Recent research achievements

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Advanced Mathematics

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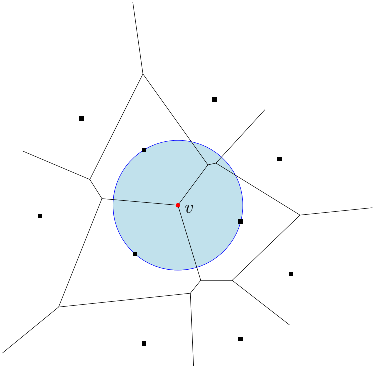
Sublinear Explicit Incremental Voronoi Diagrams

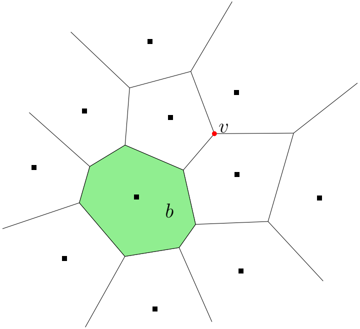
Voronoi Diagram: basics

Voronoi diagram of $S = \{s_1, \ldots, s_N\}$: subdivision of the Euclidean plane,

$$\mathrm{dist}(q, s_i) < \mathrm{dist}(q, s_j), \quad j \neq i.$$



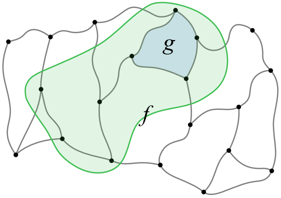
(a) Voronoi diagram, Voronoi vertex, circle



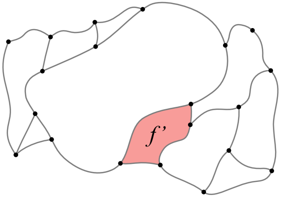
(b) v is a paw of b

Dynamic VD setting

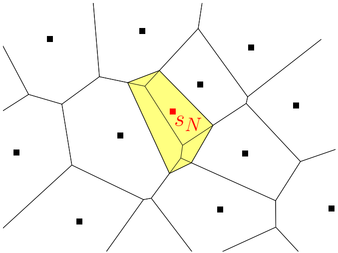
When we are inserting new site s_N, the graph of the VD undergoes operation called *flarb.*



(a) Graph before flarb



(b) Graph after flarb



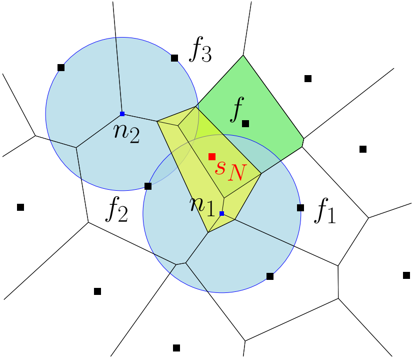
(c) Flarb on a VD

Theorem (Allen et al. 2017)

The number of cells with combinatorial changes is $O(N^{\frac{1}{2}})$ amortized, there are a constant number of combinatorial changes per cell, cells with changes are connected.

Identifying changes

Theorem: Let g be a cell adjacent to f. Cell g needs to undergo changes $\iff$ circle of either of its vertices that are paws of f encloses s_N. Allen et al. 2017



Dynamic circle reporting structure (Allen et al. 2017)

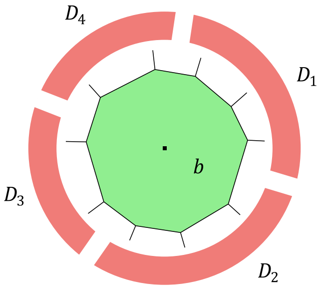
Returns all k circles enclosing given point in $\tilde{O}(k)$. Addition and deletion of a circle in $\tilde{O}(1)$.

Big cells and small cells

Definition

A cell is *big* if it has size at least $N^{\frac{1}{4}}$. Otherwise it is *small*.

Small cells can be processed by brute force. For big cells we will need DCRs.



Description of data structure

1. Graph of VD in a form of adjacency list,
2. *Dynamic nearest neighbor* structure for sites,
3. Graph of big cells stored as adjacency list,
4. For each big cell:
   - linked list of DCRs,
   - binary search tree of vertices in circular order.

Handling insertion

Initialize queue containing all the big cells and the cell of site closest to s_N. Look at cells one-by-one, on each step add to the queue unprocessed cells with changes. After all cells needing changes are in the queue, implement changes in them.

- Small cell: look at each paw to find neighboring cells needing changes, add them to the queue.
- Big cell: ask DCRs to return Voronoi circles that enclose s_N, add corresponding cells to the queue. When implementing changes use graph of big cells to identify edges that have to be deleted. Use BSTs to find the vertices that now belong to the cell of s_N.

