

# Enumerating All Convex Polyhedra Glued from Squares in Polynomial Time

Stefan Langerman  

Faculté des Sciences, Université Libre de Bruxelles

Nicolas Potvin  

Faculté des Sciences, Université Libre de Bruxelles

Boris Zolotov  

Department of Mathematics and Computer Sciences, St. Petersburg State University

## Abstract

We present an algorithm that enumerates and classifies all edge-to-edge gluings of unit squares that correspond to convex polyhedra. We show that the number of such gluings of  $n$  squares is polynomial in  $n$ , and the algorithm runs in time polynomial in  $n$  (pseudopolynomial if  $n$  is considered the only input). Our technique can be applied in several similar settings, including gluings of regular hexagons and triangles.

**2012 ACM Subject Classification** #10010061 Computational geometry

**Keywords and phrases** polyhedral metrics, alexandrov theorem, squares, edge-to-edge gluings

**Funding** *Stefan Langerman*: is Directeur de recherches du F.R.S.-FNRS

*Boris Zolotov*: is supported in part by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS” and in part by “Native towns”, a social investment program of PJSC “Gazprom Neft”.

**Acknowledgements** B. Z. wants to thank the international academic mobility & students exchange program of SPBU for offering him an exchange in ULB that allowed for cooperation needed to complete this project.

## 1 Introduction

Given a collection of 2D polygons, a *gluing* describes a closed surface by specifying how to glue each edge of these polygons onto another edge. Alexandrov’s uniqueness theorem [1] states that any valid gluing that is homeomorphic to a sphere and that does not yield a total facial angle greater than  $2\pi$  at any point, corresponds to the surface of a unique convex 3D polyhedron. The polygons of the gluing may be folded in order to glue the polyhedron.

There is no known exact algorithm for reconstructing the 3D polyhedron [8, 9]. Enumerating all possible valid gluings is also not an easy task, as the number of gluings can be exponential even for a single polygon [5]. Complete enumerations of gluings and the resulting polyhedra are only known for very specific cases such as the Latin cross [6], a single regular convex polygon [7], and a collection of regular pentagons [2].

The case when the polygons to be glued together are all identical regular  $k$ -gons, and the gluing is *edge-to-edge* was studied recently for  $k \geq 6$  [3]. The aim of this paper is to study the case of  $k = 4$ : namely, to *enumerate* all valid gluings of squares and *classify* them up to isomorphism.

## 2 Chen—Han algorithm for gluings of squares

In [7] it is shown that polyhedra are isomorphic if the lengths of shortest geodesic paths between their vertices of nonzero curvature coincide. Thus, the problem of finding out if two



© Stefan Langerman and Nicolas Potvin and Boris Zolotov;  
licensed under Creative Commons License CC-BY 4.0

Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

## XX:2 Convex Polyhedra Glued from Squares

gluings are isomorphic can be reduced to finding out the geodesic distances between vertices of a gluing. Algorithm we are using for this is the Chen—Han algorithm [4].

The idea of the algorithm is to project a cone of all possible paths from the source onto the surface of the gluing. This algorithm runs in  $O(n^2)$  time. To apply it for arbitrary edge-to-edge gluings of squares, it has to be proven that the running time is preserved. To do this, we prove the following Lemma.

► **Lemma 1.** *If  $T$  is a square of the gluing and  $\pi$  is a geodesic shortest path between two vertices of the gluing then the intersection between  $\pi$  and  $T$  is of at most 5 segments.*

The lemma implies the following Theorem.

► **Theorem 2.** *The isomorphism between two edge-to-edge gluings of at most  $n$  squares can be tested in  $O(n^2)$  time.*

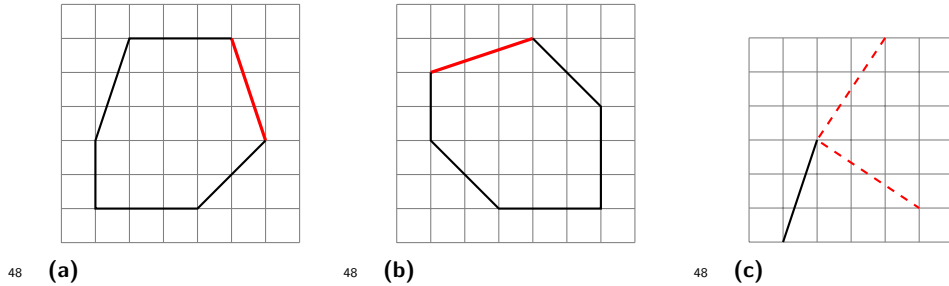
### 3 Bounds on the number of edge-to-edge gluings of squares

In this section, we prove that the number of edge-to-edge gluings of  $n$  squares is polynomial in  $n$ . This result allows to develop a polynomial algorithm to list all the gluings.

► **Theorem 3.** *There are  $O(n^{36})$  edge-to-edge gluings of at most  $n$  squares that correspond to convex polyhedra.*

**Proof.** Triangulate the polyhedron corresponding to the gluing and draw its faces on the square grid. By Gauss—Bonnet theorem, the polyhedron has no more than 8 vertices, and thus at most 18 edges. An edge shared by two faces must have the same lengths of  $x$ - and  $y$ -projections on the drawings of these faces, see Figures 1a, 1b.

