Folding Polygons to Polyhedra

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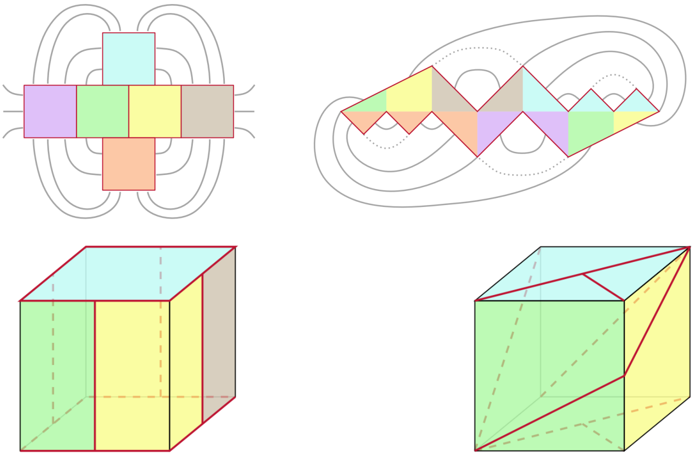
Alexandrov's Theorem

Is always brought up when it comes to gluing things.

If a gluing is homeomorphic to a sphere
and the sum of angles at each of
its vertices is $\leq 360^\circ$, there $\exists !$ convex
polyhedron that corresponds to this gluing.

Alexandrov’s Theorem — Examples

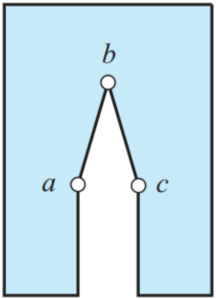


Folding a single polygon

As we can see from the previous example, a single polygon can be glued to itself, the same conditions remain.

We will be *only* considering foldability to *convex* polyhedra.

A not-foldable polygon



Consider where the reflex vertex can be glued?..

Foldability of random polygons

No matter how «a random polygon» is sampled,
we are expecting that it:

- has at least two reflex vertices on average,
- the edge lengths are drawn from a cont. density distribution.

The probability that a random n-gon can be folded into a convex polyhedron approaches 0 as $n \to \infty$.

Perimeter-halving

Any convex polygon can be folded into
a convex polyhedron by *perimeter halving*.

By varying the points half a perimeter apart we get uncountably many *incongruent* polyhedra.

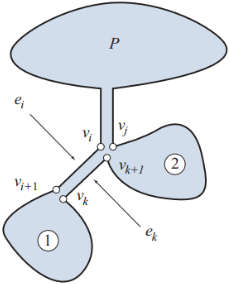
Edge-to-edge folding

Let us consider the subcase when entire edges must be glued to entire edges.

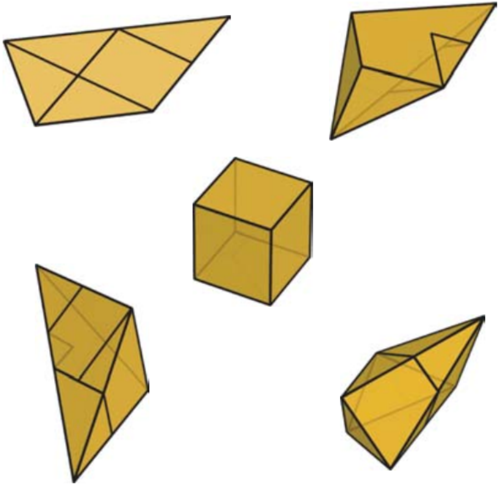
There is a dynamic programming algorithm to list all the gluings.

Dynamic programming

Idea: find a pair of matching edges, check angles, reduce to two subproblems.

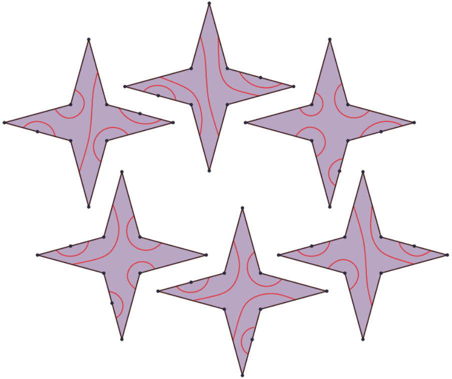


All e2e gluings of the Latin cross



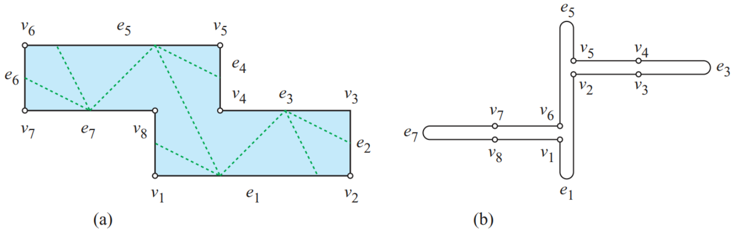
General algorithm

There is also an algorithm that lists *all* the foldings;
it has exponential running time.

