Recent research achievements

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Advanced Mathematics

Friday, December 20

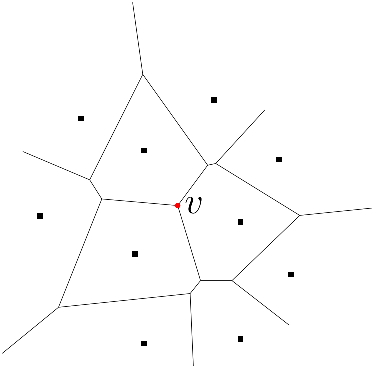
Part 1

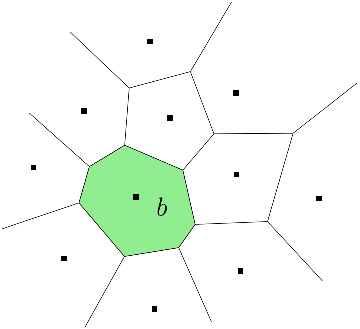
Sublinear Explicit Incremental Voronoi Diagrams

Voronoi Diagram: basics

Voronoi diagram of $S = \{s_1, \ldots, s_N\}$: subdivision of the Euclidean plane, for each q inside the *cell* of s_i

$$\mathrm{dist}(q, s_i) < \mathrm{dist}(q, s_j), \quad j \neq i.$$





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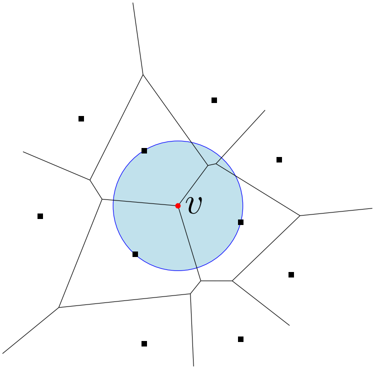

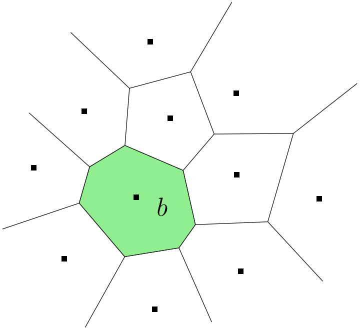

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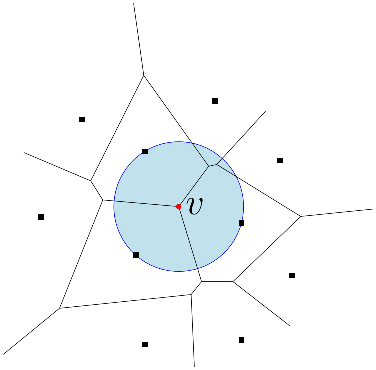

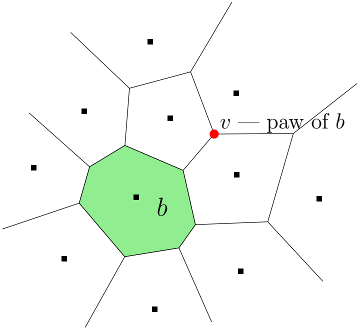

Dynamic VD: setting

Incremental Voronoi Diagram

Maintain initially empty Voronoi diagram under insertion of new sites

There is a linear-time solution, no faster solutions possible: there may be $O(N)$ changes per insertion.

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Incremental Combinatorial Voronoi Diagram

Maintain the graph of initially empty Voronoi diagram under insertion of new sites

There is a naive linear-time algorithm. Can we find a faster solution?

Dynamic VD: combinatorial changes

Theorem (Allen et al. 2017)

The number of cells with combinatorial changes is $O(N^{\frac{1}{2}})$ *amortized, there are a constant number of combinatorial changes per cell, cells with changes are connected.*

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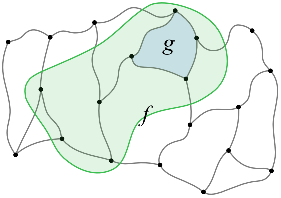
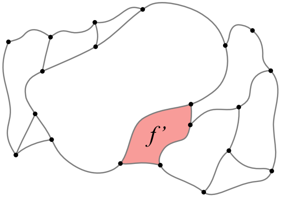
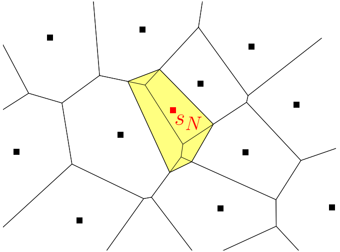
When we are inserting new site s_N, the graph of the VD undergoes operation called *flarb*.



