# Folding Polygons to Polyhedra
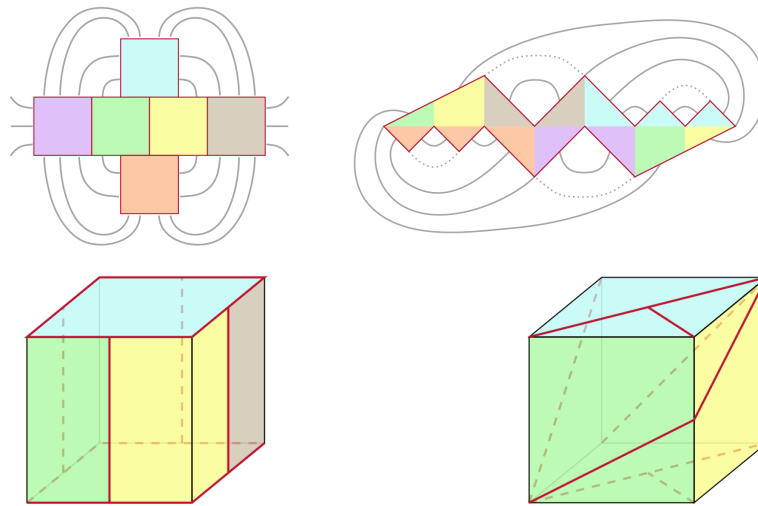
B. Zolotov, Ph. D. English Course

## Alexandrov's Theorem

Is always brought up when it comes to gluing things.

If a gluing is homeomorphic to a sphere
and the sum of angles at each of
its vertices is $\leq 360°$, there $\exists!$ convex
polyhedron that corresponds to this gluing.

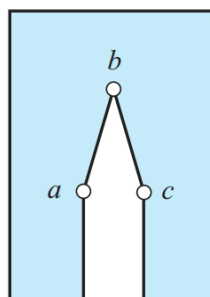## Alexandrov's Theorem — Examples



## Folding a single polygon

As we can see from the previous example,
a single polygon can be glued to itself,
the same conditions remain.

We will be *only* considering
foldability to *convex* polyhedra.

## A not-foldable polygon

Consider where the reflex vertex can be glued?..

## Foldability of random polygons

No matter how «a random polygon» is sampled,
we are expecting that it:

- has at least two reflex vertices on average,

- the edge lengths are drawn from a
  cont. density distribution.

The probability that a random $n$-gon can be folded into a convex polyhedron approaches 0 as
$n \to \infty$.

## Perimeter-halving

*Any convex* polygon can be folded into
a convex polyhedron by *perimeter halving.*

By varying the points half a perimeter apart we get
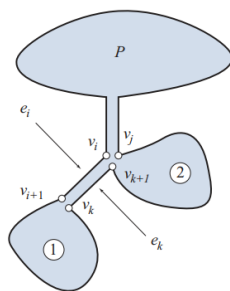uncountably many *incongruent* polyhedra.

## Edge-to-edge folding

Let us consider the subcase when entire edges
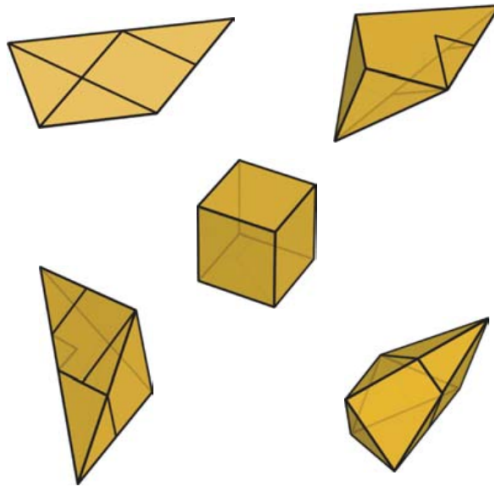must be glued to entire edges.

There is a dynamic programming algorithm
to list all the gluings.