Amortized expected runtime

$$\tilde{O}\left(sN^{\frac{1}{4}} + \sum_{i=1}^{|B|}\left(\left\lceil\frac{|b_i|}{N^{\frac{1}{4}}}\right\rceil + \ell_i\right) + N^{\frac{3}{4}} + sN^{\frac{1}{4}}\right).$$

- s is $O(N^{\frac{1}{2}})$ amortized

Allen et al. 2017,

- $\sum_{i=1}^{|B|} |b_i| \leq N$, $|B| \leq N^{\frac{3}{4}}$,
- $\sum_{i=1}^{|B|} \ell_i \leq sN^{\frac{1}{4}}$.

So this is simply

$$\tilde{O}\left(N^{\frac{3}{4}}\right) \quad \text{amortized.}$$

Ongoing research:

Õptimal-Time Incremental Voronoi Diagrams

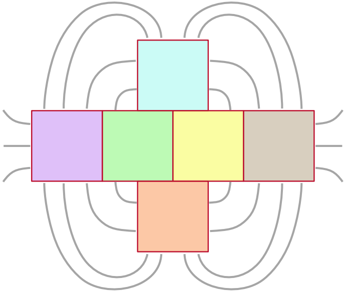
Previous research:

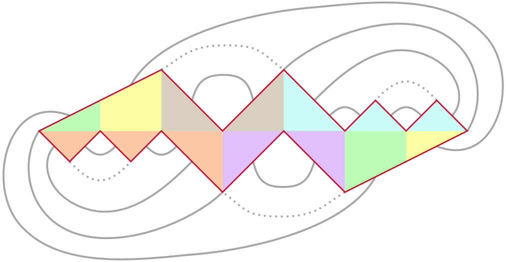
Polyhedra glued from regular pentagons

Gluings

Definition (Alexandrov 1950)

A *gluing* is a set of polygons equipped with a number of rules describing the way edges of these polygons must be glued to each other.

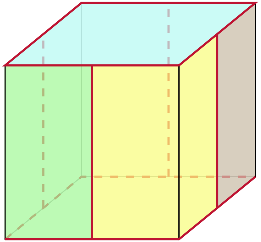


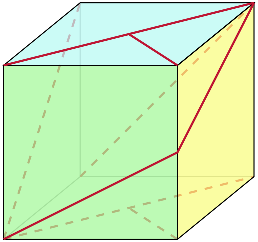


Alexandrov's Theorem

Theorem (Alexandrov 1950)

If a gluing is homeomorphic to a sphere and the sum of angles at each of its vertices is $\leq 360^{\circ}$*, there* $\exists!$ *convex polyhedron that corresponds to this gluing.*





Polyhedron Reconstruction

The proof of Alexandrov's theorem is non-constructive.

Each of the following problems is still open:

Alexandrov's Problem

Given a gluing $\mathcal{T}$ satisfying the conditions of Alexandrov's Theorem, find the convex polyhedron corresponding to it.

Cauchy Rigidity Problem (Demaine and O'Rourke 2007, 23.22)

A *poly-time* algorithm that takes as input edge lengths of a triangulated convex polyhedron, and outputs approximate coordinates of its vertices.

Skeleton Reconstruction Problem

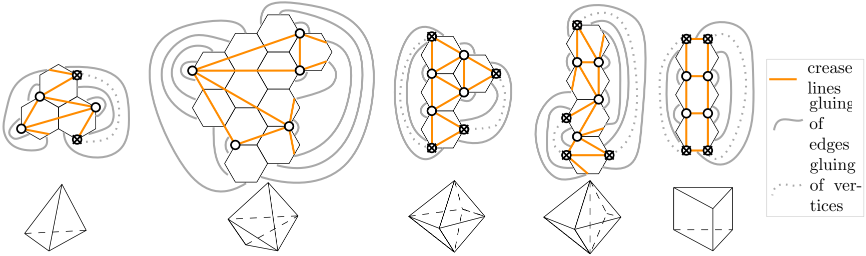
A *poly-time* algorithm that, given a net satisfying the conditions of Alexandrov's Theorem, *finds the skeleton* of the convex polyhedron corresponding to it.

Edge-to-Edge Gluings of Regular Polygons

Maybe Alexandrov's Problem becomes easier, if we start building the gluings from similar simple blocks?

Definition

A gluing is *edge-to-edge* if every edge is glued to another entire edge.

