Davenport—Schinzel Sequences

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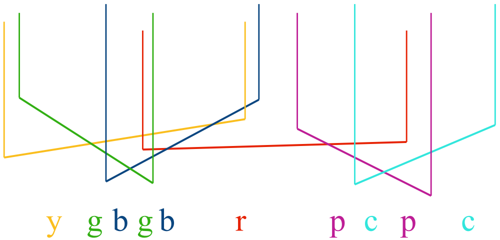
Definition

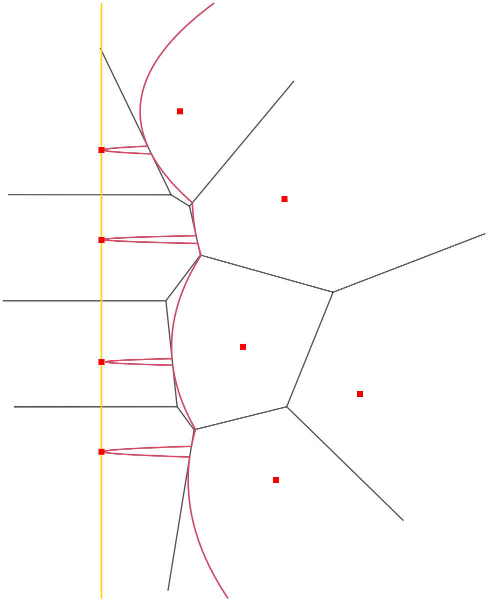
D—S sequence of order n: no *sebsequences* $a \ldots b \ldots a \ldots b \ldots$ of length $n+2$

Motivation: lower envelope of continuous functions that coincide at no more than n points.



For each segment we can construct the set of functions.

Voronoi diagrams and the beach line



Changes and cost model

$$\textbf{Insert}(a_{k+1})$$

$$\textbf{Relabel}(a_j, a_{k+1}, S)$$

$$\textbf{Delete}: \quad a_{k+1}a_{k+1} \;\longrightarrow\; a_{k+1}$$

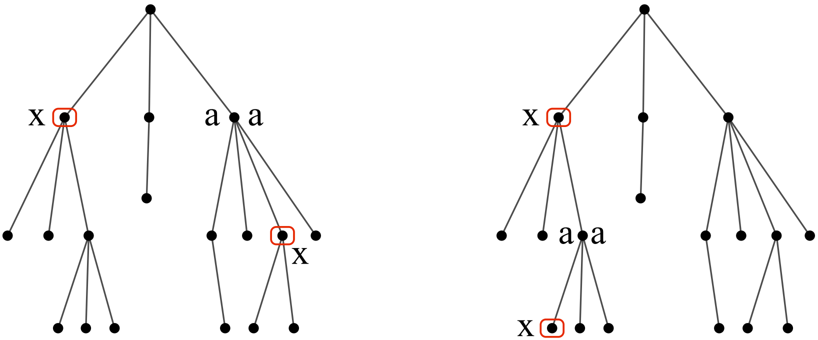
Two Inserts per step

Restate that maximal length of D—S sequence is $2n - 1$ in «amortized fashion».

- Between two occurences of letter x some letters are «trapped».
- We give 2 pennies to each letter initially.
- Idea: each letter pays for its father.
- Invariants: each letter with k instances receives at least $k - 1$ pennies from its descendants, can pay one to its father.
- Finally, all the occurences are paid for.

Location of Relabels

Letters that we relabel form a union of subtrees with their roots located on the same level and having a common father.



What happens to the tree when we relabel

We can think that relabels happen from top down: this order does not disrupt the D—S property.

What happens to the tree when we relabel

When we relabel vertices on lower levels we make their children our clidren. We can think of it as a subtree contracted into a single vertex.

Size and potential

Define size of the node as number of its children $+1$.

$$2n \geq \sum_v \mathrm{size}(v) \geq \mathrm{length}(S)$$

Define the potential function as

$$\sum_v \min\left(\mathrm{size}(v), \sqrt{n}\right)$$

$\sqrt{n}$ Relabels amortized

Each time we relabel we steal someone's children or delete a node. Let s be the number of children transferred.

$$
\begin{aligned}
& s + \Phi_{\text{new}} - \Phi_{\text{old}} \leq \\
& \leq s + \sqrt{n} - (s - \sqrt{n}).
\end{aligned}
$$

But is that really $\sqrt{n}$?