## Dynamic programming

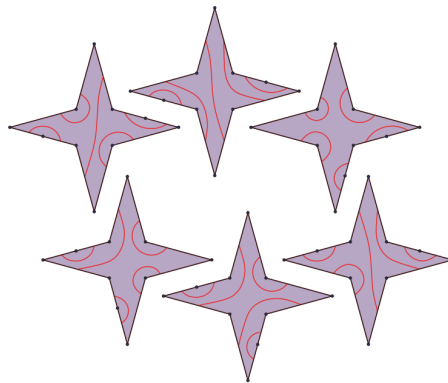Idea: find a pair of matching edges, check angles,
reduce to two subproblems.



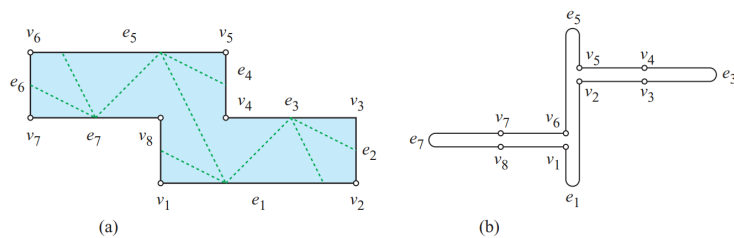## All e2e gluings of the Latin cross

## General algorithm

There is also an algorithm that lists *all* the foldings;
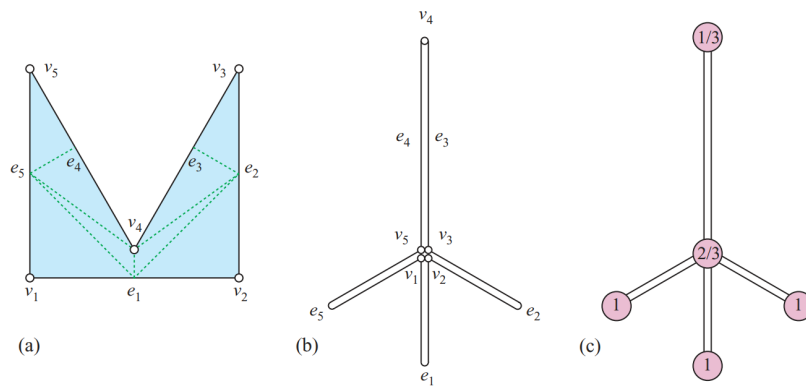it has exponential running time.



## Gluing tree

Shows how the boundary of a folded polygon
corresponds to itself.   How to find the vertices
of the polyhedron by looking at
the vertices of the gluing tree?



(a)                                    (b)

## Analyse the curvature
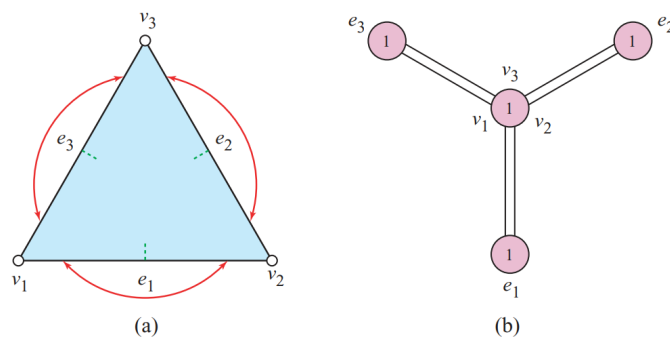
(a)    (b)    (c)

## Rolling belt

A path in the gluing tree that:

- Connects two leaves of the gluing tree
  that are either convex-vertex or
  fold-point leaves (positive deficit),

- The face angle to each side of the path is,
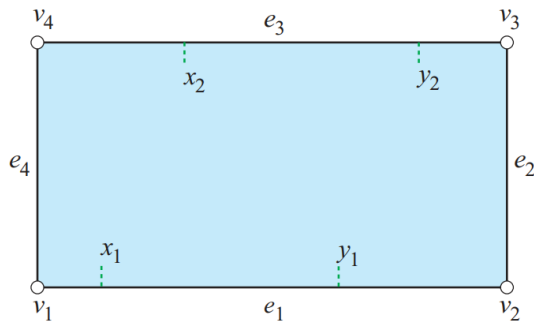  at every point, convex.