(a) Graph before flarb



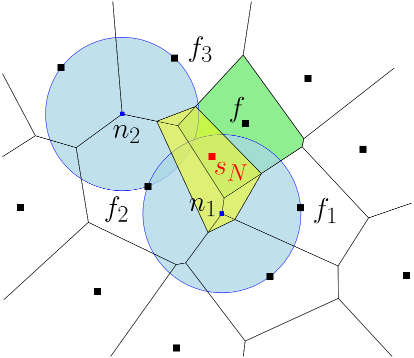
(b) Graph after flarb



(c) Flarb on a VD

Identifying changes

Theorem: Let g be a cell adjacent to f. Cell g needs to undergo changes $\iff$ circle of either of its vertices that are paws of f encloses s_N. Allen et al. 2017



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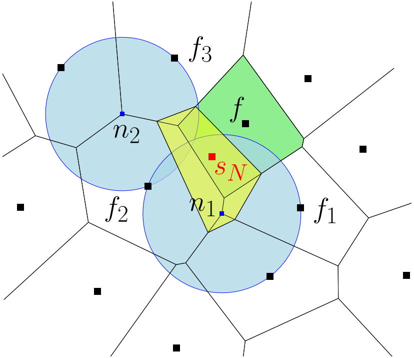

Dynamic circle reporting structure (Chan 2010; Allen et al. 2017)

Returns all k circles enclosing given point in $\tilde{O}(k)$. Addition and deletion of a circle in $\tilde{O}(1)$.

Description of data structure

Definition

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(1) Graph of VD in a form of adjacency list,

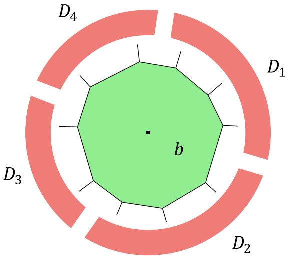
(2) *Dynamic nearest neighbor* structure for sites,

(3) Graph of big cells stored as adjacency list,

(4) For each big cell:

– linked list of DCRs,

– binary search tree of vertices in circular order.



Handling insertion

Initialize queue; look at cells one-by-one, on each step add to it unprocessed cells with changes, then implement changes.

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$$\text{Time complexity:} \quad \tilde{O}\left(sN^{\frac{1}{4}} + \sum_{i=1}^{|B|}\left(\left\lceil \frac{|b_i|}{N^{\frac{1}{4}}} \right\rceil + \ell_i\right) + N^{\frac{3}{4}} + sN^{\frac{1}{4}}\right).$$

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This is $\tilde{O}(N^{\frac{3}{4}})$ amortized.

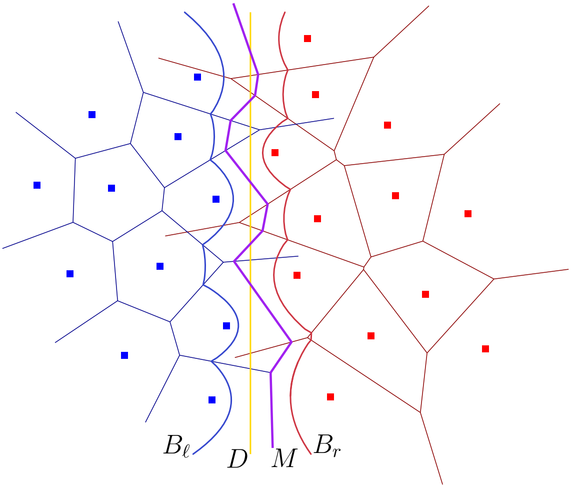
Ongoing research:

Õptimal-Time Incremental Voronoi Diagrams

Divide-and-conquer

Time complexity $\tilde{O}(N^{\frac{3}{4}})$ is not bad, but the lower bound is $N^{\frac{1}{2}}$. Can we meet it?

The idea is to use divide-and-conquer approach. It can allow us to omit doing unnecessary work.



