides do not produce valid polygons. (c) A polygon can be obtained by cutting angles of several different rectangles.

Define the length of the longer side of the rectangle by a, and length of the shorter side by b. There number of ways to choose a pair of points on each of shorter sides is

$$\left(\frac{b(b+1)}{2}\right)^2.$$

However, some sets of points do not produce valid polygons, since the rays shot from them intersect not at a vertex of the grid. An example can be seen in Figure 3b: red dashed line intersects with the line from the right side in the middle of a cell. Still, at least half the pairs of upper points on the sides are valid: for each invalid pair, the pair where left point is raised by 1 is valid. We get that the number of ways to select a valid set of points is

$$\left(\frac{b(b+1)}{2}\right)^2 \Big/ 4.$$

Note that each polyhedron has at least one horizontal edge, since otherwise a must be less than b: the sum of segments cut off horizontal edges and the sum of segments cut off vertical edges are equal. However, on the other side there may not be a horizontal edge, see Figure 3c. In this case, it is not certain from which rectangle the polygon is cut: in Figure 3c it can be either 6×4 or 6×5.

However, one polygon can be cut from at most a rectangles, since we know its horizontal dimension, and at least one horizontal side is fixed. Thus, the number of rectangles we get for a given a is at least

$$\sum_{b=1}^{a} \frac{b^2(b+1)^2}{16a} = \Omega\left(a^4\right).$$

Recall that a can vary from 1 to $\sqrt{n}/2$, the number of polygons we can produce is

$$\Omega\left(\sum_{a=1}^{\sqrt{n}/2} a^4\right) = \Omega\left(n^{\frac{5}{2}}\right).$$

□

4 Description of practical results

4.1 Listing valid gluings

The listing procedure is given a net in advance in a form of DCEL. Then it processes this list to obtain the data that will be needed for the checks: how many vertices there are, to what faces a vertex belongs and at which position it is in that vertex.

After this initialisation stage the procedure takes arbitrary coordinates for each vertex in each face and performs the following checks in the following order:

1) Each face has at least two vertices on the border of the $n \times n$ square, this is to avoid repeated listing faces that differ by a shift.

2) All the turns took by the edges of a single face are in the positive direction, and the sum of the exterior angles is at most 2π, which corresponds to each face being a convex polygon.

3) If an edge belogs to two faces, than in those faces it has the same length and the tilt angle that is the same or differing by $\frac{\pi}{2}$.

4) Sum of angles at every vertex is at most 2π.

Theorem 3. *Running time of the implemented algorithm is $O(n^{72})$.*

Proof. Once the coordinates of each vertex of each face is known, the check if the gluing is valid takes $O(1)$ time. For every vertex we need to select two coordinates, which gives $O(n^2)$ per vertex per face.

Each vertex appears as many times as many faces it belongs to, the number of those faces is equal to the degree of the vertex. Thus, we get

$$O\left(n^{2\cdot\sum\limits_{v}\deg(v)}\right) = O\left(n^{4|E|}\right) = O\left(n^{72}\right).$$

$\square$

As an attempt to speed up the running time, the unicalization of the list of vertices was performed. Originally, if a vertex appears in d faces, it will appear in the list of vertices exactly d times.

Each vertex v is represented by a list $[i_1, x_{i_1}, i_1, x_{i_2} \ldots]$, where $i_1, i_2, \ldots$ are faces to which v belongs, and $x_{i_1}, x_{i_2}, \ldots$ are the positions of v in these faces respectively. The faces are traversed in clockwise order around v.

As a modification of the program, each list was rotated in a way that the face with the smallest index takes the first position. Then the duplicates were removed from the multi-dimensional list of vertices, the code for this exact procedure was sourced from `stackoverflow.com`.

After the program achieved representations of the nets in numerical form, the graphic output was configured for an user to see the polygons and the way they are glued. TikZ code was selected as a platform for the output, drawings of several nets can be seen in **Appendix**.

5 Discussion

Appendix: examples of valid nets

Nets consisting of two hexagons

The polygons in these nets represent ones that we counted while proving Theorem 2: those are rectangles with some angles cut.

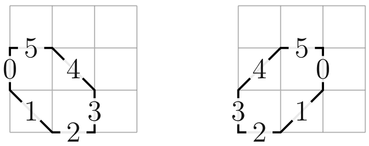


Figure 4: Net 0

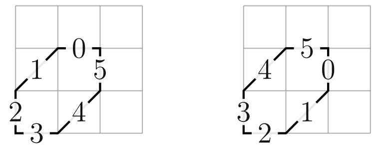


Figure 5: Net 1879605

Nets consisting of two quadrilaterals

Some rather surprising nets can be found here. While the net in Figure 6 clearly corresponds to a doubly-covered quadrilateral, for other nets it is completely unclear which polyhedron is obtain after gluing them together.