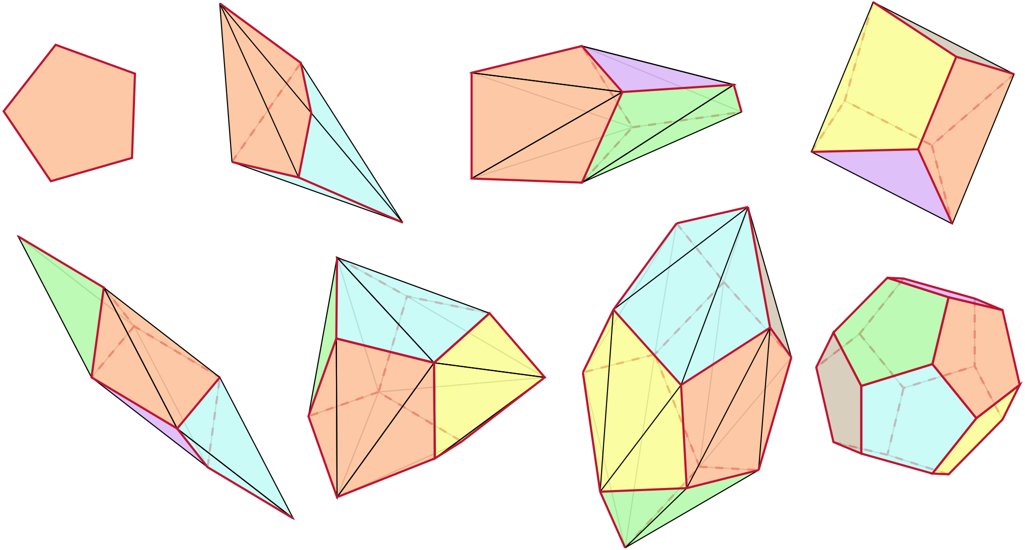


Khramtcova and Langerman 2017

Polyhedra glued from regular pentagons

One can enumerate all gluings of pentagons. Here are all the polyhedra that can be obtained:

Zolotov et al. 2019



Precision of vertex location based on the approximation

Theorem

If $\mathcal{D}\gamma < \pi/2$, then each vertex of $\mathcal{P}$ lies within an r-ball centered at the corresponding vertex of P, where

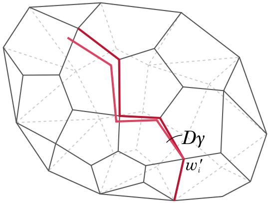
$$r = E^2 \cdot L \cdot 2\sin(\mathcal{D}\gamma/2) + E\mu.$$

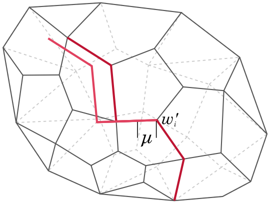
$\mathcal{P}$ — convex polyhedron corresponding to a given net,

P — approximation of $\mathcal{P}$. $\mathcal{D}$ — max degree of a vertex of P.

μ — max edge discrepancy between P and $\mathcal{P}$,

γ — max face angle discrepancy.

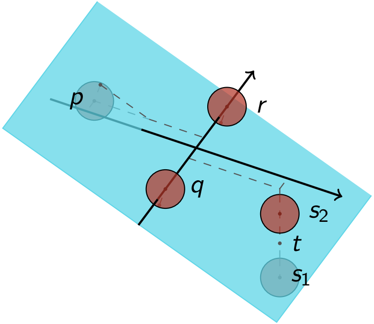




Determining the shape from the gluing

Let pqr, sqr be adjacent faces of P.
Are there edges ps, qr in $\mathcal{P}$?

Assume pqr is not vertical and P lies *below it*.
Consider three planes Π_1, Π_2, Π_3 tangent to $B_r(p)$, $B_r(q)$, $B_r(r)$.



Theorem

If s lies below Π_1, Π_2 and Π_3 and the distance from s to each of the planes Π_1, Π_2 and Π_3 is greater than r, then the edge qr must be in $\mathcal{P}$.

Implemented on *Haskell*.

Geometric methods can also be used.

Allen, S. R., Barba, L., Iacono, J., and Langerman, S. (2017). Incremental voronoi diagrams. *Discrete & Computational Geometry*, 58(4):822–848.

Demaine, E. and O'Rourke, J. (2007). *Geometric folding algorithms*. Cambridge University Press.

Khramtcova, E. and Langerman, S. (2017). Which convex polyhedra can be made by gluing regular hexagons? In *JCDCG*^{3}.

Zolotov, B., Arseneva, E., and Langerman, S. (2019). A complete list of all convex shapes made by gluing regular pentagons. In *XVIII Spanish Meeting on Computational Geometry*, page 26–29, Girona, Spain.