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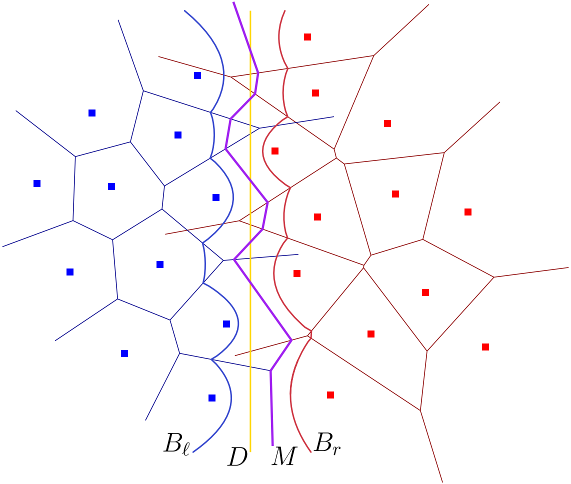


(1) Dividing line,

(2) Beach lines,

(3) Blue and red forests,

(4) Merge curve.

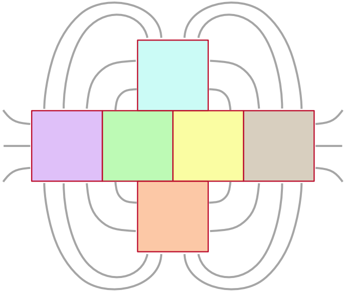
Previous research:

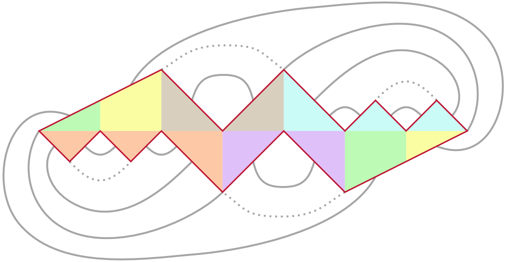
Polyhedra glued from regular pentagons

Gluings

Definition (Alexandrov 1950)

A *gluing* is a set of polygons equipped with a number of rules describing the way edges of these polygons must be glued to each other.

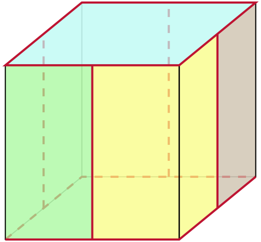


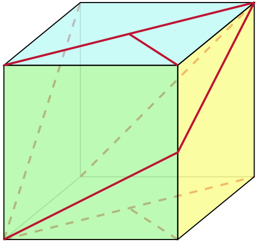


Alexandrov's Uniqueness Theorem

Theorem (Alexandrov 1950)

If a gluing is homeomorphic to a sphere and the sum of angles at each of its vertices is $\leq 360^\circ$*, there* $\exists\,!$ *convex polyhedron that corresponds to this gluing.*





Polyhedron Reconstruction

The proof of Alexandrov's theorem is non-constructive.

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Each of the following problems is still open:

Cauchy Rigidity Problem (Demaine and O'Rourke 2007, 23.22)

Given edge lengths of a triangulated convex polyhedron, output approximate coordinates of its vertices *in polynomial time.*

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Skeleton Reconstruction Problem

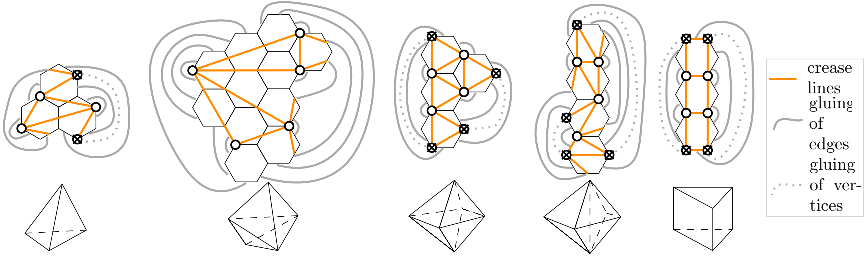
Given a net satisfying the conditions of Alexandrov's Theorem, find the skeleton of the convex polyhedron corresponding to it *in polynomial time.*

Edge-to-Edge Gluings of Regular Polygons

Maybe Alexandrov's Problem becomes easier, if we start building the gluings from similar simple blocks?