## Rolling belts: examples

Half of perimeter of any convex polygon or
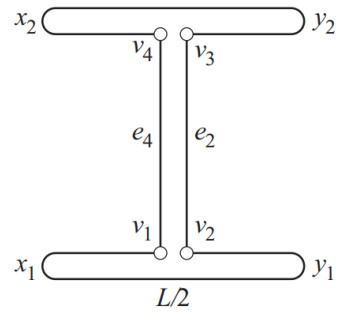


(a)    (b)

## Gluing tree properties

- Each leaf is either a vertex of the polygon,
  or a fold point.
- At most one nonvertex may be glued at any
  gluing-tree junction of degree $d \geq 3$.
- A gluing tree may have at most two rolling belts
  with distinct endpoints.
- The case of four fold-point leaves is possible
  only under special circumstances.
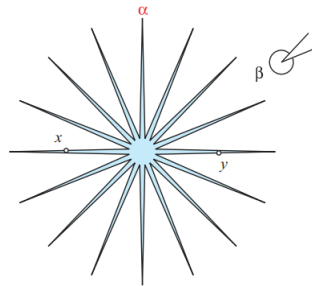
## Rectangle with two rolling belts
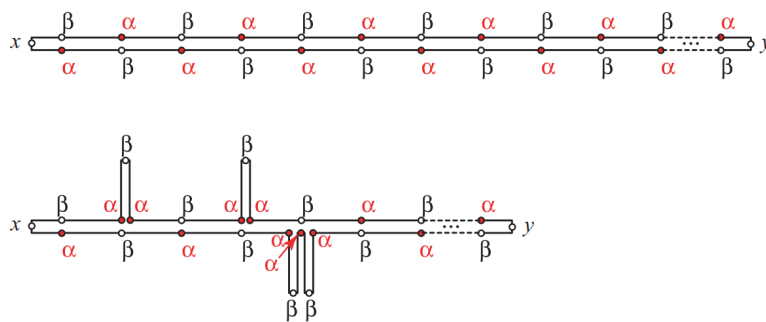
(a)            (b)

## Exponential number of gluing trees

For any even $n$, there is a polygon $P$ of $n$ vertices that has $2^{\Omega(n)}$ combinatorially distinct edge-to-edge Alexandrov gluings.



## Contracted gluing trees

There is the default gluing tree, it can be contracted in an exponential number of ways to form new trees.



## Thank you for your attention!

- Not-foldable polygon, perimeter halving

- Algorithms listing foldings

- Gluing trees, rolling belts

- Exponentially many foldings
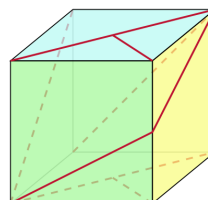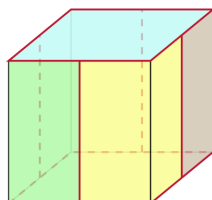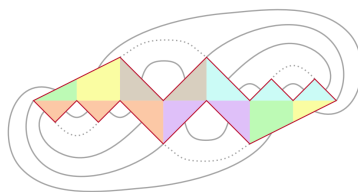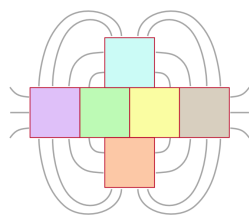
# Folding Polygons to Polyhedra

B. Zolotov, Ph. D. English Course

## Alexandrov's Theorem

Is always brought up when it comes to gluing things.

If a gluing is homeomorphic to a sphere
and the sum of angles at each of
its vertices is $\leq 360°$, there $\exists!$ convex
polyhedron that corresponds to this gluing.
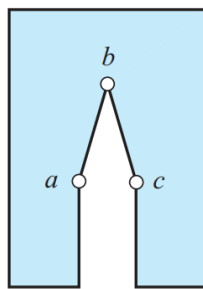
# Alexandrov's Theorem — Examples

# Folding a single polygon

As we can see from the previous example, a single polygon can be glued to itself, the same conditions remain.