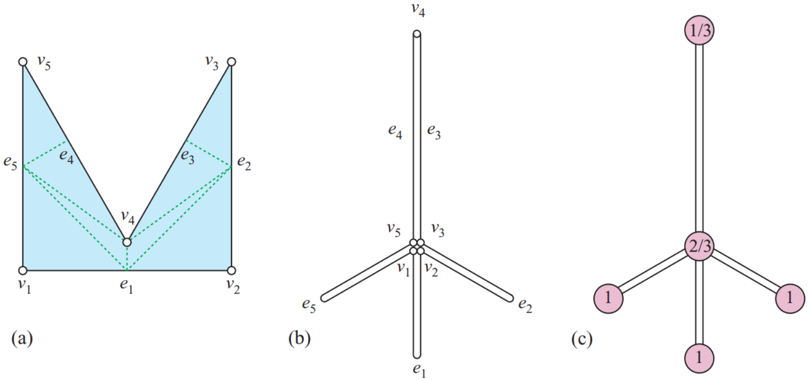


Gluing tree

Shows how the boundary of a folded polygon corresponds to itself. How to find the vertices of the polyhedron by looking at the vertices of the gluing tree?



Analyse the curvature



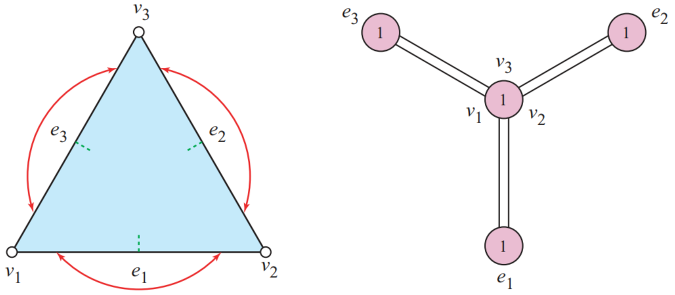
Rolling belt

A path in the gluing tree that:

- Connects two leaves of the gluing tree that are either convex-vertex or fold-point leaves (positive deficit),
- The face angle to each side of the path is, at every point, convex.

Rolling belts: examples

Half of perimeter of any convex polygon or

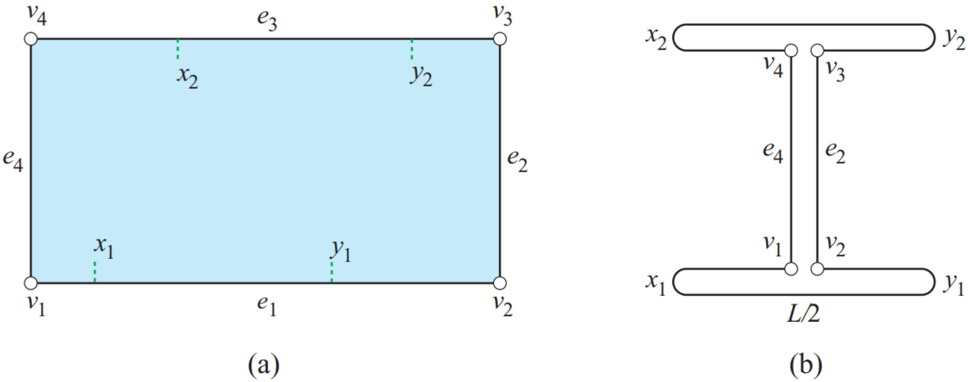


(a) (b)

Gluing tree properties

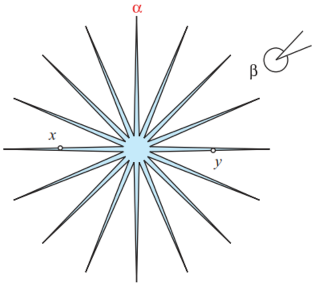
- Each leaf is either a vertex of the polygon, or a fold point.
- At most one nonvertex may be glued at any gluing-tree junction of degree $d \geq 3$.
- A gluing tree may have at most two rolling belts with distinct endpoints.
- The case of four fold-point leaves is possible only under special circumstances.

Rectangle with two rolling belts



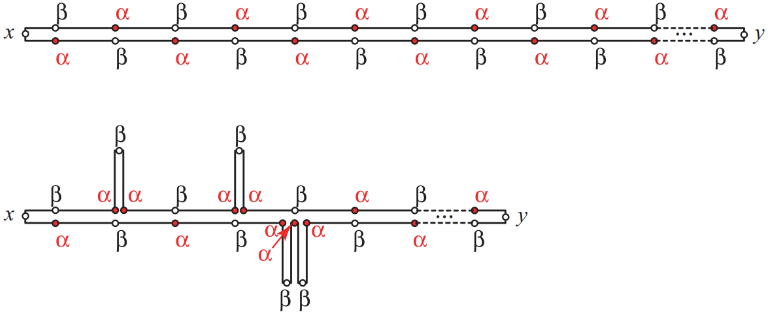
Exponential number of gluing trees

For any even n, there is a polygon P of n vertices that has $2^{\Omega(n)}$ combinatorially distinct edge-to-edge Alexandrov gluings.



Contracted gluing trees

There is the default gluing tree, it can be contracted in an exponential number of ways to form new trees.



Thank you for your attention!

- Not-foldable polygon, perimeter halving
- Algorithms listing foldings
- Gluing trees, rolling belts
- Exponentially many foldings