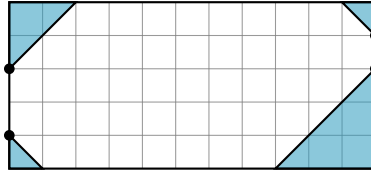
Count the number of sets of triangles satisfying this restriction and taking up at most  $n$  squares. To do so, choose the lengths of projections (that do not exceed  $n$  in length) for each of at most 18 edges. This yields the final formula. ◀



■ **Figure 1** (a), (b) Highlighted edge has the same lengths of projections on the drawings of two faces. (c) Two ways to place an edge with given projections that preserve convexity of the face.

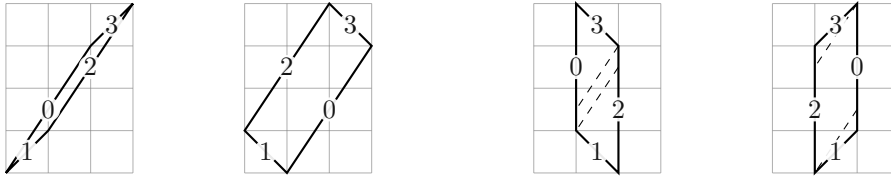
► **Theorem 4.** *There are  $\Omega(n^3)$  edge-to-edge gluings of at most  $n$  squares that correspond to convex polyhedra.*

**Proof.** To prove the theorem, we construct a series of such gluings. These gluings correspond to doubly-covered octagons, the octagons being obtained by cutting edges of a rectangle with sides no longer than  $\frac{\sqrt{n}}{2}$ , one at least twice as long as the other, see Figure 2. The bound is tight: there are  $O(n^3)$  doubly covered convex polygons that can be glued from  $n$  squares. ◀



57 **Figure 2** An example of an octagon produced by cutting angles of a rectangle

58 We implemented an algorithm that enumerates all the gluings of at most  $n$  squares for a  
 59 given graph structure of a convex polyhedron. It showed that one gluing can admit several  
 60 ways to cut itself into flat polygons, see Figure 3. Thus it can appear in the list several times.



61 (a)

61 (b)

62 **Figure 3** Doubly covered parallelogram can be cut into two flat quadrilaterals in two ways, the  
 63 latter consisting of its faces

#### 64 **4 Algorithm to classify edge-to-edge gluings of squares**

65 The algorithm consists of the following steps:

- 66 1. Generate the list of all edge-to-edge gluings of at most  $n$  squares, denote it  $L(n)$ . Due to  
 67 Theorem 3, this step takes polynomial time.
- 68 2. For each gluing in  $L(n)$ , generate matrix of pairwise distances between its vertices. Due  
 69 to Theorem 2, this step takes  $O(n^3)$  time per gluing.
- 70 3. Unicalize the list of matrices up to homothety and permutation of rows and columns,  
 71 leave only corresponding elements of  $L(n)$ . Since the matrices are of at most 8 rows and  
 72 8 columns, it takes polynomial time to remove duplicates from the list.

73 The output of this algorithm is the list of all non-isomorphic edge-to-edge gluings of at  
 74 most  $n$  squares.

#### 75 **5 Discussion**

76 The cornerstone of the technique we have been using is the possibility to draw a face of a  
 77 polyhedron glued from squares on a planar grid. It allows us to estimate the number of valid  
 78 gluings. The same technique can seemingly be applied for the cases of regular hexagons and  
 79 triangles, since these polygons also tile the plane.

#### 80 **References**

- 81 1 Alexandr Alexandrov. *Convex Polyhedra*. Springer-Verlag, Berlin, 2005.

## XX:4 Convex Polyhedra Glued from Squares

- 82 2 E. Arseneva, S. Langerman, and B. Zolotov. A complete list of all convex shapes made by  
83 gluing regular pentagons. In *XVIII Spanish Meeting on Computational Geometry*, page 1–4,  
84 Girona, Spain, 2019.
- 85 3 Elena Arseneva and Stefan Langerman. Which Convex Polyhedra Can Be Made by Glu-  
86 ing Regular Hexagons? *Graphs and Combinatorics*, page 1–7, 2019. doi:DOI:10.1007/  
87 s00373-019-02105-3.
- 88 4 Jindong Chen and Yijie Han. Shortest paths on a polyhedron. In *6-th annual symposium on*  
89 *Computational geometry*, page 360–369, Berkley, California, USA, June 1990. SCG '90.
- 90 5 Erik Demaine, Martin Demaine, Anna Lubiw, and Joseph O'Rourke. Enumerating foldings  
91 and unfoldings between polygons and polytopes. *Graphs and Combinatorics*, 18(1):93–104,  
92 2002.
- 93 6 Erik Demaine, Martin Demaine, Anna Lubiw, Joseph O'Rourke, and Irena Pashchenko.  
94 Metamorphosis of the cube. In *Proc. SOCG*, pages 409–410. ACM, 1999.
- 95 7 Erik Demaine and Joseph O'Rourke. *Geometric folding algorithms*. Cambridge University  
96 Press, 2007.
- 97 8 David Eppstein, Michael J Bannister, William E Devanny, and Michael T Goodrich. The  
98 Galois complexity of graph drawing: Why numerical solutions are ubiquitous for force-directed,  
99 spectral, and circle packing drawings. In *International Symposium on Graph Drawing*, pages  
100 149–161. Springer, 2014.
- 101 9 Daniel M Kane, Gregory N Price, and Erik D Demaine. A Pseudopolynomial Algorithm for  
102 Alexandrov's Theorem. In *WADS*, pages 435–446. Springer, 2009.