Definition

A gluing is *edge-to-edge* if every edge is glued to another entire edge.

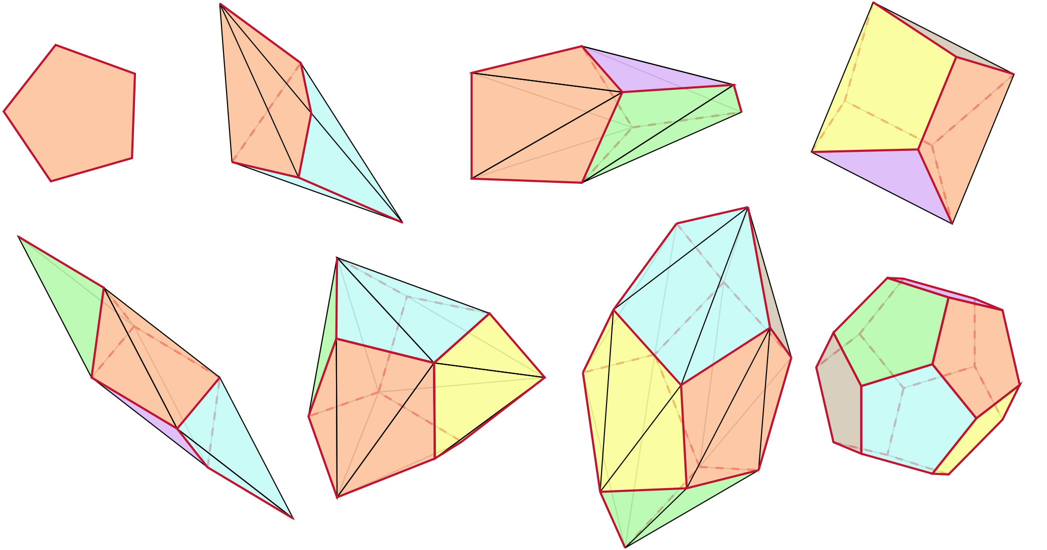


Khramtcova and Langerman 2017

Polyhedra glued from regular pentagons

One can enumerate all gluings of pentagons. Here are all the polyhedra that can be obtained:

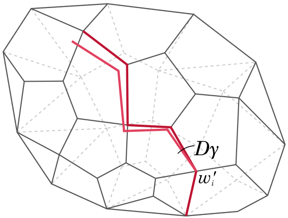
Arseneva, Langerman, **Zolotov,** 2019

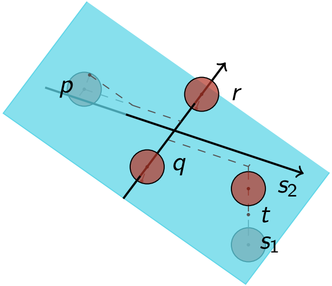


Determining the shape from the gluing

Theorem

Each vertex of $\mathcal{P}$ lies within an r-ball centered at the corresponding vertex of P, where $r = E^2 \cdot L \cdot 2\sin(\mathcal{D}\gamma/2) + E\mu$.



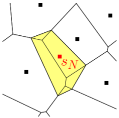


This theorem allows to create a procedure for checking whether there is a given edge in $\mathcal{P}$. The procedure was implemented.

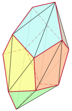
Conclusion

Results achieved:

(1) First sublinear algorithm for maintaining the graph of a Voronoi diagram

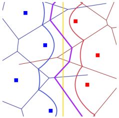


(2) Full list of the polyhedra that are made by edge-to-edge gluings of regular pentagons



Ongoing research:

(3) Algorithm for maintaining the graph of a Voronoi Diagram with the optimal time complexity



Thank you for your attention!

Allen, S. R., Barba, L., Iacono, J., and Langerman, S. (2017). Incremental voronoi diagrams. *Discrete & Computational Geometry*, 58(4):822–848.

Chan, T. M. (2010). A dynamic data structure for 3-d convex hulls and 2-d nearest neighbor queries. *J. ACM*, 57(3):16:1–16:15.

Demaine, E. and O'Rourke, J. (2007). *Geometric folding algorithms*. Cambridge University Press.

Khramtcova, E. and Langerman, S. (2017). Which convex polyhedra can be made by gluing regular hexagons? In *JCDCG*^{3}.