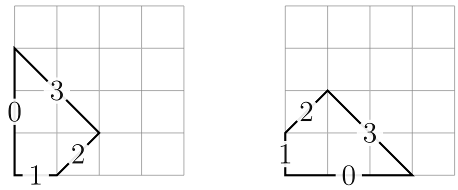


Figure 6: Net 49637490

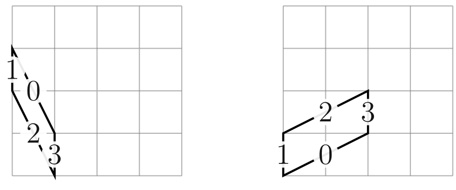


Figure 7: Net 60877856

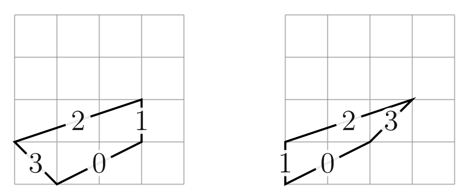


Figure 8: Net 70707676

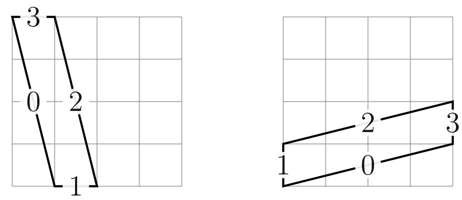


Figure 9: Net 81497520

Nets of a four-sided pyramid

We were trying to find a non-trivial way to glue a four-sided pyramid from squares, however the only polyhedron we could find complying with this net is a doubly-covered square, see Figure 12.

Net of a cube

Another net worth noting is an unusual net of cube shown in Figure 13. This way to glue the cube is rather non-trivial, and the resulting cube can be seen in Figure 14.

References

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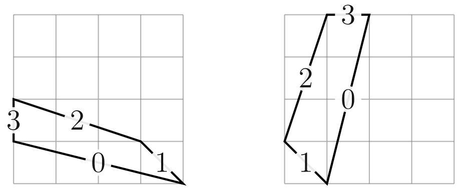

Figure 10: Net 103500700

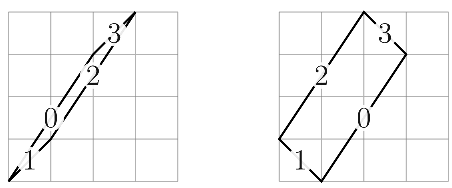

Figure 11: Net 111905612

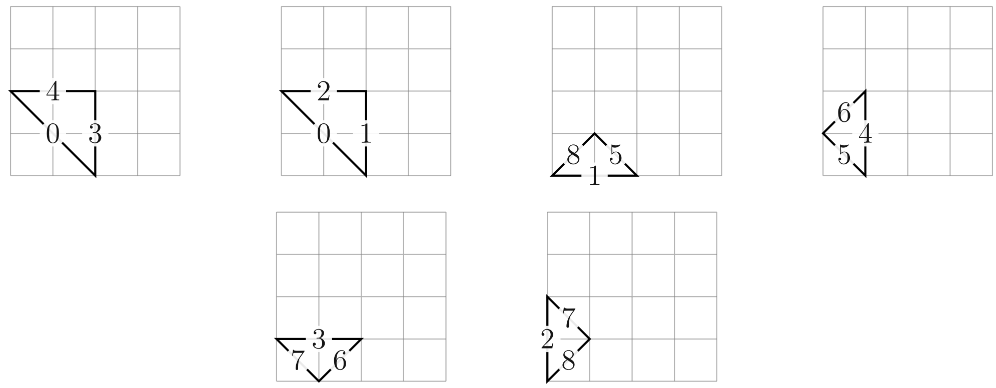

Figure 12: Net 15780252

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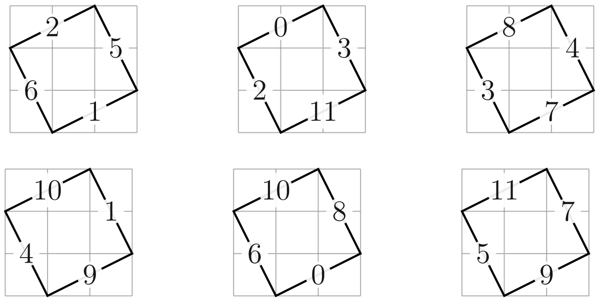

Figure 13: Net 22695

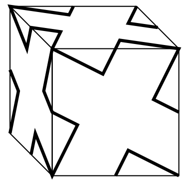

Figure 14: The way the cube is glued from the net on Figure 13

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