We will be *only* considering foldability to *convex* polyhedra.

# A not-foldable polygon



Consider where the reflex vertex can be glued?..

## Foldability of random polygons

No matter how «a random polygon» is sampled,
we are expecting that it:

- has at least two reflex vertices on average,

- the edge lengths are drawn from a
  cont. density distribution.

The probability that a random $n$-gon can be folded into
a convex polyhedron approaches 0 as $n \to \infty$.

# Perimeter-halving

*Any convex* polygon can be folded into
a convex polyhedron by *perimeter halving.*

By varying the points half a perimeter apart we get
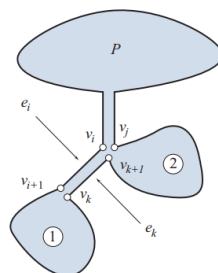uncountably many *incongruent* polyhedra.

# Edge-to-edge folding

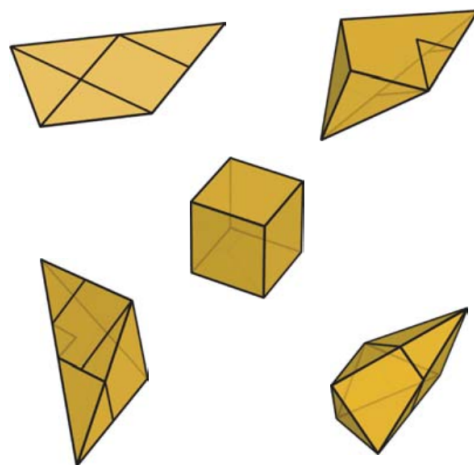Let us consider the subcase when entire edges must be glued to entire edges.

There is a dynamic programming algorithm to list all the gluings.

# Dynamic programming

Idea: find a pair of matching edges, check angles, reduce to two subproblems.
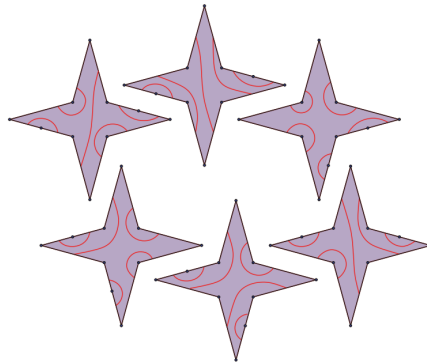
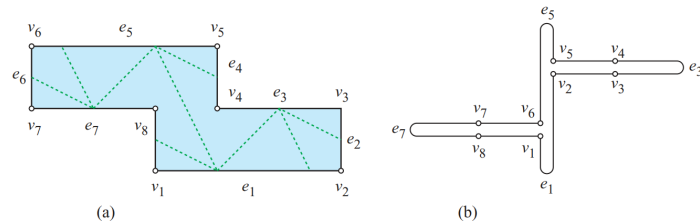# All e2e gluings of the Latin cross

# General algorithm

There is also an algorithm that lists *all* the foldings;
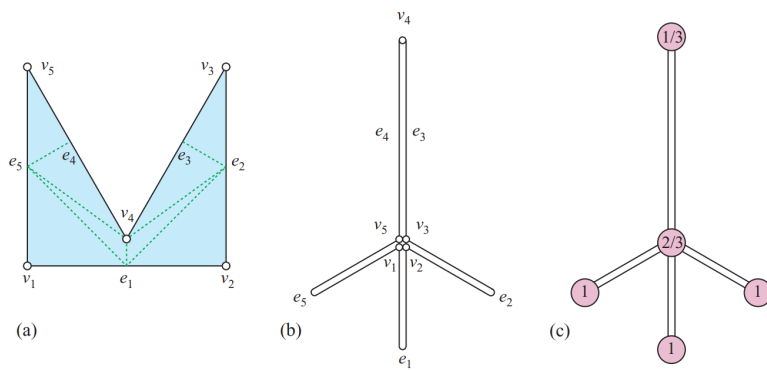it has exponential running time.

# Gluing tree

Shows how the boundary of a folded polygon
corresponds to itself.    How to find the vertices
of the polyhedron by looking at
the vertices of the gluing tree?



(a)                                    (b)

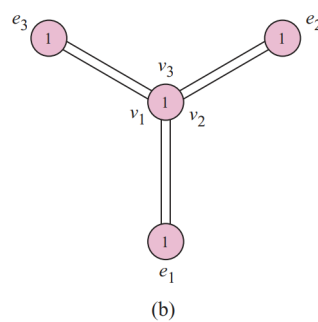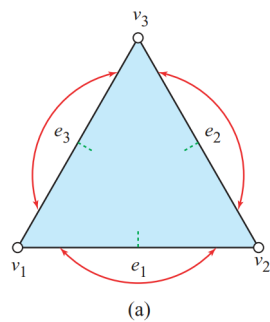# Analyse the curvature



(a)　　　(b)　　　(c)

# Rolling belt

A path in the gluing tree that:

- Connects two leaves of the gluing tree
  that are either convex-vertex or
  fold-point leaves (positive deficit),

- The face angle to each side of the path is,
  at every point, convex.
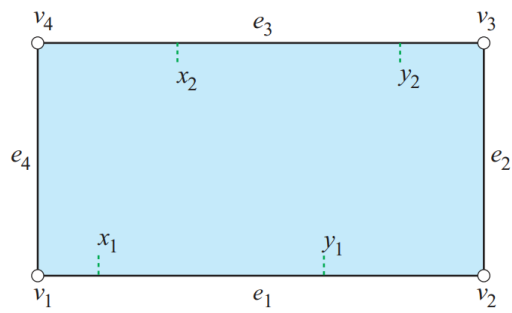
# Rolling belts: examples

## Half of perimeter of any convex polygon or
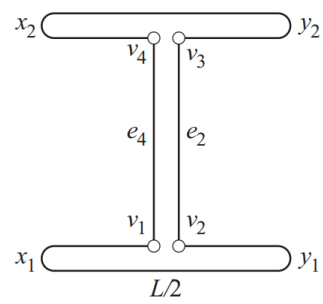


(a)                    (b)

## Gluing tree properties

- Each leaf is either a vertex of the polygon, or a fold point.
- At most one nonvertex may be glued at any gluing-tree junction of degree $d \geq 3$.
- A gluing tree may have at most two rolling belts with distinct endpoints.
- The case of four fold-point leaves is possible only under special circumstances.
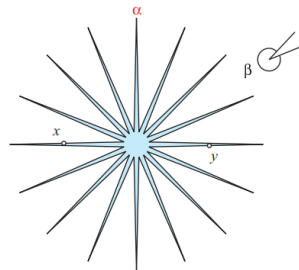
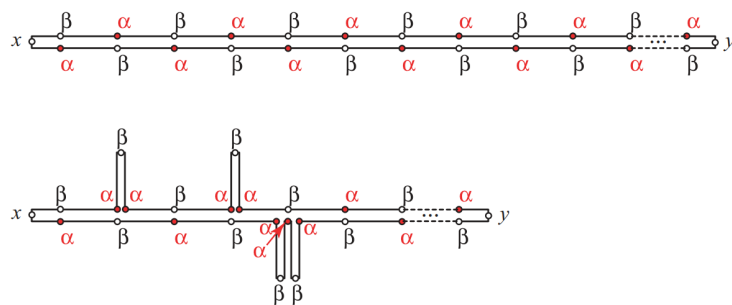# Rectangle with two rolling belts



(a)

(b)

# Exponential number of gluing trees

For any even $n$, there is a polygon $P$ of $n$ vertices that has $2^{\Omega(n)}$ combinatorially distinct edge-to-edge Alexandrov gluings.

# Contracted gluing trees

There is the default gluing tree, it can be contracted in an exponential number of ways to form new trees.

# Thank you for your attention!

- Not-foldable polygon, perimeter halving

- Algorithms listing foldings

- Gluing trees, rolling belts

- Exponentially many foldings